Pricing methods for barrier options

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The main goal of this project is to better understand barrier options and try to propose some methods to price them. We start with a review of the financial concepts necessary to develop our methods, where we explain what are derivates, options and the core of our work, barrier options. We also review the classical methods to price options which are the base for the methodologies that we propose. The two methods that we suggest to price barrier options and observe some of their properties are: a quantum mechanical approach using the path integral technique and a modification of the Monte Carlo simulation. For the first one we have considered the stock price as a free particle moving in a space bounded by two barriers and performing trajectories starting at the initial stock price and finishing at maturity time. We have developed the path integral formulation to find the probability of a certain trajectory and we have found that it depends on the inverse of the exponential of a quantity that we call action, in analogy to the path integral, which depends on the followed path by the stock price. The procedure proposed is to start from a completely random path and evolve it in order to reduce the action, thus increasing the probability of such path. We have founded that starting from a random path arising from a Gaussian distribution gives better results than starting from a uniform distribution when we compare with the classical methods. The reason is that the classical methods make the hypothesis that the dynamics of the stock price is based on the Brownian motion, which is Gaussian distributed. We call probabilistic Monte Carlo the other method because we let the stock price penetrate the barriers with some probability, with this methodology we try to observe the role of the barriers in options and the role of the drift in the Brownian motion governing the stock price. We have found that relaxing a bit the barriers is enough to recover an option without them. The reason is that, how far the stock price arrive is limited by the variance of the Brownian motion, then, if the barriers are far enough, is the same as not having them. Finally, we observe that the drift makes the stock price prone to rise, this makes the upper barrier more sensitive to changes than the lower one.

I. INTRODUCTION: DERIVATIVES AND OPTIONS

A derivative is a financial instrument which establishes a contract between two parties and its value depends on an underlying asset. In the last years derivatives have taken more and more relevance in financial markets, everyday a lot of derivatives are traded both on exchanges and over-the-counter (OTC). In OTC market the derivatives are traded directly between two parties, outside of organized markets (exchanges) . The main use of derivatives is to manage risk, for example, as we will see in this work, they are useful to hedge our portfolios. We understand hedge as an investment to cover our position in another investment, to offset potential losses, in this way we reduce the risk of the operations we are performing. A very important topic in quantitative finance is the problem of value these financial instruments, a very common way to proceed is to price derivatives in a way that we cannot make profit trading them without taking some risk. To understand how we can do this we need two key concepts that we will explain later, arbitrage-free and risk neutral valuation. There are a lot of different types of derivative such futures, forwards or options, we are going to focus on the last one.

Options are contracts that give the holder the right to exercise or not the contract, the issuer has to accept the decision of the holder. There are options for any type of underlying, we find options for stocks, bonds, commodities, currencies... There are two types of options, calls and puts. Call options gives the holder the right to buy the underlying asset at a fixed price while put options gives the right to sell it. This fixed price of buying or selling is known as the strike price K. The contracts are available for a certain time, they expire at a date known as maturity T. Depending on when we can exercise an option, there exist different style options, the most common are European and American. European options can only be exercised at maturity while American can be exercised at any time up to maturity. When an option expires we may essentially receive an amount of money or not, depending on what is stipulated on the contract, this is known as the payoff. The payoff can be so complicated as we want and this is the reason of the huge variety of existent options.

The most basic options are what we call vanilla, the payoff of these contracts is based only on the value of the underlying at maturity S_T, for call c and put p options we have,

$$g_c = \max(S_T - K, 0) \equiv (S - K)^+,$$
$$g_p = \max(K - S_T, 0) \equiv (K - S)^+. \quad (1)$$

As we said, we can complicate options as we want, options that are not vanilla are called exotic. A kind of exotic options are path-dependent, for these type the payoff depends on the history of the underlying asset. One of the most known path dependent options are barrier options, the most common are knock-out barrier options. For these kind of options, there is one or two barriers and when the price of the asset goes through the barrier the contract becomes worthless. Depending on the position of the barrier we can have:

- Up and out: the price of the asset starts below the barrier level B_u, if it rises enough to overcome the barrier the payoff becomes zero.

- Down and out: the price of the asset starts over the barrier level B_l, if it drops enough to overcome the barrier the payoff becomes zero.

- Double knock-out: there are two barrier levels, B_l and B_u, the asset price starts between them, once it overcomes any barrier the payoff becomes zero.

There also exists knock-in options, the idea is the same as for knock-out but instead, the option has a payoff different from zero when the asset price goes through the barriers.

In our work we will focus on pricing double knock-out options when the underlying is a stock. We do not have an analytic expression for double barrier and we need numerical methods to price them. First we will review the classical and most famous methods to price options: the Black-Scholes model, the binomial tree and the Monte Carlo simulation. For these we will have to do some hypothesis on the dynamics of the asset price and the behaviour of the market. Then we will propose two methods to price these options, one based on the Monte Carlo simulation and the other based on a quantum mechanical approach using the path integral technique.

II. CLASSICAL METHODS FOR OPTION PRICING

Three of the most important methods for option pricing are the Black-Scholes (BS) model, the binomial tree and the Monte Carlo simulation (MC). These models make some assumptions on the market to derive a unique and fair price. There are two key concepts that are very related, the absence of arbitrage in the market and the risk neutral valuation:

- We say that there are arbitrage opportunities in the market when we can earn money without taking risks, when an inversion has no risk of losses.

- The risk neutral valuation assumes that investors do not take more risk in their investments with the expectation of obtaining a higher return. Therefore, the expected return for any investment is the risk-free interest rate r. We assume that the world is risk-neutral.

Mixing both assumptions, the absence of arbitrage opportunities guarantees that a risk-less investment earn the risk-free interest rate.

A. The Black-Scholes model

The most famous method for pricing options is the BS model which provides a partial differential equation (PDE) for the evolution of the price of an option $V(S,t)$. The derivation of the BS equation is based on the idea of hedging a portfolio composed by a risky and a risk-less assets, a stock and a bond. Before deriving the equation we have to made some important hypothesis:

- The stock price S follows a geometric Brownian motion (GBM). A GBM is defined by the following stochastic differential equation (SDE),

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (2)$$

where μ is the drift coefficient, σ the diffusion coefficient and $W(t)$ the Wiener process or the standard Brownian motion. We assume that the drift and the volatility are constant. The GBM describes the dynamics of $S(t)$ in a way that the logarithm of this quantity follows a Brownian motion with drift.

- The return is given by the risk-free interest rate, in this way the drift in (2) becomes $\mu = r$, then we are assuming that the world is risk-neutral.

