Alternative description of the Kuramoto model of coupled oscillators

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Abstract: The Kuramoto model is one of the most representative models of coupled phase oscillators, commonly used in the description of synchronization phenomenon in complex systems. The aim of this report is to study, from a mathematical and computational point of view, three linear conversions of the Kuramoto model in the case of identical oscillators with global coupling. A comparison with the original Kuramoto dynamics is also conducted.

I. INTRODUCTION

Synchronization is the tendency of entities to cooperate in unison spontaneously. This phenomenon is witnessed in biological, chemical, physical, technological and social systems. It involves populations of interacting units, from neurons to fireflies [1], which collectively perform a global coherent activity.

As an illustrative case, one of the most paradigmatic scenario yet most fascinating spectacle of nature is the flashing in synchrony of huge congregations of fireflies observed in the forests of Southeast Asia. For years, travelers had witnessed these displays and reported stories about it [2]. However, there was a generalized sense of disbelief that some even dismissed the phenomenon as a mere coincidence. On top of that, for decades, no one was able to come up with a plausible theory. It was not until the late 1960s that the pieces begin to fall into place when an almost imperceptible clue was noticed. Synchronous fireflies not only blinked in unison, but also blinked at a constant tempo, in rhythm [3, 4]. This implied that each insect must have an inner clock, some sort of oscillator, whose timing was adjusted by advancing or delaying its internal frequency in response to the flashes of others.

Sometimes, synchronization can be dangerous. For instance, epilepsy is caused by the discharge in pathological lockstep of millions of brain cells, which cause the rhythmic convulsions associated with seizures. Perfect synchronicity could lead to extinction and, therefore, species of the same trophic level develop different circadian rhythms to increase their probability of survival. Details about these and other cases can be found in [1].

In general, research in synchronization involves interacting oscillatory elements. For example, Arthur T. Winfree spent part of his life studying the mutual synchronization of biologic oscillators, such as circadian clocks and metabolic oscillations [5]. Not only did he realize that synchronization can be understood as a threshold process, but he also was able to formulate a model with nonlinear interactions [6], except that it was hard to solve in its full generality. Nonetheless, in 1975 Yoshiki Kuramoto brought simplicity to this problem. He produced a model that was able to solve exactly for systems containing an infinite number of globally weakly coupled phase oscillators running at arbitrary intrinsic frequencies [7]. The attractive characteristic of the Kuramoto model is that it is mathematical treatable despite being sufficiently complex to be nontrivial [8].

The standard formulation of the Kuramoto model on a general network of N nodes (limit-cycle oscillators) is as follows

$$\dot{\theta}_i = w_i + K\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i), \tag{1}$$

where the solution, $\theta_i(t)$, is the state (phase) of the ith oscillator ($i = 1, ..., N$), w_i is the intrinsic angular frequency, K is the coupling constant and $A_{ij} \in \{0, 1\}$ is element of the adjacency matrix, which represents the connection between oscillators. In the absence of interaction, the phases are incoherently driven by the intrinsic frequencies. Whilst the sine coupling in the interaction term causes an attraction between the phases of two connected oscillators i and j, which makes the oscillators converge to the same phase and leads to synchronization [9].

It was also introduced the order parameter, $re^{i\phi}$, a value that gives a measure of how synchronized the system is at a point in time. Considering the oscillators as running around a unit circle in the complex plane, the order parameter is expressed as

$$re^{i\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}, \tag{2}$$

where r is the amplitude of the (complex-valued) order parameter and ϕ represents the collective phase. Notice that the order parameter ranges from 0, meaning no synchronization, to 1, which defines perfect synchronization.

II. OBJECTIVE

The dynamics of networks with many nodes and connections entails difficulties in mathematical treatment.

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In particular, once nonlinearity is introduced, the analytical study of the system's dynamics becomes extremely difficult. For this reason, many researches have introduced alternative algebraic approaches to the original Kuramoto model. The main purpose of this bachelor thesis is to study three linear conversions of the Kuramoto model and to determine the benefits and limitations that this approaches imply with respect to the original model.

