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The Consistency of the Negation of the Continuum Hypothesis

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Abstract

The purpose of this work is to prove the consistency of the negation of the Continuum Hypothesis (CH) with the $Zermelo - Fraenkel$ axiomatic system, including the Axiom of Choice (ZFC). The Continuum Hypothesis states that there is no set whose cardinality is strictly between the cardinality of the set of integers and the cardinality of the set of real numbers. It is well-known that CH is independent of ZFC: neither CH nor its negation can be proved from ZFC. In order to show the consistency of $\neg CH$, we will use the method of $forcing$ that permits us to construct a model that satisfies all the axioms of ZFC and where CH fails.

Resum

L'objectiu d'aquest treball és demostrar la consistència de la negació de la Hipòtesi del Continu (CH) amb el sistema axiomàtic de $Zermelo - Fraenkel$ amb l'Axioma d'Elecció (ZFC). La Hipòtesi del Continu diu que no existeix cap conjunt amb cardinal estrictament entre el cardinal del conjunt dels enters i el cardinal del conjunt dels nombres reals. Com és ben conegut, CH és independent de ZFC: no es pot demostrar a partir de ZFC ni que CH sigui certa ni que sigui falsa. Per a demostrar que $\neg CH$ és consistent, utilitzarem el mètode de $forcing$ amb el qual construïrem un model que satisfa tots els axiomes de ZFC i on CH falla.

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Contents

Introduction

Georg Cantor proved in his article *On a Property of the Collection of All Real Algebraic Numbers* published in 1874 that the set of real numbers is uncountable. Four years later, in 1878, Cantor raised the famous Continuum Hypothesis (CH) that states that there is no set whose cardinality is strictly between the cardinality of the set of natural numbers and the cardinality of the set of real numbers. Establishing the truth or falsehood of CH was the first problem of David Hilbert's list of twenty-three problems published in 1900 and it was selected among the ten problems he presented at the International Mathematics Congress in Paris in the same year. Paul Cohen showed in 1963, complementing the work of Kurt Gödel, that the Continuum Hypothesis is independent of the set theory axiomatic system of Zermelo-Fraenkel with the Axiom of Choice (ZFC) by means of a new technique called *forcing* that he conceived specifically to produce independence results and, in particular, this one. The independence of CH means that it is not possible to prove CH nor its negation using ordinary mathematical methods because both statements are consistent with ZFC. The aim of this work is to prove that the negation of the Continuum Hypothesis is consistent with ZFC by using the method of forcing.

With respect to the structure of this work, we will mainly follow Kunen's book [5]. In Chapter 1, we will present the basic notions of set theory: its language, the ZFC axioms and the ordinals and cardinals. In Chapter 2, we are going to study *models* of ZFC. After talking about the important concepts of *relativization*, *well − founded sets* and *absoluteness*, we are going to see how the Reflection Theorems grant us the existence of a *countable transitive model* of any finite list of axioms extending ZFC and we are going to present the *constructible universe*, from which Gödel showed that CH is consistent with ZFC. In Chapter 3, we will study the general method of forcing. Finally, in Chapter 4 we are going to use this method in order to produce a model of ZFC where CH fails.

Chapter 1

Preliminary Notions of Set Theory

1.1 The language of set theory

Set Theory is based in the first-order predicate language, which is a formal language that consists of basic symbols, rules that allow us to construct the formulas and rules of inference that will grant us the theorems of the theory by means of formal deduction. The point in formalizing the formation of formulas is the necessity of stating axioms precisely and obtaining the theorems from those axioms. Indeed, as the majority of the ZFC axioms, as we will see later, are existential ones (that establishes the existence of certain sets), we have to be sure that the formation of new sets does not lead to any contradiction. And this is ensured by this formalization. For instance, one of the axioms is the Comprehension Axiom, that certifies the existence of sets made of elements that satisfy a certain property. Now, the notion of property has to be formally defined with this first-order logic because, letting any property be written without any rigour would set up unwanted contradictions, as self-referential paradoxes. An example is Berry's paradox: consider the smallest positive integer not definable in under sixty letters.
We will be using those symbols: the logical connectors $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, the universal and existential quantifiers $\forall$ and $\exists$ and the parenthesis (,). The letters of the latin alphabet in lower or upper case (x, $y, \ldots$, A, $B, \ldots$) will be used as variables. We will also use the equality symbol $=$. The language of this theory consists of one element, the binary predicate of membership $\mathcal{L} = \{\in\}$. The atomic formulas are of the form $x \in y$ and $x = y$. Now, if φ and ψ are formulas, $\neg(\varphi)$, $(\varphi) \wedge (\psi)$ and $\exists x(\varphi)$ are also formulas. Then, $(\varphi) \vee (\psi) \equiv \neg((\neg(\varphi)) \wedge (\neg(\psi)))$, $(\varphi) \rightarrow (\psi) \equiv (\neg(\varphi)) \vee (\psi)$, $(\varphi) \leftrightarrow (\psi) \equiv ((\varphi) \rightarrow (\psi)) \wedge ((\psi) \rightarrow (\varphi))$ and $\forall x(\varphi) \equiv \neg(\exists x(\neg(\varphi)))$ are formulas as well. We won't write parentheses if it does not lead to any ambiguity.
A variable x of a formula φ is $quantified$ if it is bound to a quantifier. If x is not in

the range of any quantifier, then x is $free$. Formally, these notions are defined as follows:

Definition 1.1. *Given a formula φ,*

1. *If φ is an atomic formula, $free(\varphi)$ is the set of all the variables that appear in φ and $quant(\varphi) = \emptyset$.*
2. *$free(\neg\varphi) = free(\varphi)$, $quant(\neg\varphi) = quant(\varphi)$.*
3. *$free(\varphi \cdot \psi) = free(\varphi) \cup free(\psi)$ and $quant(\varphi \cdot \psi) = quant(\varphi) \cup quant(\psi)$, with $\cdot = \wedge, \vee, \rightarrow, \leftrightarrow$.*
4. *$free(Qx\varphi) = free(\varphi) \setminus \{x\}$ and $quant(Qx\varphi) = quant(\varphi) \cup \{x\}$ with $Q \in \{\exists, \forall\}$.*

If a formula has no free variables, it is called a *sentence*. We will write $\varphi(x)$ if x appears free in φ.
Now, given a formula φ, one may think that $\{x : \varphi(x)\}$ is a set, the set of all the elements that satisfy the formula, following the idea we discussed earlier. However, this leads to the well-known Russell's paradox. Consider $A = \{x : \neg x \in x\}$. Then $A \in A \Leftrightarrow \neg A \in A$, which is a contradiction. We need to use an axiomatic system that allows us to construct sets avoiding these kind of contradictions. Many systems have been studied during the XIXth century, but the most used one is ZFC.

1.2 The ZFC axioms

Axiom 0. Set existence: $\exists x(x = x)$.

Axiom 1. Extensionality: $\forall x\ \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

Axiom 2. Foundation: $\forall x(\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$.

Axiom 3. Comprehension Scheme:
Let φ be a formula with free variables $x, z, v_1, \ldots, v_n$.
$\forall z\ \forall v_1, \ldots, v_n\ \exists y\ \forall x(x \in y \leftrightarrow x \in z \wedge \varphi(x, z, v_1, \ldots, v_n))$.

Axiom 4. Pairing: $\forall x\ \forall y\ \exists z(x \in z \wedge y \in z)$.

Axiom 5. Union: $\forall \mathcal{F}\ \exists A\ \forall Y\ \forall x(x \in Y \wedge Y \in \mathcal{F} \rightarrow x \in A)$.

