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Marcinkiewicz-Zygmund inequalities for polynomials in Fock space

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Abstract

We study the relation between Marcinkiewicz-Zygmund families for polynomials in a weighted L^2-space and sampling theorems for entire functions in the Fock space and the dual relation between uniform interpolating families for polynomials and interpolating sequences. As a consequence we obtain a description of signal subspaces spanned by Hermite functions by means of Gabor frames.

Keywords Marcinkiewicz-Zygmund inequalities · Fock space · Reproducing kernel · Hermite function · Incomplete gamma function

Mathematics Subject Classification 30E05 · 30H20 · 41A10 · 42B30

1 Introduction

We study sampling and interpolation in Fock space and the relation to sampling and interpolation of polynomials. The Fock space $\mathcal{F}^2$ consists of all entire functions with finite norm

$$\|f\|_{\mathcal{F}^2} = \left(\int_{\mathbb{C}} |f(z)|^2 \, e^{-\pi |z|^2} dm(z) \right)^{1/2}, \tag{1}$$

where $dm(z) = dxdy$ is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$.

We denote by $\mathcal{P}_n$ the holomorphic polynomials of degree at most n. A sequence of (finite) subsets $\Lambda_n \subseteq \mathbb{C}$ is called a Marcinkiewicz-Zygmund family for the $\mathcal{P}_n$ in Fock space $\mathcal{F}^2$, if

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there exist constants $A, B > 0$, such that for all n large, $n \geq n_0$,

$$A\|p\|_{\mathcal{F}^2}^2 \leq \sum_{\lambda \in \Lambda_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} \leq B\|p\|_{\mathcal{F}^2}^2 \qquad \text{for all } p \in \mathcal{P}_n\,. \tag{2}$$

Here k_n is the reproducing kernel of $\mathcal{P}_n$, when endowed with the inner product inherited from $\mathcal{F}^2$.

This notion corresponds to the standard definition of sampling in a reproducing kernel Hilbert space $\mathcal{H}$. Let $k_\lambda(z) = k(z, \lambda)$ be the reproducing kernel of $\mathcal{H}$, i.e., $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$ at the point λ. Then a sequence Λ is a sampling set for $\mathcal{H}$, if the normalized reproducing kernels $\left\{\frac{k(z,\lambda)}{\sqrt{k(\lambda,\lambda)}} : \lambda \in \Lambda\right\}$ constitute a frame for $\mathcal{H}$. Equivalently, the sampling inequality $A\|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 k(\lambda, \lambda)^{-1} \leq B\|f\|_{\mathcal{H}}^2$ holds for all $f \in \mathcal{H}$.

In the Fock space $\mathcal{F}^2$ the reproducing kernel is $k(z, w) = e^{\pi z\overline{w}}$, and a sequence $\Lambda \subseteq \mathbb{C}$ is sampling in $\mathcal{F}^2$, if and only if

$$A\|f\|_{\mathcal{F}^2} \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-\pi|\lambda|^2} \leq B\|f\|_{\mathcal{F}^2}^2 \qquad \text{for all } f \in \mathcal{F}^2\,.$$

In this article we compare the notion of Marcinkiewicz-Zygmund families for $\mathcal{P}_n$ with sampling sequences for the Fock space $\mathcal{F}^2$. We will see that both notions are intimately connected. Roughly speaking, suitable finite sections of a sampling set for $\mathcal{F}^2$ yield a Marcinkiewicz-Zygmund family for the polynomials $\mathcal{P}_n$ in $\mathcal{F}^2$, and suitable limits of a Marcinkiewicz-Zygmund family yield a sampling set for $\mathcal{F}^2$.

A precise formulation is contained in our main result. (See Sect. 5 for an explanation of weak limits.)

Theorem 1.1 *(i) Assume that $\Lambda \subseteq \mathbb{C}$ is a sampling set for $\mathcal{F}^2$. For $\tau > 0$ set ρ_n such that $\pi\rho_n^2 = n + \sqrt{n}\tau$ and let B_{ρ_n} be the centered disk of radius ρ_n. Then for $\tau > 0$ large enough, the sets $\Lambda_n = \Lambda \cap B_{\rho_n}$ form a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $\mathcal{F}^2$.*

(ii) Conversely, every weak limit of a Marcinkiewicz-Zygmund family (Λ_n) for $\mathcal{P}_n$ in $\mathcal{F}^2$ is a sampling set for $\mathcal{F}^2$.

A dual result establishes a similar relationship between interpolating sets for $\mathcal{F}^2$ and uniformly interpolating sets for $\mathcal{P}_n$. A set Λ is interpolating in $\mathcal{F}^2$, if for every sequence $(a_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$, there exists $f \in \mathcal{F}^2$ such that $f(\lambda)e^{-\pi|\lambda|^2/2} = a_\lambda$ for all $\lambda \in \Lambda$. Of course, for polynomials of degree n every set of $n+1$ points is interpolating. In analogy to the definition of Marcinkiewicz-Zygmund families we call a family of (finite) subsets $\Lambda_n \subseteq \mathbb{C}$ a uniform interpolating family, if there exists a constant $A > 0$, such that for every sequence $a = (a_\lambda)_{\lambda\in\Lambda_n} \in \ell^2(\Lambda_n)$ there exists a polynomial $p \in \mathcal{P}_n$ such that $p(\lambda)k_n(\lambda, \lambda)^{-1/2} = a_\lambda$ for $\lambda \in \Lambda_n$ with norm control $\|p\|_{\mathcal{F}^2}^2 \leq A\|a\|_2^2$.

Theorem 1.2 *(i) Assume that $\Lambda \subseteq \mathbb{C}$ is a set of interpolation for $\mathcal{F}^2$. For $\tau > 0$ define ρ_n by $\pi\rho_n^2 = n - \sqrt{n}(\sqrt{2\log n} + \tau)$. Then for every $\tau > 0$ large enough, the sets $\Lambda_n = \Lambda \cap B_{\rho_n}$ form a uniform interpolating family for $\mathcal{P}_n$ in $\mathcal{F}^2$.*

(ii) Conversely, every weak limit of a uniform interpolating family (Λ_n) is a set of interpolation for $\mathcal{F}^2$.

Marcinkiewicz-Zygmund families and uniform interpolating families arise in several areas of analysis. They can be understood as finite-dimensional approximations of sampling theorems in reproducing kernel Hilbert spaces. Theorem 1.1(ii) shows that a Marcinkiewicz-Zygmund family can be used to prove a sampling theorem in an infinite dimensional space.

In approximation theory, a Marcinkiewicz-Zygmund family for a sequence of nested subspaces gives rise to a sequence of quadrature rules and function approximation from point evaluations, see [11] for this aspect. Random constructions of Marcinkiewicz-Zygmund families are studied in [4], and the study of deterministic point processes [1, 2, 5] uses closely related notions. In the recent advances in complexity theory and data analysis Marcinkiewicz-Zygmund families are implicit in the discretization of norms. For a nice survey see [15].

This work has several predecessors in different contexts. In [12] we have studied the analogous problem in the Bergman space and in the Hardy space in the unit disk. Indeed, our proof strategy for Theorem 1.1 is taken from [12]. Whereas in Bergman space the results can be formulated similarly to Theorem 1.1, the situation in Hardy space is rather different and the construction of Marcinkiewicz-Zygmund families needed to be based on different principles. In [10] a sampling theorem for bandlimited functions was derived via Marcinkiewicz-Zygmund families for trigonometric polynomials. The set-up of [16] is a compact manifold with a positive line bundle. Marcinkiewicz-Zygmund families for the space of holomorphic sections in powers of the line bundle are connected to sampling sequences in the tangent space.

Though line bundles appear much more complicated objects than Fock space, which even has a closed-form reproducing kernel, Fock space presents some new difficulties. It lacks compactness that made off-diagonal estimates for the reproducing kernel easier in [16]. Another source of difficulty is the behavior under translation. The Fock space $\mathcal{F}^2$ is invariant with respect to Bargmann-Fock shifts, while $\mathcal{P}_n$ endowed with the Fock norm is not.

Finally we mention the extensive work on the asymptotics of reproducing kernels for weighted polynomials in the context of *random* Marcinkiewicz-Zygmund families and determinantal point processes [1]–[5]. In [3, 4] Y. Ameur and his coauthors have studied a similar notion of sampling polynomials with respect to the discrete norm $\sum_{\lambda\in\Lambda_n}|p(\lambda)|^2e^{-\pi n|\lambda|^2}$ instead of $\sum_{\lambda\in\Lambda_n}\frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)}$. Note that all this work uses measures that depend on the polynomial degree n, quite in contrast to our set-up (2). Their choice was motivated by problems arising in random Gaussian matrix ensembles and models of the distribution of points in the one component plasma. The results in [3, 4] are not directly comparable to ours, but the common ground is the construction of point sets that are sampling for polynomials in Fock spaces. Our main interest is the connection to the infinite-dimensional sampling problem in $\mathcal{F}^2$.

One of our basic tools is the size of the reproducing kernel for the polynomials in a weighted L^2-space. Since our weight is the Gaussian weight, the kernel can be expressed explicitly in terms of the incomplete Gamma function which is a classical and well-studied object. We have collected the necessary results in the appendix for the sake of being self-contained. Estimates for the reproducing kernel have been studied in great generality in [1, 5] with potential theoretic methods—without any reference to the incomplete gamma function. Possibly these estimates could also be used in our context.

