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TWO FAMILIES OF VALUES FOR GLOBAL GAMES

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Title: Two families of values for global games

Abstract: We propose new point valued solutions for global games. We explore the implications of weakening some of the properties used by Gilboa and Lehrer (1991) in their characterization result. Our main contributions are the axiomatic characterizations of two families of values for global games.

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1 Introduction

A cooperative game is build around the assumption that each possible coalition of agents can make binding agreements and operate as a single entity. One of the main research questions is how to share the worth generated by the agents that participate in a game. There is vast literature on the topic when the information available is the (transferable) utility that each subset of players generates. The axiomatic approach initiated by the seminal paper Shapley (1953) is the most common way to address the question. It is based on discussing desirable principles, described by formal properties that a sharing rule should reasonably satisfy.

In some cases, the available information is the utility that the whole set of players can generate depending on how they are organized in coalitions. This is precisely a global (cooperative) game as introduced in Gilboa and Lehrer (1991).[1] A global game does not specify the worth of every possible coalition but the overall utility generated by the coalition structure. This can be interesting when the focus is on the public good side of the cooperation rather than on the incentives of agents or coalitions. Think for instance on the climate change problem. A lot of effort has been put in order to analyze the incentives of the countries to implement carbon reduction policies. It is well known that even if the cooperation of all countries increases the social welfare, agents have incentives to free-ride (Barrett, 1994). Most of the contributions regarding the formation and stability of international environmental agreements use non-cooperative games (Finus, 2008). Nonetheless, cooperative games have shown to be useful, for instance to study the side payments between countries (Chander and Tulkens, 2006). We believe that global cooperative games can be used to study the consequences of the potentially different commitment levels of all the countries. A follow up question is how to asses the importance of each agent's participation in the eventual formation of the grand coalition. This is our main objective. Other papers that study problems closely related to global games include Caulier et al. (2015) and Rossi (2019).

Gilboa and Lehrer (1991) characterize a point valued solution concept for global games by means of four properties. Namely linearity, efficiency, symmetry, and the

[1] Not to be mixed up with the non-cooperative games with incomplete information introduced by Carlsson and Van Damme (1993).

null player property. The first two properties are very standard and we also impose them to any sensible solution. Nonetheless, we consider different and weaker versions of the remaining two properties. According to Gilboa and Lehrer (1991) two players are symmetric if their desertion from a coalition to remain alone has the same impact on the worth of any coalition structure. Then, the property requires that two such players get the same payoff. The property may be desirable in many contexts but we could argue that leaving a small coalition is not the same as leaving a larger one. We study the implications of replacing symmetry by anonymity in their characterization. Anonymity states that the payoffs in the permuted game should be equal to the permuted payoffs in the original game. This is our second main contribution, the characterization of the family of values that satisfy linearity, efficiency, anonymity, and the null player property. Our first main contribution is the characterization of the larger family of values that satisfy efficiency, linearity, and anonymity. We provide instances of values in each of these families. Finally, we explore the possibility of weakening the null player property. Gilboa and Lehrer (1991) consider that a player is null if the global worth is not affected by her leaving a coalition to remain alone. The property states that null players should get a zero payoff. Similar to the definition of symmetric players, they only consider movements of players from being alone to participating in a coalition. We explore the consequences of contemplating more general movements, like the merging of two coalitions of arbitrary sizes in the structure. This yields a very mild null player property. We show that it is implied by efficiency and anonymity.

Our results rely on the well known lattice of partitions. Formally, a global game is just a real valued function on the set of partitions of a finite set and the set of such games with a fixed player set is a vector space. Using the finer (or coarser) relation among partitions of a finite set it is easy to identify a basis of the vector space, parallel to the well known unanimity basis of classic cooperative games. And using the Möbius inversion formula of the lattice of partitions we explicitly write the coefficients of any game in this basis. Then, the linearity property that we impose allows us to focus on the payoffs of the games in the basis. This facilitates the construction of the two families of values that we propose. To conclude, we identify other values in these families besides the one proposed by Gilboa and Lehrer (1991).

We pin down some of them and illustrate their behavior by means of examples.

The rest of the paper is organized as follows. Section 2 presents the basics of coalitional games and the Shapley value. Section 3 revises the existing results on global games. In Section 4 we introduce and characterize the family of linear, efficient, and anonymous values. In Section 5 we study the implications of imposing the null player property to the previous family of values. Section 6 discusses an alternative version of the null player property. Section 7 concludes.

2 Preliminaries

A *coalitional*[2] *game* is a pair (N, v) where N is a finite non-empty set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function* satisfying $v(\emptyset) = 0$. A coalitional game describes the worth, $v(S)$, that each coalition $S \subseteq N$ can guarantee for itself. The worth is assumed to be transferable and infinitely divisible. If the set of players N is fixed then we confuse each coalitional game with its characteristic function. We denote by $\mathcal{CG}^N$ the set of coalitional games over N. A coalitional game is said to be *zero normalized* if $v(\{i\}) = 0$ for every $i \in N$. The set of zero normalized coalitional games with set of players N is denoted by $\mathcal{CG}_0^N$. Given $S \subseteq N$ with $S \neq \emptyset$, the *coalitional unanimity game* u_S is defined by $u_S(T) = 1$ for every $T \supseteq S$ and $u_S(T) = 0$, otherwise.