Axiom 6. Replacement Scheme:

Let φ be a formula with free variables $x, y, A, v_1, \ldots, v_n$.
$\forall A \forall v_1, \ldots, v_n$

$$(\forall x \in A\ \exists! y\ \varphi(x, y, A, v_1, \ldots, v_n) \to \exists B\ \forall x \in A\ \exists y \in B\ \varphi(x, y, A, v_1, \ldots, v_n)).$$

With these axioms, we can define the concepts of *inclusion* ($\subset$), $empty - set$ ($\emptyset$), *ordinal successor* ($S(x) = x \cup \{x\}$) and $well - order$. Recall that a relation R on a set A well-orders A if and only if R totally orders A and for any non-empty $B \subset A$ there is an R-minimal element in B. The following axioms are defined with these abbreviations.

Axiom 7. Infinity: $\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$.

Axiom 8. Power Set: $\forall x \exists y \forall z(z \subset x \to z \in y)$.

Axiom 9. Choice: $\forall A\ \exists R(R\ well - orders\ A)$.

So, those ones are all the axioms of ZFC. In fact, this axiomatic system is infinite. Axioms 1, 2, 4, 5, 7, 8 and 9 are single axioms (each one consists of only one axiom), but axioms 3 and 6 are "schemes": there is a Comprehension and a Replacement axiom for each formula ϕ. So, axioms are used in order to deduce from them all the theorems of a theory. According to Hilbert's Thesis, every proof of a mathematical proposition carried out by ordinary means can be formalized in ZFC. Now, we will see how are defined the ordinals and cardinals in set theory.

1.3 Ordinals and Cardinals

Definition 1.2. *We say that y is a transitive set if and only if* $\forall x \in y(x \subset y)$.

Definition 1.3. *α is an ordinal if and only if α is a transitive set and is well-ordered by $\in$. The class of all ordinals is called* **ON**. *Moreover, an ordinal β is called:*
- *a successor ordinal $\Leftrightarrow \beta = S(\alpha)$ for some ordinal α.*
- *a limit ordinal $\Leftrightarrow \beta \neq 0$ and β is not a successor ordinal.*
- *a natural number $\Leftrightarrow$ for all $\alpha \leq \beta$, $\alpha = 0$ or α is a successor ordinal.*

We will write Greek letters such as α, β... to denote ordinals. We will use $<$ and $\leq$ symbols as follows: $\alpha < \beta \Leftrightarrow \alpha \in \beta$ and $\alpha \leq \beta \Leftrightarrow \alpha \in \beta \vee \alpha = \beta$.
Recall that, if φ is a formula, we cannot assure that $\{x : \varphi(x)\}$ is a set. We call it a class. For example, $\mathbf{ON} = \{\alpha : \alpha\ is\ an\ ordinal\}$ is the class of all the ordinals, which is not a set. We say that $\{x : \varphi(x)\}$ exists and is a set if there is one, call it A, such that $\forall x(x \in A \leftrightarrow \varphi(x))$. In this case, it is clear that $A = \{x : \varphi(x)\}$ due to the Comprehension and Extensionality Axioms.

Theorem 1.1. *All the elements of an ordinal are ordinals. If two ordinals are isomorphic (there is a bijection between them that maintains the order), then they are equal.*

Theorem 1.2. *Let x, y and z be ordinals. Then,*
a) Exactly one condition of these is true: $x = y$, $x \in y$, $y \in x$.
b) If $x \in y$ and $y \in z$ then $x \in z$.
c) Any non-empty set of ordinals has an $\in$-least element.

Definition 1.4. $\omega = \{\alpha : \alpha \text{ is a natural number}\}$.

So ω is a set and it is a limit ordinal. In fact, it is the first one. Let's see how we can compare the "size" of sets by means of cardinals.

Definition 1.5. *Let x and y be sets. If there is a bijective function from x into y, we write $x \approx y$. Moreover, $\approx$ is an equivalence relation.*

Definition 1.6. *For a set A, $|A|$ is the least ordinal α such that $A \approx \alpha$. We say that A has cardinality α.*

The Axiom of Choice ensures the existence of $|A|$.

Definition 1.7. *For any ordinal α, α is a cardinal $\Leftrightarrow |\alpha| = \alpha$.*

So a cardinal λ is an ordinal such that, for all $\alpha < \lambda$, we have that $\lambda \not\approx \alpha$. For instance, natural numbers and ω are cardinals.

Definition 1.8. *We say that a set x is finite if and only if $|x| < \omega$. If it is not finite, we call it infinite. We say that it is countable if and only if $|x| \leq \omega$. If it is not countable, we call it uncountable.*

To compare cardinals, we need to define the operations on them.

Definition 1.9. *For κ, λ cardinals, $\kappa + \lambda = |\kappa \times \lambda \cup \lambda \times \{1\}|$ and $\kappa \cdot \lambda = |\kappa \times \lambda|$.*

Theorem 1.3. *If κ and λ are infinite cardinals, then $\kappa + \lambda = \kappa \cdot \lambda = max(\kappa, \lambda)$.*

In order to formalize the Continuum Hypothesis, we we have to define the cardinal exponentiation. Recall that, for a function f, $dom(f) = \{x : \exists y(f(x) = y)\}$ and $ran(f) = \{y : \exists x(f(x) = y)\}$.

Definition 1.10. *Let A and B be sets. We define ${}^{B}A = \{f : B \to A : \ f \text{ is a function } \wedge dom(f) = B \ \wedge \ ran(f) \subset A\}$. For κ and λ cardinals, $\kappa^{\lambda} = |{}^{\lambda}\kappa|$.*

Lemma 1.1. *Let κ, λ and σ be cardinals. Then, $\kappa^{\lambda+\sigma} = \kappa^{\lambda} \cdot \kappa^{\sigma}$ and $(\kappa^{\lambda})^{\sigma} = \kappa^{\lambda+\sigma}$.*

Definition 1.11. *For any ordinal β, β^{+} is the least cardinal κ such that $\kappa > \beta$.*

We can now construct cardinals from ω, as follows.

Definition 1.12. *For every ordinal α, $\aleph_\alpha = \omega_\alpha$ is defined by recursion as:*
a) $\aleph_0 = \omega$.
b) $\aleph_{\alpha+1} = \aleph_\alpha^+$.
c) $\aleph_\gamma = sup\{\aleph_\alpha : \alpha < \gamma\}$, where γ is a limit ordinal.

Definition 1.13. *Let α, β be ordinals. If $f : \alpha \to \beta$ is a function, we say that f maps α $cofinally$ if and only if $ran(f)$ is unbounded in β.*
The cofinality of β, $cf(\beta)$, is the least α such that there is a cofinal map from α into β.

For any infinite cardinal κ, $cf(\kappa)$ is an infinite cardinal less than or equal to κ.

Definition 1.14. *A limit ordinal β is called regular if and only if $cf(\beta) = \beta$.*

So, for example, every successor cardinal is regular and $\aleph_\omega$ is not regular.

Proposition 1.1. *a) For any ordinal β, there is a cofinal strictly increasing map $f : cf(\beta) \to \beta$.*
b) If α is a limit ordinal and $f : \alpha \to \beta$ is a strictly increasing cofinal map, then $cf(\alpha) = cf(\beta)$.

Let's see a last result before presenting formally the Continuum Hypothesis. The symbol $f \restriction A$ denotes the restriction of a function f on a set A.

Definition 1.15. *Let A be a set. We say that a subset $B \subset A$ is closed under a function f on A if and only if $ran(f \restriction B) \subset B$.*
Let $\mathcal{F}$ be a set of finitary functions on A (every function in $\mathcal{F}$ is an n-ary function on A for some $n \in \omega$). Let $B \subset A$. We define the closure of B under $\mathcal{F}$ as the least subset C of A such that $B \subset C$ and C is closed under every function in $\mathcal{F}$.

Proposition 1.2. *Let κ be an infinite cardinal. Let $B \subset A$ be such that $|B| \leq \kappa$. Let $\mathcal{F}$ be a set of at most κ finitary functions on A. If C is the closure of B under $\mathcal{F}$, then $|C| \leq \kappa$.*

To end this chapter, let's define the Continuum Hypothesis, beginning with König's Theorem.

