# Classical methods of orbit determination

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**Abstract:** This work's main goal is to study the classical methods of orbit determination developed by Gauss and Laplace. To do so, simulations of objects orbiting the Earth have been programmed to test these methods and compare the results. They show poor accuracy and don't allow for predictions in the studied cases, but they improve for large enough orbits where they could be a first approach for more complex orbit determination schemes.

## I. INTRODUCTION

An orbit, in celestial mechanics, is the trajectory that an object with mass follows through space around another massive object. This is a hugely broad definition that encompasses a lot of situations, one of which is an object, such as a satellite, debris or another minor body revolving around the Earth.

The orbit of this kind of objects around the Earth also reduces the complexity of the dynamical problem, as the two masses are of very different order of magnitude and only the attraction of the Earth to the object is to be taken into account. As the object's orbit is also limited to a certain proximity of the Earth one can also neglect, at first order of approximation, the influence of the other planets and even the Sun, as the star's influence will be approximately the same for the central planet and the whole system (Earth and object) will move in solidarity.

The goal of this work is to study how the orbit of this kind of objects can be computed through astrometric observations. Although more highly accurate statistical methods are used nowadays, some of those still use classical methods of orbit determination as a preliminary result. These classical methods, such as Gauss' or Laplace's were developed as analytical solutions of the problem in hand [2] but are known to be rather inaccurate.

Nevertheless, the interest of this work is not the precision of the calculation but to understand the difficulties of the problem and how they were historically dealt with. That's why in this work I simulate the whole process, from the orbit, to its computational determination through angular observations to shed light into the problem of orbit determination and see if the results resemble the true orbit in certain conditions.

After this brief introduction, I discuss the classical methods of orbit determination derived by Gauss in 1801 and Laplace in 1780. The following section covers the methodology for testing these methods using the python programming language. Then, the initial conditions used and the types of simulations executed are detailed followed by a first analysis of the test results. Finally, a conclusion on this work is exposed as well as some ideas for further developing this work in the future.

## II. CLASSICAL METHODS OF ORBIT DETERMINATION

The dynamical problem we aspire to solve is the two-body problem, which in its most general form can be reduced to the one-body problem with the introduction of the centre of mass of the system and the relative coordinates. For the problem in hand, this step is omitted due to the huge difference in the masses of the object and the planet, as the centre of mass becomes in a highly accurate approximation the centre of the planet, and the relative coordinates become the coordinates of the object. So anyways, we end up with the one-body problem condition, that is, the fundamental equation of motion derived from Newton's laws that reads:

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu k \frac{\mathbf{r}}{r^3} \quad \rightarrow \quad \ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3} \quad \rightarrow \quad \ddot{\mathbf{r}} = -u\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the position vector of the object, $\mu$ is the mass of the system (in this case the mass of the Earth) which by the smart selection of units of earth masses, $\mu \approx 1$ and $k = \sqrt{GM}$ is the square root of the standard gravitational parameter of a celestial body of mass $M$.

The equation has been simplified in the first step by using the modified time variable $\tau \equiv k(t - t_0)$ and afterwards by the introduction of the variable $u \equiv \mu/r^3$.

It's interesting to realise that this equation has an analytical solution. Theoretically, it can be proven that this solution can be expressed as a series expansion around some epoch time [1]:

$$\mathbf{r} = \mathbf{r}_0 + \dot{\mathbf{r}}\tau + \frac{1}{2}\ddot{\mathbf{r}}_0\tau^2 + \frac{1}{6}\dddot{\mathbf{r}}_0\tau^3 + O(\tau^4) \tag{2}$$

And by the substitution of the fundamental equation (1) as soon as a $2^{\text{nd}}$ order derivative or higher appears and the introduction of the $f \equiv 1-(u_0\tau^2)/2+O(\tau^3)$ and $g \equiv \tau - (u_0\tau^3)/6 + O(\tau^4)$ series, (2) becomes:

$$\mathbf{r} = f\mathbf{r}_0 + g\dot{\mathbf{r}}_0 \tag{3}$$

This expression will prove extremely useful in the following methods of orbit determination from angular observations, that is, the geocentric coordinates of the observational stations where the observations were made

($\mathbf{R}_i$), the time of the observations ($t_i$) and a set of three topocentric angular observations ($\alpha_i, \delta_i$).

