Solitons in Bose-Einstein condensates

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Abstract: In this report we investigate the instability of a dark soliton imprinted in a two-dimensional Bose-Einstein condensate. We have shown that depending on the form of the imprinted phase step, the soliton bends and then, via a snake instability, it decays into vortices. We have performed numerical simulations of the time evolution of the imprinted dark soliton in two-dimensions. Finally, we have qualitatively interpreted the theoretical framework that describes this phenomenon in the approximation of weakly interacting bosons.

I. INTRODUCTION

Since its foundation at the beginning of the last century, quantum physics has led up to the growth of our knowledge in directions previously forbidden. After the pertinent formulation of quantum mechanics and taking into account the increasingly relevance of quantum technologies and their applications, our knowledge in those fields has grown, as have the scales at which quantum effects can be observed nowadays, starting from the subatomic and progressing to the macroscopic.

The Bose-Einstein condensate (BEC) is a form of matter that Albert Einstein predicted in 1924-1925 based on Satyendra Nath Bose's study and first observed by Eric Cornell and Carl Wieman at JILA in 1995 [1]. That state of matter consists of a system made up of bosons that become condensed when cooled near absolute zero; under those conditions, a macroscopic fraction of the particles occupy the same single-particle wave function with the lowest energy, allowing the quantum effects to be relevant on a macroscopic scale.

A many-body, weakly interacting bosonic system at very low temperatures can be described within the mean-field framework by the Gross-Pitaevskii (GP) equation, which was formulated by E. P. Gross and L. P. Pitaevskii while working on characterizing vortices in those systems [5, 7].

The GP equation is a nonlinear Schrödinger like equation, and may exhibit certain nonlinear solutions, such as solitons. Solitons are solitary waves that travel in a non-linear media without distortion as a result of a delicate balance between dissipation and the medium's nonlinearity. They are most commonly encountered in physical processes that can be explained using a series of nonlinear partial derivative equations. Bright solitons (BS) or dark solitons (DS) are a type of solution that emerges in BECs as a modulation of the density profile depending on the interparticle interaction. A BS represents a peak in the density amplitude, while a DS represents a background density localized dip with a characteristic phase step. BS appears for attractive interactions, whereas DS for repulsive ones.

Solitons in 1D are stable solutions of the GP equation. The stability of these solutions is determined, among other factors, by the dimensionality of the system under consideration. In particular, DSs for 2D are unstable against transverse modulation, a phenomenon known as snake instability (SI), which bends the nodal stripe until it breaks up into vortices and sound waves.

The report's structure is broken down as follows. Section II presents the theoretical framework we will be working with, including the mean-field approximation and the equations that govern systems of many weakly interacting bosons, as well as the DS and what it means. Section III details the specific case that will be numerically reproduced, as well as the considerations that have been taken. The results collected and the related discussion are reported in Section IV. Section V concludes with a summary of our results.

II. THEORETICAL FRAMEWORK

A. The Gross-Pitaevskii equation

Our approach for studying the behavior of these systems will be in the approximation of weakly interacting bosons, which can be described by the Gross-Pitaevskii theory. Most of the effects of two-body interactions in these dilute gases around zero temperatures may be described using this framework [3].

N identical and interacting bosons, at very low temperatures $T \simeq 0$, confined by an external potential V_{ext} are described by the following many-body Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\nabla_i^2 + V_{\text{ext}}(\mathbf{r}_i) \right] + \frac{1}{2}\sum_{i\neq j} V(r_{ij}) \,. \quad (1)$$

When the system is very dilute only binary low-energy collisions are relevant. In that situation the interaction potential between the particles can be modelized as a contact potential like $V(r_{ij}) = g\delta(\mathbf{r}_i - \mathbf{r}_j)$, where the coupling constant is $g = 4\pi\hbar^2 a_s/m$, being a_s the scattering lenght of the s-wave and m the mass of the particle.

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From Eq. (1) one can write the energy functional as:

$$E[\Psi] = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\Psi|^2 + V_{\text{ext}}(\mathbf{r}) |\Psi|^2 + \frac{g}{2} |\Psi|^4 \right] . \quad (2)$$

The GP equation can be obtained by employing the variational method on (2):

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + gN |\Psi(\mathbf{r}, t)|^2 \right] \Psi(\mathbf{r}, \mathbf{t}) , \quad (3)$$

where the wave function is normalized as $\int d\mathbf{r} |\Psi(\mathbf{r}, t)|^2 = 1$. We can associate the density profile of the condensate to $n(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2$.

