Two weakly interacting Bose-Einstein condensates: Two mode approximation

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Abstract: In this project, we study the dynamics of two weakly interacting Bose-Einstein condensates confined by a double-well potential. We obtain the two coupled equations of motion from the Gross-Pitaevskii equation, within the mean-field framework, by using a two-mode ansatz. We solve these equations numerically and study the tunneling dynamics of the system. We have obtained two different regimes: Josephson dynamics and macroscopic quantum self-trapping. Finally, we investigate the bifurcation between these two regimes and describe the dynamics of the system by using a simple pendulum analogy.

I. INTRODUCTION

Bose-Einstein condensation occurs when a system of identical bosons is cooled down at very low temperatures, as Einstein predicted in 1925. It is characterized by the macroscopic occupation of the lowest single particle state, forming the so-called Bose-Einstein condensate (BEC). This phenomenon occurs only in bosonic many-body systems at temperatures below a critical temperature characteristic of each system. In 1995, a ^{87}Rb BEC was observed for the first time using laser cooling techniques to reach temperatures around nK and magnetic trapping to confine the atoms [1].

In quantum mechanics, particles behave as waves and are characterized by the de Broglie wavelength, λ_{DB}. It increases when the temperature decreases, and at the critical temperature λ_{DB} becomes of the order of the interparticle distance. Hence, the particle waves overlap, losing their individuality, and behave coherently as a single giant wave of matter, the condensate. An important consequence is the phase coherence of the BEC [2], which may lead to tunneling and Josephson dynamics when two BECs are weakly linked, as was realized experimentally in Ref. [3]. The Josephson effect has been also exhaustively studied in Josephson junctions between two superconductors, where electrons form Cooper pairs that tunnel through the junction [4].

In this work, we investigate the tunneling dynamics between two-weakly linked BECs by using the two-mode approximation. To describe the dynamics, we derive the two-coupled equations for the particle imbalance and phase difference by assuming a two-mode ansatz. Solving these coupled equations we investigate the two dynamical regimes: Josephson oscillations and macroscopic quantum self-trapping. Finally, we characterize the critical point between these two regimes.

II. THEORETICAL FRAMEWORK

A. Gross-Pitaevskii equation

At zero temperature, a system composed by N identical bosons can be described by the following many-body Hamiltonian:

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2 \nabla_i^2}{2m} + V_{\text{ext}}(\vec{r_i}) \right] + \frac{1}{2} \sum_{i \neq j} V(\vec{r_i}, \vec{r_j}) \,, \quad (1)$$

where $V_{\text{ext}}(\vec{r_i})$ is the confining potential, and $V(\vec{r_i}, \vec{r_j})$ is the interaction potential. We can assume contact interacting particles and take the expression $g\,\delta(\vec{r_i} - \vec{r_j})$, with $g = 4\pi\hbar^2 a_s/m$ the coupling constant. The latter is the effective atomic interaction, and is proportional to the s-wave scattering length a_s. We consider only repulsive interactions ($g > 0$).

We can obtain the energy functional per particle by considering all the atoms in the same single particle state. It is a good approximation for a weakly interacting and dilute condensate with a large number of bosons N, and for temperatures close to zero:

$$\mathcal{E} = \int \mathrm{d}\vec{r} \left[-\frac{\hbar^2}{2m} |\nabla \Psi(\vec{r})|^2 + V_{ext}(\vec{r}) |\Psi(\vec{r})|^2 + \frac{Ng}{2} |\Psi(\vec{r})|^4 \right] \,, \quad (2)$$

where $\Psi(\vec{r})$ is the wave function of the condensate.

We arrive at the Gross-Pitaevskii equation (GPE), which is a non-linear Schrodinger equation, by minimizing the energy functional under variations of $\Psi(\vec{r})$. The time-dependent GPE takes the form:

$$i\hbar \frac{\partial \Psi(\vec{r})}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r}) + Ng|\Psi(\vec{r})|^2 \right] \Psi(\vec{r}) \,, \quad (3)$$

with the wave function normalized as $\int \mathrm{d}\vec{r}\, |\Psi(r)|^2 = 1$.

B. Two mode approximation

Now we consider two weakly interacting BECs within the above description. This can be obtained by confining the system with a double-well symmetric potential [3].