A *value* on $\mathcal{CG}^N$, f, is a point valued solution concept that assigns to every coalitional game $v \in \mathcal{CG}^N$ a payoff vector $f(v) \in \mathbb{R}^N$.[3] The value of player in a coalitional game is a measure of her importance in the game or what she could be willing to pay for participating in the game. One of the most popular values on $\mathcal{CG}^N$ is the *Shapley value* (Shapley, 1953), defined for every $v \in \mathcal{CG}^N$ and $i \in N$ by[4]

$$Sh_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \left[v(S \cup i) - v(S) \right].$$

Originally, Shapley (1953) introduced his value axiomatically, by stating some

[2]Cooperative transferable utility game is probably the most used term in the literature. However, we deliberately use the term coalitional to stress the fact that coalitions are the basic elements.

[3]Where $\mathbb{R}^N$ is the $|N|$-dimensional vector space with coordinates indexed by $i \in N$.

[4]We abuse notation slightly and write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively, for $S \subseteq N$ and $i \in N$. We use lowercase letters to denote the cardinality of a finite set.

properties that one may find plausible and then showing that Sh is the only value on $\mathcal{CG}^N$ that satisfies them. In order to introduce some classic properties of the Shapley value we first define some notions. A player $i \in N$ is a *null player* in $v \in \mathcal{CG}^N$ if $v(S \cup i) = v(S)$ for every $S \subseteq N \setminus i$. Two players $i, j \in N$ are *symmetric* in $v \in \mathcal{CG}^N$ if $v(S \cup i) = v(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. A *permutation* of a finite set N is a bijection $\theta : N \to N$, let Θ^N denote the set of permutations of N. Let $\theta \in \Theta^N$ and $v \in \mathcal{CG}^N$, the *permuted game* $\theta v \in \mathcal{CG}^N$ is defined by $\theta v(S) = v(\theta(S))$, for every $S \subseteq N$. Consider the following properties that a value on $\mathcal{CG}^N$, f, may satisfy.

Linearity: $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$, for every $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{CG}^N$.[5]

Efficiency: $\sum_{i \in N} f_i(v) = v(N)$, for every $v \in \mathcal{CG}^N$.

Symmetry: $f_i(v) = f_j(v)$, for every $v \in \mathcal{CG}^N$ and symmetric players i, j in v.

Anonymity: $f_i(\theta v) = f_{\theta(i)}(v)$, for every $v \in \mathcal{CG}^N$, $\theta \in \Theta^N$, and $i \in N$.

The null player property: $f_i(v) = 0$, for every $v \in \mathcal{CG}^N$ and every null player i in v.

It is well known that the Shapley value is the only efficient, linear, symmetric value on $\mathcal{CG}^N$ that has the null player property. Moreover, the symmetry property can be replaced by anonymity and the characterization results still holds.

3 Global games

A partition, or coalition structure, of a finite set N is a collection of disjoint subsets such that every $i \in N$ belongs to one of them. We denote by Π^N the set of all coalition structures of N. Let $P, Q \in \Pi^N$, we say that P is finer than Q, or that Q is coarser than P, and write $P \preceq Q$ if for all $S \in P$ there is $T \in Q$ such that $S \subseteq T$. We write $P \prec Q$ when $P \preceq Q$ but $P \neq Q$. The coarsest partition, where all players belong to the grand coalition is denoted by $\lceil N \rceil = \{N\}$ whereas the finest one, where all players form singleton coalitions is denoted by $\lfloor N \rfloor = \{\{i\} : i \in N\}$.

A *global game* is a pair (N, V) where N is a finite non-empty set of players and $V : \Pi^N \to \mathbb{R}$ is the *partition function* [6] satisfying $V(\lfloor N \rfloor) = 0$. A global game describes the worth generated by the whole set of players when they are organized

[5] Where $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$, for every $S \subseteq N$.

[6] Not to be confused by the function that describes the worth of a coalition when the complementary coalition is organized in a given partition in the framework of games with coalitional externalities.

according to a coalition structure. The worth is assumed to be transferable and infinitely divisible. If the set of players N is fixed then we confuse each global game with its partition function. We denote by $\mathcal{G}^N$ the set of global games with set of players N.

A *value* on $\mathcal{G}^N$, f, is a point valued solution concept that assigns to every global game $V \in \mathcal{G}^N$ a payoff vector $f(V) \in \mathbb{R}^N$. The value of a player in a global game is a measure of the importance of her participation in the game. Gilboa and Lehrer (1991) introduced global games and, among other things, proposed a value on $\mathcal{G}^N$ using the following transformation of a global game into a zero normalized coalitional game. Let $V \in \mathcal{G}^N$, then the associated (zero-normalized) coalitional game v^V is defined for every $S \in 2^N \setminus \{\emptyset\}$ by

$$v^V(S) = V(\lceil S \rceil \cup \lfloor N \setminus S \rfloor). \tag{1}$$

That is, the worth attached to a coalition S is the worth of the coalition structure in which S forms and the rest of players are organized in singleton coalitions. Obviously, $v^V \in \mathcal{CG}_0^N$. The value on $\mathcal{G}^N$ that they propose is obtained by applying the Shapley value to the associated coalitional game. Even if the definition of the associated game is a natural way to assess the utility that the formation of a coalition generates it implies a big loss of information. Notice that the worth generated by any coalition structure with more than one non-singleton coalition is discarded. The *Gilboa-Lehrer value*, GL, is the value on $\mathcal{G}^N$ defined for every $V \in \mathcal{G}^N$ by

$$GL(V) = Sh(v^V). \tag{2}$$

Gilboa and Lehrer (1991) characterized GL by means of four properties that can be considered parallel to the classic ones of the Shapley value. In order to present them we need some additional notations and definitions. Given a coalition structure $P \in \Pi^N$ and a player $i \in N$, we denote by P_{-i} the partition obtained from P when i leaves the coalition in which she is participating to form a singleton coalition. That is, $P_{-i} = \{S \setminus \{i\} : S \in P\} \cup \{\{i\}\}$. Let $V \in \mathcal{G}^N$, i is a *null*[7] *player in* the global