The estimates for the intrinsic reproducing kernel k_n show that (i) the L^2-energy of a polynomial of degree n is concentrated in a disk of radius $\sqrt{n/\pi}$, the so-called bulk region, and (ii) that the intrinsic kernel k_n for $\mathcal{P}_n$ is comparable to the kernel $k(z,w)=e^{\pi z\bar{w}}$ precisely in the bulk region. See Lemma 2.2 and Corollary 3.1 for the precise statements.

As a consequence of Theorem 1.1 we mention an application to time-frequency analysis. It is well-known that all problems about sampling in Fock space possess an equivalent formulation about Gabor frames in $L^2(\mathbb{R})$. To state this version, we denote the time-frequency shift of a function g by $z=(x,\xi)\in\mathbb{R}^2$ with $g_z(t)=e^{2\pi i\xi t}g(t-x)$ for $t,x,\xi\in\mathbb{R}$. The L^2-normalized Hermite functions are denoted by h_n, in particular $\phi(t)=2^{-1/4}h_0(t)=e^{-\pi t^2}$

is the Gaussian. Then Theorem 1.1(i) is equivalent to the following statement, which may be of interest in the time-frequency analysis of signal subspaces [14].

Theorem 1.3 *Assume that Λ is a sampling set for $\mathcal{F}^2$ and $\tau > 0$ large enough. Then $\{\pi(\lambda)h_0 : \pi|\lambda|^2 \leq n + \sqrt{n}\tau\}$ is a frame for $V_n = \operatorname{span}\{h_k : k = 0, \dots, n\}$ with bounds independent of n. This means that*

$$A\|f\|_2 \leq \sum_{\lambda\in\Lambda:\pi|\lambda|^2\leq n+\sqrt{n}\tau} |\langle f, \phi_\lambda\rangle|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in V_n .$$

Outlook. It is needless to say that the topic of Marcinkiewicz-Zygmund families and sampling theorems admits dozens of variations. The ultimate goal is to understand Marcinkiewicz-Zygmund families for polynomials $\mathcal{P}_n$ in a weighted Bergman space on some general domain $X \subseteq \mathbb{R}^d$ (or $\subseteq \mathbb{C}^d$). Intermediate problems would be Marcinkiewicz-Zygmund families for polynomials in Fock spaces with more general weight $e^{-Q(z)}$, or the construction of Marcinkiewicz-Zygmund families for multivariate Bergman spaces $A^2(\mathbb{B}_n)$ in n complex variables on the unit ball in $\mathbb{C}^n$. Even simple variations of the set-up yield interesting new questions.

The paper is organized as follows: In Section 2 we recall the basic facts about the Fock space and the associated reproducing kernels. Section 3 summarizes the required asymptotics of the incomplete gamma function. In Section 4 we relate sampling sets for Fock space to Marcinkiewicz-Zygmund families and prove the first part of Theorem 1.1. Section 5 covers the converse statement. In Section 6 we deal with uniform interpolating families and prove Theorem 1.2. The connection to the time-frequency analysis of signal subspaces is explained in Section 7. Finally, in the appendix we offer some elementary estimates for the zero-order asymptotics of the incomplete gamma function. These are, of course, well-known and added only to make the paper self-contained.

2 Fock space

The monomials $z \mapsto z^k$ are orthogonal in $\mathcal{F}^2$, and the normalized monomials

$$e_k(z) = \left(\frac{\pi^k}{k!}\right)^{1/2} z^k$$

form an orthonormal basis for $\mathcal{F}^2$.

Let $\mathcal{P}_n$ be the subspace of polynomials of degree at most n in $\mathcal{F}^2$. The reproducing kernel of $\mathcal{P}_n$ is given by

$$k_n(z, w) = \sum_{k=0}^{n} e_k(z)\overline{e_k(w)} = \sum_{k=0}^{n} \frac{(\pi z\bar{w})^k}{k!} . \tag{3}$$

As $n \to \infty$, this kernel converges to the reproducing kernel of $\mathcal{F}^2$:

$$k(z, w) = \lim_{n\to\infty} k_n(z, w) = e^{\pi z\bar{w}} .$$

As we have learned in our study of Marcinkiewicz-Zygmund families in Bergman spaces [12], we will need to understand the relation of the kernel k_n to k. For this purpose we will make use of the properties and the asymptotics of the *incomplete gamma function*

$$\Gamma(z, a) = \int_a^\infty t^{z-1}e^{-t}\, dt \tag{4}$$

and

$$\gamma(z,a)=\int_0^a t^{z-1}e^{-t}\,dt\,. \tag{5}$$

Denote the centered disc of radius ρ by $B_\rho=\{z\in\mathbb{C}:|z|\le\rho\}$. Then

$$\begin{aligned}\int_{B_\rho}|z|^{2k}e^{-\pi|z|^2}\,dm(z)&=2\pi\int_0^\rho r^{2k}e^{-\pi r^2}\,rdr\\&=\int_0^{\pi\rho^2}\left(\frac{u}{\pi}\right)^k e^{-u}\,du\\&=\frac{1}{\pi^k}\gamma(k+1,\pi\rho^2)\,.\end{aligned} \tag{6}$$

Lemma 2.1 *We have*

$$k_n(z,w)=e^{\pi z\bar w}\frac{\Gamma(n+1,\pi z\bar w)}{n!}$$

In particular $k_n(z,z)=e^{\pi|z|^2}\frac{\Gamma(n+1,\pi|z|^2)}{n!}$. [1]

Proof See [21, 8.4.8], or use the obvious formula

$$\frac{1}{n!}\Gamma(n+1,r)=\frac{1}{n!}\int_r^\infty t^n e^{-t}\,dt=\frac{r^n}{n!}e^{-r}+\frac{1}{(n-1)!}\Gamma(n,r)$$

repeatedly and then use analytic extension and substitute $r=\pi z\bar w$. □

The energy of a polynomial $p(z)=\sum_{k=0}^n a_k z^k\in\mathcal{P}_n$ on a disc B_ρ is

$$\begin{aligned}\int_{B_\rho}|p(z)|^2\,e^{-\pi|z|^2}\,dm(z)&=\sum_{k=0}^n|a_k|^2\int_{B_\rho}|z|^{2k}e^{-\pi|z|^2}\,dm(z)\\&=\sum_{k=0}^n|a_k|^2\frac{k!}{\pi^k}\,\frac{\gamma(k+1,\pi\rho^2)}{k!}\\&\ge\min_{0\le k\le n}\frac{\gamma(k+1,\pi\rho^2)}{k!}\sum_{k=0}^n|a_k|^2\frac{k!}{\pi^k}\quad\ge\frac{\gamma(n+1,\pi\rho^2)}{n!}\|p\|^2_{\mathcal{F}^2}\,.\end{aligned}$$

In the last inequality we have used the fact that $k\to\frac{\gamma(k+1,\pi\rho^2)}{k!}$ is decreasing and that $\|z^k\|^2_{\mathcal{F}^2}=k!/\pi^k$ by (6).

Lemma 2.2 *For every $p\in\mathcal{P}_n$ we have*

$$\int_{B_\rho^c}|p(z)|^2\,e^{-\pi|z|^2}\,dm(z)\le\frac{\Gamma(n+1,\pi\rho^2)}{n!}\|p\|^2_{\mathcal{F}^2}\,. \tag{7}$$

[1] Note that we consider $\mathcal{P}_n$ as a subspace of $\mathcal{F}^2$ and always use the fixed weight $e^{-\pi|z|^2}$. The work on determinantal point processes always uses the weight $e^{-\pi n|z|^2}$ for $\mathcal{P}_n$. The formulas and the asymptotics are therefore different.

Proof This follows immediately from the previous estimates via

$$\int_{B_\rho^c} |p(z)|^2 e^{-\pi|z|^2} dm(z) = \|p\|^2_{\mathcal{F}^2} - \int_{B_\rho} |p(z)|^2 e^{-\pi|z|^2} dm(z)$$
$$\leq (1 - \frac{\gamma(n+1, \pi\rho^2)}{n!}) \|p\|^2_{\mathcal{F}^2} = \frac{\Gamma(n+1, \pi\rho^2)}{n!} \|p\|^2_{\mathcal{F}^2}. \quad (8)$$

□

3 Asymptotics of the incomplete gamma function

The asymptotic behavior of the incomplete gamma function is well understood. We collect the properties required for Marcinkiewicz-Zygmund families in Fock space. As usual $f \asymp g$ means that there exists a constant $C > 0$ such that $C^{-1} f(x) \leq g(x) \leq C f(x)$ for all x in the domain of f and g, $f \lesssim g$ means $f(x) \leq C g(x)$, and $f \sim g$ near x_∞ means that $\lim_{x \to x_\infty} \frac{f(x)}{g(x)} = 1$.

The following result has been proved on several levels of generality [9, 19, 20, 27, 28].

The normalized incomplete gamma function admits the asymptotic expansion

$$\frac{\Gamma(a, a + \tau\sqrt{a})}{\Gamma(a)} \sim \tfrac{1}{2}\mathrm{erfc}\,(\frac{\tau}{\sqrt{2}}) + \frac{1}{\sqrt{2\pi a}} e^{-\tau^2/2} \sum_{n=0}^{\infty} \frac{C_n(\tau)}{a^{n/2}}, \quad (9)$$

where $C_0(\tau) = \frac{\tau^2-1}{3}$ *and* $\mathrm{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-t^2} dt$. *A careful interpretation of the zero order approximation implies that there exists a constant* C *independent of* a *and* τ, *such that*

$$\Big| \frac{\Gamma(a, a + \tau\sqrt{a})}{\Gamma(a)} - \tfrac{1}{2}\mathrm{erfc}(\frac{\tau}{\sqrt{2}}) \Big| \leq C(|\tau|^2 + 1) e^{-\tau^2/2} \frac{1}{\sqrt{a}} \quad (10)$$

for $|\tau| \leq a^{1/6}$.