III. METHODOLOGY

To begin with, we introduce the three linear approaches under study. In terms of the computational simulations, we consider a system of identical oscillators and, therefore, we remove the frequency term from all the following equations since for identical intrinsic frequencies, $w_i = w \ \forall i \in [1, N]$, we can simply change to a rotating frame [10] and set $w = 0$ without loss of generality. In this case, the system exhibit steady-state perfect synchronization regardless of initial conditions as we report in this work. We implement our own python code [11] using the NumPy, SciPy and SimPy libraries.

A. Linearization

In a first approximation, when the system reaches a state where the phase differences are small, the sine coupling can be replaced by its argument, the phase difference $\theta_j - \theta_i$. Within this framework, assuming intrinsic frequencies equal to zero, equation (1) can be written as

$$\dot{\theta}_i = K \sum_{j=1}^{N} A_{ij}(\theta_j - \theta_i) \tag{3}$$

or, equivalently, applying the relation between the adjacency and Laplacian matrices, $L_{ij} = k_i\delta_{ij} - A_{ij}$, where k_i is the degree of the node i,

$$\dot{\theta}_i = -K \sum_{j=1}^{N} L_{ij}\theta_j. \tag{4}$$

It is demonstrated [Albert Díaz G. (unpublished)] that this set of linear differential equations can be solved in terms of the normal modes of the Laplacian matrix such that the original coordinates can be written as

$$\theta_i = \sum_{j=1}^{N} U_{ij}\phi_j, \tag{5}$$

where U_{ij} is the matrix of the base transformation and $\phi_j = e^{-K\lambda_j t}\phi_j(0)$ are the eigenvectors of the Laplacian matrix associated with the eigenvalues λ_j.

From this solution, it is obtained that the evolution of the average of the square phase difference is

$$\left\langle (\theta_l(t) - \theta_k(t))^2 \right\rangle = 2D \sum_m (U_{lm} - U_{km})^2 \, e^{-2K\lambda_m t}, \tag{6}$$

where D is a coefficient related to the initial fluctuations.

Hence, not only the two phases approach depending on time as expected, but also on the difference among the projections on the normal modes weighted by an exponential decay. It is shown that the leading term corresponds to $m = 2$, which is the second smallest eigenvalue and usually known as algebraic connectivity [12]. So, therefore, the slope of the asymptotic exponential decay (the a plot is log-lin) is $1/\lambda_2$. In fact, we always look at the asymptotic decay towards synchronization since perfect synchronization can not be reached.

B. Linear reformulation

In 2008, the following linear reformulation of the Kuramoto model of spontaneous synchronization was proposed by D. Roberts [13], which permits its solution through an eigenvalue-eigenvector problem,

$$\dot{\psi}_i = (iw_i - \gamma)\psi_i + \sum_{j \neq i} \Omega_{ij}\psi_j, \tag{7}$$

where w_i is the intrinsic frequency, Ω_{ij} is the coupling constant of this linear model and γ is the decay constant, which is adjusted afterwards to bring the system to a steady state. In a system of globally and uniformly coupled oscillators $\Omega_{ij} = \Omega/N \ \forall i \neq j$, whose value is positive unless otherwise specified.

The solution of this linear model is

$$\boldsymbol{\psi} = \sum_{j=1}^{N} a_j \mathbf{v}_j e^{\lambda_j t}, \tag{8}$$

where a_j are constants determined by the initial conditions and $\mathbf{v}_j$ and λ_j are, respectively, the eigenvectors and eigenvalues associated with the matrix defined by the right-hand side of equation (7).

In order to observe the connection between this linear reformulation and the original Kuramoto model, it is performed a nonlinear transformation, $\psi_m(t) = R_m(t)e^{i\theta_m(t)}$, on equation (7), which leads to

$$\dot{R}_i = \gamma R_i + \frac{\Omega}{N} \sum_{j \neq i}^{N} R_j \cos(\theta_j - \theta_i), \tag{9}$$

$$\dot{\theta}_i = w_i + \frac{\Omega}{N} \sum_{j \neq i}^{N} \frac{R_j}{R_i} \sin(\theta_j - \theta_i), \tag{10}$$

which are the dynamic equations that the modulus and the argument of $\psi_m(t)$ satisfy.

We observe that, if $R_j/R_i \to 1$, in the case of uniform $\Omega_{ij} = \Omega/N$, then equation (10) becomes equation (1). Therefore, when the steady state is reached, this linear version maps onto the original model with an effective coupling constant $K = \frac{\Omega}{N}$.