We derive the BS equation starting from a portfolio $\Pi(S,t)$ which contains $\delta(S,t)$ shares of the stock $S(t)$ and $\phi(S,t)$ shares of the bond $B(t)$,

$$\Pi(S,t) = \delta(S,t)S(t) + \phi(S,t)B(t). \quad (3)$$

The evolution of the bond is deterministic, its value grows according to the risk-free interest rate,

$$B(t) = B(0)e^{rt}. \quad (4)$$

We replicate the portfolio setting $\Pi(S,t) = V(S,t)$, that is, the value of the portfolio is equal to the value of the option at any time. Therefore, its evolution has to be the same $d\Pi(S,t) = dV(S,t)$. If we define delta hedging as the variation of the option price with respect to the variation of the underlying asset, $\delta = \frac{\partial V}{\partial S}$, we obtain the following PDE,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} = rV. \tag{5}$$

This is the BS equation. What we have done is to make our portfolio risk-less using the option and because we have established a market without arbitrage opportunities, the return of the portfolio will be the risk-free interest rate.

The solution of equation (5) depends on the boundary and terminal conditions that we establish, corresponding to the particularities of the option. For vanilla European options the terminal conditions are the payoffs given by (1). With these final conditions we obtain the BS formula for the price of an European call and put vanilla options,

$$\begin{aligned} C(S,t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \\ P(S,t) &= Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \end{aligned} \tag{6}$$

where $N(x)$ is the standard normal cumulative distribution function with arguments,

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{7}$$

Here we can see that the price of an option depends on the following parameters: the stock price S, the strike price K, the risk-free interest rate r, the maturity T and the volatility σ.

Pricing exotic options solving the BS equation is harder than for vanilla ones, for barrier options we have to specify the boundary conditions, that is, a knock-out option becomes worthless if the stock price touches the barriers. We can specify this in the following way,

$$\begin{aligned} g_{up} &= (S-K)^+\theta(B_u - S), \\ g_{down} &= (K-S)^+\theta(S - B_l), \end{aligned} \tag{8}$$

where θ is the step function, equal to one when the argument is positive. There is an analytic expression for single barrier options, for double barrier we need computational methods.

B. The binomial tree

One effective numerical method to price options is the binomial tree, the idea consists in constructing a tree which represents all the possible values the stock price can take starting at S_0 and finishing at maturity time T. Then, because we know the price of the option at T we work backwards the tree in order to find the price of the option at any time for each possible value S_t until $t = 0$.

The binomial tree method is based on the same idea of the BS model, we have to hedge the same portfolio but in the discrete case. First we explain the one-step replicating portfolio. Consider that in one time step the stock price can either go up or down, $S_0 \longrightarrow S_1 \in \{S_+, S_-\}$, $S_- \le K < S_+$. Consider also that we have the portfolio defined in (3). To set a risk-less portfolio we have to find δ_0 and ϕ_0 in a way that the final value of the portfolio is equal to the value of the option at that moment, either the stock price has raised or dropped, $\Pi_1 = \max(S_1 - K, 0)$ for a call option. If we consider both cases, the stock price going up or down, we finally obtain,

$$\delta_0 = \frac{S_+ - K}{S_+ - S_-}, \qquad \phi_0 = -\frac{\delta_0 S_-}{B_0(1+r)}. \tag{9}$$

Inserting the corresponding amount of shares in (3) we finally obtain the price of the option at the initial time imposing $C_0 = \Pi_0$,

$$C = \frac{1}{1+r}\left[\tilde{p}(S_+ - K)^+ + \tilde{q}(S_- - K)^+\right], \tag{10}$$

where we have defined the risk-neutral probabilities,

$$\tilde{p} = \frac{S_0(1+r) - S_-}{S_+ - S_-}, \qquad \tilde{q} = 1 - \tilde{p}. \tag{11}$$

Under the risk-neutral probabilities the expected return of the portfolio is given by the risk-free interest rate. Equation (10) can be interpreted as the price of the option being the expected value of the payoff in a risk neutral world discounted at the risk free interest rate.

We can extend the idea of the one-step replicating portfolio to construct the binomial tree for many steps. We divide the life of an option in N steps of length $\Delta t = T/N$, at each time step the price can rise or drop proportionally to the factors u and d respectively, such that $d = 1/u$, is what we observe in FIG. 1. Once the tree has been constructed we start to go backwards on it computing the price of the option for each stock price using an expression based on (10). The price of the option at a given time-step V_n is the expected value of the option on the next step in a risk neutral world discounted at the risk-free interest rate,

$$V_n = \frac{1}{1+r}\left[\tilde{p}V_{n+1}^{+} + \tilde{q}V_{n+1}^{-}\right]. \tag{12}$$

We can introduce the volatility in the rise/drop factors in the following way, we establish $u = e^{\sigma\sqrt{\Delta t}}$, we recall $r \longrightarrow r\Delta t$ and approximate $1 + r\Delta t \simeq e^{r\Delta t}$. In the continuum limit $N \to \infty$, the binomial model converges to the Black-Scholes model. It is reasonable because we have made the same assumptions on the market and the stock price. Here S follows a random walk (at each time step it moves up or down), the discrete version of the Brownian motion.

The binomial tree method is useful for pricing barrier options, either we have only a barrier or two. For a knock-out option we proceed in the same way as if it was a vanilla, but when we are in a node located above the barrier in the case of up and out or below in the case of down and out, the corresponding option value is equal to zero.

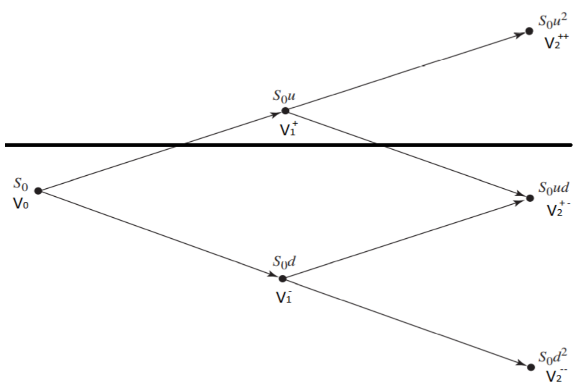

FIG. 1. The binomial tree, each node represents a possible value of the stock price in the corresponding time step. Starting at S_0, from each node, the stock price can rise proportional to u or drop proportional to d. For each node we can calculate the price of the option at that moment based on the possible option prices at the next time step, either S has risen or dropped. The value of the option at the last time step corresponds to the payoff. In the picture we can also observe an up and out barrier, for all the nodes above it we have $V_n = 0$.

C. The Monte Carlo simulation

The Monte Carlo simulation is based on the idea that we have developed for the binomial tree, that is, the price of an option is given by the expected value of the payoff in a risk-neutral world discounted at the risk-free interest rate, for European vanilla we have,

$$\begin{aligned} c &= e^{-rT}\ \mathbb{E}[g_c] = e^{-rT}\mathbb{E}\left[\max\left(S_T - K\right)\right], \\ p &= e^{-rT}\ \mathbb{E}[g_p] = e^{-rT}\mathbb{E}\left[\max\left(K - S_T\right)\right]. \end{aligned} \tag{13}$$

We basically have to simulate many different trajectories for the stock price, collect their final value S_T and calculate the corresponding payoff. We compute the mean of the obtained payoffs and we multiply it by e^{-rT}, this gives us the price of the option.

