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Stationary Reflection on $\mathcal{P}_\kappa(\lambda)$

Martha Catalina Torres Pachón

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Advisor : Prof. Joan Bagaria Pigrau

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Introduction

Throughout history, mathematicians have had to deal with infinity, always considering it in the "potential" sense, rather than an actual object. It was not until the late nineteenth century that actual infinity was the subject matter. In 1874 George Cantor published "*On a Property of the Collection of All Real Algebraic Numbers*". From the results he proved in that paper, he concluded that there were larger infinites than others, giving birth in this way to Set Theory, the study of infinite sets and the set-theoretic foundations of mathematics.

The study of infinite sets, and in particular their combinatorial properties, is not only of interest in itself, but it has numerous applications in areas such as analysis, algebra and topology (see e.g. [1; 2; 3]). Even possible applications to mathematical biology have being studied [4]. Combinatorics is always concerned about sizes, and when dealing with infinite sets there are different ways to capture the idea of how large a set is. For example, the notion of "filter" on a set A corresponds to "big" subsets of A, while positive subsets in the sense of a given filter corresponds to the notion of "not small". Stationary subsets of a cardinal κ are those that are not small in the sense of the closed and unbounded filter of κ.

The study of stationary subsets of cardinals of uncountable cofinality, and of stationary reflection, has a long history (see [5; 6; 7; 8; 9]) and it has found many applications in other areas such e.g., Abelian groups and modules (see [1]). This study has been developed very recently with the new notion of *hyperstationarity* [10; 11; 12], namely an iterated recursive definition of reflection of stationary sets. Its interest lays in its connection with the study of derived topologies on the ordinal numbers, as well as in its potential applications to other areas, such as proof theory and modal logic.

While the consistency strength of hyperstationarity is rather low in the large-cardinal hierarchy (below a measurable cardinal), its generalization to $\mathcal{P}_\kappa\lambda$ promises to be much stronger, possibly close to the level of supercompactness. Thus, the formulation of the appropriate generalization of hyperstationarity for $\mathcal{P}_\kappa\lambda$ and the development of its theory, in analogy with the notion of hyperstationarity for cardinals should allow for more interesting applications at a much higher level, in terms of consistency strength.

In the present work, we study the notion of n-stationarity in $P_\kappa(\lambda)$ proposed by Sakai in his presentation "*On generalized notions of hyperstationarity*" [13] and a slight modification of the same. We develop some of the consequences of this definitions and we look at which of the results obtained by Bagaria in his article "*Derived Topologies on Ordinals and Stationary Reflection*" [12] can be obtained within the context of $\mathcal{P}_\kappa(\lambda)$.

In Chapter 1 we provide the reader with the necessary framework to understand the following chapters. All of these notions and results are elementary and can be found in almost any text book on set theory, e.g. [6; 14].

Chapter 2 is a review of [12]. We expose, however the results and definitions in [12] in such a way that our results on their generalisation can be easily shown as such.

InChapter 3 we face our main objective; Section 3.1 places us in the context of combinatorics in $\mathcal{P}_\kappa(\lambda)$ [7; 15; 16]. Sections 3.2 and 3.3 contain our main results, e.g., Theorems 3.2.6, 3.2.10 , 3.2.12 and 3.2.14. In Section 3.4 we aim to establish some of the conjectures and to cite the known work concerning Π^1_n-indescribability in $\mathcal{P}_\kappa(\lambda)$ and its possible relation with n-stationary subsets of $\mathcal{P}_\kappa(\lambda)$, [13; 17; 18; 24].

Finally, in Chapter 4 we summarise the most important results, making clear at which point we achieve our objective of translating the results of [12] to $\mathcal{P}_\kappa(\lambda)$. We also comment on those results we could not obtain, and possible ways of sorting them out. We also conclude our work with a list of open questions for further research on this topic, some of them already proposed in [13], and some of them being the result of our work.

Chapter 1

Preliminaries

The aim of this chapter is to do a compendium of the set theoretic background we use along the main chapters of this work, as well as fixing the notation we will use from now on. We assume the reader is familiar with the standard notions of first-order logic and basic set theory. In order to avoid this chapter to be unnecessaryly long, we omit most of the proofs. Nevertheless, everything we expose in here can be found in almost any book of set theory, in particular in [6; 14; 20; 21].

1.1 Models of ZFC

In the early twentieth century Ernst Zermelo and Abraham Fraenkel proposed an axiomatic system (ZF) in order to formulate the theory of sets and along with that all formal mathematics. When adding the axiom of choice to ZF we obtained the axiomatic system ZFC, which has been the standard axiomatic theory in which almost all modern mathematics are framed out. In the present work we also work in ZFC.

Definition 1.1.1. *A* ***model*** *of (a fragment of) ZFC is a pair $\langle M, E\rangle$, where M is a non-empty set or a proper class and E is a binary relation on M such that $\langle M, E\rangle$ satisfies the (fragment of) ZFC axioms (we write $\langle M, E\rangle \models ZFC$).*

Definition 1.1.2. *Let $\langle M, E\rangle$ be a model of (a fragment of) ZFC*

1. *$\langle M, E\rangle$ is called* ***standard*** *if E is $\in$, that is, the membership relation between sets. More precisely, $E = \in \cap (M \times M)$. If $\langle M, E\rangle$ is standard, then we usually write $\in$ instead of E.*

2. *$\mathcal{M} = \langle M, E\rangle$ is* ***transitive*** *if the relation E is transitive. This is, if for every $a, b, c \in M$, aEb and bEc implies that aEc.*

3. *$\mathcal{M} = \langle M, E\rangle$ is* ***well-founded*** *if*

   (a) *E is well-founded. This is, there is no infinite descending E-chain $\ldots x_{n+1}Ex_n \ldots x_2Ex_1$ Ex_0 of elements of M,*

   (b) *E is set-like. This is, for every $x \in M$, the class $\{y \in M : yEx\}$ is a set.*

Recall that the language of Set Theory is the language of first order logic with equality plus the binary relation $\in$. Suppose that $\langle M, \in\rangle$ is a model of (a fragment of) ZFC, and $R \subseteq M$, then we use the notation $\langle M, \in, R\rangle$ when referring to the same model $\langle M, \in\rangle$, but in which the language has been expanded by adding R as a new predicate symbol.

Levy hierarchy of formulas: A formula in a language that contains the language of set theory is Σ_0 if has only bounded quantifiers $\forall x \in y$ and $\exists x \in y$. A formula is Σ_n for $n \geq 1$ if it is of the form

$$\exists x_0, \ldots, \exists x_k \varphi(x_0, \ldots, x_k, y_0, \ldots y_l)$$

where $\varphi(x_0, \dots, x_k, y_0, \dots y_l)$ is Π_{n-1}. And a formula is Π_n for $n \geq 1$ if it is of the form

$$\forall x_0, \dots, \forall x_k \varphi(x_0, \dots, x_k, y_0, \dots y_l)$$

where $\varphi(x_0, \dots, x_k, y_0, \dots y_l)$ is Σ_{n-1}.

More in general for every m, a formula in a language that contains the $m+1$- order language of set theory is Σ_0^m (or Π_0^m) if it does not have quantifiers of $m+1$-order, but it may have any number of quantifiers of order $\leq m$, and free variables of $m+1$-order. A formula is Σ_n^m for $n \geq 1$ if it is of the form

$$\exists X_0, \dots, \exists X_k \varphi(X_0, \dots, X_k, Y_0, \dots Y_l)$$

where $X_0, \dots, X_k$ are variables of order $m+1$ and $\varphi(X_0, \dots, X_k, Y_0, \dots Y_l)$ is Π_{n-1}. And a formula is Π_n for $n \geq 1$ if it is of the form

$$\forall x_0, \dots, \forall x_k \varphi(X_0, \dots, X_k, Y_0, \dots Y_l)$$

where $X_0, \dots, X_k$ are variables of order $m+1$ and $\varphi(X_0, \dots, X_k, Y_0, \dots Y_l)$ is Σ_{n-1}.

Definition 1.1.3. *Let $\langle M, \in \rangle$ and $\langle N, \in \rangle$ be two models of (a fragment of) ZFC. A function $j : M \to N$ is an elementary embedding if for every formula $\varphi(x_0, \dots, x_n)$ of the language of set theory and every $a_1, ..., a_n \in M$,*

$$\langle M, \in \rangle \models \varphi(a_0, \dots, a_n) \text{ if and only if } \langle N, \in \rangle \models \varphi(j(a_0), \dots, j(a_n)).$$

1.2 Ordinals and Cardinals

Definition 1.2.1. *An **ordinal** is a transitive set well-ordered by $\in$. This is, a set containing all elements of its elements and such that every non-empty subset of it has an $\in$-minimal element. We denote by On the class of all ordinals.*

If α is an ordinal, then the set $\alpha \cup \{\alpha\}$ is the least ordinal greater than α, and we define $\alpha + 1 := \alpha \cup \{\alpha\}$. An ordinal $\alpha > 0$ is called a *successor ordinal* whenever $\alpha = \beta + 1$ for some ordinal β, and is called a *limit ordinal* otherwise. If α is a limit ordinal, then for every $\beta < \alpha$ there is some $\gamma < \alpha$ such that $\beta < \gamma$.

A model $\langle M, \in \rangle$ of ZF is said to be an **inner model** whenever $On \subseteq M$ and M is transitive. As in the case of the natural numbers the set "*On*" of all ordinals, also satisfies a form of induction principle and recursion theorem.

Theorem 1.2.2. *(Transfinite Induction) Given a formula $\varphi(x)$ in the language of set theory, if*

1. *$\varphi(0)$,*
2. *for every ordinal α, if $\varphi(\alpha)$, then $\varphi(\alpha + 1)$,*
3. *for every ordinal α, if for each $\beta < \alpha$ it holds that $\varphi(\beta)$, then $\varphi(\alpha)$.*

Then, for all α ordinal it holds that $\varphi(\alpha)$.

Theorem 1.2.3. *(Transfinite Recursion) If G is a set-theoretic operation, there exists a unique set-theoretic operation F, such that for every ordinal α,*

$$F(\alpha) = G(F \upharpoonright \alpha)$$

Definition 1.2.4. *We say that κ is a **cardinal** if it is an ordinal and it is not bijectable with any ordinal smaller than κ.*

As in the case of ordinals, the *successor* cardinal of a given a cardinal κ is the least cardinal greater than κ, and is denoted by κ^+. If $\kappa > 0$ is not a successor cardinal, then we say it is a limit cardinal. And If κ is a limit cardinal, for every $\gamma < \kappa$ there is some μ cardinal less than κ such that $\gamma < \mu$.

It follows from the Principle of Well Ordering that every set A is bijectable with a unique cardinal. Given A, this unique cardinal is denoted by $|A|$. Moreover if $A \subseteq B$, then $|A| \leq |B|$. And if A, B are infinite sets, $|A \cup B| = \max\{|A|, |B|\}$.

Definition 1.2.5. *Let α be a limit ordinal. If $A \subseteq \alpha$, we say that A is* ***cofinal in*** *α if $\sup A = \alpha$. In particular, an increasing sequence $\langle \alpha_\xi : \xi < \beta \rangle$ where β is a limit ordinal, is cofinal in α if $\sup\{\alpha_\xi : \xi < \beta\} = \alpha$. If α is infinite, we define the* ***the cofinality of*** *α as follows,*

$$cof(\alpha) = \text{ the least ordinal } \beta \text{ such that there is an increasing sequence}$$

$$\langle \alpha_\xi : \xi < \beta \rangle \text{ such that } \sup\{\alpha_\xi : \xi < \beta\} = \alpha.$$

Intuitively, the concept of cofinality is telling us how long is the the shortest path to reach an ordinal. It is clear from the definition that for every α limit ordinal, $cof(\alpha)$ is a cardinal and that $cof(\alpha) \leq \alpha$. Notice for example that $cof(\omega) = \omega$. And for $\aleph_\omega$ the increasing sequence $\langle \aleph_n : n < \omega \rangle$ is such that $\sup\{\aleph_n : n < \omega\} = \aleph_\omega$, therefore $cof(\aleph_\omega) = \omega < \aleph_\omega$.

Definition 1.2.6. *A limit ordinal α is* ***regular*** *if and only if $cof(\alpha) = \alpha$, and it is* ***singular*** *if $cof(\alpha) < \alpha$.*

Given α limit ordinal $cof(\alpha)$ is always a regular cardinal. Note that $cof(\omega) = \omega$, so ω is a regular ordinal (in fact it is the least regular ordinal). Also since $cof(\aleph_\omega) = \omega$, we have that $\aleph_\omega$ is a singular ordinal.

Although we will study formally this in detail in the next chapter, let us informally introduce the following concepts concerning to the subsets of a given limit cardinal. Let κ be a limit ordinal of uncountable cofinality

- $T \subseteq \kappa$ is **unbounded in** κ iff for any $\beta < \kappa$ there is some $\gamma \in T$ such that $\beta \leq \gamma$.
- $C \subseteq \kappa$ is **closed in** κ iff for any $\{\beta_\xi : \xi < \gamma\} \subseteq C$ such that $\beta_\xi < \beta_\zeta$ for $\xi < \zeta \leq \gamma$, then, $\sup\{\beta_\xi : \xi < \gamma\} \in C$ whenever $\sup\{\beta_\xi : \xi < \gamma\} < \kappa$.
- $C \subseteq \kappa$ is a **club** subset of κ iff it is closed and unbounded in κ.
- $S \subseteq \kappa$ is **stationary in** κ iff for any C club in κ, $S \cap C \neq \varnothing$.

Definition 1.2.7. *A cardinal κ is said to be a* ***weakly inaccessible cardinal*** *if it is a regular uncountable limit cardinal.*

We know that $\omega = \aleph_0$ is a regular limit cardinal but it is clearly not countable. $\omega_1 = \aleph_1$ is an uncountable regular cardinal, but it is not a limit cardinal. $\aleph_\omega$ is an uncountable limit cardinal, but it is not regular. In general, we do not have an example of a weakly inaccessible cardinal. In fact, assuming ZFC is consistent from the axioms of ZFC it cannot be proved that weakly inaccessible cardinals exist.

Next proposition is a very useful characterisation of infinite regular cardinals that we will use along this work.

Proposition 1.2.8. *The following conditions are equivalent for an infinite cardinal κ.*

1. *κ is regular.*
2. *Every subset of κ of cardinality less than κ is bounded in κ.*
3. *The union of every family of less than κ sets each of cardinality less than κ is a set of cardinality less than κ.*

Definition 1.2.9. *Let κ, λ and μ be cardinals, and suppose A is a set such that $|A| \geq \kappa$. We define*

$$\lambda^\kappa := |\{f : \kappa \to \lambda : f \text{ is a function }\}|$$

$$\lambda^{<\kappa} := \sup\{\lambda^\mu : \mu \text{ is a cardinal and } \mu < \kappa\}$$

$$\mathcal{P}_\kappa(A) = [A]^\kappa := \{X \subseteq A : |X| < \kappa\}$$

It is easy to see that $|\mathcal{P}_\kappa(A)| = |A|^{<\kappa}$. In particular we have that $|\mathcal{P}_\kappa(\lambda)| = \lambda^{<\kappa}$.

Theorem 1.2.10. *(G. Cantor) For every set A, it holds that $|A| < |\mathcal{P}(A)| = 2^{|A|}$.*

Proposition 1.2.11. *Let κ, λ be infinite cardinals, and μ be any cardinal, then*

1. *If $\lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.*
2. *If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$.*
3. *If $\kappa \leq \lambda$, then $2^\lambda = \kappa^\lambda = \lambda^\lambda$.*

Definition 1.2.12. *A cardinal κ is a* ***strong limit cardinal*** *if $2^\lambda < \kappa$ for every $\lambda < \kappa$.*

Notice that every strong limit cardinal is a limit cardinal, and the converse holds under the GCH, this is, under the Generalised Continuum Hypothesis stating that for all cardinal κ, $2^\kappa = \kappa^+$. $\aleph_0$ is the least strong limit cardinal. It follows from Cantor's Theorem (1.2.10) that every strong limit cardinal is indeed a limit cardinal.

Definition 1.2.13. *A cardinal κ is (strongly)* ***inaccessible*** *if it is uncountable, regular, and strong limit.*

Every inaccessible cardinal is weakly inaccessible. Moreover, if GCH holds, κ is weakly inaccessible and $\lambda < \kappa$, then $2^\lambda = \lambda^+ < \kappa$ and so κ cardinal is inaccessible.

Definition 1.2.14. *(The cumulative hierarchy of well-founded sets)*

$$V_0 = \varnothing$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \quad \textit{for all } \alpha$$

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \textit{for all limit } \alpha$$

Since we are working in ZFC, one can prove that the universe of all sets is in fact the proper class $V = \bigcup_{\alpha \in OR} V_\alpha$.

1.3 Filters and Ideals

One of the most recurrent notions in all branches of mathematics is the notion of an ideal and the one of a filter. In all branches these concepts play a rather important role, for example the ideals in algebra or the filters in topology. The importance of these notions is that they capture the intuitive idea of smallness and bigness respectively, and these turn out to be extremely useful when studying the subsets of a given set.

Definition 1.3.1. *Let A be a non-empty set. A* ***filter*** *on A is a set F of subsets of A such that:*

1. *$A \in F$ and $\varnothing \notin F$.*
2. *If $X, Y \in F$, then $X \cap Y \in F$.*
3. *If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.*

Definition 1.3.2. *Let A be a non-empty set. An* ***ideal*** *on A is a set I of subsets of A such that:*

1. *$\varnothing \in I$.*
2. *If $X, Y \in I$, then $X \cup Y \in I$.*
3. *If $X \in i$ and $Y \subseteq X$, then $Y \in I$.*

The sets $\{A\}$ and $\{\varnothing\}$ are, respectively, trivial examples of filter an ideal on A. Moreover, the set $F = \{Y \subseteq A : X \subseteq Y\}$ of all subsets of A extending a non-empty given subset X of A constitutes a filter. A filter expressible in these terms is called a **principal** filter. Another interesting example of a filter is the *Fréchet filter* on a given cardinal κ, this is, the set $\{X \subseteq \kappa : |\kappa \setminus X| < \kappa\}$.

Proposition 1.3.3. *If F is a filter on A, then $F^* := \{A \setminus X : X \in F\}$ is an ideal on A . And if I is an ideal on A, then $I^* := \{A \setminus X : X \in I\}$ is a filter on A. Moreover, if F is a filter, then $(F^*)^* = F$. And if I is an ideal, then $(I^*)^* = I$.*

The sets F^* and I^* given by Proposition 1.3.3 are respectively called the **dual ideal** of F and the **dual filter** of I.

Let A be a set, and I be an ideal on A. We define the collection I^+ of I-**positive** subsets of A as follows

$$I^+ := \{X \subseteq A : X \notin I\}$$

If additionally, we have that $I = F^*$ for some filter F and $X \in I^+$, we also say that X is F-**positive**.

Notice that the F-positive sets with respect to the Fréchet filter F on κ are precisely the subsets of κ of cardinality κ.

Definition 1.3.4. *A filter F on a set A is called an **ultrafilter** if for every $X \subseteq A$, either $X \in F$ or $A \setminus X \in F$.*

Equivalently, a filter F is an ultrafilter if and only if F is maximal in the sense that there is no proper filter G such that $G \subseteq F$. From Zorn's Lemma it can be easily proved the following theorem.

Theorem 1.3.5. *(A. Tarski.) Every filter can be extended to an ultrafilter.*

Definition 1.3.6. *Let κ be an infinite cardinal. A filter F on a set A is called κ-**complete** if for every family $\{X_\alpha : \alpha < \gamma\}$, $\gamma < \kappa$, of elements of F, the intersection $\bigcap_{\alpha<\gamma} X_\alpha$ belongs to F. Dually, an ideal I on a set A is called κ-**complete** if for every family $\{X_\alpha : \alpha < \gamma\}$, $\gamma < \kappa$, of elements of I, the union $\bigcup_{\alpha<\gamma} X_\alpha$ belongs to I*

If F is a principal filter on A, say $F = \{Y \subseteq A : X \subseteq Y\}$ for some $X \subseteq A$, then any intersection of elements of F will still contain the set X. Thus, any principal filter on A is κ complete for every κ. Therefore, in terms of combinatorics it is more interesting to consider non-principal filters on a given set. Note that for every uncountable regukar κ the Fréchet filter on κ is κ-complete.

