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Automorphisms of Riemann Surfaces

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Introduction

In the middle of the 19th century, Bernhard Riemann began to develop the idea of the today known as Riemann Surfaces, with the aim of constructing the theory of analytic functions of complex variables in a more solid way. Similarly as in differential geometry we develop a theory of manifolds that locally looks like $\mathbb{R}^n$, we can develop an analogous theory about manifolds that locally looks like $\mathbb{C}^n$. In other words, Riemann Surfaces are manifolds of complex dimension one and we can consider them as curves. Treated as a curve, we could be interested in studying the automorphisms of compact and connected Riemann Surfaces. Already at the beginning of the 19th century, it was shown that the curves of genus $g = 0$ and $g = 1$ had an infinite number of automorphisms. However, it remained to be seen what happened for curves with $g \geq 2$. It was not until 1878 that Schwarz proved that for $g \geq 2$ the group of automorphisms was finite (Schwarz's Theorem). Later, in 1893, Hurwitz went further and gave an upper bound of $84(g-1)$ for the number of automorphisms, namely Hurwitz's Theorem. Thus, our principal objective in this work has been to prove Hurwitz's Theorem. In order to do it, we have required several objects that have themselves much importance and we could dedicate an entire paper for each of them.

Before describing how will be the work organized let's give an idea of how we will proceed. We will see that the group of automorphisms of X (compact and connected Riemann Surface) permutes Weierstrass points (W(X)) which we will prove are finite using the Wronskian. Once we will have described $Aut(X)$ and $Perm(W(X))$ we will define the morphism $\lambda : Aut(X) \to Perm(W(X))$ which is (0) or $\mathbb{Z}/2$ and prove that $Aut(X)$ is a finite group. After having proved the finiteness of $Aut(X)$, we will go to the quotient $X/Aut(X)$ and using the Riemann-Hurwitz formula find the bound of the Hurwitz Theorem.

In this work, we will start in Chapter 1 by introducing basic definitions of Riemann Surfaces and presenting the most significant examples of them such as the Riemann Sphere, the Complex Torus and Hyperelliptic Riemann Surfaces. Then, in Chapter 2 we will enter the core of the work by studying functions on Riemann Surfaces, especially meromorphic and holomorphic functions. We will as follows, describe holomorphic maps between Riemann Surfaces that will lead us to the well-known Riemann-Hurwitz formula. This formula is worthy of an entire work because it allows us to study the relationships that our discrete invariants between Riemann Surfaces have. Afterwards, in Chapter 3 we will introduce the language of different forms that will be used later on. We will continue by presenting in

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Chapter 4 finite group actions and the quotient Riemann Surface. Then, in Chapter 5 we will present principal and canonical divisors to finally come up with the Riemann-Roch Theorem. Apart from being highly versatile and generalisable it is also a crucial theorem in Hurwitz Theory. It computes the dimension of certain spaces of meromorphic differentials from properties of the so-called divisor and the genus of the Riemann Surface. However, due to the length and complexity of the proof and in order not to lengthen the work, we will not go into the proof of the theorem in detail. Next, in the following Chapter 6 we will enter the final part of the work. In this section, we will study the Wronskian to introduce the notion of Weierstrass points. These would deserve further study because they have very special properties in terms of meromorphic functions on Riemann Surfaces which make them of great interest. Finally, in the last Chapter 7, we will be ready, with all the notions we have presented, to expose and prove Schwarz's Theorem and Hurwitz's Theorem.
To develop our work, our main source of information has been [1] but it has also been completed using [2], specially to study the finiteness of the group of automorphisms for $g \geq 2$.

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Chapter 1

Riemann Surfaces: Basic Definitions

1.1 Preliminaries

We will start this work by defining precisely what are Riemann Surfaces and we will present several examples of them. Finally, we will give a few properties of Riemann Surfaces that will be used later on.
Before proceeding to the definition of Riemann Surfaces, let's recall some concepts of Complex Analysis we will need.

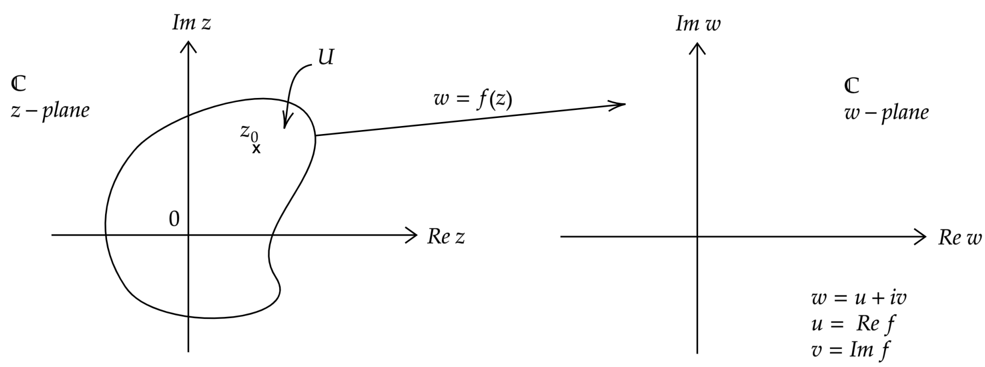

FIGURE 1.1: Holomorphic function

Definition 1.1. Recall that a function $f : \Omega \to \mathbb{C}$ defined on an open set Ω is said to be **analytic** or **holomorphic** at $z_0 \in \Omega$ if one of the following three equivalent conditions holds:

1. write $w = u + iv$ where $u = Re(f)$ and $v = Im(f)$ then we want that the first partials:
$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y} \tag{1.1}$$
exist and are continuous and further satisty the **Cauchy-Riemann Equations**:
$$u_x = v_y \text{ and } v_x = -u_y \ \forall z \text{ in a neighborhood of } z_0. \tag{1.2}$$

2. the limit $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists $\forall$ point z in a neighborhood of z_0.

3. it exists a power series of the form $\sum_{n\geq 0} a_n(z-z_0)^n$ which is convergent to $f(z)$ for each point z in a neighborhood of z_0.

Proposition 1.2. The three above conditions are equivalent.

Definition 1.3. An **injective holomorphic map** is a holomorphic isomorphism *i.e* $f : U \to \mathbb{C}$ ($U \subset \mathbb{C}$ open subset) is holomorphic and f is injective. Then $f(U)$ is open and $f^{-1} : f(U) \to U$ is also holomorphic.

1.2 Formal definition of a Riemann Surface

In this subchapter we will organize the ideas of the previous one to construct a formal definition of a Riemann Surface. To begin with, we will need to define a few concepts.

Definition 1.4. Let's start with X being a topological space:

1. A **complex chart** on X is a homeomorphism $\phi : U \to V$, where $U \subset X$ is an open set in X, and $V \subset \mathbb{C}$ is an open set in the complex plane. The open subset U is called the **domain** of the chart ϕ. We say that the chart ϕ is **centered at p** $\in U$ when $\phi(p) = 0$.

2. Let $\{U_\alpha\}_{\{\alpha\in A\}}$ be an open cover of X, i.e $\cup_{\{\alpha\in A\}} U_\alpha = X$. A **local parameter (local coordinate, coordinate chart)** is a pair (U_α, ϕ_α) of U_α with a homeomorphism $\phi_\alpha : U_\alpha \to V_\alpha$ to an open subset $V_\alpha \subset \mathbb{C}$.

3. Two charts $\phi_1 : U_1 \to V_1$, $\phi_2 : U_2 \to V_2$ are said to be **compatible** if either $U_1 \cap U_2 = \emptyset$, or the **transition function** $T = \phi_2 \circ \phi_1{}^{-1}$ is holomorphic which means that:

$$\phi_2 \circ \phi_1{}^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic as a function of complex variable (see 1.2).

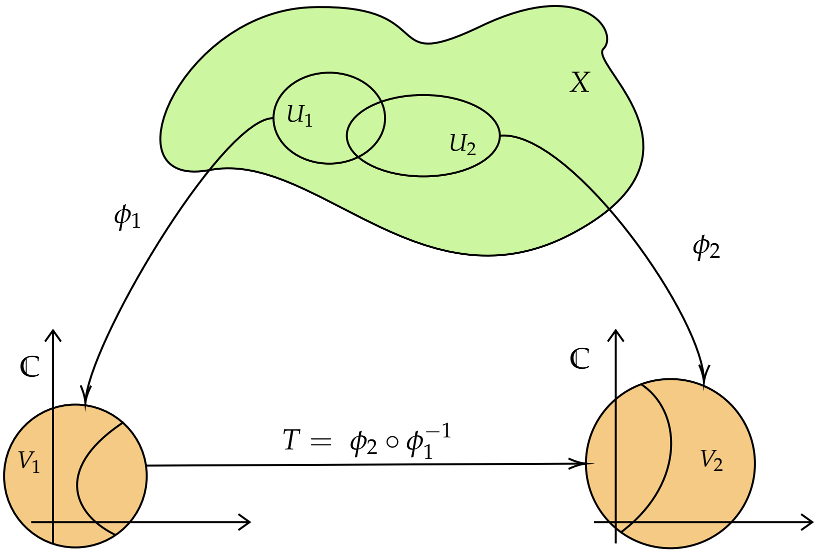

FIGURE 1.2: compatible charts

4. A **complex atlas** A on X is a collection $A = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise compatible complex charts whose domains cover X, i.e $X = \cup_\alpha U_\alpha$.

5. Two complex atlases A_1, A_2 are said to be **equivalent** if $A_1 \cup A_2$ is a complex atlas. It constitutes an equivalence relation [1].

6. A complex atlas is said to be **maximal** if it can't be included in any other one. It can be proved that any atlas is included in a single maximal atlas and that two atlases are equivalent iff they are included in the same maximal atlas.

7. A **complex structure** on X is a maximal atlas A on X, i.e, an equivalence class of complex atlases on X. It is denoted by $\sum$

8. X is **Hausdorff** if for every two distinct points $x, y \in X$, there are disjoint neighborhoods U and V of x and y, respectively.

[1] A binary relation is said to be an equivalence relation iff it is symmetric, reflexive and transitive.

9. X is **second countable** or equivalently fulfills the second axiom of countability if there exists a countable basis for its topology.

Definition 1.5. A **Riemann Surface** X is a topological space, Hausdorff, connected and second countable that has a complex structure.

Proposition 1.6. Every Riemann surface is path-connected.

Proof. By definition, a Riemann surface is a complex manifold. Hence it is connected and locally path-connected. □

Proposition 1.7. Every Riemann surface is orientable.

Proof. A possible way of showing this is to calculate the determinant of the jacobians of the transitions functions.
Suppose you have an n-dimensional complex manifolds M. That is, M is a (first countable, Hausdorff) space such that every point $p \in M$ has an open neighborhood V homeomorphic to an open subset $\Omega \subset \mathbb{C}^n$,

$$\phi : V \to \Omega \text{ homeomorphism}$$

and when two such neighborhoods intersect, the transition functions are holomorphic,

$$\psi \circ \phi^{-1} : \phi(V \cap V') \to \psi(V \cap V') \text{ biholomorphism}$$

By identifying $\mathbb{C}^n$ with $\mathbb{R}^n$ like so: $(z^1, ..., z^n) \leftrightarrow (x^1, y^1, ..., x^n, y^n)$ where $z^k = x^k + iy^k$ is the real/imaginary part decomposition, we get a real manifold structure on M. We can calculate the jacobian matrices of the transition functions for both structures.
Let's set up some notation. The first coordinate chart will be:

$$\begin{aligned} \phi : V &\to \Omega \\ p &\mapsto (z^1(p), ..., z^n(p)) \end{aligned}$$

The second coordinate chart will be:

$$\begin{aligned} \psi : V' &\to \Omega' \\ p &\mapsto (Z^1(p), ..., Z^n(p)) \end{aligned}$$

In the complex case, we have

$$Jac_{\phi(p)}(\psi \circ \phi^{-1}) = (\frac{\partial[\psi \circ \phi^{-1}]^k}{\partial z^l})_{1 \leqslant k,l \leqslant n} = (\frac{\partial Z^k(z^1, ..., z^n)}{\partial z^l})_{1 \leqslant k,l \leqslant n} = (c^k_l)_{1 \leqslant k,l \leqslant n} \in GL(n, \mathbb{C}),$$

and in the real case you'll find:

$$Jac_{\phi(p)}(\psi^{\mathbb{R}} \circ (\phi^{\mathbb{R}})^{-1}) = \begin{pmatrix} \frac{\partial X^k(x^1,y^1,...,x^n,y^n)}{\partial x^l} & \frac{\partial X^k(x^1,y^1,...,x^n,y^n)}{\partial y^l} \\ \frac{\partial Y^k(x^1,y^1,...,x^n,y^n)}{\partial x^l} & \frac{\partial Y^k(x^1,y^1,...,x^n,y^n)}{\partial y^l} \end{pmatrix}_{1\leqslant k,l\leqslant n} =$$
$$\begin{pmatrix} Re(c_l^k) & -Im(c_l^k) \\ Im(c_l^k) & Re(c_l^k) \end{pmatrix}_{1\leqslant k,l\leqslant n} \in GL(2n, \mathbb{R}),$$

using the Cauchy-Riemann equations. We will calculate the determinant of these matrices, show that it is always > 0, which is equivalent to $M^{\mathbb{R}}$ being orientable. We move on to calculating the determinants of these. Consider the $\mathbb{R}-$algebra

homomorphism

$$\rho : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$$
$$(c_l^k)_{1\leqslant k,l\leqslant n} \mapsto \begin{pmatrix} Re(c_l^k) & -Im(c_l^k) \\ Im(c_l^k) & Re(c_l^k) \end{pmatrix}_{1\leqslant k,l\leqslant n}$$

Since it is $\mathbb{R}$-linear and the spaces involved are finite dimensional, it is continuous. Also, being an algebra homomorphism, we have $det\rho(P^{-1}AP) = det(\rho(P)^{-1}\rho(A)\rho(P)) = det(\rho(A))$. Finally, the diagonalizable matrices are dense in $M_n(\mathbb{C})$, so we can restrict our calculations to diagonal matrices in $M_n(\mathbb{C})$. For these, the calculations are easy:

$$det(\rho(Diag(c_1, ..., c_n))) =$$
$$det(Diag\left(\begin{pmatrix} Re(c_1) & -Im(c_1) \\ Im(c_1) & Re(c_1) \end{pmatrix}, ..., \begin{pmatrix} Re(c_n) & -Im(c_n) \\ Im(c_n) & Re(c_n) \end{pmatrix}\right)) =$$
$$\prod_{i=1}^n det \begin{pmatrix} Re(c_i) & -Im(c_i) \\ Im(c_i) & Re(c_i) \end{pmatrix} = \prod_{i=1}^n |c_i|^2 = |det(Diag(c_1, ..., c_n)|^2,$$

so we conclude that

$$\forall A \in M_n(\mathbb{C}, det\rho(A) = |detA|^2$$

Finally, we can conclude that the transition function for the charts $\phi^{\mathbb{R}}$ for $M^{\mathbb{R}}$ have postivie determinants, thus the real underlying manifold $M^{\mathbb{R}}$ is orientable. □

1.3 Examples of Riemann Surfaces

Let's now introduce two important examples of Riemann Surfaces, with whom we will work along this TFG.