B. Dark solitons

In the nonlinear regime, the GP equation has exact analytical solutions in 1D. These solutions take the shape of solitary waves, also known as solitons, which are localized disturbances that propagate without changing their shape.

This propagation without spreading is produced by balancing the nonlinear term with the dispersion term; in our case, the corresponding terms are the nonlinear interaction $g |\Psi(\mathbf{r}, t)|^2$ and the kinetic energy one $-\frac{\hbar^2 \nabla^2}{2m}$. Our system will take a solitonic solution in the form of a DS as we will model a rubidium condensate with a repulsive atomic interaction $g > 0$, since that condition is required for DS to exist.

DS have some traits from what one might recognize them. The speed of propagation in the condensate, ν_{s}, is less than the Bogoliubov's speed of sound [8]:

$$\nu_0 = \sqrt{\frac{gn_0}{m}} , \quad (4)$$

where n_0 is the background (bulk) density. The speed and depth of the soliton, as well as the Bogoliubov speed, can be related [4]:

$$\frac{\nu_s}{\nu_0} = \cos \frac{\delta}{2} = \sqrt{1 - \frac{n_d}{n_0}} , \quad (5)$$

where n_d is the density depth of the soliton and δ the value of the phase step. For a discontinuous phase step along the BEC, $\delta = \pi$, the dark soliton has zero velocity and zero density at its center with a width of the order of the healing length, being that:

$$\xi = \frac{\hbar}{\sqrt{2 n_0 m g}} . \quad (6)$$

Looking how it evolves in terms of δ one might observe that the speed increases as δ decreases, approaching the speed of sound. The soliton grows wider and shallower, with a more progressive phase step. They do not maintain static anymore and they travel in the opposite direction of the phase gradient. Since DS behave by this characteristic phase step, a way to generate such a solitary wave is by initially imprinting a phase step, $\phi(x, y)$, on the ground state condensate wave function $\Psi \to \Psi \exp[i\phi(x, y)]$.

III. SNAKE INSTABILITY SIMULATION

In this work we consider a 2D condensate. To that end, one method of forcing a real BEC to live near two dimensions is to confine it with a three-dimensional harmonic potential with a frequency $w_z \gg w_x, w_y$; that is, a confinement in the transverse direction strong enough to freeze the dynamics in this axis, being the 2D potential that results:

$$V_{\text{ext}}(x, y) = \frac{m}{2} \left(\omega_x^2 x^2 + \omega_y^2 y^2 \right) . \quad (7)$$

To effectively describe our condensate in 2D, the GP equation (3) must be expressed with an effective coupling constant in 2D, $g \to g_{2D} = g/(\sqrt{2\pi} a_z)$, where a_z is the characteristic length, in oscillator units, for the axis perpendicular to the plane. This expression can be demonstrated reducing the GP equation from the 3D case [2].

The simulations in this report are performed for a ^{87}Rb BEC with a number of particles $N = 10^5$, a characteristic scattering length for the s-wave $a_s = 109\, a_0$ where a_0 is the Bohr radius and the trapping frequencies are $\omega_x = \omega_y = 2\pi \times 10$ Hz and $\omega_z = 2\pi \times 700$ Hz [11].