Definition 1.3.7. *Let F be a filter over a set A. A set $X \subseteq A$ is said to be F-**stationary** (or stationary with respect to the filter F) if and only if $X \cap Y = \varnothing$ for all $Z \in F$. (See [14]).*

1.4 Large Cardinals

Intuitively, a large cardinal is a cardinal that is so large that we cannot prove its existence within ZFC. In this sense, the first kind of large cardinal we mentioned was weakly inaccessible cardinals. As inaccessibility implies weakly inaccessibility, inaccessible cardinals are also large cardinals. We will state in here a list of large cardinals we will use in the next chapters. Usually large cardinals have several equivalent definitions, so we shall use the one that we find more useful for our purposes.

Definition 1.4.1. *A cardinal κ is called **weakly Mahlo** if and only if the set $\{\mu < \kappa : \mu$ is regular$\}$ is stationary in κ.*

If κ is weakly Mahlo, then it is clearly a limit uncountable cardinal. It is easy to see that κ is also regular. Hence every weakly Mahlo cardinal is in particular a weakly inaccessible cardinal.

Definition 1.4.2. *Let κ be an ordinal*

1. *κ is 0-**weakly Mahlo** if and only if κ is regular.*

2. κ *is* $\alpha+1$-***weakly Mahlo*** *if and only if the set* $\{\mu<\kappa : \mu \text{ is } \alpha \text{ Mahlo}\}$ *is stationary in* κ.

3. κ *is* α-***weakly Mahlo*** *for* α *limit, if and only if,* κ *is* β*-weakly Mahlo for all* $\beta<\alpha$.

Notice that κ is 1-weakly Mahlo if and only if it is weakly Mahlo. Moreover it can be proven by induction that if $\alpha<\beta$ and κ is a β-weakly Mahlo cardinal, then κ is also is an α-weakly Mahlo cardinal.

Definition 1.4.3. *A cardinal* κ *is called* ***Mahlo*** *if and only is the set* $\{\mu<\kappa : \mu \text{ is inaccessible}\}$ *is stationary in* κ.

Proposition 1.4.4. *If* κ *is a Mahlo cardinal, then* κ *is inaccessible.*

Definition 1.4.5. *Let* $n<\omega$, *cardinal* κ *is* Π^1_n-***indescribable*** *if for all subsets* $A\subseteq V_\kappa$ *and every* Π^1_n *sentence* φ, *if* $\langle V_\kappa,\in,A\rangle\models\varphi$, *then there is some* $\lambda<\kappa$ *such that*

$$\langle V_\lambda,\in,A\cap V_\lambda\rangle\models\varphi.$$

Proposition 1.4.6. *If* κ *is a* Π^1_n*-indescribable cardinal, then* κ *is* Π^1_m*-indescribable for all* $m<n$

Proposition 1.4.7. κ *is a* Π^1_0 *indescribable cardinal if and only if* κ *is inaccessible.*

A cardinal κ is called **weakly compact** if it is Π^1_1 indescribable. Every weakly compact cardinal is a Mahlo cardinal.

Definition 1.4.8. *An uncountable cardinal* κ *is called* ***measurable*** *if there exists a* κ*-complete non-principal ultrafilter* U *on* κ.

Proposition 1.4.9. *If* κ *is a measurable cardinal, then* κ *is weakly compact.*

Proposition 1.4.10. *If* κ *is a measurable cardinal,* U *a* κ*-complete ultrafilter on* κ *and* j *an elementary embedding from* V *to an inner model* M, *then* ${}^\kappa M\subseteq M$.

Definition 1.4.11. *Let* $\kappa\leq\lambda$. κ *is* λ-***supercompact*** *if and only if there is an elementary embedding* $j:V\to M$ *such that* $crit(j)=\kappa$, $\lambda\leq j(\kappa)$ *and* ${}^\lambda M\subseteq M$.

Proposition 1.4.12. *If* κ *is* κ*-supercompact, then* κ *is measurable.*

Definition 1.4.13. κ *is* ***supercompact*** *if and only if* κ *is* λ*-supercompact for every* $\lambda\geq\kappa$.

Chapter 2

Hyperstationary subsets of κ.

The aim of this chapter is to introduce some basic aspects of combinatorics on ordinals, to exhibit the concept of hyperstationarity and to show some of the results obtained by Bagaria in his article "*Derived Topologies On Ordinals and Stationary Reflection*" [12]. Basic definitions and results can be found in [6; 14]. Further sections are focused on the definitions and results published in [12]. Nevertheless, we present all of them in a rather different order, our objective being to make explicit the way we want to traslate what happens in κ to $\mathcal{P}_\kappa(\lambda)$.

2.1 Stationary subsets of κ.

Definition 2.1.1. *Let κ be a limit ordinal of uncountable cofinality*

1. *$T \subseteq \kappa$ is **unbounded in** κ iff for any $\beta < \kappa$ there is some $\gamma \in T$ such that $\beta \leq \gamma$.*

2. *$C \subseteq \kappa$ is **closed in** κ iff for any $\{\beta_\xi : \xi < \gamma\} \subseteq C$ such that $\beta_\xi < \beta_\zeta$ for $\xi < \zeta < \gamma$, then, $\sup\{\beta_\xi : \xi < \gamma\} \in C$ whenever $\sup\{\beta_\xi : \xi < \gamma\} < \kappa$.*

3. *$C \subseteq \kappa$ is a **club** subset of κ iff it is closed and unbounded in κ.*

4. *$S \subseteq \kappa$ is **stationary in** κ iff for any C club in κ, $S \cap C \neq \varnothing$.*

It is easy to proof that the intersection of two club subsets of κ is again a club subset of κ. Thus, if C is a club subset of κ, C is stationary. Similarly, from the fact that for each $\alpha < \kappa$ the set $\{\beta < \kappa : \alpha < \beta\}$ is closed, we conclude that if S is a stationary subset of κ, then S is unbounded. It is also well easily seen that if κ has uncountable cofinality, the set of limit ordinals smaller than κ is a club. And that if S is stationary in κ and C is closed in κ, then $S \cap C$ is stationary in κ. Proofs of these facts may be found in [6; 14].

Proposition 2.1.2. *Let κ be a limit ordinal of uncountable cofinality. S is a stationary subset of κ if and only if for all unbounded subset T of κ there is some $\beta \in S$ such that $T \cap \beta$ is unbounded in β.*

Proof : ($\Rightarrow$) Suppose $S \subseteq \kappa$ is stationary, and let $T \subseteq \kappa$ be unbounded. Let T' be the set consisting of limit points of elements of T. Clearly T' is closed, moreover, as κ has uncountable cofinality, T' is also unbounded. As S is stationary, there must exists some $\beta \in S \cap T'$. We claim that $T \cap \beta$ is unbounded in β; if $\gamma < \beta (\in T')$, then $\gamma < \sup\{\beta_\xi : \xi < \rho\}$ where $\beta_\xi \in T$. Hence, there is $\beta_\xi \in T$ such that $\gamma < \beta_\xi < \beta$.
($\Leftarrow$) Suppose that for all unbounded subset T of κ there is some $\alpha \in S$ such that $T \cap \beta$ is unbounded in β. Let C be a club subset of κ. In particular C is unbounded, so there is some $\beta \in S$ such that $C \cap \beta$ is unbounded in β. This implies that β is a limit point of C, as C is club it contains its limit points, and so, $\beta \in C$. Hence $\beta \in S \cap C \neq \varnothing$. $\square$

Given an ordinal κ, the collection of all clubs of κ gives rise to a filter. Precisely, the set $Club(\kappa) := \{X \subseteq \kappa : C \subseteq X \text{ for some club } C\}$ is a filter. Moreover if κ is a regular uncountable cardinal, then $Club(\kappa)$ is a κ-complete filter.

Let κ be a regular uncountable cardinal, and let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of subsets of κ. The ***diagonal intersection*** $\Delta_{\kappa<\kappa} X_\alpha$ of the family $\{X_\alpha : \alpha < \kappa\}$ is defined by

$$\Delta_{\alpha<\kappa} X_\alpha := \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}.$$

While κ-completeness of $Club(\kappa)$ tell us about closure for intersections of $< \kappa$-many elements in the filter, it is not always the case that the intersection of κ many elements in the filter remains in the filter. As an easy example of this fact, notice that $\bigcap_{\alpha<\kappa}\{\beta : \alpha < \beta < \kappa\} = \varnothing \notin Club(\kappa)$. However, we have that the club filter $Club(\kappa)$ is indeed closed under diagonal intersections of κ many elements. Filters with this property are called ***normal*** filters.

If N is an element of the dual ideal $Club(\kappa)^*$ then $N = \kappa \setminus C$, for some $C \in Club(\kappa)$, whence $N \cap C = \varnothing$ and so N in non-stationary. On the other hand, if N is non-stationary then $N \cap C = \varnothing$ for some $C \in Club(\kappa)$ and so $N \subseteq (\kappa \setminus C) \in Club(\kappa)^*$, whence $N \in Club(\kappa)^*$. Hence the dual filter $Club(\kappa)^*$ consists on all non-stationary subsets of κ, and it is denoted by NS_κ. Notice that by duality NS_κ is also κ-complete and normal.

An ordinal function F on a set S is called ***regressive*** if $f(\alpha) < \alpha$ for every $\alpha \in S$ with $\alpha > 0$. The following proposition is a well known result, and it follows immediately from the fact that $Club(\kappa)$ is normal.

Theorem 2.1.3. *(Fodor's theorem/Pressing-Down theorem) Let κ be a regular uncountable cardinal. If f is a regressive function on a stationary set $S \subseteq \kappa$, then there is a stationary set $T \subseteq S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$. (See [6; 14]).*

Definition 2.1.4. *Let κ be an ordinal of uncountable cofinality*

1. *If S is a stationary subset of κ, then S **reflects** at $\beta < \kappa$ if $S \cap \beta$ is stationary at β.*
2. *If S is a stationary subset of κ, then S is **reflecting** if it reflects at some $\beta < \kappa$.*
3. *κ is **stationary-reflecting** if every stationary subset of κ is reflecting.*
4. *κ is **simultaneusly-stationary-reflecting** or **s-reflecting** for short, if for every pair of stationary subsets T_1, T_2 of κ, there is $\beta < \kappa$ such that $T_1 \cap \beta$ and $T_2 \cap \beta$ are stationary in β.*

As a trivial example consider $S := \kappa \subseteq \kappa$, it is trivially stationary. Moreover, for any limit ordinal $\beta < \kappa$ we have that $S \cap \beta = \beta$, which is of course stationary in β whenever β has uncountable cofinality. That is, S is reflecting and it reflects at any such $\beta < \kappa$. However, to find an ordinal κ such that every stationary subset reflects, this is, to find a stationary-reflecting ordinal, is much harder and depends on combinatorial properties of κ.

Remark 1: If a cardinal κ is stationary-reflecting, it cannot be the successor of a regular cardinal: Towards a contradiction, suppose κ is stationary-reflecting and $\kappa = \lambda^+$ for some regular λ. Consider the stationary set $E^\kappa_\lambda := \{\beta < \kappa : cof(\beta) = \lambda\} \subseteq \kappa$. Then, there is some $\gamma < \kappa$ such that $E^\kappa_\lambda \cap \gamma$ is stationary in γ. However $C := \{\kappa < \gamma : cof(\kappa) < \lambda\} \in Club(\gamma)$, but clearly $C \cap (E^\kappa_\lambda \cap \gamma) = \varnothing$.

It is also easy to see that κ is stationary-reflecting if and only if $cof(\kappa)$ is stationary-reflecting. So, suppose κ is the least stationary-reflecting cardinal, if it is a limit cardinal, then it has to be regular and so weakly inaccessible. And if κ is a successor, then, it must be the successor of a singular cardinal.

2.2 The ξ-stationary subsets of κ.

The concept of n-stationarity in κ or higher stationarity is a generalisation of being stationary in κ, in the sense of Proposition 2.1.2. This is, by 2.1.2 we know that a set is stationary if an only if it "reflects" unbounded subsets of κ. A higher-order of stationarity, then, must be given by reflecting stationary sets of "lower level" of stationarity.

Definition 2.2.1. *Let $\delta \geq \kappa$ and $S \subseteq \delta$*

1. *We say that S is 0-**stationary** in κ if $S \cap \kappa$ is unbounded in κ.*

2. *For an ordinal $\xi > 0$, we say that S is ξ-**stationary** in κ if and only if for every $\zeta < \xi$ every $T \subseteq \kappa$ that is ζ-stationary in κ, ζ-**reflects** to some $\beta \in S$, i.e., T is ζ-stationary in β.*

3. *We say that κ is ξ-**reflecting** if κ, as a subset of δ, is ξ-stationary in κ.*

Proposition 2.2.2. *$S \subseteq \kappa$ is ξ-stationary implies that S is ζ-stationary for all $\zeta < \xi$.* $\square$

Proposition 2.2.2 follows immediately from Definition 2.2.1. Notice that if ξ is a limit ordinal then the converse is also true. This is not the case when ξ is succesor, say $\xi = \gamma + 1$. If S is ζ-stationary for all $\zeta < \gamma + 1$ (i.e. $\zeta \leq \gamma$), then we will have at most that S reflects all ζ-stationary sets for $\zeta < \gamma$. However, for S to be $\gamma + 1$-stationary, we also need that S reflects γ-stationary sets.

Also, it follows from Definition 2.2.1 and Proposition 2.1.2 that $S \subseteq \kappa$ is stationary if and only if it is 1-stationary. Then, we have that

$$S \subseteq \kappa \ \text{club} \ \rightarrow \ S \ \text{stationary} \ \leftrightarrow \ S \ \text{1-stationary} \ \rightarrow \ S \ \text{unbounded}$$

Proposition 2.2.3. *κ is stationary-reflecting if and only if κ is 2-reflecting if and only if κ is 2-stationary in κ.*

Proof : κ is stationary-reflecting if every stationary subset of κ is reflecting, if and only if for all S stationary, or equivalently 1-stationary (2.1.2). S reflects at some $\beta < \kappa$, if and only if for all S stationary there is $\beta < \kappa$ such that $S \cap \beta$ is stationary at β, if and only if κ is 2-reflecting, if and only if κ is 2-stationary in κ. $\square$

Proposition 2.2.4. *[12] For every $\xi > 0$, if S is ξ-stationary in κ and C is a club subset of κ, then $S \cap C$ is also ξ-stationary in κ. Hence if κ is ξ-reflecting, then every club subset of α is ξ-stationary.*

Proof : We proceed by induction on ξ. If $\xi = 1$ and S is 1-stationary, by 2.1.2 S is stationary and so $S \cap C$ is stationary too, and again by 2.1.2 we get that $S \cap C$ is 1-stationary. If α is limit, the result follows from 2.2.2 and the induction hypothesis. So suppose it is true for ξ, and suppose S is $\xi + 1$-stationary and C is club. We shall prove that $S \cap C$ is $\xi + 1$-stationary. From 2.2.2 and the induction hypothesis, we get that $S \cap C$ is ξ-stationary. Moreover if $T \subseteq \kappa$ is ξ-stationary in κ, by the induction hypothesis $T \cap C$ is ξ-stationary in κ. Then, there is $\beta \in S$ such that $(T \cap C) \cap \beta$ is ξ-stationary in β. In particular β is a limit point of $(T \cap C) \cap \beta$ and so a limit point of C, whence $\beta \in C$. Therefore, $\beta \in S \cap C$ is such that $(T \cap C) \cap \beta$ is ξ-stationary in β. $\square$

Then, if κ is ξ-reflecting,

$$S \subseteq \kappa \ \text{club} \ \rightarrow \ S \ \xi\text{-stationary} \ \rightarrow \ S \ \zeta\text{-stationary} \quad \text{for all} \ \zeta < \xi$$

Definition 2.2.5. *Let $\delta \geq \kappa$ and $S \subseteq \delta$*

1. *We say that S is 0-**simultaneously-stationary** in κ (0-**s-stationary** for short) if $S \cap \kappa$ is unbounded in κ.*

2. *For an ordinal $\xi > 0$, we say that S is **ξ-simultaneously-stationary** in κ (**ξ-s-stationary** for short) if and only if for every $\zeta < \xi$, every pair of subsets $T_1, T_2 \subseteq \kappa$ that are ζ-s-stationary in κ **simultaneously ζ-s-reflect** to some $\beta \in S$, i.e., S, T are both ζ-stationary in the same β.*

3. *We say that an ordinal κ is **ξ-s-reflecting** if κ, as a subset of δ, is ξ-s-stationary in κ.*

Proposition 2.2.6. *$S \subseteq \kappa$ is ξ-s-stationary implies that S is ζ-s-stationary for all $\zeta < \xi$.* □

As in the case of 2.2.2, Proposition 2.2.6 follows immediately from Definition 2.2.5. Similarly, if ξ is a limit ordinal, then, the converse is also true. And if ξ is a successor, then, the converse is not necessarily true.

Proposition 2.2.7. *$S \subseteq \kappa$ is 0-s-stationary in κ if and only if S is 0-stationary in κ. And $S \subseteq \kappa$ is 1-s-stationary in κ if and only if S is 1-stationary in κ.*

Proof : The first part is trivial from 2.2.5 and 2.2.1. The left to right implication of the second part is also trivial. Now suppose $S \subseteq \kappa$ is 1-stationary in κ, and let T_1, T_2 be 0-stationary subsets of κ. As in the proof of 2.1.2 we get that T_1', T_2' are clubs, and so is $T_1' \cap T_2'$. Then there is $\beta \in S \cap (T_1' \cap T_2')$. We claim that $T_1 \cap T_2 \cap \beta$ is 0-stationary in β. Let $\gamma < \beta$. Since $\beta \in T_1' \cap T_2'$, $\beta = \sup\{\beta_\xi^1 : \xi < \rho^1\} = \sup\{\beta_\xi^2 : \xi < \rho^2\}$, where $\beta_\xi^1 \in T_1$, $\beta_\xi^2 \in T_2$ and $\rho^1, \rho^2 < \beta$. Hence, there is $\beta_{\xi^1}^1 \in T_1$ and $\beta_{\xi^2}^2 \in T_2$ for some ξ^1, ξ^2 such that $\gamma < \beta_\xi^1, \beta_\xi^2 < \beta$. This is $T_1 \cap \beta$ and $T_2 \cap \beta$ are 0-stationary in β. □

Remark. The content of Proposition 2.2.7 does not necessarily extend to higher levels of s-stationarity. In fact, the existence of a 2-s-reflecting cardinal has higher consistency strength than the existence of a 2-reflecting cardinal [9].

Proposition 2.2.8. *[12] For every $\xi > 0$, if S is ξ-s-stationary in κ and C is a club subset of κ, then $S \cap C$ is also ξ-s-stationary in κ. Hence if κ is ξ-s-reflecting, then every club subset of α is ξ-s-stationary.*

The proof of proposition 2.2.8 is completely analogous to the proof of 2.2.4. Notice also that for all $\xi > 0$, if S is ξ-s-stationary in κ, then S is ξ-stationary in κ (take $T_1 = T_2$ in the definition of ξ-s-stationary). Then, if κ is ξ-s-reflecting, it is in particular ξ-reflecting and so

$$S \subseteq \kappa \text{ club } \rightarrow S \;\; \xi\text{-s-stationary } \rightarrow S \;\; \xi\text{-stationary } \rightarrow S \;\; \zeta\text{-stationary} \quad \text{for all } \zeta < \xi$$

2.3 The ideal of non-ξ-stationary subsets of κ

Until now we have avoided the definition of iterated topologies on ordinals, and with this some of the primary results of [12]. The reason for that is that when generalising the results to $\mathcal{P}_\kappa(\lambda)$ it is not immediately clear how to provide $\mathcal{P}_\kappa(\lambda)$ with a topology such that its isolated points correspond in some sense to a notion of higher stationarity in $\mathcal{P}_\kappa(\lambda)$. However, as we will see in this section, there is a characterisation of certain sets determining the topologies on κ, given in [12], which can be extended to $\mathcal{P}_\kappa(\lambda)$ and which will allow to define the corresponding topologies on $\mathcal{P}_\kappa(\lambda)$.

