Vacuum energy and cosmological inflation

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Abstract: We study the cosmic evolution from the perspective of the Running Vacuum Models, in which the vacuum energy density is dynamic throughout the cosmic history. In this context, we find that there is an initial period of inflation, in which the vacuum decays into radiation and there is a huge entropy production. In contrast to the standard inflaton-based mechanism of inflation, there is a smooth transition between this period and the standard ΛCDM radiation epoch. We also test the model against the Generalised Second Law, finding it is fulfilled in the current universe but not in the early stages. This is a welcome feature, however, as it may provide a possible solution to both the entropy and horizon problems, in a context where the eventual thermodynamic equilibrium of the universe is not jeopardised.

I. INTRODUCTION

The ΛCDM model has been the most successful cosmological theory, due to its ability to explain the accelerated expansion of the Universe, the Cosmic Microwave Background (CMB), the Big Bang Nucleosynthesis (BBN) and the observed abundances of elements, among others. Nevertheless, the model is not exempt of some severe limitations, as the well-known Hubble tension, for example, evinces: the incompatible measurements of the Hubble parameter H are hard to explain in this context, while a non-constant Cosmological Constant (CC), Λ, may be able to alleviate this tension [1]. But perhaps the most well-known issue of the ΛCDM model is the so-called CC problem, which arises when one tries to compare the value for the vacuum energy density, $\rho_\Lambda = \Lambda/8\pi G$, obtained via observations ($\rho_\Lambda^0 \sim 10^{-47}\mathrm{GeV}^4$) and theoretical predictions: even if we only consider the lowest contribution from the electroweak vacuum energy, the discrepancy is already [2–4]:

$$\left|\rho_\Lambda^{EW}/\rho_\Lambda^0\right| = \mathcal{O}(10^{55})\,, \tag{1}$$

which is perhaps the most disastrous prediction in the history of science. On top of that, there are the horizon and entropy problems [5, 6], which will be explained in detail later on.

In view of these issues, it has been proposed in the literature a type of Running Vacuum Models (RVMs), in which the vacuum energy density is dynamic throughout the entire cosmic history and runs with the Hubble rate, $\rho_\Lambda(H)$; this dynamical dependence comes from the calculation of quantum effects in QFT in curved space-time (see [2–6] and references therein). We will try to show that these RVMs may involve the necessary theoretical ingredients to provide a solution (or at least an alleviation) to the rest of the aforementioned problems, while at the same time complying with the laws of thermodynamics. The RVMs have also been tested against observational data from Supernovae type Ia, the CMB shift parameter and Baryonic Acoustic Oscillations [7–9]; as well as Large Scale Structure formation [6–10], with favourable results.

The aim of this work is to solve the cosmological equations of the RVMs so as to find expressions for the Hubble parameter as well as the matter and radiation densities, both in the early and current universe; we will see that these solutions provide a smooth transition between the inflation and the radiation epoch. We will also study the model from a thermodynamic point of view, for which we shall adopt the framework of the Generalised Second Law (GSL). Lastly, we will analyse the ability of the model to solve the horizon and entropy problems.

II. RUNNING VACUUM MODELS

We will focus our study on the type of models, developed in the theoretical framework of QFT in curved space-time, in which the vacuum energy density is a dynamical parameter throughout the cosmic evolution, $\dot\rho_\Lambda \neq 0$. Notice that a time-evolving gravitational constant $G = G(t)$ has also been discussed in the literature (see [2, 11]), however we will restrict ourselves to the case $G = const$. Using the FLRW metric one can obtain the following renormalisation group equation (we will be using natural units) [2]:

$$\frac{d\rho_\Lambda}{d\ln H^2} = \frac{1}{(4\pi)^2}\sum_i \left[a_i M_i^2 H^2 + b_i H^4 + \mathcal{O}\left(H^6\right)\right]\,. \tag{2}$$

The motivation and derivation of this equation can be found in [3]. Here the index i refers to the contributions of boson and fermion matter fields with masses M_i. It can be seen [2] that the solution is of the form:

$$\rho_\Lambda(H) = \frac{3}{8\pi G}\left(c_0 + \nu H^2 + \alpha\frac{H^4}{H_I^2}\right)\,, \tag{3}$$

where the dimensionless coefficients ν, α are related to the parameters a_i, b_i, M_i in (2), while H_I is the Hubble rate at the scale of inflation. A more general solution allows H^{2n+2} terms, instead of the $\mathcal{O}(H^4)$ term, to appear in (3). It leads, however, to the same conclusions we shall find, see Apendix B in [5].

