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Coverings of Generalized Petersen Graphs

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Abstract

In this project we study the coverings of Generalized Petersen Graphs that are Generalized Petersen Graphs themselves. We give a large family of such Generalized Petersen Graphs and review results of Krnc and Pisanski by focusing on Kronecker double covers. Finally, we generalize their results partially to Kronecker triple covers.

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Introduction

Graph theory is the part of discrete mathematics that studies some mathematical structure called graph. These mathematical objects are defined as a set of vertices or nodes joined by edges. Their usefulness is notorious as they can be used to model lots of real-world problems like distance travelling, network connections, molecular structure, data organization,... Also in other branches of mathematics such as geometry and topology, graph theory has shown its utility as it helped on the development of some results like in knot theory.

Covering graph is also closely related to topology as it can be defined as a discrete case of covering spaces in algebraic topology. One particular case of graph 2-covering is the well known *Kronecker double cover* (also called **bipartite double cover**) that can be defined as the tensor product $G \times K_2$.

Generalized Petersen Graphs are a famous family of graphs introduced by Coxeter et al. [Cox50] and named and formalized by [Wat69]. This graphs were created in order to generalise the usual drawing of the Petersen Graph, that is a bigger pentagon surrounding a pentagram and joining the corresponding vertex of both figures through edges.

In this bachelor thesis we study the well known family of the Generalized Petersen Graphs. We focus on the coverings of Generalized Petersen Graphs that are also Generalized Petersen Graph. It consists of 5 chapters:

The first chapter is an introduction to graph theory and algebraic graph theory, with emphasis on coverings and Generalized Petersen Graphs in order to provide a sufficient knowledge to understand the following chapters.

In Chapter 2 we provide a large family of Generalized Petersen Graph that are q-coverings of other Generalized Petersen Graph.

Chapter 3 is a synthesis of the results given by Krnc and Pisanski [KP19]. We study their Theorem focusing in the part where Generalized Petersen Graphs are Kronecker Covers of other Generalized Petersen Graphs.

We generalize the results seen at Chapter 3 partially to Kronecker triple cover in Chapter 4.

Finally, we end this thesis with conclusions in Chapter 5.

Hence, Chapter 2 and 4 are original work while Chapters 1 and 3 are known results. However, proofs on this last two chapters have been reproved in order to provide us a better understanding on the topic and to be consistent with our own notation.

Chapter 1

Graph Theory Preliminaries

This chapter is an introduction to graph theory. Its aim is to build sufficient knowledge in order to understand the following chapters. Throughout these sections, one can find proofs of some well known results in graph theory that have been reproved in these pages so as to familiarize with all the concepts. Most of the following definitions and results can be found in [Wes96] and [dD19].

1.1 Graph Theory basis

Definition 1.1. *A* ***undirected graph*** *or* ***graph*** *is an ordered pair of sets* $G = (V, E)$ *where* $E = E(G)$ *is a set of unordered pairs of elements of* $V = V(G)$*. The elements of* V *are called* ***vertices*** *and the elements of* E ***edges****.*

An **empty graph** is a graph with no vertices, and hence no edges. From now on we will consider every graph to be not empty.

Definition 1.2. *The pair of vertices forming an edge are called the* ***endpoints*** *of the edge. In this case, the vertices are* ***neighbors*** *or* ***adjacent****. It is also said that the edge* ***joins*** *or* ***connects*** *the vertices or that is* ***incident*** *on the vertices.*

Notation 1.3. *If an edge* $e \in E$ *is incident on* $u, v \in V$*, we write* $e = uv = vu$*.*

Notation 1.4. $N(v) = \{u \in V \mid uv \in E\}$ *is the set of neighbours of the vertex* v*.* $N[v] = N(v) \cup \{v\}$*.*

Definition 1.5. *If the same pair of vertices are joined by two or more edges, then the edges are called* ***multiple*** *or* ***parallel edges****.*

Definition 1.6. *An edge that connects a vertex with itself is called* ***loop****.*

Definition 1.7. *A* ***simple*** *graph is a graph with no loops and no parallel edges.*

Definition 1.8. *The **degree** of a vertex $v \in V$ is the number of edges in E that are incident to v, counting loops twice. It is denoted by $d(v)$.*

Definition 1.9. *A **cubic graph** is a graph such that $\forall v \in V$, $d(v) = 3$.*

Definition 1.10. *A **subgraph** $S = (U, D)$ of a graph $G = (V, E)$ is a graph such that $U \subset V$ and $D \subset E$.*

Definition 1.11. *A **walk of length k** is a sequence of k edges of a graph $e_1, \ldots, e_k$, such that for $1 \leq i \leq k$ the edge e_i has endpoints v_{i-1} and v_i. It can also be described as a sequence of $k+1$ vertices $v_0 \ldots v_k$ such that every two consecutive vertices are neighbours. A **trail** is a walk with no repeated edges and a **path** is a trail with no repeated vertices. For a walk or a trail if $v_0 = v_k$ we say it is **closed**.*

Remark 1.12. *A path cannot be closed as every vertex must be different.*

Definition 1.13. *If there exist a path between every pair of points in a graph G, the graph is **connected**. The **connected components** of a graph G are the inclusion maximal connected subgraphs of G.*

Definition 1.14. *Let $G = (V, E)$ be a connected graph and $u, v \in V$. A shortest path joining u and v is called **geodesic**. The **distance between u and v**, **d(u,v)**, is the length of a **geodesic**.*

Definition 1.15. *A **cycle** is a closed trail in which every vertex is different except for the first and the last one. If its length is and odd number we say it is an **odd cycle**. Otherwise we call it an **even cycle**.*

Remark 1.16. *Note that a cycle of length n has n vertices and n edges.*

Definition 1.17. *A graph is a **n-cycle** or **n-gon** if it consist of a single cycle of length n. This type of graphs are denoted as $\mathbf{C_n}$.*

Definition 1.18. *A **vertex labeling** of a graph $G = (V, E)$ is a function*

$$\xi : V \longrightarrow L$$

*such that L is a set of labels (usually integers). A **vertex k-coloring** (or a **k-coloring**), is a vertex labeling such that $|L| = k$. The labels are also called **colors**. A k-coloring is **proper** if adjacent vertices have different colors. A graph is **k-colorable** if there exist a proper k-coloring.*

Definition 1.19. *An **independent set** of a graph $G = (V, E)$ is a subset $U \subset V$ such that for every pair of vertices $u, u' \in U$, then $uu' \notin E$.*

Definition 1.20. *A **bipartite graph** $G = (V, E)$ is a graph with vertex set divided into two nonempty disjoint subsets*

$$V = A \cup B \quad \textit{such that} \quad A \cap B = \emptyset$$

*satisfying both subsets are independent sets. Hence, the edge set E has no edge with both endpoints in the same subset. In the same way, we define a **k-partite graph** as a graph such that its vertex set is the union of k nonempty disjoint independent sets. A **tripartite graph** is a 3-partite graph.*

Remark 1.21. *A k-partite graph is k-colorable as it has k disjoint independent subsets.*

The following theorem is a characterization for bipartite graphs:

Theorem 1.22. ***[König (1936):]*** *A graph is bipartite if and only if it has no odd cycles.*

Proof. Necessity: We assume $G = (A \cup B, E)$ to be a bipartite graph. Every step along a walk in G alternates between the sets A and B, so to return to the starting point, an even number of steps are needed.

Sufficiency: Let $G = (V, E)$ be a graph with no odd cycles. Let $v \in V$ be an arbitrary vertex in an arbitrary connected component $H = (W, F)$ of G. Let ξ be a 2-coloring such that $\forall u \in W$:

$$\begin{array}{rccc} \xi : & W & \longrightarrow & \{0,1\} \\ & u & \longmapsto & d(v,u) \mod 2 \end{array}$$

Let $A = \{u \in W \mid \xi(u) = 0\}$ and $B = \{u \in W \mid \xi(u) = 1\}$. One can easily see that A is the set of vertices with an even geodesic from v, and B is the set of vertices with an odd geodesic from v. Also both sets are disjoint. If two vertices $a, a' \in A$ were neighbours, then taking the geodesic from v to a, the edge aa' then the geodesic from a' to v we would have an odd closed walk in G (as both geodesics are even). As every closed walk of odd length contains an odd cycle, we have a contradiction. Now, if two vertices $b, b' \in B$ were neighbours, using the same strategy we would also have an odd cycle as both of the geodesics are odd paths. Therefore, we have two disjoint independent sets. Now as v and H are arbitrary, we have G must be bipartite. □

Definition 1.23. *A graph $G = (V, E)$ is a **complete graph** if it is a simple graph such that $\forall v \in V, N(v) = \{u \in V \mid u \neq v\}$. The **complete graph with n** vertices is denoted by $\mathbf{K_n}$.*

Definition 1.24. *A **digraph** or **directed graph** is an ordered pair of sets D = (V,A) where V is the set of vertices and A is an ordered pair of vertices. The elements of A are called **arcs**.*

Notation 1.25. *In order to make a difference on the notation of arcs and edges (see notation 1.3 for edges) we will write an arc as the tuple (u, v) meaning there is an arc that goes from u to v. The first vertex of the tuple is called **tail** and the second one **head**.*

Remark 1.26. *Let $D = (V, A)$. As an arc is an ordered tuple, for $u, v \in V$ such that if $u \neq v$ we have $a = (u, v) \neq (v, u)$.*

Definition 1.27. *Given a graph $G = (V, E)$, an **orientation of G** is a digraph with the same set of vertices $D = (V, A)$ such that $\forall e = uv \in E$ then $(u, v) \in A$ or $(v, u) \in A$.*

Definition 1.28. *Given a digraph $D = (V, A)$ we define the **underlying graph of D** as the graph with the same vertex set $G = (V, E)$ such that given $u, v \in V$, then $uv \in E$ only if $(u, v) \in A$ or $(v, u) \in A$.*

1.2 Graph Homomorphism

A graph homorphism is a function between two graphs that preserves edges. More precisely:

Definition 1.29. *A **graph homomorphism** φ between two graphs $G = (V, E)$ and $H = (W, F)$ is a function*

$$\varphi : V \longrightarrow W$$

such that if $uv \in E \Rightarrow \varphi(u)\varphi(v) \in F$.