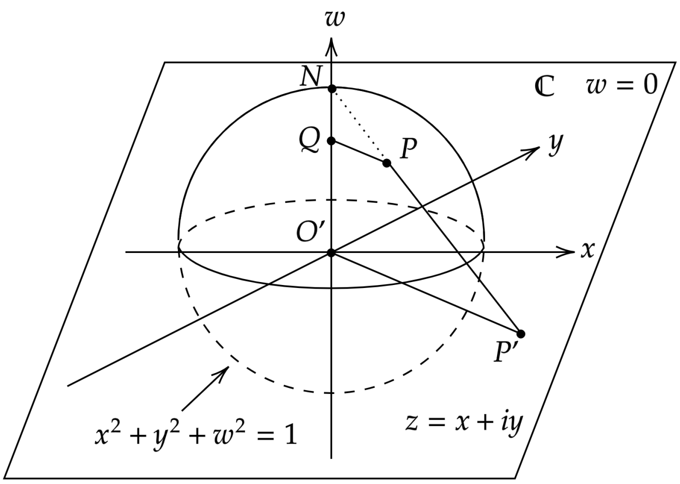

FIGURE 1.3: Riemann Sphere and Stereographic projection

Example 1.8. The unit $2-$sphere $S^2 \subset \mathbb{R}^3$ is a compact Riemann Surface which is known as **Riemann Sphere** (1.3) and is defined by:

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 | x^2 + y^2 + w^2 = 1\} \tag{1.3}$$

where we can identify the plane $w = 0$ as $\mathbb{C}$, with $(x, y, 0)$ being identified with $z = x + iy$. We define the complex charts:

$$\begin{aligned} \phi_1 &: S^2 \backslash \{(0,0,1)\} \to \mathbb{C} & \phi_2 &: S^2 \backslash \{(0,0,-1)\} \to \mathbb{C} \\ \phi_1(x,y,z) &= \frac{x}{1-w} + i\frac{y}{1-w} & \phi_2(x,y,z) &= \frac{x}{1+w} + i\frac{y}{1+w} \end{aligned}$$

with inverses:

$$\phi_1^{-1}(z) = \left(\frac{2Re(z)}{|z|^2+1}, \frac{2Im(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \qquad \phi_2^{-1}(z) = \left(\frac{2Re(z)}{|z|^2+1}, \frac{-2Im(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right)$$

We see that ϕ_1 and ϕ_2 are homeomorphism in S^2 with the induced topology in $\mathbb{R}^3$. Then, the transition function given by:

$$\begin{gathered} \phi_2 \circ \phi_1^{-1} : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\} \\ \phi_2 \circ \phi_1^{-1}(z) = \frac{1}{z} \end{gathered}$$

is holomorphic. Thus the two charts ϕ_1 and ϕ_2 are compatible. Therefore, $A = \{\phi_1, \phi_2\}$ is a complex atlas whose maximal atlas defines a complex structure over S^2, which is moreover Hausdorff, connected, compact and second numerable due to the fact that $\mathbb{R}^3$ verifies the second axiome of numerability and this topological property is hereditary. Hence, it is a Riemann surface. In this context, Riemann Sphere may be denoted by $\mathbb{C}_\infty$ or $\mathbb{C} \cup \{\infty\}$ because S^2 without a point is homeomorphic to $\mathbb{C}$ and the remaining point corresponds to the point at infinity.

Example 1.9. Next we will present the **complex torus** and see why we can consider it a Riemann Surface.
Let's start by fixing $w_1, w_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$. Define L to be the lattice, which is a subgroup of the additive group of $\mathbb{C}$.

$$L = \mathbb{Z}m_1 + \mathbb{Z}m_2 = \{m_1 w_1 + m_2 w_2 | m_1, m_2 \in \mathbb{Z}\}$$

Let $X = \mathbb{C}/L$ be the quotient group, with projection map $\pi : \mathbb{C} \to X$. Using this map, we can impose the quotient topology on X, i.e given a set $U \subset X$ we say it is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. In fact, we know that a map $f : X \to Y$ between topological spaces is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y so we can say that π is continuous and preserves the connection of spaces. Since $\mathbb{C}$ is connected, so is X. Actually π is an open mapping, that is, π takes any open set of $\mathbb{C}$ onto a open set in X.
For any $z \in \mathbb{C}$, define the closed parallelogram

$$P_z = \{z + \lambda_1 w_1 + \lambda_2 w_2 | \lambda_i \in [0, 1]\}$$

Note that any point of $\mathbb{C}$ is congruent modulo L to a point of P_z. Therefore π maps P_z onto X and since P_z is compact, so is X.
The lattice L is a discrete subset of $\mathbb{C}$, so there is an $\epsilon > 0$ such that $|w| > 2\epsilon$ for every nonzero $w \in L$. Fix such an ϵ, and fix a point $z_0 \in \mathbb{C}$. Consider the open disc $D = D(z_0, \epsilon)$ centered at z_0 and of radius ϵ. This choice of ϵ insures that no two points of $D(z_0, \epsilon)$ can differ by an element of the lattice L. For any z_0 and ϵ, $\pi|_D : D \to \pi(D)$ maps D homeomorphically onto the open set $\pi(D)$. This restriction is onto, continuous and open (since π is). It is 1-1 followed by the choice of ϵ.
We need to define now a **complex atlas** on X. Let's fix ϵ as we have done before. For each $z_0 \in \mathbb{C}$, let $D_{z_0} = D(z_0, \epsilon)$, and define $\phi_{z_0} : \pi(D_{z_0}) \to D_{z_0}$ to be the inverse of the map $\pi|_{D_{z_0}}$. By what we have seen above, ϕ's are complex charts on X.
Finally, what we have left to see is that these charts are **pairwise compatible**. Choosing two point $z_1, z_2 \in \mathbb{C}$ and considering two charts $\phi_1 = \phi_{z_1}$ and $\phi_2 = \phi_{z_2}$ in the same way as before, let U be the intersection of $\pi(D_{z_1})$ and $\pi(D_{z_2})$, i.e $U = \pi(D_{z_1}) \cap \pi(D_{z_2})$. If U is empty, there is nothing to prove. If U is not empty, let $T(z) = \phi_2(\phi_1^{-1}(z)) = \phi_2(z)$ for $z \in \phi_1(U)$.
We must check that T is holomorphic on $\phi_1(U)$. Note that $\pi(T(z)) = \pi(z)$ for all $z \in \phi_1(U)$, so $T(z) - z = w(z) \in L$ for all $z \in \phi_1(U)$. $w : \phi_1(U) \to L$ is continuous,

and L is discrete; hence, w is locally constant on $\phi_1(U)$ because it is constant on the connected components of U. Thus, locally, $T(z) = w + z$ for some fixed $w \in L$, and is therefore holomorphic as we wanted to see. As a consequence, ϕ_1 and ϕ_2 are compatible, and the collection of charts $\{\phi_z | z \in \mathbb{C}\}$ is a complex atlas on X. Hence X is a compact Riemann Surface which is called *complex torus.*

Now, let's introduce the notions we will use to characterise **Affine Plane Curves**.

Definition 1.10. Let $V \subset \mathbb{C}$ be a connected open subset of the complex plane, and let f be a holomorphic function defined on all of V. Consider the **graph** X **of** f, as a subset of $\mathbb{C}^2$:

$$X = \{(z, f(z)) | z \in V\}$$

We can generalize that to any finite collection of holomorphic functions on V.

We would like to consider a locus $X \subset \mathbb{C}^2$ which is locally a graph. One way to do it is to define X by requiring a complex polynomial of two variables $f(z, w)$ to go to zero. This would cut the dimension by one and make possible to work with a Riemann Surface. To do that, we first need to impose a condition on f based on the **Implicit Function Theorem**.

Theorem 1.11. (Implicit Function) Let $f(z, w) \in \mathbb{C}[z, w]$ be a polynomial, and let $X = \{(z, w) \in \mathbb{C}^2 | f(z, w) = 0\}$ be its zero locus. Let $p = (z_0, w_0)$ be a point of X, i.e, $f(p) = f(z_0, w_0) = 0$. Suppose that $\frac{\partial f}{\partial w}(p) \neq 0$.

Then there exists a function $g(z)$ defined and holomorphic in a neighborhood of z_0, such that, near p, X is equal to the graph $w = g(z)$.
Moreover $g' = -\frac{\partial f}{\partial z} / \frac{\partial f}{\partial w}$.

Proof. Starting from $\frac{\partial f}{\partial w}(p) \neq 0$, we can say without loss of generalization that $\frac{\partial f}{\partial w}(p) > 0$. Since f is an holomorphic function near p, we obtain that $\frac{\partial f}{\partial w}$ is continuous and so $\frac{\partial f}{\partial w} > 0$ at a neighborhood of (z_0, w_0). Therefore $f(z_0, w)$ is strictly increasing in the neighborhood of (z_0, w_0). Since $f(z_0, w_0) = 0$ there exist w_1 such that $f(z_0, w_1) > 0$ and w_2 such that $f(z_0, w_2) < 0$. Recovering the fact that f is holomorphic near p, it follows that f is continuous and then $\forall z$ near $z_0, f(z, w_1) > 0$ and $f(z, w_2) < 0$. For such an z near z_0, since $\frac{\partial f}{\partial w} > 0$, $f(z, w)$ is increasing (as a function of w). Therefore there exists an unique w such that $f(z, w) = 0$. We have just found a function of z which satisfies this property and which proves that $w = g(z)$ exists (and is unique).

To prove the second implication of the theorem we will start from $f(z, g(z)) = 0$ that we have already proven. Applying the chain rule to the above relation we obtain what we were looking for:

$$\frac{\partial f}{\partial z} \times 1 + \frac{\partial f}{\partial w} \times g'(z) = 0 \implies g'(z) = -\frac{\partial f}{\partial z} / \frac{\partial f}{\partial w}$$

where $\frac{\partial f}{\partial w} \neq 0$ by assumption. $\square$

Definition 1.12. An **affine plane curve** is the locus of zeroes in $\mathbb{C}^2$ of a polynomial $f(z, w)$.
A **polynomial** $f(z, w)$ is **nonsingular** at a root **p** if either partial derivative $\frac{\partial f}{\partial z}$ or $\frac{\partial f}{\partial w}$ is not zero at p.
The **affine plane curve** X of roots of f is **nonsingular at p** if f is nonsingular at p.
The **curve** X is **nonsingular**, or **smooth**, if it is nonsingular at each of its points.

Let X be a smooth affine plane curve defined by a polynomial $f(z, w)$.
We will now see that X is a Riemann Surface.

Let $p = (z_0, w_0) \in X$. Taking $\frac{\partial f}{\partial w}(p) \neq 0$(the case of $\frac{\partial f}{\partial z}(p) \neq 0$ is analogous), using TFI we would like to find a holomorphic function $g_p(z)$ such that in a neighborhood U of p, X is the graph $w = g_p(z)$. We observe that the projection $\pi_z : U \to \mathbb{C}$ mapping (z, w) to z is a homeomorphism from U to its image $V \subset \mathbb{C}$, which is open. By this, we obtain a complex chart on X. From the analogous case, we use the projection π_w, sending (z, w) to w near p.
Since X is smooth we know that either $\frac{\partial f}{\partial w} \neq 0$ or $\frac{\partial f}{\partial z} \neq 0$ at each point of X. Hence, the domains of these complex charts cover X.

Another point to check is that any two of these charts are compatible.
The case in which both charts are obtained by the same projection, either π_z or π_w is analogous. Then, if there is nonempty intersections with their domains, the composition of the inverse of one with the other is the identity, which is holomorphic.
If each chart is obtained from a different projection, choose a point $p = (z_0, w_0) \in U$, where U is their common domain. Assume that near p, X is locally a graph for some holomorphic function g. Then on $\pi_z(U)$ near z_0, the inverse of π_z sends z to $(z, g(z))$. Thus $\pi_w \circ \pi_z^{-1}$ sends z to $g(z)$, which is holomorphic.
Having seen that any two of the charts are compatible, we obtain a complex atlas on X.
Moreover, the space X has the properties of being second countable and Hausdorff because $X \subset \mathbb{C}^2$.