Definition 2.3.1. *Let δ be a limit ordinal. We shall define a transfinite sequence $\langle \tau_\xi : \xi \in OR \rangle$ of topologies on δ as follows*

1. *Let τ_0 be the interval topology on δ. Let $d_0 := \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be such that $d_0(S) := \{\kappa < \delta : \kappa$ is a limit point of S in the τ_0-topology$\}$.*

2. *Given τ_ξ and having defined d_ξ, let $\tau_{\xi+1}$ be the topology generated by $\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{d_\xi(S) : S \subseteq \delta\}$. And let $d_{\xi+1} := \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be such that $d_{\xi+1}(S) := \{\kappa < \delta : \kappa$ is a limit point of S in the $\tau_{\xi+1}$-topology$\}$.*

3. If ξ is a limit ordinal, let τ_ξ be the topology generated by $\mathcal{B}_\xi := \bigcup_{\zeta<\xi} \mathcal{B}$.

Notice that κ is a limit point in δ in the order topology τ_0 if and only if κ is a limit ordinal below δ. Then, limit ordinals that are smaller than κ are exactly the elements of $d_0(\kappa)$.

Proposition 2.3.2. *[12] Let $\xi > 0$ and $S \subseteq \delta$. Then, the set $d_\xi(S)$ is a closed subset of δ in the topology τ_ξ.*

Proof : We will prove that given $\xi > 0$ and $S \subseteq \delta$, the set $\delta \setminus d_\xi(S)$ is open in the topology τ_ξ. Let $\alpha \in \delta \setminus d_\xi(S)$, then α is not a limit point of S in the topology τ_ξ. Then, there is an open set $U \in \tau_\xi$ such that $\alpha \in U$ and $(U \setminus \{\alpha\}) \cap S = \varnothing$. Moreover, $U \cap d_\xi(S) = \varnothing$, for suppose $\beta \in U \cap d_\xi(S)$, then $\beta \neq \alpha$ and it is a limit point of S. Then, for the open set $U \setminus \{\alpha\}$ we have $([U \setminus \{\alpha\}] \setminus \{\beta\}) \cap S \neq \varnothing$. But this caontradict the fact that $(U \setminus \{\alpha\}) \cap S = \varnothing$. Therefore $U \cap d_\xi(S) = \varnothing$, and so $U \subseteq \delta \setminus d_\xi(S)$, this is, $\delta \setminus d_\xi(S)$ is open in τ_ξ. □

Bagaria proves that in fact $\mathcal{B}_1$ constitutes a base for the topology τ_1 (See Proposition 2.3. in [12]). Then, any open subset of τ_1 is a union of sets of the form $I \cap d_0(S_1) \cap \cdots \cap d_0(S_n)$, using this fact we can prove the following

Lemma 2.3.3. *If κ is an ordinal of uncountable cofinality and $\kappa \in U \in \tau_1$, there is a club subset C of κ such that $C \subseteq U$.*

Proof Suppose $cof(\kappa) > \omega$ and $\kappa \in U \in \tau_1$. Then, there is a basic open set $I \cap d_0(S_1) \cap \cdots \cap d_0(S_n) \subseteq U$ such that $\kappa \in I \cap d_0(S_1) \cap \cdots \cap d_0(S_n)$. We shall prove that $I \cap d_0(S_1) \cap \cdots \cap d_0(S_n)$ is a club. I must be of the form $I = (\gamma, \gamma')$ for some $\gamma < \kappa < \gamma' \leq \delta$, then $I \cap \kappa = (\gamma, \kappa)$ which is a tail subset of κ and so a club subset of κ. Now, for any $i \leq n$ the set $d_0(S_i)$ contains its limit points and therefore is closed. Moreover, $d_0(S_i)$ is unbounded in κ for any $i \leq n$. Take $\beta < \kappa$, as $\kappa \in d_0(S_i)$, there is $\beta_0 \in S_i \cap (\beta, \kappa)$. And for each $m < \omega$, let $\beta_m \in S_i \cap (\beta_{m-1}, \kappa)$. Then $\beta_\omega = \sup\{\beta_m : m < \omega\}$ is a limit point of elements of S_i, i.e., $\beta_\omega \in d_0(S_i)$. Finally, since $cof(\kappa) > \omega$, we have that $\beta < \beta_\omega < \kappa$.

Proposition 2.3.4. *[12] τ_1 is non-discrete if and only if there is $\kappa < \delta$ such that $cof(\kappa) > \omega$.*

Proof : ($\Rightarrow$) By contraposition, suppose for all $\kappa < \delta$ such that $cof(\kappa) \leq \omega$. If κ is successor or 0, clearly $\{\kappa\} \in \tau_0 \subseteq \tau_1$. If κ is limit then take $\{x_\beta : \beta < \omega\}$ cofinal, then $\{\kappa\} = d_0(\{x_\beta : \beta < \omega\}) \in \tau_1$. This is, for all $\kappa < \delta$, $\{\kappa\} \in \tau_1$ and so τ_1 is the discrete topology.

($\Leftarrow$) Suppose there is $\kappa < \delta$ such that $cof(\kappa) > \omega$ such that $cof(\kappa) > \omega$. We claim that $\{\kappa\} \notin \tau_1$. Towards a contradiction, suppose that $\{\kappa\} \in \tau_1$, but then, by lemma 2.3.3 there is a club C of κ such that $C \subseteq \{\kappa\}$ and this is nonsense.

Proposition 2.3.5. *[12] For every $S \subseteq \delta$, $d_1(S) = \{\kappa : S$ is stationary in $\kappa\}$*

Proof : ($\subseteq$) Let $\kappa \in d_1(S)$, this is, κ is a limit point of S in the τ_1 topology. If $cof(\kappa) = \omega$, there is some cofinal sequence $\{x_\beta : \beta < \omega\}$ such that $d_0(\{x_\beta : \beta < \omega\}) = \{\kappa\}$, and so $\{\kappa\} \in \tau_1$. Then, κ is an ordinal of uncountable cofinality. Let C be a club subset of κ. Then C contains its limit points, this is $d_0(C) \subseteq C$. But $d_0(C) \in \mathcal{B}_1$, and so $d_0(C) \in \tau_1$. Since κ is limit point of S we have that $S \cap (d_0(C) \setminus \{\kappa\}) \neq \varnothing$. Hence $S \cap C \supseteq S \cap (d_0(C) \setminus \{\kappa\})$ is non-empty.

($\supseteq$) Suppose S is stationary in κ. Then κ is an ordinal of uncountable cofinality. Let $U \in \tau_1$ be such that $\kappa \in U$. By lemma 2.3.3 there is a a club subset C of κ such that $C \subseteq U$. Then $S \cap C \neq \varnothing$. As $\kappa \notin C$ we also have that $S \cap (U \setminus \{\kappa\}) \supseteq S \cap (C \setminus \{\kappa\}) \neq \varnothing$, and therefore κ is a limit point of S in the τ_1 topology. □

Then to say that S is stationary in κ is equivalent to saying that $\kappa \in d_1(S)$. Using 2.3.5 we can reinterpret Definition 2.1.4 as follows

- Let S be such that $\kappa \in d_1(S)$. S **reflects** at $\beta < \kappa$ iff $\beta \in d_1(S)$.
- Let S be such that $\kappa \in d_1(S)$. S is **reflecting** iff $d_1(S) \setminus \{\kappa\} \neq \varnothing$.

- κ is **stationary-reflecting** iff for all S, $\kappa \in d_1(S)$ implies $d_1(S) \setminus \{\kappa\} \neq \varnothing$.
- κ is **s-reflecting** iff for every pair of sets T_1, T_2, if $\kappa \in d_1(T_1) \cap d_1(T_2)$, then there is $\beta < \kappa$ such that $\beta \in d_1(T_1) \cap d_1(T_2)$.

From Proposition 2.3.4 we know that the necessary and sufficient condition for τ_1 to be non-discrete is the existence of an ordinal of uncountable cofinality in δ. This is, the existence of a 1-stationary (equivalently a 1-reflecting) ordinal below δ. However, Bagaria showed that the non-discreetness of τ_2 requires more than the existence of a 2-stationary (equivalently 2-reflecting or stationary-reflecting 2.2.3) ordinal.

Proposition 2.3.6. *([12]).*

1. *An ordinal $\kappa < \delta$ is not isolated in the τ_2 topology on δ if and only if κ is s-reflecting. Thus, $\mathcal{B}_2$ generates a non-discrete topology on δ if and only if some $\kappa < \delta$ is s-reflecting.*
2. *$\mathcal{B}_2$ is a base for the τ_2 topology on δ if and only if every stationary-reflecting $\kappa < \delta$ is s-reflecting.*

In order to generalise the result obtained in Proposition 2.3.6 to topologies τ_ξ with $\xi > 2$, Bagaria uses the following

Proposition 2.3.7. *([12]).*

1. *For every ξ, $d_\xi(S) = \{\kappa : S$ is ξ-s-stationary in $\kappa\}$.*
2. *For every ξ and κ, S is $\xi+1$-s-stationary in κ if and only if $S \cap d_\zeta(T_1) \cap d_\zeta(T_1) \cap \kappa \neq \varnothing$ (equivalently, if and only if $S \cap d_\zeta(T_1) \cap d_\zeta(T_1))$ is ζ-s-stationary in κ) for every $\zeta \leq \xi$ and every pair T_1, T_2 of subsets of κ that are ζ-s- stationary in κ.*
3. *For every ξ and κ, if S is ξ-s-stationary in κ and T_i is ζ_i-s-stationary in κ for some $\zeta_i < \xi$ all $i < n$, then $S \cap d_{\zeta_1}(T_1) \cap \cdots \cap d_{\zeta_n}(T_n))$ is ξ-s-stationary in κ.*

Proposition 2.3.8. *Suppose that δ is $\xi+1$-stationary, and let $S \subseteq \delta$ be ξ-stationary. Then, $d_\xi(S) = \{\kappa < \delta : S$ ξ-s-reflects to $\kappa\}$.*

Proof : Let δ be $\xi+1$-stationary, and let $S \subseteq \delta$ be ξ-stationary. Then, from Definition 2.2.5 that S is ξ-s-stationary in $\kappa < \delta$ if and only if S ξ-s-reflects to κ. moreover, by Proposition 2.3.7 we know that $d_\xi(S)) = \{\kappa < \delta : S$ is ξ-s-stationary in $\kappa\}$. Therefore $d_\xi(S) = \{\kappa < \delta : S$ ξ-s-reflects to $\kappa\}$.

From Propositions 2.3.7, 2.3.6 and 2.2.8 it follows one of the main results of [12], which characterises the topologies τ_ξ in terms of stationary reflection. Namely

Theorem 2.3.9. *([12]). For every ξ, an ordinal $\kappa < \delta$ is not isolated in the τ_ξ topology on δ if and only if κ is ξ-s-reflecting. Thus $\mathcal{B}_\xi$ generates a non-discrete topology on δ if and only if some $\kappa < \delta$ is ξ-s-reflecting.*

Now, to study the open sets $d_\zeta(S)$ for $\zeta < \xi$ it is also useful to characterise the dual filter of the ideal of non-ξ-s stationary subsets of κ in terms of the d_ξ operator.

Definition 2.3.10. *For every ordinal ξ, $\mathcal{I}^\xi_\kappa := \{X \subseteq \kappa : X$ is not n-s-stationary in $\kappa\}$.*

As stationary sets are equivalent to 1-stationary sets. And for the cases $\xi \in \{0, 1\}$, to be ξ-sattionary is equivalent to be ξ-s-statioanry. Then, if $\xi = 1$ then $\mathcal{I}^\xi_\kappa = NS_\kappa$.

Lemma 2.3.11. *If T_1, T_2 are both not unbounded subsets of κ, then $T_1 \cup T_2$ is not unbounded either.*

Proof : T_1 and T_2 are both bounded for some $\beta_1, \beta_2 < \kappa$ respectively. Take $\beta = \max\{\beta_1, \beta_2\}$, then $T_1 \cup T_2$ is bounded by β and so $T_1 \cup T_2$ is not unbounded. □

Definition 2.3.12. *We denote by F^ξ_κ the dual filter associated to $\mathcal{I}^\xi_\kappa$, this is $F^\xi_\kappa := (\mathcal{I}^\xi_\kappa)^*$.*

Proposition 2.3.13. *([12]). Let $X \subseteq \kappa$, then $X \in F^\xi_\kappa$ if and only if there is some $\zeta < \xi$ and some ζ-s-stationary sets $T_1, T_2 \subseteq \kappa$ such that $d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa \subseteq X$.*

Then, from Proposition 2.3.13 we conclude that

$$F^\xi_\kappa = \{X \subseteq \kappa : \exists \zeta < \xi \text{ and } T_1, T_2 \subseteq \kappa \ \zeta\text{-s-stationary such that } d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa \subseteq X\}.$$

Proposition 2.3.14. *$S \subseteq \kappa$ is ξ-s-stationary if and only if S is F^ξ_κ-stationary.*

Proof : ($\Rightarrow$) Let S be ξ-s-stationary in κ, and let $X \in F^\xi_\kappa$, this is, X is such that there is $\zeta < \xi$ and $T_1, T_2 \subseteq \kappa$ ζ-s-stationary such that$d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa \subseteq X$. Since S is ξ-stationary, for T_1, T_2 there is $\beta \in S$ such that $T_1 \cap \beta$ and $T_2 \cap \beta$ are ζ-s-stationary in β. Then $\beta \in d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa \subseteq X$ and so $\beta \in S \cap X$.

($\Leftarrow$) Suppose that S is F^ξ_κ-stationary, and take $T_1, T_2 \subseteq \kappa$ ζ-s-stationary subsets of κ. Notice that $d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa$ trivially belongs to F^ξ_κ, then, there is some $\beta \in S \cap d_\zeta(T_1) \cap d_\zeta(T_2) \cap \kappa$. This is, there is $\beta \in S$ such that $T_1 \cap \beta$ and $T_2 \cap \beta$ are both ζ-stationary in β. Hence $S \subseteq \kappa$ is ξ-s-stationary.$\square$

In Section 1 we gave the standard definition of a stationary subset of κ, this definition correspond to the definition of being F-stationary 1.3.7 with respect to the filter $F = Club(\kappa)$. However, the definition of ξ-s-stationary subsets of κ we presented in section 2 was given regardless of any filter. Proposition 2.3.14 is telling us that ξ-s-stationary subsets of κ are indeed stationary with respect to some filter, the filter F^ξ_κ.

Theorem 2.3.15. *([12]). For every ξ, an ordinal κ is ξ-s-reflecting if and only if $\mathcal{I}^\xi_\kappa$ is a proper ideal, hence if and only if F^ξ_κ is a proper filter.*

2.4 Π^1_ξ-indescribability in κ.

In chapter 1 we review the well known definitions of Π^1_n and Σ^1_n formulas for $n < \omega$. As well as the concept of Π^n_1 indescribable cardinals. In 1972 R. Jensen related the notion of indescribability with the fact of simultaneously reflecting stationary sets in L. More precisely he proved that in the constructible universe L, a regular cardinal is simultaneously-reflecting if and only if it is Π^1_1-indescribable (See [22]). Recently, Bagaria, Magidor and Sakai proved that in the constructible universe L, a regular cardinal is $n+1$-simultaneously-reflecting if and only if it is Π^1_n-indescribable (See [25]). Finally, in [12] Bagaria obtained and even more general result when extending this definitions of Π^1_n formulas and Π^1_n -indescribability to the case $\xi \geq \omega$.

Definition 2.4.1. *([12]). Let $\xi \geq \gamma$. A formula is $\Sigma^1_{\xi+1}$ if it is of the form*

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Π^1_ξ. And a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k)$$

where $\varphi(X_0, \ldots, X_k)$ is Σ^1_ξ. If ξ is a limit ordinal, then we say that a formula is Π^1_ξ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Π^1_ζ for all $\zeta < \xi$, and the infinite conjunction has only finitely-many free second-order variables. And we say that a formula is Σ^1_ξ if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta$$

where φ_ζ is Σ^1_ζ for all $\zeta < \xi$, and the infinite disjunction has only finitely-many free second-order variables.

Definition 2.4.2. *([12]). A cardinal κ is Π^1_ξ-indescribable if for all subsets $A \subseteq V_\kappa$ and every Π^1_ξ sentence φ, if $\langle V_\kappa, \in, A\rangle \models \varphi$, then there is some $\lambda < \kappa$ such that*

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi.$$

Proposition 2.4.3. *([12]). Every Π^1_ξ-indescribable cardinal is $(\xi+1)$-s-reflecting. Hence, if ξ is a limit ordinal and a cardinal κ is Π^1_ζ-indescribable for all $\zeta < \xi$, then κ is ξ-s-reflecting.*

The converse of Proposition 2.4.3 is also true whenever $V = L$, and it is proved in the general case $\xi \in OR$ with the Theorem 2.4.5 that constitutes the second main result of [12].

Proposition 2.4.4. *Suppose κ is Π^1_ξ-indescribable, then κ is Π^1_ξ-indescribable in the constructible universe L*

Proof of of proposition 2.4.4 can be found in Chapter 6 of [14] for the case $\xi = n < \omega$. And as pointed out by Bagaria in [12], its generalisation to the case $\xi \geq \omega$ is straightforward.

Theorem 2.4.5. *([12]). Assume $V = L$. Suppose $\xi > 0$ and κ is a regular $(\xi+1)$-s-reflecting cardinal. Then κ is Π^1_ξ-indescribable.*

Thus, together with Theorem 2.3.9 and Theorem 2.4.5 Bagaria obtained in [12] a complete characterisation in L of the reflection of ξ stationary sets, in terms of Π^1_ξ-indescribability and of the non-discretness of the topologies τ_ξ.

Chapter 3

Hyperstationary subsets of $\mathcal{P}_\kappa(A)$.

3.1 Stationary subsets of $\mathcal{P}_\kappa(A)$

In 1971 Thomas Jech presented a generalisation of the concepts of closed unbounded and stationary set [16]. He considered the set $\langle \mathcal{P}_\kappa(\lambda), \subset\rangle$ instead of $\langle \kappa, <\rangle$, and the connection between them occurs when $\lambda = \kappa$. This is, properties defining these concepts remain when passing from $\langle \kappa, <\rangle$ to $\langle \mathcal{P}_\kappa(\kappa), \subset\rangle$. As we will see, definitions for the case $\mathcal{P}_\kappa(\lambda)$ are straightforward, and their convenience lays on the fact that results such as Fodor's Theorem remain true under this generalisation.

Definition 3.1.1. *Let κ be an uncountable regular cardinal and let A be a set of ordinals such that $|A| \geq \kappa$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is **unbounded in** $\mathcal{P}_\kappa(A)$ iff for any $X \in \mathcal{P}_\kappa(A)$ there is some $Y \in S$ such that $X \subseteq Y$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is **closed in** $\mathcal{P}_\kappa(A)$ iff for any $\{X_\xi : \xi < \beta\} \subseteq S$ with $\beta < \kappa$ and $X_\xi \subseteq X_\zeta$ for $\xi \leq \zeta < \beta$, $\bigcup_{\xi<\beta} X_\xi \in S$.*
3. *$S \subseteq \mathcal{P}_\kappa(A)$ is **club of** $\mathcal{P}_\kappa(A)$ iff S is closed and unbounded in $\mathcal{P}_\kappa(A)$,.*
4. *$S \subseteq \mathcal{P}_\kappa(A)$ is **stationary in** $\mathcal{P}_\kappa(A)$ iff for any C club in $\mathcal{P}_\kappa(A)$, $S \cap C \neq \varnothing$.*

Notice that for every $X \in \mathcal{P}_\kappa(A)$, the set $\{Y \in \mathcal{P}_\kappa(A) : X \subseteq Y\}$ is closed and unbounded. That it is unbounded is immediate, and since arbitrary increasing unions of subsets containing X do contain X, this set is also closed.