The simulation of the paths uses the fact that the dynamics of S is governed by (2), for simplicity we are going to work out this equation in order to simulate $X \equiv \ln S$ rather than S. First we use the Itô's lemma to differentiate X,

$$dX = \frac{\partial \ln S}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \ln S}{\partial S^2}dS^2. \tag{14}$$

The differential dS is given by (2), and because $\Delta t \to 0$, we can approximate $dS^2 \simeq \sigma^2 S^2 dt$, because $dW \sim dt^{1/2}$. We finally obtain,

$$dX = mdt + \sigma dW, \qquad m = \left(\mu - \frac{\sigma^2}{2}\right). \tag{15}$$

Here we can see clearly that the logarithm of the stock price follows a Brownian motion with drift m. We can also view it as a Langevin equation expressed as,

$$\dot{x} = \left(\mu - \frac{\sigma^2}{2}\right) + \sigma\eta, \tag{16}$$

where η is a white noise with the following properties,

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t'). \tag{17}$$

Now we define the log-return X as the logarithm of the relative price change,

$$X = \ln\left(\frac{S_t}{S_0}\right). \tag{18}$$

Using the return, the equation that we will use for the simulation of the paths is given by,

$$S_{i+1} = S_i \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\xi\right], \tag{19}$$

where $\xi \sim \mathcal{N}(0,1)$ is a random number taken from a Gaussian distribution to account for the Wiener process. We divide the time until maturity in N steps of length Δt and at each time step a new price is generated according to (19). To price barrier options we only have to take into account that the trajectories for which in some time step S has gone through the barrier, the payoff becomes zero, as the option automatically becomes worthless.

III. PATH INTEGRAL FORMULATION FOR PRICING OPTIONS

A. Theoretical framework

In the last years the problem of option pricing has been given a quantum mechanical interpretation and specially the path integral technique has become popular to address this kind of problems. Due to the randomness of the prices in the market, stock prices has been interpreted as quantum particles and the valuation of options as common quantum mechanical problems. The Black-Scholes model has been adapted to a quantum mechanical version where the option price satisfies an Schrödinger like equation, with the Hamiltonian governing its dynamics.

To develop the quantum mechanical version of the BS model we start doing a change of variables,

$$S = e^{x}, \qquad -\infty \leq x \leq +\infty. \tag{20}$$

This redefinition is commonly known as the return of the stock price as we have seen in (18), but here x represents a degree of freedom of the system, a quantum particle in one dimension describing the evolution of the stock price. From the BS equation (5) we can write an Schrödinger like equation in the following way,

$$\frac{\partial V}{\partial t} = \mathcal{H}_{BS} V, \tag{21}$$

with the Hamiltonian given by,

$$\mathcal{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r. \tag{22}$$

It is the same BS equation but the derivatives are taken with respect to x instead of S. Equation (21) is known as the Black-Scholes-Schrödinger (BSS) equation. Comparing with the original Schrödinger equation, the price of the option V represents the state of the system and the Hamiltonian $\mathcal{H}_{BS}$ drives the price of the option. To find the explicit form of V we can use the Feynman-Kac theorem which relates SDEs with PDEs. From (15) we know that x follows a Brownian motion with drift. The Feynman-Kac theorem applied to our case states that given a function,

$$V(t,x) = \mathbb{E}\left[e^{-r(T-t)} g(T,x)\right], \tag{23}$$

which represents the price of the option, being $g(T,x)$ the payoff with dependence on the stochastic process followed by the return. Then $V(t,x)$ satisfies the PDE (21) and the terminal condition,

$$V(T,x) = g(T,x). \tag{24}$$

So, by equation (21), the explicit form of the price of the option is given by,

$$V(t,x) = \int_{-\infty}^{\infty} dx' p(x,t|x',T) g(T,x'). \tag{25}$$

This expression gives us the price of the option in terms of the transition probability and the payoff. All we have developed can be seen from the point of view of the backward Kolmogorov equation, where we have a final condition (the payoff) and we need to evolve backward in time in order to calculate the price of the option. For the SDE (15) its backward Kolmogorov equation is given by the BSS equation (21).

Now, to determine the price of the option we need to calculate the transition probability, one way to proceed is to use the path integral formulation. In quantum mechanics is considered that the degree of freedom, when makes the transition from x to x', takes all the possible paths between the initial and the final state, in this way the path taken is indeterminate, we will assume the same for the stock price. The idea of the path integral is to consider the transition probability to be composed of the contribution of the probabilities of all the possible paths the particle can take, which can be expressed like,

$$p(x,t|x',T) = \sum_i P(\Gamma_i), \tag{26}$$

where $P(\Gamma_i)$ is the probability that the particle takes a given path Γ_i. To develop the path integral formulation of the BS model we start from its assumptions, explained in section II.A. The stock price follows a GBM and the rate of return is equal to the risk-free interest rate.

We start from equation (15), we discretize the time in N steps until maturity T, the length of the intervals is $\Delta = T/N$. The discretization of (15) leads to,

$$x_{k+1} - x_k = \left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta W_k, \tag{27}$$

where k represents the time step where we are.

Now we calculate the probability of a given trajectory $P(\Gamma)$ conditioned to start at x_0, where a trajectory is defined as the collection of points defining a path $\Gamma = \{x_0, x_1, ..., x_N\}$. Then the probability,

$$\begin{aligned} P(\Gamma) &= P\left(x_0, ..., x_N | x_0\right) \\ &= P\left(\Delta W_0, ..., \Delta W_{N-1} | x_0\right) \left|\frac{\partial \Delta W_i}{\partial x_j}\right|. \end{aligned} \tag{28}$$

We express the probability in terms of the Wiener process W_k because we know its statistical properties and due to this change of variables we have to add the jacobian determinant. Two properties are important for our development:

- W_k has independent increments ΔW_k.
- The increments ΔW_k are Gaussian, they follow a normal distribution $\Delta W_k \sim \mathcal{N}(0, \Delta t)$.