[7] Gilboa and Lehrer (1991) used the term dummy.

game V if for every $P \in \Pi^N$, $V(P) = V(P_{-i})$. Let $V \in \mathcal{G}^N$, i and j are *symmetric*[8] *players in* the global game V if for every $P \in \Pi^N$, $V(P_{-i}) = V(P_{-j})$. Let f be a value on $\mathcal{G}^N$.

LIN $f(\alpha V + \beta W) = \alpha f(V) + \beta f(W)$, for every $\alpha, \beta \in \mathbb{R}$ and $V, W \in \mathcal{G}^N$.

EFF $\sum_{i \in N} f_i(V) = V(\lceil N \rceil)$, for every $V \in \mathcal{G}^N$.

SYM $f_i(V) = f_j(V)$, for every i and j symmetric players in $V \in \mathcal{G}^N$.

NPP $f_i(V) = 0$, for every i null player in $V \in \mathcal{G}^N$.

The first property is linearity. A linear value is invariant under a change in the utility scale and is an additive function on $\mathcal{G}^N$. Namely, if a global game can be described as the sum of two, the value in the global game can be obtained by adding the values of the two global games. Taking into account that $\mathcal{G}^N$ is a vector space, it is a reasonable property.

Efficiency is the second property. It is a very sensible property to impose when the grand coalition is the coalition structure that maximizes the global worth. An efficient value, proposes a way to share the worth that the coalition structure $\lceil N \rceil$ generates.

The third property, symmetry, is an equal treatment property. It requires that the value gives the same payoff to two players whose impact to every partition when they abandon the coalition in which they participate to form a singleton coalition is equal.

The last property states that null players should get a zero payoff. Note that a player is null if her movement from being alone in the structure to participating in any existing coalition does not affect the global worth.

4 The family of LEA values

Recall that Θ^N denotes the set of permutations of N. Let $\theta \in \Theta^N$ and $V \in \mathcal{G}^N$, the *permuted game* $\theta V \in \mathcal{G}^N$ is defined by $\theta V(P) = V(\theta(P))$, for every $P \in \Pi^N$,

[8] Gilboa and Lehrer (1991) called them interchangeable.

where $\theta(P) = \{\theta(S) : S \in P\}$. Let f be a value on $\mathcal{G}^N$, the anonymity axiom says the following.

ANO $f_i(\theta V) = f_{\theta(i)}(V)$, for every $V \in \mathcal{G}^N$, $\theta \in \Theta^N$, and $i \in N$.

As Gilboa and Lehrer (1991) pointed out, in their characterization result SYM cannot be replaced by ANO, because even in the presence of LIN, EFF, and NPP, ANO is strictly weaker than SYM.

In order to introduce the family of linear, efficient, and anonymous values on $\mathcal{G}^N$ we need some machinery.

The set of global games with a fixed player set, $\mathcal{G}^N$, is an $(B_n - 1)$-dimensional vector space, where B_n is the the Bell number that counts the amount of partitions of a set of n elements. To define the values we need a basis of this vector space. Given $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$, the *global unanimity game* U_Q is defined for every $P \in \Pi^N$ by

$$U_Q(P) = \begin{cases} 1, & \text{if } Q \preceq P \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Gilboa and Lehrer (1991) showed that the set of unanimity games, $\{U_Q : Q \in \Pi^N, Q \neq \lfloor N \rfloor\}$ is a basis of $\mathcal{G}^N$. Next, we provide an explicit expression of the coefficients of any global game in this basis. Parallel to the well-known Harsanyi dividends of a coalitional game.

Proposition 4.1. *Let $V \in \mathcal{G}^N$. The coefficients of V in the unanimity basis are given by*

$$\delta_Q(V) = \sum_{M \preceq Q} (-1)^{|M|-|Q|} \binom{Q}{M} V(M),$$

where for every $M \preceq Q$ and $T \in Q$, if m_T is the number of subsets in which T is divided in M, then

$$\binom{Q}{M} = \prod_{T \in Q} (m_T - 1)!.$$

Proof. It is well known that the set of partitions of a finite set Π^N endowed with the ordering $\preceq$ is a lattice (se for instance, Stanley, 2011). Then, the coefficients are given by

$$\delta_Q(V) = \sum_{M \preceq Q} \mu(M,Q) V(M),$$

where μ is the Möbius inversion of this lattice, which is defined for every $M \preceq Q$ by

$$\mu(M,Q) = (-1)^{|M|-|Q|} \binom{Q}{M},$$

where $\binom{Q}{M} = \prod_{T \in Q} (m_T - 1)!.$ □

It is easy to check that these coefficients can also be defined recursively by

$$\delta_Q(V) = V(Q) - \sum_{P \prec Q} \delta_P(V), \tag{4}$$

taking $\delta_{\lfloor N \rfloor}(V) = 0$ by convention.

Assuming linearity as a desirable condition for a value on $\mathcal{G}^N$, in order to define a value we only need to determine the payoffs in global unanimity games, as defined in Equation (3).