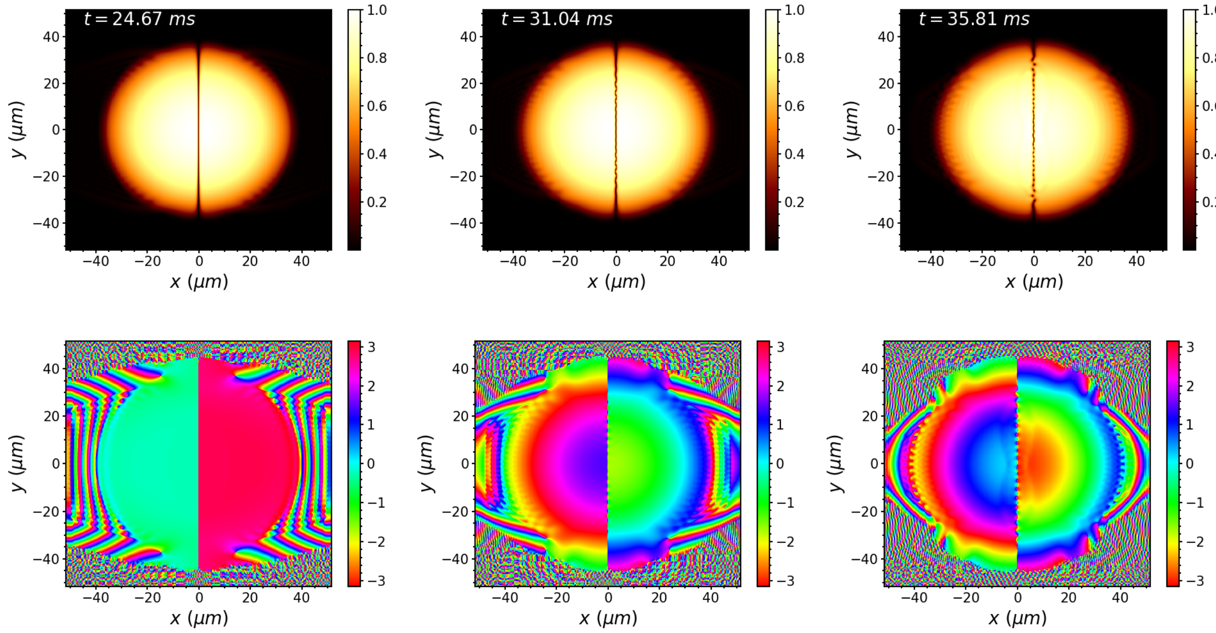
As previously mentioned, the procedure for constructing a DS is to imprint a phase step with a value of π at $x_0 = 0$ along the condensate wave function, with a phase dependence on the x axis, and then evolve the resulting wave function in imaginary time to form a DS with a nodal strip parallel to the y axis.

There is a formal difference between a DS and a gray soliton (GS): the first one has a complete depletion of the density in the nodal strip while for the GS the depletion is only partial. Even if the imprinted DS quickly decays into a GS, because it won't be able to maintain a full depletion of the density at the notch, we will still refer to the first as a DS and we will use GS for the most shallow ones [13].

Solitons are not stable solutions of a nonlinear equation beyond one dimension; in those circumstances, the stable solutions are vortices, which are phase singularities. A kind of decay of a DS to these vortices is via the instability known as the snake instability (SI), which is caused by initial disturbances modulating the amplitude of the soliton that bends along its whole length.

Because the speed of the DS (5) is related to its amplitude, its modulation has a direct effect on it, which is magnified by the subsequent disturbances caused by bending the DS, eventually tearing it into vortices [6].

The transverse extension of the condensate must be less than or of the order of the healing length for the DS to remain stable; otherwise, it will decay into vortices.



(a) (b) (c)

Figure 1: (Top panels) Snapshots of the density in the x-y plane. (Bottom panels) Snapshots of the corresponding phase in the x-y plane. The phase of the imprinted DS at $t = 0$ and $x_0 = 0$ corresponds to Eq. (8), with a value of $b \simeq 0$, very small but not null.

If the condensate's transverse extension is insufficient, the produced vortices rejoin to create the black stripe and undergo SI once again.

In the simulation, we used both sharp and smooth phase gradients because, as previously stated, the behavior of the SI will be heavily influenced by these phase gradients we will imprint DS with. While the sharp gradient can be obtained with as a Heaviside step function that extends both towards positive and negative values of the position, the impression of smooth phase can be described as follows [4, 11]:

$$\phi(x) = \frac{\pi}{2}\tanh\left(\frac{x - x_0}{b}\right). \tag{8}$$

Imprinting a phase step at x_0 is equivalent to imbuing momentum to the wave function where the phase varies. The spatial width of the phase gradient will be determined by the parameter b, see Fig. 2. The width determines the dynamics and properties that the DS and the SI will have.