See [20], Prop. 1.1 and Eq. (3.1).

We only need these estimates for $a = n + 1$ and $\tau > 0$, but their validity has been established for large domains in $\mathbb{C}$.

Proposition 3.1

(i) For every $\epsilon > 0$ *there is* $\tau > 0$, *such that*

$$\frac{\Gamma(n+1, n + \sqrt{n}\tau)}{n!} < \epsilon \qquad \forall n \geq n_0$$

(ii) For every $\tau > 0$ *there is a constant* $C(\tau) > 0$, *such that*

$$\frac{\Gamma(n+1, n + \sqrt{n}\tau)}{n!} \geq C(\tau) \qquad \forall n \geq n_0 .$$

In fact, $C(\tau)$ *can be taken as* $C(\tau) = \frac{1}{4}\mathrm{erf}(\tau/\sqrt{2})$

(iii) For $\tau > 0$

$$1 - \frac{1}{n!}\Gamma(n+1, n - \sqrt{n}\tau) \leq e^{-\tau^2/2} .$$

(iv) *For every* $x \geq 0$

$$\lim_{n\to\infty} \frac{\Gamma(n+1,x)}{n!} = 1\,.$$

The convergence is uniform on bounded sets $\subseteq \mathbb{R}^+$ *and exponentially fast.*

(v) *For all* n

$$\frac{\Gamma(n+1,n)}{n!} > 1/2\,.$$

Proof Items (i) and (ii) follow readily from (10) as follows:

(i) Choose $\tau > 0$, such that $\frac{1}{2}\mathrm{erfc}(\tau/\sqrt{2}) < \epsilon/2$. Now choose $n_0 \in \mathbb{N}$, such that $n_0 \geq \tau^6$ and the error $C(|\tau|^2+1)e^{-\tau^2/2}\frac{1}{\sqrt{n+1}} < \epsilon/2$ for $n \geq n_0$. By (10) we then have $\frac{\Gamma(n+1,n+\sqrt{n}\tau)}{n!} < \epsilon$ for all $n \geq n_0$.

(ii) Given $\tau > 0$ choose $n_0 \geq \tau^6$ such that the error $C(|\tau^2|+1)e^{-\tau^2/2}\frac{1}{\sqrt{n+1}} < \frac{1}{4}\mathrm{erfc}(\tau/\sqrt{2})$ for $n \geq n_0$. Then $\frac{\Gamma(n+1,n+\sqrt{n}\tau)}{n!} \geq \frac{1}{4}\mathrm{erfc}(\tau/\sqrt{2}) > 0$ for all $n \geq n_0$.

(iii) and (iv) are well-known.

(v) is taken from [21], formula 8.10.13.

For completeness we summarize the arguments for the zero order asymptotics in the appendix. In contrast to the full asymptotics of the incomplete Gamma function, they are elementary. □

Corollary 3.1 *If* $\pi|z|^2 \leq n+\sqrt{n}\tau$, *then* $k_n(z,z) \asymp k(z,z) = e^{\pi|z|^2}$ *with a constant depending only on* τ, *but not on* n.

We also need an off-diagonal estimate for the kernel k_n.

Lemma 3.1 *Assume that* $\pi|z|^2 < n(1-\epsilon)$ *for fixed* $\epsilon > 0$ *and* $|z-w| \leq \tau$. *Then for* n *large enough depending on* ϵ,

$$\Big|\frac{\Gamma(n+1,\pi z\bar{w})}{n!} - \frac{\Gamma(n+1,\pi|z|^2)}{n!}\Big| \leq Ce^{-\epsilon^2 n/4}\,. \tag{11}$$

Proof Since $\Gamma(n+1,\pi z\bar{w})$ is invariant with respect to the rotation $(z,w)\to(e^{i\theta}z,e^{i\theta}w)$, we may assume that $z=r\in\mathbb{R}$, $r>0$ and $w=r+\bar{u}$ with $|u|\leq\tau$. We then write

$$\begin{aligned}\Gamma(n+1,\pi z\bar{w}) &= \Gamma(n+1,\pi r^2+\pi r u) = \int_{\pi r^2}^{\infty} t^n e^{-t}\,dt + \int_{\pi r^2+\pi r u}^{\pi r^2}\dots \\ &= \Gamma(n+1,\pi r^2) + \int_{\pi r^2+\pi r u}^{\pi r^2}\dots\,.\end{aligned}$$

Let $\gamma(s) = \pi r^2 + s\pi r u$ the line segment from $\pi r^2 \in \mathbb{C}$ to $\pi r^2+\pi r u$. Then

$$\begin{aligned}\Big|\int_{\pi r^2}^{\pi r^2+\pi r u} t^n e^{-t}\,dt\Big| &= \Big|\pi r u\int_0^1 (\pi r^2+s\pi r u)^n e^{-\pi r^2 - s\pi r u}\,ds\Big| \\ &\leq \pi r|u|(\pi r^2+\pi r|u|)^n e^{-\pi r^2+\pi r|u|} \\ &\leq \pi r\tau(\pi r^2+\pi r\tau)^n e^{-\pi r^2+\pi r\tau}\,.\end{aligned}$$

Observe that $r \to (\pi r^2 + \pi r\tau)^n e^{-\pi r^2}$ is increasing, as long as $\pi r^2 + \pi r\tau \le n$. Set $x = \frac{\pi r^2}{n} \le 1-\epsilon$, then $\pi r\tau = \sqrt{\pi n x}\tau$, and by assumption $x < 1$. Using Sterling's formula, we continue with

$$\begin{aligned}\frac{1}{n!}\pi r\tau(\pi r^2+\pi r\tau)^n e^{-\pi r^2+\pi r\tau} &\le \frac{\sqrt{\pi}\tau\sqrt{nx}}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n\left(nx+\sqrt{\pi n x}\tau\right)^n e^{-nx+\sqrt{\pi x n}\tau}\\ &\lesssim \sqrt{x}\left(x+\frac{\sqrt{\pi x}\tau}{\sqrt{n}}\right)^n e^{n-nx+\sqrt{\pi x n}\tau}\\ &\le x^n\left(1+\frac{\sqrt{\pi}\tau}{\sqrt{xn}}\right)^n e^{n(1-x+\sqrt{\pi x}\tau/\sqrt{n})}\\ &\le \exp\left(n\big(1-x+\ln x+\ln(1+\frac{\sqrt{\pi}\tau}{\sqrt{nx}})+\frac{\sqrt{\pi}\tau}{\sqrt{nx}}\big)\right).\end{aligned}$$

Since $x \to 1-x+\ln x$ is increasing on $(0,1]$ and $x \le 1-\epsilon$, we have $1-x+\ln x \le \epsilon+\ln(1-\epsilon) \le -\epsilon^2/2$. Choose n so large that $\ln(1+\frac{\sqrt{\pi}\tau}{\sqrt{nx}})+\frac{\sqrt{\pi}\tau}{\sqrt{nx}} \le \epsilon^2/4$, then the latter expression is dominated by $e^{-n\epsilon^2/4}$, and this expression tends to 0 exponentially fast, as $n \to \infty$. This proves the claim. □

4 Sampling implies Marcinkiewicz-Zygmund inequalities

We summarize the main facts about sampling sets in $\mathcal{F}^2$ from the literature [17, 23–25].

(i) A set $\Lambda \subseteq \mathbb{C}$ is sampling for $\mathcal{F}^2$, if and only if it contains a uniformly separated set $\Lambda' \subseteq \Lambda$ with lower Beurling density $D^-(\Lambda') > 1$.

(ii) *Tail estimates.* Let $f \in \mathcal{F}^2$ and $\rho > 0$. The subharmonicity of $|f|^2$ implies that

$$|f(\lambda)|^2 e^{-\pi|\lambda|^2} \le c_\rho \int_{B(\lambda,\rho)} |f(z)|^2 e^{-\pi|z|^2}\,dm(z) \tag{12}$$

for all $\lambda \in \mathbb{C}$. The constant is $c_\rho = e^{\pi\rho^2}/(\pi\rho^2)$, but we will not need it.