### A. The method of Gauss

Gauss' method for orbit determination starts from a geometric view of the problem. If there are three observations of the object, there are three linearly dependent geocentric vectors $\mathbf{r}_i$ that satisfy:

$$\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3 \tag{4}$$

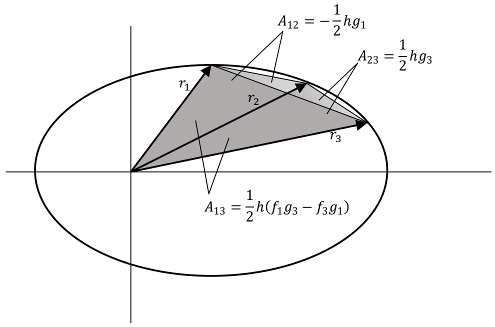


FIG. 1. Geometrical basis for the method of Gauss. The areas of the triangles formed between the respective radius can be calculated from cross products of the vectors expressed in terms of the $f$ and $g$ series using relation (3). The $f_i$ and $g_i$ series are computed with respect to the central epoch about the time of the $i^{\text{th}}$ observation. $h$ is the modulus of the vector $\mathbf{h} = \dot{\mathbf{r}}_2 \times \mathbf{r}_2$.

By making the cross products of the $\mathbf{r}_i$ vectors one can easily see a relation between the coefficients $c_1$, $c_3$ and the areas of the triangles formed between the respective radius $c_1 = A_{23}/A_{13}$ and $c_3 = A_{12}/A_{13}$. These quotients will make the $h$ value disappear.

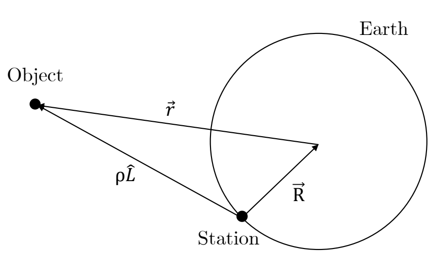


FIG. 2. Vector triangle relation.

With the introduction of the vector triangle relation $\rho\mathbf{L} = \mathbf{r} + \mathbf{R}$, where $\mathbf{L}$ is the unit vector in the direction of the station and the object known from the angular observations, equation (4) reads:

$$\sum c_i\rho_i\mathbf{L}_i = \sum c_i\mathbf{R}_i \equiv \mathbf{G} \tag{5}$$

Where $c_2 \equiv -1$. In this last three equations, all parameters are known except for $\rho_i$ and $u_2$ which is implicit in the coefficients $c_i$, so after some algebra one can obtain $\rho_i(u_2)$.

Again from the vector triangle relation is possible to obtain another equation relating $\rho_i(u_2)$ (recall that $u \equiv \mu/r^3$):

$$r_2^2 = \rho_2^2 - 2\rho_2\mathbf{L}_2 \cdot \mathbf{R}_2 + R_2^2 \tag{6}$$

Finding the intersection of functions $\rho_i(u_2)$ derived from (5) and (6) will provide us with the vectors $\mathbf{r}_i$. Finally, one can manipulate a Taylor expansion of $\mathbf{r}_i$ with respect to $\tau$ and find the velocities $\dot{\mathbf{r}}_i$ as a combination of the position vectors weighted with some coefficients that depend on the time interval between observations. This gives the set of vectors $\mathbf{r}_i, \mathbf{v}_i$ and the orbit determination scheme is completed. [1]

### B. The method of Laplace

Laplace tackled the problem of determining an orbit from angular observations by successive differentiation of the vector triangle relation $\mathbf{r} = \rho\mathbf{L} - \mathbf{R}$. This yields:

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{L} + \rho\dot{\mathbf{L}} - \dot{\mathbf{R}} \tag{7}$$

$$\ddot{\mathbf{r}} = 2\dot{\rho}\dot{\mathbf{L}} + \ddot{\rho}\mathbf{L} + \rho\ddot{\mathbf{L}} - \ddot{\mathbf{R}} \tag{8}$$

Note that the fundamental equation of motion can be applied in the left hand side of equation (8) and regrouping terms:

$$\mathbf{L}\ddot{\rho} + 2\dot{\mathbf{L}}\dot{\rho} + \left(\ddot{\mathbf{L}} + \frac{\mu}{r^3}\mathbf{L}\right)\rho = \ddot{\mathbf{R}} + \mu\frac{\mathbf{R}}{r^3} \tag{9}$$

At the central date of the observations, the derivatives $\dot{\mathbf{L}}_2, \ddot{\mathbf{L}}_2$ may be evaluated with numerical methods such as the Lagrange interpolation formula as the time between observations is known. For the central date then the above relation represents three equations with four unknowns: $\rho_2, \dot{\rho}_2, \ddot{\rho}_2$ and $r_2$.