The $\mathcal{O}(H^4)$ term dominates the vacuum energy evolution in the early inflationary universe, when the Hubble rate is comparable to H_I, while in the present universe it is negligible due to the small value at the current time H_0. This allows us to set the value for the constant parameter c_0 (which in this context would be nothing but the "actual" cosmological constant!) in terms of the current vacuum energy density ρ_Λ^0:

$$c_0 = \frac{8\pi G}{3}\rho_\Lambda^0 - \nu H_0^2 = H_0^2(\Omega_\Lambda^0 - \nu)\,. \tag{4}$$

In the last equality we have used the standard definitions $\Omega_\Lambda^0 = \rho_\Lambda^0/\rho_c^0$ and $\rho_c^0 = 3H_0^2/8\pi G$. After inflation, when $H \ll H_I$, the dynamics of ρ_Λ begin to be dominated by the $\mathcal{O}(H^2)$ term. Here, a small ν term ensures a mild evolution of the vacuum energy, or in other words a small correction to the ΛCDM model. Indeed, confronting the model with observations one finds $|\nu| \lesssim 10^{-3}$ [7, 8, 11], while a thermodynamic analysis leads us to conclude that $\nu > 0$ [5]. Therefore, the CC problem would be nothing but a consequence of living in a very low energy universe where the vacuum energy density is almost equal to $3c_0/8\pi G$, which, following the ΛCDM model, leads us to believe it has remained constant throughout the entire cosmic history!

We shall now derive some useful relations to study the evolution of the matter, radiation and vacuum energy densities in terms of the scale factor a. Applying Bianchi's identity $\nabla^\mu G_{\mu\nu} = 0$ to Einstein's equations with a CC term, $G_{\mu\nu} = 8\pi G(T_{\mu\nu} + g_{\mu\nu}\rho_\Lambda)$, one finds [6]:

$$\dot\rho + 3H(\rho + p) = -\dot\rho_\Lambda\,, \tag{5}$$

where we have used the FLRW metric to calculate the covariant derivative, and the EoS of vacuum $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. Note that $\rho = \rho_m + \rho_r$ and p refer respectively to the density and pressure of both radiation and relativistic matter. We should also recall Friedmann's equations:

$$3H^2 = 8\pi G(\rho + \rho_\Lambda)\,, \tag{6}$$

$$2\dot H + 3H^2 = -8\pi G(p + p_\Lambda)\,, \tag{7}$$

where we assume that the spatial curvature is zero or negligible.

A. Current universe

As we have mentioned, the post-inflationary universe is driven by the $\mathcal{O}(H^2)$ term. Rewriting (3) by substituting (4) we obtain:

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi G}(H^2 - H_0^2)\,. \tag{8}$$

Writing (5) for either a matter or radiation-dominated density, with $\omega_m = 0$ or $\omega_r = 1/3$:

$$\rho' + \frac{3}{a}(1+\omega)\rho = -\rho'_\Lambda\,, \tag{9}$$

where prime denotes differentiation with respect to the scale factor, and we have used the chain rule $d/dt = aHd/da$. Solving for both the matter and radiation density leads to [5]:

$$\rho_m(a) = \rho_m^0 a^{-3(1-\nu)}\,, \tag{10}$$

$$\rho_r(a) = \rho_r^0 a^{-4(1-\nu)}\,. \tag{11}$$

For $\nu = 0$ we recover the standard solutions, as expected. Inserting (10) in (9) one finds for the matter epoch:

$$\rho_\Lambda(a) = \rho_\Lambda^0 + \frac{\nu\rho_m^0}{1-\nu}(a^{-3(1-\nu)} - 1)\,. \tag{12}$$

Substituting these formulas in (6):

$$H^2(a) = H_0^2\left[1 + \frac{\Omega_m^0}{1-\nu}(a^{-3(1-\nu)} - 1)\right]\,, \tag{13}$$

where $\Omega_m^0 = \rho_m^0/\rho_c^0$ and $\Omega_m^0 + \Omega_\Lambda^0 = 1$ as usual. In these last results we verify, once again, that the ν-term seems to act as a small dynamical correction to the cosmological constant.