If Λ is relatively separated, i.e., a finite union of K uniformly discrete subsets of $\mathbb{C}$ with separation $\rho > 0$, then

$$\sum_{\lambda\in\Lambda, |\lambda|>R} |f(\lambda)|^2 e^{-\pi|\lambda|^2} \le c_\rho K \int_{|z|>R-\rho} |f(z)|^2 e^{-\pi|z|^2}\,dm(z)\,. \tag{13}$$

Theorem 4.1 *Assume that $\Lambda \subseteq \mathbb{C}$ is a sampling set for $\mathcal{F}^2$ with bounds A, B. For $\tau > 0$ set ρ_n, such that $\pi\rho_n^2 = n + \sqrt{n}\tau$. Then for $\tau > 0$ large enough, the sets $\Lambda_n = \Lambda \cap B_{\rho_n}$ form a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $\mathcal{F}^2$.*

Proof *Lower bound*: Since always $k_n(z,z) \le k(z,z) = e^{\pi|z|^2}$, we may replace k_n by k in the sampling inequalities:

$$\begin{aligned}\sum_{\lambda\in\Lambda_n}\frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} &\ge \sum_{\lambda\in\Lambda_n}\frac{|p(\lambda)|^2}{k(\lambda,\lambda)}\\ &= \sum_{\lambda\in\Lambda_n}|p(\lambda)|^2 e^{-\pi|\lambda|^2} = \sum_{\lambda\in\Lambda} - \sum_{\lambda\in\Lambda:|\lambda|>\rho_n}\dots\,.\end{aligned}$$

Since Λ is a sampling set for $\mathcal{F}^2$, the first term satisfies $\sum_{\lambda\in\Lambda}|p(\lambda)|^2 e^{-\pi|\lambda|^2} \geq A\|p\|^2_{\mathcal{F}^2}$. For the second term we observe that Λ is a finite union of uniformly discrete sets with separation $\rho > 0$ and apply (13) and (7):

$$\sum_{\lambda\in\Lambda:|\lambda|>\rho_n} |p(\lambda)|^2 e^{-\pi|\lambda|^2} \leq C\int_{|z|>\rho_n-\rho} |p(z)|^2 e^{-\pi|z|^2}\, dm(z)$$
$$\leq C\frac{\Gamma(n+1,\pi(\rho_n-\rho)^2)}{n!}\|p\|^2_{\mathcal{F}^2} \qquad \text{for } p\in\mathcal{P}_n\,.$$

Our choice of ρ_n implies that

$$\begin{aligned}\pi(\rho_n-\rho)^2 &= \pi\rho_n^2 - 2\pi\rho_n\rho + \pi\rho^2\\ &= n+\sqrt{n}\tau + \pi\rho^2 - 2\sqrt{\pi}\sqrt{n+\sqrt{n}\tau}\,\rho\\ &\geq n+\sqrt{n}\tau - 2\sqrt{\pi n}\rho(1+\frac{\tau}{2\sqrt{n}})\\ &\geq n+\sqrt{n}\tau'\end{aligned}$$

with $\tau' = \tau - 3\sqrt{\pi}\rho$ whenever $\sqrt{n} \geq \tau$. Since $a \mapsto \Gamma(x,a)$ is decreasing, we have $\frac{\Gamma(n+1,\pi(\rho_n-\rho)^2)}{n!} \leq \frac{\Gamma(n+1,n+\sqrt{n}\tau')}{n!}$. In view of Corollary 3.1(i) we may choose τ' and hence τ so that

$$\sum_{\lambda\in\Lambda:|\lambda|>\rho_n} |p(\lambda)|^2 e^{-\pi|\lambda|^2} \leq C\frac{\Gamma(n+1,n+\sqrt{n}\tau')}{n!}\|p\|^2_{\mathcal{F}^2} \leq \frac{A}{2}\|p\|_{\mathcal{F}^2} \qquad \text{for all } p\in\mathcal{P}_n$$

for large n, $n \geq n_0$, say. Combining the inequalities, we obtain $\sum_{\lambda\in\Lambda_n}\frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} \geq \frac{A}{2}\|p\|^2_{\mathcal{F}^2}$ for all $p\in\mathcal{P}_n$.

Upper inequality: For the above choice of τ Proposition 3.1(ii) says that

$$\frac{\Gamma(n+1,n+\sqrt{n}\tau)}{n!} \geq \tfrac{1}{4}\mathrm{erfc}(\tau/\sqrt{2}) = C(\tau) = C$$

for $n \geq n_0$. This implies that $k_n(\lambda,\lambda)^{-1} \leq C^{-1}k(\lambda,\lambda)^{-1} = C^{-1}e^{-\pi|\lambda|^2}$ for $\pi|\lambda|^2 \leq n+\sqrt{n}\tau$, and thus

$$\sum_{\lambda\in\Lambda_n}\frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} \leq C^{-1}\sum_{\lambda\in\Lambda,\pi|\lambda|^2\leq n+\sqrt{n}\tau} |p(\lambda)|^2 e^{-\pi|\lambda|^2} \leq C^{-1}B\|p\|_{\mathcal{F}^2} \qquad \text{for all } p\in\mathcal{P}_n\,,$$

because $\Lambda \supseteq \Lambda_n$ is a sampling set for $\mathcal{F}^2$. □

Note that the lower bound in the Marcinkiewicz-Zygmund inequalities matches the lower bound A of the sampling inequality in $\mathcal{F}^2$, whereas the upper bound is $4\,\mathrm{erfc}\,(\tau/\sqrt{2})^{-1}B$ depends also on the additional parameter τ.

Corollary 4.1 *For every $\epsilon > 0$ there exist Marcinkiewicz-Zygmund families (Λ_n) for $\mathcal{P}_n$ in $\mathcal{F}^2$ with $\#\Lambda_n \leq (1+\epsilon)(n+1)$ points.*

Proof Choose μ,δ small enough, so that $(1+2\mu)(1+\delta) < 1+\epsilon$. Let $\Lambda \subseteq \mathbb{C}$ be a (uniformly) discrete subset with $D^-(\Lambda) > 1$ and $D^+(\Lambda) < 1+\mu$. Then Λ is a sampling set for $\mathcal{F}$ by the characterizations of Lyubarskii [17] and Seip [23, 25], and for $\pi\rho_n^2 = n+\sqrt{n}\tau$ the sets

$\Lambda \cap B_{\rho_n}$ form a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$. For n large enough and $\tau/\sqrt{n} < \delta$, we find that

$$\#\Lambda_n = \#(\Lambda \cap B_{\rho_n}) \leq (1+2\mu)|B_{\rho_n}| = (1+2\mu)(n+\sqrt{n}\tau) < (1+\epsilon)(n+1)\,.$$

□

For a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ we need at least $\dim \mathcal{P}_n = n+1$ points in each layer Λ_n. The construction above yields Marcinkiewicz-Zygmund families for Fock space with nearly optimal cardinality.

5 Marcinkiewicz-Zygmund inequalities imply sampling

We first formulate a few properties of the distribution of Marcinkiewicz-Zygmund families.

Lemma 5.1 *Assume that (Λ_n) is a Marcinkiewicz-Zygmund family for $\mathcal{F}^2$ with bounds A, B. Let $\varepsilon > 0$ and $\pi\sigma_n^2 = n(1-\varepsilon)$.*

(i) *Then $\#(\Lambda_n \cap B^c_{\sigma_n}) \leq B\frac{n+1}{(1-\varepsilon)^n}$. This holds also for $\epsilon = 0$.*
(ii) *Let $B(z,\rho)$ be a disc in B_{σ_n}. Then*

$$\#(\Lambda_n \cap B(z,\rho)) \leq C\,.$$

Consequently, every $\Lambda_n \cap B_{\sigma_n}$ is a union of at most L separated sets with uniform separation $\delta > 0$ independent of n.

Proof (i) For $\pi|w|^2 \geq n$ and $k \leq n$, we have

$$\frac{(\pi|w|^2)^k}{k!} \leq \frac{(\pi|w|^2)^n}{n!}\,,$$

so the reproducing kernel satisfies the estimate

$$k_n(w,w) = \sum_{k=0}^{n}\frac{(\pi|w|^2)^k}{k!} \leq (n+1)\frac{(\pi|w|^2)^n}{n!}\,. \tag{14}$$

If $\pi|z|^2 \geq (1-\varepsilon)n$, then, using $\sqrt{1-\varepsilon}\,w = z$, we have $\pi|w|^2 \geq n$, and as before we obtain:

$$k_n(z,z) \leq \frac{n+1}{(1-\varepsilon)^n}\frac{(\pi|z|^2)^n}{n!},$$

To estimate $\#(\Lambda_n \cap B^c_{\sigma_n})$, we test the Marcinkiewicz-Zygmund inequalities for the monomial $p_n(z) = \frac{\pi^{n/2}}{n!^{1/2}}z^n$. Then $\|p_n\|_{\mathcal{F}^2} = 1$. (14) implies that

$$\frac{|p_n(\lambda)|^2}{k_n(\lambda,\lambda)} \geq \frac{(1-\varepsilon)^n}{n+1} \qquad \text{for } \pi|\lambda|^2 \geq n(1-\varepsilon)\,,$$

and therefore

$$\frac{(1-\varepsilon)^n}{n+1}\#(\Lambda_n \cap B^c_{\sigma_n}) \leq \sum_{\lambda\in\Lambda_n:\pi|\lambda|^2\geq n(1-\varepsilon)}\frac{|p_n(\lambda)|^2}{k_n(\lambda,\lambda)} \leq B\|p_n\|_{\mathcal{F}^2} = B\,,$$

so we obtain $\#(\Lambda_n \cap B^c_{\sigma_n}) \leq \frac{B(n+1)}{(1-\varepsilon)^n}$.