By considering the vector triangle relation dotted with itself as in the Gauss method (6) an independent geometrical relation is obtained relating $r_2$ and $\rho_2$. The solution of the problem of Laplace now hinges upon the simultaneous solution of equations (6) and (9). This will give a value for $r_2$, and then, $\rho_2$ and $\dot{\rho}_2$ can also be computed

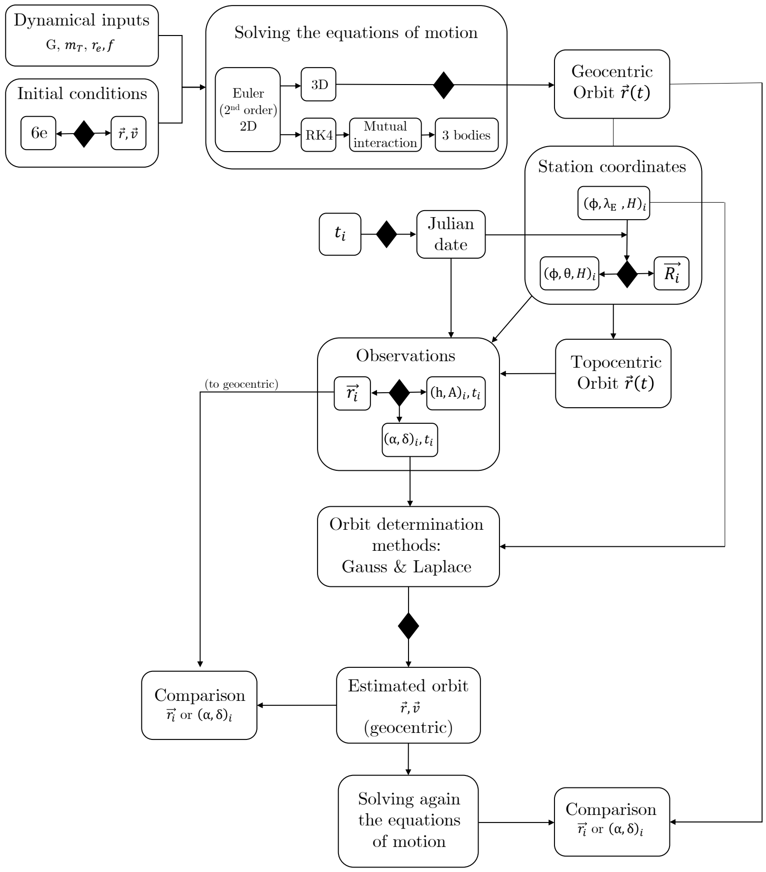


FIG. 3. Flux diagram of the programmed methodology for testing the Gauss and Laplace methods. Rhombus imply some code going on.

so $\mathbf{r}_2$ through the vector triangle relation and $\dot{\mathbf{r}}_2$ through (7) can be obtained and the orbit determination scheme is completed.

This method is known to be poorly accurate when trying to compute NEOs as the inclusion of higher-order derivatives of $\mathbf{L}_2$ must be taken into account, but it performs rather well in the determination of the very distant heliocentric planetary orbits that are not treated in this work. [1]

The expectations for the results of this method are therefore quite low, with the hope of finding minimally decent results in the larger orbits simulated.

## III. METHODOLOGY FOR TESTING THE GAUSS AND LAPLACE METHODS

The purpose of this work is to test these methods of orbit determination. To do so, I implemented in a python environment the scheme presented in figure 3. All the code is available at `https://github.com/Calic1999/TFG`.

The program starts by choosing randomly from a selected range of values some initial conditions for the orbit that is going to be simulated. This initial conditions can be in the form of a position and velocity vector or the set of six orbital elements. Either way, two scripts allow the conversion between these two sets of values. With these initial conditions and the necessary dynamical constants of the problem a numerical solution for the equations of movement (1) can be made.