Since we have two weakly interacting BECs, one on the left (L) well and the other on the right (R), we can approximate the total wave function as the superposition of left and right [5]:

$$\Psi(\vec{r},t) = \psi_L(t)\Phi_L(\vec{r}) + \psi_R(t)\Phi_R(\vec{r}) \,, \qquad (4)$$

with $\Phi_{R(L)}$, the time-independent wave function, mainly located on the right (left) side of the trap. This ansatz provides a good approximation if the barrier of the double well is high enough to allow weak interactions but low enough to ensure tunneling between the two BECs.

Since a condensate is a coherent state, each BEC has a well defined phase ϕ_i with $(i = L, R)$. Then, $\psi_i(t) = \sqrt{N_i(t)}\, e^{\phi_i(t)}$, being N_i the number of particles on each side of the trap. Moreover, one can assume that Φ_i are real functions. Inserting Eq. (4) in the GPE (3) and projecting it into each mode, it follows the two coupled equations:

$$\begin{aligned} i\hbar\frac{\partial\psi_L(t)}{\partial t} &= \left[\varepsilon_L + U_L N_L\right]\psi_L(t) - \kappa_{LR}\psi_R(t) \,, \\ i\hbar\frac{\partial\psi_R(t)}{\partial t} &= \left[\varepsilon_R + U_R N_R\right]\psi_R(t) - \kappa_{RL}\psi_L(t) \,, \end{aligned} \qquad (5)$$

where we have neglected the overlapping terms $\Phi_R\Phi_L$, and

$$\begin{aligned} \varepsilon_i &= \int \mathrm{d}\vec{r}\left[\frac{\hbar^2}{2m}|\Phi_i(\vec{r})| + |\Phi_i(\vec{r})|V_{ext}(\vec{r})\right] \,, \\ \kappa_{ij} &= -\int \mathrm{d}\vec{r}\left[\frac{\hbar^2}{2m}\nabla\Phi_i(\vec{r})\nabla\Phi_j(\vec{r}) + \Phi_i(\vec{r})V_{ext}\Phi_j(\vec{r})\right] \,, \\ U_i &= g\int \mathrm{d}\vec{r}|\,\Phi(\vec{r})|^4 \,. \end{aligned} \qquad (6)$$

Equations (5) correspond to the well known standard two-mode approximation (S2M). They can be rewritten in terms of two new variables:

$$\begin{aligned} z(t) &= \frac{N_L - N_R}{N} \,, \\ \delta\phi(t) &= \phi_R - \phi_L \,, \end{aligned} \qquad (7)$$

the imbalance, and the phase difference between each side of the barrier, respectively. These new coupled equations are:

$$\begin{aligned} \frac{dz(t)}{dt} &= -\sqrt{1-z^2(t)}\sin\delta\phi(t) \\ \frac{d\delta\phi((t)}{dt} &= \Lambda z(t) + \frac{z(t)}{\sqrt{1-z^2(t)}}\cos\delta\phi(t) \,, \end{aligned} \qquad (8)$$

where the time is in units of the inverse of the Rabi frequency $\omega_R = 2\kappa/\hbar$. Since V_{ext} is symmetric, it follows that $\varepsilon \equiv \varepsilon_L = \varepsilon_R$, $\kappa \equiv \kappa_{LR} = \kappa_{RL}$, and $U \equiv U_L = U_R$. We have defined the dimensionless parameter $\Delta \equiv NU/(\hbar\omega_R)$ that quantifies the strength of the interaction of particles: it is the ratio between the interaction of the particles of the same mode and the coupling term κ [6]. The total number of bosons is $N = N_R(t) + N_L(t)$ and it is constant in time.

One can obtain the Hamiltonian in the two-mode approximation, as a function of the new variables, from the energy functional (2). It yields [7]:

$$H = \frac{\Lambda}{2}z(t)^2 - \sqrt{1-z(t)^2}\cos\delta\phi(t) \,. \qquad (9)$$

The variables $z(t)$ and $\delta\phi(t)$ behave as canonical conjugate variables:

$$\frac{\partial H}{\partial z} = \frac{d\delta\phi}{dt}, \qquad \frac{\partial H}{\partial\delta\phi} = -\frac{dz}{dt} \,. \qquad (10)$$

Using them, one can recover Eqs. (8). This system of equations can be solved numerically by providing an initial set of values for the population imbalance $z_0 \equiv z(0)$, and phase difference $\delta\phi_0 \equiv \delta\phi(0)$.