(ii) Let $\kappa_{n,z}(w) = k_n(w,z)/k_n(z,z)^{1/2}$ be the normalized reproducing kernel of $\mathcal{P}_n$ and $B(z,\rho) \subseteq B_{\sigma_n}$ be an arbitrary disc inside B_{σ_n}. Recall that $k_n(z,w) = e^{\pi\bar{z}w}\Gamma(n+1,\pi\bar{z}w)/n!$ and that for $\pi|z|^2 \le n(1-\epsilon)$ we have $\Gamma(n+1,\pi|z|^2)/n! \ge 1/2$ by Proposition 3.1(v). So after substituting the formulas for the kernel, we obtain

$$\begin{aligned}\sum_{\lambda\in\Lambda_n\cap B(z,\rho)} \frac{|\kappa_{n,z}(\lambda)|^2}{k_n(\lambda,\lambda)} &= \sum_{\lambda\in\Lambda_n\cap B(z,\rho)} \frac{|k_n(z,\lambda)|^2}{k_n(z,z)k_n(\lambda,\lambda)} \\ &= \sum_{\lambda\in\Lambda_n\cap B(z,\rho)} |e^{\pi\bar{z}\lambda}|^2 e^{-\pi|z|^2} e^{-\pi|\lambda|^2} \frac{|\Gamma(n+1,\pi\bar{z}\lambda)|^2}{\Gamma(n+1,\pi|z|^2)\Gamma(n+1,\pi|\lambda|^2)} \\ &= \sum_{\lambda\in\Lambda_n\cap B(z,\rho)} e^{-\pi|\lambda-z|^2} \frac{|\Gamma(n+1,\pi\bar{z}\lambda)|^2}{\Gamma(n+1,\pi|z|^2)\Gamma(n+1,\pi|\lambda|^2)} = (*)\,.\end{aligned}$$

Now note that by Lemma 3.1 $|\Gamma(n+1,\pi\bar{z}\lambda)|^2/n! \ge 1/4$ for n large, whereas $|\Gamma(n+1,\pi|z|^2)/n! \le 1$, so that the last sum finally is bounded below by

$$(*) \ge e^{-\pi\rho^2} \sum_{\lambda\in\Lambda_n\cap B(z,\rho)} \frac{1}{4} = \frac{1}{4} e^{-\pi\rho^2}\,\#(\Lambda_n\cap B)\,.$$

Reading backwards, we obtain

$$\frac{1}{4} e^{-\pi\rho^2}\,\#(\Lambda_n\cap B) \le \sum_{\lambda\in\Lambda_n\cap B} \frac{|\kappa_{n,z}(\lambda)|^2}{k_n(\lambda,\lambda)} \le B\|\kappa_{n,z}\|^2_{\mathcal{F}} = B\,,$$

which was claimed. □

For completeness we mention that the number of points in the transition region $C_{n,\tau} = \{z\in\mathbb{C} : n-\sqrt{n}\tau \le \pi|z|^2 \le n+\sqrt{n}\tau\}$ is bounded by

$$\#(\Lambda_n\cap C_{n,\tau}) \lesssim \sqrt{n}e^{\tau^2/2}\,.$$

This can be shown as above by testing against the monomial z^n.

Before stating our main theorem, we recall that a sequence of sets $\Lambda_n \subseteq \mathbb{C}$ converges weakly to $\Lambda \subseteq \mathbb{C}$, if for all compact disks $B \subseteq \mathbb{C}$

$$\lim_{n\to\infty} d\big((\Lambda_n\cap B)\cup\partial B, (\Lambda\cap B)\cup\partial B\big) = 0\,,$$

where $d(\cdot,\cdot)$ denotes the Hausdorff distance between two compact sets in $\mathbb{C}$. If every Λ_n is the union of at most K uniformly separated sets with fixed separation δ, then

$$\sum_{\lambda\in\Lambda_n\cap B} \frac{|f(\lambda)|^2}{k(\lambda,\lambda)} \to \sum_{\lambda\in\Lambda\cap B} \frac{|f(\lambda)|^2}{k(\lambda,\lambda)} m(\lambda)\,, \tag{15}$$

with multiplicities $\mu(\lambda) \in \{1,\dots,K\}$.

Theorem 5.1 *Assume that (Λ_n) is a Marcinkiewicz-Zygmund family for the polynomials $\mathcal{P}_n$ in $\mathcal{F}^2$. Let Λ be a weak limit of (Λ_n) or of some subsequence (Λ_{n_k}). Then Λ is a sampling set for $\mathcal{F}^2$.*

Proof The assumption that Λ_n is a Marcinkiewicz-Zygmund family for $\mathcal{P}_n$ in $\mathcal{F}^2$ means that there exist $A, B > 0$ such that $A\|p\|_{\mathcal{F}^2} \le \sum_{\lambda\in\Lambda_n} \frac{|p(\lambda)|^2}{k_n(\lambda,\lambda)} \le B\|p\|^2_{\mathcal{F}^2}$ for all polynomials $p \in \mathcal{P}_n$.

(i) Let $B = \bar{B}(w, \rho)$ be a closed disc. By Lemma 5.1(ii) $\#(\Lambda_n \cap \bar{B}(w, \rho)) \leq C$ for some constant C independent of n and B, provided that n is big enough. Since Λ is a weak limit of Λ_n, we know that $\#(\Lambda \cap \bar{B}(w, \rho)) \leq C$. This means that Λ is a union of K uniformly separated sets with separation $\delta > 0$.
(ii) It follows immediately from (13) that Λ satisfies the upper bound in the sampling inequality for $\mathcal{F}^2$.
(iii) **Lower bound.** Fix a polynomial $p \in \mathcal{P}_N$ (of degree N) and choose $r > 0$ such that

$$\int_{|z| \geq \sqrt{r/\pi}} |p(z)|^2 e^{-\pi |z|^2}\, dm(z) < \frac{A}{4c_\delta K} \|p\|^2_{\mathcal{F}^2} ,$$

where c_δ is the constant in (12) for separation δ. To avoid the ugly notation in subscript, we write $\nu = \sqrt{r/\pi}$, $\rho_n = \sqrt{n/\pi}$, and $\sigma_n = \sqrt{n(1-\varepsilon)/\pi}$.

For $p \in \mathcal{P}_N$ the Marcinkiewicz-Zygmund inequalities are satisfied for every $n \geq N$, therefore

$$\begin{aligned} A\|p\|^2_{\mathcal{F}^2} &\leq \sum_{\lambda \in \Lambda_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} \\ &= \sum_{\lambda \in \Lambda_n, |\lambda| < \nu + \delta} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} \\ &\quad + \sum_{\lambda \in \Lambda_n, \nu+\delta \leq |\lambda| < \sigma_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} + \sum_{\lambda \in \Lambda_n, |\lambda| \geq \sigma_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} = \\ &= A_n + B_n + C_n . \end{aligned}$$

If $|\lambda| \leq \sigma_n$, then $k_n(\lambda, \lambda) \geq \frac{1}{2} k(\lambda, \lambda) = \frac{1}{2} e^{\pi |\lambda|^2}$ as a consequence of Lemma 2.1 and Proposition 3.1(v). Thus in the expressions for A_n and B_n we may replace the kernel k_n for polynomials by the kernel $k(z, z) = e^{\pi |z|^2}$ for Fock space. Consequently

$$A\|p\|^2_{\mathcal{F}^2} \leq 2 \sum_{\lambda \in \Lambda_n, |\lambda| \leq \nu+\delta} |p(\lambda)|^2 e^{-\pi|\lambda|^2} + B_n + C_n. \tag{16}$$

Since in this sum all points λ lie in the compact set $\bar{B}(0, \nu + \delta)$, the weak convergence (including multiplicities $m(\lambda) \in \{1, \ldots, K\}$) implies the convergence to Λ and

$$\begin{aligned} \lim_{n\to\infty} A_n \leq 2 \lim_{n\to\infty} \sum_{\lambda \in \Lambda_n, |\lambda| \leq \nu+\delta} |p(\lambda)|^2 e^{-\pi|\lambda|^2} = \\ = 2 \sum_{\lambda \in \Lambda \cap \overline{B_{\nu+\delta}}} |p(\lambda)|^2 e^{-\pi|\lambda|^2} m(\lambda) \end{aligned}$$

For the term B_n, we recall that every $\Lambda_n \cap B_{\sigma_n}$ is a finite union of at most K uniformly separated sequences with separation δ and apply the tail estimate (13). Our choice of r and $\nu = \sqrt{r/\pi}$ yields

$$\begin{aligned} B_n &\leq 2 \sum_{\lambda \in \Lambda_n, \nu+\delta \leq |\lambda| < \sigma_n} \frac{|p(\lambda)|^2}{k(\lambda, \lambda)} \leq 2c_\delta K \int_{|z| > \nu} |p(z)|^2 e^{-\pi|z|^2}\, dm(z) \\ &\leq 2c_\delta K \frac{A}{4c_\delta K} \|p\|^2_{\mathcal{F}^2} = \frac{A}{2} \|p\|^2_{\mathcal{F}^2} . \end{aligned}$$

To treat C_n, recall that p has degree $N < n$. We use the trivial estimate

$$|p(\lambda)|^2 = |\langle p, k_N(\lambda, \cdot)\rangle|^2 \le \|p\|^2_{\mathcal{F}^2} k_N(\lambda, \lambda)$$

and substitute into C_n to obtain

$$C_n = \sum_{\lambda \in \Lambda_n, |\lambda| > \sigma_n} \frac{|p(\lambda)|^2}{k_n(\lambda, \lambda)} \le \|p\|^2_{\mathcal{F}^2} \#(\Lambda_n \cap B^c_{\sigma_n}) \sup_{|z| \ge \sigma_n} \frac{k_N(z, z)}{k_n(z, z)} .$$

By Lemma 5.1(ii) $\#\Lambda_n \cap B^c_{\sigma_n} \le \frac{B(n+1)}{(1-\varepsilon)^n}$, whereas the ratio of the different reproducing kernels is

$$\frac{k_N(z, z)}{k_n(z, z)} = \frac{e^{\pi|z|^2}\Gamma(N+1, \pi|z|^2)n!}{e^{\pi|z|^2}\Gamma(n+1, \pi|z|^2)N!} .$$

For simplicity set $\pi|z|^2 = R > n(1-\varepsilon)$. Then

$$\begin{aligned}\Gamma(n+1, R) &= \int_R^\infty t^n e^{-t}\, dt \\ &\ge R^{n-N} \int_R^\infty t^N e^{-t}\, dt \\ &= R^{n-N}\Gamma(N+1, R) ,\end{aligned}$$

so that

$$\sup_{\pi|z|^2 > n(1-\varepsilon)} \frac{k_N(z, z)}{k_n(z, z)} \le (n(1-\varepsilon))^{N-n} \frac{n!}{N!} .$$

Altogether

$$C_n \le \|p\|^2_{\mathcal{F}^2} \frac{B(n+1)}{(1-\varepsilon)^n} (n(1-\varepsilon))^{N-n} \frac{n!}{N!} \to 0 ,$$

as $n \to \infty$ by Stirling's formula, provided that we choose ε such that $(1-\varepsilon)^2 > 1/e$.