In a similar way, we can define a graph isomorphism between two graphs as a bijective function that preserves vertex adjacency and non-adjacency:

Definition 1.30. *A **graph isomorphism** φ between two graphs $G = (V, E)$ and $H = (W, F)$ is a bijective function*

$$\varphi : V \longrightarrow W$$

such that $uv \in E \iff \varphi(u)\varphi(v) \in F$.

*If there exists an isomorphism between G and H we say G **is isomorphic to** H and we denote it by $G \cong H$.*

In particular, as opposed to group isomorphism, bijective homomorphisms are not isomorphisms in general. For example, as shown in Figure 1.1 let $G = (V, E)$ be the graph with vertex set $V = \{u, v\}$ and edge set $E = \emptyset$ and $H = (W, F)$ the graph such that $W = \{a, b\}$ and $F = \{ab\}$. It is clear that

$$\varphi : \quad V \quad \longrightarrow \quad W$$

such that $\varphi(u) = a$ and $\varphi(v) = b$ is a bijective homomorphism but graphs are clearly not isomorphic.

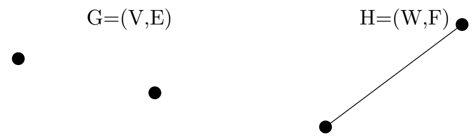

Figure 1.1: There is a bijective homomorphism between G and H but they are not isomorphic graphs

Proposition 1.31. *φ is an isomorphism if and only if it is a bijective homomorphism and its inverse φ^{-1} is also a homomorphism.*

Proof. Necessity: It follows from the definition of graph isomorphism.

Sufficiency: Let $G = (V, E)$, $H = (W, F)$ be two graphs. $\varphi : V \longrightarrow W$ be a bijective homomorphism such that its inverse $\varphi^{-1} : W \longrightarrow V$ is also a homomorphism. Now as φ is a homomorphism, by definition we have that $uv \in E \Rightarrow \varphi(u)\varphi(v) \in F$. Moreover, as it is bijective an φ^{-1} is also an isomorphism we have $\varphi(u)\varphi(v) \in F \Rightarrow \varphi^{-1}(\varphi(u))\varphi^{-1}(\varphi(v)) = uv \in E$.

□

1.2.1 Graph Automorphism

Definition 1.32. *A **graph automorphism** is a bijective homomorphism between a graph $G = (V, E)$ and itself:*

$$\varphi : V \longrightarrow V$$

Remark 1.33. *A graph automorphism can also be thought as a permutation of the vertices of a graph that preserves the adjacency and non-adjacency.*

Remark 1.34. *Given a graph $G = (V, E)$, the set of all the automorphisms of G with the composition of functions has group structure. It is called the **automorphism group of the graph G** and is denoted by $Aut(G)$.*

Notation 1.35. *Given 2 automorphism ω, τ we denote the composition $\omega \circ \tau = \tau\omega$.*

Remark 1.36. *Given a graph $G = (V, E)$ such that $|V| = n$, $Aut(G)$ is a subgroup of the permutation group S_n.*

Remark 1.37. *The automorphism group of the graph C_3 is $Aut(C_3) \cong D_3$.*

Definition 1.38. *A graph $G = (V, E)$ is **vertex-transitive** if $\forall\ u, v \in V$, there exists an automorphism φ such that*

$$\varphi(u) = v$$

*We say it is **symmetric** if additionally for every $u, v, a, b \in V$, such that $uv, ab \in E$ there exist an automorphism*

$$\varphi : V \longrightarrow V$$

satisfying $\varphi(u) = a$ and $\varphi(v) = b$.

Definition 1.39. *Let $G = (A \cup B, E)$ be a bipartite graph. We say an automorphism φ is **color-preserving** if $\forall a \in A$ and $\forall b \in B$, we have $\varphi(a) \in A$ and $\varphi(b) \in B$. Otherwise, if $\forall a \in A$ and $\forall b \in B$, we have $\varphi(a) \in B$ and $\varphi(b) \in A$ we say that φ is **color-reversing**.*

Remark 1.40. *For a bipartite graph an automorphism is always color-preserving or color-reversing.*

1.2.2 Covering map of a Graph

Definition 1.41. *Let $G = (V, E)$, $H = (W, F)$ be two graphs and $\varphi : V \longrightarrow W$ a function between their vertex sets. φ is a **covering map** if it is a surjective homomorphism such that $\forall v \in V$ the restriction of φ to $N[v]$ is a bijection onto $N[\varphi(v)]$. If there exists a covering map between G and H, we say that G is a **covering graph** or a **lift** of H. Also if $|V| = k|W|$ we say it is a **k-covering**. We also say that H is the **base graph** or the **quotient** of G.*

Remark 1.42. *From the definition one can see there might be a relationship between algebraic topology and graph theory. In fact, graph covering is a discrete case of covering space in topology.*

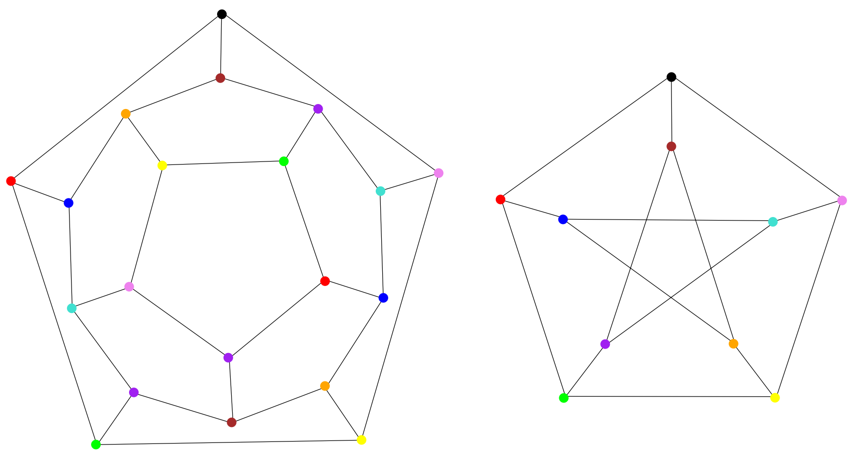

Figure 1.2: Dodecahedron graph as a 2-covering of the Petersen Graph

As can be seen in Figure 1.5 the Dodecahedron graph is a 2-covering of the Petersen Graph. Vertex are represented with different colors in order to show that the restriction of the homomorphism is locally bijective.

For finite graphs, one can use Integer Linear Programming in order to check the existence of a k-covering between two graphs. The following code has been used throughout the research in order to find some coverings:

```
def is_kcover_of(G, H, k, core=False, solver=None, verbose=0):
    #checks if there is homomorphism from G to H such that all fibers are of size exactly k such that is a local isomorphism.
        G._scream_if_not_simple()
        from sage.numerical.mip import MixedIntegerLinearProgram, MIPSolverException
        p = MixedIntegerLinearProgram(solver=solver, maximization=False)
        b = p.new_variable(binary=True) #b[ug,uh]=1 <=> phi maps G to H

        # Each vertex has an image
        for ug in G:
            p.add_constraint(p.sum(b[ug,uh] for uh in H) == 1)
        for uh in H:
            p.add_constraint(p.sum(b[ug,uh] for ug in G) == k)#k-cover

        nonedges = H.complement().edges(labels=False)

        for ug,vg in G.edges(labels=False):
            # Two adjacent vertices cannot be mapped to the same element
            for uh in H:
                p.add_constraint(b[ug,uh] + b[vg,uh] <= 1)

            # Two adjacent vertices cannot be mapped to no adjacent vertices
            for uh,vh in nonedges:
                p.add_constraint(b[ug,uh] + b[vg,vh] <= 1)
                p.add_constraint(b[ug,vh] + b[vg,uh] <= 1)

        for ug in G.vertices():
            for vg in G.neighbors(ug):
                for wg in G.neighbors(ug):
                    if wg != vg:
                        #the end of a path of lenght 2 has to go two vertices
                        for uh in H:
                            p.add_constraint(b[vg,uh] + b[wg,uh] <= 1)

        try:
            p.solve(log = verbose)
            b = p.get_values(b)
            mapping = dict(x[0] for x in b.items() if x[1])
            return mapping

        except MIPSolverException:
            return False
```

Figure 1.3: Integer Linear Program code used in order to find q-coverings between graphs

1.3 Kronecker cover

Definition 1.43. *Let G and H be two graphs. The **tensor product** of G and H (denoted by $G \times H$) is a graph satisfying the following two conditions:*

- *The vertex set $V(G \times H)$ is the Cartesian product $V(G) \times V(H)$.*
- *$(g_1, h_1)(g_2, h_2) \in E(G \times H)$, if and only if $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$.*

*It is also called **Kronecker product** or direct product.*

Definition 1.44. *The **Kronecker cover KC(G)** (or **bipartite double cover**) of a simple undirected graph G is a bipartite covering graph with twice as many vertices as G. It can* *be defined as the Kronecker product $G \times K_2$. Therefore, is the graph with set of vertices*

$$V(KC(G)) = \{v', v''\}_{v \in V(G)}$$

and with edges set

$$E(KC(G)) = \{u'v'', u''v'\}_{uv \in E(G)}$$

Remark 1.45. *The 2 sets of bipartition are naturally constructed. One can easily see that $V(KC(G)) = V' \cup V''$.*