Theorem 1.4 (König)**.** *If $\kappa \geq \omega$ then $\kappa^{cf(\kappa)} > \kappa$ and $cf(2^\kappa) > \kappa$.*

It follows that $cf(2^\omega) > \omega = cf(\aleph_\omega)$, and then $2^\omega \neq \lambda$ for any cardinal λ of cofinality ω.
Now we can formally express the Continuum Hypothesis (CH) and the Generalized Continuum Hypothesis (GCH).

Definition 1.16. *The Continuum Hypothesis states that $2^\omega = \omega_1$. The Generalized Continuum Hypothesis states that $\forall\alpha(2^{\omega_\alpha} = \omega_{\alpha+1})$.*

We recall that the purpose of this work is to prove the consistency of the negation of CH with the ZFC axiom system. We will obtain this result using *models* of ZFC, that we present and study in the next chapter.

Chapter 2

Models of Set Theory

2.1 Relativization

We will now establish a way to restrict the study of formulas to a class. This will be helpful to state some consistency results.

Definition 2.1. *Let* $\mathbf{M}$ *be a class. We define for any formula* ϕ *the relativization of* ϕ *to* $\mathbf{M}$*, in symbols* $\phi^{\mathbf{M}}$*, by recursion as follows:*

1. $(x = y)^{\mathbf{M}}$ *is* $x = y$.
2. $(x \in y)^{\mathbf{M}}$ *is* $x \in y$.
3. $(\phi \wedge \psi)^{\mathbf{M}}$ *is* $\phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$.
4. $(\neg\phi)^{\mathbf{M}}$ *is* $\neg(\phi^{\mathbf{M}})$.
5. $(\exists x\ \phi)^{\mathbf{M}}$ *is* $\exists x(x \in \mathbf{M} \wedge \phi^{\mathbf{M}})$.

Lemma 2.1. *For any class* $\mathbf{M}$ *and formulas* ϕ *and* ψ*:*
a) $(\phi \vee \psi)^{\mathbf{M}}$ *is* $\phi^{\mathbf{M}} \vee \psi^{\mathbf{M}}$.
b) $(\forall x\ \phi)^{\mathbf{M}}$ *is* $\forall x(x \in \mathbf{M} \to \phi^{\mathbf{M}})$.

Proof. a) $(\phi \vee \psi)^{\mathbf{M}} \equiv (\neg(\neg\phi \wedge \neg\psi))^{\mathbf{M}}$. By Definition 2.1, this is $\neg(\neg\phi \wedge \neg\psi)^{\mathbf{M}}$ which is $\neg(\neg(\phi^{\mathbf{M}}) \wedge \neg(\psi^{\mathbf{M}})) \equiv (\phi^{\mathbf{M}} \vee \psi^{\mathbf{M}})$.
b) $(\forall x\ \phi)^{\mathbf{M}} \equiv (\neg\exists x(\neg\phi)^{\mathbf{M}})$. By Definition 2.1, this is $\neg(\exists x \neg\phi)^{\mathbf{M}}$ which is $\neg\exists x(x \in \mathbf{M} \wedge \neg\phi^{\mathbf{M}}) \equiv \forall x(\neg x \in \mathbf{M} \vee \phi^{\mathbf{M}}) \equiv \forall x(x \in \mathbf{M} \to \phi^{\mathbf{M}})$. □

Intuitively, we think of a *model* as a class where a list of sentences is satisfied. We formally define this concept by means of the relativization of those formulas to the class.

Definition 2.2. *Let* $\mathbf{M}$ *be a class.*
a) Let ϕ *be a sentence. If* $\phi^{\mathbf{M}}$ *holds, we say that* ϕ *is true in* $\mathbf{M}$.
b) Let S *be a set of sentences. We say that* $\mathbf{M}$ *is a model of* S *if and only if each sentence of* S *is true in* $\mathbf{M}$.

The next Theorem presents an important result about relative consistency based on the notion of model that we will use for the proof of the consistency of $\neg CH$ with ZFC, which is the aim of this work. A set of sentences S is consistent, in symbols $Con(S)$, if and only if it does not lead to any contradiction.

Theorem 2.1. *Let* S *and* T *be two sets of sentences. If for some class* $\mathbf{M}$ *we can prove from* T *that* $\mathbf{M} \neq 0$ *and* $\mathbf{M}$ *is a model of* S*, then* $Con(T) \to Con(S)$.

Proof. If S were not consistent, there would be a sentence ϕ for which we could prove $\phi \wedge \neg\phi$ from S. Since we can prove from T that any sentence of S is true in $\mathbf{M}$, then $\phi^{\mathbf{M}} \wedge \neg\phi^{\mathbf{M}}$ which is a contradiction from T. Hence, T is not consistent. □

We will use this Theorem with $T = ZFC$ and $S = ZFC + \neg CH$: we will assume the consistency of ZFC and, from a model of these axioms whose existence will be settled by the $Reflection\ Theorems$ as we will see later, we will construct another model of ZFC and $\neg CH$ by means of a technique called forcing.
For now, let's continue talking about the relativization of formulas. Basic formulas are easy to relativize to a class $\mathbf{M}$, but, if we use abbreviations and defined notions, we must first see how these notions are relativized to a class.
It is the case of the inclusion. Recall that $x \subset y$ formally means that $\forall z(z \in x \to z \in y)$. Then $(x \subset y)^{\mathbf{M}}$ means that $\forall z \in \mathbf{M}(z \in x \to z \in y)$, which is equivalent to $\forall z(z \in \mathbf{M} \wedge z \in y \to z \in x)$, which is finally equivalent to $y \cap \mathbf{M} \subset x$.

Next, we are going to present a specific class, WF, and see that it is a model of ZFC.

2.2 The Well-Founded Sets

As we know, the Universe $\mathbf{V} = \{x : x\ is\ a\ set\}$ is the class of all sets and is not a set. Our goal is to describe $\mathbf{V}$. To do so, we will define WF, the class of all well-founded sets, which are those that are constructed from the empty set following a recursive definition, and prove that $\mathbf{V} = WF$ by means of the Axiom of Foundation. This will allow us to restrict our reasonings to WF, leaving apart troublesome objects that could be considered as sets, as for example any x such that $x = \{x\}$, that the axiom of Foundation does not let be a set as it is not well-founded. We will then have a clearer picture of what $\mathbf{V}$ is.

Definition 2.3. *For all $a \in ON$, we define $V(\alpha)$ as follows:*

1. $V(0) = 0$.

2. $V(\alpha + 1) = \mathcal{P}(V(\alpha))$.

3. $V(\alpha) = \bigcup_{\beta<\alpha} V(\beta)$ *if α is a limit ordinal.*

Lemma 2.2. *a) If A is a transitive set, then $\mathcal{P}(A)$ is transitive.*
b) The union of transitive sets is transitive.

Proof. a) Let $a \in \mathcal{P}(A)$. Then $a \subset A$. If $b \in a$, then $b \in A$ and $b \subset A$ by the transitivity of A. Thus, $b \in \mathcal{P}(A)$. Hence, $a \subset \mathcal{P}(A)$.
b) Let $\{A_i\}_{i \in I}$ be a family of transitive sets. Let $A = \bigcup\{A_i : i \in I\}$. If $a \in A$, then $a \in A_{i_0}$ for some $i_0 \in I$. Thus $a \subset A_{i_0}$ as A_{i_0} is transitive, and so $a \subset A$. □

Proposition 2.1. *For each ordinal α, $V(\alpha)$ is transitive and $\forall \beta \leq \alpha(V(\beta) \subset V(\alpha))$.*

Proof. For $\alpha = 0$: $V(0) = 0$ clearly is transitive.
If $\alpha = \beta + 1$ is a successor ordinal, and we suppose that the proposition holds for all $\gamma \leq \beta$, then, by Lemma 2.2 a), $V(\alpha) = \mathcal{P}(V(\beta))$ is transitive since $V(\beta)$ is transitive . Now, $V(\beta) \subset V(\alpha) = \mathcal{P}(V(\beta))$ so, for $\gamma \leq \alpha$, if $\gamma = \alpha$ or $\gamma = \beta$ it is clear that $V(\gamma) \subset V(\alpha)$, and if $\gamma < \beta$ then $V(\gamma) \subset V(\beta) \subset V(\alpha)$.
If α is a limit ordinal and we suppose that the proposition holds for all $\beta < \alpha$, by Lemma 2.2 b) $V(\alpha)$ is transitive. The second part is obvious. □

Now that we have defined those $V(\alpha)$ and showed their basic properties, let's see how WF is defined from them.