Finally, the last thing to check to prove that X is a Riemann Surface is its connectivity. An important theorem to see that is the following.

Theorem 1.13. If $f(z, w)$ is an irreducible polynomial, then its locus of roots X is connected. Hence if f is nonsingular and irreducible, X is a Riemann Surface.

Proof. The proof of this theorem is based on covering spaces but is very extensive and not trivial. Thus it can be found in Chapter II, Section 2 of [3]. □

Another extensive source of examples for compact Riemann Surfaces are the so-called **smooth projective plane curves**.
Let $F(x, y, z)$ be a homogeneous (all the terms of the equation of the same degree in the variables), nonsingular (no common solutions in $\mathbb{P}^2$ to $F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$). Therefore:

$$X = \{[x, y, z] \in \mathbb{P}^2 | F(x, y, z) = 0\} \tag{1.4}$$

is a compact Riemann Surface. Additionally, taking the three open sets that cover $\mathbb{P}^2$ as U_0, U_1, U_2:

$$U_0 = \{[x, y, z] | x \neq 0\}; U_1 = \{[x, y, z] | y \neq 0\}; U_2 = \{[x, y, z] | z \neq 0\}$$

the intersections X_i of X with U_i for $i = \{0, 1, 2\}$ are affine plane curves looked at $\mathbb{C}^2$.

Let's finally present the example of **Hyperelliptic Riemann Surfaces**.
Let start with $h(x)$ being a polynomial with $\deg(h) = 2g + 1 + \epsilon$, where ϵ can be 0 or 1. Assume $h(x)$ has distinct roots. Then, let X be the smooth affine plane curve defined by the equation $y^2 = h(x)$ where $U = \{(x, y) \in X | x \neq 0\} \subset X$ is an open subset. However, when compacting, one has to be careful with the singularities appearing on the line at infinity. This is solved through a normalization process as described in Chapter II, Section 3 of [3].
Similarly, let Y be the smooth affine plane curve defined by the equation $\omega^2 = k(z) = z^{2g+2}h(1/z)$ where $V = \{(z, \omega) \in Y | z \neq 0\} \subset Y$ is an open subset. Observe that the polynomial $k(z)$ has distinct roots since h does.
According to this, we can define the isomorphism $\phi : U \to V$ by:

$$\phi(x, y) = (z, \omega) = \left(\frac{1}{x}, \sqrt{k(z)}\right) = \left(\frac{1}{x}, z^{g+1} \cdot y\right) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right)$$

The compact Riemann Surface Z we obtain via the glueing[2] of X and Y together along U and V via ϕ is called **hyperelliptic Riemann surface**.
Z has genus g and the holomorphic map $\pi : Z \to \mathbb{C}_\infty$ has degree 2.
Given the automorphism $\sigma : (x, y) \to (x, -y)$ named **hyperelliptic involution**, we say a Riemann surface X is hyperelliptic if it has an involution σ such that $\mathbb{Z}/\sigma = \mathbb{P}^1$. Equivalently, there exists a nonconstant morphism $\mathbb{Z} \to \mathbb{P}^1 \cong S^2$ of degree 2.

[2]see [1] p 59

Chapter 2

Functions and maps

2.1 Functions on Riemann Surfaces

To study functions on Riemann Surfaces we will need to use complex charts to transport them to the complex plane and check the properties of the functions there. Therefore, it is important to make sure that the property we are studying does not depend on the choice of the chart.

Let X be a Riemann surface, let p be a point of X, and let f be a complex-valued function defined in a neighborhood W of p.

Definition 2.1. f is said to be **holomorphic at p** if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. We say f is **holomorphic in W** if it is holomorphic at every point of W.

Lemma 2.2. The previous definition leads us to:

1. f is holomorphic at p if and only if for every chart $\phi : U \to V$ with $p \in U$, the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$;

2. f is holomorphic in W if and only if there exists a set of charts $\{\phi_i : U_i \to V_i\}$ with $W \subset \cup_i U_i$, such that $f \circ \phi_i^{-1}$ is holomorphic on $\phi_i(W \cap U_i)$ for each i;

3. if f is holomorphic at p, f is holomorphic in a neighborhood of p.

Definition 2.3. If $W \subset X$ is an open subset of a Riemann surface X, we will denote the set of holomorphic functions on W by $\mathcal{O}_X(W)$:

$$\mathcal{O}_X(W) = \mathcal{O}(W) = \{f : W \to \mathbb{C} | f \text{ is holomorphic}\}.$$

$\mathcal{O}(W)$ is a $\mathbb{C}$-algebra.

Definition 2.4. Taking f holomorphic in a punctured neighborhood[1] of $p \in X$.

1. f has a **removable singularity** at p if and only if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1}$ has a removable singularity at $\phi(p)$.

   Investigating the behaviour of $f(x)$ for x near p, if $|f(x)|$ is bounded in a neighborhood of p, then f has a removable singularity at p. Furthermore, the limit $\lim_{x\to p} f(x)$ exists, and if we define $f(p)$ to be this limit, f is holomorphic at p.

2. f has a **pole** at p if and only if there exists a chart $\phi : U \to V$ with $p \in U$, such that $f \circ \phi^{-1}$ has a pole at $\phi(p)$.

   As before, if $|f(x)|$ approaches ∞ as x approaches p, then f has a pole at p.

3. We say f has an **essential singularity at p** if and only if there exists a chart $\phi : U \to V$ with $p \in U$, such that $f \circ \phi^{-1}$ has an essential singularity at $\phi(p)$.

   Regarding f, we say it has an essential singularity at p if $|f(x)|$ has no limit as x approaches p.

Definition 2.5. A function f on X is said to be **meromorphic at a point** $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole, at p according to the definitions we have presented before. If f is meromorphic at every point of an open set W we say that f is meromorphic at W.
Similarly as we have done previously, being $W \subset X$ an open subset of a Riemann Surface X, we denote the **set of meromorphic functions on W** by $\mathcal{M}_X(W)$:

$$\mathcal{M}_X(W) = \mathcal{M}(W) = \{f : W \to \mathbb{C} | f \text{ is meromorphic}\}.$$

Example 2.6. Any rational function of the form $f(z) = p(z)/q(z)$ is meromorphic on all the Riemann Sphere $\mathbb{C}_\infty$.
Indeed, given a Riemann Surface X, any function h which is meromorphic at a point $p \in X$ is locally the ratio of two holomorphic functions.

Definition 2.7. Let's take f a defined and holomorphic function in a punctured neighborhood of $p \in X$, a chart $\phi : U \to V$ on X with $p \in U$ and the local coordinate $z = \phi(x)$ for x near p.
If $f \circ \phi^{-1}$ is holomorphic in a neighborhood of $z_0 = \phi(p)$ we can define the **Laurent series for** f **about** p **with respect to** ϕ (or respect to z) as the following expansion around z_0:

$$f(\phi^{-1}(z)) = \sum_n c_n (z - z_0)^n \tag{2.1}$$

[1] a set of the form $U - \{p\}$, where U is a neighborhood of p

The respective coefficients $\{c_n\}$ of this series are known as the **Laurent coefficients**.

Moreover, the Laurent series is a useful tool to study the singularities of f.

Lemma 2.8.
1. f has a removable singularity at p if and only if one of it Laurent series has no negative terms.
2. f has a pole at p if and only if one of it Laurent series finitely many, but not zero as the first case, negative terms.
3. f has an essential singularity at p if and only if any one of its Laurent series has infinitely many negative terms.

Definition 2.9. Let f be a meromorphic function at a point $p \in X$. Given the Laurent series we have presented in 2.7, we can define the **order of** f **at** p such as:

$$\operatorname{ord}_p(f) = \min\{n | c_n \neq 0\}.$$

The $\operatorname{ord}_p(f)$ is well defined, i.e independent of the choice of local coordinate to define the Laurent series.

Lemma 2.10. Suppose f is meromorphic at p.
Then f is holomorphic at p if and only if $\operatorname{ord}_p(f) \geq 0$.

1. $f(p) = 0$ if and only $\operatorname{ord}_p(f) = n \geq 1$ and we say f has a zero of order n at p.
2. f has neither a zero nor a pole if and only if $\operatorname{ord}_p(f) = 0$.
3. On the other hand, $\operatorname{ord}_p(f) = -n < 0$ if and only if f has a pole of order n at p.

We will now present several theorems involving meromorphic and holomorphic functions which are analogous to those we have working with functions defined on open sets $U \subset \mathbb{C}$.

Theorem 2.11. (Discreteness of zeroes and poles) Let f be a meromorphic function defined on a connected open set W of X and which is not identically zero. Then the zeroes and the poles of f form a discrete subset of W.
If the Riemann surface is compact, then the number of zeroes and poles is finite.

Theorem 2.12. Suppose that f is holomorphic on all of a compact Riemann surface. Then f is a constant function.

Proof. Since f is holomorphic in X, we know that $|f|$ is a continuous function. Also, since X is compact, $|f|$ achieves its maximum at some $p \in X$. Since X is connected, by the Maximum Modulus Theorem we have that f must be constant in X. $\square$

The above theorem is analogous to Liouville's Theorem on complex domains.

2.2 Examples of Meromorphic Functions

Theorem 2.13. Any meromorphic function on $\mathbb{C}_\infty$ is a rational function. We have already seen in 2.6 that the converse is also true. Therefore, we have that $\{$meromorphic functions$\} = \{$rational functions$\}$ for functions on $\mathbb{C}_\infty$.

Corollary 2.14. If f is a meromorphic function on $\mathbb{C}_\infty$, as a rational function on $\mathbb{C}_\infty$ is a quotient of two polynomials with the same degree, then:

$$\sum_p \operatorname{ord}_p(f) = 0$$

We will see in 2.38 that this is true in general, for any meromorphic function on a compact Riemann Surface.

Lemma 2.15. Let f be any nonconstant meromorphic function on a complex torus $X = \mathbb{C}/L$. Then:

$$\sum_p \operatorname{ord}_p(f) = 0$$

In both cases, this statement is significative and useful because it means that the number of zeroes must be equal to the number of poles according to the order we have set.

On Smooth Plane Curves $X \subset \mathbb{C}^2$ that were defined by $f(x, y) = 0$ we present the following constraint for the construction of meromorphic functions.

Proposition 2.16. Let X be a smooth affine plane curve defined by an irreducible nonsingular polynomial $f(x, y) = 0$. Then any ratio of polynomials $r = g(x, y)/h(x, y)$ is a meromorphic function on X as long as f does not divide the denominator h.

2.3 Holomorphic Maps Between Riemann Surfaces

In the first section of this chapter we have already presented functions on Riemann Surfaces but functions between Riemann Surfaces remain to be seen.

Definition 2.17. Let X and Y be Riemann Surfaces. A mapping $F : X \to Y$ is **holomorphic at** $p \in X$ if and only if there exists charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$ such that $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$ (FIG. 2.1).
If F is defined on an open set $W \subset X$, then we say F is **holomorphic on** W if F is holomorphic at each point of W.
In particular, F is a **holomorphic map** if and only if F is holomorphic on all of X.

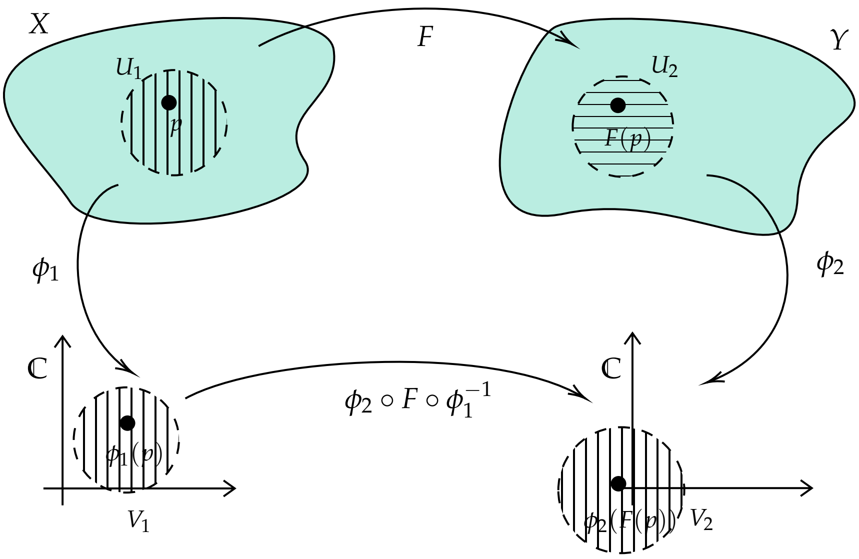

FIGURE 2.1: Holomorphic Maps between Riemann Surfaces

Example 2.18.

1. The identity mapping $id : X \to X$ is holomorphic for any Riemann Surface X
2. The holomorphic map $F : X \to Y$ with Y being the complex plane is simply a holomorphic function on X, as we have seen in the previous sections.

Remark 2.19. Similarly as we have seen in the previous subchapter, the holomorphicity of a map can be checked with **any pair of charts**.

Lemma 2.20. Let $F : X \to Y$ be a holomorphic map between Riemann Surfaces.

1. F is continuous and $\mathcal{C}^\infty$.
2. The composition of holomorphic maps is holomorphic.
3. The composition of a holomorphic map with a holomorphic function is holomorphic.