B. Early universe

For the early inflationary universe we can neglect the constant term c_0, which is relevant only for the current universe and essentially determines the value of the measured CC, see equation (4). Combining (6), (7) and substituting (3) we find:

$$aHH' + 2H^2 = 2\left(\nu H^2 + \alpha\frac{H^4}{H_I^2}\right)\,. \tag{14}$$

Solving this equation leads [6]:

$$H(a) = \sqrt{\frac{1-\nu}{\alpha}}\frac{H_I}{\sqrt{1 + Da^{4(1-\nu)}}}\,, \tag{15}$$

where D must be a positive constant so that the Hubble rate decreases with the expansion. Notice that $H(0) = \sqrt{(1-\nu)/\alpha}H_I$ is finite (as long as $\alpha \neq 0$) and therefore there is no singular initial point in the RVMs. That is, if there were such an initial singularity, it was erased by the process of inflation, together with the spatial curvature and any other previous information of the Universe. It is also important to note that in the initial period $Da^{4(1-\nu)} \ll 1$, $H(a) \simeq H(0)$ and thus integrating the scale factor we find $a(t) = a(0)e^{H(0)t}$, i.e. the scale factor increases exponentially during the early inflationary period.

Substituting (15) in Friedmann's equations we find for the vacuum and radiation density [5]:

$$\rho_r(\hat a) = \hat\rho_I\frac{(1-\nu)\hat a^{4(1-\nu)}}{[1 + \hat a^{4(1-\nu)}]^2}\,, \tag{16}$$

$$\rho_\Lambda(\hat a) = \hat\rho_I\frac{1 + \nu\hat a^{4(1-\nu)}}{[1 + \hat a^{4(1-\nu)}]^2}\,, \tag{17}$$

where we have re-scaled $\hat{H}_I = H(0)$, $\hat{\rho}_I = 3\hat{H}_I^2/8\pi G$ and $\hat{a} \equiv a/a_*$. Here $a_* \equiv D^{-1/[4(1-\nu)]}$ essentially corresponds to the vacuum-radiation equality point (not to be mistaken with the radiation-matter equality point which occurs in a much later stage of the cosmic evolution). We can immediately see that $\rho_r(0) = 0$ and $\rho_\Lambda(0) = \hat{\rho}_I$, thus in the beginning there is only vacuum energy. Vacuum then decays into radiation, however ρ_Λ remains essentially constant in the very first instants, causing a short inflationary period. The radiation density continues to increase until the equality point a_* is reached, after which it starts to decay as well. It is also important to realise that $|\rho_\Lambda/\rho_r| \propto |\nu| \ll 1$ after the equality point, and therefore we would have a smooth transit to the radiation epoch of the ΛCDM picture, where BBN can occur [6]. All these results are represented in Figure 1.

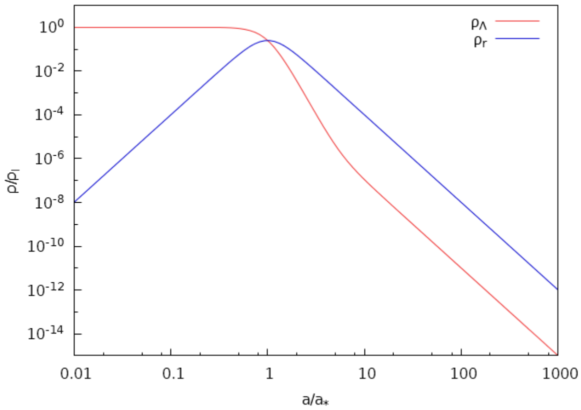

FIG. 1: Evolution of $\rho_\Lambda(\hat{a})$ and $\rho_r(\hat{a})$ in the early universe (normalised to ρ_I), represented in double-logarithmic scale. We set $\nu = 10^{-3}$.

III. GENERALISED SECOND LAW

For our thermodynamic discussion we shall adopt the perspective of the GSL, where one takes the apparent horizon as the characteristic length and studies the evolution of the entropy inside it, $S_\mathcal{V}$, with contributions from both matter and radiation; as well as the entropy of the horizon area, $S_\mathcal{A}$ [12]. The following conditions must be fulfilled:

$$S'_\mathcal{V} + S'_\mathcal{A} \geq 0\,, \qquad S''_\mathcal{V} + S''_\mathcal{A} < 0\,. \tag{18}$$