The simulations we ran made use of the Trotter-Suzuki package (TS), created by Wittek and Calderaro [12]. This package offers a solution for the Schöringer equation that can simulate the evolution of interacting BECs given by the GP equation. To lower the computational cost of evolving the GP, the software applies a generalization of the Trotter formula [10]. An exponential operator can be roughly approximated using that formula. According to Trotter's formula, in the case of an exponential with only two operators:

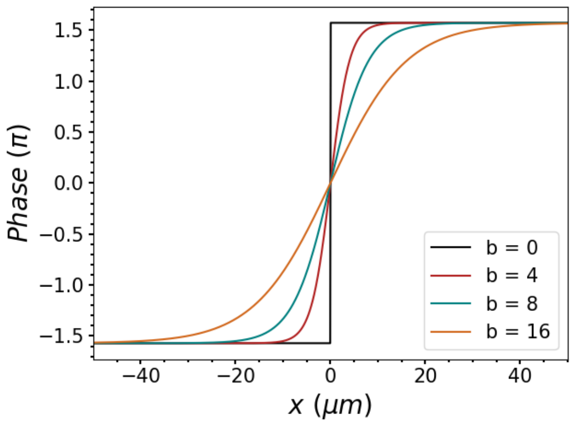


Figure 2: Phase step function, Eq.(8), along the x axis, centered at $x_0 = 0$, according to different values of the parameter b, which defines the amplitude of the imprinted soliton.

$$\exp(A + B) = \lim_{n\to\infty}\left(\exp\frac{A}{n} + \exp\frac{A}{n}\right)^n \tag{9}$$

Where A and B are arbitrary real or complex matrices $M \times M$. The exponential of an operator is expensive to calculate numerically, but when expressed as the sum of operators this cost is reduced. This formula provides a way to reasonably approximate these exponentials of operators. It should be noted that another approximation

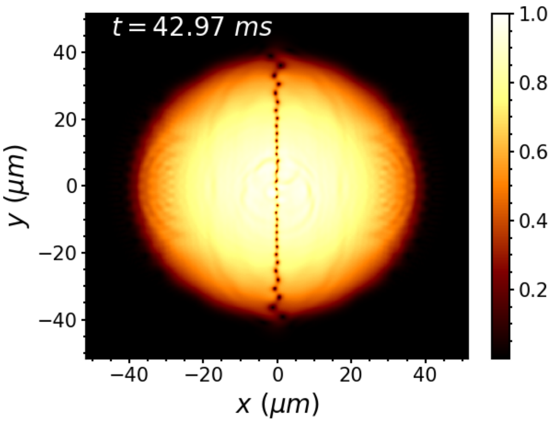


Figure 3: Density snapshot in the x-y plane at $t = 42.97$ ms and the imprinted DS at $x_0 = 0$. The SI has already been broken into numerous vortices

is necessary and decide where to truncate n, as an infinity limit is required and this is not numerically appropriate. Suzuki provided a good approximation of the error assumed depending on how n is truncated [9].

IV. RESULTS

We performed different simulations by imprinting a DS centered at x_0 using the phase step, Eq. (8), with different widths of the phase gradient, i.e. modifying the values of b. We have investigated the time evolution of the DS and the possible appearance of the SI.

If we imprint the phase step at $x_0 = 0$, with a small value of b, into the ground state of the wave function and let it evolve we can see how a DS emerges in the center, followed by sound waves and GSs that can be identified by their speed. Being the velocity of the sound waves the speed of sound, the velocity of the GSs depends on the depth of their notch and will be lower than the speed of sound. Then, the SI start to form at the center of the DS and evolves to the edges of the BEC.

If we increase b in the phase imprinted DS we see that the SI will develop at the same time at the center and at the edges of the BEC, propagating to the intermediate regions. Increasing further the parameter b the SI develops first at the edges, evolving to the center.

This case can be seen in the panels in Fig. 1 as an imprinted phase step of value π develops a DS in the center of the BEC, see Fig. 1.a. This DS lasts until transverse disturbances destabilize it, eventually leading to the development of the SI. It can be observed how it begins on the borders, due to the non null value of b, and progresses to the center, see Fig. 1.b. Eventually, see Fig. 1.c, the integrity of the SI is compromised, resulting in the formation of numerous vortices, see Fig. 3. Finally for very large b values the system does not decay through a SI but directly into vortices, because the created DS is shallow enough to have a velocity near the speed of sound.