   For every open set $W \subset Y$, F induces a $\mathbb{C}$-algebra homomorphism:

   $$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W))$$

   where $F^*(g) = g \circ F$ taking g as a holomorphic function on W.
4. The composition of a holomorphic map with a meromorphic function is meromorphic. The same construction as before is analogous for meromorphic functions:

   $$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W))$$

   where now g is a meromorphic function on W.

Definition 2.21. An **isomorphism** between Riemann Surfaces is a holomorphic map $F : X \to Y$ which is bijective and has a holomorphic inverse.
As we already know from previous courses if $Y = X$ the isomorphism is called **automorphism**.

We will now present several results concerning holomorphic maps that would be useful later and are consequences of well-known theorems on functions on 1-complex variable.
Let X and Y be Riemann surfaces:

Proposition 2.22. (Open Mapping Theorem) Let $F : X \to Y$ be a nonconstant holomorphic map. Then F is an open mapping, which means that it maps open sets to open sets.

Proposition 2.23. (Identity Theorem) Let F, G two holomorphic maps between X and Y. If $F = G$ on $S \subset X$ with a limit point in X, then $F = G$.

Proposition 2.24. Let X be compact and $F : X \to Y$ a nonconstant holomorphic map. Then Y is compact and F is onto.

Proof. Since X is open and F is holomorphic, we have by 2.22 that $F(X)$ is open in Y.
Additionally, since X is compact, $F(X) \subset Y$ must be compact and closed because Y is Hausdorff by the definition of Riemann surface.
Finally, as $F(X)$ is at the same time open and closed it must be all of Y. This proves that F is onto and Y is compact. □

Proposition 2.25. (Discreteness of preimages) Let $F : X \to Y$ be a nonconstant holomorphic map. Then $\forall y \in Y, F^{-1}(y)$ is a discrete subset of X.
In the case of X, Y being compact, $F^{-1}(y)$ is a nonempty finite set for every $y \in Y$.

2.4 Global Properties of Holomorphic Maps

We will start this part introducing the fact that a holomorphic map between two Riemann surfaces has a standard normal form in some local coordinates. The following result will let us express any $F : X \to Y$ locally.
Let X and Y be Riemann surfaces.

Proposition 2.26. (Local Normal Form) Let $F : X \to Y$ be a holomorphic map defined at $p \in X$ and which is not constant. Then $\exists! m \in \mathbb{Z}, m \geq 1$ such that for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$, there exist a chart $\phi_1 : U_1 \to V_1$ on X centered at p fulfilling $\phi_2(F(\phi_1^{-1}(z))) = z^m$.

Proof. Let fix a chart ϕ_2 on Y centered at $F(p)$ and any chart $\psi : U \to V$ on X centered at p. Then the function $T(\omega) = \phi_2(F(\psi^{-1}(\omega)))$ has a Taylor series of the form:

$$T(\omega) = \sum_{i=m}^{\infty} c_i \omega^i \text{ with } c_m \neq 0 \text{ and } m \geq 1$$

Then, we can write $T(\omega)$ as the product $T(\omega) = \omega^m S(\omega)$ where $S(\omega)$ is a holomorphic function at $\omega = 0$, and $S(0) \neq 0$. Thus, there exists a function $R(\omega)$ such that $R(\omega)^m = S(\omega)$. Therefore, taking that into account, we can write $T(\omega) = \omega^m S(\omega) = \omega^m R(\omega)^m = (\omega R(\omega))^m = \eta(\omega)^m$. Let's now derive $\eta(\omega)$: $\eta'(\omega) = (\omega R(\omega))' = R(\omega) + \omega R'(\omega)$ and applying it to $\omega = 0$, we obtain $\eta'(0) = R(0) \neq 0$.
Then, applying the Implicit Function Theorem (1.11), as $\eta'(0) = R(0) \neq 0$, we have that near 0 $\eta(\omega)$ is invertible and holomorphic.

Finally, using the chart $\psi : U \to V$ on X, we can construct another chart on X $\phi_1 = \eta \circ \psi$ which is defined and centered at p. If we see η as defining a new coordinate $z = \eta(\omega) = \omega R(\omega)$, z and ω are related $z = \omega R(\omega)$. Thus,

$$\begin{aligned}\phi_2(F(\phi_1^{-1}(z))) &= \phi_2(F(\psi^{-1}(\eta^{-1}(z)))) \\ &= T(\eta^{-1}(z)) = T(\omega) = (\omega R(\omega))^m = z^m\end{aligned}$$

□

Definition 2.27. The **multiplicity** of F at p, denoted $\text{mult}_p(F)$, is the unique integer m such that there are local coordinates near p and $F(p)$ with F having the form $z \mapsto z^m$. Recall that $\text{mult}_p(F) \geq 1$.

Definition 2.28. Let $F : X \to Y$ be a nonconstant holomorphic map.
A point $p \in X$ is a **ramification point** for F if $\text{mult}_p(F) \geq 2$.
A point $y \in Y$ is a **branch point** for F if it is the image of a ramification point.

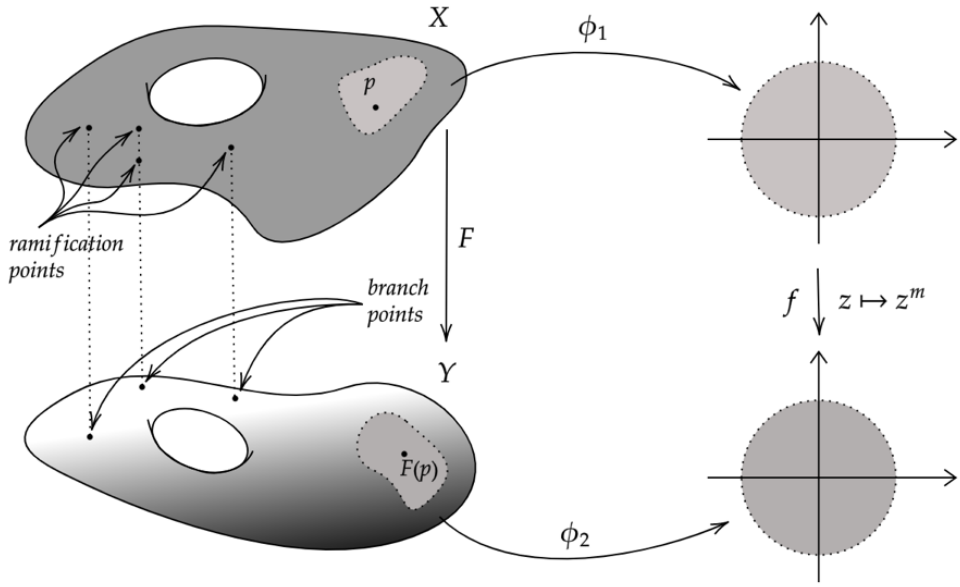

FIGURE 2.2: Ramification and branch points

Lemma 2.29. The points of the domain where F has multiplicity at least two form a discrete set.
Therefore, the ramification and branch points for a holomorphic map form discrete subsets of the domain and range respectively.

Lemma 2.30. Let X be a smooth affine plane curve defined by $f(x,y) = 0$. Define $\pi : X \to \mathbb{C}$ by $\pi(x,y) = x$. Then π is ramified at $p \in X$ if and only if $(\partial f/\partial y)(p) = 0$.

The following lemma will help us to relate the multiplicity defined for a holomorphic map between Riemann surfaces to the order which is defined for a meromorphic function.

Lemma 2.31. Let f be a meromorphic function on a Riemann surface X, with associated holomorphic map $F : X \to \mathbb{C}_\infty$.

1. If $p \in X$ is a zero of f, then $\mathrm{mult}_p(F) = \mathrm{ord}_p(f)$.
2. If p is a pole of f, then $\mathrm{mult}_p(F) = -\mathrm{ord}_p(f)$.
3. If p is neither a zero or a pole of f, then $\mathrm{mult}_p(F) = \mathrm{ord}_p(f - f(p))$.

We will now introduce the concept of degree of a holomorphic map between compact Riemann surfaces X, Y which is motivated by the following property.

Proposition 2.32. Let $F : X \to Y$ be a nonconstant holomorphic map. For each $y \in Y$, define $d_y(F)$ to be the sum of the multiplicities of F at the points of X mapping to y:

$$d_y(F) = \textstyle\sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F)$$

Thus $d_y(F)$ is **constant and independent of y.**

Proof. The idea is to see that $y \mapsto d_y(F)$ is a locally constant function from Y to $\mathbb{Z}$ and since Y is connected, a locally constant function must be constant. □

Definition 2.33. The **degree of** F, denoted $\deg(F)$, is the integer $d_y(F)$ for any $y \in Y$.

Corollary 2.34. A holomorphic map between Riemann Surfaces is an isomorphism if and only if it has degree one.

Proposition 2.35. If X is a compact Riemann Surface having a meromorphic function f with a single simple pole, then X is isomorphic to $\mathbb{C}_\infty$

Remark 2.36. Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann Surfaces. It is sometimes called a **branched covering** because it is a covering map away from the branch points, which is a finite set. Over these points the map behaves in a good way.

Proposition 2.37. We admit without a proof that every compact Riemann Surface admits nonconstant meromorphic functions on X and therefore nonconstant holomorphic maps from X to $\mathbb{C}_\infty$.

Now using the theory of the degree we would be able to prove the generalization of results such as the ones presented in corollary 2.14 and lemma 2.15

Proposition 2.38. Let f be a nonconstant meromorphic function on a compact X.

$$\textstyle\sum_p \operatorname{ord}_p(f) = 0.$$

Proof. From 2.37, let $F : X \to \mathbb{C}_\infty$ be the associated nonconstant holomorphic map to $\mathbb{C}_\infty$.
Let $\{x_i\}$ be the points of X mapping to 0 (i.e the zeroes of f) and $\{y_j\}$ those mapping to ∞ (i.e the poles of f).
Let $d = \deg(F)$ and by its definition we obtain that:

$$d = \sum_i \operatorname{mult}_{x_i}(F) \quad \text{and} \quad d = \sum_i \operatorname{mult}_{y_j}(F)$$

As we know, the only points of X where f has nonzero order are its zeroes and poles, in our case $\{x_i\}$ and $\{y_j\}$ resp. Additionally, according to the lemma 2.31, we have:

$$\sum_p \operatorname{ord}_p(f) = \sum_i \operatorname{ord}_{x_i}(f) + \sum_j \operatorname{ord}_{y_j}(f) = \sum_i \operatorname{mult}_{x_i}(F) - \sum_j \operatorname{mult}_{y_j}(F) = 0$$

□

2.5 Hurwitz's Formula

Due to the constancy of the degree for a holomorphic map between compact Riemann Surfaces (2.32), a formula using the Euler number theory relates the genus of the domain and range of the map with its degree and ramification. This formula was named after Hurwitz and will be explained in this section. Let's first review the following notions.
Let X be a compact $2-$manifold.

Definition 2.39.

1. A **triangulation** of X is a decomposition of X into closed subsets and homeomorphic to a triangle, such that any two of them are either disjoint, meet only at a single vertex, or at a single edge.

2. Given a triangulation of X with v vertices, e edges and t triangles, the **Euler number** of X with respect to the chosen triangulation is the integer $\chi = v - e + t$.

3. By the Classification Theorem of Surfaces, any connected and compact surface is homeomorphic to the connected sum of g tori (if S is orientable, which is the case of X) or of g projective planes (if S is not orientable). This constant is defined as the **genus** of S.

Remark 2.40. We can also say that in particular, a surface of genus g is topologically a sphere with g *handles*. For the Riemann Sphere and the Complex Torus we have respectively $g = 0$ and $g = 1$ (FIG. 2.3).

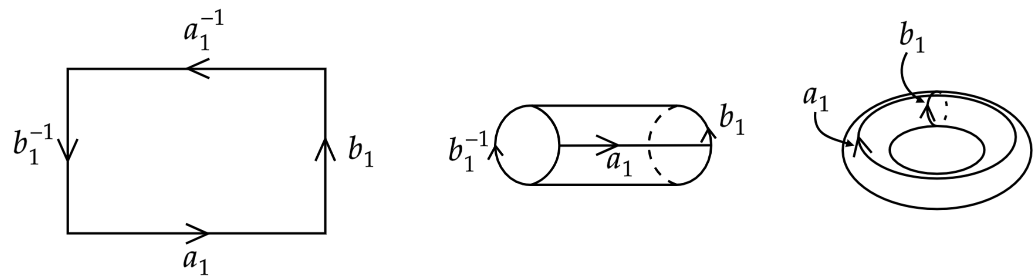

FIGURE 2.3: Surface of genus 1

Proposition 2.41. χ is independent of the choice of triangulation. For a compact orientable 2-manifold without boundary of topological genus g, $\chi = 2 - 2g$.

Proof. It belongs to a previous course in topology. ☐

Theorem 2.42. *(Hurwitz's formula)* Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then:

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\text{mult}_p(F) - 1] \tag{2.2}$$

Remark 2.43. We can sometimes see this formula divided by 2 or in terms of the Euler number as:

$$\chi(X) = \deg(F)\chi(Y) - \sum_{p \in X} [\text{mult}_p(F) - 1]$$

Remark 2.44. The number $\text{mult}_p(F) - 1$ is sometimes called the **branch number of F at p** and it is denoted by $b_F(p)$.
Moreover, $B = \sum_{p \in X} [\text{mult}_p(F) - 1] = \sum_{p \in X} b_F(p)$ is called the **total branching number**.
Rewriting 2.2, and dividing by 2 we obtain:

$$g(X) = \deg(F)(g(Y) - 1) + \frac{B}{2}$$

Proof. 2.42
What we will want to prove is that the right part of 2.2 corresponds to the Euler number of X with a negative sign. To do so, we will like to find the number of edges, triangles and vertices of X.

