Exact solution to the exit-time problem for an undamped free particle driven by Gaussian white noise

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(Received 26 October 1995)

In a recent paper [Phys. Rev. Lett. **75**, 189 (1995)] we have presented the exact analytical expression for the mean exit time, $T(x,v)$, of a free inertial process driven by Gaussian white noise out of a region $(0,L)$ in space. In this paper we give a detailed account of the method employed and present results on asymptotic properties and averages of $T(x,v)$.

PACS number(s): 02.50.−r, 05.40.+j, 05.60.+w, 05.20.Dd

I. INTRODUCTION

The noise-induced escape problem from a stable state is ubiquitous in many branches of mathematics, physics, chemistry, and engineering [1–5]. The problem has been the subject of intense research since the end of the last century when Arrhenius law was first published and most especially after the landmark work of Kramers in 1940 [6].

From a dynamical point of view the escape problem is closely related with the evaluation of the mean exit time (MET) out of an interval of a diffusing particle moving in a potential $V(x)$, under the influence of a heat bath. In many cases the dynamics of the system is governed by the following stochastic differential equation for the position $X(t)$ of the particle,

$$\ddot{X} + \beta\dot{X} + f(X) = \xi(t), \tag{1.1}$$

where β is the damping constant, $f(x) = -V'(x)$ is a deterministic force, and the thermal fluctuations are represented by $\xi(t)$, a zero-centered Gaussian white noise with correlation function

$$\langle \xi(t)\xi(t')\rangle = D\delta(t-t'). \tag{1.2}$$

The evaluation of MET's for second-order processes such as (1.1) is an extremely difficult task and there exists an enormous literature on different approximation schemes to this problem, ranging from strong and weakly damped systems to small noise intensities [7,8]. Among these approximations the most frequently used stochastic model is that of a high-damping system where inertial effects can be neglected. This simplification allows us to deal only with the position variable $X(t)$, without having to worry about the associated velocity variable $\dot{X}(t)$. This approximation therefore reduces system (1.1) to an one-dimensional random process for which the MET is exactly known [8,9].

In a recent paper [10] we have presented for the first time the exact expression of the MET out of an interval $(0,L)$ for a particular, but relevant, inertial process. The model describes the motion of an undamped free particle under the influence of a random acceleration:

$$\ddot{X} = \xi(t), \tag{1.3}$$

where $X(t)$ represents the displacement of the particle and $\xi(t)$ is Gaussian white noise with correlation function given by Eq. (1.2). Let us denote by v the velocity of our undamped free inertial process. The mean exit time of process (1.3) is a function of the displacement and the velocity, $T = T(x,v)$, and obeys the partial differential equation [10]

$$\frac{1}{2}D\frac{\partial^2 T}{\partial v^2} + v\frac{\partial T}{\partial x} = -1, \tag{1.4}$$

with boundary conditions

$$T(L,v) = 0 \quad \text{if } v \geq 0; \qquad T(0,v) = 0 \quad \text{if } v \leq 0. \tag{1.5}$$

Boundary value problems such as this are known in the mathematics literature as a "problem of Fichera" and also as "Kolmogorov's exit problem." It was shown in the late 1950's that they are well posed boundary value problems [11]. In physics, the boundary conditions given by Eq. (1.5), in the phase space for the joint probability density of the displacement and the velocity, were first introduced by Wang and Uhlenbeck [12]. Although the half range problem ($L = \infty$) when $f(X)$ is constant has been solved recently [13] and several approximations have also been obtained [14], any attempt to solve the exit time problem out of a finite interval $(0,L)$, even in the simple case (1.4) and (1.5), has failed to result in closed and exact expressions for $T(x,v)$ [15,16]. The reason for this difficulty lies in the special form of the boundary conditions (1.5) with data on a nonsmooth boundary at $v=0$. Indeed, problem (1.4),(1.5) is defined in the strip $0 \leq x \leq L$ of phase space. However, we only know the value of $T(x,v)$ on the half lines ($x = 0, v < 0$) and ($x = L, v > 0$). This results in a discontinuous boundary contour at $v = 0$ as is shown in Fig. 1.

Our aim in this paper is to present a full account of the method and the results briefly outlined in our previous work [10] along with some new results on averages and the asymptotics of the mean exit time. Our method for solving the boundary value problem (1.4) and (1.5) basically consists in decomposing $T(x,v)$ in the two regions of phase space where $v > 0$ and $v < 0$. Having done this, a formal solution to the problem can be written as a functional of the derivative of the mean exit time at

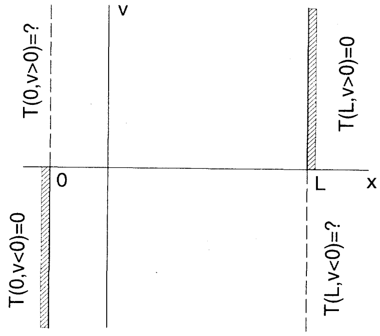

FIG. 1. Boundary conditions (1.5) for $T(x,v)$ in the position-velocity (x,v) phase space.

zero velocity. The matching at $v = 0$ of the decomposition then results in a singular integral equation whose solution allows us to obtain the explicit expression for $T(x, v = 0)$, which in turn leads to the complete mean exit time $T(x,v)$. We develop this technique in the following sections.

II. ANALYSIS

We will first obtain a formal solution to the boundary-value problem (1.4) and (1.5) in the Laplace domain. To this end we observe from Eqs. (1.4) and (1.5) that the mean exit time $T(x,v)$ satisfies the fundamental symmetry relation

$$T(x,v) = T(L-x,-v). \tag{2.1}$$

This equation implies the following continuity conditions at $v = 0$

$$T(x,0) = T(L-x,0),$$

$$\left.\frac{\partial T(x,v)}{\partial v}\right|_{v=0} = -\left.\frac{\partial T(L-x,v)}{\partial v}\right|_{v=0}. \tag{2.2}$$

We note that Eq. (2.1) allows us to write the solution $T(x,v)$ for all v once we know the solution of (1.4) and (1.5) for, say, $v \le 0$. We thus assume that $v \le 0$ and write

$$T_1(x,v) \equiv T(x,v) \qquad \text{if} \quad v \le 0. \tag{2.3}$$

Hence, in dimensionless units defined by

$$u = x/L, \quad y = -(2/LD)^{1/3}v,$$
$$T_1'(u,y) = (D/2L^2)^{1/3} T_1(x,v), \tag{2.4}$$

the mean exit time $T_1'(u,y)$ obeys the equation

$$\frac{\partial^2 T_1'}{\partial y^2} - y\frac{\partial T_1'}{\partial u} = -1, \tag{2.5}$$

with boundary conditions

$$T_1'(0,y) = 0 \quad (y \ge 0), \qquad T_1'(u,\infty) = 0 \quad (0 \le u \le 1). \tag{2.6}$$

