The toric code, anyons and quantum computation

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Abstract: The aim of this project is to study the toric code, a particular example of a system that realizes quantum computation. We explain the role of topology in determining the ground state of the system and its degeneracy, as well as the surprising fact that excitations are neither bosonic nor fermionic, but rather anyonic. As a novel result, we deduce the full spectrum of the toric code.

I. Introduction

In recent years, one of the main challenges of quantum computation (QC) has been the study of fault-tolerant quantum computation and the development of quantum error-correcting codes (QECCs). Proposed by Kitaev in 2003 (cf. [1]), a simple example to realize QC is the toric code (TC); a system of $\frac{1}{2}$-spin particles on a lattice embedded in a torus, which uses the topological properties of the surface to encode and protect information. The purpose of the article is to examine this quantum system in detail.

In chapter II, we begin introducing the concept of anyonic statistics as the particle statistics that naturally emerges in $\mathbb{R}^2$. Although it might seem disconnected from QC at first, we will later see how it is deeply connected with the excitations of the TC. In chapter III, we focus on the TC by studying the energy and degeneracy of the ground and excited states, hence computing the full spectrum of the Hamiltonian. We also analyse the behaviour of its low-energy excitations, which turn out to be anyonic. To conclude, in chapter IV we briefly review how the TC can be used for the realization of QC.

II. Particle statistics. Anyons

Consider the exchange of $N \geqslant 2$ indistinguishable particles in $\mathbb{R}^d$ (). We study the effect of this exchange in the wave function of the system, ψ_N, and the different particle statistics that emerge depending on $d > 1$. For this purpose we consider the configuration space, defined as

$$C_N^d(\mathbb{R}) := \mathrm{Conf}_N(\mathbb{R}^d) := \frac{(\mathbb{R}^d)^N \backslash \Delta}{S_N}, \tag{1}$$

where S_N is the symmetric group of order N and $\Delta := \{(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mid r_i = r_j \text{ for some } i \neq j\}$.

We want to study the changes in ψ_N when it evolves from the point $p_i := (x_i, t_i)$ to $p_f := (x_f, t_f)$. Because the particles are indistinguishable, we can consider $x_i = x_f$, that is, closed loops in the configuration space. Using the path-integral formulation (cf. [2, p. 8]) we can show that the change in ψ_N depends solely on some parameters $\chi(\alpha)$, where $[\alpha] \in \Pi_1(C_N^d(\mathbb{R}))$. In fact, these $\chi(\alpha)$ are representations of $\Pi_1(C_N^d(\mathbb{R}))$.

Summarizing, the variations of ψ_N under exchange depend only on the fundamental group of the configuration space, and its representations give us the different particle statistics that emerge in that space. Hence, to obtain these different behaviours depending on d, we need the following result (cf. [3]):

$$\Pi_1(C_N^d(\mathbb{R})) = \begin{cases} S_N, & d > 2, \\ B_N, & d = 2 \end{cases}, \tag{2}$$

where B_N is the braid group of order N. Let's focus on the 1-dimensional representations of these groups.

We begin with the most well-studied case, $d > 2$. There are only two 1-dimensional representations of the symmetric group; the identity, $\mathbb{I}$, and the sign representation, sgn. The former gives place to bosons and the latter to fermions. More specifically, we obtain $\chi_{\mathbb{I}}(\sigma) = 1$ and $\chi_{\mathrm{sgn}}(\sigma) = \pm 1$, $\forall \sigma \in S_N$, which define the statistics of the wave function under exchange, depending on the particle type.

Now for $d = 2$, we have that there are infinite 1-dimensional representations of the braid group. We will call these representations *abelian anyons* (higher dimensional representations correspond to *non-abelian anyons*, but we don't consider them here). The term *abelian* is due to the commutation of the 1-dimensional representations, that are just complex numbers. The term *anyon* means that these particles have *any* statistics, as there are infinite representations of B_N. We can also assign values to $\chi(\alpha) \in \mathbb{C}$, $\forall \alpha \in B_N$ (cf. [2, p. 18]), that give rise to an infinite number of statistics under exchange.