Let $n \in \mathbb{N}$ that represents the amount of players, i.e., $|N| = n$. A *partition of the integer* n is a tuple, $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$, satisfying

1. $t_1, \ldots, t_p, \lambda_1, \ldots, \lambda_p \in \mathbb{N}$,
2. $t_1 < \cdots < t_p$,
3. $\sum_{k=1}^{p} \lambda_k t_k = n$.

For every $k \in \{1, \ldots, p\}$, the integer λ_k is called the *multiplicity* of t_k, which represents the cardinality of a coalition in the structure. The set of partitions of n is denoted by $\Pi(n)$. In any partition of n we omit the multiplicity if it equals one. For instance,

$$\Pi(4) = \left\{(4), (1,3), \left(2^2\right), \left(1^2, 2\right), \left(1^4\right)\right\}.$$

Let $P \in \Pi^N$, the *norm* of P is defined as the partition of n:

$$\|P\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \tag{5}$$

such that P consists of λ_k coalitions of cardinality t_k for every $k = 1, \ldots, p$.

As the reader may anticipate, in order to obtain an anonymous value, the payoffs in a global unanimity game can only depend on the norm of the underlying coalition structure. The next definition formalizes this idea.

Definition 4.1. *A unanimity function over* $\Pi(n)$ *is a mapping* α *satisfying for all* $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \setminus \{(1^n)\}$

1. $\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \mathbb{R}^p$,

2. $\sum_{k=1}^{p} \alpha_k \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = 1.$

The set of unanimity functions is denoted by F_n.

A unanimity function describes the importance of each coalition in the underlying partition of a unanimity game. By convenience, we do not assign weights to $(1^n) \in \Pi(n)$ because the game $U_{\lfloor N \rfloor}$ does not belong to the unanimity basis of $\mathcal{G}^N$. The first condition allows us to associate a coefficient to coalitions of a given size. The second is a normalization condition that will be useful to obtain an efficient value.

We are now in the position to introduce the family of LEA (linear, efficient, and anonymous) values.

Definition 4.2. *Let* $\alpha \in F_n$, $Q \in \Pi^N$ *with* $Q \neq \lfloor N \rfloor$, *and* $i \in N$. *The* α*-value,* Φ^α, *is the linear extension of the value defined for unanimity games by*

$$\Phi_i^\alpha \left(U_Q\right) = \frac{\alpha_{k(i)}(\|Q\|)}{\lambda_{k(i)} \cdot t_{k(i)}},$$

where $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$ *and* $k(i) \in \{1, \ldots, p\}$ *is such that* $|T| = t_{k(i)}$, *with* $i \in T$.

Hence, if $V \in \mathcal{G}^N$ and $\alpha \in F_n$ then Φ^α is determined by

$$\Phi^\alpha(V) = \sum_{Q \in \Pi^N : Q \neq \lfloor N \rfloor} \delta_Q(V) \Phi^\alpha(U_Q). \tag{6}$$

Next we show that the members of this family are characterized by means of linearity, efficiency, and anonymity.

Theorem 4.1. *A value on $\mathcal{G}^N$ satisfies* LIN, EFF, *and* ANO *if and only if it is an α-value for some $\alpha \in F_n$.*

Proof. On the one hand, we prove that all the values in the family satisfy the properties. Let $\alpha \in F_n$.

LIN: Φ^α is linear by construction.

EFF: Let $V \in \mathcal{G}^N$, by Definition 4.2

$$\sum_{i \in N} \Phi_i^\alpha(V) = \sum_{i \in N} \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V) \Phi_i^\alpha(U_Q) = \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V) \sum_{i \in N} \Phi_i^\alpha(U_Q)$$

Given $Q \in \Pi^N$ with $\|Q\| = \left(t_1^{\lambda_1}, \cdots, t_p^{\lambda_p}\right)$, for every $k \in \{1, \ldots, p\}$ we define

$$T_k = \bigcup_{T \in Q : |T| = t_k} T.$$

Obviously, $\{T_k : k = 1, \ldots, p\} \in \Pi^N$. Then,

$$\sum_{i \in N} \Phi_i^\alpha(U_Q) = \sum_{k=1}^{p} \sum_{i \in T_k} \Phi_i^\alpha(U_Q) = \sum_{k=1}^{p} \sum_{i \in T_k} \frac{\alpha_k(\|Q\|)}{\lambda_k \cdot t_k} = \sum_{k=1}^{p} \alpha_k(\|Q\|) = 1,$$

where the third equality holds because $|T_k| = \lambda_k \cdot t_k$ and last equality is by point 2. of Definition 4.1.

Finally, by Proposition 4.1

$$\sum_{i \in N} \Phi_i^\alpha(V) = \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V) = V(\lceil N \rceil)$$

ANO: Let $\theta \in \Theta^N$ and $i \in N$. We first show that Φ^α satisfies anonymity on global unanimity games. Let $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$. First we prove that

$$\delta_Q(\theta V) = \delta_{\theta Q}(V). \tag{7}$$

In fact, as $|\theta(Q)| = |Q|$ y $\binom{\theta(Q)}{\theta(M)} = \binom{Q}{M}$ for all $M \preceq Q$ we have

$$\begin{aligned}
\delta_Q(\theta V) &= \sum_{M \preceq Q} (-1)^{|M|-|Q|} \binom{Q}{M} \theta V(M) \\
&= \sum_{\theta(M) \preceq \theta(Q)} (-1)^{|\theta(M)|-|\theta(Q)|} \binom{\theta(Q)}{\theta(M)} V(\theta(M)) \\
&= \sum_{M \preceq \theta(Q)} (-1)^{|M|-|\theta(Q)|} \binom{\theta(Q)}{M} V(M) = \delta_{\theta(Q)}(V).
\end{aligned}$$