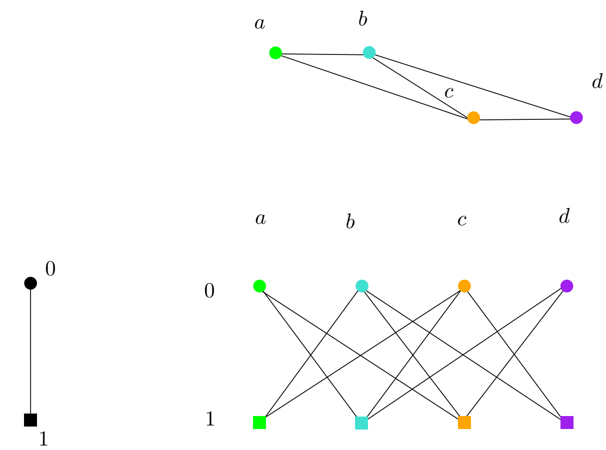

Figure 1.4: A graph and its Kronecker cover

Notation 1.46. *As can be seen in Figure 1.4 vertex set of $G \times K_2$ can be written as $V(G) \times \{0,1\}$, we may also use the following notation to represent the vertices on a Kronecker cover of a graph G:*

Given a vertex $v \in V(G)$:

$$v' = (v, 0) \quad \textit{and} \quad v'' = (v, 1)$$

1.3.1 Kronecker Invoultion

Definition 1.47. *A function $f : A \longrightarrow A$ is an **involution** or **self-inverse function** if it is such that*

$$(f \circ f)(x) = f(f(x)) = x$$

or equivalently,

$$f(x) = f^{-1}(x)$$

Definition 1.48. *Let $G = (V, E)$ be a bipartite graph and $Aut(G)$ its automorphism group. We say $\omega \in Aut(G)$ is a **Kronecker involution** if it is a color-reversing involution such that $\forall v \in V, v\omega(v) \notin E$*

The following result from [KP19] relates the Kronecker cover with the Kronecker involution:

Theorem 1.49. *Let $G = (A \cup B, E)$ be a bipartite graph. Then, there exist a graph $H = (W, F)$ such that $KC(H) \cong G$ if and only if $Aut(G)$ admits a Kronecker involution. Furthermore, the corresponding quotient graph may be obtained by contracting all pairs of vertices, naturally coupled by a given involution.*

Proof. Necessity: Let us assume $G = (V, E)$ be a bipartite graph and $H = (W, F)$ a simple graph such that $KC(H) \cong G$. Hence, $V = W \times \{0, 1\}$ and by definition we have $|V| = 2|W|$. Let $\tau \in Aut(G)$ and given $(w, i) \in V$, such that $w \in W, i \in \{0, 1\}$, we define $\tau(w, i) = (w, i + 1 \mod 2)$. We can clearly see that it is an involution as $(\tau \circ \tau)(w, i) = \tau(\tau(w, i)) = \tau(w, i + 1 \mod 2) = (w, i + 2 \mod 2) = (w, i)$. Moreover, it is also easy to see that is a color-reversing automorphism as $i \not\equiv i + 1 \mod 2$. If we prove that $\forall (w, i) \in V$, $(w, i)\tau(w, i) \notin E$ we are done. Let us do it by contraposition: Suppose there exists $(w, i) \in V$, such that $(w, i)\tau(w, i) \in E$. Then, $(w, i)(w, i + 1 \mod 2) \in E$ and hence, $ww \in W$, which contradicts the fact of H being a simple graph.

Sufficiency: Now we assume $G = (A \cup B, E)$ to be a bipartite graph and $\omega \in Aut(G)$ be a Kronecker involution. Let us define the following graph $H = (A, F)$ such that $a_1, a_2 \in F$ if and only if $a_1, \omega(a_2) \in E$. Note that if $a \in A$, $\omega(a) \in B$ by definition of Kronecker involution. If we see that $H \times K_2$ is isomorphic to G we are done. Let us define the following homomorphism:

$$\begin{array}{rccc} \varphi : & A \times \{0, 1\} & \longrightarrow & A \cup B \\ & (a, i) & \longmapsto & \omega^i(a) \end{array}$$

Where $\omega^0(a) = Id(a) = a$. Note that φ is a function between the vertex set of KC(H) and the vertex set of G. It is clearly a homomorphism: by definition of the Kronecker cover $(a_1, i)(a_2, j)$ is an edge of $A \times \{0, 1\}$ if and only if $i \neq j$ and $a_1a_2 \in F$. Now, by definition of H, $a_1a_2 \in F$ if and only if $a_1\omega(a_2) \in E$. Finally, if $(a_1, i)(a_2, j)$ is an edge of $A \times \{0, 1\}$ then $i \neq j$ and we have, $\varphi(a_1, i)\varphi(a_2, j) = \omega^i(a_1)\omega^j(a_2) \in E$. Now, let us check that it is an injective homomorphism: Let us assume $\varphi(a_1, i) = \varphi(a_2, j)$. Therefore, $\omega^i(a_1) = \omega^j(a_2)$, and hence $i = j$, and $(a_1, i) = (a_2, j)$. Now, to prove it is a surjective homomorphism, as ω is a Kronecker involution, we trivially have $|A| = |B|$, and consequently $|A \times \{0, 1\}| = |A \cup B|$. If we prove that φ^{-1} is also a homomorphism, by Proposition 1.31 we will have finished. As ω is a Kronecker involution and by definition of φ we have:

$$\begin{array}{rccc} \varphi^{-1} : & A \cup B & \longrightarrow & A \times \{0, 1\} \\ & v & \longmapsto & \begin{cases} (v, 0) & if\ v \in A \\ (\omega(v), 1) & if\ v \in B \end{cases} \end{array}$$

As G is bipartite, if $e \in E$, $e = ab$, such that $a \in A$ and $b \in B$. Hence, by definition of the Kronecker involution and the construction of H, we have $a\omega(b) \in F$ and therefore, by definition of the cross product $\varphi^{-1}(a)\varphi^{-1}(b) = (a,0)(\omega(b),1) \in F \times \{0,1\}$. So, φ^{-1} is a homomorphism, and hence φ is an isomorphism and we have $KC(H) \cong G$. □

1.4 Generalized Petersen Graphs

The *Petersen Graph* is a famous graph, not only because it appears on the cover of lots of books, but also due to its usefulness in several graph problems as example or counterexample.

As can be seen in Figure 1.5, this graph is usually drawn as a *pentagon* surrounding a *smaller pentagram* and joining the vertices of the pentagon with the corresponding vertex of the pentagram through *spokes*.

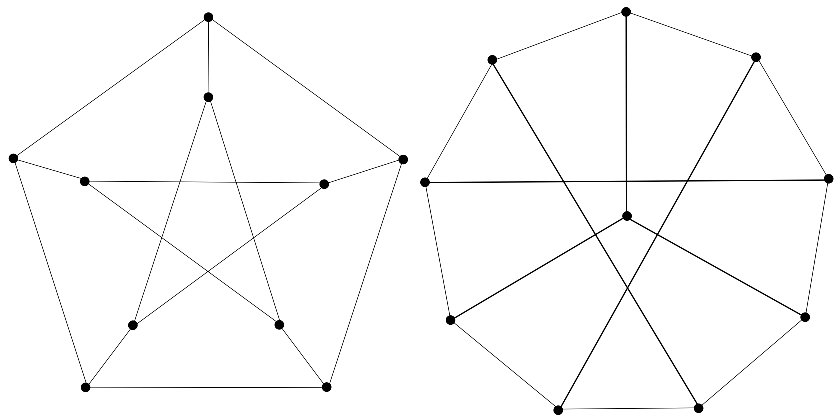

Figure 1.5: 2 different drawings of the Petersen Graph G(5,2)

In order to generalize this concept, H. S. M. Coxeter, in [Cox50], constructed a family of graphs that years later would be named as *Generalized Petersen Graph* and formalized by M. E. Watkins [Wat69]. The idea was the following: draw a n-cycle (n-gon) surrounding a star n-gon, and joining their corresponding vertices through spokes. More precisely, and using M. E. Watkins notation:

Definition 1.50. *Given $n, k \in \mathbb{Z}$ such that $n \geq 3$ and $1 \leq k < \frac{n}{2}$. The* ***Generalized Petersen Graph G(n, k)*** *is the graph with vertex set:*

$$V(G(n,k)) = \{u_0, \ldots, u_{n-1}, v_0 \ldots, v_{n-1}\}$$

and edge set:

$$E(G(n,k)) = \{u_i u_{i+1}\}_{i=0}^{n-1} \cup \{v_i v_{i+k}\}_{i=0}^{n-1} \cup \{u_i v_i\}_{i=0}^{n-1}$$

Remark 1.51. *In the edge set, the subscripts of the vertices are written in modulus n.*

Remark 1.52. *The three subsets in which is divided $E(G(n,k))$ would define the n-gon (also called outer rim), the star n-gon (the inner rims) and the spokes respectively.*

Remark 1.53. *Due to its construction, one can easily see that all Generalized Petersen Graphs are cubic graphs.*

Remark 1.54. *Let $G(n,k)$ be a Generalized Petersen Graph. The inner rim(s) consist of $gcd(n,k)$ $\frac{n}{gcd(n,k)}$-cycles.*

The condition $1 \leq k < \frac{n}{2}$ is due to the following lemma one can find on [Wat69].

Lemma 1.55. *$G(n,k)$ and $G(n,n-k)$ are isomorphic.*

Another interesting result on isomorphic Generalized Petersen Graph from [SS09] that follows from [Wat69]:

Theorem 1.56. *$G(n,k)$ and $G(n,l)$ are isomorphic if and only if $k = l$, $k = n - l$ or $kl \equiv \pm 1 (mod\ n)$.*

The following result characterize the bipartite Generalized Petersen Graphs.

Theorem 1.57. *A Generalized Petersen Graph $G(n,k)$ is bipartite if and only if n is even and k is odd.*

Proof. Necessity: Using König's Theorem (1.22) we know G is a bipartite graph if and only if it has no odd cycles. We will prove it by contraposition: Let $G(n,k)$ be a Generalized Petersen Graph and n be an odd number. Looking at the cycle that forms the outer rim we have $u_0 u_1 \dots u_{n-1} u_0$ is an odd cycle and hence it won't be bipartite. Now let k be an even number. The cycle $v_0 v_k u_k u_k \dots u_0 v_0$ is an odd cycle. Looking at the edges we have the k-trail $u_k u_k \dots u_0$ that has an even number of edges, $u_0 v_0$ and $u_k v_k$ are the 2 spokes, and $v_0 v_k$ is the only edge of the inner rims of the cycle. So we have a (k+3)-cycle, and as k is even we have an odd cycle. Therefore, it won't be bipartite.