With this properties we can express the probability of the path as the product of the probabilities of the increments ΔW_k,

$$P(\Gamma) = \prod_{k=0}^{N-1} \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(\Delta W_k)^2}{2\Delta t}\right) \left|\frac{\partial \Delta W_i}{\partial x_j}\right|. \quad (29)$$

From equation (27) we can calculate the determinant of the jacobian,

$$\left[\frac{\partial \Delta W_i}{\partial x_j}\right| = \sigma^{-N}. \quad (30)$$

Using this result in (29) we finally obtain the expression,

$$P(\Gamma) = \left(\frac{1}{2\pi\Delta t\sigma^2}\right)^{N/2} e^{-\mathcal{S}[path]}. \quad (31)$$

Where we have defined the action $\mathcal{S}[path]$ in the following way,

$$\mathcal{S}[path] = \sum_{k=0}^{N-1} \frac{\Delta t}{2\sigma^2} \left[\frac{x_{k+1} - x_k}{\Delta t} - r + \frac{\sigma^2}{2}\right]^2. \quad (32)$$

This is the discrete version of the path integral, in the continuum limit $\Delta t \to 0$ the action is given by,

$$\mathcal{S}[path] = \int_0^T \frac{1}{2\sigma^2} \left[\dot{x} - r + \frac{\sigma^2}{2}\right]^2 dt. \quad (33)$$

Up to this moment we only have the probability of a given path. Our goal is to calculate the transition probability so we need to consider all the possible paths between the initial and final conditions. In the discrete we can use equation (26), for the continuous case we can use the following transformation,

$$\sum_i \longrightarrow \prod_{t=0}^{T} \int_{-\infty}^{\infty} dx(t) \equiv \int DX. \quad (34)$$

This can be seen that for each infinitesimal time step we integrate over all the space, it is a functional integral. Therefore, the transition probability in the continuous reads as follows,

$$p(x, t|x', T) = \int DX e^{\mathcal{S}[path]}. \quad (35)$$

All we have developed is valid for vanilla options but we can adapt this model to exotic options such as barrier ones. As we have explained, barrier options restrict the space where the price can move, when S go through the barrier the option becomes worthless. It is easy to see that we only have to change the domain of integration in (34) to price this kind of options, restricting the movement of the particle according to the constraints imposed by the contract. At each time step we integrate over all the possible values that the stock price can take. For a double barrier we have,

$$\prod_{t=0}^{T} \int_{B_l}^{B_u} dx(t). \quad (36)$$

B. Computational implementation of the path integral

The path integral formulation considers all the possible paths starting at the initial stock price $x_0 = \ln S_0$ and finishing at maturity T. The possible paths have different probabilities determined by the action $\mathcal{S}[path]$ as we can see in (31), which contains the information of the trajectory of the stock, then the paths with the highest probability will determine the price of the option while the contribution of the others can be neglected. Following this argument we propose a Metropolis-Hastings-like algorithm to price barrier options. The idea is to create completely random paths and evolve them in order to decrease the action and consequently increase the probability. In this way we will find the most likely trajectories followed by the stock.

First we create a completely random path starting at x_0, we discretize the time in N steps of length Δt and at each one we assign a random number from a uniform distribution with the constraints imposed by the option we are dealing with. For example, for a double knock out option with B_l the lower barrier and B_u the upper barrier, the distribution used will be,

$$x \sim U(x_{min} = \ln B_l, x_{max} = \ln B_u). \quad (37)$$

Then, the algorithm to find the most likely paths consists in, for each time step:

- Calculate the action of the given path $\mathcal{S}_i$.
- Propose a change $x \to x'$ at one time step.
- Calculate the new action of the trajectory $\mathcal{S}_f$.

- If $\mathcal{S}_f - \mathcal{S}_i < 0$ accept the change, otherwise reject it.

Repeat this algorithm until the number of changes stabilizes at a very low value. The action is calculated using the discrete equation (32).

The change we propose is to choose a number from a uniform distribution centered in x in an interval which will be reduced at each iteration n of the algorithm to increase the precision and find the best value for x'. If we define the parameters,

$$L_0 = \frac{x_{max} - x_{min}}{2}, \qquad L_n = L_0 \left(\frac{N-n}{N} \right). \tag{38}$$

The value of x' will be taken from a uniform distribution in the following way,

$$x' \sim U(\max{(x - L_n, x_{min})}, \min{(x + L_n, x_{max})}). \tag{39}$$

If the interval crosses the lower barrier, the lower limit of the distribution will be x_{min}. If the interval crosses the upper barrier, the upper limit of the distribution will be x_{max}.

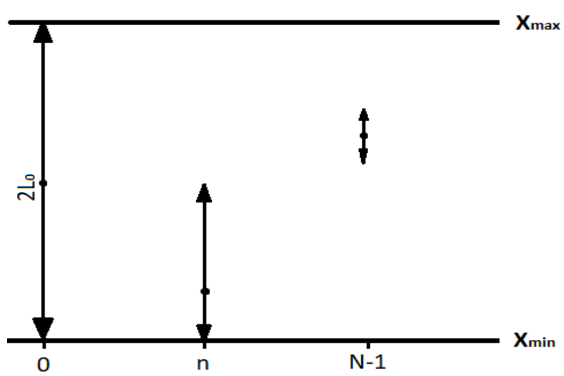

FIG. 2. Situation of how the uniform distribution changes as more iterations have been performed. The dot represents the value of the return x at some iteration, the arrows represent the range where we can propose a change of the price. We start covering all the space allowed $2L_0$, then we reduce a bit after each iteration. The limits of the distribution cannot surpass the barriers. The last iteration covers a tiny interval.

If we repeat this process we will obtain different paths and we can use their payoffs to calculate the value of the option using (13).

Once we have established the algorithm we have to test its behavior comparing with another method which we know that performs correctly. From the well known methods explained in section II, we think that the most natural way to proceed is to use the Monte Carlo simulation as a reference model. The reason is that it is based in the creation of different random paths and obtains the value of the option using the equation (13). The main difference with our model is that the Monte Carlo method uses the discretization of the SDE which governs the dynamics of the underlying asset to create directly the most likely paths, while our method tries to search these paths correcting completely random trajectories. The study will consist in fixing all the parameters except one that we will vary. The parameter that we are going to vary is the strike price K. Each option price is calculated over 200 paths and each path iterated 500 times, enough for the number of changes to stabilize near zero.

What we can observe firstly is that the path integral method gives a logic result, the price of the option decreases as the strike approaches the initial stock price, so the proposed method makes sense without going to details. Comparing with the Monte Carlo simulation we observe that the difference between prices increase as the distance between the barriers increase, we can observe it in FIG. 3.

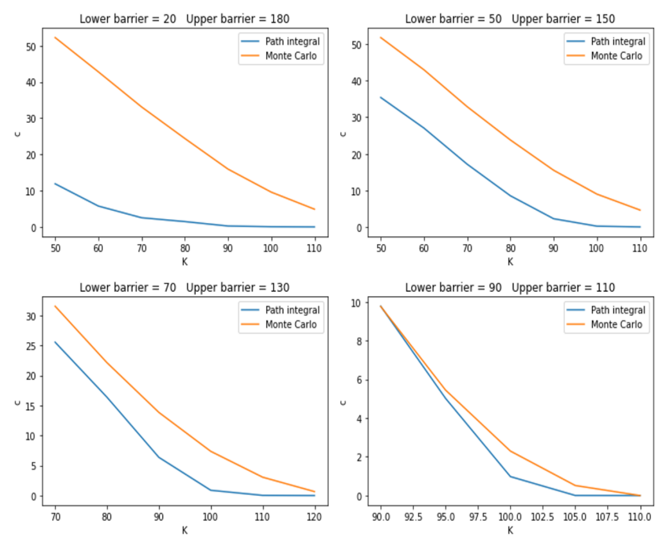

FIG. 3. Price of a double knock-out barrier call option for different strike prices K and barriers separation using the path integral algorithm starting from a uniform distributed path (blue) and the MC simulation (orange). The parameters of the option are: $S_0 = 100$, $T = 0.5$, $r = 0.1$, $N = 50$, $\sigma = 0.25$.