In spite of the fact that the range of u is bounded by an upper bound, it is still permissible to define the Laplace transform of $T_1'(u,y)$ with respect to u [17]:

$$\hat{T}_1(s,y) \equiv \int_0^\infty e^{-su} T_1'(u,y)\,du.$$

This transformation leads to the following inhomogeneous Airy equation for $\hat{T}_1(s,y)$

$$\frac{d^2\hat{T}_1}{d\zeta^2} - \zeta\hat{T}_1 = -s^{-5/3}, \tag{2.7}$$

where $\zeta \equiv s^{1/3}y$. The general solution of Eq. (2.7) under the condition $\hat{T}_1(s,\infty) = 0$ reads

$$\hat{T}_1(s,y) = s^{-5/3}\alpha(s)\mathrm{Ai}(\zeta) + \pi s^{-5/3} \times \left[\mathrm{Bi}(\zeta)\int_\zeta^\infty \mathrm{Ai}(t)dt + \mathrm{Ai}(\zeta)\int_0^\zeta \mathrm{Bi}(t)dt\right], \tag{2.8}$$

where $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ are Airy functions [18], and $\alpha(s)$ is an unknown quantity independent of ζ. Let $\hat{\phi}(s)$ be the Laplace transform of the derivative of $T_1'(u,y)$ with respect to y at $y = 0$

$$\phi(u) \equiv \left.\frac{\partial T_1'(u,y)}{\partial y}\right|_{y=0}. \tag{2.9}$$

If we take into account the following properties of the Airy functions [18],

$$\mathrm{Ai}'(0) = -\frac{\mathrm{Bi}'(0)}{\sqrt{3}} = -\frac{3^{-1/3}}{\Gamma(1/3)}, \qquad \int_0^\infty \mathrm{Ai}(t)dt = \frac{1}{3}, \tag{2.10}$$

then the substitution of Eq. (2.8) into Eq. (2.9) allows us to write $\alpha(s)$ in terms of $\hat{\phi}(s)$:

$$\alpha(s) = \frac{\pi}{\sqrt{3}} - 3^{1/3}\Gamma(1/3)s^{4/3}\hat{\phi}(s).$$

We substitute this expression into Eq. (2.8) with the result

$$\hat{T}_1(s,y) = -3^{1/3}\Gamma(1/3)s^{-1/3}\hat{\phi}(s)\mathrm{Ai}(\zeta) + \pi s^{-5/3}\left[\mathrm{Bi}(\zeta)\int_\zeta^\infty \mathrm{Ai}(t)dt + \mathrm{Ai}(\zeta)\int_0^\zeta \mathrm{Bi}(t)dt + 3^{-1/2}\mathrm{Ai}(\zeta)\right].$$

We recall that $\zeta = ys^{1/3}$ and write this equation in the form

$$\hat{T}_1(s,y) = -3^{1/3}\Gamma(1/3)\hat{\phi}(s)s^{-1/3}\mathrm{Ai}(ys^{1/3}) + \hat{R}(s,y), \tag{2.11}$$

where

$$\hat{R}(s,y) = \pi s^{-5/3}\left[\mathrm{Bi}\left(ys^{1/3}\right)\int_{ys^{1/3}}^{\infty}\mathrm{Ai}(t)dt + \mathrm{Ai}\left(ys^{1/3}\right)\int_0^{ys^{1/3}}\mathrm{Bi}(t)dt + 3^{-1/2}\mathrm{Ai}\left(ys^{1/3}\right)\right]. \tag{2.12}$$

If we now take into account that Airy functions are related to modified Bessel functions [18],

$$\mathrm{Ai}(ys^{1/3}) = \frac{1}{\pi}\left(\frac{ys^{1/3}}{3}\right)^{1/2} K_{1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right), \tag{2.13}$$

the Laplace inversion formula [19]

$$\mathcal{L}^{-1}\{s^{-\nu/2}K_\nu(as^{1/2})\} = a^{-\nu}(2u)^{\nu-1}e^{-a^2/4u}, \tag{2.14}$$

and use the convolution theorem, we finally obtain

$$T_1'(u,y) = -\frac{1}{3^{1/3}\Gamma(2/3)}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}\phi(u-z)dz + R(u,y), \tag{2.15}$$

where $R(u,y)$ is the Laplace inversion of Eq. (2.12). Equation (2.15) gives the mean exit time as a functional of the derivative of the mean exit time at zero velocity [cf. Eq. (2.9)]. Note that Eq. (2.15) is only a formal expression as long as the function $\phi(u)$ remains unknown. We have thus finished the first step of our method. In the next section we will find $\phi(u)$.

III. MEAN EXIT TIME AT ZERO VELOCITY

Let us now first obtain the explicit expression for $\phi(u)$. To this end we start from Eq. (2.15) and set $y=0$

$$T_1'(u,0) = -\frac{1}{3^{1/3}\Gamma(2/3)}\int_0^u \frac{\phi(u-z)}{z^{2/3}}dz + R(u,0).$$

If we take into account Eq. (2.12), use the property [18]

$$\mathrm{Ai}(0) = \frac{\mathrm{Bi}(0)}{\sqrt{3}} = \frac{3^{-2/3}}{\Gamma(2/3)},$$

and invert the Laplace transform we obtain

$$R(u,0) = \frac{\pi}{3^{1/6}\Gamma^2(2/3)}u^{2/3}.$$

Hence,

$$T_1'(u,0) = -\frac{1}{3^{1/3}\Gamma(2/3)}\int_0^u \frac{\phi(z)}{(u-z)^{2/3}}dz + \frac{\pi}{3^{1/6}\Gamma^2(2/3)}u^{2/3}. \tag{3.1}$$

We see from the first of the matching conditions given by Eq. (2.2) that

$$T_1'(u,0) = T_1'(1-u,0).$$

Therefore, from Eq. (3.1) we get

$$\frac{1}{3^{1/3}\Gamma(2/3)}\left[\int_0^u \frac{\phi(z)}{(u-z)^{2/3}}dz - \int_u^1 \frac{\phi(1-z)}{(z-u)^{2/3}}dz\right] = \frac{\pi}{3^{1/6}\Gamma^2(2/3)}\left[u^{2/3}-(1-u)^{2/3}\right]. \tag{3.2}$$

But, from the second matching condition (2.2) and from Eq. (2.9), one can easily see that

$$\phi(u) = -\phi(1-u).$$

Using this and the reflection formula of the gamma function $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\pi\nu)$ [18], we see from Eq. (3.2) that the unknown function $\phi(u)$ satisfies the integral equation

$$\int_0^1 \frac{\phi(z)}{|u-z|^{2/3}}dz = \frac{3^{2/3}\Gamma(1/3)}{2}[u^{2/3}-(1-u)^{2/3}]. \tag{3.3}$$

In Appendix A we show that the solution of Eq. (3.3) is given by

$$\phi(u) = \frac{3^{1/6}}{2\Gamma^2(5/6)}u^{-1/6}\frac{d}{du}\int_u^1 \frac{t^{1/3}dt}{(t-u)^{1/6}}\frac{d}{dt}\int_0^t \frac{\tau^{-1/6}}{(t-\tau)^{1/6}}\left[(1-\tau)^{2/3}-\tau^{2/3}\right]d\tau. \tag{3.4}$$