III. The toric code

Consider a $2-$dimensional square lattice with $L \times L$ edges with periodic boundary conditions so that it can be embedded in a torus, and with $\frac{1}{2}$-spin particles in the middle of each link, as in FIG. 1. Clearly, we have $2L^2$ spins, and the Hilbert space of the system, $\mathcal{H}$, has dimension $\dim \mathcal{H} = 2^{2L^2}$. Consider a Hamiltonian for this system given by:

$$H := -\sum_{v \in V} A_v - \sum_{p \in P} B_p, \tag{3}$$

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where A_v and B_p are called vertex and plaquette operators acting on vertex v and plaquette p of the lattice, respectively. They are defined as

$$A_v := \bigotimes_{j=1}^{L^2} P_j, \quad P_j = \begin{cases} X, & \text{if} \quad j \in v \\ I, & \text{if} \quad j \notin v \end{cases}, \\ B_p := \bigotimes_{j=1}^{L^2} Q_j, \quad Q_j = \begin{cases} Z, & \text{if} \quad j \in p \\ I, & \text{if} \quad j \notin p \end{cases}, \tag{4}$$

where I, X and Z are the usual identity and Pauli operators, and $\otimes$ denotes the Kronecker product. V and P are the complete set of vertices and plaquettes of the lattice, respectively. It is easy to check that we have L^2 A_v operators, and the same number of B_p operators.

When referring to any operator acting on the lattice, we will only reference the operators different from I (e.g. $B_p \equiv \bigotimes_{j\in p} Z_j$, where Z_j is the operator Z acting on the spin s_j). To conclude with the definitions, we define $|0\rangle$ and $|1\rangle$ as the states of a spin such that $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$, $X|0\rangle = |1\rangle$, and $X|1\rangle = |0\rangle$.

The aim of this section is to study H and its properties. To do so, one can first check that, $\forall v, v' \in V$, $\forall p, p' \in P$ and for any given eigenstates $|\psi\rangle, |\psi'\rangle \in \mathcal{H}$ of a A_v or B_p operator, the following properties hold:

1. $A_v|\psi\rangle = \pm|\psi\rangle$ and $B_p|\psi'\rangle = \pm|\psi'\rangle$,
2. $\prod_{v\in V} A_v = \prod_{p\in P} B_p = I$,
3. $[A_v, A'_v] = [B_p, B'_p] = [A_v, B_p] = 0$,
4. $[A_v, H] = [B_p, H] = 0$.

The first property follows from the fact that A_v and B_p are all hermitian and idempotent. The second follows because X and Z are idempotent and they appear twice in each spin in the product. For the third property; $[A_v, A'_v] = 0 = [B_p, B'_p]$ is trivial because $[X, X] = [Z, Z] = 0$. Also, if v and p share a spin, then they share two. Using $[X \otimes X, Z \otimes Z] = 0$, we have that $[A_v, B_p] = 0$. The fourth property is immediate from the third one. It also follows that all terms in the Hamiltonian can be simultaneously diagonalised. With these basic properties, let's study the states of the Hamiltonian, together with their energies and degeneracies.

A. The ground states

Let's study the ground states of the Hamiltonian. We need to find a configuration such that the energy of H is minimal, thus, for which the eigenvalues of A_v and B_p are maximal. Before studying the complete Hamiltonian, consider first the restricted case

$$\hat{H} := -\sum_{p\in P} B_p. \tag{5}$$

Consider the subspace $\mathcal{H}_B := \{|\xi\rangle \in \mathcal{H} \ : \ \lambda_p(\xi) = +1, \forall p \in P\}$, and let $\mathcal{B}_{\mathcal{H}_B}$ be the Z-product basis of that space, where $\lambda_p(\xi)$ is the eigenvalue of $|\xi\rangle$ when B_p is applied to it.