To explain this fact we can take a look at the paths created by both methods, in FIG. 4 we can see how they look. First we can observe that the paths created by our algorithm are smoother than those computed with the Monte Carlo method (MC), which points out that our trajectories are not following a geometric Brownian motion. If we look at what really affects the price of the option, the value of the stock at maturity S_T, we observe that for a geometric Brownian motion its values cover a range centered in S_0. This happens for the path integral but only when the barriers are close enough, if not the paths deviate downwards and cover a range centered in a value below S_0.

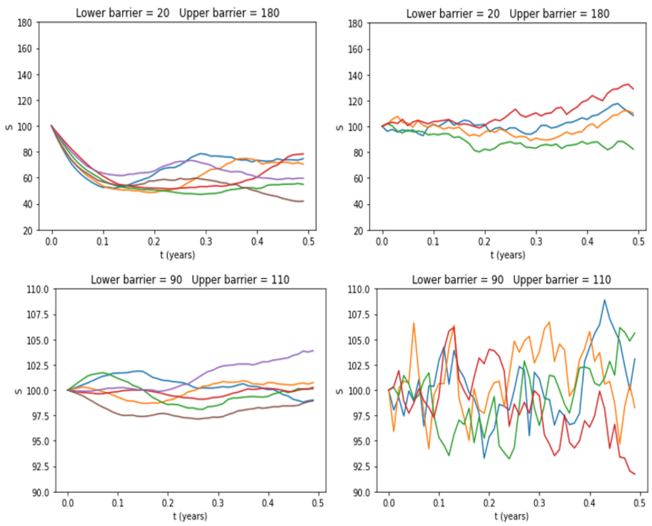

FIG. 4. Paths created by the path integral algorithm (left) and the Monte Carlo simulation (right) for two different barrier separation. The parameters are: $S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

First we try to understand why our method gives smoother trajectories. Remember that the goal of the algorithm is to minimize the action to increase the probability of the path. If we take a look at the expression of the action (32), we see that its dependency on the path is given by the difference of the returns at each time step, then the only way to decrease the action is reducing these differences and therefore make the trajectory smoother.

Now we want to understand why as we increase the distance between barriers the price computed moves away from which calculated with MC. Because the action depends on the sum of the differences between returns at each time step and not on their actual values we think that the problem is not in the algorithm, maybe the issue is related with the initial uniform distribution of the path, which is a very rough approximation. We know that the return follows a Brownian motion with drift which is characterized by a Gaussian distribution, $\mathcal{N}(X_0\ ,\ \sigma^2 t)$ if it starts at X_0, and is what we observe in the MC simulation in FIG. 4 , take a look at how the variance grows in time. Let's try this distribution for the initial path, for each time step we assign a random number taken from $X \sim \mathcal{N}(X_0\ ,\ \sigma^2 k\Delta t)$ with $k = 1, ..., N$ to the return. We expect that if the initial random path approaches the trajectories created by the MC simulation, with the variance growing in time and centered in S_0, the results will be better. In FIG. 5 we can see the prices computed by the path integral method but now starting from a Gaussian distributed path.

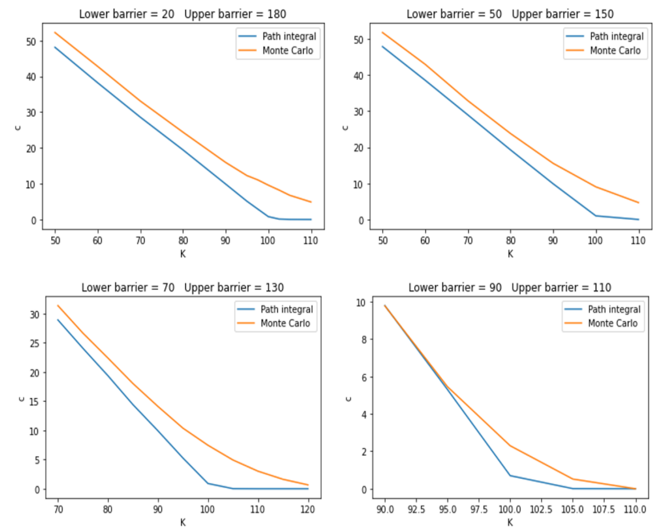

FIG. 5. Price of a double knock-out barrier call option for different strike prices K and barriers separation using the path integral algorithm starting from a Gaussian distributed path (blue) and the MC simulation (orange). The parameters of the option are: $S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

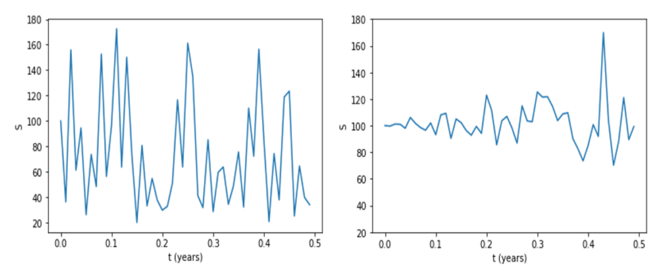

FIG. 6. Initial random paths for the path integral algorithm taken from a uniform distribution (left) and from a Gaussian distribution (right). The parameters of the option are: $S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

If we compare FIG. 5 with FIG. 3 we can see how the results have improved, in particular for the situation where the barriers are far away from each other. When the strike is far from S_0 the price computed is really close to the MC method but when K approaches it the difference increases. However we have seen how a little change in the initial path is enough for improve our algorithm.