Definition 2.4. *A set is well-founded if and only if it is in some $V(\alpha)$ for an ordinal α. $WF = \bigcup\{V(\alpha) : \alpha \in ON\}$ is the class of the well-founded sets.*

Definition 2.5. *For $A \in WF$, $rank(A) = min\{\alpha \in \mathbf{ON} : A \in V(\alpha + 1)\}$.*

We can redefine $V(\alpha)$ by means of $rank(A)$.

Proposition 2.2. *For any ordinal α, $V(\alpha) = \{A \in WF : rank(A) < \alpha\}$.*

Proof. If $A \in WF$, then $rank(A) < \alpha$ if and only if $rank(A) + 1 \leq \alpha$ if and only if $A \in V(rank(A) + 1) \subset V(\alpha)$ by Proposition 2.1. Hence, we have just showed that $A \in V(\alpha)$ if and only if $rank(A) < \alpha$, and so we obtain the desired equality. □

The next proposition says that WF is transitive with increasing rank.

Proposition 2.3. *Let $A \in WF$. Then, for all $B \in A$ we have that $B \in WF$ and $rank(B) < rank(A)$.*

Proof. Note that $A \in V(rank(A)+1) = \mathcal{P}(V(rank(A)))$ and so, if $B \in A$ then $B \in V(rank(A))$. Thus, $B \in WF$ and $rank(B) < rank(A)$ by Proposition 2.2. □

This proposition shows that the objects considered at the beginning of this section, like some x such that $x = \{x\}$, are not well-founded, since we would have that $rank(x) < rank(x)$.

Proposition 2.4. *For any set $A \in WF$, $A \subset V(rank(A))$.*

Proof. If $A \in WF$, then if $B \in A$, by Proposition 2.3, $rank(B) < rank(A)$ and so $V(rank(B)) \subset V(rank(A))$ by Proposition 2.1. Then, as $B \in V(rank(B)+1)$, $B \in V(rank(A))$. □

Now we will see a way to compute ranks more easily.

Lemma 2.3. *If $A \in WF$, then $rank(A) = sup\{rank(B)+1 : B \in A\}$.*

Proof. Let $\alpha = sup\{rank(B)+1 : B \in A\}$. By Proposition 2.3, $\forall B \in A(rank(B) < rank(A))$ and so $\alpha \leq rank(A)$. But also if $B \in A$, then $rank(B) < \alpha$. So $B \in V(\alpha)$ and thus $A \subset V(\alpha)$, which means that $A \in V(\alpha+1)$. Hence, $rank(A) \leq \alpha$ by Proposition 2.2. □

Proposition 2.5. *For any set A, $A \in WF$ if and only if $A \subset WF$.*

Proof. By Proposition 2.3, if $A \in WF$ then for any $B \in A$ we have that $B \in WF$, and so $A \subset WF$. On the other hand, suppose that $A \subset WF$. Then, if $B \in A$, we have that $B \in WF$. Thus, we can define $\alpha = sup\{rank(B)+1 : B \in A\}$. As $rank(B) < \alpha$, then, by Proposition 2.2, $B \in V(\alpha)$. Then $A \subset V(\alpha)$, so $A \in V(\alpha+1)$ by Proposition 2.2. □

The two next propositions will show that ordinals are well-founded and WF is closed under the standard set theory constructions (the power set of a well-founded set is well-founded, the union of two well-founded sets is well-founded, etc).

Proposition 2.6. *If α is an ordinal, then $\alpha \in WF$ and $rank(\alpha) = \alpha$.*

Proof. For $\alpha = 0$: $0 = V(0) \in WF$ and $rank(0) = 0$ since $0 \notin V(0)$ and $0 \in V(1) = \{0\}$.
If α is a successor or a limit and we assume the proposition holds for every $\beta < \alpha$ then, if $\beta \in \alpha$, by inductive hypothesis and by Propositions 2.2 and 2.1 we have that $\beta \in V(\beta+1) \subset V(\alpha)$ and so $\beta \in V(\alpha)$. This leads to $\alpha \subset V(\alpha)$ and, thus, $\alpha \in V(\alpha+1)$. Hence, $\alpha \in WF$. Now, by Lemma 2.3, $rank(\alpha) = sup\{rank(\beta)+1 : \beta \in \alpha\} = \alpha$. □

Proposition 2.7. *Let A, $B \in WF$. Then:*
a) $\{A\}$, $\mathcal{P}(A)$ and $\bigcup A \in WF$.
b) $\{A, B\}$, (A, B), $A \cup B$, $A \cap B$, $A \times B$ and ${}^{B}A \in WF$.

Proof. a) If $A \in WF$ then $\{A\} \subset WF$ and so $\{A\} \in WF$ by Proposition 2.5. Analogous for the other sets.
b) We assume that $A, B \in WF$. Let $C \in A \cup B$. If $C \in A$ then $C \in WF$ by Proposition 2.5. Similarly, if $C \in B$ then $C \in WF$. Hence, $A \cup B \subset WF$ and so $A \cup B \in WF$ by Proposition 2.5. We proceed in an analogous way for the other sets. □

2.3 Well-Founded Relations

In this section, we will prove that $V = WF$ following the concept of well-founded relation.

Definition 2.6. *Let A be a set and R a relation on A. We say that R is well-founded on A if and only if $\forall B \subset A(B \neq 0 \rightarrow \exists b_0 \in B(\neg \exists b \in B(b\ R\ b_0)))$. Moreover, we say that b_0 is an R-minimal element in B.*

Lemma 2.4. *a) If $A \in WF$ then $\in$ is well-founded on A.*
b) If A is a transitive set and $\in$ is well-founded on A, then $A \in WF$.

Proof. a) Let $0 \neq B \subset A$. Let $b \in B$ be an element of B with the least rank. Then b is $\in$-minimal in B. If it was not, there would be a $b' \in B$ such that $b' \in b$ and so $rank(b') < rank(b)$, which is a contradiction with the minimality of the rank of b in B.

b) Suppose that $A \notin WF$. Then, by Proposition 2.5, $A \not\subset WF$ which implies that $A_0 = \{a \in A : a \notin WF\} \subset A$ is non-empty. Obviously, $A_0 \cap WF = \emptyset$. As $\in$ is well-founded on A, A_0 has an $\in$-minimal element, B. For any $b \in B$, $b \in A$ because A is transitive, but $b \notin A_0$ because B is $\in$-minimal in A_0, and thus $b \in WF$. Hence, $B \subset WF$ and, by Proposition 2.5, $B \in WF$, which is a contradiction with the fact that $B \in A_0$. □

The preceding proposition shows the necessity to work with transitive sets, as the fact that $\in$ is well-founded on A implies that A is well-founded needs A to be transitive. For that reason, we will define the transitive closure of A.

Definition 2.7. *For a set A , we define by recursion on $n \in \omega$: $\bigcup^0 A = A$ and $\bigcup^{n+1} A = \bigcup(\bigcup^n A)$. We now define A^+, the transitive closure of A, as the union of these unions: $A^+ = \bigcup\{\bigcup^n A : n \in \omega\}$.*

From now on, A^+ denotes the transitive closure of A (and not the next cardinal as in Definition 1.11). The two following propositions describe A^+ stating some of its properties. In particular, the next one shows that A^+ is the least transitive set containing all the elements of A.