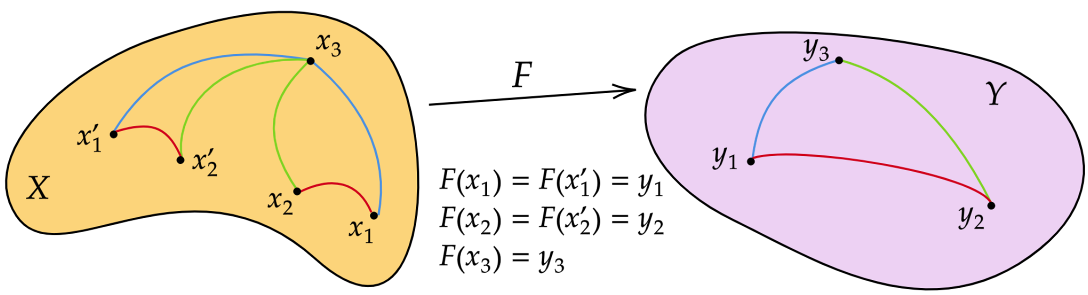

FIGURE 2.4: Scenario of the proof of Hurwitz's formula

Let's begin by noting that since X is compact, the set of ramification points is finite and so is the sum which is restricted to the ramification points of F.

We take a particular triangulation of Y with e edges, v vertices and t triangles in which each branch point of F corresponds to a vertex. This choice of the triangulation can be done as we have proved that the Euler number is independent of the triangulation.
We lift this triangulation of Y to X via the map F and we obtain a triangulation with v', e' and t' on X. Due to the way we have chosen the triangulation of Y, we observe that every vertex on X corresponds to a ramification point of F. This particular location of the ramification points (exclusively over the vertices of the triangles of X) provides that each triangle of Y lifts to $\deg(F)$ triangles in X.
Therefore, having t in Y, we obtain that:

$$t' = \deg(F)t$$

Similarly,

$$e' = \deg(F)e$$

It does not happen the same with v'. Taking a vertex $q \in Y$, its number of preimages in X is given by $|F^{-1}(q)|$ and can be written as:

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = \deg(F) + \sum_{p \in F^{-1}(q)} [1 - \text{mult}_p(F)].$$

This is fullfied for a single vertex of Y. Therefore, taking all the vertices of Y and making its preimages we obtain:

$$v' = \sum_{\text{vertex q of Y}} F^{-1}(q) = \sum_{\text{vertex q of Y}} (\deg(F) + \sum_{p \in F^{-1}(q)} [1 - \text{mult}_p(F)]) =$$
$$\deg(F)v - \sum_{\text{vertex q of Y}} (\sum_{p \in F^{-1}(q)} [\text{mult}_p(F) - 1]) = \deg(F)v - \sum_{\text{vertex p of X}} [\text{mult}_p(F) - 1]$$

Now that we have computed t', e', v', we can express $-\chi(X)$ as:

$$\begin{aligned}
2g(X) - 2 = -\chi(X) &= -v' + e' - t' \\
&= -\deg(F)v + \sum_{\text{vertex p of X}} [\text{mult}_p(F) - 1] + \deg(F)e - \deg(F)t \\
&= -\deg(F)\chi(Y) + \sum_{\text{vertex p of X}} [\text{mult}_p(F) - 1] \\
&= \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\text{mult}_p(F) - 1]
\end{aligned}$$

□

Example 2.45. Consider the Fermat curve given by:

$$F_3 = \{[x, y, z] \in \mathbb{CP}^2 : x^3 + y^3 + z^3 = 0\} \tag{2.3}$$

F_3 is a compact Riemann surface and we have a natural mapping:

$$F : F_3 \to \mathbb{CP}^1$$
$$[x, y, z] \mapsto [x, y]$$

Fix $[x, y] \in \mathbb{CP}^{\mathbb{1}}$. Then either:

1. $[x, y] \in \{[1, -1], [1, -\omega], [1, -\omega^2]\}$ where $w = \exp(2\pi i/3)$, in which case

$$F^{-1}(\{[x, y]\}) = \{[x, y, 0]\}$$

   or

2. $F^{-1}(\{[x, y]\}) = \{[x, y, -\alpha], [x, y, -\omega\alpha], [x, y, -\omega^2\alpha]\}$
   where $\alpha^3 = x^3 + y^3 \neq 0$

The degree of F is 3 and there exists exactly 3 ramification points which are $[1, -1, 0], [1, -\omega, 0], [1, -\omega^2, 0]$, each of them having ramification index 3.
Taking into account that $\mathbb{CP}^1 \approx \mathbb{S}^2$ has genus 0, 2.2 gives us:

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1]$$
$$2g(F_3) - 2 = 3(20 - 2) + [(3 - 1) + (3 - 1) + (3 - 1)] = 0$$

Therefore, we obtain $g(F_3) = 1$.
More generally, the implication of 2.2 for the genus of a Fermat curve given by:

$$F_d = \{[x, y, z] \in \mathbb{CP}^2 : x^d + y^d + z^d = 0\}$$

is that

$$g = \tfrac{(d-1)(d-2)}{2}$$

In fact, this formula generally holds for a smooth projective curve of degree d.

Proposition 2.46. A Riemann Surface X can never map (nontrivially) to a Riemann Surface Y of a higher genus. That means we always have $g(X) \geq g(Y)$

Proof. • If $g(Y) = 0$, then it is clear that $g(X) \geq g(Y)$ because we always have that $g \geq 0$.
• In the case of $g(Y) \geq 1$, using 2.2 and solving for $g(X)$ we obtain that:

$$g(X) \geq g(Y) + (\deg(F) - 1)(g(Y) - 1) + \frac{\sum_{p \in X} [\mathrm{mult}_p(F) - 1]}{2}$$

We are done because $\deg(F) \geq 1$ and $g(Y) - 1 \geq 0$. □

Chapter 3

Differential forms

3.1 Basic definitions

The aim of this chapter is to introduce an object with which we can integrate on Riemann Surfaces. These objects are called *forms*.
1-forms allows us to consider line integrals when we want to integrate around a path.
Similarly, if we want to compute a surface integral over a suitable 2-dimensional piece of a Riemann surface we need to work with 2-forms. However, we will not comment on them in this work.

Remark 3.1. In this chapter, we may use *form, differential form* or *differential* indistinctly to refer to the same object.

Definition 3.2. A ***0-form*** on X is a function on X.

Definition 3.3. A ***1-form*** ω on X is an (ordered) assignment of two continuous functions f and g to each local coordinate $z(= x + iy)$ on X such that:

$$f(x, y)dx + g(x, y)dy \tag{3.1}$$

is invariant under coordinate changes.

Remark 3.4. The condition of invariance under coordinate changes means that if $\tilde{z}$ is another local coordinate on X such as its domain intersects non-trivially the domain of z and ω assigns the functions $\tilde{f}, \tilde{g}$ to $\tilde{z}$, then:

$$\begin{bmatrix} \tilde{f}(\tilde{z}) \\ \tilde{g}(\tilde{z}) \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial y}{\partial \tilde{x}} \\ \frac{\partial x}{\partial \tilde{y}} & \frac{\partial y}{\partial \tilde{y}} \end{bmatrix} \begin{bmatrix} \tilde{f}(z(\tilde{z})) \\ \tilde{g}(z(\tilde{z})) \end{bmatrix} \tag{3.2}$$

on the intersection of the domains of z and $\tilde{z}$. Observe that the 2×2 matrix is the Jacobian matrix of the coordinate change $\tilde{z} \mapsto z$

3.2 Holomorphic 1-forms

Definition 3.5. A **holomorphic 1-form** in the coordinate z on an open set $V \subset \mathbb{C}$ is a 1-form

$$w = f(z)dz$$

where f is a holomorphic function on V.

We now want to work with this object on Riemann surfaces. To do so we need complex charts that have compatibility conditions whenever two of them have overlapping domains.

Definition 3.6. Let V_1 and V_2 be open sets, $\omega_1 = f(z)dz$ a holomorphic 1-form in the coordinate z defined on V_1 and $\omega_2 = g(w)dw$ a holomorphic 1-form in the coordinate w on V_2.
We say that ω_1 transforms to ω_2 under a holomorphic mapping T from V_2 to V_1 defined by $z = T(w)$ (and $dz = T'(w)dw$) if $g(w) = f(T(w))T'(w)$.

Given that condition we are able to define holomorphic 1-form on X

Definition 3.7. A **holomorphic 1-form on** X is a collection of holomorphic 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the coordinate of V.
Given two charts $\phi_i : U_i \to V_i$ (for $i = 1, 2$) if they have overlapping domains, then ω_{ϕ_1} transforms to ω_{ϕ_2} under the transition function $T = \phi_1 \circ \phi_2^{-1}$

In order to simplify the work, to define a holomorphic 1-form on X it is enough to give holomorphic 1-form on the charts of some atlas as we will see at 3.8

Lemma 3.8. Let $\mathcal{A}$ be a complex atlas on X. Suppose that holomorphic 1-forms are given for each chart of $\mathcal{A}$, which transform to each other on their common domains. Then there exists a unique holomorphic 1-form on X extending these holomorphic 1-forms on each of the charts of $\mathcal{A}$.

3.3 Meromorphic 1-forms

Definition 3.9. A **meromorphic 1-form** in the coordinate z on an open set $V \subset \mathbb{C}$ is a 1-form

$$w = f(z)dz$$

where f is a meromorphic function on V.

We require the same compatibility conditions that we have asked for at the previous section.

Definition 3.10. Let V_1 and V_2 be open sets, $\omega_1 = f(z)dz$ a meromorphic 1-form in the coordinate z defined on V_1 and $\omega_2 = g(w)dw$ a meromorphic 1-form in the coordinate w on V_2.
We say that ω_1 transforms to ω_2 under a meromorphic mapping T from V_2 to V_1 defined by $z = T(w)$ (and $dz = T'(w)dw$) if $g(w) = f(T(w))T'(w)$.

As before, to transport these notions to a Riemann surface we would have the following:

Definition 3.11. A **meromorphic 1-form on** X is a collection of meromorphic 1-forms $\{\omega_\phi\}$, one for each chart $\phi : U \to V$ in the coordinate of V.
Given two charts $\phi_i : U_i \to V_i$ (for $i = 1, 2$) if they have overlapping domains, then ω_{ϕ_1} transforms to ω_{ϕ_2} under the transition function $T = \phi_1 \circ \phi_2^{-1}$

Finally, similarly as the case of holomorphic 1-forms, given an atlas $\mathcal{A}$, it is only necessary to use the charts in $\mathcal{A}$ to define a meromorphic 1-form in X.

Lemma 3.12. Let $\mathcal{A}$ be a complex atlas on X. Suppose that meromorphic 1-forms are given for each chart of $\mathcal{A}$, which transform to each other on their common domains. Then there exists a unique meromorphic 1-form on X extending these meromorphic 1-forms on each of the charts of $\mathcal{A}$.

Let ω be a meromorphic 1-form efined in a neighborhood of a point p. Choosing a local coordinate centered at p,we may write $\omega = f(z)dz$ where f is a meromorphic function at $z = 0$.

Definition 3.13. Similarly as we have done in 2.9 we can define the **order of ω at** p, denoted by $\text{ord}_p(\omega)$, as the order of f at 0 in the way we have defined it at 2.9.

Remark 3.14. We can define a meromorphic or holomorphic 1-form ω on X by giving a single formula in a single chart. This can be done by the Identity Theorem for meromorphic functions and forms [1]. However, it is important to check that the formula transforms uniquely to give a meromorphic 1-form on all of X.

Remark 3.15. Let f be a meromorphic function on X which can be used as a local coordinate at a point p where the function f is holomorphic and has $\text{ord}_p(f - f(p)) = 1$. This is the case for all the points on X with the exception of a discrete set.
Given such an f, any meromorphic 1-form ω can be written as $g(z)df$ for a suitable meromorphic function g.

[1] if two 1-forms agree on an open set, they must be identical

3.4 The Residue Theorem for 1-forms

Definition 3.16. Let ω be a 1-form on X which is meromorphic at a point $p \in X$. The **residue of** ω **at** p, denoted by $\mathrm{Res}_p(\omega)$, is the coefficient c_{-1} in the following Laurent series for ω at p:

$$w = f(z)dz = \left(\sum_{n=-M}^{\infty} c_n z^n \right) dz \tag{3.3}$$

where z is the local coordinate centered at p and $c_{-M} \neq 0$, so that $\mathrm{ord}_p(\omega) = -M$.

The following lemma will help us to see how is the coefficient c_{-1} well defined and so is the residue of the meromorphic 1-form ω at p.

Lemma 3.17. Let ω be a meromorphic 1-form defined in a neighborhood of p. Let γ be a small path on X enclosing just p and any other pole of ω. Then:

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \int_\gamma \omega \tag{3.4}$$

We can conclude that the residue of ω is well defined because the integral in 3.4 is independent of the choice of the chart and by extension of the chosen local coordinate used z in 3.3.

Lemma 3.18. Let f be a meromorphic function at $p \in X$. Then df/f is a meromorphic 1-form at p, and

$$\mathrm{Res}_p(df/f) = \mathrm{ord}_p(f) \tag{3.5}$$

Theorem 3.19. *(The Residue Theorem)*
Let ω be a meromorphic 1-form on a compact Riemann surface. Then:

$$\sum_{p \in X} \mathrm{Res}_p(\omega) = 0 \tag{3.6}$$

Chapter 4

Group Actions on Riemann Surfaces

4.1 Finite Group Actions

In this part of the work we will study group actions G on compact Riemann Surfaces X and focus on the Riemann Surface given by X/G.
For this chapter, let's take G as a finite group.