Now we are going to take a look at how fast we reach the stabilization of the path for different barrier separations. What we first observe in FIG. 7 is that for all cases the number of changes necessary to stabilize the path is lower starting from a random Gaussian distributed path

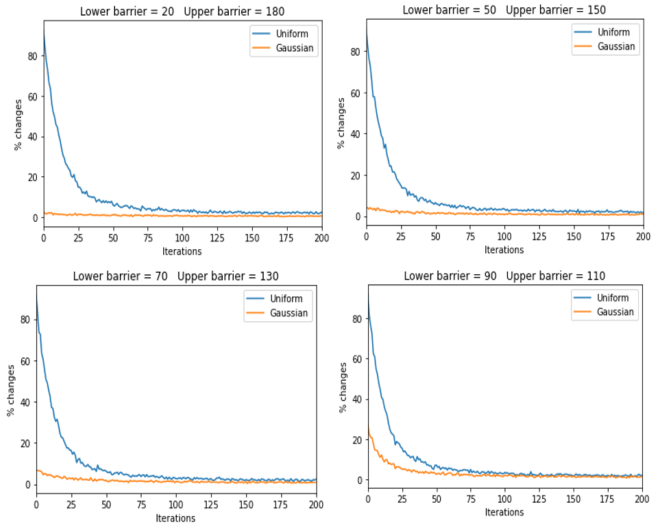

FIG. 7. Number of changes done with respect the total possible changes (N) in percentage as a function of the number of iterations performed. In blue starting from a random uniform distributed path and in orange starting from a random Gaussian distributed path. The parameters of the option are: $S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

than departing from a uniform one. This is something that we could expect as the Gaussian resembles more the paths that the stock price follows, compare FIG. 6 with the right panel of FIG. 4. Is easier to go from the right panel of FIG. 6 to the right panel of FIG. 4 than from the left panel of FIG. 6. Another thing that we notice is, when the difference between barriers is big, almost no changes are needed, that is, the random path created is really close to those the stock price follows. Otherwise, when the distance between barriers decrease we need to perform some changes. We can observe this comparing another time FIG. 4 and FIG. 6, the initial random Gaussian path approaches more the MC simulated path when the distance between barriers is big rather than when it is small, where the growing variance of S is cut by the barriers.

IV. PROBABILISTIC MONTE CARLO

Now we want to see how the price of the option varies as we relax the barrier constraints. To do it we take the Monte Carlo method and we modify it. In the original method, at each time step the stock price evolves according to the expression (2), if the proposed price have not gone through the barrier we accept it, otherwise we reject it, in this way only the paths that fulfills the constraints are used to price the option. To see the effect of the barriers in the option price we can relax this condition and allow the possibility to penetrate the barrier. In this modified MC the acceptation of the price is probabilistic and we fix a percentage of allowed penetration that tells us until which value the price can arrive. When we are away from the barrier we always accept the price, as we approach the threshold value the probability of acceptation starts to decrease until 0 at the limit. The probability that we will use for the acceptation is the survival probability. Before developing the algorithm that we will use we take a look at the theoretical concepts we need.

Consider the log-return of the stock price as a particle following a Brownian motion in one dimension with two absorbing barriers representing the constraints of the option. To find the probability distribution function (PDF) for the particle we have to solve the Fokker-Planck (FP) equation with the corresponding initial and boundary conditions,

$$\frac{\partial p(x,t|x_0)}{\partial t} = \left(\frac{\sigma^2}{2} - r\right)\frac{\partial p(x,t|x_0)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t|x_0)}{\partial x^2}, \tag{40}$$

$$p(x, t = 0|x_0) = \delta(x - x_0), \tag{41}$$

$$p(x = x_{min}, t|x_0) = 0 \qquad p(x = x_{max}, t|x_0) = 0. \tag{42}$$

We are interested in the survival probability $\mathbb{S}(t|x_0)$, which tells us the probability that the particle (the return of S) has not penetrated the barriers up to time t. We have to treat both barriers separately, for the upper barrier we will consider a case in which a particle can move in one dimensional space with an absorbing barrier located at x_{max} and starting below it $x_0 < x_{max}$. For the lower barrier the same but the barrier is located at x_{min} and the particle starts moving above it at $x_0 > x_{min}$. For each case we define the survival probability as,

$$\begin{aligned}\mathbb{S}_{up}(t|x_0) &= \int_{-\infty}^{x_{max}} p(x,t|x_0)dx \\ \mathbb{S}_{down}(t|x_0) &= \int_{x_{min}}^{\infty} p(x,t|x_0)dx\end{aligned} \tag{43}$$

The analytic expression for these quantities for an absorbing barrier located at $x = 0$ and initial condition $\delta(x - x_0)$ is given by,

$$\mathbb{S}(t|x_0) = N\left(\frac{mt + x_0}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2x_0 m}{\sigma^2}\right) N\left(\frac{mt - x_0}{\sigma\sqrt{t}}\right). \tag{44}$$

We can find this expression and its development in Ref. [1] , we have adapted it to our boundary conditions and we obtained,

$$\mathbb{S}_{up}(t|x_0) = N\left(\frac{x_{max} - x_0 - mt}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2(x_{max} - x_0)m}{\sigma^2}\right) N\left(\frac{x_0 - mt - x_{max}}{\sigma\sqrt{t}}\right). \quad (45)$$

For the upper barrier we have made a translation and changed the sign of the drift because in our case the barrier is located above the initial level. For the lower barrier only a translation is necessary,

$$\mathbb{S}_{down}(t|x_0) = N\left(\frac{mt + x_0 - x_{min}}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2(x_0 - x_{min})m}{\sigma^2}\right) N\left(\frac{mt - x_0 + x_{min}}{\sigma\sqrt{t}}\right). \quad (46)$$

In a rough approximation, we can consider that both events are independent and take their product as the probability of remaining between the barriers up to time t,

$$\mathbb{S}(t|x_0) = \mathbb{S}_{up}(t|x_0) \cdot \mathbb{S}_{down}(t|x_0). \quad (47)$$

With this theoretical framework we can develop an algorithm to test the effect of the barrier as we have commented. For simplicity we will neglect the drift term, we can justify it taking a look at the equation that governs the dynamics of the return,

$$\Delta x = m\Delta t + \sigma\Delta W, \qquad m = r - \frac{\sigma^2}{2}, \quad (48)$$

where $\Delta W \sim (0, \Delta t)$ is the standard Brownian motion. For the typical values that we will use for the simulations we have $\frac{\mu\Delta t}{\sigma|\Delta W|} << 1$. Then we can use the standard Brownian motion (diffusion) for our simulations. The survival probability that we will use comes from solving the FP equation (40) without the drift term and integrating the corresponding PDF for each case. This results in,

$$\mathbb{S}_{up}(t|x_0) = \operatorname{erf}\left(\frac{x_{max} - x_0}{\sigma\sqrt{2t}}\right), \qquad \mathbb{S}_{down}(t|x_0) = \operatorname{erf}\left(\frac{x_0 - x_{min}}{\sigma\sqrt{2t}}\right), \quad (49)$$

being $\operatorname{erf}(x)$ the error function. Now we can develop the algorithm that we are going to use, we modify the MC method in the following way:

- Start a path from S_0.
- Propose a new stock price according to the equation (19) at each time step.
- Accept the new price with probability: $p = p_{up} \cdot p_{down}$
- Collect where the path finishes and compute the payoff.