C. Josephson effect and macroscopic quantum self-trapping

Considering repulsive interactions, then $\Lambda > 0$, Eqs. (8) lead to two distinct dynamical behaviors depending on the strength of the interactions Λ and the initial conditions $(z_0, \delta\phi_0)$. These regimes are the Josephson effect (JE) regime and the macroscopic quantum self-trapping (MQST) regime [3].

The dynamics in the Josephson regime, is characterized by a fast oscillating tunneling of the bosons through the barrier, consequently, z and $\delta\phi$ oscillate sinusoidally with time. The system evolves following closed trajectories in the $(z, \delta\phi)$ plane, around a maximum or a minimum point of the system. Moreover, the population imbalance oscillates around zero, so the mean population imbalance over time in the JE is equal to zero $\langle z\rangle_t = 0$.

The MQST dynamics occurs when tunneling is strongly suppressed. In this case, the particles remain a majority on one side of the trap. In this regime $\langle z\rangle_t \neq 0$ and $\delta\phi$ doesn't oscillate, but increases or decreases with time.

We can compare these two regimes with the simple pendulum case. The simple pendulum Hamiltonian is given by:

$$H = \frac{{P_\theta}^2}{2ml^2} - mgl\cos\theta, \qquad (11)$$

where m and l are the mass and the length of the pendulum, g is the gravity acceleration, θ is the swing angle and P_θ the generalized momentum. We realize a close similitude between (11) and the two-mode hamiltonian (9). Taking $P_\theta \to z$, $\theta \to \delta\phi$, and $l \propto \sqrt{1-{P_\theta}^2}$ we can make the analogy between the pendulum and the two weakly interacting BECs system to understand the two regimes. In the simple pendulum, the mass takes the role of the interaction parameter. Then, when $H(z_0, \delta\phi_0) < H(0, \pi)$ we have oscillations around the minimum energy, so the BECs system is in the JE. When $H(z_0, \delta\phi_0) > H(0, \pi)$ there is enough energy to do laps without oscillations. In this case θ ($\delta\phi$) and P_θ (z) doesn't change their sign so that corresponds to MQST regime. We can calculate the critical value Λ_C, between the two regimes, by imposing $H(z_0, \delta\phi_0) = H(0, \pi)$. We arrive at:

$$\Lambda_C = \frac{1 + \sqrt{1 - {z_0}^2} \cos \delta\phi_0}{{z_0}^2/2} \,. \tag{12}$$

When the interaction parameter exceeds the critical value, $\Lambda > \Lambda_C$, MQST dominates. On the contrary, when the interaction parameter is $\Lambda < \Lambda_C$, JE dominates. These regimes have been experimentally obtained in Ref. [6].

We can analyse the energy of the system and obtain the stationary points [5]:

$$\left.\frac{\partial H}{\partial z}\right|_{z_0, \delta\phi_0} = 0, \quad \left.\frac{\partial H}{\partial \delta\phi}\right|_{z_0, \delta\phi_0} = 0\,. \tag{13}$$

Studying the Hessian matrix we can distinguish the stationary points between maximum, minimum and saddle point, see the summary in Table I.

TABLE I: Stationary points and stability of the system.

$(z_0, \delta\phi_0)$	Stationary	Minimum	Maximum	Saddle
(0,0)	$\forall\Lambda$	$\forall\Lambda$	–	–
(0,$\pm\pi$)	$\forall\Lambda$	–	$\Lambda < 1$	$\Lambda > 1$
$(\pm\sqrt{1 - 1/\Lambda^2}, \pm\pi)$	$\Lambda > 1$	–	$\Lambda > 1$	–

III. RESULTS

A. JE and MQST

We solve the two coupled differential equations of motion (8) obtained within the S2M approximation, by using a Runge-Kutta method. We fix a set of initial parameters $(z_0, \delta\phi_0)$, and we use different values of the interaction parameter Λ to investigate the dynamical regimes. We take the interaction parameter Λ above and below the critical case Λ_C. In order to appreciate the two regimes clearly, we will use values of Λ far from Λ_C.