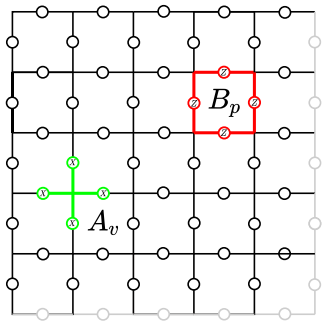

FIG. 1: Example of a $L = 5$ toric code lattice. The circles in the middle of each link simulate the $\frac{1}{2}$-spin particles, and the grey edges and circles represent the periodic boundary conditions of the lattice. Notice also the representation of a A_v and a B_p operator.

Now, because given a plaquette p the spins s_j on that plaquette are either $|0\rangle$ or $|1\rangle$, we can count the number of 0 and 1 spins in each plaquette, which we call the parity of the plaquette p:

$$\nu_p(\xi) := \sum_{j\in p} s_j \pmod 2. \tag{6}$$

Notice that, given $|\xi\rangle \in \mathcal{B}_{\mathcal{H}_B}$, $\nu_p(\xi) = 0 \pmod 2$, $\forall p \in P$.

Consider two adjacent columns of vertical links (a column of plaquettes from the bottom to the top of the lattice where we have eliminated the horizontal links), c_1 and c_2. It is easy to check that $\nu_{c_1}(\xi) := \sum_{j\in c_1} s_j \pmod 2 = \sum_{j\in c_2} s_j \pmod 2 =: \nu_{c_2}(\xi)$, $|\xi\rangle \in \mathcal{B}_{\mathcal{H}_B}$. Because this can be done with any two given adjacent columns of links, we have that $\nu_c := \nu_{c_i}(\xi)$ is a homological invariant called *topological number*, where i is any given column. A similar argument considering rows instead of columns leads us to another invariant, ν_r.

Because ν_c and ν_r can either be 0 or 1 for any given $|\xi\rangle \in \mathcal{B}_{\mathcal{H}_B}$, we have four possible combinations of topological numbers; $(\nu_r, \nu_c) = \{(0,0), (0,1), (1,0), (1,1)\}$. Define $\Omega_{(\nu_1,\nu_2)} = \{|\xi\rangle \in \mathcal{B}_{\mathcal{H}_B} \ : \ (\nu_r, \nu_c)(\xi) = (\nu_1, \nu_2)\}$.

Going back to $\hat{H}$, it is clear that a ground state can be expressed as the following linear combination:

$$|\nu_1, \nu_2\rangle = \sum_{|\xi\rangle\in\Omega_{(\nu_1,\nu_2)}} \alpha_\xi |\xi\rangle . \tag{7}$$

where the $\alpha_\xi \in \mathbb{C}$ are arbitrary parameters. To prove that such a state exists, it suffices to consider a lattice with rows of $|0\rangle$-spins and columns of $|1\rangle$-spins.

Consider now the full Hamiltonian given by (3). We claim that if we take $\alpha_\xi = 1$, $\forall |\xi\rangle \in \Omega_{(\nu_1,\nu_2)}$, then $|\nu_1, \nu_2\rangle$ is also a ground state of H. First notice that given $|\xi\rangle \in \Omega_{(\nu_1,\nu_2)}$, for any A_v, we have that $|\xi'\rangle := A_v|\xi\rangle \in \Omega_{(\nu_1,\nu_2)}$:

$$B_p|\xi'\rangle = B_p A_v|\xi\rangle = A_v B_p|\xi\rangle = A_v|\xi\rangle = |\xi'\rangle , \tag{8}$$

where we have used property 3. Moreover, because A_v will flip an even number of spins of the column c_i and row r_j it acts on, the numbers (ν_1, ν_2) won't change. It's also easy to see that $A_v |\xi'\rangle \neq A_v |\xi''\rangle$ for any A_v and for any two different states $|\xi'\rangle, |\xi''\rangle \in \Omega_{(\nu_1,\nu_2)}$, because $A_v^2 = I$.