The first condition ensures the increase of entropy while the second guarantees the system will reach thermodynamic equilibrium. In the case of a FLRW universe with no spatial curvature, the apparent horizon has the form $l_h(t) = H^{-1}(t)$, and surface entropy [5]:

$$S_\mathcal{A} = \frac{A}{4G} = \frac{\pi l_h^2}{G} = \frac{\pi}{GH^2}\,. \tag{19}$$

A. Current universe

By simply substituting (13) in (19) we find:

$$S_\mathcal{A}(a) = \frac{\pi}{GH_0^2\left[1 + \frac{\Omega_m^0}{1-\nu}(a^{-3(1-\nu)} - 1)\right]}\,. \tag{20}$$

To determine $S_\mathcal{V}$ we must revisit equation (10). Since $\rho_m = mn$, if we assume the particle mass remains constant during the cosmic history $m = m_0$, then the number density cannot follow the standard dilution law ($n \sim a^{-3}$) but rather an "anomalous" one:

$$n(a) = n_0 a^{-3(1-\nu)}\,. \tag{21}$$

This equation, given $\nu > 0$, implies particle production through vacuum decay is still active even in our current universe. In our present universe the entropy will be dominated by the matter contribution, thus:

$$S_\mathcal{V} = n\sigma V_h = \frac{4\pi}{3} l_h^3 n\sigma = \frac{4\pi\sigma}{3}\frac{n(a)}{H^3(a)}\,, \tag{22}$$

with σ being the specific particle entropy. For the sake of simplicity we can set $\sigma = k_B$ (keeping in mind that Boltzmann's constant is the unit of entropy), and in natural units $k_B = 1$. Using (21) and (13) we find:

$$S_\mathcal{V}(a) = \frac{4\pi n_0 a^{-3(1-\nu)}}{3H_0^3\left[1 + \frac{\Omega_m^0}{1-\nu}(a^{-3(1-\nu)} - 1)\right]^{3/2}}\,. \tag{23}$$

Notice that $S_\mathcal{V}(a \to \infty) = 0$, i.e. the volume entropy is actually decreasing with the expansion, and therefore cannot fulfil the GSL by itself. Its numerical significance, however, is negligible in front of $S_\mathcal{A}$, since:

$$\frac{n_0}{H_0^3} \sim \frac{\rho_m^0}{mH_0^3} \sim \frac{1}{GmH_0} \sim \left(\frac{H_0}{m}\right)\frac{1}{GH_0^2} \ll \frac{1}{GH_0^2}\,, \tag{24}$$

where we used $H_0^2 \sim G\rho_m^0$, from (6), and in the last inequality we considered $H_0 \sim 10^{-42}$GeV which is much smaller than any known particle mass. Hence it seems $S_\mathcal{A}$ should be able to continue increasing the total entropy and bring the universe to thermodynamic equilibrium. Differentiating equations (20) and (23) one indeed finds (see [5] for details): $S'_{total}(a \to \infty) > 0$ and $S''_{total}(a \to \infty) < 0$. Thus the GSL is fulfilled despite the decreasing $S_\mathcal{V}$. In Figure 2 we can observe these results. See also [13] for a comparison between the GSL predictions and observations, with positive conclusions.

B. Early universe

To study the entropy evolution in the early universe we must rewrite (15) using the re-scaled parameters we introduced in the previous section:

$$H(\hat{a}) = \frac{\hat{H}_I}{\sqrt{1 + \hat{a}^{4(1-\nu)}}}\,. \tag{25}$$

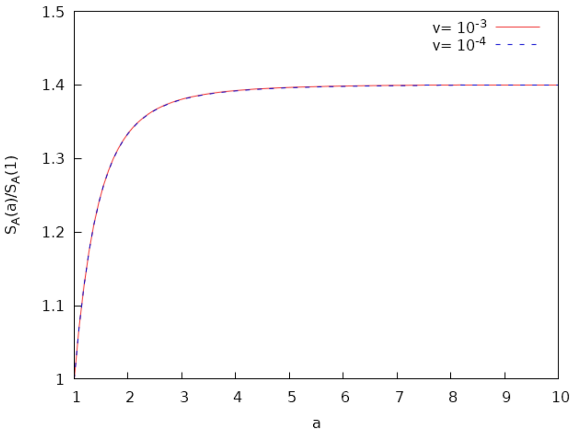

FIG. 2: Evolution of $S_{\mathcal{A}}(a)$ for $\nu = 10^{-3}$ and $\nu = 10^{-4}$, normalised to the current value, from the present universe into the future. Recall that $S_{\mathcal{V}}$ is numerically insignificant.