Proposition 2.8. *For any set A:*
a) $A \subset A^+$.
b) A^+ is transitive.
c) If T is a transitive set such that $A \subset T$, then $A^+ \subset T$.

Proof. a) $A = \bigcup^0 A \subset A^+$.
b) Let $a \in A^+$. Then $a \in \bigcup^n A$ for some $n \in \omega$ and $a \subset \bigcup A^{n+1} \subset A^+$.
c) We will prove it by induction. Obviously, $\bigcup^0 A = A \subset T$. We assume now that $\bigcup^k A \subset T$ for all $k < n$. Then $\bigcup^n A = \bigcup\{\bigcup^{n-1} A\} \subset \bigcup T$. But for any $a \in \bigcup T$, there is a $T_a \in T$ such that $a \in T_a$. But, as T is transitive, $T_a \subset T$ and so $a \in T$. Hence, $\bigcup^n A \subset T$. We can now conclude that $A^+ \subset T$. □

Proposition 2.9. *For any set A:*
a) If A is transitive, then $A^+ = A$.
b) If $B \in A$, then $B^+ \subset A^+$.
c) $A^+ = A \cup \bigcup\{B^+ : B \in A\}$.

Proof. a) Clearly, $A \subset A^+$. As A is transitive, taking A as the T of the Proposition 2.8 c), $A^+ \subset A$.
b) As $A \subset A^+$, if $B \in A$ then $B \in A^+$ and so $B \subset A^+$ because A^+ is transitive. Taking A^+ as the T of Proposition 2.8 c), $B^+ \subset A^+$.
c) Let $T = A \cup \bigcup\{B^+ : B \in A\}$. First, let's show that T is transitive. Let $a \in T$. If $a \in A$, then $a^+ \subset T$ and so $a \subset T$. If $a \in B^+$ for some $B \in A$, then $a \subset B^+ \subset T$ because B^+ is transitive. Thus, T is transitive. By Proposition 2.8 c), $A^+ \subset T$. On the other hand, let $a \in T$. If $a \in A$, then, as $A \subset A^+$, $a \in A^+$. If $a \in B^+$ for some $B \in A$, then, by b) $a \in A$ and so $a \in A^+$. Hence, $T \subset A^+$. □

As A^+ is transitive, by Lemma 2.4 we have that:

Corollary 2.1. *For any set A, $A^+ \in WF$ if and only if $\in$ is well-founded on A^+.* □

Theorem 2.2. *For any set A, $A \in WF$ if and only if $A^+ \in WF$.*

Proof. If $A \in WF$, then $\bigcup^0 = A \in WF$ and, assuming $\bigcup^n A \in WF$ for an $n \in \omega$, we also have that $\bigcup^{n+1} A = \bigcup\{\bigcup^n A\} \in WF$ by Proposition 2.7 and so $A^+ \in WF$. Conversely, if $A^+ \in WF$, we have that $A \subset A^+ \subset WF$ and thus $A \in WF$ by Proposition 2.5. □

Theorem 2.3. $\mathbf{V} = WF$.

Proof. Trivially, $WF \subset \mathbf{V}$. Now, by the preceding proposition, if $A \in V$ then $\in$ is well-founded on A by the Axiom of Foundation, and so it is on A^+. Thus, $A^+ \in WF$ by Corollary 2.1 and it follows from Theorem 2.2 that $A \in WF$. Hence, $V \subset WF$. □

We just showed that the Universe, in fact the union of the $V(\alpha)$'s, is constructed iteratively from 0. Now that we have a clearer picture of it, let's return to work with general classes with the concept of an $absolut\ formula$ as it is an important notion on model theory that will help us to show that WF, and hence $\mathbf{V}$, is a model of ZFC.

2.4 Absoluteness

We say that a formula ϕ is $absolute$ for two classes $\mathbf{M}$ and $\mathbf{N}$ where $\mathbf{M} \subset \mathbf{N}$ if ϕ is true in $\mathbf{M}$ whenever ϕ is true in $\mathbf{N}$ and vice-versa.

Definition 2.8. *Let ϕ be a formula with all free variables among $x_1, \ldots, x_n$. For any classes $\mathbf{M}, \mathbf{N}$ with $\mathbf{M} \subset \mathbf{N}$, we say that*
a) ϕ is absolute for $\mathbf{M}, \mathbf{N}$ if and only if

$$\forall x_1, \ldots, x_n \in \mathbf{M}(\phi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \phi^{\mathbf{N}}(x_1, \ldots, x_n)).$$

b) ϕ is absolute for $\mathbf{M}$ if and only if ϕ is absolute for $\mathbf{M}, \mathbf{V}$, which means that

$$\forall x_1, \ldots, x_n \in \mathbf{M}(\phi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \phi(x_1, \ldots, x_n)).$$

If $\mathbf{M}$ and $\mathbf{N}$ are clear from context, we will simply say that a formula is absolute. Next, we will prove that some general formulas and notions are absolute for some sort of classes. To do this, after seeing preliminary results, we will introduce the concept of Δ_0 formulas, that, intuitively, are formulas for which their quantifiers are all *bounded* existential ones, of the form $\exists x \in y...$.

Lemma 2.5. *If ϕ has no quantifiers, then ϕ is absolute for any $\mathbf{M}, \mathbf{N}$ with $\mathbf{M} \subset \mathbf{N}$.*

Proof. First, $(x = y)^{\mathbf{M}} \equiv x = y \equiv (x = y)^{\mathbf{N}}$ and $(x \in y)^{\mathbf{M}} \equiv x \in y \equiv (x \in y)^{\mathbf{N}}$, so $x = y$ and $x \in y$ are absolute for $\mathbf{M}, \mathbf{N}$. Now, we assume that ϕ has no quantifiers and is absolute for $\mathbf{M}, \mathbf{N}$. We know that $(\neg\phi)^{\mathbf{M}} \equiv \neg\phi^{\mathbf{M}}$ and, by the inductive hypothesis, $\phi^{\mathbf{M}} \leftrightarrow \phi^{\mathbf{N}}$. Thus, $(\neg\phi^{\mathbf{M}} \leftrightarrow \neg\phi^{\mathbf{N}})$.
If we assume that ϕ and ψ have no quantifiers and are absolute for $\mathbf{M}, \mathbf{N}$, as $(\phi \wedge \psi)^{\mathbf{M}} \equiv \phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$ and since $\phi^{\mathbf{M}} \leftrightarrow \phi^{\mathbf{N}}$ and $\psi^{\mathbf{M}} \leftrightarrow \psi^{\mathbf{N}}$ by the inductive hypothesis, then $(\phi \wedge \psi)^{\mathbf{M}} \leftrightarrow \phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}} \leftrightarrow \phi^{\mathbf{N}} \wedge \psi^{\mathbf{N}} \leftrightarrow (\phi \wedge \psi)^{\mathbf{N}}$.
Since the logical operators $\neg$ and $\wedge$ build any quantifier-free formula from the atomic ones $x = y$ and $x \in y$, any formula of this kind is absolute for $\mathbf{M}, \mathbf{N}$. □

It is also clear that if ϕ is the negation of an absolute formula or the conjunction of two absolute formulas, then ϕ is also absolute. Next, we are going to see that for transitive classes a formula of the form $\exists x \in y\ \phi$ with ϕ being an absolute formula is also absolute. Note that $\exists x \in y\ \phi$ is equivalent to $\exists x(x \in y \wedge \phi)$.

Lemma 2.6. *Let* $\mathbf{M}, \mathbf{N}$ *be transitive classes with* $\mathbf{M} \subset \mathbf{N}$ *and* ϕ *an absolute formula for* $\mathbf{M}, \mathbf{N}$. *Then,* $\exists x(x \in y \wedge \phi)$ *is absolute for* $\mathbf{M}, \mathbf{N}$.