With the arguments above, if we take $\alpha_\xi = 1$, we have

$$A_v |\nu_1, \nu_2\rangle = \sum_{|\xi\rangle \in \Omega_{(\nu_1,\nu_2)}} A_v |\xi\rangle = \sum_{|\xi'\rangle \in \Omega_{(\nu_1,\nu_2)}} |\xi'\rangle = |\nu_1, \nu_2\rangle , \tag{9}$$

$\forall v \in V$, so that the eigenvalue of $|\nu_1, \nu_2\rangle$ under any A_v is $+1$, and so it's a ground state of H, as expected.

Because of the different possible combinations of (ν_r, ν_c), we have four different ground states:

$$|\nu_1, \nu_2\rangle = \sum_{|\xi\rangle \in \Omega_{(\nu_1,\nu_2)}} |\xi\rangle . \tag{10}$$

These 4 vectors, in fact, form a base of the ground state vector space. This means that we have a 4-fold degenerate ground space, with an explicit way to build its base.

To conclude this section, let's compute the energy of the ground states. Because all the A_v and B_p operators have eigenvalue $+1$ and thanks to property 4, the energy it's just a sum of these eigenvalues. Because we have $2L^2$ operators, the energy of the ground state, E_{GS}, is:

$$E_{GS} = -2L^2. \tag{11}$$

B. Excited states

1. Energy of the excited states

Let's study the excited states of the Hamiltonian. Consider a ground state $|\Psi\rangle$, and let $|\psi_X\rangle := X |\Psi\rangle$ and $|\psi_Z\rangle := Z |\Psi\rangle$. Because of the anti-commutation relations $\{Z, A_v\} = \{X, B_p\} = 0$, we have $A_v |\psi_Z\rangle = - |\psi_Z\rangle$ and $B_p |\psi_X\rangle = - |\psi_X\rangle$. Thus, applying X or Z to any spin will create an excited state. In particular, $|\psi\rangle$ will have eigenvalue -1 under an operator A_v (B_p) if we apply Z (X) to an odd number of spins in interaction with A_v (B_p). Because applying a Pauli matrix to one spin will always change two eigenvalues, the energy of the first excited state is given by:

$$E_{1st} = E_{GS} + 4 = -2L^2 + 4. \tag{12}$$

In general, if we have $2l$ operators with eigenvalue -1, the energy of the excited state will be:

$$E_{2l} = E_{GS} + 4l = -2L^2 + 4l. \tag{13}$$

Finally, observe that because any flipped spin will always affect an even number of operators of the same type, we have the restriction $2l \leqslant 2L^2$ if L is even, and $2l \leqslant 2L^2 - 2$ if L is odd. Hence, the energy of the highest excited state will be $E_{2L^2} = 2L^2$ if L is even, and $E_{2L^2-2} = 2L^2 - 4$ if L is odd.

2. Degeneracy of the excited states

Let's move to the degeneracy of the excited states. We proceed in a similar way as for the ground state; we first study the degeneracy of $\hat{H}$, and then that of H.

Let $\mathcal{B}_\mathcal{H}$ be the Z-product basis, $|\xi\rangle \in \mathcal{B}_\mathcal{H}$, and fix the eigenvalues $\lambda_1, ..., \lambda_{L^2-1}$ of $|\xi\rangle$ for $L^2 - 1$ B_p operators (all except for one), where a of them are -1-eigenvalues. Because we have $2L^2$ edges and $L^2 - 1$ restrictions for the B_p operators, we can freely choose the state of $2L^2 - (L^2 - 1) = L^2 + 1$ spins so that the state $|\xi\rangle$ has the specified eigenvalues in the given order. Because we are interested in the degeneracy of H, we can count the number of states having a -1-eigenvalues in a random order as $2^{L^2+1}\binom{L^2-1}{a}$. If we include the last plaquette, the possible combinations increase to $2^{L^2+1}\binom{L^2}{2k}$, where $2k = a$ if a is even, and $2k = a + 1$ if a is odd. Thus we know the number of states in $\mathcal{B}_\mathcal{H}$ having $2k$ -1-eigenvalues for the B_p operators, thus the degeneracy of $\hat{H}$.