Combining the estimates for A_n, B_n, and C_n and letting n go to ∞, we obtain the lower sampling inequality

$$\begin{aligned}\sum_{\lambda \in \Lambda} |p(\lambda)|^2 m(\lambda) e^{-\pi|\lambda|^2} \ge \sum_{\lambda \in \Lambda, |\lambda| \le \nu + \delta} |p(\lambda)|^2 m(\lambda) e^{-\pi|\lambda|^2} - \limsup_{n\to\infty} B_n - \lim_{n\to\infty} C_n \ge \\ \ge \frac{A}{2} \|p\|^2_{\mathcal{F}^2} .\end{aligned}$$

As the multiplicities satisfy $1 \le m(\lambda) \le K$ for $\lambda \in \Lambda$, we may omit them by changing the lower sampling constant to $A/(2K)$.

Since polynomials are dense in $\mathcal{F}^2$, this estimate extends to all of $\mathcal{F}^2$. □

6 Uniform interpolation

In a sense the dual problem to sampling is the interpolation of function values. A set $\Lambda \subseteq \mathbb{C}$ is interpolating for $\mathcal{F}^2$, if for every $a = (a_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ there exists $f \in \mathcal{F}^2$, such that $f(\lambda)e^{-\pi|\lambda|^2/2} = a_\lambda$. Equivalently, the set of normalized reproducing kernels $\kappa_\lambda = k_\lambda/\|k_\lambda\|_{\mathcal{F}^2} = k_\lambda/k(\lambda,\lambda)^{1/2}$ is a Riesz sequence, i.e., there exists $A, B > 0$, such that

$$A\|a\|_2^2 \le \|\sum_{\lambda\in\Lambda} a_\lambda \kappa_\lambda\|_{\mathcal{F}^2}^2 \le B\|a\|_2^2 \tag{17}$$

for all $a \in \ell^2(\Lambda)$. It suffices to require (17) only for all a with finite support.

In analogy to Marcinkiewicz-Zygmund families for sampling, we define uniform families for interpolation as follows. We denote the normalized reproducing kernels in $\mathcal{P}_n$ by $\kappa_{n,\lambda} = k_{n,\lambda}/\|k_{n,\lambda}\|_{\mathcal{F}^2}$.

Definition 6.1 A sequence of finite sets $\Lambda_n \subseteq \mathbb{C}$ is a uniform interpolating family for $\mathcal{P}_n$ in $\mathcal{F}^2$, if there exist constants $A, B > 0$ independent of n, such that for n large enough, $n \ge n_0$,

$$A\|a\|_2^2 \le \|\sum_{\lambda\in\Lambda_n} a_\lambda \kappa_{n,\lambda}\|_{\mathcal{F}^2}^2 \le B\|a\|_2^2 \qquad \text{for all } a \in \ell^2(\Lambda_n)\,. \tag{18}$$

Equivalently, for every $a \in \ell^2(\Lambda_n)$ there exists a polynomial $p \in \mathcal{P}_n$, such that

$$\frac{p(\lambda)}{k_n(\lambda,\lambda)^{1/2}} = a_\lambda \qquad \text{and} \qquad \|p\|_{\mathcal{F}^2}^2 \le A\|a\|_2^2\,.$$

A further equivalent condition is that the associated Gram matrix with entries $G_{\mu,\lambda} = \langle \kappa_{n,\lambda}, \kappa_{n,\mu}\rangle$ has the smallest eigenvalue $\lambda_{min} \ge A$ [18, Sect. 2.3 Lem. 2].

The relation between sets of interpolation for $\mathcal{F}^2$ and uniform interpolating families is similar to the case of sampling.

Theorem 6.1 *Assume that $\Lambda \subseteq \mathbb{C}$ is a set of interpolation for $\mathcal{F}^2$. For $\tau > 0$ define ρ_n via $\pi\rho_n^2 = n - \sqrt{n}(\sqrt{2\log n} + \tau)$. Then for every $\tau > 0$ large enough, the sets $\Lambda_n = \Lambda \cap B_{\rho_n}$ form a uniform interpolating family for $\mathcal{P}_n$ in $\mathcal{F}^2$.*

Proof Since $D^+(\Lambda) < 1$ is necessary for an interpolating set in $\mathcal{F}^2$ by [23], the definition of ρ_n implies that

$$\#(\Lambda \cap B_{\rho_n}) \le 1 \cdot |B_{\rho_n}| \le n$$

for $n \ge n_0$ large enough. Consequently $\Lambda_n = \Lambda \cap B_{\rho_n}$ contains at most n points.

We show that we can choose $\tau > 0$ in such a manner that, for $a \in \ell^2(\Lambda)$ with finite support and all $n \in \mathbb{N}$ sufficiently large,

$$\|\sum_{\lambda\in\Lambda_n} a_\lambda(\kappa_\lambda - \kappa_{n,\lambda})\|_{\mathcal{F}^2}^2 \le \frac{A}{4}\|a\|_2^2. \tag{19}$$

Then via the triangle inequality $\frac{A}{4}\|a\|_2^2 \le \|\sum_{\lambda\in\Lambda_n} a_\lambda\kappa_{n,\lambda}\|_{\mathcal{F}^2}^2 \le (B + \frac{A}{4} + \sqrt{AB})\|a\|_2^2$.

Denote the difference of the kernels by $e_\lambda = \kappa_\lambda - \kappa_{n,\lambda}$ and the Gram matrix of e_λ by E with entries $E_{\lambda,\mu} = \langle e_\mu, e_\lambda\rangle$, $\lambda, \mu \in \Lambda_n$. Then (19) amounts to saying the $\|E\|_{\mathrm{op}} \le A/4$. Since E is positive (semi-)definite, it suffices to bound the trace of E. To do this, consider the diagonal elements of E first. We see that

$$\begin{aligned} E_{\lambda,\lambda} &= \|\kappa_\lambda - \kappa_{n,\lambda}\|^2_{\mathcal{F}^2} \\ &= 2 - 2\,\mathrm{Re}\,\langle \kappa_\lambda, \kappa_{n,\lambda}\rangle \\ &= 2\Big(1 - \frac{\langle k_\lambda, k_{n,\lambda}\rangle}{k(\lambda,\lambda)^{1/2} k_n(\lambda,\lambda)^{1/2}}\Big) \\ &= 2\Big(1 - \frac{k_n(\lambda,\lambda)^{1/2}}{k(\lambda,\lambda)^{1/2}}\Big). \end{aligned}$$

Since $k_n(\lambda,\lambda) < k(\lambda,\lambda)$, the estimate for the diagonal elements simplifies to

$$E_{\lambda,\lambda} \le 2\Big(1 - \frac{k_n(\lambda,\lambda)}{k(\lambda,\lambda)}\Big) = 2\Big(1 - \frac{\Gamma(n+1, \pi|\lambda|^2)}{n!}\Big).$$

If $x \le n - \sqrt{n}\tau_n^2$ (with τ_n depending on n), then by Proposition 3.1(iii).

$$1 - \frac{\Gamma(n+1,x)}{n!} \le 1 - \frac{\Gamma(n+1, n - \sqrt{n}\tau_n)}{n!} \le e^{-\tau_n^2/2}$$

Combining these observations, we arrive at

$$\begin{aligned} \|E\|_{\mathrm{op}} &\le 2 \sum_{\lambda \in \Lambda \cap B_{\rho_n}} \Big(1 - \frac{\Gamma(n+1, \pi|\lambda|^2)}{n!}\Big) \\ &\le 2ne^{-\tau_n^2/2} \end{aligned}$$

By choosing $\tau_n = \sqrt{2\log n} + \tau$, with $\tau > 0$ large enough, we achieve $\|E\|_{\mathrm{op}} \le A/4$ for $n \ge n_0$. As we have seen, this suffices to conclude that $\kappa_{n,\lambda}$ is a Riesz sequence in $\mathcal{P}_n$ with lower constant independent of the degree n. □

Similar to the case of Marcinkiewicz-Zygmund families for sampling, we obtain uniform families for interpolation with the correct cardinality.