Proof. Without loss of generality, we consider $\phi = \phi(x, y, w_1, \ldots, w_n)$ with all its free variables listed. Then, for any $y, w_1, \ldots, w_n \in \mathbf{M}$,

$$(\exists x(x \in y \wedge \phi(y, w_1, \ldots, w_n))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}(x \in y \wedge \phi^{\mathbf{M}}(y, w_1, \ldots, w_n)) \leftrightarrow$$
$$\exists x \in \mathbf{N}(x \in y \wedge \phi^{\mathbf{N}}(y, w_1, \ldots, w_n)) \leftrightarrow (\exists x(x \in y \wedge \phi(y, w_1, \ldots, w_n))^{\mathbf{N}}.$$

The first and the last equivalences come from the transitivity of $\mathbf{M}$ and $\mathbf{N}$ (for the second formula we should have written $\exists x(x \in \mathbf{M} \wedge x \in y\ ...)$, but, as $y \in \mathbf{M}$, if $x \in y$ then directly $x \in \mathbf{M}$; the same holds for $\mathbf{N}$). The middle one follows from the absoluteness of ϕ and the fact that $M \subset N$. □

Definition 2.9. *We define the* Δ_0 *formulas as follows:*
a) $x \in y$ *and* $x = y$ *are* Δ_0.
b) If ϕ, ψ *are* Δ_0, *then* $\neg\phi$, $\phi \wedge \psi$ *and* $\exists x(x \in y \wedge \phi)$ *are also* Δ_0.

Proposition 2.10. *If* $\mathbf{M}, \mathbf{N}$ *are transitive classes with* $\mathbf{M} \subset \mathbf{N}$ *and* ϕ *is* Δ_0, *then* ϕ *is absolute for* $\mathbf{M}, \mathbf{N}$.

Proof. If ϕ is quantifier-free, by Lemma 2.5 ϕ is absolute for $\mathbf{M}, \mathbf{N}$. It is also clear that if ϕ and ψ are absolute then $\neg\phi$ and $\phi \wedge \psi$ is absolute. If $\phi = \exists x(x \in y \wedge \psi)$, then by Lemma 2.6 ϕ is absolute for $\mathbf{M}, \mathbf{N}$. □

Observe that Proposition 2.10 is true if ϕ is equivalent to a Δ_0 formula. In particular, if the quantifiers appearing in ϕ are all *bounded*, then $Qx \in y$ for $Q \in \{\exists, \forall\}$, ϕ is equivalent to a Δ_0 formula and is absolute for any transitives $\mathbf{M}, \mathbf{N}$ with $\mathbf{M} \subset \mathbf{N}$. Note that, if S is any set of sentences and ϕ is a formula, $S \vdash \phi$ means that we can deduce, prove, ϕ from S.

Proposition 2.11. *Let* $\mathbf{M}, \mathbf{N}$ *with* $\mathbf{M} \subset \mathbf{N}$ *be models of a set of sentences* S *such that* $S \vdash \forall x_1, \ldots, x_n(\phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n))$. *Then* ϕ *is absolute for* $\mathbf{M}, \mathbf{N}$ *if and only if* ψ *is.*

Proof. For any $x_1, \ldots, x_n$, if we suppose that ψ is absolute for $\mathbf{M}, \mathbf{N}$, then

$$\phi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \ldots, x_n) \leftrightarrow \phi^{\mathbf{N}}(x_1, \ldots, x_n).$$

The first and the last equivalences follow from the hypothesis. The middle one follows from the absoluteness of ψ for $\mathbf{M}, \mathbf{N}$. We proceed in an analogous way for the other implication. □

So if a formula is absolute, any equivalent formula is also absolute. We will use this result to see a criterion to discuss the absoluteness of defined notions.

Definition 2.10. *Let* $\mathbf{M}, \mathbf{N}$ *be classes with* $\mathbf{M} \subset \mathbf{N}$. *Let* $F(x_1, \ldots, x_n)$ *be a defined function. Then* F *is absolute for* $\mathbf{M}, \mathbf{N}$ *if and only if the formula* $F(x_1, \ldots, x_n) = y$ *is.*

Proposition 2.12. *Let* F *be defined as* $F(x_1, \ldots, x_n) = y \leftrightarrow \phi(x_1, \ldots, x_n, y)$. *Then* F *is absolute for* $\mathbf{M}, \mathbf{N}$ *if and only if* ϕ *is.*

Proof. Taking $F(x_1, \ldots, x_n) = y$ as ψ and ϕ as ϕ in Proposition 2.11, the result is direct. □

We have stated some ways to verify that a formula is absolute for some classes $\mathbf{M}$. We will use these results to prove that basic set notions are absolute for transitive models of $ZF^- - P - Inf$, that is ZF without the Foundation, Power Set and Infinity Axioms. Note that these absoluteness results are true if $\mathbf{M}$ is transitive and satisfy more axioms than those ones.

Theorem 2.4. *The following relations and functions are absolute for any transitive class* $\mathbf{M}$ *such that* $\mathbf{M}$ *is a model of* $ZF^- - P - Inf$.

1. $x \in y$,
2. $x = y$,
3. $x \subset y$,
4. $\{x\}$,
5. $\{x, y\}$,
6. (x, y),
7. 0,
8. $x \cup y$,
9. $x \cap y$,
10. $x \setminus y$,
11. $S(x)$ $(= x \cup \{x\})$,
12. x *is transitive*,
13. $\bigcup x$,
14. $\bigcap x$ $(\bigcap 0 = 0)$.

Proof. 1. and 2. were proved in Lemma 2.5. For 3., as discussed above $(x \subset y)^{\mathbf{M}}$ is $x \cap \mathbf{M} \subset y$. But, as $\mathbf{M}$ is transitive, $x \cap \mathbf{M} = x$ and so $(x \subset y)^{\mathbf{M}} \leftrightarrow x \subset y$. For the other defined functions and relations, we will use Proposition 2.12 with the definition of each function, showing that the formulas that define them have all their quantifiers bounded and are thus equivalent to some Δ_0 formulas.

For example, for 4., $\{x\}$ is defined as $y = \{x\} \leftrightarrow (x \in y \wedge \forall z(z \in y \rightarrow z = x))$. The formula on the right of the double implication connector is equivalent to a Δ_0 formula since it is made of the conjunction of an atomic formula with a formula for which the only quantifier is bounded. Thus, by Proposition 2.10, it is absolute for $\mathbf{M}$..

For 12., *x is transitive* $\leftrightarrow (\forall y \in x\ \forall z \in y (z \in x))$.

The formula at the right of the equivalence connector is equivalent to a Δ_0 formula since all its connectors are bounded and so, by Proposition 2.10.

For 13., $y = \bigcup x \leftrightarrow (\forall z \in x(z \subset y) \wedge \forall u \in y\ \exists z \in x(u \in z))$. Since, by clause 3., $y \subset z$ is absolute and $\forall z \in x(z \subset y)$ is equivalent to a Δ_0 formula because the quantifier appearing in it is bounded, by Proposition 2.10 this formula is absolute for $\mathbf{M}$. Moreover, as all the quantifiers appearing in the formula $\forall u \in y\ \exists z \in x(u \in z))$ are bounded, this formula is equivalent to a Δ_0 one and thus is absolute for $\mathbf{M}$. We can conclude that the formula that defines $\bigcup x$ is absolute for $\mathbf{M}$ because it is the conjunction of two absolute formulas, and hence $\bigcup x$ is absolute for $\mathbf{M}$.

We proceed on an analogous way for the rest of conditions. □

We will now show that absolute notions are closed under composition. We can use this result to produce new absoluteness results about functions and relations.

Proposition 2.13. *Let* $\mathbf{M} \subset \mathbf{N}$. *If* $\phi(x_1, \ldots, x_n)$, $F(x_1, \ldots, x_n)$, $G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m)$ *are all absolute for* $\mathbf{M}, \mathbf{N}$, *then* $\phi(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m))$ *and* $F(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m))$ *are also absolute for* $\mathbf{M}, \mathbf{N}$.