Sufficiency: Now we assume n is even and k is odd. If we show that $G(n,k)$ is 2-colorable, then we would have that $G(n,k)$ is bipartite. Let $V = V(G(n,k)) = \{u_0, \dots, u_{n-1}, v_0 \dots, v_{n-1}\}$. Let us define the following labeling:

$$\begin{array}{rccc} \xi : & V & \longrightarrow & \{0,1\} \\ & u_i & \longmapsto & i \mod 2 \\ & v_i & \longmapsto & i+1 \mod 2 \end{array}$$

for every $i \in \mathbb{Z}/n\mathbb{Z}$. We have that for every vertex u_i, its neighbours are u_{i+1}, u_{i-1}, v_i and for every vertex v_i, its adjacent vertex are v_{i+k}, v_{i-k}, u_i (all subscript are in modulo n). It is easy to check that

$$\xi(u_i) = i \mod 2 \neq \xi(u_{i+1}) = \xi(u_{i-1}) = \xi(v_i) = i + 1 \mod 2$$

and as k is odd:

$$\xi(v_i) = i + 1 \mod 2 \neq \xi(v_{i+k}) = \xi(u_{i-k}) = \xi(u_i) = i \mod 2$$

So, $G(n,k)$ is 2-colorable and as consequence, bipartite.

□

The following result one can found in [KP19] that follows from the work of [FGW71], [LS97] and [PZ09]:

Theorem 1.58. *Let $G(n,k)$ be a Generalised Petersen Graph. Then*

a) it is symmetric if and only if

$$(n,k) \in \{(4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)\}$$

b) it is vertex-transitive if and only if $k^2 \equiv \pm 1 \mod n$, or $n = 10$ and $k = 2$

c) it is a Cayley graph if and only if $k^2 \equiv 1 \mod n$

The following is a list of a few members of the family of the Generalized Petersen Graphs (a part from the Petersen graph) that are well known in Graph Theory (see also Figure 4.2):

- for $n \geq 3$ G(n,1) is the n-prism, in particular the cube G(4,1)
- the Dürer graph, G(6,2)
- the Möbius-Kantor graph G(8,3)
- the Dodecahedron graph G(10,2)
- the Desargues graph G(10,3)
- the Nauru graph G(12,5)

Also mention that it is such a famous family of graphs that there are also generalitzations like the Supergeneralized Petersen Graphs [SPP07] or the I-graphs [PZ09].

Figure 1.6: "*Melancolía-I*". painting by Alberto Dürer. $G(6,2)$ can be found as the skeleton of the left convex polyhedron.

1.4.1 Automorphism Group of Generalized Petersen Graph

Let us start this section defining the following permutations one can find in [FGW71]:

Definition 1.59. *Let $G(n,k)$ a generalized Petersen graph and $i \in \mathbb{Z}/n\mathbb{Z}$. We define the permutations on vertex set $V(G(n,k))$ α, β, γ as:*

$$\begin{array}{ll} \alpha(u_i) = u_{i+1} & \alpha(v_i) = v_{i+1} \\ \beta(u_i) = u_{-i} & \beta(v_i) = v_{-i} \\ \gamma(u_i) = v_{ki} & \gamma(v_i) = u_{ki} \end{array}$$

The following result can also be found in [FGW71]:

Theorem 1.60. *For every not symmetric Generalized Petersen Graph $G(n,k)$ we have:*

- *if $k^2 \equiv 1 \mod n$, then,*

$$Aut(n,k) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^2 = \gamma^2 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

- *if $k^2 \equiv -1 \mod n$, then,*

$$Aut(n,k) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^2 = \gamma^4 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma^k, \beta\gamma = \gamma\beta \rangle$$

In this case $\beta = \gamma^2$

- *Otherwise $G(n,k)$ is not vertex-transitive and:*

$$Aut(n,k) = \langle \alpha, \beta | \alpha^n = \beta^2 = 1, \alpha\beta = \beta\alpha^{-1} \rangle$$

Remark 1.61. *α, β, γ are automorphisms that preserve the spokes. Therefore, the automorphism group can not be generated only by α, β, γ.*

Remark 1.62. *α can be thought as a rotation of the vertices of the graph and β may be thought as an axial symmetry. As for any G(n,k) $\alpha, \beta \in Aut(n,k)$ it is obvious that the dihedral group D_n of order $2n$ is a subgroup of $Aut(n,k)$.*

Remark 1.63. *Given a bipartite Generalized Petersen Graph $G(n,k)$, α, γ are color-reversing automorphism while β is color-preserving.*

Let us rewrite the following lemma from [KP19] in order to preserve the notation of composition of automorphisms (see Notation 1.35):

Lemma 1.64. *Let $G(n,k)$ be a not symmetric Generalized Petersen Graph. Then, any automorphism $\omega \in Aut(n,k)$ may be associated a unique triple $(c,b,a) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ such that $\omega = \gamma^c \beta^b \alpha^a$*

Proof. 1. $\alpha^a \beta = \beta \alpha^{-a}$:

$$\alpha^a(\beta(u_i)) = \alpha^a(u_{-i}) = u_{-i+a} = \beta(u_{-(-i+a)}) = \beta(u_{i-a}) = \beta(\alpha^{-a}(u_i))$$

$$\alpha^a(\beta(v_i)) = \alpha^a(v_{-i}) = v_{-i+a} = \beta(v_{-(-i+a)}) = \beta(v_{i-a}) = \beta(\alpha^{-a}(v_i))$$

2. $\alpha^a \gamma = \gamma \alpha^{ak}$ if $k^2 \equiv 1 \mod n$

$$\alpha^a(\gamma(u_i)) = \alpha^a(v_{ki}) = v_{ki+a} = \gamma(u_{i+ka}) = \gamma(\alpha^{ak}(u_i))$$

Where we have used in $\gamma(u_{i+ka}) = v_{ki+k^2a} = v_{ki+a}$ if $k^2 \equiv 1 \mod n$. Analogously we have:

$$\alpha^a(\gamma(v_i)) = \alpha^a(u_{ki}) = u_{ki+a} = \gamma(v_{i+ka}) = \gamma(\alpha^{ak}(v_i))$$

3. $\alpha^a \gamma = \gamma \alpha^{-ak}$ if $k^2 \equiv -1 \mod n$

$$\alpha^a(\gamma(u_i)) = \alpha^a(v_{ki}) = v_{ki+a} = \gamma(u_{i-ka}) = \gamma(\alpha^{-ak}(u_i))$$

Where we have used in $\gamma(u_{i+ka}) = v_{ki+k^2a} = v_{ki-a}$ if $k^2 \equiv -1 \mod n$. Analogously we have:

$$\alpha^a(\gamma(v_i)) = \alpha^a(u_{ki}) = u_{ki+a} = \gamma(v_{i-ka}) = \gamma(\alpha^{-ak}(v_i))$$

4. $\beta\gamma = \gamma\beta$:
$$\beta(\gamma(u_i)) = \beta(v_{ki}) = v_{-ki} = \gamma(u_{-i}) = \gamma\beta(u_i)$$
$$\beta(\gamma(v_i)) = \beta(u_{ki}) = u_{-ki} = \gamma(v_{-i}) = \gamma\beta(v_i)$$

Now looking at the automorphisms groups of the Generalized Petersen Graphs we see that if $k \equiv -1 \mod n$, then $\beta = \gamma^2$, hence $\gamma^3 = \gamma\beta$. And now, by using any of these rules conveniently we get the result. $\square$

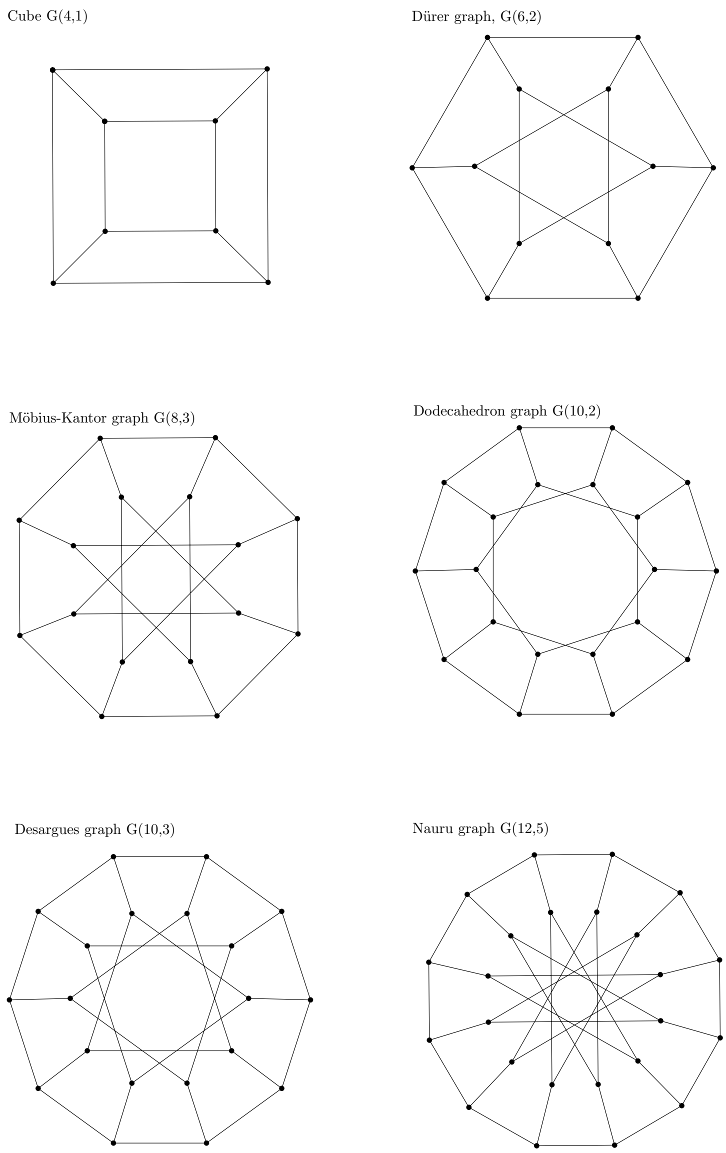

Figure 1.7: Some famous Generalized Petersen Graphs

Chapter 2

Other q-coverings of Generalized Petersen Graph

As we have seen in Chapter 1 the Dodecahedron $G(10,2)$ is indeed a 2-covering of the Petersen graph.