After some amount of algebra involving the Gauss hypergeometric function, $F(a,b;c;z)$, this solution can be written in the following more explicit and convenient form (see Appendix A)

$$\phi(u) = Mu^{-1/6}(1-u)^{-1/6}\left[F\left(1,-\frac{2}{3};\frac{5}{6};1-u\right) - F\left(1,-\frac{2}{3};\frac{5}{6};u\right)\right], \tag{3.5}$$

where

$$M = \frac{3^{1/6}\Gamma(3/2)}{2\Gamma(5/6)\Gamma(4/3)}.$$

Having obtained the explicit expression of $\phi(u)$ we are now in the position to evaluate $T(x, v = 0)$. In effect, the substitution of Eq. (3.5) into Eq. (3.1) yields, after lengthy calculations detailed in Appendix B, the exact expression of $T(x, 0)$. In the original units [cf. Eq. (2.4)] this expression reads

$$T(x,0) = N\left(\frac{2L^2}{D}\right)^{1/3}\left(\frac{x}{L}\right)^{1/6}\left(1-\frac{x}{L}\right)^{1/6}\left[F\left(1,-\frac{1}{3};\frac{7}{6};\frac{x}{L}\right)+F\left(1,-\frac{1}{3};\frac{7}{6};1-\frac{x}{L}\right)\right], \tag{3.6}$$

where $N = (4/3)^{1/6}/\Gamma(7/3)$. In order to check the correctness of our calculations we compare the numerical values of $T(x, 0)$ with simulation results. Figure 2 shows the complete agreement between the expression of $T(x, 0)$ given by Eq. (3.6) and simulation data. Monte Carlo values were obtained by simulating a free inertial system driven by Markovian dichotomous noise [20] of value $\pm a$ and average switching time λ^{-1}. This noise is known to converge in distribution to a Gaussian white noise of intensity D when $a \to \infty$ and $\lambda \to \infty$, provided $D = a^2/\lambda$ [21]. Several simulations were run for growing values of a and λ and checked to converge.

IV. COMPLETE MEAN EXIT TIME

We will now obtain the exact expression of the complete mean exit time $T(x, v)$ for all values of position x and velocity v. The starting point of this derivation is Eq. (2.15):

$$T_1'(u,y) = -\frac{3^{1/6}\Gamma(1/3)}{2\pi}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}\phi(u-z)dz + R(u,y), \tag{4.1}$$

where the function $R(u, y)$ is given by the inverse Laplace transform of Eq. (2.12). In Appendix C we show that the expression for $R(u, y)$ is

$$R(u,y) = \frac{3^{1/3}}{2\Gamma(1/3)}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}(u-z)^{1/3}dz + \frac{(\pi y)^{1/2}}{6}\int_0^u \frac{e^{-y^3/18z}}{z^{1/2}}\left[I_{-1/6}\left(\frac{y^3}{18z}\right)+I_{1/6}\left(\frac{y^3}{18z}\right)\right]dz. \tag{4.2}$$

Since $T_1'(u, y)$ is the dimensionless MET for negative velocities [cf. Eqs. (2.3) and (2.4)] then from the symmetry relation (2.1) we see that the total dimensionless MET reads

$$T'(u,y) = T_1'(1-u,|y|)\Theta(-y) + T_1'(u,|y|)\Theta(y),$$

where $T_1'(u, y)$ is given by Eqs. (4.1) and (4.2) and $\Theta(y)$ is the Heaviside step function. Therefore, in the original units [cf. Eq. (2.4)], the complete mean exit time can be written in the form

$$T(x,v) = \left(\frac{2L^2}{D}\right)^{1/3}\left[A\left(\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(-v) + A\left(1-\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(v)\right], \tag{4.3}$$

where $A(u, y) = T_1'(u, y)$, that is,

$$A(u,y) = -\frac{3^{1/6}\Gamma(1/3)}{2\pi}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}\phi(u-z)dz + R(u,y), \tag{4.4}$$

where $\phi(u)$ is given by Eq. (3.5) and $R(u, y)$ is given by Eq. (4.2). We plot the complete solution (4.3) and (4.4) in Fig. 3. In Fig. 4 we plot $T(x, v)$ as a function of v for two particular values of x. Note that at $x = 0$, $T(0, v)$ is zero for negative velocities and reaches the maximum value at some positive velocity.

Next we focus in the behavior of $T(x, v)$ when $|v| \to \infty$. We first observe that if $|v| \to \infty$ then $|y| \to \infty$ and the first integral on the right-hand side of Eq. (4.4) is exponentially small. In this case the only contribution comes from the Bessel functions. Taking into account that as $|y| \to \infty$ [18],

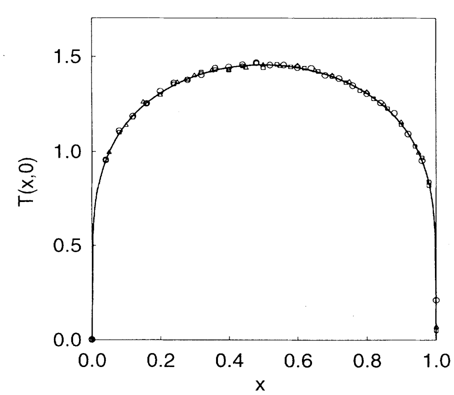

FIG. 2. $T(x, 0)$ as a function of x, for noise intensity $D = 1$ and $L = 1$ (solid line). Simulation data correspond to $a = 10$ and $\lambda = 100$ (circles), $a = 100$ and $\lambda = 10^4$ (squares), and $a = 200$ and $\lambda = 4 \times 10^4$ (triangles). Error bars are of the order of symbol sizes ($\sigma \sim 10^{-2}$).

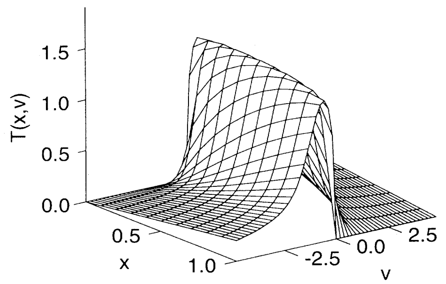

FIG. 3. Plot of the exact expression for $T(x,v)$ as a function of x and v, for $D=1$ and $L=1$.

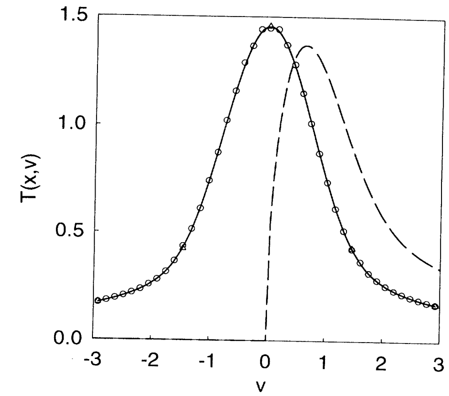

FIG. 4. $T(L/2,v)$ as a function of v, for $D=1$ and $L=1$ (solid line), along with simulation data for $a=10$ and $\lambda=100$ (circles), and $a=100$ and $\lambda=10^4$ (triangles). The dashed line represents $T(0,v)$ for $D=L=1$.

$$I_{\pm 1/6}\left(\frac{y^3}{18z}\right) \sim \frac{e^{y^3/18z}}{(\pi y^3/9z)^{1/2}}\left[1+\frac{2z}{y^3}+O\left(\frac{1}{y^6}\right)\right] ,$$

we get from Eqs. (4.2) and (4.4) that

$$A(u,y) \sim \frac{u}{y}\left[1+\frac{u}{y^3}+O\left(\frac{1}{y^6}\right)\right] \qquad (y\to\infty). \quad (4.5)$$

Hence the MET decreases as $1/|v|$, that is,

$$T(x,v) \sim \frac{x}{|v|}\Theta(-v)+\frac{L-x}{|v|}\Theta(v) \qquad (|v|\to\infty). \quad (4.6)$$

Note that the quantities $x/|v|$ and $(L-x)/|v|$ correspond to the ballistic times to cross the lower and upper boundaries under constant speed. We thus see that for sufficiently large velocities the mean exit time is not influenced by the input noise. This is not a surprising result if one recalls that a finite acceleration has negligible effects when the initial velocity is large enough. Thus, for instance, in the case of a free particle driven by a constant acceleration a, an elementary calculation shows that the time to reach a given position, $x=L$, starting from $x_0=0$ and $v_0>0$, is approximately given by L/v_0 as long as $v_0 \gg (aL)^{1/2}$.