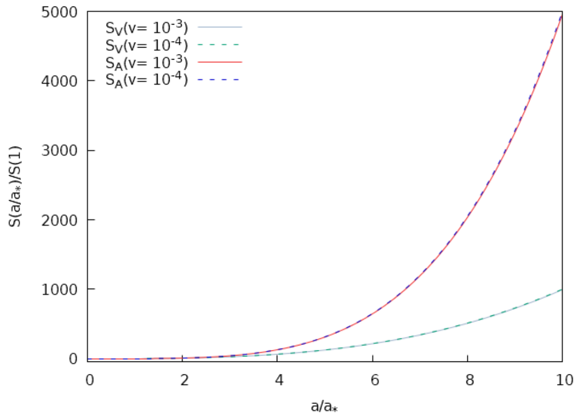

FIG. 3: Evolution of $S_{\mathcal{A}}(\hat{a})$ and $S_{\mathcal{V}}(\hat{a})$, for $\nu = 10^{-3}$ and $\nu = 10^{-4}$, normalised to the value at the equality point, in the early inflationary universe. There is an unrestricted growth of entropy due to $S''_{total}(\hat{a}) > 0$.

Substituting in (20):

$$S_{\mathcal{A}}(\hat{a}) = \frac{\pi(1+\hat{a}^{4(1-\nu)})}{G\hat{H}_I^2} \,. \tag{26}$$

The entropy inside the horizon, on the other hand, will be determined by the radiation contribution $S_r = \frac{p_r+\rho_r}{T_r}V_h^3 = \frac{4}{3}\frac{\rho_r}{T_r}V_h^3$, where the radiation temperature T_r is related to its density via: $\rho_r = \pi^2 g_* T_r^4/30$, with g_* being a factor counting the number of effectively massless degrees of freedom [5]. Using (16) and (25) we find:

$$\begin{aligned} S_{\mathcal{V}}(\hat{a}) &= \frac{4}{3}\left(\frac{\pi^2 g_*}{30}\right)^{1/4}\rho_r^{3/4}(\hat{a})V_h^3 = \\ &= \frac{8\pi^3}{135}g_*\left(\frac{\hat{T}_I}{\hat{H}_I}\right)^3(1-\nu)^{3/4}\hat{a}^{3(1-\nu)} \,. \end{aligned} \tag{27}$$

In the last expression we have conveniently defined $\hat{T}_I = \left(\frac{30\hat{\rho}_I}{\pi^2 g_*}\right)^{1/4}$. Notice that $S_{\mathcal{A}} \sim \hat{a}^{4(1-\nu)}$ and $S_{\mathcal{V}} \sim \hat{a}^{3(1-\nu)}$ and thus we have a huge entropy production caused by inflation. In the next section we shall see this result might provide a solution to the ΛCDM entropy problem. Both the surface and volume entropy are represented in Figure 3.

Differentiating equations (26) and (27) it can be seen that (see [5] for details): $S'_{total}(\hat{a}) > 0$ and $S''_{total}(\hat{a}) > 0$. That is, the GSL is violated in the early universe. We shouldn't be too alarmed by this result as the cosmic evolution will eventually slow down the growth of entropy and lead to equilibrium in the final de Sitter epoch, as we have seen in the previous section. In fact, rewriting (20):

$$S_{\mathcal{A}}(a) = \frac{\pi(1-\nu)}{H_0^2[\Omega_m^0 a^{-3(1-\nu)} + \Omega_\Lambda^0 - \nu]} \,, \tag{28}$$

we see that as the $\sim a^{-3(1-\nu)}$ term disappears with the expansion, the denominator becomes $H_0^2(\Omega_\Lambda^0 - \nu) = c_0$ and the entropy is dominated by the cosmological constant. Therefore, the presence of c_0, which is negligible in the early universe but becomes the dominant term in the current universe, is responsible for decelerating the entropy production and setting the universe on the path towards thermodynamic equilibrium and the fulfillment of the GSL.