On the other hand, equation (7) can be written as

$$\dot{\psi}_i = \sum_{j=1}^{N} (\Omega_{ij} - \gamma\delta_{ij})\psi_j, \tag{11}$$

where it has been taken $w_i = 0$ as we study the case of identical oscillators, i.e., $w_i = w\ \forall i$.

From this expression, we can identify the matrix $\left[\frac{\Omega}{N}(1-\delta_{ij}) - \gamma\delta_{ij}\right]$ and find the corresponding eigenspectrum. We observe that there are $N-1$ degenerate eigenvalues, each equal to $-\gamma - \Omega/N$, and one unique eigenvalue

$$\lambda_N = -\gamma + \Omega(N-1)/N, \tag{12}$$

which has associated the eigenvector $\mathbf{v}_N = (1, 1, 1, ..., 1)$.

Regarding the constraint underlying the constant γ, we impose that $\mathrm{Re}(\lambda_N) = 0$ to force each $R_i(t)$ to go to a steady state for large times. In other words, we tune the constant γ so that

$$\gamma = \Omega(N-1)/N, \tag{13}$$

and, therefore, the phase difference $\left\langle(\psi_j(t) - \psi_i(t))^2\right\rangle$ decays as $\sim e^{-2\Omega}$ to synchronization.

It should be noticed that in this analysis it has been adopted the convention of ordering the eigenvalues by their real part, from λ_1 (least) to λ_N (greatest).

C. Complex-valued matrix formulation

In 2021, a complex-valued matrix formulation of the Kuramoto model was introduced by L. Muller and coworkers [14], whose argument is claimed to coincide with the original Kuramoto dynamics. It is based on the addition of an imaginary component to the coupling term which leads to the next linear equation,

$$\dot{\mathbf{x}} = (\mathrm{diag}[i\mathbf{w}] + K\mathbf{A})\,\mathbf{x}, \tag{14}$$

where $\mathbf{x} = e^{i\boldsymbol{\theta}}$ with $\theta_i \in \mathbb{C}$, $\mathbf{w}$ is a vector that contains the intrinsic frequencies, K is the coupling constant, and $\mathbf{A}$ is the adjacency matrix. The detailed procedure followed to obtain the above equation can be found in [14].

In this case, when dealing with homogeneous intrinsic frequencies a scaling of the coupling strength is required such that $\gamma = 2K/\pi$. So, equation (14) is written as

$$\dot{\mathbf{x}} = \gamma\mathbf{A}\mathbf{x}, \tag{15}$$

whose general solution is

$$\mathbf{x} = e^{\gamma t\mathbf{A}}\mathbf{x}(0). \tag{16}$$

In fact, it should be mentioned that we accept the relation between the coupling constant K and the new constant γ since a clear explanation could not be found.

It is also established that since the decomposition of $\boldsymbol{\theta}$ into its real and imaginary parts can be written as $\boldsymbol{\theta} = \boldsymbol{\theta}_{re} + i\boldsymbol{\theta}_{im}$, then $\boldsymbol{\theta}_{re}$ must be the argument of the analytical solution of $\mathbf{x}$. In particular, it is taken $\boldsymbol{\theta}_{re} \in [-\pi, \pi]$.

With the aim of reversing the change, following the procedure presented in the last model, we perform a nonlinear transformation of the form $x_i(t) = R_i(t)e^{i\alpha_i(t)}$ on equation (15) and find the dynamic equations that the modulus and the argument of $x_i(t)$ obey,

$$\dot{R}_i = \gamma \sum_{j=1}^{N} A_{ij}R_j \cos(\alpha_j - \alpha_i), \tag{17}$$

$$\dot{\alpha}_i = \gamma \sum_{j=1}^{N} A_{ij}\frac{R_j}{R_i} \sin(\alpha_j - \alpha_i). \tag{18}$$

IV. PARTICULAR CASE

In this case, if $R_j/R_i \to 1$, then equation (18) becomes equation (1) as long as $R_i(t)$ never diverges, given that if we compare the transformation realized of $x_i(t)$ with the decomposition of $\theta_i(t)$ in the complex plane we observe that

$$R_i(t) = e^{-i\mathrm{Im}(\theta_i(t))}, \tag{19}$$

$$\alpha_i(t) = \mathrm{Re}(\theta_i(t)). \tag{20}$$

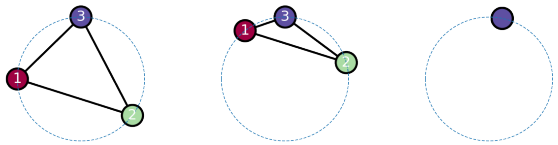
Therefore, the argument of $x_i(t)$ corresponds to the real part of the phase of the oscillator only if $R_i(t)$ converges to a finite value.