It follows immediately from Definition 3.1.1 that $\mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$ for all κ. Also the following result follows directly from the definition.

Proposition 3.1.2. *If $S \subseteq \mathcal{P}_\kappa(A)$ is club in $\mathcal{P}_\kappa(A)$, then it is stationary in $\mathcal{P}_\kappa(A)$. And if $S \subseteq \mathcal{P}_\kappa(A)$ is stationary in $\mathcal{P}_\kappa(A)$, then it is unbounded in $\mathcal{P}_\kappa(A)$.*

Proof : Let S be be a club of $\mathcal{P}_\kappa(A)$, and pick any club C of $\mathcal{P}_\kappa(A)$. It is clear that $S \cap C$ is closed, so we will prove that it is unbounded in $\mathcal{P}_\kappa(A)$. Let $X_0 \in \mathcal{P}_\kappa(A)$, as S, C are unbounded in $\mathcal{P}_\kappa(A)$, we may construct the following ω-sequence

$$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n \subsetneq X_{n+1} \subsetneq \cdots$$

Where $X_i \in S$ if $i > 0$ is even and $X_i \in C$ otherwise. Then, $\bigcup_{i<\omega} X_{2i} \in S$ and $\bigcup_{i<\omega} X_{2i+1} \in C$, but $\bigcup_{i<\omega} X_{2i} = \bigcup_{i<\omega} X_{2i+1}$, therefore $\bigcup_{i<\omega} X_i \in S \cap C$.

For the second statement take $X \in S$, consider the club subset $C = \{Y \in \mathcal{P}_\kappa(A) : X \subseteq Y\}$. Pick $Z \in S \cap C$, then $Z \in S$ and $X \subseteq Z$, this is S is stationary in $\mathcal{P}_\kappa(A)$. $\square$

Proposition 3.1.3. *$C \subseteq \mathcal{P}_\kappa(A)$ is closed if and only if for every 2-directed set $X \subseteq C$ of cardinality $< \kappa$, $\bigcup X \in C$.*

Proof : ($\Rightarrow$) We prove this direction by induction on $|X| = \gamma$. Suppose that $X = \{A_\alpha : \alpha < \gamma\}$. If $\gamma = \omega$, choose $n_0 < n_1 < \cdots$ such that $n_0 = 0$ and $A_{n_{i+1}} \supseteq A_i \cup A_{n_i}$. So that $A_{n_0} \subseteq A_{n_1} \subseteq \cdots$ is an increasing sequence with union $A_0 \cup A_1 \cup \cdots$. If $\gamma > \omega$, notice that if X is a directed system and $Y \subseteq X$, there is a directed subsystem $Y \subseteq Z \subseteq X$ such that $|Y| \leq \max\{|Z|, \omega\}$. Then, we can see X as an increasing, union $X = \bigcup\{X_\alpha : \alpha < \gamma\}$ of directed systems X_α of smaller cardinality. By our inductive hypothesis we get $\bigcup X_\alpha \in C$ for every $\alpha < \gamma$, so finally this holds for X because$C$ is closed and $\{X_\alpha : \alpha < \gamma\}$ is increasing.

($\Leftarrow$) Let C be a closed set of $\mathcal{P}_\kappa(A)$, and suppose $\{X_\xi : \xi < \beta\} \subseteq C$ whit $\beta < \kappa$ and $X_\xi \subseteq X_\zeta$ for $\xi \leq \zeta < \beta$. Let $X_{\xi_1}, X_{\xi_2} \in \{X_\xi : \xi < \beta\}$, then w.l.g. we may assume $X_{\xi_1} \subseteq X_{\xi_2}$, then $X_{\xi_1} \cup X_{\xi_2} \subseteq X_{\xi_2}$. This is, $\{X_\xi : \xi < \beta\}$ is a 2-directed subset of C of cardinality $\beta < \kappa$, then by hypothesis we have that $\bigcup_{\xi<\beta} X_\xi \in S$.

If $|A| = |B|$, then $\langle \mathcal{P}_\kappa(A), \subseteq \rangle$ is isomorphic to $\langle \mathcal{P}_\kappa(B), \subseteq \rangle$. Thus considering the case $\mathcal{P}_\kappa(A)$ is equivalent to considering the case $\mathcal{P}_\kappa(\lambda)$, where $|A| = \lambda \geq \kappa$. Now, every ordinal $< \kappa$ is identified with an element of $\mathcal{P}_\kappa(\lambda)$ determined by itself, so that $\kappa \subseteq \mathcal{P}_\kappa(\kappa)$.

Proposition 3.1.4. *A set $S \subseteq \kappa$ is unbounded (or closed, or stationary) in the sense of κ if and only if, it is unbounded (or closed, or stationary) in the sense of $\mathcal{P}_\kappa(\kappa)$.*

Proof : Suppose $S \subseteq \kappa$ is unbounded in κ. Let $X \in \mathcal{P}_\kappa(\kappa)$ and take $\alpha = \sup X$. Then $X \subseteq \alpha < \kappa$. So there is $\gamma \in S$ such that $\alpha < \gamma < \kappa$. But then $X \subseteq \gamma \in S$. Now suppose that $S \subseteq \kappa$ is unbounded in $\mathcal{P}_\kappa(\kappa)$. Let $\alpha < \kappa$, then $\alpha \in \mathcal{P}_\kappa(\kappa)$, and so, there is $X \in S$ such that $\alpha \subseteq X$. Since $X \in S \subseteq \kappa$, $X = \gamma$ for some $\gamma \in \kappa$. Then there is $\gamma < \kappa$ such that $\alpha \leq \gamma < \kappa$.

Suppose $S \subseteq \kappa$ is closed in κ, and take $\{X_\xi : \xi < \beta\} \subseteq S$ with $\beta < \kappa$ and $X_\xi \subseteq X_\zeta$ for $\xi \leq \zeta < \beta$. As $S \subseteq \kappa$, each element in the sequence is in fact an ordinal less than κ, namely $\{X_\xi : \xi < \beta\} = \{\alpha_\xi : \xi < \beta\}$. Then $\bigcup_{\xi<\beta} X_\xi = \bigcup_{\xi<\beta} \alpha_\xi \in S$. Now suppose that $S \subseteq \kappa$ is closed in $\mathcal{P}_\kappa(\kappa)$, and $\{\alpha_\xi : \xi < \beta\}$ is an increasing sequence of ordinals less than κ. Clearly $\{\alpha_\xi : \xi < \beta\}$ is also an increasing sequence of elements of $\mathcal{P}_\kappa(\kappa)$. Then $\bigcup_{\xi<\beta} \alpha_\xi \in S$. Moreover $\alpha := \bigcup_{\xi<\beta} \alpha_\xi$ is an ordinal, and so $\alpha = \sup_{\xi<\beta} \alpha_\xi$.

Suppose $S \subseteq \kappa$ is stationary in κ, and take C a club subset of $\mathcal{P}_\kappa(\kappa)$. We claim that the set $C \cap \kappa \subseteq \kappa$ is a club of κ, if so, then $\varnothing \neq (C \cap \kappa) \cap S \subseteq C \cap S$ and we are done. Closure is trivial, so we are left to prove unboundedness. Let $\alpha < \kappa$, then, there is $X_0 \in C$ such that $\alpha \subseteq X_0$. For each $i \in \{1, \ldots, n\}$, let X_i be an element of C such that $\sup X_{i-1} \subseteq X_i$. Then $\bigcup_{n<\omega} X_n \in C$ is an ordinal less than κ, and $\alpha \subseteq \bigcup_{n<\omega} X_n$. This is, for $\alpha < \kappa$ there is $\beta := \bigcup_{n<\omega} X_n \in C \cap \kappa$ such that $\alpha \leq \beta$. hence $C \cap \kappa$ is unbounded in κ. Now suppose $S \subseteq \kappa$ is stationary in $\mathcal{P}_\kappa(\kappa)$, and take C a club subset of κ. As $C \subseteq \kappa$ we may apply previous items obtaining that C is also closed unbounded in $\mathcal{P}_\kappa(\kappa)$, and so $S \cap C \neq \varnothing$. □

Proposition 3.1.4 is the reason why we say that, considering definitions in 3.1.1, $\langle \mathcal{P}_\kappa(\lambda), \subset \rangle$ is indeed a generalisation of $\langle \kappa, < \rangle$. Also, notice that as in the case of $\langle \kappa, < \rangle$, the union of less than κ many bounded subsets of $\mathcal{P}_\kappa(\lambda)$ is bounded in $\mathcal{P}_\kappa(\lambda)$.

The ***closed unbounded filter on*** $\mathcal{P}_\kappa(A)$ is the filter generated by the closed unbounded sets. In the case $A = \kappa$, the set $\kappa \subseteq \mathcal{P}_\kappa(\kappa)$ is a club of $\mathcal{P}_\kappa(\kappa)$ and so $Club(\kappa)$ is the restriction of the club filter on $\mathcal{P}_\kappa(A)$ to $\mathcal{P}_\kappa(\kappa)$. Thus, there is a dual ideal corresponding to $Club(\kappa)$, we sat that it is the ideal of ***non-stationary subsets*** of $\mathcal{P}_\kappa(A)$ and it is denoted by $NS_{\kappa,A}$.

Proposition 3.1.5. *The intersection of $\gamma < \kappa$ many club subsets of $\mathcal{P}_\kappa(\lambda)$ is again a club subset of $\mathcal{P}_\kappa(\lambda)$. Hence the club filter on $\mathcal{P}_\kappa(\lambda)$ is κ-complete. (See [6; 14].)*

Let $\langle X_a : a \in A\rangle$ be a sequence of subsets of $\mathcal{P}_\kappa(A)$, its diagonal intersection is defined by

$$\Delta_{a\in A} X_a := \{X \in \mathcal{P}_\kappa(A) : X \in \bigcap_{a\in X} X_a\}.$$

Proposition 3.1.6. *If $\langle C_a : a \in A\rangle$ is a sequence of club subsets of $\mathcal{P}_\kappa(A)$, then its diagonal intersection $\Delta_{a\in A} C_a$ is a club subset of $\mathcal{P}_\kappa(A)$. (See [6].)*

Last proposition is the key to prove the generalisation of Fodor's theorem to $\mathcal{P}_\kappa(\lambda)$ given by Jech in [16]. He considered choice functions instead of regressive functions.

Theorem 3.1.7. ***(T. Jech)*** *If f is a function on a stationary set $S \subseteq \mathcal{P}_\kappa(\lambda)$ and if $f(x) \in x$ for every nonempty $x \in S$, then there exists a stationary set $T \subseteq S$ and some $a \in A$ such that $f(x) = a$ for all $a \in T$.*

Proof : Follows from Proposition 3.1.6 analogously to the case of κ. (See [6; 7].) $\square$

3.2 The n-stationary subsets of $\mathcal{P}_\kappa(\lambda)$.

The main objective of this work is to investigate the most suitable approach to a concept of hyperstationarity in $\mathcal{P}_\kappa(\lambda)$ in such a way that the results obtained by Bagaria in [12] may be extended to the case of $\mathcal{P}_\kappa(\lambda)$. The primary incentive for doing this is that we expect the consistency strength of hyperstationarity in $\mathcal{P}_\kappa(\lambda)$ to be much stronger than the one for κ, possibly close to the level of supercompactness.

As far as we are concerned, the unique attempt to define n-stationary sets in $\mathcal{P}_\kappa(\lambda)$ was made for Hiroshi Sakai, Sakaé Fuchino and Hazel Brickhil as exposed the talk " *On generalised notion of higher stationarity* " [13]. We take this definition as a starting point.

Definition 3.2.1. ***(H. Sakai)*** *Let κ be a regular cardinal, $\kappa \subseteq A$, and $n < \omega$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is 0-**stationary in** $\mathcal{P}_\kappa(A)$ iff S is unbounded in $\mathcal{P}_\kappa(A)$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is n-**stationary in** $\mathcal{P}_\kappa(A)$ iff for all $m < n$ and for all $T \subseteq \mathcal{P}_\kappa(A)$ m-stationary in $\mathcal{P}_\kappa(A)$, there is $B \in S$ such that*
   - *$\mu := B \cap \kappa$ is a regular cardinal.*
   - *$T \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$*
3. *$\mathcal{P}_\kappa(A)$ is n-**stationary** if it is n-stationary in $\mathcal{P}_\kappa(A)$ as a subset of $\mathcal{P}_\kappa(A)$.*

For the sake of readability, whenever the context is clear we will say "S is n-stationary" instead of "S is n-stationary in $\mathcal{P}_\kappa(A)$".

Proposition 3.2.2. *$S \subseteq \mathcal{P}_\kappa(A)$ being 1-stationary implies S is unbounded.*

Proof : Suppose that $S \subseteq \mathcal{P}_\kappa(A)$ 1-stationary and let $X \in \mathcal{P}_\kappa(A)$. The set $U_X := \{Y \in \mathcal{P}_\kappa(A) : X \subseteq Y\}$ is clearly unbounded in $\mathcal{P}_\kappa(A)$. Then there is $B \in S$ such that $\mu := B \cap \kappa$ is regular and $U_X \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$. Note that $\bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$, because if $b \in B$, then $\{b\} \in \mathcal{P}_\mu(B)$ and so there is $Y \in U_X \cap \mathcal{P}_\mu(B)$ such that $\{b\} \subseteq Y$. Thus, $b \in Y \in U_X \cap \mathcal{P}_\mu(B)$ and $b \in \bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$. Now we will see that $X \subseteq B$. Let $x \in X$. Then $x \in Y$ for all $Y \in U_X$, in particular $x \in Y$ for all $Y \in U_X \cap \mathcal{P}_\mu(B)$. Hence $x \in \bigcup(U_X \cap \mathcal{P}_\mu(B)) = B$. $\square$

Next Proposition is the analogous of 2.2.2 in the case of $\langle\kappa, <\rangle$, so it is a good sign that 3.2.1 behaves well as a generalisation to the $\langle\mathcal{P}_\kappa(A), \subset\rangle$ case.

Proposition 3.2.3. *$S \subseteq \mathcal{P}_\kappa(A)$ being n-stationary implies S is m-stationary for all $m < n$.*

Proof : We proceed by induction. The case $n = 0$ is precisely Proposition 3.2.2. Suppose we have the result for all $k < n$, and that $S \subseteq \mathcal{P}_\kappa(A)$ n-stationary. Let $m < n$ and take $T \subseteq \mathcal{P}_\kappa(A)$ to be l-stationary for some $l < m$. As S is n-stationary, there is some $B \in S$ such that $\mu := B \cap \kappa$ is regular and $T \cap \mathcal{P}_\mu(B)$ is l-stationary in $\mathcal{P}_\mu(B)$. Therefore, S is m-stationary. □

It is straightforward that if $S' \subseteq S \subseteq \mathcal{P}_\kappa(A)$ and S' is n-stationary, then S is n-stationary as well. So the least condition for the existence of a n-stationary subset of $\mathcal{P}_\kappa(A)$ is to ask $\mathcal{P}_\kappa(A)$ to be n-stationary itself. In the previous chapter we saw that the fact that κ being 1-stationary in κ, is due to the fact that $cof(\kappa) > \omega$. For $\mathcal{P}_\kappa(A)$ to be 1-stationary we also have a necessary condition on the largeness of κ.

Proposition 3.2.4. *If $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$, then κ is weakly Mahlo.*

Proof : Suppose that $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$. $\kappa$is regular uncountable cardinal, in order to prove that κ is weakly Mahlo, it is enough to prove that the set $E := \{\mu < \kappa : \mu \text{ is a regular cardinal}\}$ is stationary in κ.

Let C be a club subset of κ and consider the set $T := \{X \in \mathcal{P}_\kappa(A) : \exists \alpha \in C \text{ s.t. } \; X \cap \kappa \subsetneq \alpha \leq |X|\}$.

T is unbounded in $\mathcal{P}_\kappa(A)$: Suppose $Y \in \mathcal{P}_\kappa(A)$ and let $\alpha \in C$ be such that $Y \cap \kappa \subsetneq \alpha$. Consider $\widetilde{\alpha} := \{\delta \setminus \{0\} : \delta \in \alpha\}$, clearly $\widetilde{\alpha} \cap \kappa = \{\varnothing\}$. Now $Z := Y \cup \{\widetilde{\alpha}\}$ is such that $Z \cap \kappa = (Y \cup \{\widetilde{\alpha}\}) \cap \kappa = (Y \cap \kappa) \cup (\{\widetilde{\alpha}\} \cap \kappa) = Y \cap \kappa \subsetneq \alpha$. Moreover $\alpha \leq |\alpha| = |\widetilde{\alpha}| \leq |Y \cup \widetilde{\alpha}| = |Z|$, whence $Z \in T$. Hence, for every $Y \in \mathcal{P}_\kappa(A)$ there is $Z \in T$ such that $Y \subseteq Z$.

Since $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$, and T is unbounded in $\mathcal{P}_\kappa(A)$, there is $B \in \mathcal{P}_\kappa(A)$ such that

- $\mu := B \cap \kappa$ is a regular cardinal.
- $T \cap \mathcal{P}_\mu(B)$ is 0-stationary in $\mathcal{P}_\mu(B)$.

We claim that $\mu \in C$, to see this we shall prove that $C \cap \mu$ is unbounded in $\mu < \kappa$. As C is closed, that would imply that $\mu \in C$.

- $C \cap \mu$ is unbounded in μ : Let $\gamma < \mu$, then $\gamma \in \mu = B \cup \kappa \subseteq B$, and so $\gamma \in \mathcal{P}_\mu(B)$. Then, there is $X \in T \cap \mathcal{P}_\mu(B)$ such that $\gamma \subseteq X$ (and so $\gamma \subseteq X \cap \kappa$). As $X \in T$, there is some $\alpha \in C$ such that $X \cap \kappa \subsetneq \alpha \leq |X|$. But then $\gamma \subsetneq X \cap \kappa \subsetneq \alpha \leq |X| < \mu$. This is $\alpha \in C \cap \mu$ and $\gamma < \alpha$.

Therefore $\mu \in C \cap E$, this shows that the set $E = \{\mu < \kappa : \mu \text{ is a regular cardinal}\}$ is stationary in κ. Hence κ is weakly Mahlo. □

In $\langle \kappa, < \rangle$, the condition $cof(\kappa) > \omega$ was also a sufficient condition for κ to be 1-stationary in κ. So it is natural to ask if "κ weakly Mahlo" is also a sufficient condition for $\mathcal{P}_\kappa(A)$ to be 1-stationary. Unfortunately, for Sakai's definition as given in 3.2.1 we do not have an answer. The main obstacle is that the conditions $\mu := B \cap \kappa$ regular and $T \cap \mathcal{P}_\mu(B)$ 0-stationary in $\mathcal{P}_\mu(B)$ seem difficult to be satisfied simultaneously. However, a slight modification of 3.2.1 yields that "κ weakly Mahlo" is also a sufficient condition.

Definition 3.2.5. *Let κ be a regular cardinal, $\kappa \subseteq A$ and $n < \omega$.*

1. *$S \subseteq \mathcal{P}_\kappa(A)$ is 0-**stationary in** $\mathcal{P}_\kappa(A)$ iff S is unbounded in $\mathcal{P}_\kappa(A)$.*
2. *$S \subseteq \mathcal{P}_\kappa(A)$ is n-**stationary** in $\mathcal{P}_\kappa(A)$ iff for all $m < n$ and for all $T \subseteq \mathcal{P}_\kappa(A)$ m-stationary in $\mathcal{P}_\kappa(A)$, there is $B \in S$ and $\mu < \kappa$ regular cardinal such that*
    - *$\mu \subseteq B \cap \kappa$,*
    - *$T \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$*
3. *$\mathcal{P}_\kappa(A)$ is n-**stationary** if it is n-stationary in $\mathcal{P}_\kappa(A)$ as a subset of $\mathcal{P}_\kappa(A)$.*

A very simple but useful fact is that if $S \subseteq \mathcal{P}_\kappa(A)$ is n-stationary in $\mathcal{P}_\kappa(A)$ and $S \subseteq S' \subseteq \mathcal{P}_\kappa(A)$, then S' is also n-stationary in $\mathcal{P}_\kappa(A)$. To prove this, suppose $S \subseteq \mathcal{P}_\kappa(A)$. If S is n-stationary in $\mathcal{P}_\kappa(A)$, then for all $m < n$ and all $T \subseteq \mathcal{P}_\kappa(A)$ m-stationary there is $B \in S \subseteq S'$ (and so there is $B \in S'$) and μ regular such that $\mu \subseteq B \cap \kappa$ and $T \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$. But this is precisely to say that S' is n-stationary in $\mathcal{P}_\kappa(A)$.