The following result presents a large family of q-coverings:

Theorem 2.1. *Let k, n, q be integers such that $n \geq 3$, $1 \leq k < \frac{n}{2}$ and $q \geq 1$. Let $j \in \mathbb{Z}$. Then:*

- *$G(qn, jn+k)$ is a q-cover of $G(n,k)$ for $0 \leq j \leq \frac{q-1}{2}$*
- *$G(qn, jn-k)$ is a q-cover of $G(n,k)$ for $1 \leq j \leq \frac{q}{2}$*

Proof. First note that for $0 \leq j \leq \frac{q-1}{2}$, as $1 \leq k < \frac{n}{2}$ we have $k \leq jn+k \leq \frac{(q-1)n}{2} + k < \frac{(q-1)n}{2} + \frac{n}{2} = \frac{qn}{2}$. Now, let us prove the first statement:

Let $V = \{u_0, \ldots, u_{qn-1}, v_0, \ldots, v_{qn-1}\}$ be the vertex set of $G(qn, jn+k)$ and $W = \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ the vertex set of $G(n,k)$. Let us define the following function:

$$\begin{array}{rccc} f: & V & \longrightarrow & W \\ & u_i & \longmapsto & a_{i \bmod n} \\ & v_i & \longmapsto & b_{i \bmod n} \end{array}$$

It is clearly a surjection by definition. We have to see it is a local isomorphism:
We have that for any $u_i, v_i \in V$, $N(u_i) = \{u_{i+1}, u_{i-1}, v_i\}$ and $N(v_i) = \{v_{i+(jn+k)}, v_{i-(jn+k)}, u_i\}$. Analogously, $\forall a_i, b_i \in W$, $N(a_i) = \{a_{i+1}, a_{i-1}, b_i\}$ and $N(b_i) = \{b_{i+k}, b_{i-k}, b_i\}$.

For any $i \in \{0, \ldots, 2n-1\}$ we have:

$$
\begin{aligned}
f(u_i) &= a_{i(mod\ n)} \\
f(u_{i+1}) &= a_{i+1(mod\ n)} = a_{(i(mod\ n)+1(mod\ n))(mod\ n)} = a_{(i(mod\ n)+1)(mod\ n)} \\
f(u_{i-1}) &= a_{i-1(mod\ n)} = a_{(i(mod\ n)-1(mod\ n))(mod\ n)} = a_{(i(mod\ n)-1)(mod\ n)} \\
f(v_i) &= b_{i(mod\ n)} \\
f(v_{i+(jn+k)}) &= b_{i+(jn+k)(mod\ n)} = b_{(i(mod\ n)+(jn+k)(mod\ n))(mod\ n)} \\
&= b_{(i(mod\ n)+((jn(mod\ n)+k(mod\ n))(mod\ n))(mod\ n)} = b_{(i(mod\ n)+k)(mod\ n)} \\
f(v_{i-(jn+k)}) &= b_{i-(jn+k)(mod\ n)} = b_{(i(mod\ n)-(jn+k)(mod\ n))(mod\ n)} \\
&= b_{(i(mod\ n)-((jn(mod\ n)+k(mod\ n))(mod\ n))(mod\ n)} = b_{(i(mod\ n)-k)(mod\ n)}
\end{aligned}
$$

Remember that every subscript of a vertex in V is supposed to be modulo qn while for every vertex in W it is supposed to be modulo n. Therefore we have:

$$N(f(u_i)) = N(a_i) = \{a_{i+1}, a_{i-1}, b_i\} = \{f(u_{i+1}), f(u_{i-1}), f(v_i)\}$$

$$N(f(v_i)) = N(b_i) = \{b_{i+k}, b_{i-k}, a_i\} = \{f(v_{i+(jn+k)}), f(v_{i-(jn+k)}), f(u_i)\}$$

Hence, we have that $\forall v \in V$ the restriction of f to $N[v]$ onto $N[f(v)]$ is a bijection as we wanted to prove.

Now, let us prove the second statement:

For $1 \leq j \leq \frac{q}{2}$ and as $1 \leq k < \frac{n}{2}$ we have $\frac{n}{2} < n-k \leq jn-k \leq \frac{qn}{2} - k < \frac{qn}{2}$. Let $j' = j-1$ and $k' = n-k$. From the first statement we know that $G(qn, j'n+k')$ is a q-covering of $G(n,k')$. As $j'n + k' = j'n + (n-k) = (j'+1)n - k = jn-k$ we have that $G(qn, jn-k)$ is a q-covering of $G(n, n-k)$. Finally, using lemma 1.55 we have $G(n, n-k)$ isomorphic to $G(n,k)$. Note that we have used a little abuse on the notation as $n-k > \frac{n}{2}$.

□

Figures 2.1 and 2.2 are examples of coverings found by using Theorem 2.1

In theorem 2.1 we have seen a large family of Generalized Petersen Graph that are 2-coverings of other Generalized Petersen Graphs. Indeed, we know that for every $n \geq 3$ and $1 \leq k < \frac{n}{2}$, $G(2n,k)$ is a 2-covering of $G(n,k)$. However, are they also Kronecker Covers of these graphs? By theorem 1.57 we know that $G(2n,k)$ is bipartite if and only if k is odd. Hence, we can ensure that for even k it is not a Kronecker cover. For example, as can be seen in Figure 2.1, $G(16,2)$ is a 2-covering of $G(8,2)$. Also, as k is even we know that $G(16,2)$ is not bipartite, so it can not be a Kronecker Cover.

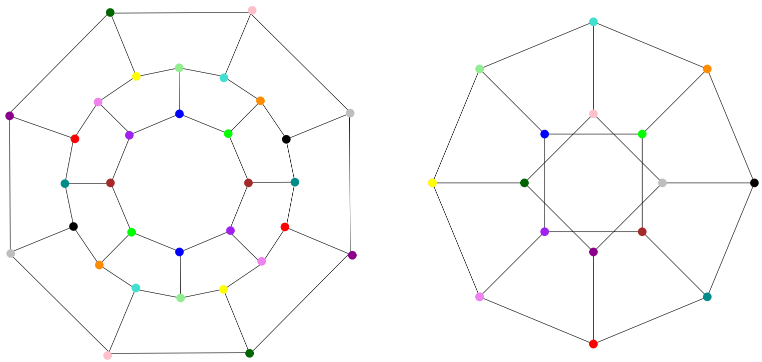

Figure 2.1: G(16,2) is a 2-covering of G(8,2)

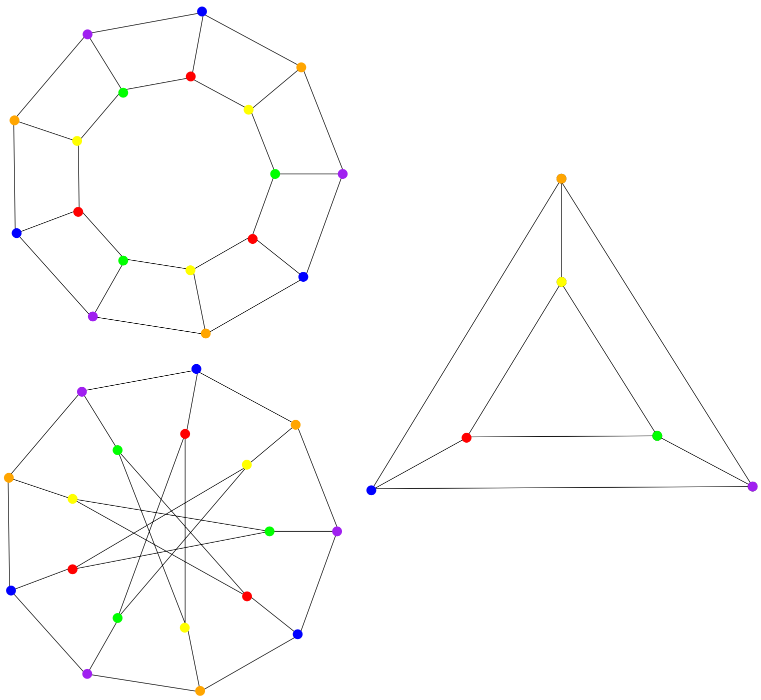

Figure 2.2: G(9,1) and G(9,4) are 3-coverings of G(3,1)

The following step was trying to see if $gcd(p,q) = 1$ was a sufficient condition for $G(qn, pk)$ to be a q-cover of $G(n,k)$. However, by using SageMath software [The21] we found the following counterexample. $G(12,3)$ is not a 2-covering of $G(6,1)$.

Chapter 3

Generalized Petersen Graph that are Kronecker covers of other Generalized Petersen Graph

The main goal of this chapter is to show which Generalized Petersen Graph are Kronecker covers of other members of the same family. This question is answered by Krank and Pizanski in [KP19]. Their main result states that $G(10,3)$ is the only Generalized Petersen Graph with two non-isomorphic quotients. Moreover, they prove which Generalized Petersen Graphs are Kronecker covers and which is their single quotient graph. However, we will only target the cases were the quotient is also a Generalized Petersen Graph.