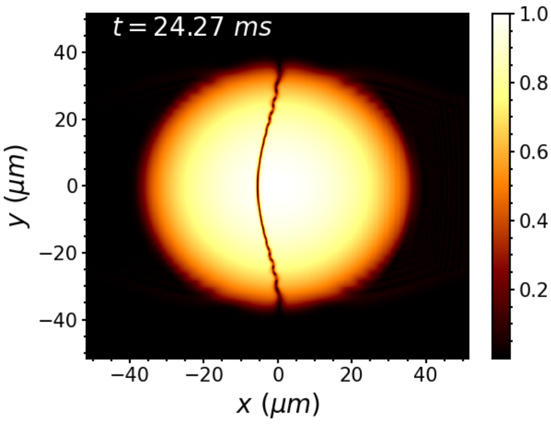


Figure 4: Density snapshot of the density in the x-y plane at $t = 24.27$ ms. The imprinted DS at $x_0 = 0$ has a value of $b \simeq 10^{-3}$ large enough to provide a bending of the DS while the SI is developing.

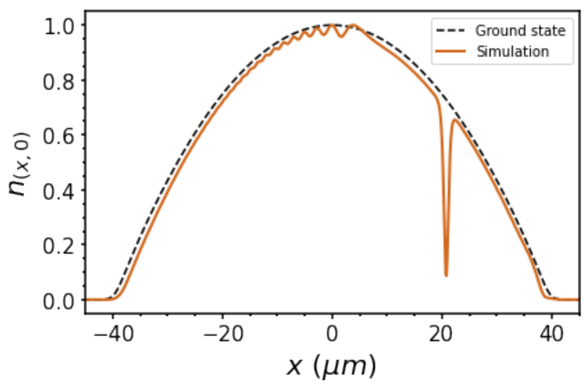


Figure 5: Snapshot of the normalized density profile at $y = 0$ and $t = 11.94$ ms after having imprinted the phase step at $x_0 = 24.1$ μm. The profile of the GS is shown.

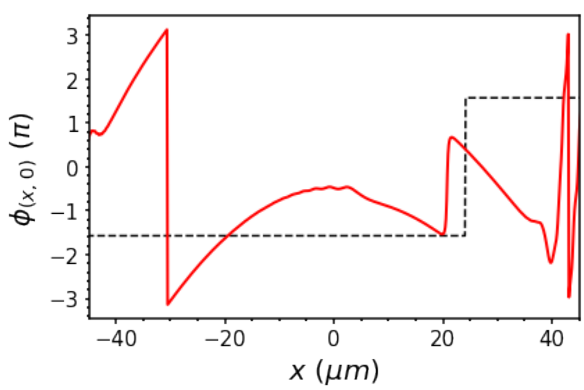


Figure 6: Snapshot of the phase profile. Dashed line is the phase profile of the initially imprinted soliton at $x_0 = 24.1\mu$m, and $y = 0$. Solid line is the phase profile at $y = 0$ and $t = 11.94$ ms

Another effect of acting with a smooth gradient is that the soliton, independently of how the SI develops, tends to bend like a bow, see Fig. 4.

When the imprinted phase is off-center in the condensate, see Fig. 5 and Fig. 6, the resulting DS does not remain static and instead moves along the condensate in the opposite direction of the phase gradient. This movement will be slower than the speed of sound, identifying it as a DS versus GSs, as well as sound waves that will form. Moving on a background of non-homogeneous density will cause the SI to be more difficult to develop, whether it comes to form, it will be shredded into vortices directly as it travels along the BEC.

V. CONCLUSIONS

We have addressed the phenomenon of snake instability of a dark soliton in a two-dimensional Bose-Einstein condensate with repulsive interactions. To understand it, we have analyzed how the phase imprinted on the ground state density to generate the dark soliton affects this instability. We have also investigated this phenomenon numerically.

The numerical simulations we have performed reproduce the expected behavior that derives from the theory. We have been able to see how the parameters that determine the given system and its evolution are: the value of the phase step, the amplitude of the gradient of the phase step, and the relative position where it is printed in the density profile.

Finally, we have seen that the snake instability opens up the door to form a condensate full of vortices exhibiting a rich dynamics. This dynamics deserves further investigation.

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