Corollary 6.1 *For every $\epsilon > 0$ there exist uniform interpolating families (Λ_n) for $\mathcal{P}_n$ in $\mathcal{F}^2$ with $\#\Lambda_n \ge (1-\epsilon)(n+1)$ points.*

Proof The proof is similar to the one of Corollary 4.1. □

Theorem 6.2 *Assume that (Λ_n) is a uniform interpolating family for the polynomials $\mathcal{P}_n$ in $\mathcal{F}^2$. Let Λ be a weak limit of (Λ_n) or of some subsequence (Λ_{n_k}). Then Λ is a set of interpolation for $\mathcal{F}^2$.*

Proof Let Λ be a weak limit of Λ_n (or some subsequence), and let $a \in \ell^2(\Lambda)$ with finite support in some disk B_{ρ_N} say. Enumerate $\Lambda \cap B_{\rho_N} = \{\lambda_j : j = 1, \dots, L\}$. By weak convergence, for every $\lambda_j \in \Lambda \cap B_{\rho_N}$ there is a sequence $\lambda_j^{(n)} \in \Lambda_n$, such that $\lim_{n\to\infty} \lambda_j^{(n)} = \lambda_j$.

We show that

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^{L} a_{\lambda_j} (\kappa_{\lambda_j} - \kappa_{n,\lambda_j^{(n)}}) \Big\|^2_{\mathcal{F}^2} = 0. \tag{20}$$

Consequently,

$$\Big\| \sum_{\lambda\in\Lambda\cap B_{\rho_N}} a_\lambda \kappa_\lambda \Big\|_{\mathcal{F}^2} = \lim_{n\to\infty} \Big\| \sum_{j=1}^{L} a_{\lambda_j}\kappa_{n,\lambda_j^{(n)}} \Big\|_{\mathcal{F}^2} \geq A \Big\| a \Big\|_2^2 ,$$

because (Λ_n) is a uniform interpolating family. Thus $\{\kappa_\lambda : \lambda \in \Lambda\}$ is a Riesz sequence in $\mathcal{F}^2$.

To show (20), we set $e_j = \kappa_{\lambda_j} - \kappa_{n,\lambda_j^{(n)}}$ and consider the associated Gramian with entries $E_{jk} = \langle e_k, e_j\rangle$. Again we use

$$\begin{aligned}\|E\|_{\mathrm{op}} \leq \operatorname{tr} E &= \sum_{j=1}^{L} \|e_j\|^2_{\mathcal{F}^2} = \sum_{j=1}^{L} \|\kappa_{\lambda_j} - \kappa_{n,\lambda_j^{(n)}}\|^2_{\mathcal{F}^2} \\ &= 2\sum_{j=1}^{L}\Big(1 - \operatorname{Re}\langle \kappa_{\lambda_j}, \kappa_{n,\lambda_j^{(n)}}\rangle\Big) .\end{aligned}$$

Consider a single term of this sum and write $\lambda_j^{(n)} = \lambda$ and $\lambda_j = \mu$ for fixed j. Note that $|\lambda - \mu| \leq 1$ for n large enough and that $\pi|\mu|^2 \leq N$ by the assumption that $\operatorname{supp} a \subseteq B_{\rho_N}$. Now

$$\begin{aligned}\operatorname{Re}\langle \kappa_{\lambda_j}, \kappa_{n,\lambda_j^{(n)}}\rangle &= \operatorname{Re} \frac{k_n(\lambda,\mu)}{k_n(\lambda,\lambda)^{1/2} k(\mu,\mu)^{1/2}} \\ &= e^{-\pi|\lambda-\mu|^2/2} \frac{\Gamma(n+1,\pi|\lambda|^2) + \big[\Gamma(n+1,\pi\lambda\bar{\mu}) - \Gamma(n+1,\pi|\lambda|^2)\big]}{\Gamma(n+1,\pi|\lambda|^2)^{1/2}\, n!^{1/2}} \\ &= e^{-\pi|\lambda-\mu|^2/2} \frac{\Gamma(n+1,\pi|\lambda|^2)^{1/2}}{n!^{1/2}} + e(n,\lambda,\mu) .\end{aligned}$$

Since

$$\frac{\Gamma(n+1,\pi\lambda\bar{\mu}) - \Gamma(n+1,\pi|\lambda|^2))}{n!} \leq e^{-n\eta}$$

for some $\eta > 0$ by Lemma 3.1 and $\frac{\Gamma(n+1,\pi|\lambda|^2)}{n!} \to 1$, the term $e(n,\lambda,\mu)$ tends to zero, as $n \to \infty$. By a similar reasoning, as $n \to \infty$ and thus $\lambda = \lambda_j^{(n)} \to \lambda_j = \mu$, we have

$$1 - e^{-\pi|\lambda-\mu|^2/2} \frac{\Gamma(n+1,\pi|\lambda|^2)^{1/2}}{n!^{1/2}} \to 0$$

for finitely many terms. Unraveling the notation, this means that $\operatorname{tr} E \to 0$ and (20) is proved. □

Proposition 6.1 *There is no Marcinkiewicz-Zygmund family (Λ_n) for $\mathcal{P}_n$ in $\mathcal{F}^2$ with $\#\Lambda_n = n+1$.*

Proof A set Λ_n with $n+1$ points is both sampling and interpolating for $\mathcal{P}_n$ with the same constants for interpolation as for sampling. By Theorem 5.1 any weak limit Λ of a Marcinkiewicz-Zygmund family is a sampling set for $\mathcal{F}^2$, and by Theorem 6.2 Λ is a set of interpolation for $\mathcal{F}^2$. This is a contradiction, since $\mathcal{F}^2$ does not admit any sets that are simultaneously sampling and interpolating. See, e.g., [23, Lemma 6.2]. □

7 Gabor frames for subspaces spanned by Hermite functions

By using the well-known connection between sampling in Fock space and the theory of Gaussian Gabor frames we may rephrase the main results in the language of Gabor frames for subspaces.

Recall that the Bargman transform is defined to be

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi z t - \pi t^2} \, dt \; e^{-\pi z^2/2}$$

for $z \in \mathbb{C}$. It maps functions and distributions on $\mathbb{R}$ to entire functions.

We use the following properties of the Bargman transform. See e.g., [8].

(i) The Bargman transform is unitary from $L^2(\mathbb{R})$ onto Fock space $\mathcal{F}^2$.

(ii) Let $\phi_z(t) = e^{-2\pi i y t} e^{-\pi(t-x)^2}$ denote the time-frequency shift of the Gaussian by $z = x + iy$. Then

$$B\phi_z(w) = k_z(w) = e^{\pi \bar{z} w}$$

is the reproducing kernel of $\mathcal{F}^2$.

(iii) B maps the normalized Hermite functions h_k,

$$h_k(t) = c_k e^{\pi t^2} \tfrac{d^k}{dt^k}(e^{-2\pi t^2}), \quad \|h_k\|_2 = 1,$$

to the monomials $e_k(z) = \left(\frac{\pi^k}{k!}\right)^{1/2} z^k$. With the Bargman transform all questions about the spanning properties of time-frequency shifts ϕ_z of the Gaussian can be translated into questions about the reproducing kernels k_z in Fock space. For instance, $\{\phi_\lambda : \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R})$, if and only if Λ is a sampling set for $\mathcal{F}^2$. Almost all statements about Gaussian Gabor frames have been obtained via complex analysis methods, notably the complete characterization of Gaussian Gabor frames by Lyubarski [17] and Seip [23] and many subsequent detailed investigations [6, 7]. To this line of thought we add a statement about Gabor frames for distinguished subspaces spanned by Hermite polynomials. Constructions of this type have been used in signal processing [14].

Theorem 7.1 *Assume that Λ is a sampling set for $\mathcal{F}^2$, or equivalently $\mathcal{G}(h_0, \Lambda) = \{\phi_\lambda : \lambda \in \Lambda\}$ is Gabor frame in $L^2(\mathbb{R})$, then $\{\phi_\lambda : \pi|\lambda|^2 \le n + \sqrt{n}\tau\}$ is a frame for $V_n = \operatorname{span}\{h_k : k = 0, \dots, n\}$ with bounds independent of n, i.e.,*

$$A\|f\|_2 \le \sum_{\lambda \in \Lambda : \pi|\lambda|^2 \le n + \sqrt{n}\tau} |\langle f, \phi_\lambda \rangle|^2 \le B\|f\|_2^2 \qquad \text{for all } f \in V_n \,.$$

Proof The statement is equivalent to Theorem 4.1 via the Bargman transform.[2] □

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[2] Since $\phi_\lambda \notin V_n$, some authors use the term "pseudoframe" for this situation.