Definition 4.1.

1. An **action** of G on X is a map $G \times X \to X$, which we will denote by $(g, p) \mapsto g \cdot p$, which satisfies
   (a) $(gh) \cdot p = g \cdot (h \cdot p)$ for $g, h \in G$ and $p \in X$, and
   (b) $e \cdot p = p$ for $p \in X$, where $e \in G$ is the identity
2. The **orbit** of a point $p \in X$ is the set $G \cdot p = \{g \cdot p | g \in G\}$. If we consider a subset $A \subset X$, then the set of orbits of points in A is denoted by $G \cdot A = \{g \cdot a | g \in G, a \in A\}$.
3. The **stabilizer** of a point $p \in X$ is the subgroup $G_p = \{g \in G | g \cdot p = p\}$. We often call it the **isotropy subgroup of** p.

   Note that points in the same orbit have conjugate stabilizers: $G_{g \cdot p} = gG_pg^{-1}$. Considering that G is finite:

$$|G \cdot p||G_p| = |G|$$

4. The **kernel** of an action of G on X is the subgroup $K = \{g \in G | g \cdot p = p$ for all $p \in X\}$. It is the intersection of all stabilizer subgroups. We will usually consider that the kernel is trivial. In this case, we call it an **effective action**.

5. We say that the action is **continuous** (resp. **holomorphic**), if for every $g \in G$, the bijection $p \mapsto g \cdot p$ is a continuous (resp. holomorphic) map from X to X. We get an automorphism of X if it is holomorphic.

6. The **quotient space** X/G is the set of orbits.

   The quotient map that sends a point to its orbit is given by $\pi : X \to X/G$. We give the quotient topology to the quotient space by declaring a subset $U \subset X/G$ to be open if and only if $\pi^{-1}(U)$ is open in X.

In the following sections we would like to provide X/G with a complex structure so that π is a holomorphic map.
Before doing that, let's present two characteristics of the Stabilizer Subgroups. Let's take G a group acting holomorphically and effectively on X.

Proposition 4.2. Fix a point $p \in X$.
Suppose that G_p is finite. Then G_p is a finite cyclic group.
Particularly, if G is also finite, then all stabilizer subgroups are finite cyclic subgroups.

Proposition 4.3. Let G be finite. Then the points of X with nontrivial stabilizers are discrete.

4.2 The Quotient Riemann Surface

Throughout this section, let G be a finite group acting holomorphically and effectively on X.
As we said before, we would like to provide X/G with a complex structure. To do that we must find complex charts and the following proposition will point the way towards defining charts on X/G.

Proposition 4.4. Let G be a finite group acting holomorphically and effectively on X. Fixing a point $p \in X$, there is an open neighborhood U of p such that:

1. For $g \in G_p$, where G_p is the stabilizer of p, we have that $g \cdot u \in U$ for $u \in U$.

2. For $g \notin G_p$, $g \cdot U$ and U are disjoint.

3. $\alpha : U/G_p \to X/G$ is a homeomorphism onto an open subset of X/G.

4. p is the only point in U fixed by any element $g \in G_p$.

According to this, to define charts in X/G we will need to define charts in U/G_p and transport them to X/G via the homeomorphism α. Let $p \in X$ and $\bar{p} \in X/G$ the orbit of this point. Suppose $m = |G_p| \geq 2$. We are interested in finding an appropiate function from a neighborhood W of $\bar{p}$ to $\mathbb{C}$. Additionally, we may assume that, away from p and taking U as a G_p-invariant neighborhood of p, the projection $\pi : U \to U/G_p$ is $m-$to-1. The function h defined as follows would be a G_p-invariant on a neighborhood of p:

$$\begin{aligned} h : U &\xrightarrow{\pi} U/G_p \xrightarrow{\alpha} W \xrightarrow{\phi} \mathbb{C} \\ h(z) &= \prod_{g \in G_p} g(z) \end{aligned} \tag{4.1}$$

where z is a local coordinate centered at p. Given h, that has multiplicity m at p, we can shrink U to the neighborhood of p where h is G_p-invariant. By construction, h is holomorphic and G_p invariant and we can define the projection $\bar{h} : U/G_p \to \mathbb{C}$ which is a homeomorphism since it is 1-1, continuous and open. Finally, we are able to give the chart map ϕ on W as:

$$\phi : \; W \xrightarrow{\alpha^{-1}} U/G_p \xrightarrow{\bar{h}} V \subset \mathbb{C}$$

where α^{-1} is a homeomorphism since α is a homeomorphism.

Theorem 4.5. Taking the complex charts from above on X/G, X/G becomes a Riemann surface.
Also, $\pi : X \to X/G$ is holomorphic of degree $|G|$ and $\text{mult}_p(\pi) = |G_p|$ for any point $p \in X$.

Proof. The charts we have defined for X/G cover X/G. What we have to do now is check if these complex charts are all compatible and give a complex structure on X. To do it, we will separate the proof in different cases according to the value of m.

- If $m \geq 2$, since the points with nontrivial stabilizers are discrete (4.3), we may assume that no two charts domains meet.
- If the two charts are constructed in the $m = 1$ case, these charts are compatible since the original charts on X are compatible.
- Let's now suppose that one chart $\phi_1 : \overline{U}_1 \to V_1$ is constructed with $m = 1$ and the other $\phi_2 : \overline{U}_2 \to V_2$ is constructed with $m \geq 2$, where U_1 and U_2 are the associated open sets in X used to construct these charts.

Let $\bar{r} \in \overline{U}_1 \bigcap \overline{U}_2$ and lift $\bar{r}$ to $r \in U_1 \bigcap U_2$. If $U_1 \bigcap U_2 = \emptyset$, then we can replace U_1 by a translate U_1' under the group so that $U_1' \bigcap U_2 \neq \emptyset$.

Now, let ω be the local coordinate in U_1 and $\overline{U}_1$. On the other hand, let z be the local coordinate in U_2 and $h(z)$ the local coordinate in $\overline{U}_2$, where h is the one we have defined in 4.1. As we have seen before, the function h is holomorphic and since z and ω are themselves compatible, we obtain that ϕ_1 and ϕ_2 are compatible.

Finally, since G is finite and X is Hausdorff (Riemann Surface), we have that X/G is also Hausdorff. Moreover, since X is connected and $\pi : X \to X/G$ is onto, X/G is also connected. Then, as X/G is Hausdorff and connected, we have that these charts make X/G a Riemann Surface.

The projection π is holomorphic and $\deg(\pi) = |G|$ by the way we have defined the charts on X/G. Finally, $\text{mult}_p(\pi) = \text{mult}_p(h) = |G_p|$.

□

Lemma 4.6. Let $\pi : X \to Y = X/G$. For every branch point (see 2.28) $y \in Y$ there is an integer $r \geq 2$ such that $\pi^{-1}(y)$ has $|G|/r$ points on X and at each of these, the multiplicity of π is r.

Corollary 4.7. Let $y_1, ..., y_k$ be k branch points in $Y = X/G$ with π having multiplicity r_i at each of the $|G|/r_i$ points above y_i. Then, applying the Hurwitz's formula (see 2.2):

$$
\begin{aligned}
2g(X) - 2 &= \deg(\pi)(2g(X/G) - 2) + \sum_{\pi^{-1}(y_i) \in X} [\text{mult}_{\pi^{-1}(y_i)}(\pi) - 1] \\
&= |G|(2g(X/G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) \\
&= |G|[2g(X/G) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{r_i})] = |G|[2g(X/G) - 2 + R]
\end{aligned}
$$

Lemma 4.8. Suppose the integers $r_1, ..., r_k$ with $r_i \geq 2$, as stated in 4.6. Then, R can take different values in the following cases:

1.

$$
R < 2 \iff k, \{r_i\} = \begin{cases} k = 1, & \text{any } r_1; \\ k = 2, & \text{any } r_1, r_2; \text{or} \\ k = 3, & \{r_i\} = \{2, 2, \text{any } r_3\}; \text{or} \\ k = 3, & \{r_i\} = \{2, 3, 3\}, \{2, 3, 4\} \quad \text{or} \{2, 3, 5\} \end{cases}
$$

2.

$$R = 2 \iff k, \{r_i\} = \begin{cases} k = 3, \quad \{r_i\} = \{2,2,6\}, \{2,4,4\} \quad \text{or } \{3,3,3\} \text{ or} \\ \\ k = 4, \qquad \{r_i\} = \{2,2,2,2\} \end{cases}$$

3. If $R > 2$, then in fact $R \geq 2 + \frac{1}{42}$

Proof. The cases 1. and 2. can be computed easily taking into account that the highest $1 - \frac{1}{r_i}$ is $\frac{1}{2}$ (for $r_i = 2$) and as we take higher r_i, $1 - \frac{1}{r_i}$ decreases.
The value obtained in 3. is achieved when $k = 3$ for $r_1 = 2, r_2 = 3$ and $r_3 = 7$ as $R = \frac{1}{2} + \frac{2}{3} + \frac{6}{7} = \frac{85}{42} = 2 + \frac{1}{42}$ (≈ 2.02). It is the lowest value, higher than 2, that we can reach because:

$$\frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 2$$
$$\frac{1}{2} + \frac{2}{3} + \frac{7}{8} = \frac{49}{24} \approx 2.04$$

□

Chapter 5

Divisors and Riemann-Roch Theorem

This chapter is crucial for the construction of the rest of the paper. We will first introduce the divisors and discuss their properties and then present the Riemann-Roch Theorem. Finally, we will end up by discussing some of the consequences of this theorem.

Definition 5.1. We define the group of functions $D : X \to \mathbb{Z}$ as the group $\mathbb{Z}^X$ which is a group under pointwise addition.
The points $p \in X$ where $D(p) \neq 0$ are said to be the **support of** D.
Then D is a **divisor** on X if its support is a discrete subset of X. We denote by $\mathrm{Div}(X)$ the divisors on X that form a group under pointwise addition. We denote D with the following summation notation:

$$D = \sum_{p \in X} D(p) \cdot p$$

which is a finite sum because the set of points p where $D(p) \neq 0$ is a discrete subset of X, according to the definition.

Remark 5.2. In the particular case of a compact Riemann Surface a function D is said to be a divisor if and only if its support is finite.

Definition 5.3. On a compact Riemann surface we can define the **degree** of D as:

$$\deg(D) = \sum_{p \in X} D(p).$$

which is a group homomorphism. Its kernel is the subgroup denoted by $\mathrm{Div}_0(X)$ formed by the divisors of degree 0.

Down below we will describe two types of divisors having an interest in our work. We will firstly introduce **principal divisors** and then **canonical divisors** which are, respectively, the divisors of meromorphic functions and the divisors of meromorphic 1-forms.

Definition 5.4. A **principal divisor** on X is the divisor defined by the order function (see 2.9) as:

$$\operatorname{div}(f) = \sum_p \operatorname{ord}_p(f) \cdot p$$

where f is a meromorphic function on X. We denote the set of principals divisors on X as $\operatorname{PDiv}(X)$.

Lemma 5.5. Let f and g be nonzero meromorphic functions on X. Then, we have the following properties related with principal divisors:

1. $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$
2. $\operatorname{div}(f/g) = \operatorname{div}(f) - \operatorname{div}(g)$
3. $\operatorname{div}(1/f) = -\operatorname{div}(f)$

We see from this lemma that $\operatorname{PDiv}(X)$ is a subgroup of $\operatorname{Div}(X)$.
If in particular X is compact, then we have a stronger property which is that $\operatorname{PDiv}(X)$ is a subgroup of $\operatorname{Div}_0(X)$.

We have an equivalent lemma to 2.38 with divisors.

Lemma 5.6. Let f be a nonzero meromorphic function on a compact Riemann surface, then $\deg(\operatorname{div}(f)) = 0$.

Definition 5.7. We can define the **divisors of zeroes of** f and the **divisors of poles of** f, denoted respectively by $\operatorname{div}_0(f)$ and $\operatorname{div}_\infty(f)$ as:

$$\operatorname{div}_0(f) = \sum_{p \text{ with} \operatorname{ord}_p(f)>0} \operatorname{ord}_p(f) \cdot p$$
$$\operatorname{div}_\infty(f) = \sum_{p \text{ with} \operatorname{ord}_p(f)<0} (-\operatorname{ord}_p(f)) \cdot p$$

In terms of the divisor of f we have the expression:

$$\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f) \tag{5.1}$$

Let's now introduce the divisors of meromorphic $1 - forms$ in a similar way.

Definition 5.8. Let ω be a meromorphic 1-form on X. A **canonical divisor** on X, denoted by $\mathrm{div}(\omega)$ is a divisor defined by the order function:

$$\mathrm{div}(\omega) = \sum_p \mathrm{ord}_p(\omega) \cdot p.$$

The set of canonical divisors on X is denoted by $\mathrm{KDiv}(X)$.

Lemma 5.9. Given two meromorphic 1-forms on X, with ω_1 different to 0, there is a unique meromorphic function f on X with $\omega_2 = f\omega_1$.

Corollary 5.10. The difference of any two canonical divisors is principal. Therefore, for any nonzero ω we have:

$$\mathrm{KDiv}(X) = \mathrm{div}(\omega) + \mathrm{PDiv}(X)$$

Moreover, we also have the concept of $\mathrm{div}_0(\omega)$ and $\mathrm{div}_\infty(\omega)$ which are defined similarly to 5.7.
Let now $F : X_1 \to X_2$ be a nonconstant holomorphic map between Riemann surfaces.