Figure 1 shows the dynamics in the JE regime. It has been obtained with the initial conditions $(z_0, \delta\phi_0) = (0.3, 0)$ and an interaction $\Lambda = 1$. We can appreciate the oscillation of the particles between the two BECs around the imbalance value $z = 0$, therefore $\langle z \rangle_t = 0$. Also, we observe the oscillation in the phase difference of the two BECs satisfies that $\langle \delta\phi \rangle_t = 0$. Thus, the phase diagram corresponds to a closed orbit around the energy minimum, $(z, \delta\phi) = (0, 0)$. Under other initial conditions, the closed orbit could be around the maximum. Under these initial conditions, the critical point corresponds to $\Lambda_C \approx 43$ well above the value we have used for Λ. We observe that the Josephson regime occurs for z_0 close to 0 and small values of Λ, as we expected.

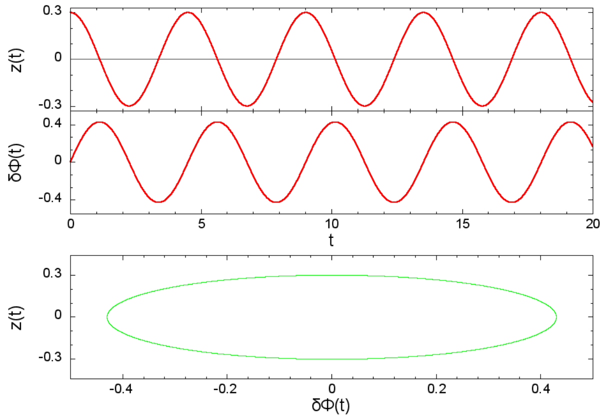

FIG. 1: Top panel: imbalance and phase difference as a function of time. Bottom panel: trajectory in the z - $\delta\phi$ plane. Initial conditions: $(z_0, \delta\phi_0) = (0.3, 0)$ and an interaction parameter $\Lambda = 1$. Time is in dimensionless units.

In JE, considering small oscillations around the minimum $(z, \delta\phi) = (0, 0)$, we can estimate the oscillation frequency. We can approximate $\sin\delta\phi \approx \delta\phi$, and considering the equations of motion (8) we arrive at $d^2z/dt^2 = -[1 + \cos(\delta\phi)\Lambda]z(t)$. We recover the expression of a harmonic oscillator with frequency $\omega_J = \omega_R\sqrt{1 + \cos\delta\phi\,\Lambda}$.

We can observe MQST regime in Fig. 2. In this example we have taken the initial conditions $(z_0, \delta\phi_0) =$

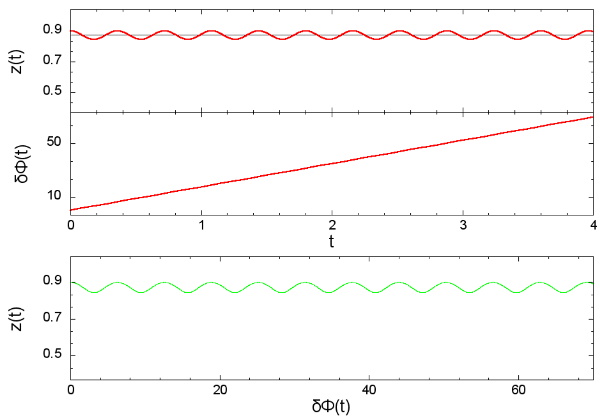

FIG. 2: Top panel: imbalance and phase difference as a function of time. Bottom panel: trajectory in the z - $\delta\phi$ plane. Initial conditions: $(z_0, \delta\phi_0) = (0.9, 0)$ and an interaction parameter $\Lambda = 20$. Time is in dimensionless units.

$(0.9, 0)$ and an interaction parameter $\Lambda = 20$. Under these initial conditions, the critical point corresponds to $\Lambda_C \approx 3.5 \ll \Lambda$. The top panel shows slight oscillations in the imbalance around a non-zero value, $\langle z \rangle_t \neq 0$. The phase difference, however, increases linearly with time so the trajectories do not correspond to closed orbits in the $z - \delta\phi$ plane. We observe that the MQST regime occurs for z_0 close to 1 (or -1) and for strong interaction between particles. Therefore, this regime takes place when one of the condensates is mostly populated than the other, and the interparticle interaction is heavy enough, according to Eq. (12).

B. Bifurcation

When the initial conditions are close to the critical point, the behavior starts to separate from the two dynamical regimes shown in Figs. 1 and 2. Now, we are going to analyze what happens near this point.