The dynamical constants are the standard gravitational parameter $k^2 = Gm_T$, the equatorial radius of the Earth $r_e$, the flattening of the same $f$ and the average rotation velocity of the planet $d\theta/dt$. The values used correspond to the ones presented in Escobal. This constants are used in many other points of the scheme but are not made explicit for the sake of clarity.

During the subject of *Física Computacional* at 3$^{\text{rd}}$ course of the degree, different methods were presented to solve differential equation, such as Euler's or Runge-Kutta's, that I implemented in this scenario to obtain the orbit of the object that the chosen initial conditions represent.

The first solver for this equations that I implemented used Euler's method at 2nd order for a two dimensional situation of an object revolving around a centre of coordinates with the mass of the Earth. This program was then upgraded on different levels.

On one hand, this same dynamical situation was programmed using the method of Runge-Kutta and afterwards, mutual interaction between the object and the planet originally at the centre of coordinates was implemented. Finally this chain of improvement culminated in the design of a numerical solver for the three body problem.

On the other hand, the first Euler method was also upgraded to handle the 3 dimensional case and this was the code used to solve the equations of motion so a geocentric orbit $\mathbf{r}(t)$ is obtained.

Next step is the selection of three stations for the observations (that for the most general case are different) and three different epochs. The stations were chosen randomly around all the Earth and between sea level and 4 km of altitude with the condition that the object at the epoch times were over 10 degrees of elevation, for which a code was programmed to obtain the azimuth-elevation coordinates of an object. The dates of the observation were chosen equally spaced in time for a given fraction of the orbit.

The station coordinates and the sideral time of the observations allow the geocentric orbit we had to be expressed as a topocentric orbit from which topocentric right ascension and declination coordinates can be obtained with a simple script.

The methods of orbit determination need the station coordinates, the times of observations and the right ascension-declination angles at those times (as well as the dynamical inputs). With those quantities simulated, the code for each method of orbit determination is executed yielding the estimated orbit as a position and velocity vectors at the central epoch of observation. This result can be directly compared with the vector used as an observation.

Given that the methods of orbit determination produce the six parameters needed to define an orbit, the equation of motion can be solved again (using the same 2nd order Euler's method in 3D) to obtain a estimated orbit to be compared with the original simulated orbit.

## IV. SIMULATIONS AND INITIAL CONDITIONS

The simulations depend now on the initial conditions that are given, so different range on these have been used, but not all the parameters will affect the results in the same way. The most relevant are the size of the orbit, represented by the semimajor axis $a$, and in which part of the orbit the observations are made, as it will not be the same to have closely spaced observations or really far apart ones. It may also effect the results if the observations are made near the periapsis or near the apoapsis for high eccentric orbits. Other parameters, such as the inclination of the orbit, should not have any effect on the results of the orbit determination.

The semimajor axis values used in the simulation range from $10^7$ to $5\times10^9$ meters, covering orbits lower than the moon as well as orbits as far away as the L2 point. The arc of observations covered has been simulated to cover from the 5% to the 50% of the period of the orbit.

As the definition of "observations near the periapsis/apoapsis" is not very clear and even less when the observations make up to half the orbit, this variable effect has been left for future studies, and only circular orbits have been simulated (even though the code is perfectly prepared to handle closed eccentric orbits).

## V. PRELIMINARY TEST ANALYSIS EVALUATION

As the scheme in figure 3 suggests, once the results of the Gauss and Laplace methods are obtained, I compared this value of the position vector at the central date of observation with the original position vector of the simulated orbit. This comparison can be made through the vector itself or the angular coordinates they belong to. While the later comparison can be made simply by computing the angle between the two vectors, the former is evaluated through the relative error.

| $a$ ($10^7$ m) | $\varepsilon_{r_2}$ (G) | $\varepsilon_{r_2}$ (L) | $\theta_G$ (º) | $\theta_L$ (º) |
|---|---|---|---|---|
| 1 | 1.5 | 1.5 | 59.86 | 48.92 |
| 5 | 1.0 | 1.3 | 35.79 | 9.29 |
| 10 | 2.3 | 1.4 | 16.59 | 8.07 |
| 50 | 1.5 | 1.1 | 17.79 | 1.00 |
| 100 | 0.8 | 1.6 | 0.70 | 0.66 |
| 500 | 2.5 | 1.5 | 0.65 | 0.14 |

TABLE I. Comparison of the $\mathbf{r}_2$ vector from the original simulation and the estimation by the orbit determination methods. $G$ stands for Gauss method and $L$ for Laplace's. Each computed as the average over 10 orbits with the same $a$ and $e$ but the other orbital elements being random. No error is shown as the standard deviation is of the same order as the results, so instead the precision shown is the needed but not more than sufficient to clearly visualize the results.