Definition 5.11. The **ramification divisor** of F is the divisor on X_1 denoted by R_F and defined by:

$$R_F = \sum_{p \in X} [\mathrm{mult}_p(F) - 1] \cdot p$$

The **branch divisor** of F is the divisor on X_2 denoted by B_F and defined by:

$$B_F = \sum_{y \in X_2} [\sum_{p \in F^{-1}(y)} (\mathrm{mult}_p(F) - 1)] \cdot y$$

Remark 5.12. The Hurwitz's formula in 2.2 can be formulated in terms of the degree of the ramification divisor:

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \deg(R_F)$$

Let's introduce now the concept of spaces of functions and forms associated to a divisor. A main use of divisors is to organize the meromorphic functions on X by employing the order function. To do so, we need to establish that $\mathrm{ord}_p(f) = \infty$ if f is identically 0 in a neighborhoord of p where $\infty > n$ for $n \in \mathbb{Z}$.

Definition 5.13. Let D be a divisor on X. The **space of meromorphic functions with poles bounded by D** is the set of meromorphic functions (can be thought of as a complex vector space) denoted by $L(D)$ and defined by:

$$L(D) = \{f \in \mathcal{M}(X) | \operatorname{div}(f) \geq -D\}.$$

For $f \in L(D)$ one of the following two conditions must be fulfilled. At a discrete set of points of X, either poles are being allowed to specified order and no worse or zeroes are being required to at least some specified order.

Definition 5.14. We can define a **partial ordering on the set** $\operatorname{Div}(X)$. We can write that $D \geq 0$ if $D(p) \geq 0$ for all $p \in X$.
For $D_1, D_2 \in \operatorname{Div}(X)$, we have that $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.
The definition is equivalent for ">", "<" and "$\leq$".
Given two nonnegative divisors P and N with disjoint support, any divisor D can be written as:

$$D = P - N \tag{5.2}$$

Proposition 5.15. Let $D_1 \leq D_2$ where D_1, D_2 are two divisors on X. As any function with poles bounded in D_1 has poles bounded in D_2, we have that:

$$L(D_1) \subset L(D_2) \tag{5.3}$$

Proposition 5.16. Generally, recalling that a meromorphic function is holomorphic if and only if $\operatorname{div}(f) \geq 0$, then:

$$L(0) = \mathcal{O}(X) = \{\text{holomorphic functions on } X\}.$$

In the particular case of X being compact, where the only holomorphic functions are the constant ones:

$$L(0) = \{\text{constant functions on } X\} \cong \mathbb{C} \tag{5.4}$$

Proposition 5.17. If D is a divisor on a compact X with $\deg(D) < 0$, then $L(D) = \{0\}$.

We have an analogous definition of $L(D)$ for meromorphic 1-forms.

Definition 5.18. We denote by $L^{(1)}(D)$ the space of meromorphic 1-forms with poles bounded by D. It is defined by the set:

$$L^{(1)}(D) = \{\omega \in \mathcal{M}^{(1)}(X) |\, \mathrm{div}(\omega) \geq -D\}.$$

As before, $L^{(1)}(D)$ determines a complex vector space.

Proposition 5.19. $L^{(1)}(0) = \Omega^1(X)$, the space of global holomorphic 1-forms on X.

As it follows, we present a few useful results concerning the bounds of $L(D)$.

Lemma 5.20.
1. Let D be a divisor on X and a point $p \in X$. Then either $L(D-p) = L(D)$ or $L(D-p)$ has codimension one in $L(D)$.
2. In the particular case of X being compact, $L(D)$ has a finite dimension. Writing D as in 5.2, $\dim L(D) \leq 1 + \deg(P)$.

   For D being a nonnegative divisor, $\dim L(D) \leq 1 + \deg(D)$

This shows that for compact Riemann surfaces, the spaces $L^{(1)}(D)$ are finite-dimensional.

Remark 5.21. For commodity, $\dim L(D)$ and $\dim L(K-D)$ would be written as $l(D)$ and $l(K-D)$, respectively.

The following is a deep theorem that we will admit without a deep proof.

Theorem 5.22. (Riemann-Roch Theorem) Let g be the genus of an algebraic curve. Then for any divisor D and any canonical divisor K, we have:

$$l(D) = l(K-D) + \deg(D) + 1 - g \tag{5.5}$$

Let's introduce some particular cases to illustrate the use of this theorem.

Corollary 5.23.

1. If $D = 0$, as we have seen in 5.4,

   $$L(0) = \{f \in \mathcal{M}(X) |\, \mathrm{div}(f) \geq 0\} \cong \mathbb{C}.$$

   Therefore, $l(0) = 1$ and using 5.5:

   $$l(0) = l(K) + \deg(0) + 1 - g \implies l(K) = g \tag{5.6}$$

2. If $D = K$, 5.5 turns to:

$$l(K) = l(0) + \deg(K) + 1 - g$$

using that $l(0) = 1$ and 5.6, we obtain:

$$g = l(K) = 1 + \deg(K) + 1 - g \implies \deg(K) = 2g - 2$$

Corollary 5.24. We have previously seen that either $l(D + p) = l(D)$ or $l(D + p) = l(D) + 1$.
Let's take D as $D + p$ in 5.5

$$\begin{aligned} l(D+p) &= l(K-D-p) + \deg(D+p) + 1 - g \\ &= l(K-D-p) + \deg(D) + \deg(p) + 1 - g \\ &= l(K-D-p) + \deg(D) + 2 - g \end{aligned} \tag{5.7}$$

1. If $l(D + p) = l(D)$, using 5.7 and 5.5 for $l(D)$ we obtain that:

$$\begin{aligned} l(K-D-p) + \deg(D) + 2 - g &= l(K-D) + \deg(D) + 1 - g \\ \implies l(K-D-p) &= l(K-D) - 1 \end{aligned}$$

2. If $l(D + p) = l(D) + 1$, using 5.7 and 5.5 for $l(D) + 1$ we obtain that:

$$\begin{aligned} l(K-D-p) + \deg(D) + 2 - g &= l(K-D) + \deg(D) + 2 - g \\ \implies l(K-D-p) &= l(K-D) \end{aligned}$$

Definition 5.25. We say that $n \in \mathbb{N}$ with $n > 0$ is a **gap number** for $|D|$ at p if:

$$l(D - np) + 1 = l(D - (n-1)p)$$

Taking $D = K$ as the canonical divisor, where $l(K) = g$,

$$\begin{gathered} l(K - np) + 1 = l(K - (n-1)p) \\ l(np) + 2g - 2 - n + 1 - g + 1 = l((n-1)p) + 2g - 2 - n + 1 + 1 - g \end{gathered}$$

where we have used 5.5 to compute the last equation. Finally, comparing terms side to side, we obtain:

$$l((n-1)p) = l(np)$$

Definition 5.26. The set of gap numbers for $|D|$ at p is denoted by $G_p(|D|)$.

Remark 5.27. A linear system Q is called a g_d^r if $\dim Q = r$ and $\deg(Q) = d$.

Lemma 5.28. Let Q be a nonempty g^r_d on X. Fixing a point $p \in X$ we have that $G_p(Q)$ is a finite set that has cardinal $1 + r$.

Definition 5.29. Let $p \in X$, we say that:

$$1 < \alpha_1 < \alpha_2 < ... < \alpha_g = 2g \tag{5.8}$$

be the first g **"non-gaps"**.

Proposition 5.30. For each integer j, $0 < j < g$, we have

$$\alpha_j + \alpha_{g-j} \geqslant 2g \tag{5.9}$$

Proof. To prove this condition, we will suppose that $\alpha_j + \alpha g - j < 2g$. Thus for each $k \leqslant j$, we would also have $\alpha_k + \alpha_{g-j} < 2g$. Since the sum of "non-gaps" is a "non-gap", we would have at least j "non-gaps" strictly between α_{g-j} and α_g. Thus at least $(g-j)+j+1 = g+1 \leqslant 2g$ "non-gaps". This contradicts the fact that there are only g "non-gaps" so $\alpha_j + \alpha_{g-j} \geqslant 2g$, as we wanted to prove. □

Proposition 5.31.

1. If $\alpha_1 = 2$, then $\alpha_j = 2j$ and $\alpha_j + \alpha_{g-j} = 2g$ for $0 < j < g$.
2. If $\alpha_1 > 2$, then for some j with $0 < j < g$, we have $\alpha_j + \alpha_{g-j} > 2g$.

Corollary 5.32. From the previous proposition, we obtain that:

$$\sum_{j=1}^{g-1} \alpha_j \geqslant g(g-1) \tag{5.10}$$

Finally, let's see some results of the application of the Riemann-Roch Theorem.

Lemma 5.33. Let X be a compact Riemann surface. If $l(p) > 1$, for some $p \in X$, then X is isomorphic to $\mathbb{C}_\infty$.

Proof. By the hypothesis, we have that there must be a nonconstant meromorphic function f in $L(p)$. This function f has a unique simple pole allowed in p and the associated holomorphic map $F : X \to \mathbb{C}_\infty$ has degree one. Therefore X is isomorphic to the Riemann Sphere. □

Proposition 5.34. Every X of genus 0 is isomorphic to $\mathbb{C}_\infty$.

Proof. Let fix any point $p \in X$. Then, being K a canonical divisor with $\deg(K) = 2g - 2 = -2$ (by 5.23), we have that $\deg(K - p) = -3$. Due to the fact that this degree is negative, we have that $L(K - p) = 0$. Finally, let apply 5.5 to $D = p$:

$$l(p) = l(K - p) + \deg(p) + 1 - g = 0 + 1 + 1 - 0 = 2$$

Finally, according to 5.33, we conclude that $X \cong \mathbb{C}_\infty$. $\square$

Proposition 5.35. Every compact Riemann surface of genus 1 is isomorphic to the complex torus.

Proof. See [1]. $\square$

Proposition 5.36. Every compact Riemann surface of genus 2 is hyperelliptic.

Proof. Consider a canonical divisor K with $\deg(K) = 2g - 2 = 4 - 2 = 2$ (by 5.23). By 5.6 we have that $l(K) = g = 2$ so we can assume that $K > 0$. Thus, there is a nonconstant function $f \in L(K)$ and the associated holomorphic map $F : X \to \mathbb{C}_\infty$ has degree 2. Then, X is hyperelliptic. $\square$

Chapter 6

The Wronskian and Weierstrass Points

First of all, to give a proper definition of the Wronskian, which will lead us to Weierstrass Points, we need to introduce higher-order forms on X.

Definition 6.1. A **meromorphic n-fold differential** (instead of form to avoid confusion) in the coordinate z on an open set V, where f is a meromorphic function, is an expression of the form

$$\mu = f(z)(dz)^n$$

As long as we have done in this work, there is a compatibility condition for these objects. Given two meromorphic n-fold differential $\mu_1 = f(z)(dz)^n$ and $\mu_2 = g(w)(dw)^n$, in a coordinate z on an open set V_1 and in a coordinate w on an open set V_2 resp., we can define a holomorphic mapping $z = T(w)$ from V_1 to V_2.
μ_1 transforms to μ_2 under T if $g(w) = f(T(w))T'(w)^n$.

Given that, we are able to understand the Wronskian as higher order forms on X. Taking the divisor as the canonical one, K, we fix a local coordinate z centered at $p \in X$ and any basis $\{\varphi_{k_i}\}$ for $L(K)$. Setting $g_{k_i} = z^{K(p)}\varphi_{k_i}$ for each i between 1 and $r+1$ (where $r = g-1 = \dim|K| = l(K)-1$). Explicitly, g_{k_i} is given by:

$$\begin{aligned} g_{k_1} &: \alpha_0 + \alpha_1\varphi_{k_1} + ... + \alpha_g\varphi_{k_g} = 0 \\ g_{k_2} &: z^{k_1}(\alpha_1\frac{a_{-k_1}}{z^{k_1}} + ...) = 0 \\ &[...] \\ g_{k_g} &: z^{k_g}(\alpha_g\frac{a_{-k_g}}{z^{k_g}} + ...) = 0 \end{aligned}$$

where every g_{k_i} for $i \in \{1, ..., g\}$ is holomorphic at p. As we have taken $D = K$, we have that $l(D) = l(K) = g$ and thus $g = r + 1$. Due to this, we can also write $i \in \{1, ..., r + 1\}$.

Remark 6.2. Do not confuse the g of the genus with the g_{k_i} of the basis.

Definition 6.3. Given this, we can define the **Wronskian** to be the function:

$$W_z(g_{k_1}, ..., g_{k_{r+1}})(z) = \det \begin{pmatrix} g_{k_1}(z) & g'_{k_1}(z) & g^{(2)}_{k_1}(z) & ... & g^{(r)}_{k_1}(z) \\ g_{k_1}(z) & g'_{k_1}(z) & g^{(2)}_{k_1}(z) & ... & g^{(r)}_{k_2}(z) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ g_{k_{r+1}}(z) & g'_{k_{r+1}}(z) & g^{(2)}_{k_{r+1}}(z) & ... & g^{(r)}_{k_{r+1}}(z) \end{pmatrix} \tag{6.1}$$

which is holomorphic because every g_{k_i} is holomorphic.

Definition 6.4. A point $p \in X$ with local coordinate z is a Weierstrass point for $|K|$ if and only if for any basis $\{\varphi_{k_1}, ..., \varphi_{k_{r+1}}\}$ for $L(K)$, the Wronskian $W_z(z^{K(p)}\varphi_{k_1}, ...,$-$z^{K(p)}\varphi_{k_{r+1}})$ is zero at p.

Lemma 6.5. The Wronskian is well defined by $|K|$ itself, and not by the choice of basis.