We generate 10000 sample paths and calculate the mean of the payoff, using equation (13) we obtain the price of the option. The probabilities that we will use are the following,

$$p_{up} = \operatorname{erf}\left(\frac{\ln\left[B_u \cdot (1+\alpha)\right] - \ln S_k}{\sigma\sqrt{2\Delta t}}\right), \qquad p_{down} = \operatorname{erf}\left(\frac{\ln S_k - \ln\left[B_l \cdot (1-\alpha)\right]}{\sigma\sqrt{2\Delta t}}\right). \quad (50)$$

The parameter α accounts for the quantity that we let the price to penetrate into the barrier, in this way, if the upper barrier is located at $B_u = 110$ and the lower at $B_l = 90$ and we let penetrate the barrier a 10%, that is $\alpha = 0.1$, then the absorbing barriers are located at $B_u^* = 121$ and the lower at $B_l^* = 81$. In FIG. 8 we can see how the probability of remaining between the barriers given by the product of the expressions in (50) evolve as the allowed penetration increases. For no penetration the probability decays to zero just at the given barriers $B_l = 90$ and $B_u = 110$, as we let penetrate more we see that the probability of acceptation broadens away from the original barriers.

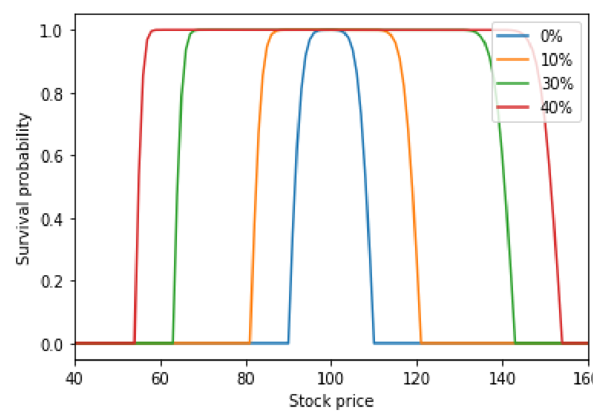

FIG. 8. Survival probabilities calculated as the product of the expressions in (15) for different S_k and for different allowed penetration values α into the barriers. The parameters used are: $B_l = 90,\ B_u = 110,\ T = 0.5,\ \ N = 50,\ \sigma = 0.25$.

We calculate the option's price for different strike values and for different allowed penetrations in the barrier, we can see the obtained results in FIG. 9. As we could expect, the computed prices differ more from the original MC as we allow the return to penetrate more in the barrier. Obviously, as we relax the constraints more and more, the paths created by the algorithm will differ more from the original, there is more space to explore despite as we approach barriers the probability of being there decreases.

Another observation is that the deviation is upwards, as we let the return penetrate more in the barrier the price of the option increases. Because we are treating a call, this implies that the paths are reaching higher values each time. We can observe this in FIG. 10, for the original MC most of the paths finish at a value centered between the barriers, but as we allow more penetration, the paths tend to move upwards and most of them finish at a higher value than $S_T = 100$. When it is allowed, the return is prone to explore the space over the upper barrier, this is caused by having a positive drift in the Brownian motion. The drift parameter gives an idea of the growth of the stock price, we can see it better taking a look at the expected value of the stock price at a given time t,

$$\mathbb{E}[S_t] = S_0 e^{\mu t}. \tag{51}$$

In this equation we can see that for a $\mu > 0$, S_t will grow in time and this effect is what we observe in FIG. 9 and FIG. 10.

We also observe saturation from 30-40% on, this is because the paths can explore a limited space until maturity defined by their drift and volatility. In the saturation limit we expect to recover the price for an option without barriers, a vanilla option, we will explore this now.

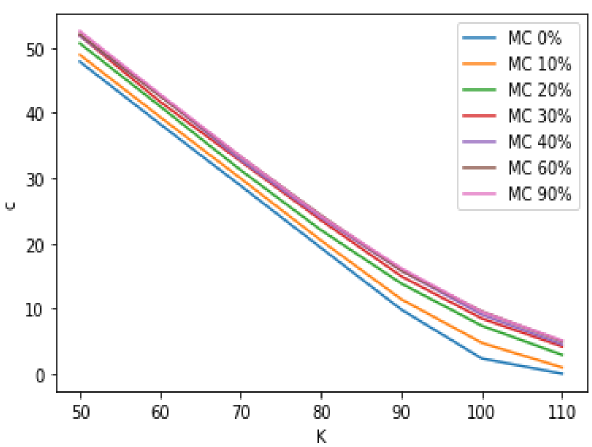

FIG. 9. Call option prices for different strikes and penetrations α calculated using the modified MC algorithm. The parameters of the option are: $B_l = 90$, $B_u = 110$, $S_0 = 100$, $T = 0.5$, $r = 0.1$, $N = 50$, $\sigma = 0.25$.

We ask ourselves at which point the effect of the barriers disappear and recover the result for a vanilla option given by the BS formula (6). To do it we compare the results of FIG. 9 with the results obtained for an option without barriers, we can see the comparison in FIG. 11. As we could expect the relaxation of the constraints lead to the BS result for a vanilla option.

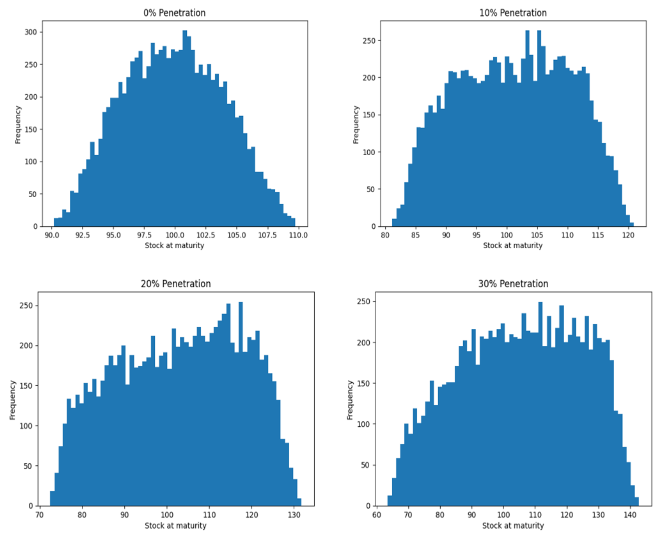

FIG. 10. Histograms for the frequency of different stock prices at maturity S_T for different α values. The parameters of the option are: $B_l = 90$, $B_u = 110$, $S_0 = 100$, $T = 0.5$, $r = 0.1$, $N = 50$, $\sigma = 0.25$.

Allowing the price to penetrate a 40 % in the barriers is enough to suppress their effect. If we look at FIG. 8 we observe that for 40 % the price is allowed to explore approximately a space limited by $60 < S < 140$ with 100 % probability and if we look at the top right panel of FIG. 4 we see that starting at $S_0 = 100$ the price explore more or less the space limited by $80 < S < 140$, so it seems reasonable that for 40 % the price converges to the vanilla one.