Let's now study the degeneracy of H. Again, we consider the same elements $|\xi\rangle$ of the Z-product basis with the $\lambda_1, ..., \lambda_{L^2-1}$ eigenvalues fixed for the first $L^2 - 1$ B_p operators. First, using the same argument as in equation (8), we have that multiplying any of these states $|\xi\rangle$ by any A_v operator, the new state $|\xi'\rangle$ has the same set of eigenvalues for the B_p operators than $|\xi\rangle$. Because we have 2^{L^2-1} possible products of the different A_v operators and 2^{L^2+1} states with fixed eigenvalues for the B_p's, we get $\frac{2^{L^2+1}}{2^{L^2-1}} = 4$ sets of states with the same eigenvalues and so that, multiplying a state from a set for any A_v operator, the new state will belong to the same set.

In addition, if we want $2i$ -1-eigenvalues for the A_v operators, we can use projector operators, defined as

$$\Pi_{v,\pm} := \frac{I \pm A_v}{2}, \quad \forall v \in V. \tag{14}$$

A projector $\Pi_{v,\pm}$ transforms any state $|\xi\rangle$ to an eigenstate of the A_v operator with eigenvalue ± 1. Hence, the different combinations of projectors that we have to build eigenstates with $2i$ -1-eigenvalues for the L^2 A_v operators is given by $\binom{L^2}{2i}$.

Finally, we conclude that the number of states having $2l = 2i + 2k$ -1-eigenvalues for the A_v and B_p operators is given by the following sum:

$$\deg(E_{2l}) = 4 \sum_{2k+2i=2l} \binom{L^2}{2k}\binom{L^2}{2i}, \tag{15}$$

where the 4 is justified by the four sets of states defined above (similar to the ground state degeneracy).

Our objective is now to compute this sum. Define $a(x) := (1 + x)^n (1 + x)^n$ and $b(x) := (1 + x)^n (1 - x)^n$. From the binomial theorem, $a(x)$ and $b(x)$ are equivalent

to:

$$a(x) = \sum_{i,j=0}^{n} \binom{n}{i}\binom{n}{j} x^{i+j},\ b(x) = \sum_{i,j=0}^{n} (-1)^j \binom{n}{i}\binom{n}{j} x^{i+j}. \tag{16}$$

If we consider $n = L^2$, $p(x) := 2(a(x) + b(x))$, and a given x^r of $p(x)$, $r = 2l$ even, from (16) we get that the coefficient p_{2l} is:

$$\begin{aligned} p_{2l} &= 2 \sum_{i+j=2l} \binom{L^2}{i}\binom{L^2}{j} + 2 \sum_{i+j=2l} (-1)^i \binom{L^2}{i}\binom{L^2}{j} \\ &= 4 \sum_{2i+2j=2l} \binom{L^2}{2i}\binom{L^2}{2j} = \deg(E_{2l}), \end{aligned} \tag{17}$$

where in the second identity we have used that the summands will eliminate unless i is even, and thus $2i+j = 2l$ will only be satisfied if j is also even.

If we now rewrite $a(x) = (1+x)^{2n}$ and $b(x) = (1-x^2)^n$, we can also derive the relations:

$$a(x) = \sum_{i=0}^{2n} \binom{2n}{i} x^i,\ \ b(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j x^{2j}, \tag{18}$$

from which it follows that, taking $n = L^2$, the coefficient p_{2l} of the polynomial $p(x)$ defined before is

$$\deg(E_{2l}) = p_{2l} = 2\left[\binom{2L^2}{2l} + (-1)^l \binom{L^2}{l}\right]. \tag{19}$$

The previous result gives us an analytical solution for the degeneracy of the eigenvalues of H. It is also easy to check that if we sum the degeneracy of all energy states, we get $\sum_{0\leqslant 2l\leqslant 2L^2} \deg(E_{2l}) = 2^{2L^2}$, as expected. With this result we conclude the study of the Hamiltonian H.