Remark. Since the conditions of Definition 3.2.5 are weaker than the ones of Definition 3.2.1, it is immediate to see that 3.2.2, 3.2.3 and 3.2.4 remain true under the new definition.

From now on, when talking about n-stationarity of a subset of $\mathcal{P}_\kappa(A)$ we will refer to this new definition. However, we will have still in mind 3.2.1, to justify why we think 3.2.5 is more suitable to our purposes. And we will clarify whenever a result is also valid with 3.2.1.

Theorem 3.2.6. *If κ is weakly Mahlo, then $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$.*

Proof : Suppose that κ is weakly Mahlo. Then, the set $E = \{\mu < \kappa : \mu \text{ is a regular cardinal}\}$ is stationary in κ. Let $T \subseteq \mathcal{P}_\kappa(A)$ be 0-stationary in $\mathcal{P}_\kappa(A)$, and construct the following transfinite sequence

$X_0 \in T$.
$X_{\alpha+1} \in T$ is such that $X_{\alpha+1} \supsetneq X_\alpha \cup \alpha$.
$X_\gamma \in T$ is such that $X_\gamma \supsetneq \bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha]$, for $\gamma < \kappa$ limit.

This sequence is well defined. The successor step may be performed since T is unbounded, $|X_\alpha|, |\alpha| < \kappa$ and so $X_\alpha \cup \alpha \in \mathcal{P}_\kappa(A)$. And limit step because T is unbounded, κ is regular and $\bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha] \in \mathcal{P}_\kappa(A)$. So defined $\{X_\alpha : \alpha < \kappa\} \subseteq T$ is an strict ascending chain. Now, consider the set $U := \{\alpha < \kappa : \exists \beta < \kappa \text{ s.t. } |X_\beta| = \alpha\}$.

Claim. U is unbounded in κ : Let $\delta < \kappa$. As κ is a regular limit cardinal $|\delta|^+ < \kappa$. Then $X_{|\delta|^++1} \supseteq X_{|\delta|^+} \cup |\delta|^+$. Note that $\delta < |\delta|^+ \leq |X_{|\delta|^++1}| < \kappa$. Then, for $\alpha := |X_{|\delta|^++1}| < \kappa$, there exists $\beta := |\delta|^+ + 1 < \kappa$ such that $|X_\beta| = \alpha > \delta$. Thus $\alpha \in U$ and $\delta < \alpha < \kappa$.

Now, since E is stationary in κ, from the claim above we get that, there is $\mu \in E$ such that $U \cap \mu$ is unbounded in μ. We may now construct the following subsequence:

Pick $\delta < \mu$. Then, there is $\delta_0 \in U \cap \mu$ such that $\delta < \delta_0$, and so there is $\beta_0 < \kappa$ such that $|X_{\beta_0}| = \delta_0 < \mu$. Given X_{β_α} let $X_{\beta_{\alpha+1}}$ be such that $|X_{\beta_\alpha}| < |X_{\beta_{\alpha+1}}| < \mu$; and for $\alpha < \mu$ limit, let X_{β_α} be such that $|\bigcup_{\xi<\alpha} X_{\beta_\xi}| < |X_{\beta\alpha}| < \mu$. Notice that $\beta_\alpha \neq \beta_{\alpha'}$ for $\alpha \neq \alpha'$ and since $\{X_{\beta_\alpha} : \alpha < \mu\} \subseteq \{X_\alpha : \alpha < \kappa\}$, we have that $\{X_{\beta_\alpha} : \alpha < \mu\}$ is also a chain. Since $|X_{\beta_\alpha}| < \kappa$, for all $\alpha < \mu < \kappa$ and κ is regular, $\bigcup_{\alpha<\mu} X_{\beta_\alpha} \in \mathcal{P}_\kappa(A)$.

Let $B := \bigcup_{\alpha<\mu} X_{\beta_\alpha}$, and notice that since $\{X_{\beta_\alpha} : \alpha < \mu\}$ forms a strictly ascending chain, $|B| \geq \mu$. Moreover, B is the union of at most μ many sets of cardinality less than μ, so that $|B| = \mu$. To conclude the proof we will show that B is as we wanted, this is, there is $\mu < \kappa$ regular such that

(i) $\mu \subseteq B \cap \kappa$: First notice that, if $\alpha < \alpha'$ then $X_{\beta_\alpha} \subsetneq X_{\beta_{\alpha'}}$, and since $\{X_\alpha : \alpha < \kappa\}$ is strict ascending, this implies $\beta_\alpha < \beta_{\alpha'}$. Now, we claim that

(a) $\beta_\alpha \subseteq B$ for all $\alpha < \mu$: let $\alpha < \mu$, as μ is limit $\alpha + 1 < \mu$, and so there is $\beta_{\alpha+1}$, and $\beta_\alpha \subseteq X_{\beta_{\alpha+1}} \subseteq B$.

(b) $\alpha \leq \beta_\alpha$ for all $\alpha < \mu$: by induction, $0 \leq \beta_0$. If $\alpha \leq \beta_\alpha$ we have that $\alpha \leq \beta_\alpha < \beta_{\alpha+1}$, whence $\alpha + 1 \leq \beta_{\alpha+1}$. For $\alpha < \mu$ limit, given that $\delta < \beta_\delta$ for all $\delta < \alpha$, we have $\alpha = \sup_{\delta<\alpha} \delta \leq \sup_{\delta<\alpha} \beta_\delta$. Also, for all $\delta < \alpha$ we have that $\beta_\delta < \beta_\alpha$ and so $\sup_{\delta<\alpha} \beta_\delta \leq \beta_\alpha$. Thus $\alpha \leq \sup_{\delta<\alpha} \beta_\delta \leq \beta_\alpha$.

From (a) we have that $\sup_{\alpha<\mu} \beta_\alpha = \bigcup_{\delta<\alpha} \beta_\delta \subseteq B$. From (b) we conclude that $\mu = \sup_{\alpha<\mu} \alpha \leq \sup_{\alpha<\mu} \beta_\alpha$. Therefore $\mu \leq \sup_{\alpha<\mu} \beta_\alpha \subseteq B$.

(ii) $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$: Let $X \in \mathcal{P}_\mu(B)$. Then $X \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha}$ and $|X| < \mu$. As $|B| = \mu$ is regular, we get that X is not unbounded in B. Then $X \subseteq X_{\beta_\alpha}$ for some $\alpha < \mu$. But $X_{\beta_\alpha} \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha} = B$ and $|X_{\beta_\alpha}| < \mu$. So, there is $X_{\beta_\alpha} \in T \cap \mathcal{P}_\mu(B)$ such that $X \subseteq X_{\beta_\alpha}$. □

It is worth noting a couple of things about the proof of Theorem 3.2.6. For every T unbounded subset of $\mathcal{P}_\kappa(A)$, we needed a set $B \in \mathcal{P}_\kappa(A)$ and a regular μ witnessing the reflection of T. We construct B and μ simultaneously and they are strongly related, however we could no guarantee that $\mu = B \cap \kappa$, as in Sakai's original definition (Def. 3.2.1). The problem is that despite the condition $|X_{\beta_\alpha}| < \mu$, it could perfectly be the case that for some $\gamma > \mu$, $\gamma \in X_{\beta_\alpha}$, so that $\gamma \in B \cap \kappa \setminus \mu$.

It is also interesting that in the proof of Theorem 3.2.6, we actually proved a little bit more than required. We proved an extra condition for B, namely $|B| = \mu$. That was due to the fact that B is the union of at most μ many sets of cardinality less than μ.

Therefore, from Proposition 3.2.4 and Theorem 3.2.6, we get a complete characterisation of 1-stationarity for $\mathcal{P}_\kappa(A)$, namely

Corollary. *$\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$ if and only if κ is weakly Mahlo.* □

Thus, stationary subsets of $\mathcal{P}_\kappa(A)$ exists only in the case that κ is weakly Mahlo. Moreover, for higher level of stationary subsets of $\mathcal{P}_\kappa(A)$, we need to require stronger conditions over κ. For instance, if κ is the least weakly Mahlo cardinal, then $\mathcal{P}_\kappa(A)$ does not contain 2-stationary subsets.

Proposition 3.2.7. *Let κ be the least weakly Mahlo cardinal, then $\mathcal{P}_\kappa(A)$ is not 2-stationary.*

Proof : Towards a contradiction, suppose that $\mathcal{P}_\kappa(A)$ is 2-stationary. As κ is weakly Mahlo, by Theorem 3.2.6 we have that $\mathcal{P}_\kappa(A)$ is 1-stationary. Then, there is $B \in \mathcal{P}_\kappa(A)$ and $\mu \subseteq B \cap \kappa$ such that $\mathcal{P}_\kappa(A) \cap \mathcal{P}_\mu(B)$ is 1-stationary in $\mathcal{P}_\mu(B)$. From $B \in \mathcal{P}_\kappa(A)$ and $\mu \subseteq B \cap \kappa$ we get that $\mu < \kappa$. But $\mathcal{P}_\kappa(A) \cap \mathcal{P}_\mu(B) = \mathcal{P}_\mu(B)$, and then $\mathcal{P}_\mu(B)$ is 1-stationary in $\mathcal{P}_\mu(B)$, but again by Proposition 3.2.4 this implies μ weakly Mahlo. □

Recall that for $\langle \kappa, < \rangle$, we have

$$S \subseteq \kappa \ \text{ club } \ \rightarrow \ S \ \text{ stationary } \ \leftrightarrow \ S \ \text{ 1-stationary } \ \rightarrow \ S \ \text{ unbounded} \tag{3.1}$$

In the case of $\langle \mathcal{P}_\kappa(A), < \rangle$ by Proposition 3.1.2, we have that

$$S \subseteq \mathcal{P}_\kappa(A) \ \text{ club } \ \rightarrow \ S \ \text{ stationary } \ \rightarrow \ S \ \text{ unbounded .}$$

We would like to have in the case $\mathcal{P}_\kappa(A)$, a similar diagram, as in Eq. 3.1, relating in this way 1-stationarity and stationarity in $\mathcal{P}_\kappa(A)$. Unfortunately, it is not that immediate to see how these concepts are linked in $\mathcal{P}_\kappa(A)$. However, for 1-stationarity we do have the following result

Proposition 3.2.8. *If κ is weakly Mahlo, then $C \subseteq \mathcal{P}_\kappa(A)$ club implies C is 1-stationary.*

Proof : Suppose that κ is weakly Mahlo, we may then perform a quite similar construction of what we did in 3.2.6. For each unbounded T of $\mathcal{P}_\kappa(A)$, we will however, construct the sequence of elements X_α inside $\bar{T} \cap C$. In this way we may guarantee $B \in C$.

Since κ is weakly Mahlo, the set $E = \{\mu < \kappa : \mu \text{ is regular}\}$ is stationary in κ. Let $T \subseteq \mathcal{P}_\kappa(A)$ be 0-stationary in $\mathcal{P}_\kappa(A)$, then $\bar{T} \cap C$ is a club of $\mathcal{P}_\kappa(A)$. Construct the following transfinite sequence

$X_0 \in T$. And $Y_0 \in C$ such that $X_0 \subseteq Y_0$
$X_{\alpha+1} \in T$ is such that $X_{\alpha+1} \supsetneq X_\alpha \cup \alpha \cup Y_\alpha$. And $Y_{\alpha+1} \in C$ such that $X_{\alpha+1} \subseteq Y_{\alpha+1}$
$X_\gamma \in T$ is such that $X_\gamma \supsetneq \bigcup_{\alpha<\gamma}[X_\alpha \cup \alpha \cup Y_\alpha]$ for $\gamma < \kappa$ limit.

As in the proof of 3.2.6 this sequence is well defined. In fact, the justification of this is completely analogous. We also may use exactly the same reasoning as in 3.2.6 to prove that the set $U = \{\alpha < \kappa : \exists\beta < \kappa \text{ s.t. } |X_\beta| = \alpha\}$ is unbounded in κ.

Now, since E is stationary and U is unbounded in κ, there is $\mu \in E$ such that $U \cap \mu$ is unbounded in μ. We may now construct an analogous subsequence as in 3.2.6. This is, a sequence $\{X_{\beta_\alpha} : \alpha < \mu\}$ such that $|X_{\beta_\alpha}| < |X_{\beta_{\alpha+1}}| < \mu$ for each $\alpha < \mu$. Thus $B := \bigcup_{\alpha<\mu} X_{\beta_\alpha} \in \mathcal{P}_\kappa(A)$, and $|B| = \mu$.

(i) $\mu \subseteq B \cap \kappa$: as in 3.2.6

(ii) $T \cap \mathcal{P}_\mu(B)$ is unbounded in $\mathcal{P}_\mu(B)$: Let $X \in \mathcal{P}_\mu(B)$. Then $X \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha}$ and $|X| < \mu$. As $|B| = \mu$ is regular, we get that X is not unbounded in B. Then $X \subseteq X_{\beta_\alpha}$ for some $\alpha < \mu$. But $X_{\beta_\alpha} \subseteq \bigcup_{\alpha<\mu} X_{\beta_\alpha} = B$ and $|X_{\beta_\alpha}| < \mu$. So, there is $X_{\beta_\alpha} \in T \cap \mathcal{P}_\mu(B)$ such that $X \subseteq X_{\beta_\alpha}$.

To conclude the proof it is enough to show that $B \in C$. We claim that $\bigcup_{\alpha<\mu} X_{\beta_\alpha} = \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$. Let $z \in \bigcup_{\alpha<\mu} X_{\beta_\alpha}$, this is $z \in X_{\beta_\alpha}$ for some $\alpha < \mu$. But by construction $X_{\beta_\alpha} \subseteq Y_{\beta_\alpha}$, then $z \in Y_{\beta_\alpha} \subseteq \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$. Conversely, if $z \in \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$ then $z \in Y_{\beta_\alpha}$ for some $\alpha < \mu$. Notice that for all $\alpha < \mu$, $X_{\beta_\alpha} \subsetneq X_{\beta_{\alpha+1}}$, hence $X_{\beta_\alpha+1} \subseteq X_{\beta_{\alpha+1}}$. Moreover, by construction (successor step) we have that $Y_{\beta_\alpha} \subseteq X_{\beta_\alpha+1} \subseteq X_{\beta_{\alpha+1}}$. Whence $z \in X_{\beta_{\alpha+1}}$ and so $z \in \bigcup_{\alpha<\mu} X_{\beta_\alpha}$.

Now $\{Y_{\beta_\alpha} : \alpha < \mu\}$ is clearly an ascending sequence of element of C. Then, as C is closed, we get that $\bigcup_{\alpha<\mu} Y_{\beta_\alpha} \in C$. But $B = \bigcup_{\alpha<\mu} X_{\beta_\alpha} = \bigcup_{\alpha<\mu} Y_{\beta_\alpha}$, then $B \in C$. $\Box$

Hence, by propositions 3.2.2 and 3.2.8, for κ weakly Mahlo we have

$$S \subseteq \mathcal{P}_\kappa(A) \text{ club } \rightarrow S \text{ 1-stationary } \rightarrow S \text{ unbounded .}$$

We shall also prove that the 1-stationarity of $\mathcal{P}_\kappa(A)$ is at least a stronger notion than stationarity of $\mathcal{P}_\kappa(A)$. The converse seems not to be true though. We may however, modify a bit more our definition of n-stationarity in order to have the equivalence. We will introduce this definition in the final chapter, since for us, having the equivalence between 1-stationarity and stationarity appears to be less important than having a definition of n-stationarity for $\mathcal{P}_\kappa(A)$ as close as possible to the definition of n-stationarity in ordinals [12].

Proposition 3.2.9. *If $S \subseteq \mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$, then S is stationary in $\mathcal{P}_\kappa(A)$.*

Proof : Suppose that S is 1-stationary in $\mathcal{P}_\kappa(A)$, and let C be a club subset of $\mathcal{P}_\kappa(A)$. In particular C is unbounded in $\mathcal{P}_\kappa(A)$ and so, there is some $B \in S$ such that

- $\mu \subseteq B \cap \kappa$ is a regular cardinal.
- $C \cap \mathcal{P}_\mu(B)$ is 0-stationary in $\mathcal{P}_\mu(B)$

Then, for each $x \in B$, there is some $Y_x \in C \cap \mathcal{P}_\mu(B)$ such that $\{x\} \subseteq Y_x$. As well, for each couple $Z = \{Y, Y'\}$ of elements of C, there is an element $W \in C \cap \mathcal{P}_\mu(B)$ such that $Y, Y' \subseteq W$. Using axiom of choice, we may pick one of these W for each pair Z. We will denote this choice by Y_Z.

We construct a sequence of subsets of $\mathcal{P}_\kappa(A)$. Let $T_0 := \{Y_x : x \in B\}$, then $|T_0| = |B| < \kappa$. $T_1 := T_0 \cup \{Y_Z : Z \in [T_0]^2\}$, then $|T_1| = \max\{|T_0|, |\{Y_Z : Z \in [T_0]^2\}|\} = \max\{|B|, |[T_0]^2|\} = \max\{|B|, |B|\} = |B| < \kappa$. Suppose that for each $i \in 0, \ldots, n-1$ the set $T_{i+1} = T_i \cup \{Y_Z : Z \in [T_i]\}$ is such that $|T_{i+1}| < \kappa$. Then $T_n := \{Y_Z : Z \in [T_{n-1}]\}$ and clearly $|T_n| = \max\{|T_{n-1}|, |\{Y_Z : Z \in [T_{n-1}]^2\}|\} = \max\{|T_{n-1}|, |[T_{n-1}]^2|\} = |T_{n-1}| < \kappa$.

Now, consider $T := \bigcup_{n<\omega} T_n$, then $|T| = \sup\{|T_n| : n < \omega\}$. Since each $|T_n| < \kappa$ and κ has uncountable cofinality, we conclude that $|T| < \kappa$. Moreover by the way we constructed T it is straightforward that it is finitely directed. Since C is a closed subset of $\mathcal{P}_\kappa(A)$ and T is finitely directed, by Proposition 3.1.3 we have that $\bigcup T \in C$.

We claim that $\bigcup T = B$. If $x \in \bigcup T$, then $x \in Y_Z$ for some $Y_Z \in T_n$ and $n < \omega$. But since $Y_Z \in \mathcal{P}_\mu(B)$, $x \in Y_Z \subseteq B$. Also, if $x \in B$, there is $Y_x \in T_0 \subseteq T$ such that $\{x\} \subseteq Y_x$, this is, $x \in \bigcup T$. Therefore $B \in S \cap C$ and so $S \cap C \neq \varnothing$. $\square$

Remark. Notice that in the proof of Proposition 3.2.9 we do not use the particular condition "$\mu \subseteq B \cap \kappa$", then, this result is also valid for Definition 3.2.1 in which "$\mu = B \cap \kappa$".

What can we say about higher levels of stationarity in $\mathcal{P}_\kappa(A)$? In the case of $\langle \kappa, < \rangle$, Proposition 2.2.3 tells us that α is a 2-stationary set if and only α is stationary reflecting, and we also saw that this fact implies that α is not the successor of a regular cardinal. Does 2-stationarity over $\mathcal{P}_\kappa(A)$ implies some condition over κ?