From the results shown in table 1, one can see that the magnitude of the position vector is not accurate at all and the size of the orbit doesn't affect this result. The angular position though is drastically improved as the orbits get bigger. The different ranges in between observations do not translate in any tendency of the results.

In all cases, the estimated velocity vector is way too big, so when this results are used as initial conditions for solving again the equations of motion, the object supposedly moves away from the Earth rapidly, until it's so far away that it's movement is not perceptible from the angular coordinates we would see from the Earth. This effect is usually more exaggerated in the Laplace method.

Interestingly, this incorrectly estimated movement follows the the same angular path the object really takes when the original orbit is big enough, disagreeing more and more when the orbit is smaller, so for orbits of $a = 10^7$ m the estimated angular path doesn't have anything to do with the real orbit, but for orbits of $a = 10^9$ m the estimated angular path follows almost exactly the same it should, even though the progression is much different as the estimated trajectory is not closed but tends to go to infinity. This behaviour doesn't allow for predictions to be made.

It's interesting to point out that sometimes, apparently randomly, the prediction of the $\mathbf{r}_2$ vector points it almost exactly in the opposite side of the sky. The previous discussion still holds for this kind of results, but this cases have not been taken into account when making table 1, as then the angular discrepancies would not show any tendency either.

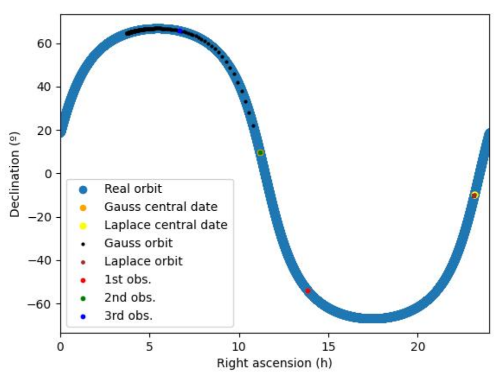


FIG. 4. Orbit simulated with orbital elements $a = 5 \times 10^9$ m, $e = 0$, $i = 247^{\text{o}}$, $\Omega = 352^{\text{o}}$, $\omega = 66^{\text{o}}$, $\nu = 352^{\text{o}}$ which shows the behaviour discussed with decent results as this is a big orbit. Gauss prediction is seemingly correct but the magnitude of the velocity vector doesn't allow for correct predictions and the orbit is not closed. For the Laplace method, the angular position is exactly $180^{\text{o}}$ from its true position but it stays there for the same reason.

## VI. CONCLUSIONS AND FUTURE WORK

The orbit determination methods of Gauss and Laplace, although theoretically correct do not translate to accurate results under the particular case studied. For the Gauss method this could be due to the approximations on the $f$ and $g$ series as well as the triangle approximation for the area under the vectors. Some texts [3] also suggest implementing least-squares techniques to provide even better estimates. For the Laplace method, which performs even worse in this simulations, the inaccurate results likely come from not applying the method in its ideal work range. It's interesting though how different the approaches to solving the problem of orbit determination can be. The simplicity of the Laplace's method is what seems to hold it back in favour of the method of Gauss, but with the possibility of implementing higher-order derivatives and/or more observations to make the method more precise, the intelligibility of this scheme may result more attractive.

In future studies I would like to use this same testing scheme developed to see the results for more cases and different types of orbit, mainly distant heliocentric ones for which the implementation of more precise simulations of the equations of motion and adding the perturbations of other celestial bodies would be needed. Another future project is studying the effect the eccentricity of the orbit may introduce to the results. This work leaves the perfect setting for broadening this field of study to the modern orbit determination schemes as some of them begin by using these classical methods to initialise the computation.

## VII. ACKNOWLEDGEMENTS

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[3] F. R. Moulton, *An Introduction to Celestial Mechanics*, The Macmillan Company, New York, 1914

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