In this study, we consider a system of three identical oscillators, i.e., all with the same natural frequency. Taking into consideration that the oscillators are uniformly coupled, the corresponding adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{21}$$

Then, the Laplacian matrix with its corresponding eigenvalues and eigenvectors is

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}; \quad \begin{cases} \lambda_1 = 0\ ;\ v_1 = (1,1,1) \\ \lambda_2 = 3\ ;\ v_2 = (-1,0,1) \\ \lambda_3 = 3\ ;\ v_3 = (-1,1,0) \end{cases} \tag{22}$$



(a) (b) (c)

FIG. 1: Dynamical evolution of three interacting identical oscillators with coupling constant $K = 1$ in a unit circle. Results from the forth order Runge-Kutta method with a time step $dt = 0.0001$ s and initial conditions $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$. (a) Initial state. (b) Intermediate state. (c) Final state.

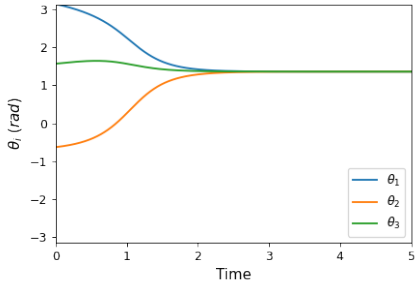


FIG. 2: Time evolution of the phases for a system of three identical oscillators coupled with $K = 1$. Results obtained by the RK4 integration method with a time step $dt = 0.0001$ and initial conditions $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$ rad.

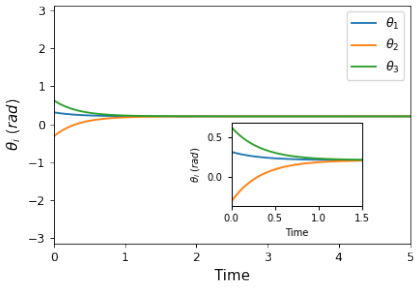


FIG. 3: Evolution of the phases for three identical oscillators coupled with $K = 1$ under the linearized equations of motion. RK4 with $dt = 0.0001$ and $\boldsymbol{\theta_0} = (\pi/10, -\pi/10, \pi/5)$ rad. An amplification of the first steps is depicted in the insert.

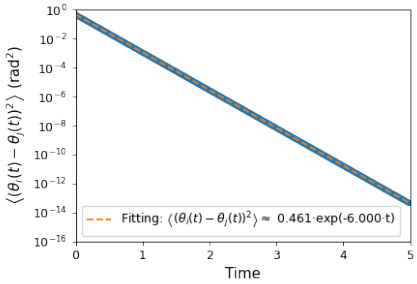


FIG. 4: Evolution of the average of the square phase difference for three identical oscillators following the linearized equations of motion. It is adjusted an exponential fitting. Notice that the y-axis is in logarithmic scale while the x-axis in linear scale.

V. RESULTS AND DISCUSSION

In order to study the linear approaches, we first obtain the original Kuramoto dynamics (showed in figures 1 and 2) via the numerical integration of equation (1) using the forth order Runge-Kutta (RK4) method. From this simulation we can observe that the phases arrive to synchronization after some transient time as expected. Another remarkable fact is that with the integration of the linearized equations of motion (3) we are able to approximately reproduce the same behavior (see figure 3). It should be noticed that, as a consequence of the approximation taken, in order to guarantee the convergence of the phases it is necessary to impose restrictions to the initial conditions; indeed, we take initial values of the phases next to zero. Furthermore, if we represent the evolution of the average of the square difference (figure 4), calculated as $\frac{1}{N}\sum_{j>i}^{N}(\theta_j(t) - \theta_i(t))^2$, we observe that decays as $\sim e^{-6t}$. Therefore, it coincides with what was predicted by the theory (equation (6)) since in the case of three oscillators the second smallest eigenvalue of the Laplacian matrix is $\lambda_2 = 3$.