Also, since $\|Q\| = \|\theta(Q)\|$ and $k(i) = k(\theta(i))$, by Definition 4.2

$$\Phi_i^\alpha(\theta U_Q) = \Phi_{\theta(i)}^\alpha(U_Q). \tag{8}$$

Besides, we get the equality

$$\theta U_Q = U_{\theta^{-1}(Q)}, \tag{9}$$

because for all $P \in \Pi^N$ we obtain

$$\theta U_Q(P) = U_Q(\theta(P)) = \left\{ \begin{array}{ll} 1, & \text{if } Q \preceq \theta P \\ 0, & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{ll} 1, & \text{if } \theta^{-1}(Q) \preceq P \\ 0, & \text{otherwise} \end{array} \right\} = U_{\theta^{-1}(Q)}(P).$$

Finally, let $V \in \mathcal{G}^N$. Then,

$$\begin{aligned}
\Phi_i^\alpha(\theta V) &= \sum_{\substack{\theta^{-1}(Q) \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_{\theta^{-1}(Q)}(\theta V) \Phi_i^\alpha\left(U_{\theta^{-1}(Q)}\right) = \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V) \Phi_i^\alpha\left(\theta U_Q\right) \\
&= \sum_{\substack{Q \in \Pi^N \\ Q \neq \lfloor N \rfloor}} \delta_Q(V) \Phi_{\theta(i)}^\alpha\left(U_Q\right) = \Phi_{\theta(i)}^\alpha(V),
\end{aligned}$$

where the second equality is due to Eq. (7) and Eq. (9), and the third equality hold by Eq. (8).

On the other hand, let f be a value on $\mathcal{G}^N$ satisfying the three properties. By LIN we only need to find a unanimity function α such that for every $i \in N$ and $Q \in \Pi^N$, $Q \neq \lfloor N \rfloor$,

$$f_i(U_Q) = \frac{\alpha_{k(i)}(\|Q\|)}{\lambda_{k(i)} \cdot t_{k(i)}}. \tag{10}$$

Let $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \setminus \{(1^n)\}$ and $Q \in \Pi^N$ such that $\|Q\| = \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right)$. For every $k = 1, \ldots, p$, we define

$$\alpha_k \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \lambda_k \cdot t_k \cdot f_i(U_Q)$$

where $i \in T \in Q$ with $|T| = t_k$. Note that since f satisfies ANO, all players that belong to coalitions T of a given cardinality get the same payoff in U_Q. This guarantees that α is well defined. Obviously, the function α satisfies the required equality. It only remains to check that α is a unanimity function. By definition $\alpha\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \mathbb{R}^p$ for all $\left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \setminus \{(1^n)\}$. By EFF and ANO of f, we have

$$\sum_{k=1}^{p} \alpha_k \left(t_1^{\lambda_1}, \ldots, t_p^{\lambda_p}\right) = \sum_{k=1}^{p} \lambda_k \cdot t_k \cdot f_i(U_Q) = \sum_{i \in N} f_i(U_Q) = U_Q(\lceil N \rceil) = 1,$$

which concludes the proof. □

Notice that Equation (10) provides a method to obtain the unanimity function associated with a LEA value from the payoffs in global unanimity games.

We conclude the section by describing some particular LEA values.

Example 4.1. *Obviously, the Gilboa-Lehrer value belongs to the family because it satisfies* LIN, EFF, *and* ANO. *Recall that it is defined as the Shapley value of the (zero-normalized) coalitional game associated with a global game, see Equation (2). Note that the coalitional game associated with a global unanimity game is a coalitional unanimity game. Indeed, let $Q \in \Pi^N$ with $Q \neq \lfloor N \rfloor$ and define*

$$R_Q = \bigcup_{T \in Q : |T| > 1} T.$$

Then, by Equation (1), $v^{U_Q} = u_{R_Q}$ because $Q \preceq \lceil S \rceil \cup \lfloor N \setminus S \rfloor$ if and only if $R_Q \subseteq S$. Using Equation (2) we can write

$$GL\left(U_Q\right) = Sh\left(u_{R_Q}\right).$$

Hence $GL = \Phi^\alpha$ *with*

$$\alpha\left(t_1^{\lambda_1},\ldots,t_p^{\lambda_p}\right)=\begin{cases}\frac{1}{n}(\lambda_1 t_1,\ldots,\lambda_p t_p) & \text{if } t_1>1\\ \frac{1}{n-\lambda_1}(0,\lambda_2 t_2,\ldots,\lambda_p t_p) & \text{if } t_1=1.\end{cases} \tag{11}$$

Finally, consider the global unanimity game of the partition $P = \{\{1\},\{2,3\},\{4,5,6\}\}$*. The Gilboa-Lehrer value, gives a zero payoff to player* 1 *and treats the remaining five players equally. Then,*

$$GL\left(U_P\right)=\left(0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right).$$