Theorem 3.2.10. *If $\mathcal{P}_\kappa(A)$ is 2-stationary in $\mathcal{P}_\kappa(A)$, then κ is 2-weakly Mahlo i.e. the set $\{\alpha < \kappa : \alpha \text{ is weakly mahlo }\}$ is stationary in κ.*

Proof : Suppose that $\mathcal{P}_\kappa(A)$ is 2-stationary in $\mathcal{P}_\kappa(A)$, we shall prove that the set $E := \{\mu < \kappa : \mu \text{ is weakly mahlo}\}$ is stationary in κ. By Proposition 3.2.3 the fact that $\mathcal{P}_\kappa(A)$ is 2-stationary implies $\mathcal{P}_\kappa(A)$ is 1-stationary and so κ is weakly Mahlo. Let C be a club subset of κ and consider the set $T := \{X \in \mathcal{P}_\kappa(A) : \exists \alpha \in C \text{ s.t. } \; X \cap \kappa \subseteq \alpha \leq |X|\}$.

- T is unbounded in $\mathcal{P}_\kappa(A)$: Suppose $Y \in \mathcal{P}_\kappa(A)$. Let $\alpha \in C$ be such that $Y \cap \kappa \subseteq \alpha$. Consider $\widetilde{\alpha} := \{\delta \setminus \{0\} : \delta \in \alpha\}$, clearly $\widetilde{\alpha} \cap \kappa = \{\varnothing\}$. Now $Z := Y \cup \{\widetilde{\alpha}\}$ is such that $Z \cap \kappa = (Y \cup \{\widetilde{\alpha}\}) \cap \kappa = (Y \cap \kappa) \cup (\{\widetilde{\alpha}\} \cap \kappa) = Y \cap \kappa \subseteq \alpha$. Moreover $\alpha \leq |\alpha| = |\widetilde{\alpha}| \leq |Y \cup \widetilde{\alpha}| = |Z|$, whence $Z \in T$. Hence, for $Y \in \mathcal{P}_\kappa(A)$ there is $Z \in T$ such that $Y \subseteq Z$.

- T is closed in $\mathcal{P}_\kappa(A)$: Let $\{X_\beta : \beta < \mu\}$ be an ascending sequence of elements of T. Notice that, for each X_β there is some α_β such that $X_\beta \cap \kappa \subseteq \alpha_\beta \leq |X|$. Consider $\alpha := \sup\{\alpha_\beta : \beta < \mu\}$. As C is closed, $\alpha \in C$. Moreover, from $X_\beta \cap \kappa \subseteq \alpha$ for each $\beta < \mu$, we get that $(\bigcup_{\beta<\mu} X_\beta) \cap \kappa \subseteq \sup\{\alpha_\beta : \beta < \mu\} = \alpha$. Also from $\alpha_\beta \leq |X_\beta|$ for each $\beta < \mu$, we get that $\alpha \leq \sup\{|X_\beta| : \beta < \mu\} = |\sup\{X_\beta : \beta < \mu\}| = |\bigcup_{\beta<\mu} X_\beta|$. This is, $(\bigcup_{\beta<\mu} X_\beta) \cap \kappa \subseteq \alpha \leq |\bigcup_{\beta<\mu} X_\beta|$, so that $\bigcup_{\beta<\mu} X_\beta \in T$.

Hence T is a club subset of $\mathcal{P}_\kappa(A)$, and so it is 1-stationary (3.2.8,). Now, since $\mathcal{P}_\kappa(A)$ is 2-stationary, there are $B \in \mathcal{P}_\kappa(A)$ and μ regular such that

- $\mu \subseteq B \cap \kappa$ is a regular cardinal.
- $T \cap \mathcal{P}_\mu(B)$ is 1-stationary in $\mathcal{P}_\mu(B)$.

But $T \cap \mathcal{P}_\mu(B)$ being 1-stationary in $\mathcal{P}_\mu(B)$ implies $\mathcal{P}_\mu(B)$ 1-stationary in $\mathcal{P}_\mu(B)$ and so μ is weakly Mahlo (3.2.4). Moreover, we claim that $\mu \in C$. To see that, we shall prove that $C \cap \mu$ is unbounded in $\mu < \kappa$. As C is closed, that will imply $\mu \in C$.

- $C \cap \mu$ is unbounded in μ : Let $\gamma < \mu$, then $\gamma \in \mathcal{P}_\mu(B)$. So, there is $X \in T \cap \mathcal{P}_\mu(B)$ such that $\gamma \subseteq X$ (and so $\gamma \subseteq X \cap \kappa$). As $X \in T$, there is some $\alpha \in C$ such that $X \cap \kappa \subseteq \alpha \leq |X|$. But then $\gamma \subseteq X \cap \kappa \subseteq \alpha \leq |X| < \mu$. This is, $\alpha \in C \cap \mu$ and $\gamma \leq \alpha$.

Therefore $\mu \in C \cap E$, whence E is stationary in κ. This shows κ is 2-weakly Mahlo. $\square$

So we have that κ being 2-weakly Mahlo is a necessary condition whenever $\mathcal{P}_\kappa(\lambda)$ is 2-stationary. Is this also a sufficient condition? In other words, do we have an analogous of Theorem 3.2.6 ? The fact of having a sufficient condition over κ for $\mathcal{P}_\kappa(A)$ to be 2-stationary, is equivalent to having a sufficient condition over κ to guarantee the existence of 2-stationary subsets of $\mathcal{P}_\kappa(A)$. Recall that in the case of κ, we get the existence of 1-stationary and 2-stationary sets respectively, jumping from the condition $cof(\kappa) \geq \omega_1$ to the condition of κ being simultaneous-reflecting. This suggests that the condition of κ being 2-weakly-Mahlo is too weak as a sufficient condition for 2-stationarity.

In general, we are interested in the conditions we have to ask of κ in order that $\mathcal{P}_\kappa(\lambda)$ reflects m-stationary sets for all $m < n$. In fact, we would like to know the least such condition, as in Theorem 3.2.6 . We will slowly approach this question by first considering what happens in $\mathcal{P}_\kappa(\kappa)$, and then with a general but probably too strong answer in the general case $\mathcal{P}_\kappa(\lambda)$.

Lemma 3.2.11. *Let κ be regular and let $\mu < \kappa$. Then the formula $\varphi_n(S)$: "$S \subseteq \mathcal{P}_\kappa(\kappa)$ is n-stationary in $\mathcal{P}_\kappa(\kappa)$" is Π^1_n over $\langle V_\kappa, \in, S\rangle$. Moreover, if $B \in \mathcal{P}_\kappa(\kappa)$, then $\varphi'_n(T)$: "$T \subseteq \mathcal{P}_\mu(B)$ is n-stationary in $\mathcal{P}_\mu(B)$" is a Π^1_0 sentence over $\langle V_\kappa, \in\rangle$, in the parameters T, μ, B.*

Proof : First we will show that $\mathcal{P}_\kappa(\kappa) \in V_{\kappa+1} \setminus V_\kappa$ and $\mathcal{P}_\mu(B) \in V_\kappa$. If $X \in \mathcal{P}_\kappa(\kappa)$, then $X \subseteq \alpha$ for some $\alpha < \kappa$. So we have $\text{rank}(X) \leq \text{rank}(\alpha) < \text{rank}(\kappa) = \kappa$, this is $X \in \{z : \text{rank}(z) < \kappa\} = V_\kappa$, whence $\mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$ and so $\mathcal{P}_\kappa(\kappa) \in V_{\kappa+1}$. Since $\kappa \subseteq \mathcal{P}_\kappa(\kappa)$, $\kappa = \text{rank}(\kappa) \leq \text{rank}(\mathcal{P}_\kappa(\kappa))$, and this implies $\mathcal{P}_\kappa(\kappa) \notin V_\kappa$. Moreover, if $B \in S \subseteq \mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$, $B \in V_\alpha$ for some $\alpha < \kappa$. So that $\mathcal{P}(B) \in V_{\alpha+1} \subseteq V_\kappa$, and so $\mathcal{P}_\mu(B) \in V_\kappa$.

Notice that $X \in \mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \psi(X)$ where $\psi(X) : \exists\alpha(OR(\alpha) \wedge X \subseteq \alpha)$. So defined $\psi(X)$ is a Π^1_0 formula. In fact, $\psi(X)$ is a Σ_1 formula with X as a free variable.

We will now prove the lemma by simultaneous induction. Let $n = 0$. $S \subseteq \mathcal{P}_\kappa(\kappa)$ is 0-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_0(S)$ where

$$\varphi_0(S) : \ \forall X \ (\psi(X) \rightarrow \exists Y \in S \ (X \subseteq Y)\ )$$

X is a first-order variable, because it ranges over elements of $\mathcal{P}_\kappa(\kappa) \subseteq V_\kappa$. Thus $\varphi_0(S)$ is first order, i.e., Π^1_0.

Given $\mu < \kappa$ and $B \in \mathcal{P}_\kappa(\kappa)$, we have that $T \subseteq \mathcal{P}_\mu(B)$ is 0-startionary in $\mathcal{P}_\mu(B)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi'_0(T, \mu, B)$ where

$$\varphi'_0(T, \mu, B) : \ \forall X \ (X \in \mathcal{P}_\mu(B) \rightarrow \exists Y \in T \ (X \subseteq Y)\ )$$

Since $T \subseteq \mathcal{P}_\mu(B) \in V_\kappa$ and $X \in \mathcal{P}_\mu(B) \in V_\kappa$, $\varphi'_0(T; \mu, B)$ is a Π_1 formula, and so it is Π^1_0 in the parameters T, μ, B.

For $n = 1$, $S \subseteq \mathcal{P}_\kappa(\kappa)$ is 1-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_1(S)$ where

$$\varphi_1(S) : \ \forall X \ \phi_1(S, X)$$

$$\phi_1(S, X) : (\forall Z(Z \in X \rightarrow \psi(Z)) \wedge \varphi_0(S)) \ \rightarrow \ \sigma_1(S, X)$$

$$\sigma_1(S, X) : \exists B \exists \mu(B \in S \wedge Reg(\mu) \wedge \mu \subseteq B \wedge \varphi'_0(X \cap \mathcal{P}_\mu(B)))$$

X is a second order variable because its possible values are subsets of $\mathcal{P}_\kappa(\kappa)$. Note that Z ranges over elements of V_κ ($X \in V_{\kappa+1}$ and $Z \in X$ implies $Z \in V_\kappa$). Then, as $\varphi'_0(X \cap \mathcal{P}_\mu(B))$ is Π^1_0, so is $\sigma_1(S, X)$. Together with the fact that $\psi(Z)$ and $\varphi_0(S)$ are also Π^1_0, we get that $\varphi_1(S)$ is Π^1_1.

Given $\mu < \kappa$ and $B \in \mathcal{P}_\kappa(\kappa)$, we have that $T \subseteq \mathcal{P}_\mu(B)$ is 1-stationary in $\mathcal{P}_\mu(B)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi'_1(T; \mu, B)$ where

$$\varphi'_1(T; \mu, B) : \ \forall X \ \phi'_1(X, T; \mu, B)$$

$$\phi'_1(T; \mu, B) : (X \subseteq \mathcal{P}_\mu(B) \wedge \varphi'_0(X; \mu, B)) \ \rightarrow \ \sigma'_1(T, X)$$

$$\sigma'_1(T, X) : \exists B' \exists \mu'(B' \in T \wedge Reg(\mu') \wedge \mu' \subseteq B \wedge \varphi'_0(X \cap \mathcal{P}_{\mu'}(B'); \mu', B'))$$

Here X is a first-order variable because its possible values are subsets of $\mathcal{P}_\mu(B) \in V_\kappa$, and $\varphi'_0(X; \mu, B), \varphi'_0(X \cap \mathcal{P}_{\mu'}(B'); \mu', B')$ are Π_1 formulas. Then, $\sigma'_1(T, X)$ is a Σ_2 formula, whence $\varphi'_1(T; \mu, B)$ is a Π_3 formula and so a Π^1_0 formula .

Suppose now, that $S \subseteq \mathcal{P}_\kappa(\kappa)$ is m-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in\rangle \models \varphi_m(S)$, where $\varphi_m(S)$ is a Π^1_m formula for all $m < n$. Then $\varphi_m(S)$ is of the form $\forall\, \mathbf{Y}^m_1 \ \exists\, \mathbf{Y}^m_2 \ \ldots \ Q\, \mathbf{Y}^m_m \ \phi_m(S, \mathbf{Y}^m_1, \ldots, \mathbf{Y}^m_m)$

where $Q = \forall$ if m is odd, $Q = \exists$ if m is even, $\mathbf{Y}^m_j = Y_1, \ldots, Y_{k_j}$ for $j \in \{1, \ldots, n\}$ and $\phi_m(S, \mathbf{Y}^m_1, \ldots, \mathbf{Y}^m_m)$ is a Π^1_0 formula.

Let us prove the result for n. We have $S \subseteq \mathcal{P}_\kappa(\kappa)$ is n-stationary in $\mathcal{P}_\kappa(\kappa)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi_n(S)$ where

$$\varphi_n(S) : \varphi_{n-1}(S) \ \wedge \ \forall X((\forall Z(Z \in X \to \psi(Z)) \wedge \varphi_{n-1}(S)) \to \sigma_n(S, X) \ )$$

From the inductive hypothesis, we know that $\varphi_{n-1}(S)$ is of the form $\forall\, \mathbf{Y}^{n-1}_1 \, \exists\, \mathbf{Y}^{n-1}_2 \ \ldots \ Q\, \mathbf{Y}^{n-1}_{n-1}$ $\phi_{n-1}(S, \mathbf{Y}^{n-1}_1, \ldots, \mathbf{Y}^{n-1}_{n-1})$, and so we have that

$$\forall X((\forall Z(Z \in X \to \psi(Z)) \wedge \varphi_{n-1}(S)) \to \sigma_n(S, X) \ ) \equiv \forall X \ \ \exists\, \mathbf{Y}^{n-1}_1 \ \forall\, \mathbf{Y}^{n-1}_2 \ \cdots$$
$$\bar{Q}\, \mathbf{Y}^{n-1}_{n-1}((\forall Z(Z \in X \to \psi(Z)) \wedge \phi_{n-1}(S, \mathbf{Y}^{n-1}_1, \ldots, \mathbf{Y}^{n-1}_{n-1})) \to \sigma_n(S, X) \ )$$

where $\bar{Q} = \forall$ if $Q = \exists$ and $\bar{Q} = \exists$ if $Q = \forall$. And the first order formula

$$\sigma_n(S, X) : \exists B \exists \mu (B \in S \wedge Reg(\mu) \wedge \mu \subseteq B \wedge \varphi'_{n-1}(X \cap \mathcal{P}_\mu(B)))$$

Therefore, if $(\mathbf{X}_1 := X, \mathbf{Y}^1_1, \ldots, \mathbf{Y}^{n-1}_1), \ldots, (\mathbf{X}_i := \mathbf{Y}^i_i, \ldots, \mathbf{Y}^{n-1}_i, \mathbf{Y}^{n-1}_{i-1}), \ldots, (\mathbf{X}_n := \mathbf{Y}^{n-1}_{n-1})$, we may write $\varphi_n(S)$ in the following form

$$\varphi_n(S) \equiv \forall \mathbf{X}_1 \ \ \exists\, \mathbf{X}_2 \ \forall\, \mathbf{X}_3 \ \ldots \ \bar{Q}\, \mathbf{X}_n(\phi_1(S, \mathbf{Y}_1) \wedge \phi_2(S, \mathbf{Y}_1, \mathbf{Y}_2) \wedge \cdots \wedge \phi_{n-1}(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1})$$
$$\wedge \ ((\forall Z(Z \in X \to \psi(Z)) \wedge \phi_{n-1}(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_{n-1})) \to \sigma_n(S, X)) \ )$$

Since $\phi_j(S, \mathbf{Y}_1, \ldots, \mathbf{Y}_i)$ and $\sigma_n(S, X)$ are Π^1_0 formulas for $j \in \{1, \ldots, n-1\}$, we get that $\varphi_n(S)$ is a Π^1_n formula.

Suppose now, that for $\mu < \kappa$ and $B \in \mathcal{P}_\kappa(\kappa)$, $T \subseteq \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi'_m(T, \mu, B)$, where $\varphi'_m(T, \mu, B)$ is a Π^1_0 formula for all $m < n$.

$T \subseteq \mathcal{P}_\mu(B)$ is n-stationary in $\mathcal{P}_\mu(B)$ if and only if $\langle V_\kappa, \in \rangle \models \varphi'_n(T, \mu, B)$, where

$$\varphi'_n(T, \mu, B) : \varphi'_{n-1}(T, \mu, B) \ \wedge \ \forall X((X \subseteq \mathcal{P}_\mu(B) \wedge \varphi'_{n-1}(X, \mu, B)) \ \to \ \sigma'_n(T, X) \ )$$

and where

$$\sigma'_n(T, X) : \exists B' \exists \mu' (B' \in T \wedge Reg(\mu') \wedge \mu' \subseteq B \wedge \varphi'_{n-1}(X \cap \mathcal{P}_{\mu'}(B'); \mu', B')).$$

Here, X is a first-order variable because its possible values are subsets of $\mathcal{P}_\mu(B) \in V_\kappa$, and $\varphi'_{n-1}(X \cap \mathcal{P}_\mu(B), \mu', B')$ and $\sigma'_n(T, X)$ are first-order formulas. Then $\varphi'_n(T, \mu, B)$ is a first-order formula and so it is Π^1_0. $\square$

Theorem 3.2.12. *Let $n < \omega$. If κ is Π^1_n indescribable, then $\mathcal{P}_\kappa(\kappa)$ is $n+1$ stationary.*

Proof : Suppose κ is Π^1_n indescribable. Let $S \subseteq \mathcal{P}_\kappa(\kappa)$ be m-stationary, some $m < n+1$. Consider the Π^1_m sentence in $\langle V_\kappa, \in, S \rangle$. Then, we have

$$\langle V_\kappa, \in, S \rangle \models \varphi_m(S)$$

As κ is Π^1_n indescribable and $m \leq n$, there is some $\mu < \kappa$ such that

$$\langle V_\mu, \in, S \cap V_\mu \rangle \models \varphi_m(S \cap V_\mu).$$

Now, note that $\mathcal{P}_\kappa(\kappa) \cap V_\mu = \mathcal{P}_\mu(\mu)$. For if $X \in \mathcal{P}_\kappa(\kappa) \cap V_\mu$ then $X \subseteq \kappa \cap V_\mu = \mu$. Also $|X| < \mu$, otherwise $\mathrm{rank}(X) = \mu$ and so $X \notin V_\mu$. Hence $X \in \mathcal{P}_\mu(\mu)$.

Thus, since $S = S \cap \mathcal{P}_\kappa(\kappa)$, we have that $S \cap V_\mu = S \cap \mathcal{P}_\kappa(\kappa) \cap V_\mu = S \cap \mathcal{P}_\mu(\mu)$. Therefore, we have $\langle V_\mu, \in, S \cap \mathcal{P}_\mu(\mu) \rangle \models \varphi_m(S \cap \mathcal{P}_\mu(\mu))$, and so $S \cap \mathcal{P}_\mu(\mu)$ is m-stationary in $\mathcal{P}_\mu(\mu)$. $\square$

Remark. Notice that Lemma 3.2.11 is also valid when considering Definition 3.2.1. To do this, in every formula we used, we should; quit μ as a parameter and replace the condition "$\mu \subseteq B$" by the condition "$\mu = B \cap On$ $(= B \cap \kappa)$". It is straightforward to check that everything else still works after doing this. Therefore, Theorem 3.2.12 is also true for definition 3.2.1.

Corollary. *If κ is totally indescribable, then $\mathcal{P}_\kappa(\kappa)$ is n-stationary for any $n \in \mathbb{N}$.*

Now, κ is Π^1_1-indescribable if and only if it is weakly compact (See [14]). Then, if κ is weakly compact, $\mathcal{P}_\kappa(\kappa)$ is 2-stationary.