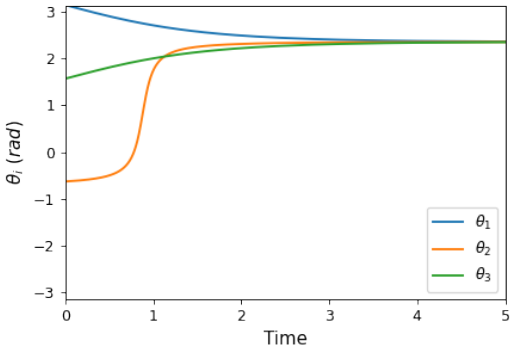


FIG. 5: Time evolution of the phases for a system of three identical oscillators coupled with $K = 1/3$ ($\Omega = 1$) under the linear reformulated equations of motion. Simulation realized with time step $dt = 0.0001$ s and initial conditions $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$.

Meanwhile, for the linear reformulations of the Kuramoto model (equation (1)) we obtain that the phases evolve as showed in figures 5 and 6. In both cases, we also observe the convergence of the phases. However, it can be seen that synchronization is reached before in the context of the complex-valued formulation than with the linear reformulation. This can be understood by taking into account that what sets the time scale to attain full synchronization is the coupling strength; in fact, the smaller the coupling constant is, the longer the time scale.

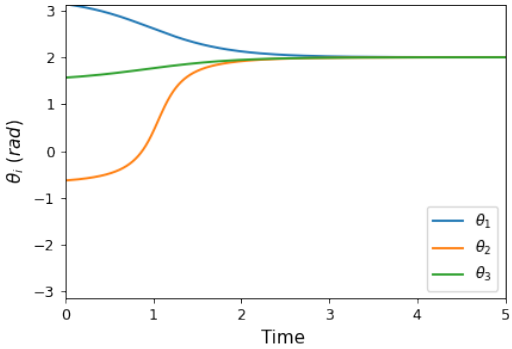


FIG. 6: Eevolution of the phases for a system of three identical oscillators coupled with $K = 1$ under the complex-valued formulation of the equations of motion. Simulation realized with time step $dt = 0.0001$ and initial conditions $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$ rad.

In an effort to do a further study of this linear models, we solve the system of coupled equations composed by equations (9)-(10) and equations (17)-(18), respectively to obtain the dynamical evolution for the modulus of the variable defined in each linear formulation. First, we analyze figure 7 and observe that for large times all the modulus of $\psi_i(t)$ tend to the same value. Therefore, it is satisfied the condition $R_j/R_i \to 1$. Said otherwise, it is true that equation (7) maps onto equation (1) once a steady state is reached. So, consequently, the proposed linear reformulation reproduces the original Kuramoto dynamics as long as all $R_i(t)$ go to a steady state. Nevertheless, in the case of the complex-valued formulation (figure 8), we observe that the modulus of $x_i(t)$ grows

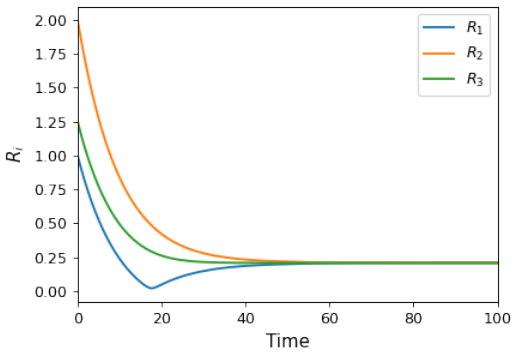


FIG. 7: Evolution of the modulus of $\psi_i(t)$ for a system of three identical oscillators coupled with $K = 1/3$. Results from the RK4 method with time step $\mathrm{d}t = 0.0001$ and initial conditions $\mathbf{R_0} = (1, 2, 1.25)$ and $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$ rad.