Appendix

For completeness we present some elementary estimates for the zero order asymptotics of the incomplete gamma function that imply Corollary 3.1. Using Stirling's formula for $\left(\frac{n}{e}\right)^n\sqrt{2\pi n} \le n! \le e\sqrt{n}\left(\frac{n}{e}\right)^n$, we write

$$\frac{\Gamma(n+1, n+\sqrt{n}\tau)}{n!} = \frac{1}{n!}\int_{n+\sqrt{n}\tau}^{\infty} t^n e^{-t}\, dt \asymp \frac{1}{\sqrt{n}}\int_{n+\sqrt{n}\tau}^{\infty} \left(\frac{t}{n}\right)^n e^{n-t}\, dt\,.$$

(The constants in the equivalence are in $[e^{-1}, (2\pi)^{-1/2}]$.) Using the substitution $u = \frac{t}{n} - 1$ we obtain

$$\begin{aligned}\frac{\Gamma(n+1, n+\sqrt{n}\tau)}{n!} &\asymp \sqrt{n}\int_{\tau/\sqrt{n}}^{\infty}(1+u)^n e^{-nu}\, du \\ &= \sqrt{n}\int_{\tau/\sqrt{n}}^{\infty} e^{-n(u-\ln(1+u))}\, du = \sqrt{n}\Big(\int_{\tau/\sqrt{n}}^{1}\cdots + \int_{1}^{\infty}\ldots dt\Big)\,.\end{aligned} \tag{21}$$

Using the inequality $u - \ln(1+u) \ge (1-\ln 2)u$ valid for $u \in [1,\infty)$, the latter integral is bounded by

$$\sqrt{n}\int_1^{\infty} e^{-n(u-\ln(1+u))}\, du \le \sqrt{n}\int_1^{\infty} e^{-n(1-\ln 2)u}\, du \le \frac{\sqrt{n}}{n(1-\ln 2)} = \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\,. \tag{22}$$

In the first integral in (21) we use the power series of ln and obtain, for $u \in [0, 1]$,

$$u - \ln(1+u) = \sum_{k=2}^{\infty}\frac{(-1)^k}{k}u^k \ge \frac{u^2}{2} - \frac{u^3}{3} \ge \frac{u^2}{6}\,.$$

Consequently,

$$\begin{aligned}\sqrt{n}\int_{\tau/\sqrt{n}}^{1}(1+u)^n e^{-nu}\, du &\le \sqrt{n}\int_{\tau/\sqrt{n}}^{1} e^{-nu^2/6}\, du \\ &= \sqrt{n}\sqrt{\frac{6}{n}}\int_{\tau/\sqrt{6}}^{\sqrt{n/6}} e^{-v^2}\, dv \lesssim \operatorname{erf}(\tau/\sqrt{6}) \le e^{-\tau^2/6}\,.\end{aligned}$$

From $u - \ln(1+u) \le u^2/2$ we obtain the lower bound

$$\begin{aligned}\sqrt{n}\int_{\tau/\sqrt{n}}^{1}(1+u)^n e^{-nu}\, du &\ge \sqrt{n}\int_{\tau/\sqrt{n}}^{1} e^{-nu^2/2}\, du \\ &= \sqrt{n}\sqrt{\frac{2}{n}}\int_{\tau/\sqrt{2}}^{\sqrt{n/2}} e^{-v^2} \gtrsim \operatorname{erf}(\tau/\sqrt{2}) \gtrsim \frac{1}{\tau}e^{-\tau^2/2}\,,\end{aligned}$$

for n large enough. Items (i) and (ii) of Proposition 3.1 now follow easily from these estimates.

A weaker version of item (v) follows by setting $\tau = 0$ in the above estimates.

(iii) Similarly,

$$\begin{aligned}1 - \frac{\Gamma(n+1, n-\sqrt{n}\tau)}{n!} &\le \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n\int_0^{n-\sqrt{n}\tau} t^n e^{-t}\, dt = \\ &= \frac{1}{\sqrt{2\pi n}}\int_0^{n-\sqrt{n}\tau}\left(\frac{t}{n}\right)^n e^{n-t}\, dt\,.\end{aligned}$$

With the substitution $u = 1 - \frac{t}{n}$ we obtain

$$\frac{1}{\sqrt{2\pi n}} \int_0^{n-\sqrt{n}\tau} \left(\frac{t}{n}\right)^n e^{n-t}\, dt = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\tau/\sqrt{n}}^1 (1-u)^n e^{nu}\, du =$$
$$= \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\tau/\sqrt{n}}^1 e^{n(u+\ln(1-u))}\, du = \frac{\sqrt{n}}{\sqrt{2\pi}} \Big(\int_{\tau/\sqrt{n}}^{1/2} \cdots + \int_{1/2}^1 \ldots dt \Big). \tag{23}$$

Since $u + \ln(1-u) \le 1/2 - \ln 2 < 0$ on the interval $[1/2, 1]$, the second term decays exponentially in n. For the first term we use $u + \log(1-u) = -\sum_{k=2}^\infty \frac{u^k}{k} \le -u^2/2$ and obtain

$$\frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\tau/\sqrt{n}}^{1/2} e^{n(u+\ln(1-u))}\, du \le \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{\tau/\sqrt{n}}^{1/2} e^{-nu^2/2}\, du$$
$$= \frac{1}{\sqrt{2\pi}} \int_\tau^{\sqrt{n}/2} e^{-v^2/2}\, du \le \tfrac{1}{2} e^{-\tau^2/2}.$$

(iv) follows from (iii). If $n - \sqrt{n}\tau \ge x$, then

$$\frac{\Gamma(n+1, x)}{n!} \ge \frac{\Gamma(n+1, n-\sqrt{n}\tau)}{n!} \ge 1 - e^{-\tau^2}.$$

Since this holds arbitrary τ and n large, we obtain $\lim_{n\to\infty} \frac{\Gamma(n+1,x)}{n!} = 1$.

References

1. Ameur, Y.: Near-boundary asymptotics for correlation kernels. J. Geom. Anal. **23**(1), 73–95 (2013)
2. Ameur, Y., Hedenmalm, H.K., Makarov, N.: Berezin transform in polynomial Bergman spaces. Comm. Pure Appl. Math. **63**(12), 1533–1584 (2010)
3. Ameur, Y., Ortega-Cerdà, J.: Beurling-Landau densities of weighted Fekete sets and correlation kernel estimates. J. Funct. Anal. **263**(7), 1825–1861 (2012)
4. Ameur, Y., Romero, J. L.: The planar low temperature Coulomb gas: separation and equidistribution. To appear in Rev. Mat. Iberoam. https://doi.org/10.4171/RMI/1340
5. Ameur, Y., Cronvall, J.: Szegö type asymptotics for the reproducing kernel in spaces of full-plane weighted polynomials. 2021. Preprint, arXiv:2107.11148
6. Ascensi, G., Lyubarskii, Y., Seip, K.: Phase space distribution of Gabor expansions. Appl. Comput. Harmon. Anal. **26**(2), 277–282 (2009)
7. Belov, Y., Borichev, A., Kuznetsov, A.: Upper and lower densities of Gabor Gaussian systems. Appl. Comput. Harmon. Anal. **49**(2), 438–450 (2020)
8. Folland, G.B.: Harmonic Analysis in Phase Space. Princeton Univ. Press, Princeton, NJ (1989)
9. Gil, A., Segura, J., Temme, N.M.: Efficient and accurate algorithms for the computation and inversion of the incomplete gamma function ratios. SIAM J. Sci. Comput. **34**(6), A2965–A2981 (2012)
10. Gröchenig, K.: Irregular sampling, Toeplitz matrices, and the approximation of entire functions of exponential type. Math. Comp. **68**(226), 749–765 (1999)
11. Gröchenig, K.: Sampling, Marcinkiewicz-Zygmund inequalities, approximation, and quadrature rules. *J. Approx. Theory*, 257:105455, 20, (2020)
12. Gröchenig, K., Ortega-Cerdà, J.: Marcinkiewicz-Zygmund inequalities for polynomials in Bergman and Hardy spaces. J. Geom. Anal. **31**(7), 7595–7619 (2021)
13. Gröchenig, K., Ortega-Cerdà, J., Romero, J.L.: Deformation of Gabor systems. Adv. Math. **277**, 388–425 (2015)
14. Hlawatsch, F.: Time-frequency Analysis and Synthesis of Linear Signal Spaces: Time-Frequency Filters Signal Detection and Estimation, and Range-Doppler Estimation. Kluwer Academic Publishers, Boston (1998)
15. Kashin, B., Kosov, E., Limonova, I., Temlyakov, V.: Sampling discretization and related problems. *J. Complexity* 71, (2022), Paper No. 101653

16. Lev, N., Ortega-Cerdà, J.: Equidistribution estimates for Fekete points on complex manifolds. J. Eur. Math. Soc. (JEMS) **18**(2), 425–464 (2016)
17. Lyubarskiĭ, Y. I.: Frames in the Bargmann space of entire functions. In *Entire and subharmonic functions*, pages 167–180. Amer. Math. Soc., Providence, RI, (1992)
18. Meyer, Y.: *Wavelets and operators.* Cambridge University Press, (1992)
19. Nemes, G.: The resurgence properties of the incomplete gamma function. I. Anal. Appl. (Singap.) **14**(5), 631–677 (2016)
20. Nemes, G., Olde Daalhuis, A.B.: Asymptotic expansions for the incomplete gamma function in the transition regions. Math. Comp. **88**(318), 1805–1827 (2019)
21. *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.2 of 2021-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds
22. Ortega-Cerdà, J., Saludes, J.: Marcinkiewicz-Zygmund inequalities. J. Approx. Theory **145**(2), 237–252 (2007)
23. Seip, K.: Density theorems for sampling and interpolation in the Bargmann-Fock space. I. J. Reine Angew. Math. **429**, 91–106 (1992)
24. Seip, K.: Interpolation and sampling in spaces of analytic functions. University Lecture Series, vol. 33. American Mathematical Society, Providence, RI (2004)
25. Seip, K., Wallstén, R.: Density theorems for sampling and interpolation in the Bargmann-Fock space. II. J. Reine Angew. Math. **429**, 107–113 (1992)
26. Simon, B.: *Basic complex analysis*. A Comprehensive Course in Analysis, Part 2A. American Mathematical Society, Providence, RI, (2015)
27. Temme, N.M.: Uniform asymptotic expansions of the incomplete Gamma functions and the incomplete beta function. Math. Comp. **29**(132), 1109–1114 (1975)
28. Tricomi, F.G.: Asymptotische Eigenschaften der unvollständigen Gammafunktion. Math. Z. **53**, 136–148 (1950)