Another interesting limiting behavior is provided by the case of weak noise where $D\to 0$. The governing equations of a random process are very difficult to handle in the $D\to 0$ limit, and sophisticated techniques involving path integrals have been used recently to deal with this limit [22,23]. However, from our expressions the weak noise behavior of $T(x,v)$ is easily obtained. In effect, if we first assume that $v\neq 0$ then from the dynamical equation (1.3) we see that, as $D\to 0$, the mean exit time converges to the ballistic times to cross the boundaries under constant velocity. Indeed, we note from Eq. (4.3) that the limit $D\to 0$ is equivalent to the limit $|v|\to\infty$. Therefore, from Eq. (4.5) we get

$$T(x,v) \sim \frac{x}{|v|}\left(1+D\frac{x}{2|v|^3}\right)\Theta(-v) + \frac{L-x}{|v|}\left(1+D\frac{L-x}{2|v|^3}\right)\Theta(v)+O\left(D^2\right) \qquad (D\to 0), \quad (4.7)$$

and the lowest order of this expansion yields the ballistic time to cross the boundaries. At $v=0$ the MET presents a completely different behavior because in this case $T(x,0)$ diverges as $D\to 0$. In effect, we see from Eq. (3.6) that the mean exit time at zero velocity grows as

$$T(x,0) \sim \frac{1}{D^{1/3}} \qquad (D\to 0)$$

$(x\neq 0,L)$. It is interesting to compare this with the faster growth of the MET for a first-order process, $\dot{X}(t)=\xi(t)$, where $T(x)\sim 1/D$ as $D\to 0$. This greater divergence is due to the fact that accelerated particles move faster than when no acceleration is present (see below).

Let us finally obtain the asymptotic behavior of $T(x,v)$ as $L\to\infty$. Note that when L is very large and $|v|$ is finite we have from Eq. (4.4) that

$$\lim_{L\to\infty} A[u_L,(2/LD)^{1/3}|v|] = \lim_{L\to\infty} A\left(u_L,0\right), \quad (4.8)$$

where $u_L=x/L$ or $u_L=1-x/L$. Hence,

$$T(x,v) \sim \left(\frac{2L^2}{D}\right)^{1/3}\left[A\left(\frac{x}{L},0\right)\Theta(-v)+A\left(1-\frac{x}{L},0\right)\Theta(v)\right] \qquad (L\to\infty), \quad (4.9)$$

but, taking into account the continuity of the mean exit time at $v = 0$, we see that

$$A\left(\frac{x}{L}, 0\right) = A\left(1 - \frac{x}{L}, 0\right), \tag{4.10}$$

and Eq. (4.9) clearly shows that the behavior of $T(x, v)$ when L is large equals that of the mean exit time at zero velocity, i.e.,

$$T(x, v) \sim T(x, 0) \qquad (L \to \infty), \tag{4.11}$$

provided that x and $|v|$ are both finite. Therefore, from Eq. (3.6) we get

$$T(x, v) \sim N\left(\frac{2L^2}{D}\right)^{1/3}\left(\frac{x}{L}\right)^{1/6}\left[1 + F\left(1, -\frac{1}{3}; \frac{7}{6}; 1\right)\right],$$

that is,

$$T(x, v) \sim \frac{2^{2/3}3^{5/6}}{\Gamma(1/3)D^{1/3}} x^{1/6}L^{1/2} \qquad (L \to \infty), \tag{4.12}$$

a valid expression for all finite values of the velocity. If we recall that the mean first passage time (MFPT) to a given value, say $x = 0$, can be obtained as the limit $L \to \infty$ from the mean exit time out of the interval $(0, L)$ [9], we see from Eq. (4.12) that the MFPT of our free inertial particle is ∞. We observe that an analogous situation arises for free one-dimensional processes driven by Gaussian white noise, $\dot{X}(t) = \xi(t)$, where the MFPT to a given label is ∞ and the MET out of an interval $(0, L)$ is $T(x) = x(L - x)/D$. However, in this latter case the MET grows linearly as L increases, i.e., $T(x) \sim L$ while for our inertial process the MET grows as $T(x, v) \sim L^{1/2}$. This slower growth can be explained if one takes into account that a free undamped particle moves faster when it is accelerated than when no acceleration is present [in the first case $\langle X^2(t)\rangle \sim t^3$, while in the second case $\langle X^2(t)\rangle \sim t$ [24]]. As a consequence, the MET is smaller for inertial processes.

V. AVERAGES OF THE MEAN EXIT TIME

We will now evaluate some averages of the mean exit time. We first assume that the initial velocity v of the particle is a random variable with a given probability density function $p(v)$. Then the averaged mean exit time $\overline{T}_v(x)$ over all initial velocities is

$$\overline{T}_v(x) = \int_{-\infty}^{\infty} p(v)T(x, v)dv. \tag{5.1}$$

In what follows we will assume that $p(v)$ is an even function of v, that is, $p(-v) = p(v)$. Now from Eq. (4.3) and Eq. (5.1) we have

$$\overline{T}_v(x) = \left(\frac{2L^2}{D}\right)^{1/3}\int_0^{\infty} p(v)\left[A\left(\frac{x}{L}, (2/LD)^{1/3}|v|\right) + A\left(1 - \frac{x}{L}, (2/LD)^{1/3}|v|\right)\right]dv. \tag{5.2}$$

In order to proceed further we need to specify $p(v)$. The commonest probability distribution by far is given by the Gaussian density (which, in dimensionless units, we assume to be of zero mean and unit variance)

$$p(v) = \frac{(2/LD)^{1/3}}{\sqrt{2\pi}} \exp\left[-\frac{v^2(2/LD)^{2/3}}{2}\right]. \tag{5.3}$$

Unfortunately the substitution of this density into Eq. (5.2) does not allow us to obtain an explicit expression for the averaged mean exit time. However, we have carried out this evaluation numerically and the result is plotted in Fig. 5.