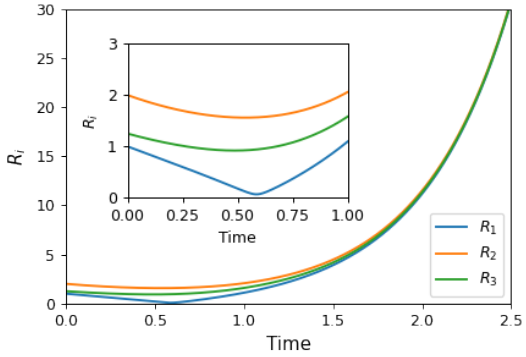


FIG. 8: Time evolution of the modulus of $\psi_i(t)$ for a system of three identical oscillators coupled with $K = 1$. Results from the RK4 method with time step $\mathrm{d}t = 0.0001$ and initial conditions $\mathbf{R_0} = (1, 2, 1.25)$ and $\boldsymbol{\theta_0} = (\pi, -\pi/5, \pi/2)$ rad. An amplification of the first time steps is depicted in the insert.

exponentially and, therefore, diverges. In other words, since $R_i(t)$ does not converge we can not guarantee that the argument of $x_i(t)$ corresponds to the real part of the phase of the oscillator.

VI. CONCLUSIONS

From the studied linear conversions of the Kuramoto model, we have observed that a simple first approximation of the sinus coupling when the phase differences are small is good enough to show the decay towards synchronization. On the other hand, the proposed linear reformulation although it is completely solvable, it is only capable of reproducing the original dynamics when steady state is reached, in general for large times. Finally, the complex-valued formulation should be further studied with more detail. Even though at first seems that presents the same dynamical behavior as the original Kuramoto model, the necessary condition to establish the relation between the proposed variable and the original coordinates is not satisfied and, therefore, we can not guarantee that the reproduction of the original dynamics is completely correct. Notwithstanding all the above mentioned, it should be bear in mind that linear approximations to nonlinear dynamics is still of great advantage since it permits a less complex analytical insight into the Kuramoto model.

Acknowledgments

I would like to express my most sincere gratitude to Albert Díaz for his guidance and support through this journey and, above all, for introducing me to the marvelous paradigm of complex systems.

[1] Steven H. Strogatz. *Sync: The Emerging Science of Spontaneous Order*. Hyperion, New York, 2003.

[2] Richard Hakluyt. *A Selection of the Principal Voyages, Traffiques and Discoveries of the English Nation*. 1589.

[3] E Kaempfer, J G Scheuchzer, S Delboe, H Gibben, and W Ramsden. *The History of Japan: Together with a Description of the Kingdom of Siam, 1690-92*. Number v. 1. J. MacLehose and sons, 1906.

[4] John Bonner Buck. Synchronous Rhythmic Flashing of Fireflies. *https://doi.org/10.1086/394562*, 13(3):301–314, sep 2015.

[5] Arthur T. Winfree. *The Geometry of Biological Time*. New York, 1980.

[6] Arthur T. Winfree. Biological rhythms and the behavior of populations of coupled oscillators. *Journal of Theoretical Biology*, 16(1):15–42, jul 1967.

[7] Yoshiki Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. In Huzihiro Araki, editor, *International Symposium on Mathematical Problems in Theoretical Physics*, pages 420–422, Berlin, Heidelberg, 1975. Springer Berlin Heidelberg.

[8] Juan A. Acebrón, L. L. Bonilla, Conrad J.Pérez Vicente, Félix Ritort, and Renato Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Reviews of Modern Physics*, 77(1):137–185, jan 2005.

[9] Steven H. Strogatz. *Nonlinear dynamics and chaos : with applications to physics, biology, chemistry and engineering*. 1994.

[10] Steven H. Strogatz and Renato E. Mirollo. Collective synchronisation in lattices of nonlinear oscillators with randomness. *Journal of Physics A: Mathematical and General*, 21(13):L699–L705, jul 1988.

[11] Maria Li López. Kuramoto python codes. `https://github.com/itsmarialb/Kuramoto`.

[12] Ravindra. B. Bapat. Algebraic Connectivity. In *Graphs and Matrices*, pages 81–94. Springer, London, 2010.

[13] David C. Roberts. Linear reformulation of the kuramoto model of self-synchronizing coupled oscillators. *Phys. Rev. E*, 77:031114, Mar 2008.

[14] Lyle Muller, Ján Mináč, and Tung T. Nguyen. Algebraic approach to the Kuramoto model. *Phys. Rev. E*, 104:L022201, Aug 2021.