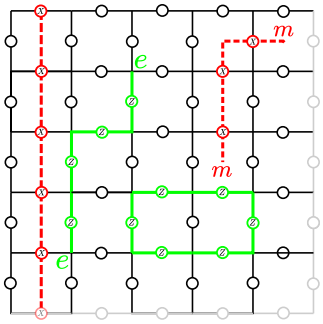

FIG. 2: Example of 4 strings. Two of them are open strings, one creating two charges and the other creating two monopoles. The other two are closed strings; a contractible and a non-contractible loop ($\overline{X}_v$). The dotted lines represent the path followed by the X-strings, which can be thought as a path in the dual space of the lattice.

3. *Anyonic behaviour of the excited states*

Our goal now is to examine the behaviour of the low-energy excitations. We will see that excitations have particle behaviour, so given an eigenstate $|\psi\rangle \in \mathcal{H}$ of an operator A_v with eigenvalue -1, we will say that the vertex v is an *e-excitation* or *charge*. Similarly, the plaquette p is a *m-excitation* or *monopole*, and if we have a pair of charges and a pair of monopoles where the v and p share spins, we say that we have two pairs of ϵ-*excitations* or *dyons*. Again, because a spin belongs to two v and two p, excitations always come in pairs.

Let $|\Psi\rangle$ be a ground state. Let's formally define the concept of string, already used in chapter II. Two spins are Z, X or Y-adjacent if they share a A_v, B_p, or A_v and B_p operator, respectively. Let $\alpha = \{X, Y, Z\}$, and define a α-string as a set of operators acting on α-adjacent spins:

$$\Gamma_\alpha := \bigotimes_{j\in\gamma_\alpha} Q_j, \tag{20}$$

where $\gamma_\alpha = \{s_1, ..., s_n\}$ is any finite set of α-adjacent spins and $Q_j = \alpha$, $\forall j \in \gamma_\alpha$. Notice that, for any single string, we will always have only two excitations. Clearly, Γ_X strings create a pair of e, Γ_Z a pair of m, and Γ_Y a pair of ϵ (see FIG. 2).

If we consider two α-strings with one edge in common, we can define an operation of concatenation in these two strings in the following way. Given Γ_α and Γ'_α with a common spin s_n,

$$\Gamma_\alpha \cup \Gamma'_\alpha := \bigotimes_{j\in\gamma_\alpha\cup\gamma'_\alpha} Q_j, \tag{21}$$

where again the $Q_j = \alpha$. We note that, as long as there is only one edge in common, there is no increase nor decrease of energy in the concatenation of strings. Hence, we can move the two excitations around the lattice without any exchange of energy. However, if we create a closed string (loop) by concatenating strings, we annihilate the excitations and recover the ground state energy.

Another way of annihilating or combining excitations is by applying more than one Pauli operator to the same spin. This creates what we call *fusion rules*:

- $e \times e = m \times m = \epsilon \times \epsilon = 1$,
- $e \times m = \epsilon, \quad e \times \epsilon = m, \quad m \times \epsilon = e$,

where 1 represents the absence of excitation. The fusion rules are a direct consequence of the products of Pauli matrices (up to a phase).

Let's complete this section studying what happens when we exchange two excitations. Observe that exchanging two particles can be thought of as winding a particle around the other for half a turn. Hence, a complete turn can be thought of as two complete exchanges of the particles.

Consider two pairs of excitations of any type created by strings Γ_α and Γ'_β with no edge in common, where $\alpha, \beta \in \{X, Y, Z\}$, and let $|\Psi\rangle$ be any ground state. Clearly, $[\Gamma_\alpha, \Gamma'_\beta] = 0$. Consider the initial state $|\psi_i\rangle = \Gamma'_\beta\Gamma_\alpha |\Psi\rangle$. Suppose we want to wind an excitation created by string Γ_α around an excitation created

by string Γ'_β. This can be represented by concatenating Γ_α with a closed loop Γ''_α. Clearly, Γ''_α also has an edge in common with Γ'_β. With this construction, the final state after the winding can be represented by

$$|\psi_f\rangle = \Gamma''_\alpha |\psi_i\rangle = \Gamma''_\alpha \Gamma'_\beta \Gamma_\alpha |\Psi\rangle . \tag{22}$$

Studying the commutation relation $[\Gamma''_\alpha, \Gamma'_\beta]$ for the different α, β will give us the exchange behaviour of the excitations, that we call *braiding relations*.