Let us analyze the asymptotics of $\overline{T}_v(x)$ as $L \to \infty$. Suppose we have any even density $p(v)$, then we see from Eq. (5.2) that when L is large

$$\overline{T}_v(x) \sim \left(\frac{2L^2}{D}\right)^{1/3}\left[A\left(\frac{x}{L}, 0\right) + A\left(1 - \frac{x}{L}, 0\right)\right] \times \int_0^{\infty} p(v)dv$$

$(x \neq 0, L)$ but from Eq. (4.10) and taking into account the normalization of $p(v)$ we have

$$\overline{T}_v(x) \sim 2\left(\frac{2L^2}{D}\right)^{1/3} A\left(\frac{x}{L}, 0\right) \qquad (L \to \infty),$$

that is [cf. Eq. (4.12)]

$$\overline{T}_v(x) \sim T(x, 0) \sim L^{1/2} \qquad (L \to \infty). \tag{5.4}$$

Therefore, the averaged MET over all initial velocities has the same asymptotic behavior, when $L \to \infty$, as that of the complete MET.

We now evaluate the averaged mean exit time $\overline{T}_x(v)$ over all initial positions x. This time is defined by

$$\overline{T}_x(v) = \int_0^L p(x)T(x, v)dx, \tag{5.5}$$

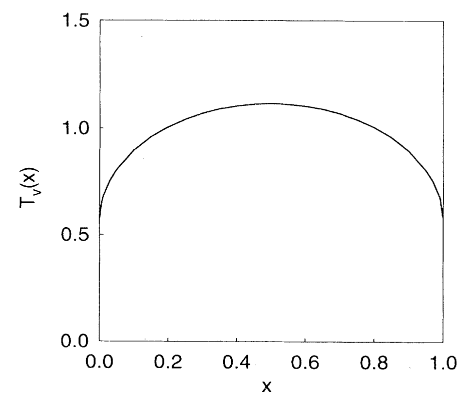

FIG. 5. $\overline{T}_v(x)$ as a function of x ($D = L = 1$). The distribution of initial velocities is given by the Gaussian density (5.3).

where $p(x)$ is the probability density function of the initial displacement x. The substitution of Eq. (4.3) into Eq. (5.5) yields

$$\overline{T}_x(v) = \left(\frac{2L^2}{D}\right)^{1/3} C_L(v), \tag{5.6}$$

where

$$C_L(v) \equiv \int_0^1 f(u,L)\{A[u,(2/LD)^{1/3}|v|]\Theta(-v) + A[1-u,(2/LD)^{1/3}|v|]\Theta(v)\}du, \tag{5.7}$$

and $f(u,L) \equiv Lp(Lu)$ is the transformed probability density function of the initial position over the interval $0 < u < 1$. If we assume that the initial position of the particle is uniformly distributed on the interval $(0, L)$, then

$$\overline{T}_x(v) = \frac{1}{L}\int_0^L T(x,v)dx, \tag{5.8}$$

and Eq. (5.7) reduces to

$$C_L(v) = \int_0^1 A[u,(2/LD)^{1/3}|v|]du. \tag{5.9}$$

If we substitute Eq. (4.4) into this equation and exchange the order of integrations we have

$$\begin{aligned} C_L(v) \equiv{}& \frac{3^{1/6}\Gamma(1/3)}{2\pi}\int_0^1 \frac{e^{-y^3/9z}}{z^{2/3}}\left[\int_0^{1-z}\phi(u)du + \frac{\pi 3^{7/6}}{4\Gamma^2(1/3)}(1-z)^{4/3}\right]dz \\ &+(\pi^{1/2}/6)y^{1/2}\int_0^1 z^{-1/2}(1-z)e^{-y^3/18z}\left[I_{-1/6}\left(\frac{y^3}{18z}\right) + I_{1/6}\left(\frac{y^3}{18z}\right)\right]dz, \end{aligned} \tag{5.10}$$

where $y \equiv (2L/D)^{1/3}|v|$. In Fig. 6 we plot $\overline{T}_x(v)$ for the uniform distribution.

There is a simple expression for $\overline{T}_x(v)$ when $v = 0$ and $p(x)$ is the uniform density. This expression can be obtained from Eqs. (5.6) and (5.10) but it is simpler to proceed as follows. In effect, when $v = 0$ the substitution of Eq. (3.6) into Eq. (5.8) yields

$$\overline{T}_x(0) = 2N\left(\frac{2L^2}{D}\right)^{1/3}\int_0^1 u^{1/6}(1-u)^{1/6} \times F\left(1,-\frac{1}{3};\frac{7}{6};u\right)du,$$

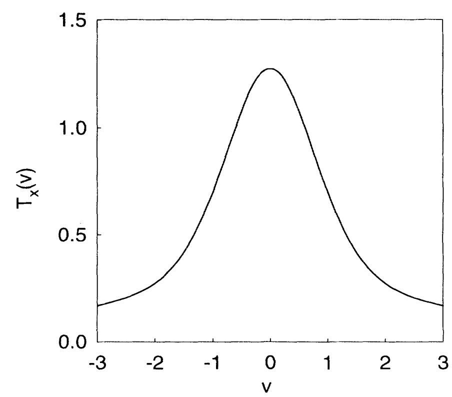

FIG. 6. $\overline{T}_x(v)$ as a function of v when the initial position is uniformly distributed in the interval $(0, 1)$ with $D = 1$ [cf. Eqs. (5.6) and (5.9)].

but [25]

$$\int_0^1 u^{1/6}(1-u)^{1/6}F\left(1,-\frac{1}{3};\frac{7}{6};u\right)du = \frac{\Gamma(1/6)\Gamma(7/6)}{10\Gamma(4/3)}.$$

Therefore,

$$\overline{T}_x(0) = \frac{3^{7/3}\Gamma(1/6)}{40\sqrt{\pi}}\left(\frac{2L^2}{D}\right)^{1/3}. \tag{5.11}$$

An interesting feature of this expression is that the dependence of $\overline{T}_x(0)$ on the size L of the interval is $L^{2/3}$. We recall that the dynamical exponent ν of a random process $X(t)$, which we assume zero centered, is given by

$$\langle X^2(t)\rangle \sim t^{2\nu}.$$

We have shown elsewhere [24] that for free inertial processes driven by white noise, the dynamical exponent of the displacement $X(t)$ is $\nu = 3/2$. As a consequence, Eq. (5.11) clearly demonstrates the reciprocity between exponents. This reciprocity had been conjectured in a previous work by a scaling argument [20].