With that purpose in mind, let us focus on the following part of their theorem:

Theorem 3.1. ***[KP19]:*** *Let $G \cong G(n,k)$ be a Generalized Petersen Graph. If $n \equiv 2\,(mod\,4)$, then G is a Kronecker cover. In particular:*

1. *If $4k < n$ its corresponding quotient graph is $G(\frac{n}{2},k)$.*
2. *If $n < 4k < 2n$ the quotient graph is $G(\frac{n}{2},\frac{n}{2}-k)$.*

Remark 3.2. *$G(10,3)$ holds the second case. However, as we said earlier in this chapter, M. Krns and T. Pisanksi proved in [KP19] that the Desargues graph is the only graph among the Generalized Petersen Graph family that is a Kronecker cover of two non-isomorphic graphs. One of those being the Petersen Graph $G(5,2)$.*

3.1 Previous results

This section will give some previous results from [KP19] that will be useful in order to prove Theorem 3.1.

Let us rewrite the following result in order to be consistent with our notation:

Proposition 3.3. *For a bipartite Generalized Petersen Graph $G(n,k)$, the following statements hold:*

1. *α^a is a Kronecker involution if and only if $a = \frac{n}{2}$ and $n \equiv 2 \mod 4$.*
2. *$\beta\alpha^a$ is not a Kronecker involution.*
3. *if $k^2 \equiv -1 \mod n$, then neither $\gamma\alpha^a$ nor $\gamma\beta\alpha^a$ are a Kronecker involution for any $a \in \mathbb{Z}/n\mathbb{Z}$.*

Proof.

1. *Necessity:* Let $\omega = \alpha^a$ be a Kronecker Involution. Hence, $\omega^2 = Id$ and also it has to be color-reversing satisfying that for every vertex v, $v\omega(v)$ is not an edge. As $\alpha^n = Id$ we trivialy have $a = \frac{n}{2}$. Now ω color-reversing if a is odd, so we have $n \equiv 2 \mod 4$. We also have $\omega(u_i) = u_{i+a}$ and $\omega(v_i) = v_{i+a}$, and both satisfying $u_i\omega(u_i), v_i\omega(v_i) \notin E(G(n,k))$

   Sufficiency: Now if $a = \frac{n}{2}$ and $n \equiv 2 \mod 4$ we have a is odd so $\omega = \alpha^a$ is color-reversing. And also we have $\omega^2 = \alpha^{2a} = \alpha^{2\frac{n}{2}} = \alpha^n = Id$. And hence ω is a Kronecker involution.

2. Let us suppose that $\omega = \beta\alpha^a$ is a Kronecker involution. By definition, as before we have a must be an odd number. In the proof of lemma 1.64 we have seen $\alpha^a\beta = \beta\alpha^{-a}$. As $\beta^2 = Id$ we also have $\omega = \beta\alpha^a = \alpha^{-a}\beta$.Now for $i = \frac{a-1}{2}$ we have $\alpha^{-a}(\beta(u_i)) = \alpha^{-a}(\beta(u_{\frac{a-1}{2}})) = \alpha^{-a}(u_{\frac{-a+1}{2}}) = u_{\frac{-a+1}{2}+a} = u_{\frac{2a-a+1}{2}} = u_{\frac{a+1}{2}} = u_{i+1}$. Therefore, $u_i\omega(u_i) = u_iu_{i+1} \in E(G(n,k))$ which contradicts the fact of ω being a Kronecker Involution.

3. Now we assume $k^2 \equiv -1 \mod n$. Let us suppose first $\omega_1 = \gamma\alpha^a$ is a Kronecker involution. Note that $\omega_1^2 = \gamma\alpha^a\gamma\alpha^a$ and using lemma 1.64 $\gamma\alpha^a\gamma\alpha^a = \gamma\gamma\alpha^{-ak}\alpha^a = \alpha^{-ak+a}\beta \neq Id$ contradicting the fact that ω_1 is an involution. Now let us suppose $\omega_2 = \gamma\beta\alpha^a$ is a Kronecker involution. Again, by lemma 1.64 we have:

   $$\begin{aligned}\gamma\beta\alpha^a\gamma\beta\alpha^a &= \gamma\beta\gamma\alpha^{-ak}\beta\alpha^a = \gamma\beta\gamma\beta\alpha^{ak}\alpha^a = \gamma\gamma\beta\beta\alpha^{ak+a} = \gamma^2\beta^2\alpha^{a(k+1)} \\ &= \gamma^2\alpha^{a(k+1)} = \beta\alpha^{a(k+1)} \neq Id\end{aligned}$$

   again contradicting the fact of ω_2 being a Kronecker Involution.

□

3.2 Proof of the theorem

From 1, and 2 of the Proposition 3.3 above it follows that the only Kronecker involution in the Dihedral group is α^a when $a = \frac{n}{2}$ and $n \equiv 2 \mod 4$. Also, by

Theorem 1.57, in order to be bipartite $G(n,k)$, n must be an even number and k an odd number. Finally, we have from Theorem 1.49 that under these conditions, there exists a Graph $H=(W,F)$ such that $KC(H)\cong G(n,k)$. If we show H is the quotient graph described in Theorem 3.1 we will have the proof.

Note that we already know from Theorem 2.1 that $G(2n,k)$ and $G(2n,n-k)$ are 2-covers of $G(n,k)$. However, we can only affirm that they are not Kronecker covers of $G(n,k)$ for an even k.

In order to proof the theorem, and following [KP19], we will prove the following proposition:

Proposition 3.4. *Let n be an odd number and an integer k such that* $1\leq k<\frac{n}{2}$. *Then,*

$$KC(G(n,k))\cong\begin{cases}G(2n,k) & if\ k\ is\ odd\\ G(2n,n-k) & if\ k\ is\ even\end{cases}$$

Proof. First of all if n and k are odd integers such that $1\leq k<\frac{n}{2}$, then $G(2n,k)$ is bipartite, $2n\equiv 2 \mod 4$ and α^n is the Kronecker involution of the automorphism group $Aut(2n,k)$. However, if n is an odd integer but k is an even integer satisfying $1\leq k<\frac{n}{2}$, then $G(2n,n-k)$ is bipartite and α^n is the Kronecker involution of the automorphism group $Aut(2n,n-k)$. Also note that $n-k$ is odd an as $k<\frac{n}{2}$, $0<k<\frac{n}{2}<n-k<n=\frac{2n}{2}$.

As both proofs are similar, let us define

$$k'=\begin{cases}k & if\ k\ is\ odd\\ n-k & if\ k\ is\ even\end{cases}$$

Let $G\cong G(2n,k')$, $H\cong G(n,k)$ such that $G=(V,E')$ and $H=(W,F)$. We have written E' to emphasise that the edge set is different depending on k'. As G and H are Generalized Petersen Grpahs let us define $V=\{u_0,\dots,u_{2n-1},v_0,\dots,v_{2n-1}\}$, $W=\{a_0,\dots,a_{n-1},b_0,\dots,b_{n-1}\}$. Remember that throughout the hole proof any subscript of vertex in V is written modulo $2n$ while any subscript of the vertex in W is written modulo n. Let us define the following homomorphism:

$$\begin{array}{rccc}\varphi: & V & \longrightarrow & W\times\{0,1\}\\ & u_i & \longmapsto & (a_i, i \mod 2)\\ & v_i & \longmapsto & (b_i, i+1 \mod 2)\end{array}$$

Let us proof first it is a homomorphism: As G is a Generalized Petersen Graph, its edges are of the form u_iu_{i+1}, $v_iv_{i+k'}$ or u_iv_i. The edges of H are of the form a_ia_{i+1}, b_ib_{i+k} or a_ib_i and by definition of the Kronecker product we have $E(KC(H))=E(H\times K_2)=\{(u,j \mod 2)(v,j+1 \mod 2)|uv\in F\}$. Hence, we have:

$$\varphi(u_i)\varphi(u_{i+1})=(a_i,i \mod 2)(a_{i+1},i+1 \mod 2)$$

$$\varphi(v_i)\varphi(v_{i+k'}) = (b_i, i+1 \mod 2)(b_{i+k'}, i+k'+1 \mod 2)$$

$$\varphi(u_i)\varphi(v_i) = (a_i, i \mod 2)(b_i, i+1 \mod 2)$$

As $a_i a_{i+1} \in F$ and clearly, $i \not\equiv i+1 \mod 2$, $\varphi(u_i)\varphi(u_{i+1}) \in E(KC(H))$. Now it is clear that if k is odd $k' = k$ and $b_i b_{i+k} \in F$ and $(i+1) \not\equiv (i+1)+k \mod 2$. If k is even, $k' = n-k$ is odd and as $k' \equiv -k \mod n$ we have $b_i b_{i-k} = b_{i-k} b_i \in F$. Therefore $\varphi(v_i)\varphi(v_{i+k'}) \in E(KC(H))$. Finally, as $a_i b_i \in F$ and $i \not\equiv i+1 \mod 2$, we have $\varphi(u_i)\varphi(v_i) \in E(KC(H))$. Hence we conclude it is a homomorphism. We will prove that φ is injective by contraposition: Let $i, j \in \mathbb{Z}/2n\mathbb{Z}$. Clearly $u_i \neq v_j$ and $\varphi(u_i) = (a_i, i \mod 2) \neq (b_j, j+1 \mod 2) = \varphi(v_j)$ as $a_i \neq b_j$. Now let $u_i, u_j \in V$ such that $u_i \neq u_j$. We have $\varphi(u_i) = (a_i, i \mod 2)$ and $\varphi(u_j) = (a_j, j \mod 2)$. If $i \not\equiv j \mod n$, $ai \neq a_j$ and hence, $\varphi(u_i) \neq \varphi(u_j)$. If $i \equiv j \mod n$ we have $a_i = a_j$. However, as $u_i \neq u_j$ we have $i = j+n$ or $i = j-n$. As n is an odd number $i \not\equiv j \mod 2$ and we also have $\varphi(u_i) \neq \varphi(u_j)$. Analogously, let $v_i, v_j \in V$ such that $v_i \neq v_j$. We have $\varphi(v_i) = (b_i, i+1 \mod 2)$ and $\varphi(v_j) = (b_j, j \mod 2)$. If $i \not\equiv j \mod n$, $bi \neq b_j$ and hence, $\varphi(v_i) \neq \varphi(v_j)$. If $i \equiv j \mod n$, then $bi = b_j$. However, as $v_i \neq v_j$ we have $i = j+n$ or $i = j-n$. As n is an odd number $i \not\equiv j \mod 2$. It is also a surjective homomorphism as it is injective and we have $|V| = |W \times \{0,1\}|$. Let us see now its inverse function is a homomorphism too:

$$\begin{aligned} \varphi^{-1}: \quad W \times \{0,1\} &\longrightarrow \quad V \\ (a_i, j) &\longmapsto \begin{cases} u_i & if\ i \equiv j \mod 2 \\ u_{i+n} & if\ i \not\equiv j \mod 2 \end{cases} \\ (b_i, j) &\longmapsto \begin{cases} v_{i+n} & if\ i \equiv j \mod 2 \\ v_i & if\ i \not\equiv j \mod 2 \end{cases} \end{aligned}$$