Definition 6.6. A point p is a **Weierstrass point** on X if and only if $L(gp)$ has a nonconstant function in it, or, equivalently, if and only if $l(gp) \geqslant 2$.

Definition 6.7. The **Weierstrass weight** of a point p is:

$$w(p) = \sum_{i=1}^{g} (n_i - i) \tag{6.2}$$

We have that p is a Weierstrass point if and only if $w(p) > 0$.

Lemma 6.8. Taking $|K|$ on X with $\dim |K| = r = g - 1$, we have:

$$\deg(\mathrm{div}(W(|K|)) = \sum_p \mathrm{ord}_p(W(|K|)) = r(r+1)(g-1) = (g-1)g(g-1) = g(g-1)^2$$

Lemma 6.9. Taking $D = K$, the order of the Wronskian at $p \in X$ can be expressed as:

$$\begin{aligned} \mathrm{ord}_p(W_z(z^{K(p)}\varphi_{k_1}, ..., z^{K(p)}\varphi_{k_{r+1}})) &= \mathrm{ord}_p(z^{(r+1)K(p)}W_z(\varphi_{k_1}, ...\varphi_{k_{r+1}})) \\ &= (r+1)K(p) + \mathrm{ord}_p(W(|K|)) \\ &= gK(p) + \mathrm{ord}_p(W(|K|)) \end{aligned}$$

Lemma 6.10. We have that the Weierstrass weight of a point p is defined as the order of the Wronskian at p, as we have defined before in 6.9. So:

$$\mathrm{ord}_p(W_z(z^{K(p)}\varphi_{k_1},...,z^{K(p)}\varphi_{k_{r+1}})) = w(p) = \sum_{i=1}^{g}(n_i - i)$$

We will now see which is the total weight of the Weierstrass points.

Theorem 6.11. Taking the same conditions as before ($\deg(K) = d = 2g-2$), we obtain that:

$$\begin{aligned}\sum_{p\in X} w(p) &= (r+1)(d+rg-r) = g(2g-2+g(g-1)-g+1) \\ &= g(2g-2+g^2-g-g+1) = g(g^2-1) = g^3-g\end{aligned} \tag{6.3}$$

Proof. As it has been said in 6.10, we can compute the following:

$$\begin{aligned}\sum_p w(p) &= \sum_p \mathrm{ord}_p(W_z(z^{K(p)}\varphi_{k_1},...,z^{K(p)}\varphi_{k_{r+1}})) \\ &= \sum_p (r+1)K(p) + \mathrm{ord}_p(W(|K|))(\text{see } 6.9) \\ &= (r+1)d + g(g-1)^2(\text{see } 6.8) = g(2g-2) + g(g-1)^2 \\ &= g(2g-2+(g-1)^2) = g(2g-2+g^2-2g+1) = g(g^2-1) \\ &= g^3-g\end{aligned}$$

□

Theorem 6.12. For $g \geqslant 2$, the weight of a point is $\leqslant g(g-1)/2$. This bound is attained only for a point p where the "non-gap" sequence begins with 2.

Proof. Let $2 \leqslant \alpha_1 < \alpha_2 < ... < \alpha_g = 2g$ be the first g "non-gaps" at p and $1 = n_1 < n_2 < ... < n_g < 2g$ be the $g-$"gaps" at p. The α_i's and the n_i's are complementary in $\{1, ..., 2g\}$. Then, by the definition 6.7, we have:

$$\begin{aligned}w(p) &= \sum_{i=1}^{g}(n_i - i) = \sum_{i=1}^{2g} i - \sum_{i=1}^{g}\alpha_i - \sum_{i=1}^{g} i \\ &= \sum_{i=g+1}^{2g-1} i - \sum_{i=1}^{g-1}\alpha_i \leqslant \frac{3g}{2}(g-1) - g(g-1) = g(g-1)/2\end{aligned}$$

where we have used 5.10 and with equality holding if and only if $\alpha_1 = 2$ (hyperelliptic case). □

Once we have found the weights of Weierstrass points, it is time to compute how many of them we can have.

Proposition 6.13. For X with $g \geqslant 2$, there are between $2g+2$ and $g^3 - g$ Weierstrass points. The lower bound, $2g+2$, occurs only in the hyperelliptic case.

Proof. To compute the number of Weierstrass points, we will use that:

$$\#\text{Weierstrass points} = \frac{\sum\limits_{p \in X} w(p)}{w(p)}$$

According to 6.3 and 6.12, we obtain the lower bound. The upper bound is obtained because the minimum weight of a point is 1. $\square$

Chapter 7

Hurwitz's Theorem

From all we have seen, given X, we can say that Weierstrass points are points intrinsically defined to X. This property, among others, will help us in this chapter to construct the necessary to prove Schwarz's Theorem and Hurwitz's Theorem.

Proposition 7.1. Let X be a compact Riemann surface with genus $g \geq 2$. If $1 \neq T \in Aut(X)$, then T has at most $2g+2$ fixed points.

Proof. First, since X is compact and the fixed point set of T is discrete, the fixed point set of T is finite. Let $P \in X$ be a not fixed point of X. Then, there is a meromorphic function f on X whose divisor of poles of f (5.7) is P^r with $1 \leq r \leq g+1$. Considering the function $h = f - f \circ T$, its divisor of poles is $P^r(T^{-1}P)^r$ and the function h has therefore $2r \leq 2g+2$ zeros. Since, as we can see, each point of T is a zero of h, we can conclude that T has at most $2g+2$ fixed points. □

Defining $W(X)$ to be the set of Weierstrass points on X, we have from 6.13 that this set is finite.

Proposition 7.2. If $T \in Aut(X)$, then $T(W(X)) = W(X)$. Specifically, we define $Perm(W(X))$ as the permutation group of the Weierstrass points.

Proof. The demonstration derives from the fact that the gap sequences at $P \in X$ and at TP are the same. □

Proposition 7.3. Let X be compact with genus g. It is hyperelliptic if and only if there exists a conformal involution J on X ($J \in Aut(X)$ with $J^2 = 1$) that fixes $2g+2$ points. J can also be called hyperelliptic involution.

Proof. If X is hyperelliptic, there exists a conformal involution J such that $X/J = \mathbb{P}^1$. Using Riemann-Hurwitz formula (2.2) we have that there are $2g+2$ ramification points and thus $2g+2$ fix points of J.
Conversely, let J be a conformal convolution of X with $2g+2$ fixed points. Let consider the subgroup $<J>$ of order 2 and the two sheeted covering $X \to X/<J>$ which is branched at the $2g+2$ fixed points of J. Applying again (2.2) we obtain that $g(M/<J>) = 0$ and thus X has a meromorphic function of degree 2 and X is hyperelliptic. □

Corollary 7.4. The homomorphism

$$\lambda : Aut(X) \to Perm(W(X)) \tag{7.1}$$

is injective unless X is hyperelliptic, in which case $\mathrm{Ker}(\lambda) =< J >$.

Proof. Let's focus on the case where X is not hyperelliptic. We have seen in 6.13 that in this case the number of Weierstrass points is strictly higher than $2g+2$. However, we know from 7.1, that if $1 \neq T \in Aut(X)$, then T has at most $2g+2$ fixed points. Therefore, the only possibility remaining is to be the identity map and thus we obtain the injectivity. □

Theorem 7.5. (Schwarz's Theorem) Let X be a Riemann surface with genus $g \leq 2$, then $Aut(X)$ is a finite group.

Proof. We have defined in 7.4 the homomorphism λ that maps $Aut(X)$ to the permutation group of the Weierstrass points, $Perm(W(M))$.
We know that $Perm(W(M))$ is finite because the set of Weierstrass points is finite.
To prove the finiteness of $Aut(X)$, let's recall that generally, given

$$\lambda : Aut(X) \to Perm(W(X)),$$
$$\implies \dim(Aut(X)) = \dim(\mathrm{Ker}(\lambda)) + \dim(\mathrm{Im}(\lambda))$$

and by definition $\dim(\mathrm{Im}(\lambda)) \leq \dim(Perm(W(M)))$.
Therefore, if $\mathrm{Ker}(\lambda)$ is finite and $Perm(W(M))$ either is, as we have stated, we have that $Aut(X)$ is also finite.
If X is hyperelliptic, then by 7.4 we had that $Ker(\lambda) =< J >$ which is finite because $J^2 = 1$.
In the case of X being not hyperelliptic, the finiteness of $Ker(\lambda)$ is fulfilled because λ was injective.

□

Theorem 7.6. **(Hurwitz's Theorem)** Let G be a finite group acting holomorphically and effectively on a compact Riemann surface X of genus $g \geq 2$. Then, we can find the following bound:

$$|G| \leq 84(g-1) \tag{7.2}$$

Since $Aut(X)$ acts effectively and holomorphically on X, we can replace G by $Aut(X)$.

Proof. To start with this proof, we will recall 4.7:

$$2g-2 = |G|[2g(X/G) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{r_i})] = |G|[2g(X/G) - 2 + R] \tag{7.3}$$

where $R = \sum_{i=1}^{k}(1 - \frac{1}{r_i})$. From the Riemann-Hurwitz's formula, we claimed that $g(X) \geq g(X/G)$. Because of this, let's divide the proof into two cases.

1. Assume $g(X/G) = 0$. Then:

   $$2g-2 = |G|[R-2] \tag{7.4}$$

   Observe that in 7.4, the left part, $2g-2 \geq 2$, because $g \geq 2$. Due to this, from the right part of 7.4, we find that $R > 2$ and in 4.8 we obtained that in this case, $R \geq 2 + \frac{1}{42}$.

   Therefore, to sum up:

   $$|G| = \frac{2g-2}{R-2} = \frac{2(g-1)}{R-2} \leq 2 \cdot 42(g-1) = 84(g-1)$$

2. Assume $g(X/G) \geq 1$. We can suppose two cases: $R = 0$ and $R \neq 0$. Let's study both.

   (a) If $R = 0$ there is no ramification to the quotient map. We have from 7.3 that:

   $$2g-2 = |G|[2g(X/G) - 2] \implies g-1 = |G|[g(X/G) - 1] \tag{7.5}$$

   From 7.5, we see that the case $g(X/G) = 1$ can't be taken because we will find $g = 1$!! This leads us to $g(X/G) \geq 2$, which implies that $|G| \leq g-1$ (and $|G| \leq 84(g-1)$ either).

   (b) If $R \neq 0$, this forces $R \geq 1/2$. Taking into account that we have also assumed $g(X/G) \geq 1$, we obtain from 7.3:

   $$2g(X/G) - 2 + R \geq 2 - 2 + 1/2 = 1/2$$

   Applying this into 7.3 we conclude that:

   $$|G| \leq 4(g-1)$$

□

Example 7.7. Finally, a well-known example of a curve of genus 3 that attains the upper bound of Hurwitz's Theorem is **Klein's quartic**, $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$. It has exactly $84(3-1) = 168$ automorphisms.
Let's explicitly describe $Aut(X)$. Let $\zeta = \exp\frac{2\pi i}{7}$ be the primitive $7-$th root of unity. Firstly:

$$S : [t_0, t_1, t_2] \to [\zeta t_0 : \zeta^2 t_1 : \zeta^4 t_2]$$

defines an automorphism of order 7.
Then, let U be the cyclic permutation of coordinates of order 3 defined by:

$$U : [t_0 : t_1 : t_2] \to [t_1 : t_2 : t_0]$$

We observe that the subgroup generated by S and U is a semidirect product of order 21 ($USU^{-1} = S^4$). Finally, let T the automorphism of order 2 described by the matrix:

$$T = \frac{i}{\sqrt{7}} \begin{pmatrix} \zeta - \zeta^6 & \zeta^2 - \zeta^5 & \zeta^4 - \zeta^3 \\ \zeta^2 - \zeta^5 & \zeta^4 - \zeta^3 & \zeta - \zeta^6 \\ \zeta^4 - \zeta^3 & \zeta - \zeta^6 & \zeta^2 - \zeta^5 \end{pmatrix}$$

The group generated by U and T is the dihedral group of order 6 ($TUT^{-1} = U^2$).
To sum up, using Sylow's Theorem and by Hurwitz's Theorem we conclude that $Aut(X) = G =< S, T, U >$. This group is named **projective special linear group** $PSL(2,7)$, isomorphic to $PSL(3,2)$.

Definition 7.8. A compact Riemann Surface of genus g for which the maximum of 7.2 is achieved, is named **Hurwitz surface** or **Hurwitz curve**. The Fuchsian group of a Hurwitz Surface is a finite index torsionfree normal subgroup of the (2,3,7) triangle group, as we have defined in 7.7. The finite quotient group is precisely the automorphism group of these Hurwitz surfaces.
However, it does not exist Hurwitz surfaces for each genus. The sequence of allowable values of the genus g for which we find Hurwitz surfaces is given by:

$$3, 7, 14, 17, 118, 129, 146, 385, ... \text{ (sequence A179982 in the OEIS)}$$

After Klein's quartic, the following well-known Hurwitz surface with $g = 7$ is **Macbeth surface**, with 504 automorphisms.

Bibliography

[1] R. Miranda, Algebraic Curves and Riemann Surfaces, Amer. Math. Soc. (1995).

[2] H. M. Farkas, I. Kra, Riemann Surfaces, Springer-Verlag (1980)

[3] P. A. Griffiths Introduction to Algebraic Curves, Amer. Math. Soc. (1989)

[4] O. Foster, Lectures on Riemann Surfaces, Graduate Texts in Mathematics , **81** (1999).

[5] J. Jost, Compact Riemann Surfaces, Springer (2006).