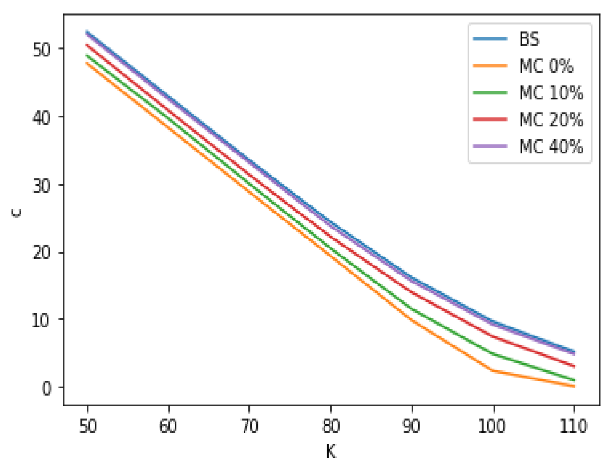

FIG. 11. Call option prices for different strikes and penetrations α calculated using the modified MC algorithm compared with the prices given by the Black-Shcoles formula (blue). The parameters of the option are: $B_l = 90$, $B_u = 110$, $S_0 = 100$, $T = 0.5$, $r = 0.1$, $N = 50$, $\sigma = 0.25$.

Now we are going to test what happens when we relax only a barrier and let the other fixed, that is breaking the symmetry between barriers. Because we know that the price tends to increase due to what we have commented about the role of the drift (51), we expect that the effect of relaxing the lower barrier is less important than affecting the upper barrier. Take a look at the results in FIG. 12 and FIG. 13.

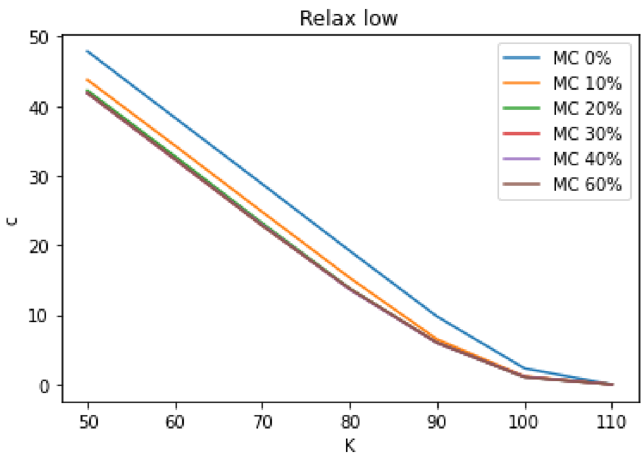

FIG. 12. Call option prices for different strikes and penetrations α calculated using the modified MC algorithm. This picture shows the results when the upper barrier is fixed at 110 and the lower is relaxed. The parameters of the option are: $B_l = 90,\ B_u = 110,\ S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

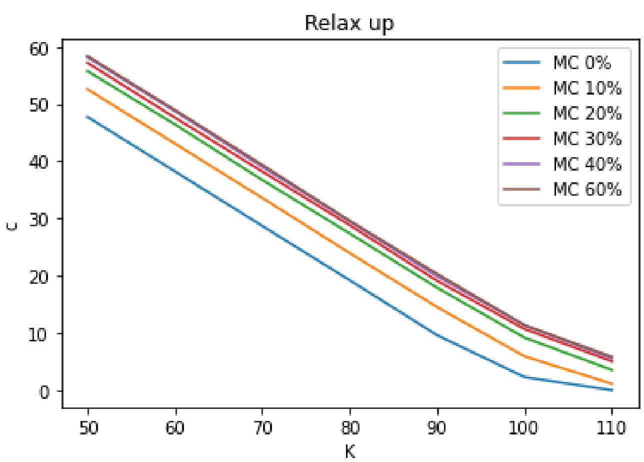

FIG. 13. Call option prices for different strikes and penetrations α calculated using the modified MC algorithm. This picture shows the results when the lower barrier is fixed at 90 and the upper is relaxed. The parameters of the option are: $B_l = 90,\ B_u = 110,\ S_0 = 100,\ T = 0.5,\ r = 0.1,\ N = 50,\ \sigma = 0.25$.

When we let penetrate the lower barrier but not the upper, what we first observe is that, opposite to what we have seen in FIG. 9 the prices obtained deviate downwards with respect to the original MC. This is reasonable as we break the symmetry between the barriers and despite the effect of the drift, the price can explore more space downwards than moving upwards because there is a rigid boundary that we cannot penetrate. However, we see that this phenomenon saturates at 20 %, we can increase more the allowed penetration but the price will not explore more space downwards, this effect is due to the drift which tends to push the price upwards. On the other hand, when we fix the lower barrier and we let penetrate the upper, the pattern obtained is similar to what we had obtained at FIG. 6. The main difference is that the upwards deviation with respect the original MC is bigger, here the effect of the drift is enhanced by the effect of having the lower barrier fixed, the price can reach higher values. Despite, there is saturation approximately at 30 %, because the stock price cannot grow indefinitely, remember that it grows proportionally to volatility and time.

V. CONCLUSIONS

Now we are going to review the main conclusions extracted from the proposed numerical methods. In the path integral implementation we have treated the return of the stock price as a quantum particle and modelled the problem of option pricing assuming that this particle performs a transition from a known initial state to a final unknown state. The key quantity to compute was the transition probability and we have used the path integral technique to obtain the probabilities of the paths. We have implemented a numerical method in which we depart from a random path based on a given distribution and let it evolve in order to find the most probable paths. From the computational implementation proposed we conclude:

- Starting from a path based on a uniform distribution, the calculation of the option price becomes worse as we increase the distance between barriers, the price calculated is always below the real one. The reason is that the paths created using the path integral do not recreate the properties exhibited by the GBM, fundamentally, we do not observe the variance of the path growing proportionally to σt, the paths created with our algorithm cannot reach such high values of S. For this reason, when the separation between barriers is little the result is correct, because our paths are closer to those created with the MC.

- When we modify the initial distribution of the path, using a Gaussian instead of a uniform, the path from which we start approaches more the reality, the Brownian motion followed by the return, consequently the results improve. The number of changes is negligible when starting from a Gaussian random path, meaning that the random initial path is very close to those created with the SDE (2).

The probabilistic MC has been used mainly to observe two things, the role of the drift in the Brownian motion followed by the stock price and the effect of the barriers in the option price. The extracted conclusions are:

- When we relax the barriers and the stock price can penetrate them, the calculated option price is higher than if there were fixed barriers. The reason is that the stock price has more space to explore, it is not constrained by the fixed barriers, and due to the drift term, S tends to grow, reaching higher prices.

- A 30/40% relaxation of the barriers is enough to recover a vanilla option. This is related with the variance of the Brownian motion, which limits the growth/decrease of the stock price. In this way, having a barrier far away is the same of not having it.

- The price of the option is more affected by changes in the upper barrier than in the lower one. Another time this is consequence of the drift term. Because the tendency of the stock is to rise, upper barriers are more affected.

VI. ACKNOWLEDGEMENTS

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