Remember that $E(KC(H)) = \{(u, j \mod 2)(v, j+1 \mod 2) | uv \in F\}$. Therefore we have to check the following cases:

- Let us first suppose $i \equiv j \mod 2$:
  - $\varphi^{-1}(a_i, j \mod 2)\varphi^{-1}(a_{i+1}, j+1 \mod 2) = u_i u_{i+1} \in E'$
  - $\varphi^{-1}(a_i, j \mod 2)\varphi^{-1}(b_i, j+1 \mod 2) = u_i v_i \in E'$
  - $\varphi^{-1}(b_i, j \mod 2)\varphi^{-1}(b_{i+k}, j+1 \mod 2) = \begin{cases} v_{i+n} v_{i+n+k} & if\ k\ is\ odd \\ v_{i+n} v_{i+k} & if\ k\ is\ even \end{cases}$

  Where at the last case we have used that if k is odd, as $i \equiv j \mod 2$, $i+k \equiv j+1 \mod 2$ and if k is even, $i+k \not\equiv j+1 \mod 2$. Moreover, for the case when k is even, as $v_{i+n}v_{i+k} = v_{i+k}v_{i+n} = v_{i+k}v_{i+k+(n-k)}$. Therefore, in both cases we have:

  $$\varphi^{-1}(b_i, j \mod 2)\varphi^{-1}(b_{i+k}, j+1 \mod 2) \in E'$$

- Now, if $i \not\equiv j \mod 2$:
  - $\varphi^{-1}(a_i, j \mod 2)\varphi^{-1}(a_{i+1}, j+1 \mod 2) = u_{i+n}u_{i+n+1} \in E'$
  - $\varphi^{-1}(a_i, j \mod 2)\varphi^{-1}(b_i, j+1 \mod 2) = u_{i+n}v_{i+n} \in E'$
  - $\varphi^{-1}(b_i, j \mod 2)\varphi^{-1}(b_{i+k}, j+1 \mod 2) = \begin{cases} v_i v_{i+k} & if\ k\ is\ odd \\ v_i v_{i+k+n} & if\ k\ is\ even \end{cases}$

    Where at the last case we have used that if k is odd, as $i \not\equiv j \mod 2$, $i+k \not\equiv j+1 \mod 2$ and if k is even, $i+k \equiv j+1 \mod 2$. Moreover, for the case when k is even, and as the subscripts are in modulo $2n$ we have $v_i v_{i+k+n} = v_{i+k+n}v_i = v_{i+k+n}v_{i+(k-k+2n)} = v_{i+k+n}v_{i+k+n+(n-k)}$. Therefore, in both cases we have:

$$\varphi^{-1}(b_i, j \mod 2)\varphi^{-1}(b_{i+k}, j+1 \mod 2) \in E'$$

.

Hence, φ^{-1} is also a homomorphism and we have $KC(G(n,k)) \cong G(2n,k')$. $\square$

Chapter 4

Generalization of the Kronecker cover

In this chapter we will generalize the Kronecker cover to a 3-cover.

4.1 Kronecker triple cover

Remember that by definition of the Kronecker product, for every vertex $v \in G$ we had two vertices in $KC(G)$ and the same with the edges. Therefore, in order to generalize the Kronecker cover to 3-covering we need to find a graph such that for every vertex we get three vertices, and for every edge we also get three edges.

Definition 4.1. *Let D and H be two directed graph. The **directed tensor product** of D and H (denoted by $D \times H$) is a digraph satisfying the following two conditions:*

- *The vertex set $V(D \times H)$ is the Cartesian product $V(D) \times V(H)$.*
- *$(g_1, h_1)(g_2, h_2) \in A(D \times H)$, if and only if $g_1 g_2 \in A(D)$ and $h_1 h_2 \in A(H)$.*

Remark 4.2. *The notation used is the same as the tensor product of two undirected graphs although they mean different things.*

Definition 4.3. *Let $D = (V, A)$ be a simple digraph. We define the **Kronecker Triple cover of D**, and denote by KTC(D) the underlying of the directed tensor product $D \times \overrightarrow{C_3}$. Therefore, it is the underlying graph of the digraph with vertex set*

$$V() = \{v', v'', v'''\}_{v \in V(D)}$$

and arc set

$$A(D \times \overrightarrow{C_3}) = \{(u', v''), (u'', v'''), (u''', v')\}_{(u,v) \in A(D)}$$

Remark 4.4. *$KTC(D)$ is clearly tripartite as $V(KTC(D)) = V' \cup V'' \cup V'''$ where V', V'', V''' are 3 independent sets.*

Notation 4.5. *In order to simplify and unify the notation we will write the vertex set of $KTC(D)$ as $V \times \{0,1,2\}$. We may also use the following notation to represent the vertices on a Kronecker triple cover of a digraph D:*

Given a vertex $v \in V(D)$:

$$v' = (v,0), \quad v'' = (v,1), \quad \text{and} \quad v''' = (v,2)$$

In this way, the arc set of $D \times \overrightarrow{C_3}$ may be written as:

$$A(D \times \overrightarrow{C_3}) = \{(u, i \mod 3)(v, i+1 \mod 3)\}_{(u,v)\in A(D)}$$

where $i \in \{0,1,2\}$.

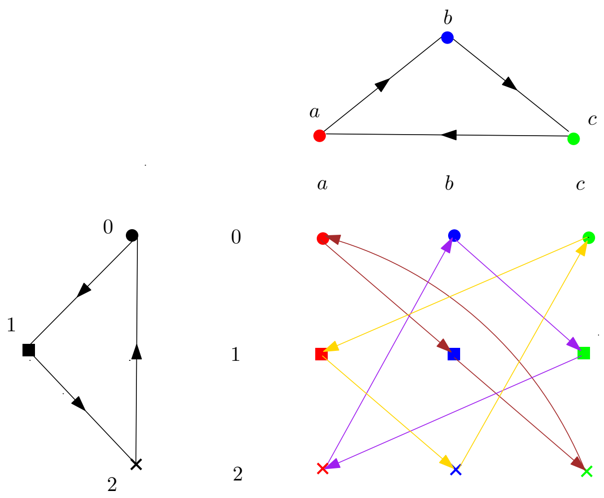

Figure 4.1: $\overrightarrow{C_3} \times \overrightarrow{C_3}$.

Remark 4.6. *As can be seen in Figure 4.1, $KTC(\overrightarrow{C_3}$ has 3 connected component.*

Theorem 4.7. *The Kronecker Triple cover of D is a 3-cover of the underlying graph of D.*

Proof. We will prove a stronger statment: $D \times \overrightarrow{C_3}$ is a 3-cover of D: Let $D = (V, A)$. By definition of the cross product we have, $D \times \overrightarrow{C_3} = (V \times \{0,1,2\}, B)$. Let us define the following homomorphism:

$$\begin{array}{rccc} \varphi: & V \times \{0,1,2\} & \longrightarrow & V \\ & (v,i) & \longmapsto & v \end{array}$$

It is clearly a homomorphism: $\forall(u,i),(v,j) \in V \times \{0,1,2\}$ if $((u,i),(v,j)) \in B$, by definition we have $j = i+1 \mod 3$ and $(u,v) \in A$ and as $(\varphi(u,i), \varphi(v,i+1)) = (u,v) \in A$ we have the result. Surjection is also clear by definition of φ. Now, let $(u,i) \in V$. Also we have, $N(u,i) = \{(v,i+1),(w,i-1)|(u,v),(w,v) \in A\}$ and $N(u) = \{v,w|(u,v),(w,v) \in A\}$. For φ restricted to $N[v]$ we have $\varphi(v,i+1) = v$ and $\varphi(w,i+1) = w$ and hence we have the local isomorphism. $\square$

Theorem 4.8. *Let $D = (V,A)$ be a digraph. If there exists a homomorphism φ such that*

$$\varphi : V(D) \longrightarrow V(\overrightarrow{C_3})$$

then the Kronecker Triple Cover KTC(D) is not a connected graph.