In Fig. 3, the initial conditions correspond to $(z_0, \delta\phi_0) = (0.8, 0)$, and the interaction parameter is the critical one $\Lambda = \Lambda_C = 5$. In the top panel, we observe a particular behavior, the imbalance decreases to zero and is maintained there for some time. Then starts to increase until the initial value is reached and decreases with the same behavior. Sometimes the imbalance tends to z_0 and other to $-z_0$. We note that the phase difference increases or decreases by 2π each time the imbalance goes and returns to zero.

We have seen that the two regimes are characterized by a different time average of the imbalance. It is $\langle z \rangle_t = 0$ in JE, and $\langle z \rangle_t \neq 0$ in MQST. Hence we can use this to identify the dynamical regime.

We fixe the initial conditions ($z_0 = \pm 0.9\, , \delta\phi_0 = 0$) and run the numerical Runge-Kutta program for different interaction parameters Λ. In Fig. 4 we show the temporal mean value of the imbalance. For $\Lambda \lesssim 3.15$, the imbalance mean value is close to 0, and for $\Lambda > 3.15$ takes a non-zero value, meaning that the dynamical regimes are JE and MQST, respectively. We can appreciate the positive branch, which corresponds to initial value $z_0 = +0,9$ and the negative branch which corresponds to the initial value $z_0 = -0,9$. Both of them have the same behavior for $\Lambda \lesssim 3.15$ as we expected. The critical interaction parameter calculated in analogy with the simple pendulum corresponds to $\Lambda_C \approx 3.55$ which is in good agreement but slightly larger than the numerical value. This is an accordance to what we have seen in Fig. 3. Where as we noted, in the critical point the mean value is different from zero.

We have seen that in JE the phase difference exhibits also sinusoidal oscillations around the zero value. Since these oscillations vanish in MQST, we can perform an analogous study computing now the temporal mean value $\langle \delta\phi \rangle_t$. The bifurcation must be located at the same value of Λ. In Fig. 5, we can appreciate the bifurcation at the same point as in Fig. 4. For $\Lambda \lesssim 3.14$, the temporal mean value is $\langle \delta\phi \rangle_t = 0$, hence the dynamics correspond to JE, whereas for $\Lambda > 3.15$ to MQST. Notice that in the latter, $\langle \delta\phi \rangle_t$ increases linearly with time, due to the linear behavior of $\delta\phi(t)$. In Figs. 4 and 5 we have computed the temporal mean value for different final times. Cyan line is calculated for a time average 10000 times larger than the red dashed line. We have repeated the calculation using different initial conditions ($z_0 \simeq \pm 1, \delta\phi_0 = 0$) and we have obtained the bifurcation in the same value $\Lambda_C \simeq 3.15$.

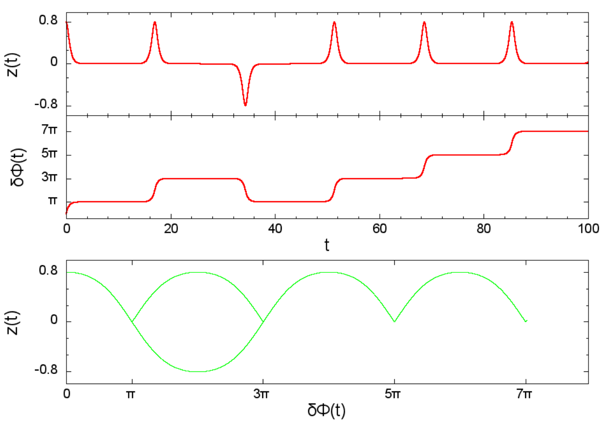

FIG. 3: Top panel: imbalance and phase difference as a function of time. Bottom panel: trajectory in the z - $\delta\phi$ plane. Initial conditions: $(z_0, \delta\phi_0) = (0.8, 0)$ and an interaction parameter $\Lambda = 5$. Time is in dimensionless units.

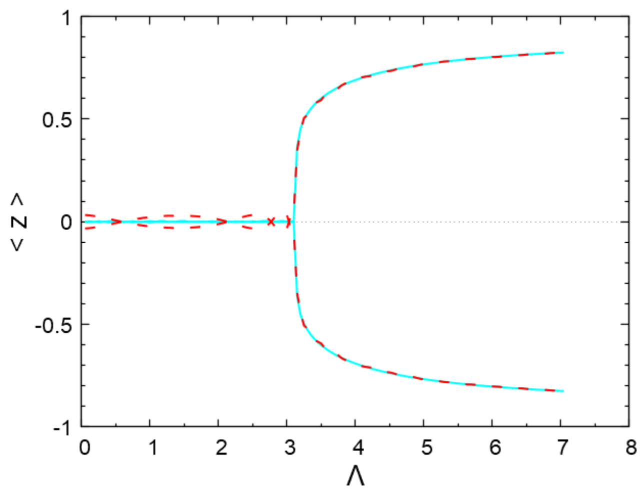

FIG. 4: Imbalance mean value as a function of Λ. Initial conditions $z_0 = \pm 0.9, \delta\phi_0 = 0$. Cyan line is calculated for 10000 times more than red dashed line.