Let us finally see that the asymptotic dependence on L of the averaged MET, $\overline{T}_x(v)$, for all values of the velocity and for any given density $p(x)$, it is also given by a power law with the same exponent $2/3$. Indeed, when $L \to \infty$ and $|v| < \infty$ we see from Eq. (5.7) that [see Eq. (4.10)]

$$\lim_{L\to\infty} C_L(v) = \int_0^1 A(u,0)[f(u)\Theta(-v) + f(1-u)\Theta(v)]du,$$

where $f(z)$ is the limit when $L \to \infty$ of the density $f(z,L)$ defined above. Note that this limit exists because of the normalization of $p(x)$ over the interval $(0,L)$. Therefore, the above integral does not depend on L and from Eq. (5.6) we obtain

$$\overline{T}_x(v) \sim L^{2/3} \quad (L \to \infty). \tag{5.12}$$

VI. CONCLUSIONS

We now briefly summarize the main results achieved. The mean exit time out of an interval for the motion of a free undamped particle under the influence of a random acceleration (modeled as Gaussian white noise) has been exactly obtained up to quadrature by

$$T(x,v) = \left(\frac{2L^2}{D}\right)^{1/3} \left[A\left(\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(-v) + A\left(1-\frac{x}{L},(2/LD)^{1/3}|v|\right)\Theta(v)\right],$$

where $A(u,y)$ is given by Eq. (4.4). The explicit expression of the mean exit time at zero velocity is simpler and it is given by

$$T(x,0) = N\left(\frac{2L^2}{D}\right)^{1/3}\left(\frac{x}{L}\right)^{1/6}\left(1-\frac{x}{L}\right)^{1/6}\left[F\left(1,-\frac{1}{3};\frac{7}{6};\frac{x}{L}\right)+F\left(1,-\frac{1}{3};\frac{7}{6};1-\frac{x}{L}\right)\right].$$

We have shown that the asymptotic dependence of the mean exit time on the length L of the interval is given by

$$T(x,v) \sim L^{1/2} \qquad (L \to \infty)$$

for all values of the velocity provided that $|v| < \infty$. This asymptotic relation clearly proves that the mean first-passage time to a given label of the system (1.3) is infinity. Moreover, the asymptotic behavior of $T(x,v)$ as $D \to 0$ (weak noise intensity) is

$$T(x,v) \sim \frac{x}{|v|}\left(1+D\frac{x}{2|v|^3}\right)\Theta(-v) + \frac{L-x}{|v|}\left(1+D\frac{L-x}{2|v|^3}\right)\Theta(v) + O\left(D^2\right) \qquad (D \to 0)$$

if $v \neq 0$, and $T(x,0) \sim D^{-1/3}$.

We have also obtained several averages of the mean exit time corresponding to a random initial velocity and to a random initial position. Thus, when the velocity is a random variable with a given density $p(v)$, the asymptotic dependence of the averaged MET, $\overline{T}_v(x)$, is also given by

$$\overline{T}_v(x) \sim L^{1/2} \qquad (L \to \infty).$$

Nevertheless, when the initial position is randomized over the interval $(0,L)$, with a given density $p(x)$, we have shown that the resulting mean exit time satisfies the following asymptotic relation for large L

$$\overline{T}_x(v) \sim L^{2/3} \qquad (L \to \infty).$$

This relation becomes exact at $v = 0$ for the uniform density.

We finally observe that the method employed for solving the boundary value problem (1.4) and (1.5) may open a new way of dealing with a variety of similar problems with relevant physical implications. In this direction we mention that the study of the mean exit time when a linear damping term is added to Eq. (1.3) is under present investigation and some results are expected soon.

ACKNOWLEDGMENTS

This work has been supported in part by Dirección General de Investigación Científica y Técnica under Contract No. PB93-0812, and by Societat Catalana de Física (Institut d'Estudis Catalans). We thank R. F. Pawula for a careful reading of the manuscript.

APPENDIX A: SOLUTION TO AN INTEGRAL EQUATION

Equation (3.3) is a particular case of more general integral equations of the form

$$\int_0^1 \frac{\phi(z)}{|u-z|^\alpha} = f(u) \qquad (0 \le u \le 1), \tag{A1}$$

where $0 < \alpha < 1$. Equation (A1) is a weakly singular Fredholm equation of first kind with the positive kernel

$$k(u-z) = \frac{1}{|u-z|^\alpha} \qquad (0 < \alpha < 1).$$

Positive kernels define positive operators K which, under general circumstances, can be factorized in the product of two Volterra adjoint operators, $K = HH^*$. Thus, if

ϕ satisfies the integral equation $K\phi = f$, where K is a positive operator and we can find a Volterra operator H such that $K = HH^*$ then the original equation can be expressed as the coupled pair $H\psi = f$ and $H^*\phi = \psi$. An explicit solution ϕ follows if these two equations can be solved in closed form. The above kernel can be represented in the form [26]

$$\frac{1}{|u-z|^\alpha} = \frac{1}{B(\alpha,(1-\alpha)/2)} \times \int_0^{\min(u,z)} \frac{(uz)^{(1-\alpha)/2}dt}{t^{1-\alpha}(u-t)^{(1+\alpha)/2}(z-t)^{(1+\alpha)/2}},$$

where $B(\mu,\nu) = \Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)$ is the Beta function. The substitution of this representation into Eq. (A1) and the exchange of the order of integration yields the factorization

$$\int_0^u \frac{dt}{t^{1-\alpha}(u-t)^{(1+\alpha)/2}} \int_t^1 \frac{z^{(1-\alpha)/2}}{(z-t)^{(1+\alpha)/2}}\phi(z)dz = g(u),$$

where $g(u) = B(\alpha,(1-\alpha)/2)u^{(\alpha-1)/2}f(u)$. We thus see that Eq. (A1) is equivalent to the following coupled pair of Volterra equations of Abel type:

$$\int_t^1 \frac{z^{(1-\alpha)/2}}{(z-t)^{(1+\alpha)/2}}\phi(z)dz = \psi(t), \tag{A2}$$

$$\int_0^u \frac{\psi(t)}{t^{1-\alpha}(u-t)^{(1+\alpha)/2}}dt = g(u). \tag{A3}$$

These equations can be solved using a standard procedure [26] with the result

$$\phi(u) = -\frac{\cos(\pi\alpha/2)}{\pi} u^{-(1-\alpha)/2}\frac{d}{du}\int_u^1 \frac{\psi(t)}{(t-u)^{(1-\alpha)/2}}dt, \tag{A4}$$

and

$$\psi(\xi) = B(\alpha,(1-\alpha)/2)\frac{\cos(\pi\alpha/2)}{\pi}\xi^{1-\alpha}\frac{d}{d\xi} \times \int_0^\xi \frac{u^{-(1-\alpha)/2}}{(\xi-u)^{(1-\alpha)/2}}f(u)du. \tag{A5}$$

Substituting this equation into Eq. (A4) and taking into account some properties of the gamma and beta functions we see that the solution to Eq. (A1) reads

$$\phi(u) = -\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{\pi\Gamma^2((1+\alpha)/2)}u^{-(1-\alpha)/2}\frac{d}{du}\int_u^1 \frac{t^{1-\alpha}dt}{(t-u)^{(1-\alpha)/2}}\frac{d}{dt}\int_0^t \frac{f(\tau)}{\tau^{(1-\alpha)/2}(t-\tau)^{(1-\alpha)/2}}d\tau. \tag{A6}$$

One can easily see that when $\alpha = 2/3$ and

$$f(\tau) = \frac{3^{2/3}\Gamma(1/3)}{2}\left[\tau^{2/3} - (1-\tau)^{2/3}\right],$$

Eq. (A6) reduces to Eq. (3.4).