Example 4.2. *Another interesting LEA value that does not satisfy* NPP *is the* Equal division *value, defined for every* $V \in \mathcal{G}^N$ *and* $i \in N$ *by*

$$ED_i(V)=\frac{V(\lceil N\rceil)}{n}.$$

Note that, when applied to unanimity games, for every $Q \in \Pi^N$ *with* $Q \neq \lfloor N \rfloor$*,*

$$ED_i(U_Q)=\frac{1}{n}.$$

It is easy to check that it is a LEA value. Indeed, $ED = \Phi^\alpha$ *where*

$$\alpha\left(t_1^{\lambda_1},\ldots,t_p^{\lambda_p}\right)=\frac{1}{n}\left(\lambda_1 t_1,\ldots,\lambda_p t_p\right).$$

In the framework of coalitional games van den Brink (2007) conducted an axiomatic comparison of the Shapley value and the Equal division value. We conclude by calculating the payoffs in the global unanimity game of the partition $P = \{\{1\},\{2,3\},\{4,5,6\}\}$*. Obviously, all players get the same fraction of* 1*, i.e.,*

$$ED\left(U_P\right)=\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right).$$

5 The family of LEAN values

In this section we characterize the family of values on $\mathcal{G}^N$ that satisfy LIN, EFF, ANO, and NPP. Obviously, this family is different to the LEA values because the Equal division value defined above does not satisfy NPP. In other words, we impose the null player property used by Gilboa and Lehrer (1991), NPP, to the family of values introduced in Definition 4.2. We have already mentioned that not all values in the family satisfy NPP. So, we restrict the unanimity functions presented in Definition 4.1 by requiring one more condition. The condition states that the coefficient associated to a coordinate of cardinality one in a partition of an integer should be equal to zero. We formalize this idea in our second main result, where we characterize the family of LEAN (linear, efficient, anonymous, and null player property) values.

Theorem 5.1. *Let* $\alpha \in F_n$ *be such that for every* $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi_n^N \setminus \{(1^n)\}$ *with* $t_1 = 1$,

$$\alpha_1\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = 0.$$

Then, the α*-value satisfies* NPP. *Moreover, the* α*-values associated with unanimity functions satisfying this condition are the only values on* $\mathcal{G}^N$ *satisfying* LIN, EFF, ANO, *and* NPP.

Proof. For the existence, take an $\alpha \in F_n$ satisfying the above condition. To show that Φ^α satisfies NPP, the following claim will be useful.

Claim: Let $V \in \mathcal{G}^N$ and $i \in N$ a null player in the global game V. Then, for every $Q \in \Pi^N$ with $\{i\} \notin Q$ and $Q \neq \lfloor N \rfloor$, $\delta_Q(V) = 0$.

We prove the Claim by induction on the rank of the partition. The rank of Q is given by $r(Q) = n - (|Q| - 1)$. Since $Q \neq \lfloor N \rfloor$, take $Q \in \Pi^N$ with $r(P) = 2$ and $\{i\} \notin Q$. Then, $Q = \lceil \{i,j\} \rceil \cup \lfloor N \setminus \{i,j\} \rfloor$. Since i is a null player in V, $V(Q) = V(Q_{-i})$. But $Q_{-i} = \lfloor N \rfloor$ and by definition $V(\lfloor N \rfloor) = 0$. Then, $V(Q) = 0$. Moreover, $\lfloor N \rfloor$ is the only partition which is finer than Q. Then, using the recursive definition of the coefficients in Equation (4) and the fact that $\delta_{\lfloor N \rfloor}(V) = 0$,

$$\delta_Q(V) = V(Q) - \delta_{\lfloor N \rfloor}(V) = 0 - 0 = 0.$$

Take $Q \in \Pi^N \setminus \{\lfloor N \rfloor\}$ with $r(Q) = 3$. Then, two cases can arise[9].

[9] If $|N| = 3$, only case 1 appears.

1. $Q = \lceil\{i,j,k\}\rceil \cup \lfloor N \setminus \{i,j,k\}\rfloor$. Then, the partitions $P \prec Q$ are

$$\begin{aligned} P_1 &= \lfloor N \rfloor, & P_2 &= \lceil\{i,j\}\rceil \cup \lfloor N \setminus \{i,j\}\rfloor, \\ P_3 &= \lceil\{i,k\}\rceil \cup \lfloor N \setminus \{i,k\}\rfloor, & P_4 &= \lceil\{j,k\}\rceil \cup \lfloor N \setminus \{j,k\}\rfloor \end{aligned}$$

having rank 2 the partitions P_2, P_3, P_4 and rank 1 the partition P_1. Since $\{i\} \notin P_2$ and $\{i\} \notin P_3$, we have already seen that the coefficients associated with these partitions are equal to zero. The coefficient associated with P_1 is zero by convention. Moreover, note that $Q_{-i} = P_4$ and by the recursive definition of the coefficients, $\delta_{P_4}(V) = V(P_4) = V(Q_{-i})$. Then, using again the recursive definition of the coefficients and the fact that i is a null player in V we can write

$$\delta_Q(V) = V(Q) - \sum_{r=1}^{4} \delta_{P_r}(V) = V(Q) - V(Q_{-i}) = 0.$$

2. $Q = \lceil\{i,j\}\rceil \cup \lceil\{k,l\}\rceil \cup \lfloor N \setminus \{i,j,k,l\}\rfloor$. Then, the partitions $P \prec Q$ are

$$\begin{aligned} P_1 &= \lfloor N \rfloor, & P_2 &= \lceil\{i,j\}\rceil \cup \lfloor N \setminus \{i,j\}\rfloor, \\ P_3 &= \lceil\{k,l\}\rceil \cup \lfloor N \setminus \{k,l\}\rfloor \end{aligned}$$

having rank 2 the partitions P_2, P_3 and rank 1 the partition P_1. We have already seen that $\delta_{P_2}(V) = 0$ because $\{i\} \notin P_2$. Recall, that $\delta_{P_1}(V) = \delta_{\lfloor N \rfloor}(V) = 0$. Moreover, note that $Q_{-i} = P_3$ and by the recursive definition of the coefficients, $\delta_{P_3}(V) = V(P_3) = V(Q_{-i})$. Then, using again the recursive definition of the coefficients and the fact that i is a null player in V we can write

$$\delta_Q(V) = V(Q) - \sum_{r=1}^{3} \delta_{P_r}(V) = V(Q) - V(Q_{-i}) = 0,$$

where the second equality holds by the induction hypothesis.