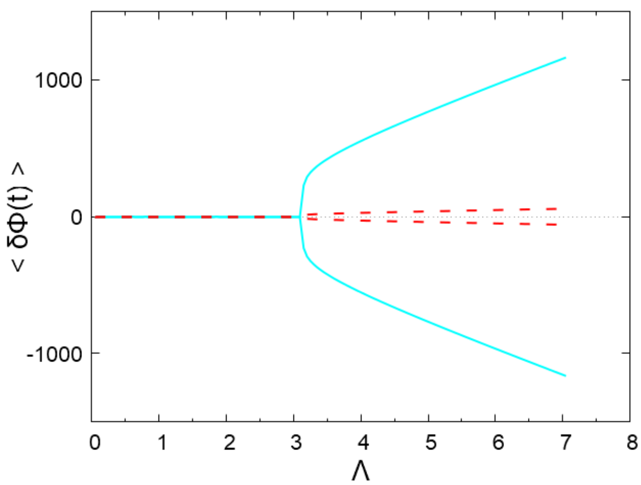

FIG. 5: Phase difference mean value as a function of Λ. Initial conditions $z_0 = \pm 0.9\,, \delta\phi_0 = 0$. Cyan line is calculated for 10000 times more than red dashed line.

IV. CONCLUSIONS

In this work we have studied the tunneling dynamics between two weakly linked Bose-Einsteins condensates confined by a double-well trap. We have used as theoretical framework the GP equation. Using the standard two mode approximation we have arrived at two coupled differential equations of motion in terms of the imbalance and phase difference. We have studied the two dynamical regimes: Josephson effect and Macroscopic quantum self trapping.

We have observed the oscillations around the minimum in JE, noting the fast tunneling of the bosons trough the barrier between the two BECs. We have observed the imbalance slight oscillations in the MQST regime, which occurs around a value different to 0. In MQST regime, we have observed that the phase difference increases (or decreases) linearly with time in contrast with JE. Also we have analyzed what happens in the critical case between the two regimes. We have worked with the critical interaction parameter, which depends on the initial conditions, calculated with the analogy of the simple pendulum system.

Finally, we have studied the bifurcation between the two regimes. We have used the fact that in JE the oscillations are around $z = 0$, and $\delta\phi = 0$, whereas in MQST not. We have used a Runge-Kutta method to solve the two coupled differential equations with initial conditions $z_0 = \pm 0.9\,, \delta\phi_0 = 0$. Then, we have analyzed different interaction parameters Λ, and calculated the average values of $z(t)$ and $\delta\phi$ with time. With this approach we have investigated the bifurcation between the two regimes.

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[1] C. F. Barenghi and N. G. Parker *A Primer on Quantum Fluids*, Springer, Aug 2016.
[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and Sandro Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., Vol. 71, No. 3, April 1999.
[3] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler,*Direct Observation of Tunneling and Nonlinear Self-Trapping in a Single Bosonic Josephson Junction*, Phys. Rev. Lett. **95**, 010402 (2005).
[4] Neil W. Ashcroft, N. David Mermin, Cornell University *Solid State Phiysics* 1976 by Harcourt. Inc. College edition.
[5] M. Melé-Messeguer, B. Juliá-Díaz, M. Guilleumas, A Polls, and A Sanpera, *Weakly linked binary mixtures of F = 1 87Rb Bose–Einstein condensates*, New J. Phys. **13**, 033012 (2011).
[6] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, *Classical Bifurcation at the Transition from Rabi to Josephson Dynamics*, Phys. Rev. Lett. **105**, 204101 (2010).
[7] A. Smerzi, S. Fantoni, S. Giovanazzi, and S. R. Shenoy, *Quantum Coherent Atomic Tunneling between Two Trapped Bose-Einstein Condensates*, Phys. Rev. Lett. **79**, 4950 (1997).