Proof. If $x_1, \ldots, x_n$ are in $\mathbf{M}$, then

$$\begin{gathered}(\phi(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m)))^{\mathbf{M}} \leftrightarrow \\ \phi^{\mathbf{M}}(G_1^{\mathbf{M}}(x_1, \ldots, x_m), \ldots, G_n^{\mathbf{M}}(x_1, \ldots, x_m)) \leftrightarrow \\ \phi^{\mathbf{N}}(G_1^{\mathbf{N}}(x_1, \ldots, x_m), \ldots, G_n^{\mathbf{N}}(x_1, \ldots, x_m)) \leftrightarrow \\ (\phi(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m)))^{\mathbf{N}}.\end{gathered}$$

The first and last equivalences are by definition. The one in the middle follows from the hypothesis that ϕ and G_i for $i = 1, \ldots, n$ are absolute.
Taking ϕ as $y = F(G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m))$, the second part of the proposition follows from the first one. □

Theorem 2.5. *The following relations and functions are absolute for any transitive* $\mathbf{M}$ *such that* $\mathbf{M}$ *is a model of* $ZF^- - P - Inf$.

1. *z is an ordered pair*
2. $A \times B$
3. *R is a relation*
4. $dom(R)$
5. $ran(R)$
6. *F is a function*
7. $F(x)$
8. *F is an injective function.*

Proof. The procedure will always be the same: write the definition using formulas of each notion and see that the formula is a composition of absolute functions and/or absolute relations.
For 1., z *is an ordered pair* $\leftrightarrow (\exists x \in \bigcup z\ \exists y \in \bigcup z(z = (x, y)))$. By Theorem 2.4, $\bigcup z$ and (x, y) are absolute for **M**. Now, the formula at the right of the double implication is $\phi(\bigcup z, \bigcup z, z)$ with $\phi(x_1, x_2, x_3) = \exists x \in x_1\ \exists y \in x_2(x_3 = (x, y))$, which is a Δ_0 formula since the only quantifiers that appear in it are existential bounded ones, and is thus absolute by Proposition 2.10, and the formula $x_3 = (x, y)$ is absolute. Then, $\phi(\bigcup z, \bigcup z, z)$ is absolute by Proposition 2.13..

For 3., let $\phi(z)$ be the formula z *is an ordered pair*. Then $\phi(z)$ is absolute by 1. Moreover, R *is a relation* $\leftrightarrow \forall z \in R\ \phi(z)$, which is equivalent to a Δ_0 formula since its quantifier is bounded and thus, by Proposition 2.10, is absolute.
We proceed in an analogous way for the rest of conditions. □

Now we will see some absoluteness results with the condition of **M** being a model of $ZF - P$. Note that **M** can be a model of a larger set of axioms, as for example ZFC.

Lemma 2.7. *In $ZF - P$, α is an ordinal if and only if α is transitive and totally ordered by $\in$.*

Proof. Assume that α is transitive and totally ordered by $\in$. We have to show that $\in$ well-orders α. Let X be a non-empty subset of α. By the axiom of Foundation, there is an $\in$-minimal element of X and so, as $\in$ totally orders α, $\in$ well-orders α. □

Theorem 2.6. *The following relations and functions are absolute for any transitive model* **M** *of $ZF - P$.*

1. *α is an ordinal,*
2. *α is a successor ordinal,*
3. *α is a limit ordinal,*
4. *α is a finite ordinal,*
5. *ω,*
6. *Any natural number.*

Proof. The formula that defines any of these relations and formulas is equivalent to a Δ_0 formula. For the first one, by Lemma 2.7, we have to see that α *is transitive* and $\in$ *totally orders* α are absolute for **M**. But, by Theorem 2.4, α *is transitive* is absolute for **M** as **M** is transitive and a model in particular of $ZF^- - P - Inf$. Now, $\in$ *totally orders* $\alpha \leftrightarrow \forall \beta \in \alpha\ \forall \gamma \in \alpha(\beta \in \gamma \vee \gamma \in \beta \vee \beta = \gamma)$. As the formula at the right of the double implication connector is a Δ_0 formula since its quantifiers are all bounded, by Proposition 2.10 it is absolute for $\mathbf{M}$. Hence, by Proposition 2.11, $\in$ *totally orders* α is absolute for $\mathbf{M}$.

For 3., x *is a limit ordinal* $\leftrightarrow$ x *is an ordinal* $\wedge\, x \neq 0 \wedge \forall y \in x\ \exists z \in x(y \in z)$. The formula at the right of the equivalence operator is a conjunction of two absolute formulas, and is then absolute. Indeed, x *is ordinal* is absolute and $\forall y \in x\ \exists z \in x(y \in z)$ is equivalent to a Δ_0 formula and so, by Proposition 2.10, is absolute.

For 5., $x = \omega \leftrightarrow x$ *is a limit ordinal* $\wedge\, \forall y \in x(y$ *is not a limit ordinal*$)$. The formula at the right of the equivalence operator is a conjunction of two absolute formulas, and is then absolute. Indeed, as the formula x *is a limit ordinal* is absolute, and so it is its negation, and $\forall y \in x(y$ *is not a limit ordinal*$)$ is equivalent to a Δ_0 formula (since its only quantifier is bounded), by Proposition 2.10, this formula is absolute.
We proceed in an analogous way for the rest of conditions. □

Proposition 2.14. *If* $\mathbf{M}$ *is a transitive model of* $ZF - P$*, then every finite subset of* $\mathbf{M}$ *is an element of* $\mathbf{M}$*.*

Proof. Formally, we have to prove that $\forall x \subset \mathbf{M}(|x| = n \rightarrow x \in \mathbf{M})$. We will prove it by induction on n.
If $n = 0$, $x = 0$ and $x \in \mathbf{M}$ since $\mathbf{M} \in WF$.
Now, we assume that the proposition holds for n. Let $x \subset \mathbf{M}$ have cardinality $n+1$. Since x is non-empty, let $y \in x$. As $\mathbf{M}$ is transitive, $y \in \mathbf{M}$ and $x \setminus \{y\} \subset \mathbf{M}$ with cardinality n. By the inductive hypothesis, $x \setminus \{y\} \in \mathbf{M}$ and $\{y\} \in \mathbf{M}$. By Theorem 2.4, $\cup, \{y\}$ and $\setminus$ are absolute for $\mathbf{M}$ and hence $x = \{y\} \cup (x \setminus y) \in \mathbf{M}$. □

Theorem 2.7. *The following are absolute for any transitive model* $\mathbf{M}$ *of* $ZF - P$ *and* $A, R \in \mathbf{M}$*.*

1. *x is finite,*
2. *A^n,*
3. *$A^{<\omega}$ $(= \bigcup\{A^n : n \in \omega\})$,*
4. *R well-orders A,*
5. *$type(A, R)$.*

Proof. For 1., x is finite $\leftrightarrow \exists f \phi(x, f)$ where $\phi(x, f) = f$ *is a function* $\wedge\, dom(f) = x \wedge ran(f) \subsetneq \omega \wedge f$ *is injective*. By Theorem 2.4 and Theorem 2.5, $\phi(x, f)$ is absolute and so $(\exists f\ \phi(x, f))^{\mathbf{M}} \leftrightarrow \exists f \in \mathbf{M}\ \phi(x, f)$. We then have to see that

$\exists f \in \mathbf{M}\ \phi(x,f) \leftrightarrow \exists f \phi(x,f)$. The left to right implication is obvious. For the reciprocal one, note that, for any $x \in \mathbf{M}$, $f = \{(a,b) : a \in x \wedge b \in \omega\}$ because $\phi(x,f)$ holds. But since $\mathbf{M}$ is transitive and $x, \omega \in \mathbf{M}$, then $a, b \in \mathbf{M}$ and so f is a finite set of ordered pairs of elements of $\mathbf{M}$. Since, by Theorem 2.5 being an ordered pair is absolute, then by Proposition 2.14 $f \in \mathbf{M}$.