Let us assume that the result is true for every $Q \in \Pi^N \setminus \{\lfloor N \rfloor\}$ with $1 \leq r(Q) = r < n$. Take $Q \in \Pi^N \setminus \{\lfloor N \rfloor\}$ with $r(Q) = r + 1$. By the induction hypothesis, $\delta_P(V) = 0$ for every P such that $\{i\} \notin P$ and $r(P) \leq r(Q) - 1$. Applying the recursive definition of the coefficients of Equation (4) twice,

$$\delta_Q(V) = V(Q) - \sum_{\substack{P \prec Q \\ \{i\} \in P}} \delta_P(V) = V(Q) - \sum_{P \preceq Q_{-i}} \delta_P(V) = V(Q) - V\left(Q_{-i}\right) = 0.$$

Which concludes the proof of the Claim.

Let $V \in \mathcal{G}^N$ and $i \in N$ a null player in the global game V. Using the Claim, the linearity of Φ^α, and the decomposition of V in global unanimity games we can write,

$$\Phi_i^\alpha(V) = \sum_{\substack{Q \in \Pi^N \setminus \{\lfloor N \rfloor\} \\ \{i\} \in Q}} \delta_Q(V) \Phi_i^\alpha\left(U_Q\right).$$

Finally, from Definition 4.2 $k(i) = 1$ and the condition imposed on the unanimity function α implies that $\Phi_i^\alpha\left(U_Q\right) = 0$ for every $Q \in \Pi^N$ such that $Q \neq \lfloor N \rfloor$ and $\{i\} \in Q$. Therefore, $\Phi_i^\alpha(V) = 0$, which concludes the proof of the existence of a solution satisfying the four properties.

For the uniqueness, let f be a value on $\mathcal{G}^N$ satisfying LIN, EFF, ANO, and NPP. By Theorem 4.1 we already know that there is an $\alpha \in F_n$ such that $f = \Phi^\alpha$. Then, it only remains to check that for every $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi(n) \setminus \{(1^n)\}$ with $t_1 = 1$, the unanimity function α satisfies

$$\alpha_1\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = 0.$$

Note that the above partition of n is associated with $P \in \Pi^N$ such that $P \neq \lfloor N \rfloor$ and $\{i\} \in P$. Note that $\{i\} \in P$ implies that $U_P(Q) = U_P\left(Q_{-i}\right)$. Then, by NPP $\Phi_i^\alpha\left(U_P\right) = 0$. Finally, by Definition 4.2 and the fact that $t_{k(i)} = 1$, the unanimity function α satisfies the desired condition. □

To conclude the section we illustrate the family of LEAN values by presenting an instance which is not the Gilboa-Lehrer value.

Example 5.1. *In the lattice of partitions $(\Pi^N, \preceq)$, each element $P \in \Pi^N$ covers exactly*

$$\sum_{S \in P} 2^{|S|-1} - |P| = \sum_{S \in P} \left(2^{|S|-1} - 1\right)$$

partitions. Consider the global unanimity game U_P, with $P \in \Pi^N \setminus \{\lfloor N \rfloor\}$. The idea is to split 1 *equally among the agents in the coalitions whose union gives a coalition in P. That is, let φ be the value on $\mathcal{G}^N$ defined for every $i \in S \in P$ by*

$$\varphi_i\left(U_P\right) = \frac{2^{|S|-1} - 1}{|S| \sum_{T \in P} (2^{|T|-1} - 1)}.$$

Note that if $\{i\} \in P$, then $\varphi_i\left(U_P\right) = 0$. Additionally, for every $i, j \in S \in P$, $\varphi_i\left(U_P\right) = \varphi_j\left(U_P\right)$. Moreover, $\varphi_i\left(U_P\right) = \varphi_j\left(U_P\right)$ whenever i and j belong to two different coalitions of P with the same sizes, i.e., $i \in S \in P$, $j \in T \in P$, and $|S| = |T|$.

It can be checked that the linear extension of this value belongs to the LEAN family. Indeed, $\varphi = \Phi^\alpha$ for the unanimity function defined for every $\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) \in \Pi_n^N \setminus \{(1^n)\}$ by

$$\alpha_{k(i)}\left(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_p^{\lambda_p}\right) = \lambda_{k(i)} \frac{2^{t_{k(i)}-1} - 1}{\sum_{r=1}^{p} \lambda_r \left(2^{t_r - 1} - 1\right)}$$

We conclude by illustrating the behavior of this value in the global unanimity game of partition $P = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$. Then,

$$\varphi\left(U_P\right) = \left(0, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

Note that, compared to the Gilboa-Lehrer value, this value favors the players who form larger coalitions.

6 The complete null player property

In this section we discuss an alternative null player property. In general, a player is null if it does not contribute to the creation of worth. The notion of null player introduced by Gilboa and Lehrer (1991), NPP, requires that the worth does not change when the player moves from being alone in the structure to joining a coalition. Implicitly, they assume that a contribution is a movement of this kind. But it is not difficult to consider more general notions of what a contribution is in a global game.