We will now see how Eq. (3.5) is obtained. We write Eq. (3.4) in the form

$$\phi(x) = \frac{3^{1/6}}{2\Gamma^2(5/6)}x^{-1/6}[\phi_2(x) - \phi_1(x)], \tag{A7}$$

where

$$\phi_1(x) \equiv \frac{d}{dx}\int_x^1 \frac{t^{1/3}dt}{(t-x)^{1/6}}\frac{d}{dt}\int_0^t \frac{z^{1/2}}{(t-z)^{1/6}}dz \tag{A8}$$

and

$$\phi_2(x) \equiv \frac{d}{dx}\int_x^1 \frac{t^{1/3}dt}{(t-x)^{1/6}}\frac{d}{dt}\int_0^t \frac{(1-z)^{2/3}}{z^{1/6}(t-z)^{1/6}}dz. \tag{A9}$$

Let us first evaluate $\phi_1(x)$. Taking into account [25]

$$\int_0^t z^{\mu-1}(t-z)^{\nu-1}dz = t^{\mu+\nu-1}B(\mu,\nu),$$

we have

$$\phi_1(x) = \frac{4}{3}B\left(\frac{3}{2},\frac{5}{6}\right)\frac{d}{dx}\left\{(1-x)^{5/6}\int_0^1 (1-\xi)^{-1/6}[1-(1-x)\xi]^{2/3}d\xi\right\}.$$

Using the following properties of the Gauss hypergeometric function [18],

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt \quad (c > b > 0), \tag{A10}$$

and

$$\frac{d}{dx}[x^c F(a,b;c+1;x)] = cx^{c-1}F(a,b;c;x), \tag{A11}$$

we get

$$\phi_1(x) = -\frac{4}{3}B\left(\frac{3}{2},\frac{5}{6}\right)(1-x)^{-1/6}F\left(-\frac{2}{3},1;\frac{5}{6};1-x\right). \tag{A12}$$

The evaluation of $\phi_2(x)$ is much more involved. Taking into account Eqs. (A10) and (A11) we write Eq. (A9) in the form

$$\phi_2(x) = \frac{\Gamma^2(5/6)}{\Gamma(2/3)} J(x), \tag{A13}$$

where

$$J(x) \equiv \frac{d}{dx}\int_x^1 (t-x)^{-1/6} F\left(-\frac{2}{3},\frac{5}{6};\frac{2}{3};t\right) dt.$$

We use the Gauss hypergeometric series [18]

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{A14}$$

and Eqs. (A10) and (A11) to write

$$J(x) = -\sum_{n=0}^{\infty} \frac{(-2/3)_n (5/6)_n}{(2/3)_n}(1-x)^{-1/6} F\left(-n,1;\frac{5}{6};1-x\right),$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol. From the linear transformation formula [18]

$$F(a,b;c;1-x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;x) + x^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;x), \tag{A15}$$

and from the fact that the gamma function $\Gamma(-n)$ diverges for $n = 0, 1, 2, \ldots$, we have

$$J(x) = -(1-x)^{-1/6}\sum_{n=0}^{\infty} \frac{(-2/3)_n(-1/6)_n}{(2/3)_n n!} F\left(-n,1;1+\frac{1}{6}-n;x\right). \tag{A16}$$

On the other hand, from Eq. (A14) we see that

$$J(x) = -(1-x)^{-1/6}\sum_{k=0}^{\infty}\frac{(-2/3)_k}{(2/3)_k}x^k \sum_{n=0}^{\infty}\frac{(k-2/3)_n(-1/6)_n}{(k+2/3)_n n!},$$

which, after using some properties of the Gauss hypergeometric functions, reads

$$J(x) = -\frac{\Gamma(3/2)\Gamma(2/3)}{\Gamma(4/3)\Gamma(5/6)}(1-x)^{-1/6}F\left(-\frac{2}{3},1;\frac{5}{6};x\right).$$

Substituting this into Eq. (A13) yields

$$\phi_2(x) = -\frac{\Gamma(3/2)\Gamma(5/6)}{\Gamma(4/3)}(1-x)^{-1/6}F\left(-\frac{2}{3},1;\frac{5}{6};x\right). \tag{A17}$$

The substitution of Eqs. (A12) and (A17) into Eq. (A7) proves Eq. (3.5).

APPENDIX B: DERIVATION OF EQ. (3.6)

We introduce Eq. (3.5) into Eq. (3.1) with the result

$$T_1'(u,0) = \frac{\pi}{3^{1/6}\Gamma^2(2/3)}\left(u^{2/3} + \frac{\Gamma(2/3)\Gamma(3/2)}{2\pi\Gamma(5/6)\Gamma(4/3)}\left[\psi_1(u) - \psi_2(u)\right]\right), \tag{B1}$$

where

$$\psi_1(u) = \int_0^u (u-z)^{-2/3} z^{-1/6}(1-z)^{-1/6} F\left(1, -\frac{2}{3}; \frac{5}{6}; z\right), \tag{B2}$$

and

$$\psi_2(u) = \int_0^u (u-z)^{-2/3} z^{-1/6}(1-z)^{-1/6} F\left(1, -\frac{2}{3}; \frac{5}{6}; 1-z\right). \tag{B3}$$

1. Evaluation of Eq. (B2)

Starting from Eq. (B2) and using the definition of the Gauss hypergeometric series (A14) we have

$$\psi_1(u) = u^{1/6} \sum_{n=0}^{\infty} \frac{(-2/3)_n}{(5/6)_n} u^n \times \int_0^1 t^{n-1/6}(1-t)^{-2/3}(1-ut)^{-1/6} dt.$$

From Eq. (A10) we get

$$\psi_1(u) = \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(7/6)} u^{1/6} \times \sum_{n=0}^{\infty} \frac{(-2/3)_n}{(7/6)_n} u^n F\left(\frac{1}{6}, n+\frac{5}{6}; n+\frac{7}{6}; u\right), \tag{B4}$$

and, taking into account the linear transformation formula [18]

$$F(a,b;c;u) = (1-u)^{c-a-b} F(c-a, c-b; c; u), \tag{B5}$$

we have

$$\psi_1(u) = \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(7/6)} u^{1/6}(1-u)^{1/6} \times \sum_{n=0}^{\infty} \frac{(-2/3)_n}{(7/6)_n} u^n F\left(1+n, \frac{1}{3}; \frac{7}{6}+n; u\right).$$

We finally use the property [27]

$$\sum_{n=0}^{\infty} \frac{(a)_n (b')_n}{(c)_n n!} x^n F(a+n, b; c+n; x) = F(a, b+b'; c; x),$$

to obtain

$$\psi_1(u) = \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(7/6)} u^{1/6}(1-u)^{1/6} F\left(1, -\frac{1}{3}; \frac{7}{6}; u\right). \tag{B6}$$

2. Evaluation of Eq. (B3)

From Eqs. (B3), (A14), and (A10) we have

$$\psi_2(u) = \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(7/6)} u^{1/6} \times \sum_{n=0}^{\infty} \frac{(-2/3)_n}{(5/6)_n} F\left(\frac{1}{6} - n, \frac{5}{6}; \frac{7}{6}; u\right).$$

Using the linear transformation formulas (A15) and (B5), taking into account the expression [27]

$$\Gamma(\alpha - n) = (-1)^n \frac{\Gamma(\alpha)}{(1-\alpha)_n},$$

and some cancellation of terms, yield

$$\psi_2(u) = \chi(u) + \frac{\Gamma(1/3)\Gamma(-1/6)}{\Gamma(1/6)} \sum_{n=0}^{\infty} \frac{(-2/3)_n}{(7/6)_n} (1-u)^{n+1/6} F\left(\frac{1}{6}, \frac{5}{6}+n; \frac{7}{6}+n; 1-u\right), \tag{B7}$$

where

$$\chi(u) \equiv \Gamma(5/6) u^{1/6} \sum_{n=0}^{\infty} \frac{(-2/3)_n \Gamma(n+1/6)}{(5/6)_n \Gamma(n+1)} F\left(\frac{1}{6} - n, \frac{5}{6}; \frac{5}{6} - n; 1-u\right). \tag{B8}$$