Lemma 3.2.13. *Let f be an isomorphism between $\mathcal{P}_\kappa(\lambda)$ and $\mathcal{P}_\kappa(\delta)$, then, $S \subseteq \mathcal{P}_\kappa(\lambda)$ is m-stationary in $\mathcal{P}_\kappa(\lambda)$ if and only if $f[S]$ is m-stationary in $\mathcal{P}_\kappa(\delta)$.*

Theorem 3.2.14. *If κ is λ-supercompact and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_\kappa(\lambda)$ is n-stationary for any $n \in \mathbb{N}$.*

Proof : Let $n < \omega$ and take $S \subseteq \mathcal{P}_\kappa(\lambda)$ be m-stationary for a given $m < n$. Suppose that κ is λ-supercompact, this is, there is an elementary embedding $j : V \preceq M$ such that $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$ and ${}^\lambda M \subseteq M$, where M is transitive.

Recall that $j``x = \{j(y) : y \in x\}$, we claim that $j``\alpha \in M$, for all $\alpha \leq \lambda$. We prove this by induction on OR, $j``0 = 0 \in M$ because $j|_\kappa = Id|_\kappa$. If $j``\alpha \in M$ for $\alpha < \lambda$, then $j``(\alpha + 1) = j``\alpha \cup \{j(\alpha)\} \in M$. And if $\alpha \leq \lambda$ limit and $j``\beta \in M$ for all $\beta < \alpha$ then $j``\alpha = \{j``\beta : \beta < \alpha\}$ which is a sequence of $\alpha \leq \lambda$ elements of $M$, whence $j``\alpha \in^\lambda M \subseteq M$.

Since $j \restriction_\kappa = Id \restriction_\kappa$, we have that, $j``\kappa = \{j(\alpha) : \alpha < \kappa\} = \{\alpha : \alpha < \kappa\} = \kappa \in M$. Then, it follows that $\mathcal{P}_{j``\kappa}(j``\lambda) = \mathcal{P}_\kappa(j``\lambda) \subseteq M$. Moreover, as $|j``\lambda| = |\lambda|$, then $|\mathcal{P}_\kappa(j``\lambda)| = |j``\lambda|^{<\kappa} = \lambda^{<\kappa} = \lambda$, and so $\mathcal{P}_\kappa(j``\lambda) \in M$. Now, notice that there is an isomorphism f between $\mathcal{P}_\kappa(\lambda)$ and $\mathcal{P}_\kappa(j''\lambda)$ given by $X \mapsto j''X$.

By hypothesis, we have that $S \subseteq \mathcal{P}_\kappa(\lambda)$ is m-stationary in $\mathcal{P}_\kappa(\lambda)$, so applying Lemma 3.2.13 we get that, $f[S] = j``S \subseteq \mathcal{P}_\kappa(j``\lambda)$ is m-stationary in $\mathcal{P}_\kappa(j``\lambda)$. Therefore, as $j''S \subseteq j(S)$ we have that

$$V \models \text{“ } j(S) \cap \mathcal{P}_\kappa(j``\lambda) \text{ is } m\text{-stationary in } \mathcal{P}_\kappa(j``\lambda) \text{ ”}$$

Since $\mathcal{P}_\kappa(j``\lambda) \in M$, we have that $\mathcal{P}(\mathcal{P}_\kappa(j``\lambda)) \subseteq M$. So, since being m-stationary depends only on the subsets of $\mathcal{P}_\kappa(j``\lambda)$.

$$M \models \text{“ } j(S) \cap \mathcal{P}_\kappa(j``\lambda) \text{ is } m\text{-stationary in } \mathcal{P}_\kappa(j``\lambda) \text{ ”.}$$

In M we have that κ is regular and such that $\kappa < j(\kappa)$. If we define $B := j``\lambda$, then $\kappa = j``\kappa \subseteq j``\lambda = B$, and so $\kappa \subseteq B \cap j(\kappa)$. In fact $\kappa = B \cap j(\kappa)$ because if $\alpha \in (B \cap j(\kappa)) \setminus \kappa$, then $\alpha = j(\beta)$ for some $\kappa < \beta < \lambda$ and $\alpha < j(\kappa)$, but $\kappa < \beta$ implies $j(\kappa) < j(\beta) = \alpha$, and this is a contradiction. Besides, as $|j``\lambda| = \lambda < j(\kappa)$, we have that $B \in \mathcal{P}_{j(\kappa)}(j(\lambda))$. Hence the following holds, witnessed by $\mu = \kappa$ and $B = j``\lambda$

$M \models \exists\mu, \exists B(\, Reg(\mu) \wedge B \in \mathcal{P}_{j(\kappa)}(j(\lambda)) \wedge \mu = B \cap j(\kappa) \wedge \text{“}j(S) \cap \mathcal{P}_\mu(B) \text{ is } m\text{-stationary in } \mathcal{P}_\mu(B)\text{”} \,)$.

As j is an elementary embedding we get that

$$V \models \exists\mu, \exists B(\, Reg(\mu) \wedge B \in \mathcal{P}_{j^{-1}(j(\kappa))}(j^{-1}(j(\lambda))) \wedge \mu = B \cap j^{-1}(j(\kappa)) \wedge$$
$$\text{“}j^{-1}(j(S)) \cap \mathcal{P}_\mu(B) \text{ is } m\text{-stationary in } \mathcal{P}_\mu(B)\text{”} \,).$$

and since $j^{-1}(j(\kappa)) = \kappa$, $j^{-1}(j(\lambda)) = \lambda$ and $j^{-1}(j(S)) = S$,

$$V \models \exists\mu, \exists B(\, Reg(\mu) \wedge B \in \mathcal{P}_\kappa(\lambda) \wedge \mu = B \cap \kappa \wedge \text{“}S \cap \mathcal{P}_\mu(B) \text{ is } m\text{-stationary in } \mathcal{P}_\mu(B)\text{”} \,).$$

This is, for each $m < n$ if $S \subseteq \mathcal{P}_\kappa(\lambda)$ is m-stationary, there is $B \in \mathcal{P}_\kappa(\lambda)$ and $\mu < \kappa$ regular such that

- $\mu \subseteq B \cap \kappa$.

- $S \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$

and this is precisely to say that $\mathcal{P}_\kappa(\lambda)$ is n-stationary. $\square$

Remark. Notice that in the previous proof the B we obtained is such that $\mu = B \cap \kappa$. Then, the theorem 3.2.14 also holds when considering the original definition of Sakai 3.2.1.

3.3 The ideal of non-n-stationary subsets of $\mathcal{P}_\kappa(\lambda)$.

In the previous chapter we reviewed how in [12] Bagaria defined an increasing sequence of topologies on δ in such a way that the non-discreteness of each topology τ_ξ corresponds to the existence of a ξ simultaneously reflecting cardinal (See Theorem 2.3.15). Each one of the topologies defined in 2.3.1 depends on an operator d_ξ acting on subsets of δ, which take the limit points of each set in each topology. Defining an analogous sequence of topologies in $\mathcal{P}_\kappa(A)$ is however not that immediate, and the main obstacle is precisely to determine which points from $\mathcal{P}_\kappa(A)$ will we consider as the limit points.

In [12], Bagaria obtained a characterisation of limit points in τ_ξ in terms of the points where a given set is ξ-s-stationarity (See Proposition 2.3.7), which for the case $\xi \in \{1, 0\}$ is equivalent to ξ-stationarity. And by Proposition 2.3.8, when δ is $\xi + 1$-s-stationary this characterisation extends to the points in which a given set ξ-s-reflect. These suggest a possible way of defining the operator d_n and so the adequate topologies in $\mathcal{P}_\kappa(A)$.

Definition 3.3.1. *We say that an n-stationary subset $X \subseteq \mathcal{P}_\kappa(A)$ n-**reflects** at $B \in \mathcal{P}_\kappa(A)$ iff there is $\mu < \kappa$ regular such that $\mu \subseteq B \cap \kappa$ and $X \cap \mathcal{P}_\mu(B)$ is n-stationary in $\mathcal{P}_\mu(B)$.*

Remark. We also may perform a similar definition, asking the stronger condition "$\mu = B \cap \kappa$ and $X \cap \mathcal{P}_\mu(B)$ is n-stationary in $\mathcal{P}_\mu(B)$", instead of, "there is $\mu < \kappa$ regular such that $\mu \subseteq B \cap \kappa$ and $X \cap \mathcal{P}_\mu(B)$ is n-stationary in $\mathcal{P}_\mu(B)$". So it is convenient to bear in mind that the results and definitions in these chapter can also be done with Definition 3.2.1.

Notice that if κ is weakly Mahlo, then every unbounded subset T of $\mathcal{P}_\kappa(A)$ 0-reflects to some element of $\mathcal{P}_\kappa(A)$. More in general, if $\mathcal{P}_\kappa(A)$ is n-stationary, then every m-stationary subset S of $\mathcal{P}_\kappa(A)$ for $m < n$, m-reflects to some $B \in \mathcal{P}_\kappa(A)$.

Definition 3.3.2. *Let $\mathcal{P}_\kappa(A)$ be n-stationary. Given an m-stationary subset $S \subseteq \mathcal{P}_\kappa(A)$ with $m < n$, let*

$$d_m(S) := \{X \in \mathcal{P}_\kappa(A) : S \text{ } m\text{-reflects at } X\}.$$

This definition, gives an example of a nontrivial n-stationary subset of $\mathcal{P}_\kappa(A)$. More precisely, we have that

Proposition 3.3.3. *Suppose that $\mathcal{P}_\kappa(A)$ is n-stationary, then $d_m(\mathcal{P}_\kappa(A))$ is an n-stationary proper subset of $\mathcal{P}_\kappa(A)$.*

Proof : To see that $d_m(\mathcal{P}_\kappa(A)) \neq \mathcal{P}_\kappa(A)$, it is enough to take $X \in \mathcal{P}_\kappa(A)$ such that $X \cap \kappa < \omega$. Now, let S be m-stationary for some $m < n$. As $\mathcal{P}_\kappa(A)$ is n-stationary, there is $B \in \mathcal{P}_\kappa(A)$ and $\mu < \kappa$ regular such that $\mu \subseteq B \cap \kappa$ and $S \cap \mathcal{P}_\mu(B)$ is m-stationary in $\mathcal{P}_\mu(B)$. But this is exactly to say that S m-reflects to B, and so $B \in d_m(\mathcal{P}_\kappa(A))$. $\square$

Proposition 3.3.4. *Let S be an n-stationary subset of $\mathcal{P}_\kappa(A)$ and let $m < n$, then $d_n(S) \subseteq d_m(S)$.*

Proof: Follows immediately from the definition 3.3.2. $\square$

Definition 3.3.5. *Let $NS^n_{\kappa,A}$ be the set of non n-stationary subsets of $\mathcal{P}_\kappa(A)$, this is $NS^n_{\kappa,A} := \{S \subseteq \mathcal{P}_\kappa(A) : S$ is not n-stationary in $\mathcal{P}_\kappa(A)\}$. Moreover let $F^n_{\kappa,A} := \{\mathcal{P}_\kappa(A) \setminus X : X \in NS^n_{\kappa,A}\}$.*

Notice that whenever $NS^n_{\kappa,A}$ is an ideal $F^n_{\kappa,A}$ is in fact the dual filter associated to $NS^n_{\kappa,A}$, this is $F^n_{\kappa,A} := (NS^n_{\kappa,A})^*$.

Proposition 3.3.6. *Let $\mathcal{P}_\kappa(A)$ be n-stationary and let $X \in \mathcal{P}_\kappa(A)$. Then $X \in F^n_{\kappa,A}$ if and only if there is $T_X \subseteq \mathcal{P}_\kappa(\lambda)$ m-stationary for some $m < n$ such that $d_m(T_X) \subseteq X$.*

Proof : ($\Rightarrow$) Let $X \in F^n_{\kappa,A}$. Then $X = \mathcal{P}_\kappa(A) \setminus Y$ for some $Y \in NS^n_{\kappa,A}$. Since Y is not n-stationary, there is $T_X \subseteq \mathcal{P}_\kappa(\kappa)$ m-stationary with $m < n$ such that, for all $B \in Y$ and all $\mu \subseteq B \cap \kappa$ regular, $T_X \cap \mathcal{P}_\mu(B)$ is not m-stationary in $\mathcal{P}_\mu(B)$ (*).

We claim that $d_m(T_X) \subseteq X$. To see this it is enough to prove that $d_m(T_X) \cap Y = \varnothing$. Towards a contradiction, suppose that $W \in d_m(T_X) \cap Y$. Then, $W \in Y$ and T_X m-reflects at W. This is, $W \in Y$ and there is $\mu < \kappa$ regular such that $\mu \subseteq W \cap \kappa$ and $T_X \cap \mathcal{P}_\mu(W)$ is m-stationary in $\mathcal{P}_\mu(W)$, but this is a contradiction to (*).

($\Leftarrow$) Suppose that $X \in \mathcal{P}_\kappa(A)$ is such that there is $T_X \subseteq \mathcal{P}_\kappa(\lambda)$ m-stationary for some $m < n$ such that $d_m(T_X) \subseteq X$. Let us consider $Y := \mathcal{P}_\kappa(A) \setminus X$. We shall prove that $Y \in NS^n_{\kappa,\lambda}$. By contradiction, suppose Y is n-stationary. Then, for the m-stationary set $T_X \subseteq \mathcal{P}_\kappa(A)$, there is $B \in Y$ and $\mu \subseteq B \cap \kappa$ such that $T_X \cap \mathcal{P}_\mu(B)$is m-stationary in $\mathcal{P}_\mu(B)$. From the latter, we conclude that $B \in d_m(T_X) \subseteq X$. But B is also an element of Y, this is $B \in \mathcal{P}_\kappa(\lambda) \setminus X$, contradicting the fact that $B \in X$. □

From 3.3.6, we conclude that in analogy with the case $\langle \kappa, z \rangle$ (Proposition 2.3.13), whenever $\mathcal{P}_\kappa(A)$ is n-stationary,

$$F^n_{\kappa,A} = \{X \subseteq \mathcal{P}_\kappa(A) : \exists T_X \subseteq \mathcal{P}_\kappa(\lambda) \;\; m\text{-stationary for some } m < n, \text{ such that } d_m(T_X) \subseteq X\}.$$

Notice that if S is an m-stationary subset of $\mathcal{P}_\kappa(A)$ for $m < n$, then $d_m(S) \in F^n_{\kappa,A}$.

Lemma 3.3.7. *If T_1, T_2 are both not unbounded subsets of $\mathcal{P}_\kappa(A)$, then $T_1 \cup T_2$ is not unbounded either.*

Proof : Suppose $T_i \subseteq \mathcal{P}_\kappa(A)$ is not unbounded for $i \in \{1, 2\}$, then, there is $X_i \in \mathcal{P}_\kappa(A)$ such that for all $Y \in T_i$, $X_i \nsubseteq Y$. Towards a contradiction, suppose that $T_1 \cup T_2$ is unbounded in $\mathcal{P}_\kappa(A)$. Then, there is $Y_1 \in T_1 \cup T_2$ such that $X_1 \subseteq Y_1$. Notice that $Y_1 \notin T_1$. Also, there is $Y_2 \in T_1 \cup T_2$ such that $Y_1 \cup X_2 \subseteq Y_2$. Then $X_1 \subseteq Y_1 \cup X_2 \subseteq Y_2$. So, if $Y_2 \in T_1$ then $X_1 \subseteq Y_2$ contradicts that for all $Y \in T_1$, $X_1 \nsubseteq Y$. Similarly if $Y_2 \in T_2$ then $X_2 \subseteq Y_2$ contradicts that for all $Y \in T_2$, $X_2 \nsubseteq Y$. Hence $Y_2 \notin T_1 \cup T_2$, which is a contradiction. □

Proposition 3.3.8. *If $\mathcal{P}_\kappa(A)$ has the property that for all T_1, T_2 m^*-stationary, there is some T m-stationary such that $d_m(T) \subseteq d_{m^*}(T_1) \cap d_{m^*}(T_2)$, where $m \leq m^*$. Then, the set $NS^n_{\kappa,A}$ is an ideal over $\mathcal{P}_\kappa(A)$. Moreover $\mathcal{P}_\kappa(A)$ is n-stationary if and only if $NS^n_{\kappa,A}$ is a proper ideal.*

Proof : To prove that $NS^n_{\kappa,A}$ is an ideal we need to show; (i) $\varnothing \in NS^n_{\kappa,A}$, (ii) $X_1, X_2 \in NS^n_{\kappa,A}$ implies $X_1 \cup X_2 \in NS^n_{\kappa,A}$ and (iii) $X \in NS^n_{\kappa,A}$ and $Y \subseteq X$ then $Y \in NS^n_{\kappa,A}$. We will proceed by induction on n:

(i) Suppose $\varnothing$ is n-stationary in $\mathcal{P}_\kappa(A)$, then for the unbounded set $\mathcal{P}_\kappa(A)$ it must exist some element $B \in \varnothing$ witnessing the requirements of the definition. However, $B \in \varnothing$ is a contradiction. Hence $\varnothing \in NS^n_{\kappa,A}$.

(iii) Let $X \in NS^n_{\kappa,A}$ and let $Y \in \mathcal{P}_\kappa(A)$ such that $Y \subseteq X$. If Y is n-stationary in $\mathcal{P}_\kappa(A)$, then X is n-stationary in $\mathcal{P}_\kappa(A)$, which is a contradiction, hence $Y \in NS^n_{\kappa,A}$.

* (ii) The case $n = 0$ is precisely Lemma 3.3.7. Suppose that we have the result for all $m < n$, and let $X_1, X_2 \in NS^n_{\kappa,A}$. Then $\mathcal{P}_\kappa(A) \setminus X_1, \mathcal{P}_\kappa(A) \setminus X_2 \in F^n_{\kappa,A}$, by Proposition 3.3.6, there are T_{X_1} m_1-stationary and T_{X_2} m_2-stationary with $m_1, m_2 < n$, such that $d_{m_1}(T_{X_1}) \subseteq \mathcal{P}_\kappa(A) \setminus X_1$ and $d_{m_2}(T_{X_2}) \subseteq \mathcal{P}_\kappa(A) \setminus X_2$. But $d_{m_1}(T_{X_1}) \cap d_{m_2}(T_{X_2}) \subseteq (\mathcal{P}_\kappa(A) \setminus X_1) \cap (\mathcal{P}_\kappa(A) \setminus X_2) = \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. And using 3.3.4 we get that $d_{m^*}(T_{X_1}) \cap d_{m^*}(T_{X_2}) \subseteq \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. Now, applying the hypothesis

we get that there is $m \leq m^* < n$ and T m-stationary such that $d_m(T) \subseteq d_{m^*}(T_{X_1}) \cap d_{m^*}(T_{X_2})$ but this implies that $d_m(T) \subseteq \mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2)$. By 3.3.6, we conclude that $\mathcal{P}_\kappa(A) \setminus (X_1 \cup X_2) \in F^n_{\kappa,A}$ and so $X_1 \cup X_2 \in NS^n_{\kappa,A}$.

Finally, suppose that $\mathcal{P}_\kappa(A)$ is n-stationary , then $\mathcal{P}_\kappa(A) \notin NS^n_{\kappa,A}$ and so $NS^n_{\kappa,A}$ is non-trivial. $\square$

Proposition 3.3.9. *The ideal of non-1-stationary subsets of $\mathcal{P}_\kappa(A)$ is contained in the ideal of non-stationary subsets of $\mathcal{P}_\kappa(A)$. This is, $NS_{\kappa,A} \subseteq NS^1_{\kappa,A}$.*

Proof : Let $X \in NS_{\kappa,A}$, this is, X is not stationary in $\mathcal{P}_\kappa(A)$. By contraposition of Proposition 3.2.9 we have that X is not 1-stationary in $\mathcal{P}_\kappa(A)$, then $X \in NS^1_{\kappa,A}$. $\square$

Following proposition shows us that our definition of n-stationarity (3.2.5) does in fact correspond to the common notion of stationarity with respect to a filter, in this case $F^n_{\kappa,A}$.