In the classic theory of coalitional games a contribution can be identified with a link in the Boolean lattice $\left(2^N, \subseteq\right)$. We follow a parallel approach in the lattice of partitions $\left(\Pi^N, \preceq\right)$. The links in this lattice can be considered an indivisible step in the formation process of the grand coalition. Each link represents a union of two coalitions of arbitrary sizes in one. Let $P, Q \in \Pi^N$, we say that P *covers* Q if there are two different coalitions $T_1, T_2 \in Q$ with $P = (Q \setminus \{T_1, T_2\}) \cup \lceil T_1 \cup T_2 \rceil$. Then, we define a *contribution* in a global game, $V \in \mathcal{G}^N$, as the change in the global worth when two coalitions join in one, i.e., $V(P) - V(Q)$, for every $P, Q \in \Pi^N$ where P covers Q. Note that, several players are involved in a contribution. We consider that all of them participate in the contribution. Formally, a player $i \in N$ is *active* in the contribution $V(P) - V(Q)$ if $i \in T_1 \cup T_2$. Next, we introduce a new notion of null player for global games.

Definition 6.1. *We say that $i \in N$ is a* completely null player *in $V \in \mathcal{G}^N$ if all the contributions in which she is active are null. That is, if $V(P) = V(Q)$ for every $P, Q \in \Pi^N$ such that P covers Q and i changes her affiliation from Q to P.*

Note that a completely null player is in particular a null player as defined by Gilboa and Lehrer (1991) but not the other way around. The next global game has a null player who is not completely null.

Example 6.1. *Consider the global game where $N = \{1,2,3\}$ and $V(\lceil N \rceil) = V(\{\{1\},\{2,3\}\}) = 1$, and $V(P) = 0$ otherwise. Player 1 is a null player in V because*

$$
\begin{aligned}
&V(\lceil N \rceil) = V(\{\{1\},\{2,3\}\}),\\
&V(\{\{1,2\},\{3\}\}) = V(\lfloor N \rfloor), \text{ and}\\
&V(\{\{1,3\},\{2\}\}) = V(\lfloor N \rfloor).
\end{aligned}
$$

Player 1 is not a completely null player because $\lceil N \rceil$ covers $\{\{1,2\},\{3\}\}$, 1 changes her affiliation between these two partitions, and

$$V(\lceil N \rceil) \neq V(\{\{1,2\},\{3\}\}).$$

The notion of a completely null player is very demanding. We use this notion to define a new and very weak null player property that we call *the complete null player property.*

CNP $f_i(V) = 0$, for every i completely null player in $V \in \mathcal{G}^N$.

We show that, if a global game has a completely null player then all the players are completely null and the game itself is null.

Proposition 6.1. *Let $V \in \mathcal{G}^N$. Then, there is a completely null player in V if and only if V is the null game, i.e., if $V(Q) = 0$ for every partition $Q \in \Pi^N$.*

Proof. The implication to the left is trivial, in a null game all players are completely null. Then, we only have to show the implication to the right. Let $h(P)$ be the distance from $P \in \Pi^N$ to $\lceil N \rceil$ in the lattice of partitions. That is, $h(Q) = h$ if there is a sequence of partitions $Q_0, \ldots, Q_h$ such that $Q_0 = Q$, $Q_h = \lceil N \rceil$ and Q_k covers to Q_{k-1} for all $k = 1, \ldots, h$. Let $V \in \mathcal{G}^N$ and i a completely null player in V. We show the result by induction on $h(Q)$. If $h(Q) = 1$ then $\lceil N \rceil$ covers Q, and all players are active in the contribution $V(\lceil N \rceil) - V(Q)$, in particular i. Thus $V(Q) = V(\lceil N \rceil)$. Suppose that for every partition Q with $h(Q) = h - 1$, $h > 1$, $V(Q) = V(\lceil N \rceil)$. Take now $Q \in \Pi^N$ with $h(Q) = h$. Since $h > 1$ there exists $S \in Q$ with $i \in S$ and there is also $T \in Q$ with $T \neq S$. Consider the partition $P = (Q \setminus \{S, T\}) \cup \lceil S \cup T \rceil$. Since P covers Q, $h(P) = h - 1$. Moreover, player i changes her affiliation from P to Q. Then, as i is a completely null player $V(Q) = V(P) = V(\lceil N \rceil)$. The result follows because $V(\lfloor N \rfloor) = 0$ by convention. □

Consequently, all members of the LEA family satisfy the complete null player property, CNP. Indeed, note that EFF and ANO imply CNP. We have just seen that the existence of a completely null player implies the global game to be null. By ANO, all players in a null game get the same payoff and by EFF this payoff equals zero.

7 Conclusions

We have contributed to the scarce theoretical literature on global games by studying two families of values in detail. We believe that any sensible Shapley-like value for global games should lie within the family of LEA values. We have illustrated the families by providing new values that we plan to study in more detail in the near future.

Our study provides the necessary theoretical framework that eases the application to real problems. Indeed, we have provided a method to identify a value of the family that can better fit a particular situation by only specifying the desired payoffs in global unanimity games. For instance, we believe that it can shed light to the problem of assessing fair transfers or penalties to the countries in international environmental agreements. In the future, we would like to study if the values proposed here could be used to avoid free-riding or at least to minimize the gains from this behavior.

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