If we recall that $\Gamma(-1/6)/\Gamma(1/6) = -\Gamma(5/6)/\Gamma(7/6)$, then from Eq. (B4) we see that the last term on the right-hand side of Eq. (B7) coincides to $-\psi_1(1-u)$. Therefore,

$$\psi_2(u) = \chi(u) - \psi_1(1-u). \tag{B9}$$

Let us now evaluate $\chi(u)$. From Eqs. (B5) and (B8) we write

$$\chi(u) = \Gamma(5/6)\Gamma(1/6)\sum_{n=0}^{\infty}\frac{(-2/3)_n(1/6)_n}{(5/6)_n n!}F\left(\frac{2}{3},-n;\frac{5}{6}-n;1-u\right).$$

The use of Eq. (A14) and some manipulations yield

$$\chi(u) = \Gamma(5/6)\sum_{k=0}^{\infty}\frac{(1-u)^k}{k!}\ (2/3)_k S(k),$$

where

$$S(k) = \Gamma(1/6)\frac{(-2/3)_k}{(5/6)_k}\sum_{m=0}^{\infty}\frac{(1/6)_m(k-2/3)_m}{(k+5/6)_m m!}.$$

Note that [cf. Eq. (A14)]

$$\sum_{m=0}^{\infty}\frac{(1/6)_m(k-2/3)_m}{(k+5/6)_m m!} = \frac{\Gamma(k+5/6)\Gamma(4/3)}{\Gamma(k+2/3)\Gamma(3/2)}.$$

Hence,

$$\chi(u) = \frac{\Gamma(1/6)\Gamma^2(5/6)\Gamma(4/3)}{\Gamma(2/3)\Gamma(3/2)}\sum_{k=0}^{\infty}\frac{(-2/3)_k}{k!}(1-u)^k,$$

but

$$(1-z)^\alpha = \sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}z^n \quad (|z|<1),$$

whence

$$\chi(u) = \frac{\Gamma(1/6)\Gamma^2(5/6)\Gamma(4/3)}{\Gamma(2/3)\Gamma(3/2)}\ u^{2/3}. \tag{B10}$$

Therefore $\psi_2(u)$ is given by Eq. (B9) along with Eqs. (B6) and (B10). Finally the substitution of the expression of $\psi_2(u)$ thus obtained into Eq. (B1) reads

$$T_1'(u,0) = \frac{\Gamma(1/6)\Gamma(3/2)}{4\pi 3^{1/6}\Gamma(2/3)\Gamma(4/3)}\left[\psi_1(u)+\psi_1(1-u)\right],$$

which, after using Eq. (B6), proves Eq. (3.6).

APPENDIX C: DERIVATION OF EQ. (4.2)

We write Eq. (2.12) in the form

$$\hat{R}(s,y) = \hat{G}(s,y) + \hat{H}(s,y), \tag{C1}$$

where

$$\hat{G}(s,y) = \pi s^{-4/3}\left[\mathrm{Bi}(ys^{1/3})\int_y^{\infty}\mathrm{Ai}(y_0 s^{1/3})dy_0 + \mathrm{Ai}(ys^{1/3})\int_0^{y}\mathrm{Bi}(y_0 s^{1/3})dy_0\right], \tag{C2}$$

and

$$\hat{H}(s,y) = 3^{-1/2}\pi s^{-5/3}\mathrm{Ai}(ys^{1/3}). \tag{C3}$$

Let us proceed to invert these transforms. We start by inverting $\hat{H}(s,y)$. Note that the convolution theorem allows us to write

$$H(u,y) = \frac{3^{-1/2}\pi}{\Gamma(4/3)}\int_0^u h(z,y)(u-z)^{1/3}dz,$$

where

$$h(u,y) = \mathcal{L}^{-1}\{s^{-1/3}\mathrm{Ai}(ys^{1/3})\}.$$

From Eqs. (2.13) and (2.14) we easily see that

$$h(u,y) = (3^{-1/6}/2\pi)\ \frac{e^{-y^3/9u}}{u^{2/3}}.$$

Therefore,

$$H(u,y) = \frac{3^{1/3}}{2\Gamma(1/3)}\int_0^u \frac{e^{-y^3/9z}}{z^{2/3}}(u-z)^{1/3}dz. \tag{C4}$$

We will now evaluate $G(u,y)$. Taking into account Eq. (2.13) and the expression [18]

$$\mathrm{Bi}(ys^{1/3}) = \left(\frac{ys^{1/3}}{3}\right)^{1/2}\left[I_{-1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right) + I_{1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right)\right], \tag{C5}$$

we can write $G(u,y)$ in the form

$$\hat{G}(s,y) = \frac{y^{1/2}}{3s}\left\{\int_y^\infty y_0^{1/2}K_{1/3}\left(\frac{2}{3}y_0^{3/2}s^{1/2}\right)\left[I_{-1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right)+I_{1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right)\right]dy_0 + \int_0^y y_0^{1/2}K_{1/3}\left(\frac{2}{3}y^{3/2}s^{1/2}\right)\left[I_{-1/3}\left(\frac{2}{3}y_0^{3/2}s^{1/2}\right)+I_{1/3}\left(\frac{2}{3}y_0^{3/2}s^{1/2}\right)\right]dy_0\right\}.$$

From the inversion formula [19,25]

$$\mathcal{L}^{-1}\left\{I_{\pm\nu}(a\sqrt{s})K_\nu(b\sqrt{s})\right\} = \frac{e^{-(a^2+b^2)/4u}}{2u}I_{\pm\nu}\left(\frac{ab}{2u}\right) \tag{C6}$$

$(0<a<b)$, we get

$$\mathcal{L}^{-1}\{s\hat{G}(s,y)\} = \frac{y^{1/2}}{6u}\int_0^\infty y_0^{1/2}e^{-(y^3+y_0^3)/9u}\left[I_{-1/3}\left(\frac{2y^{3/2}y_0^{3/2}}{9u}\right)+I_{1/3}\left(\frac{2y^{3/2}y_0^{3/2}}{9u}\right)\right]dy_0.$$

But [19]

$$\int_0^\infty y_0^{1/2}e^{-y_0^3/9u}I_\nu\left(\frac{2y^{3/2}y_0^{3/2}}{9u}\right)dy_0 = (\pi u)^{1/2}e^{y^3/18u}I_{\nu/2}\left(\frac{y^3}{18u}\right).$$

Hence

$$G(u,y) = \frac{\pi^{1/2}}{6}y^{1/2}\int_0^u z^{-1/2}e^{-y^3/18z}\left[I_{-1/6}\left(\frac{y^3}{18z}\right)+I_{1/6}\left(\frac{y^3}{18z}\right)\right]dz. \tag{C7}$$

The substitution of Eqs. (C4)–(C7) into the inverse Laplace transform of Eq. (C1) proves Eq. (4.2).

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