First, it is trivial to see that if we exchange e with another e, or an m with another m, all the operators commute because $\alpha = \beta$, and thus the final state is the same as the initial state. We conclude that e and m particles behave like bosons. In other words, the exchange statistics is $\chi(e, e) = \chi(m, m) = 1$.

What if we want to wind a particle m around e? Clearly the initial state is given by $|\psi_i\rangle = \Gamma'_X |\psi_Z\rangle$, where $|\psi_Z\rangle \equiv \Gamma_Z |\Psi\rangle$. Now, if we apply string Γ''_Z, we obtain

$$|\psi_f\rangle = \Gamma''_Z |\psi_i\rangle = -\Gamma'_X \Gamma''_Z |\psi_Z\rangle = -\Gamma'_X |\psi_Z\rangle = - |\psi_i\rangle , \tag{23}$$

where we have used that Γ'_X and Γ''_Z anticommute, and that Γ''_Z is a closed loop so it doesn't change the state $|\psi_Z\rangle$. Thus, when winding m around e, $|\psi_f\rangle = - |\psi_i\rangle$. Taking the root (because an entire turn is equivalent to two exchanges), we obtain that $\chi(e, m) = i$. This exchange relation is neither bosonic nor fermionic, so we finally find particles presenting anyonic behaviour.

Similar remarks can be done to obtain the rest of the braiding relations. A not so trivial relation is $\chi(\epsilon, \epsilon)$. Because $\alpha = \beta$, one would expect the dyons to have boson behaviour. However, it can be shown (cf. [4, p. 6]) that this is not true and, in fact, they exhibit fermionic behaviour. Summarizing, we obtain:

- $\chi(e, e) = \chi(m, m) = 1, \quad \chi(\varepsilon, \varepsilon) = -1,$
- $\chi(e, m) = \chi(\epsilon, e) = \chi(\epsilon, m) = i.$

These braiding relations, together with the fusion rules, fully describe the behaviour of TC excitations, which turns out to be anyonic. This shows the connection between these -apparently- unrelated fields.

IV. Quantum computation

We conclude this paper by briefly mentioning how to use the TC to realize QC. The lattice is embedded on a torus, so we have four classes of non-trivial loops. These are vertical and horizontal X and Z-loops; $\overline{X}_v$, $\overline{X}_h$, $\overline{Z}_v$, $\overline{Z}_h$.

If we consider the four different ground states defined on chapter II, the non-contractible loops act as operators on these states as usual Pauli matrices acting on a system of 2 spins (e.g. $\overline{X}_h |00\rangle = |10\rangle$). This translates, in the computational language, as having two logical qubits encoded in the system, with their correspondent logical Pauli operators. Moreover, the system is stable against local perturbations (cf. [5, p. 19]); namely, if we consider a local perturbation V acting on less than L spins, then the ground state degeneracy isn't lifted. This means that, the largest the lattice, the more protected is the system.

V. Conclusions

We conclude this paper with a brief review of what we have studied, and possible directions towards which further research could lead.

We have extensively studied the TC and its properties. We have provided a detailed description of its spectrum, a result not present in the literature, as far as we have been able to investigate. We have also studied the behaviour of its low-energy excitations, that turn out to be anyonic. Due to the intrinsic connection between anyons and braid groups, we have provided the fusion and braiding rules of these excitations. We have also shown how the TC has topological order, which makes it a good candidate to realize QC.

Finally, we point out that the homological arguments used to study the degeneracy of the ground state and its protection from local operators, are now widely used in research and in the study of surface codes (for instance, cf. [6]). This reinforces the idea that using the topological properties of systems in a smart way is fundamental to build promising QECCs, and for QC in general.

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