Proposition 3.3.10. *Let $\mathcal{P}_\kappa(A)$ be n-stationary. Then $S \subseteq \mathcal{P}_\kappa(A)$ is n-stationary if and only if S is $F^n_{\kappa,A}$-stationary.*

Proof : ($\Rightarrow$) Let S be n-stationary in $\mathcal{P}_\kappa(\lambda)$, and let $X \in F^n_{\kappa,A}$, this is, X is such that there is $T_X \subseteq \mathcal{P}_\kappa(\lambda)$ m-stationary for some $m < n$ such that $d_m(T_X) \subseteq X$. Since S is n-stationary, for T_X there are $B \in S$ and $\mu \subseteq B \cap \kappa$ regular such that $T_X \cap \mathcal{P}_\mu(B)$ is m-sationary in $\mathcal{P}_\mu(B)$, whence $B \in d_m(S)$. Therefore $B \in S \cap d_m(S) \subseteq S \cap X$.

($\Leftarrow$) Suppose that S is $F^n_{\kappa,A}$-stationary, and take $T \subseteq \mathcal{P}_\kappa(A)$ to be m-stationary for some $m < n$. Recall that $d_m(T) \in F^n_{\kappa,A}$. Then, $S \cap d_m(T) \neq \varnothing$. Thus, there is $B \in S$ and $\mu \subseteq B \cap \kappa$ such that $T \cap \mathcal{P}_\mu(B)$ is m-sationary in $\mathcal{P}_\mu(B)$. Therefore S is n-stationary. $\square$

Recall that any filter is closed under finite intersections and arbitrary unions, therefore, when added the empty set, any filter constitutes a topology. So we have now a way of defining topologies in $\mathcal{P}_\kappa(A)$ which is directly linked with the n-stationarity of $\mathcal{P}_\kappa(A)$.

Definition 3.3.11. *For each $n < \omega$ we define in $\mathcal{P}_\kappa(A)$ the following topology $\tau_n := F^n_{\kappa,A} \cup \{\varnothing\}$.*

Proposition 3.3.12. *Let $\mathcal{P}_\kappa(A)$ be n-stationary. If $X \notin d_m(\mathcal{P}_\kappa(A))$ for no $m < n$, then, X is a limit point in the topology τ_n.*

Proof : Suppose that $\mathcal{P}_\kappa(A)$ is n-stationary, and $X \notin d_m(\mathcal{P}_\kappa(A))$ for all $m < n$. We claim that $\mathcal{P}_\kappa(A) \setminus \{X\}$ is n-stationary in $\mathcal{P}_\kappa(A)$. Let T be m-stationary in $\mathcal{P}_\kappa(A)$ for some $m < n$, as $\mathcal{P}_\kappa(A)$ is n-stationary, T m-reflects to some B, this is $B \in d_m(\mathcal{P}_\kappa(A))$. But $X \notin d_m(\mathcal{P}_\kappa(A))$, then $B \neq X$. Then $B \in \mathcal{P}_\kappa(A) \setminus \{X\}$. This is, for every $m < n$, T m-reflects to some point of $\mathcal{P}_\kappa(A) \setminus \{X\}$. Hence $\mathcal{P}_\kappa(A) \setminus \{X\}$ is n-stationary in $\mathcal{P}_\kappa(A)$.

Therefore, $\mathcal{P}_\kappa(A) \setminus \{X\} \notin NS^n_{\kappa,A}$ and so $\{X\} \notin F^n_{\kappa,A}$. Whence $\{X\} \notin \tau_n$, this is, X is not an isolated point of τ_n. That is equivalent to say that X is a limit point of $\mathcal{P}_\kappa(A)$ of τ_n. $\square$

3.4 Π^1_n-indescribability in $\mathcal{P}_\kappa(\lambda)$

In his article "*Derived Topologies on Ordinals and Stationary Reflection*" [12], Bagaria proved that in the constructible universe L, a regular cardinal in $\xi + 1$-simultaneusly-reflecting if and only if it is Π^1_ξ-indescribable. Baumgartner in [23] defined a generalized notion of Π^1_n-indescribability in $\mathcal{P}_\kappa(A)$. In this section we will study how this notion is related w ith the notion of n-stationarity 3.2.5.

For a regular cardinal κ and a set $A \supseteq \kappa$ we define $V_\alpha(\kappa, A)$ by induction on α as follows

- $V_0(\kappa, A) := A$,

- $V_{\alpha+1}(\kappa, A) := \mathcal{P}_\kappa(V_\alpha(\kappa, A)) \cup V_\alpha(\kappa, A)$,

- $V_\alpha(\kappa, A) := \bigcup_{\beta<\alpha} V_\beta(\kappa, A)$ for a limit α.

Definition 3.4.1. *(Baumgartner) Suppose κ is a regular cardinal κ, $A \supseteq \kappa$ and $n < \omega$. A set $S \subseteq \mathcal{P}_\kappa(A)$ is* Π^1_n***-indescribable in*** $\mathcal{P}_\kappa(A)$ *if for all $P \subseteq V_\kappa(\kappa, A)$ and all Π^1_n-sentence φ with $\langle V_\kappa(\kappa, A), \in, P\rangle \models \varphi$, there is $B \in S$ such that $B \cap \kappa = \mu$ and $\langle V_\mu(\mu, A), \in, P \cap V_\mu(\mu, A)\rangle \models \varphi$ where $\mu := |B \cap \kappa|$.*

In its presentation "*On generalized notion of higher stationarity*" [13], Sakai stated the following proposition due to Donna Carr and Yoshihiro Abe.

Proposition 3.4.2. *([13]). $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ is Π^1_1-indescribable implies that κ is λ-supercompact.*

Although we shall not get into the details of the proof, we will sketch how it is obtained from the results in [17; 18; 24]. First, we need to define a very useful a combinatoric concept introduced in [18] and inspired in works from Shelah and Carr [18; 19].

Definition 3.4.3. *We say that $S \subseteq \mathcal{P}_\kappa(\lambda)$ is* ***Shelah*** *if for every $\langle f_X : X \to X : X \in \mathcal{P}_\kappa(\lambda)\rangle$ there is $f : \lambda \to \lambda$ such that for every $Y \in \mathcal{P}_\kappa(\lambda)$ the set $\{X \in S \cap \{Z \in \mathcal{P}_\kappa(\lambda) : Y \subsetneq Z\} : f \restriction_Y = f_X \restriction Y\}$ is unbounded in $\mathcal{P}_\kappa(\lambda)$. We say that κ is* λ***-Shelah*** *if $\mathcal{P}_\kappa(\lambda)$ is Shelah.*

Theorem 3.4.4. *(Carr [18]). If κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact.*

Theorem 3.4.5. *(Carr [17]). If $X \subseteq \mathcal{P}_\kappa(\lambda)$ is Π^1_1-indescribable, then X is Shelah.*

Then, if we have that $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ is Π^1_1-indescribable, then $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ is Shelah. But by definition 3.4.3 this means that κ is $2^{\lambda^{<\kappa}}$-Shelah. Then by 3.4.4 we get that κ is λ-supercompact.

Proposition 3.4.6. *([13]). If κ is λ-supercompact, then $\mathcal{P}_\kappa(\lambda)$ is Π^1_n-indescribable for all $n \in \omega$.*

Proposition 3.4.7. *([13]). If S is Π^1_n-indescribable in $\mathcal{P}_\kappa(\lambda)$, then S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$.*

Chapter 4

Conclusions and open questions

The present chapter is devoted to present the results obtained on this work, as well as the questions that arise and that remains still unsolved to us.

We defined in $\mathcal{P}_\kappa(A)$ two different notions of n-stationarity 3.2.1 and 3.2.5. Being 3.2.5 weaker than 3.2.1. Definition 3.2.1 was first proposed by Sakai et al. in [13], and it corresponds to the idea of n-satationarity in $\langle \kappa, < \rangle$ in the following way :

1. $S \subseteq \mathcal{P}_\kappa(A)$ being n-stationary implies S is m-stationary for all $m < n$. See 2.2.2 and 3.2.3.
2. (Stated in [13]) The existence of 1-stationary subsets on $\mathcal{P}_\kappa(A)$ demands a condition on κ, namely κ weakly Mahlo 3.2.4. In $\langle \kappa, < \rangle$ we required κ to have uncountable cofinality.
3. Being 1-stationary in $\mathcal{P}_\kappa(A)$ is at least a stronger condition than being stationary in $\mathcal{P}_\kappa(A)$. See 2.1.2 and 3.2.9.
4. (Stated in [13]) The set formed by the n-stationary subsets of $\mathcal{P}_\kappa(A)$ constitutes an ideal on $\mathcal{P}_\kappa(A)$. See 2.3.15 and 3.3.8.
5. For each $n < \omega$, there is an operator d_n acting on subsets of $\mathcal{P}_\kappa(A)$, taking out the points in which some given set n-reflects. See 2.3.8 and 3.3.2.
6. There is a characterisation of the dual filter $F^n_{\kappa,A}$ in terms of the operators d_m for $m < n$. See 2.3.13 and 3.3.6.
7. Under certain condition on $\mathcal{P}_\kappa(A)$, we have that $\mathcal{P}_\kappa(A)$ is n-stationary if and only if $NS^n_{\kappa,A}$ is a proper ideal and so if and only if $F^n_{\kappa,A}$ is a proper filter. See 2.3.15 and 3.3.8.
8. n-stationarity in $\mathcal{P}_\kappa(A)$ is in fact stationarity with respect to a filter, namely $F^n_{\kappa,A}$. See 2.3.14 and 3.3.10.
9. There is a natural ascending sequence of topologies $\langle \tau_n : n < \omega \rangle$ in $\mathcal{P}_\kappa(A)$ each of them generated by the previous one and the operator d_n. See 2.3.1 and 3.3.11.
10. (Stated in [13]) If S is Π^1_n-indescribable in $\mathcal{P}_\kappa(\lambda)$, then S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$. See 2.4.3 and 3.4.7.

However there are still some crucial questions concerning to this correspondence we still don't have the answer

Q.1. Is there a least condition to guarantee the existence of 1-stationary subsets on $\mathcal{P}_\kappa(A)$. In $\langle \kappa, < \rangle$, κ having uncountable cofinality was also sufficient.

Q.2. If $C \subseteq \mathcal{P}_\kappa(A)$ is a club subset of $\mathcal{P}_\kappa(A)$, then, is C a 1-stationary subset of $\mathcal{P}_\kappa(A)$? Under which conditions on κ is this the answer affirmative? (See 2.2.4).

Q.3. More in general, if $C \subseteq \mathcal{P}_\kappa(A)$ is a club subset of $\mathcal{P}_\kappa(A)$, is then C an n-stationary subset of $\mathcal{P}_\kappa(A)$ for all $n < \omega$? Under which conditions on κ is the answer affirmative? (See 2.2.4).

Q.4. Is 1-stationarity equivalent to stationarity in $\mathcal{P}_\kappa(A)$, in other word does the converse of 3.2.9 hold? (See 3.2.9).

Q.5. Does the existence of 2-stationary subsets on $\mathcal{P}_\kappa(A)$ demand some stronger condition on κ than in the case of 1-stationarity? (See *Remark 1.*)

Q.6. Given $m < n$ and $S \subseteq \mathcal{P}_\kappa(A)$ n-stationary, is the set $d_m(S)$ closed? (See 2.3.2).

Q.7. How does discreteness of τ_n in $\mathcal{P}_\kappa(A)$ relate with reflection of n-stationary sets in $\mathcal{P}_\kappa(A)$? (See 2.3.9).

Q.8. Does the converse of (11) holds in the constructible universe L? This is, if $V = L$, does $n+1$-stationarity of S implies S is Π^1_n-indescribable? (See 2.3.9).

We defined a weaker version of n-stationarity (Definition 3.2.5). We in fact developed (3) to (11) using 3.2.5 and just observing they remain true with 3.2.1. However, there are some answers to our previous questions we could only solve with 3.2.5, namely

11. (Answer to Q.1.) If κ weakly Mahlo, then $\mathcal{P}_\kappa(A)$ is 1-stationary in $\mathcal{P}_\kappa(A)$. (See 3.2.6).
12. (Answer to Q.2.) $C \subseteq \mathcal{P}_\kappa(A)$ being club implies C is 1-stationary, whenever κ is weakly Mahlo. (See 3.2.8).
13. (Answer to Q.5.) If there is a 2-stationary subset of $\mathcal{P}_\kappa(A)$, then κ is 2-weakly Mahlo. (See 3.2.10).

In correspondence with results in [12] we worked with both definitions. New interesting questions arose. Inspired by the result we obtained in 3.2.6, we wonder what is the least condition we need on κ in order to guarantee the existence of n-stationary subsets. We found two partial answers that work for both 3.2.1 and 3.2.5.

14. If κ is Π^1_n indescribable, then $\mathcal{P}_\kappa(\kappa)$ is $n+1$ stationary. (See 3.2.12).
15. If κ is λ-supercompact and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_\kappa(\lambda)$ is n-stationary for any $n < \omega$. (See 3.2.14).

Clearly, (14) provides a stronger answer than (15), however it refers to the particular case $\kappa = \lambda$. In (15) we have a much general answer, and if it is the case that the converse also holds it would give the exact consistency strength of n-stationarity in $\mathcal{P}_\kappa(A)$.

Concerning the relation between indescribability and hyperstationary reflection in $\mathcal{P}_\kappa(A)$, we stated the following result, which is a consequence of works done by Carr ([17; 18]).

16. $\mathcal{P}_\kappa(2^{\lambda^{<\kappa}})$ is Π^1_1-indescribable implies that κ is λ-supercompact. (See 3.4.2).

Also, with respect to indescribability in $\mathcal{P}_\kappa(A)$ we pointed out a pair of propositions stated by Sakai in [12].

17. If κ is λ-supercompact, then $\mathcal{P}_\kappa(\lambda)$ is Π^1_n-indescribable for all $n \in \omega$. (See 3.4.6).
18. If S is Π^1_n-indescribable in $\mathcal{P}_\kappa(\lambda)$, then S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$. (See 3.4.7).

We expect that further work in these topic will elucidate the answer to questions Q.2. to Q.4 and Q.6. to Q.8. It should also make clear if it is 3.2.1 or 3.2.5, the right notion of n-stationarity that corresponds to a natural sequence of topologies in $\mathcal{P}_\kappa(A)$ as in [12]. That is, in a way such that isolated points correspond exactly to points in which the entire set ξ-reflects.

Furthermore, work in this direction should also approach to following open questions proposed by Sakai et. al. in [12]

- Is $\mathcal{P}_\kappa(\lambda)$ n-stationary for all $n < \omega$ assuming κ is λ-strongly compact?
- Is the following jointly consistent?
  - For all regular κ, all $\lambda \geq \kappa$, all $S \subseteq \mathcal{P}_\kappa(\lambda)$, and all $n < \omega$, S is Π^1_n-indescribable iff S is $n+1$-stationary in $\mathcal{P}_\kappa(\lambda)$.
  - There is a supercompact cardinal
- For $n \geq 3$, is it consistent that there is a cardinal $\kappa \leq 2^\omega$ such that $\mathcal{P}_\kappa(\lambda)$ is n-stationary for all $\lambda \geq \kappa$?.

Bibliography

[1] Paul C. Eklof and Alan H. Mekler. *Almost free modules*, Revised edition, North-Holland Mathematical Library, vol. 65, North-Holland Publishing Co., Amsterdam, 2002.

[2] W. T. Gowers. *An infinite Ramsey theorem and some Banach-space dichotomies.* Annals of Mathematics, 156 (2002), 797–833.

[3] E. Odell, T. Schlumprecht. *Trees and branches in Banach spaces.* Transactions of the American Mathematical Society, 354(10) (2002), 4085–4108.

[4] Saharon Shelaha,b, Lutz Strungmann. *Infinite combinatorics in mathematical biology.* BioSystems. (2021) 204. 104392.

[5] James Cummings, Matthew Foreman, and Menachem Magidor. *Squares, scales and stationary reflection*, J. Math. Log. 1 (2001), no. 1, 35–98.

[6] Thomas Jech. *Set theory, Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[7] Thomas Jech. *Stationary sets, Handbook of set theory.* Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 93–128.

[8] Menachem Magidor. *Reflecting stationary sets*, J. Symbolic Logic 47 (1982), no. 4, 755–771 (1983).

[9] Alan H. Mekler and Sharon Shelah. *The consistency strength of "every stationary set reflects"*, Israel J. Math. 67 (1989), no. 3, 353–366.

[10] Joan Bagaria, Menachem Magidor, and Salvador Mancilla. *The consistency strength of hyperstationarity*, J. Math. Log. 1 (2020), no. 20, 2050004

[11] Joan Bagaria. *Topologies on Ordinals and the Completeness of Polymodal Provability Logics* , Circulated manuscript (2011).

[12] Joan Bagaria. *Derived Topologies On Ordinals and Stationary Reflection.* Trans. Amer. Math. Soc., (2019), 371(3):1981–2002.

[13] Hiroshi Sakai. *On generalized notion of higher stationarity*, Reflections on Set Theoretic Reflection 2018, November 16–19. [Slides] http://www.ub.edu/RSTR2018/slides/Sakai.pdf

[14] Akihiro Kanamori. *The Higher Infinite, Large Cardinals in Set Theory from Their Beginnings*, Springer Berlin, Heidelberg, 2009. Second Edition - Springer Monographs in Mathematics.

[15] Péter Komjáth, Vilmos Totik. *Problems and Theorems in Classical Set Theory*, Springer New York, NY, 2006. - Problem Books in Mathematics

[16] Thomas Jech. *The closed unbounded filter over* $P_\kappa(\lambda)$. Notices Am. Math. Soc. 18 (1971) 663.

[17] D. M. Carr. *The structure of ineffabilty properties of* $\mathcal{P}_\kappa(\lambda)$, Acta Math. Hung. 47 (1986), 325–332.

[18] D. M. Carr. *$\mathcal{P}_\kappa(\lambda)$-generalization of weak compactness*, Z. Math. Logik Grundlag. Math. 31 (1985), 393–401.

[19] Saharon Shelah. *Weakly compact cardinals: a combinatorial proof*, J. Symb. Logic 44 (1979), 559-562.

[20] Joan Bagaria. *Combinatorial set theory* [Class handout]. https://www.icrea.cat/web/ scientificstaff/joan-bagaria-i-pigrau-119

[21] Joan Bagaria. *Basic set theory* [Class handout]. https://www.icrea.cat/web/ scientificstaff/joan-bagaria-i-pigrau-119

[22] R. Björn Jensen. *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308; erratum, ibid. 4 (1972), 443. With a section by Jack Silver.

[23] J. Baumgartner. *Ineffability properties of cardinals I, Infinite and finite sets*, P. Erdos 60th Birthday Colloquium, Keszthely, Hungary, 1973, Colloquia Mathematica Societatis Janos Bolyai, vol.10, North-Holland, Amsterdam (1975), 109–130.

[24] Yoshihiro Abe. *Combinatorial characterization of Π^1_1-indescribability in $\mathcal{P}_\kappa(\lambda)$*, Arch. Math. Log. 37 (4) (1998) 261–272.

[25] Joan Bagaria, Menachem Magidor, and Hiroshi Sakai. *Reflection and indescribability in the constructible universe*, Israel J. Math. 208 (2015), no. 1, 1–11.

[26] Lev Beklemishev and David Gabelaia. *Topological completeness of the provability logic GLP*, Ann. Pure Appl. Logic 164 (2013), no. 12, 1201–1223.

[27] S. N. Artemov and L. D. Beklemishev. *Provability Logic. Handbook of Philosophical Logic*, 2nd ed., vol. 13 (D. Gabbay and F. Guenthner, eds.), Springer, Dordrecht, 2004, pp. 189–360.

[28] L. D. Beklemishev. *Reflection schemes and provability algebras in formal arithmetic* (Russian, with Russian summary), Uspekhi Mat. Nauk 60 (2005), no. 2(362), 3–78; English transl., Russian Math. Surveys 60 (2005), no. 2, 197–268.

[29] L. D. Beklemishev. *On GLP spaces*, (2009). Circulated manuscript.

[30] Lev Beklemishev and David Gabelaia. *Topological interpretations of provability logic*, Leo Esakia on duality in modal and intuitionistic logics, Outst. Contrib. Log., vol. 4, Springer, Dordrecht, 2014, pp. 257–290. M

[31] G. K. Japaridze. *The modal logical means of investigation of provability*, (1986). Thesis in Philosophy, in Russian. Moscow.