IV. HORIZON AND ENTROPY

Next we will discuss the potential of the RVMs to solve the horizon and entropy problems afflicting the ΛCDM model, namely the apparent inability of the latter to explain neither how the universe evolved from a low entropy value to a large one (giving rise to the arrow of time) [6], nor the homogeneity of the universe, as it predicts a decreasing horizon moving into the past. Let us first define the concept of particle horizon, the visible region for a co-moving observer at a given time [5]:

$$l_p(a) = a\int_{t_i}^{t}\frac{dt}{a(t)} = a\int_0^a \frac{da'}{a'^2 H(a')} \,. \tag{29}$$

In the ΛCDM model the integral tends to 0 for $a \to 0$ both in the radiation and matter epochs, with $H \sim a^{-2}$ and $H \sim a^{-3/2}$ respectively. This means that the observers become isolated in the past, or in other words that the number of causally disconnected regions increases. In the context of the RVMs, however, we can substitute (25) in the previous equation to find:

$$\frac{l_p(a)}{a} = \lim_{\epsilon\to 0}\frac{1}{\hat{H}_I}\int_\epsilon^a \frac{da'}{a'^2}\sqrt{1+\hat{a}'^{4(1-\nu)}} \to \infty \,. \tag{30}$$

Therefore, the particle horizon increases faster than the scale factor and becomes infinite (i.e. there is no horizon at all) in the primitive universe. In conclusion, the RVMs seem to have no horizon problem.

In the case of the entropy problem, in the context of the ΛCDM model the present value enclosed in our horizon is [6]:

$$S_0 = \frac{2\pi^2}{45} g_{*,0} T_{r0}^3 H_0^{-3} \sim 10^{88}, \tag{31}$$

with T_{r0} being the current temperature of the CMB. This result evolves in time as $(TH^{-1})^3$, and for an adiabatic evolution $T \sim a^{-1}$, so the entropy evolves as $S \sim a^3$ and $S \sim a^{3/2}$ for the radiation and matter epochs, respectively [6]. At the time of recombination ($a_{rec} \sim 10^{-3}$), for example, the entropy would be $S_{rec} \sim 10^{83}$, and at the primordial nucleosynthesis epoch ($a_{BBN} \sim 10^{-9}$) it would be $S_{BBN} \sim 10^{61}$. This would imply that our present horizon contains $\sim 10^5$ regions which were causally disconnected at recombination, and $\sim 10^{27}$ at primordial nucleosynthesis. We have stumbled upon a re-phrased horizon problem. In the context of the RVMs, however, there is a huge entropy production driven by the $\mathcal{O}(H^4)$ term during the early inflationary period, which is then transferred to the radiation epoch and preserved throughout the cosmic evolution, accounting for the enormous value at present (31). In summary, the RVMs might also provide a possible solution to the entropy problem.

V. CONCLUSIONS

- In this work we have studied some aspects of the Running Vacuum Models (RVMs), we have solved its cosmological equations and studied some of the thermodynamic implications from the early universe until the present time. We have explored also its asymptotic regime, in particular we have considered if the RVMs universe tends eventually to a state of thermodynamic equilibrium. From the confrontation of the RVMs predictions with the observational data in the present universe, it has been suggested that the vacuum energy is not necessarily a strict constant, but may be mildly dynamical due to the $\sim \nu H^2$ term [6, 8, 9]. The early universe, however, is driven by an $\mathcal{O}(H^4)$ term, which causes a short period of rapid inflation, with an exponential increase of the scale factor. During inflation, the vacuum energy decays into relativistic particles, increasing the radiation density ρ_r, until the vacuum-radiation equality point is reached, after which they both start decaying (Figure 1). Nevertheless, during this post-inflationary decay $|\rho_\Lambda/\rho_r| \ll 1$, and therefore the main features of the ΛCDM radiation epoch are preserved, such as the Big Bang Nucleosynthesis.

- We introduced the Generalised Second Law (GSL), which studies the surface entropy of the apparent horizon as well as the entropy contained inside it; and tried to determine whether or not the RVMs comply with it. We found that it is fulfilled in the present universe, i.e. the current universe is on the path towards thermodynamic equilibrium (Figure 2). On the other hand, it was not satisfied in the early universe as there is an unlimited entropy growth (Figure 3). We saw, however, that this apparent violation of the GSL was not worrying at all, as the presence of the constant term c_0 in $\rho_\Lambda(H)$, which was negligible in the initial period of the cosmic evolution but became the dominant term in the late universe, would eventually slow down the entropy growth until $S''_{total} < 0$, setting off towards equilibrium.

- We determined that this enormous entropy production during inflation, which is preserved throughout the entire cosmic history up until the present, may be able to provide a solution to the ΛCDM entropy problem. The RVMs might also solve the horizon problem, as they have no particle horizon at all in the primitive universe and hence there are no causality issues.

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