Proof. Let us assume that there exists a homomorphism $\varphi : V(D) \longrightarrow V(\overrightarrow{C_3})$. For easier notation let us put for every $v \in V$, $\varphi(v) \in \{0,1,2\}$. Hence, for $v \in V$ such that $(v,u) \in A$ we have $(\varphi(v), \varphi(u)) \in A(\overrightarrow{C_3})$. Therefore, $\varphi(u) = \varphi(v) + 1 \mod 3$. Let us define the following function:

$$\begin{array}{rccc} f: & V \times \{0,1,2\} & \longrightarrow & \{0,1,2\}^2 \\ & (v,i) & \longmapsto & (\varphi(v), i) \end{array}$$

Let us prove it is a homomorphism:

For $v \in V$, then $(v, i \mod 3)(u, i+1 \mod 3)$ if $(v,u) \in A)$. Then, we have $(f(v,i), f(u,i+1)) = ((\varphi(v),i),(\varphi(u),i+1)) = ((\varphi(v),i),(\varphi(v)+1,i+1))$, that is indeed an arc of $\overrightarrow{C_3} \times \overrightarrow{C_3}$. Note that $\varphi(v) - i \mod 3$ is an invariant for every connected vertices of $\overrightarrow{C_3} \times \overrightarrow{C_3}$ meaning that $\overrightarrow{C_3} \times \overrightarrow{C_3}$ is not connected (see Figure 4.1). Hence, as f is a homomorphism, it preserve edges which means $D \times \overrightarrow{C_3}$ is also disconnected. $\square$

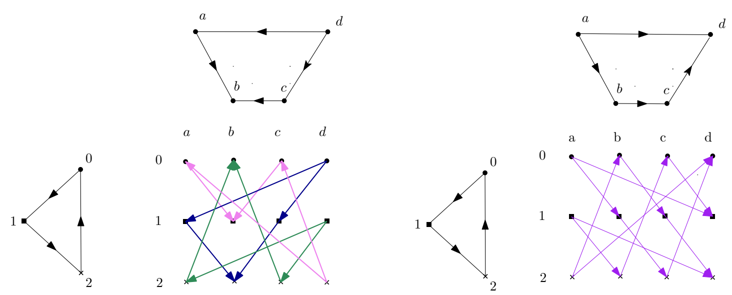

Figure 4.2: The KTC of two different orientations of the same graph can result on two non isomorphic graphs

4.1.1 3-Kronecker Automorphism

Definition 4.9. *An automorphism $f : A \longrightarrow A$ is said to be of **order 3** if*

$$(f \circ f \circ f)(x) = f^3(x) = f(f(f(x))) = x$$

or equivalently,

$$f^{-1}(x) = f^2(x)$$

Definition 4.10. *Let $G = (V, E)$ be a tripartite graph and $Aut(G)$ its automorphism group. We say $\omega \in Aut(G)$ is a **3-Kronecker automorphism** if it is an automorphism of order 3 swapping the 3 sets of tripartition such that $\forall v \in V$, $v\omega(v) \notin E$*

Theorem 4.11. *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a simple tripartite graph. Then, there exist a digraph $H = (W, F)$ such that $KTC(H) \cong G$ if and only if $Aut(G)$ admits a 3-Kronecker automorphism.*

Proof. Necessity: Let us assume $G = (V, E)$ a tripartite graph, $H = (W, F)$ a simple digraph such that $KTC(D) \cong G$. By definition of the $KTC(D)$ we have $V = W \times \{0, 1, 2\}$. Let us define $\tau \in Aut(G)$ as:

$$\begin{array}{rccc} \tau : & W \times \{0,1,2\} & \longrightarrow & W \times \{0,1,2\} \\ & (w, i) & \longmapsto & (w, i+1 \mod 3) \end{array}$$

We can clearly see that $(\tau \circ \tau \circ \tau)(w, i) = \tau(\tau(\tau(w, i))) = \tau(\tau(w, i+1 \mod 3)) = \tau(w, i+2 \mod 3) = (w, i+3 \mod 3) = (w, i)$. Hence, τ has order 3 and changes all sets of tripartition as $i \not\equiv i+1 \mod 3 \not\equiv i+2 \mod 3 \not\equiv i \mod 3$. Now, if $(w, i), \tau(w, i) \in V$ then $(w, i)(w, i+1) \in E$ and by definition of the $KTC(D)$ there is an arc $(w, w) \in F$ which is contradicting the fact of H being simple. Therefore, τ is 3-Kronecker automorphism.

Sufficiency: Now we assume $G = (A \cup B \cup C, E)$ to be a tripartite graph and $\omega \in Aut(G)$ be a 3-Kronecker automorphism. Let us define the digraph $H = (A, F)$ such that $(a_1, a_2) \in F \iff a_1\omega(a_2) \in E$. Note that if $a \in A$, by definition of 3-Kronecker automorphism it swaps the sets of tripartition. Hence we have $\omega(a) \in B \cup C$ and as G is tripartite we have A, B and C are 3 independent sets. For simple notation we will suppose $\omega(a) \in B$ and $\omega^2(a) \in C$. Also, as ω is an autmorphism, if $a_1\omega(a_2) \in E$ we also have $\omega(a_1)\omega^2(a_2) \in E$ and $\omega^2(a_1)a_2 \in E$. Now, let us define the following homomorphism:

$$\begin{array}{rccc} \varphi : & A \times \{0,1,2\} & \longrightarrow & A \cup B \cup C \\ & (a, i) & \longmapsto & \omega^i(a) \end{array}$$

Where $A \times \{0, 1, 2\}$ is the vertex set of KTC(H). Let us first check it is a homomorphism. $(a_1, i)(a_2, j) \in E(KTC(H))$ if and only if $j = i + 1$ and $(a_1, a_2) \in F$. As we have previously seen $(a_1, a_2) \in F$ if and only if $a_1\omega(a_2) \in E$ by definition of H. Now if $j = i + 1$ and $(a_1, a_2) \in F$, as we have seen before and by definition of the 3-Kronecker automorphism we have $\varphi(a_1, i)\varphi(a_2, j) = \omega^i(a_1)\omega^j(a_2) \in E$. It is also an injection: if $\varphi(a_1, i) = \varphi(a_2, j)$ we have $\omega^i(a_1) = \omega^j(a_2)$ and trivially we have $i = j$ and $a_1 = a_2$. Therefore, $(a_1, i) = (a_2, j)$ and φ is injective. It is also a surjection by definition: For $i = 0$ and $\forall a \in A$, we have $\varphi(a, 0) = a \in A$. Moreover, ω is a 3-Kronecker automorphism, hence $|A| = |B| = |C|$ and we also have $|A \cup B \cup C| = |A \times \{0, 1, 2\}|$. Finally, we will check φ^{-1} is a homomorphism.

$$\begin{array}{rccc} \varphi^{-1}: & A \cup B \cup C & \longrightarrow & A \times \{0,1,2\} \\ & v & \longmapsto & \begin{cases} (v,0) & if\ v \in A \\ (\omega^2(v),1) & if\ v \in B \\ (\omega(v),2) & if\ v \in C \end{cases} \end{array}$$

As G is tripartite, if $e \in E$, then we have $e = ab = ba$ such that $a \in A, b \in B$ or $e = bc = cb$ such that $b \in B, c \in C$ or $e = ca = ac$ such that $a \in A, c \in C$. We remember that by the definition of the 3-Kronecker utomorphism, $e = bc$ is the same as there exisist $a_1, a_2 \in A$ such that $b = \omega(a_1)$, $c = \omega^2(a_2)$ and $e = \omega(a_1)\omega^2(a_2)$. Also for $e = ac$ there exisist $a_3 \in A$ such that $c = \omega^2(a_3)$ and $e = a\omega^2(a_3)$. Let us check the three cases:

- $e = ab$ such that $a \in A, b \in B$:

  $\varphi^{-1}(a)\varphi^{-1}(b) = (a, 0)(\omega^2(b), 1)$. As ω has order 3, we have $e = a\omega^3(b) = a\omega(\omega^2(b)) \in E$. As $a, \omega^2(b) \in A$, by construction of H we have $(a, \omega^2(b)) \in F$ and therefore, by definition of the KTC we have $\varphi^{-1}(a)\varphi^{-1}(b) = (a, 0)(\omega^2(b), 1) \in E(KTC(H))$.

- $e = bc$ such that $b \in B, c \in C$:

  $\varphi^{-1}(b)\varphi^{-1}(c) = (\omega^2(b), 1)(\omega(c), 2)$. As ω has order 3, we have $e = \omega^3(b)\omega^3(c) = \omega(\omega^2(b))\omega(\omega^2(c)) \in E$. As ω is an automorphism we also have $\omega^2(b)\omega(\omega(c)) \in E$. Note that $\omega^2(b), \omega(c) \in A$. Hence, by construction of H we have $(\omega^2(b), \omega(c)) \in F$ and therefore, $\varphi^{-1}(b)\varphi^{-1}(c) = (\omega^2(b), 1)(\omega(c), 2) \in E(KTC(H))$.

- $e = ca$ such that $a \in A, c \in C$:

  $\varphi^{-1}(a)\varphi^{-1}(c) = (a, 0)(\omega(c), 2)$. As ω is an automorphism we also have $\omega(c)\omega(a) = \in E$. By construction of H we have $(\omega(c), a) \in F$ and therefore, by definition of the KTC we have $\varphi^{-1}(a)\varphi^{-1}(c) = (\omega(c), 2)(a, 0) \in E(KTC(H))$.

Therefore, for every edge $uv \in E$ we have $\varphi^{-1}(u)\varphi^{-1}(v) \in E(KTC(H))$ so φ^{-1} is also a homomorphism. So $KTC(H) \cong G$. $\square$

Chapter 5

Conclusion

In order to finish this bachelor thesis, we will summarize and highlight the key items of each chapter:

We have started this thesis by introducing some basic knowledge of graph theory and more specific ones of algebraic graph theory such the k-covering. Also, we have defined the family of Generalized Petersen Graphs and introduced the Kronecker Cover. We have also reproved some known theorems such as Theorem 1.49, that sets a relation between the Kronecker Cover and the Kronecker Involution of a graph G.

Immediately after this introduction, we have presented a family of Generalized Petersen Graph that are q-coverings of other Generalized Petersen Graphs. Due to the lack of time, we haven't found any other Generalized Petersen Graph that satisfy being a q-covering of another Generalized Petersen Graph although it might be some other. It would be interesting to use Theorem 1.56 in order to extend our result.

Afterwards, we have seen which Generalized Petersen Graphs are Kronecker Covers from other Generalized Petersen Graphs. These are known results from Krnc and Pisanski [KP19] that we have reproved in order to preserve our notation. Also, all their results have been useful in order to prepare us for the next step.

Finally, we have introduced the Kronecker Triple Cover and the 3-Kronecker automorphism. We have generalized the Kronecker cover to a 3-covering with the aim of finding which Generalized Petersen Graphs are Kronecker Triple Covers of other Generalized Petersen Graph. Although we have proved the relation between the existence of a 3-Kronecker automorphism and the Kronecker Triple Cover we have not characterized which Generalized Petersen Graphs are KTC of other such graphs due to there might exist more that one 3-Kronecker automorphism. Due to lack of time, it also remains as an open problem to generalize the Kronecker Cover to a q-covering and relate it to the q-Kronecker automorphism, encouraging researchers to follow our results.

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