For 2., for each n, $A^n = F(A,n) = \{f : \phi(A,n,f)\}$ with $F(A,x) = 0$ if $x \notin \omega$ and $\phi(A,n,f) = f\ is\ a\ function \wedge dom(f) = n \wedge ran(f) \subset A$. Let $x \in \mathbf{M}$. If $x \notin \omega$, then $F^{\mathbf{M}}(A,x) = 0 = F(A,x)$. If $x = n \in \omega$, then $F^{\mathbf{M}}(A,n) = \{f \in \mathbf{M} : \phi^{\mathbf{M}}(A,n,f)\}$. By Theorem 2.4 and Theorem 2.5, $\phi(A,n,f)$ is absolute and so $F^{\mathbf{M}}(A,n,f) = \{f \in \mathbf{M} : \phi(A,n,f)\}$. But, for $A \in \mathbf{M}$ and $n \in \omega$, as $\phi(A,n,f)$ holds and $\mathbf{M}$ is transitive, then $f \subset \mathbf{M}$ and so by Proposition 2.14 $F(A,n) = F^{\mathbf{M}}(A,n)$.

We proceed in an analogous way for 3.

For 4., we have to see that $(R\ well-orders\ A)^{\mathbf{M}} \leftrightarrow R\ well-orders\ A$.
For the right to left implication, we write $R\ well-orders\ A$ as $R\ totally\ orders\ A \wedge \forall X\ \phi(X,A,R)$, where $\phi(X,A,R) = X \subset A \wedge X \neq 0 \rightarrow \exists y \in X\ \forall z \in X((z,y) \notin R)$. In proof of the absoluteness of $x\ is\ an\ ordinal$ (see Theorem 2.6), we have seen that $\in totally\ orders\ A$ is absolute. Then, as $R\ is\ a\ relation$ is absolute, $R\ totally\ orders\ A$ is also absolute and so $(R\ totally\ orders\ A)^{\mathbf{M}}$. Additionally, as $\phi(X,A,R)$ is a composition of absolute formulas, by Proposition 2.13 $\phi(X,A,R)$ is absolute for $\mathbf{M}$ and so
$(\forall X\ \phi(X,A,R))^{\mathbf{M}} \leftrightarrow \forall X \in \mathbf{M}\ \phi(X,A,R) \leftrightarrow \forall X\ \phi(X,A,R)$ since $R\ well-orders\ A$.
On the other hand, to show the left to right implication, we use the fact that a well-ordering is isomorphic to an ordinal.
Supposing that $(R\ well-orders\ A)^{\mathbf{M}}$, there are $f, \alpha \in \mathbf{M}$ such that
$(\alpha\ is\ an\ ordinal\ \wedge f\ is\ an\ isomorphism\ from\ \langle A,R\rangle\ to\ \alpha)^{\mathbf{M}}$. By Theorem 2.6, $\alpha\ is\ an\ ordinal$ is absolute and
$f\ is\ an\ isomorphism \leftrightarrow f\ is\ injective \wedge f\ is\ surjective$. But f being surjective means that $ran(f) = \alpha$, which is clearly absolute. Since, from Theorem 2.5, $f\ is\ injective$ is absolute, then $f\ is\ an\ isomorphism$ is also absolute and thus α is in fact an ordinal and f an isomorphism.

We can prove 5. by means of an argument similar to the one given in 4.

□

Proposition 2.15. *Let α and β be ordinals. The ordinal operations $\alpha + \beta$ and $\alpha \cdot \beta$ are absolute for any transitive model* $\mathbf{M}$ *of* $ZF - P$.

Proof. Both are defined from *type* and are then absolute by Theorem 2.7. □

If we consider a relation R as the set of the pairs that satisfy a formula $R(x,y)$, $R = \{(x,y) : R(x,y)\}$, then for any class $\mathbf{M}$ the relativization of R to $\mathbf{M}$ is $R^{\mathbf{M}} = \{(x,y) \in \mathbf{M} \times \mathbf{M} : R(x,y)^{\mathbf{M}}\}$ and so R is absolute for $\mathbf{M}$ if and only if $R^{\mathbf{M}} = R \cap (\mathbf{M} \times \mathbf{M})$. If $F : \mathbf{V} \to \mathbf{V}$ is a function, the same holds (we can consider G to be $\{(x,y) : F(x,y)\}$, with $F(x,y)$ some function). Nonetheless, $F^{\mathbf{M}}$ being a function needs that $(\forall x\ \exists! y(F(x) = y))^{\mathbf{M}}$ or, with our notation, $(\forall x\ \exists! y\ F(x,y))^{\mathbf{M}}$. If this holds, then $F^{\mathbf{M}} : \mathbf{M} \to \mathbf{M}$ is absolute for $\mathbf{M}$ if and only if $F^{\mathbf{M}} = F \restriction \mathbf{M}$.

Definition 2.11. *Let R be a relation on a class $\mathbf{A}$. We say that R is a set-like relation on $\mathbf{A}$ if and only if, for all $x \in \mathbf{A}$, $\{y \in \mathbf{A} : yRx\}$ is a set.*

The following proposition states how to define a function by $transfinite\ recursion$. It is a generalization of the natural recursion. We put $pred(\mathbf{A}, x, R) = \{y \in \mathbf{A} : yRx\}$.

Proposition 2.16. *Let R be a well-founded and set-like relation on $\mathbf{A}$ and let $F : \mathbf{V} \to \mathbf{V}$ be a function. Then, there is a unique function $G : \mathbf{A} \to \mathbf{V}$ such that*
$\forall x \in \mathbf{A}(G(x) = F(x, G \restriction pred(\mathbf{A}, x, R)).$ □

Proposition 2.17. *Let R be a well-founded and set-like relation on $\mathbf{V}$ and $F : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ be a function. We take $pred(x) = pred(V, x, R)$. Let $G : \mathbf{V} \to \mathbf{V}$ be a function defined by transfinite recursion as $\forall x(G(x) = F(x, G \restriction pred(x)))$. Let $\mathbf{M}$ be a transitive model of $ZF - P$. If F is absolute for $\mathbf{M}$, then G is also absolute for $\mathbf{M}$.*

Proof. First, R is absolute means that $R^{\mathbf{M}} = R \cap (\mathbf{M} \times \mathbf{M})$, and so $(R\ is\ well - founded)^{\mathbf{M}}$ is $\forall X \in \mathbf{M}(X \neq 0 \to \exists y \in X \cap \mathbf{M}(\neg \exists z \in X \cap \mathbf{M}((z,y) \notin R \cap (\mathbf{M} \times \mathbf{M}))))$. But, since $\mathbf{M}$ is transitive, $X \cap \mathbf{M} = X$, and so $(R\ is\ well - founded)^{\mathbf{M}}$ is $\forall X \in \mathbf{M}(X \neq 0 \to \exists y \in X \cap \mathbf{M}(\neg \exists z \in X \cap \mathbf{M}((z,y) \notin R \cap (\mathbf{M} \times \mathbf{M}))))$, which holds since R is well-founded. Thus, we can define the relativization of G to $\mathbf{M}$ by transfinite induction as $(\forall x(G(x) = F(x, G \restriction pred(x))))^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}(G^{\mathbf{M}}(x) = F^{\mathbf{M}}(x, G^{\mathbf{M}} \restriction pred^{\mathbf{M}}(x)))$.
But $pred^{\mathbf{M}}(x) = pred(x)$ because $\mathbf{M}$ is transitive and, by the absoluteness of F, $F^{\mathbf{M}}(x, G^{\mathbf{M}} \restriction pred^{\mathbf{M}}(x)) = F(x, G^{\mathbf{M}} \restriction pred(x))$. Suppose that $G^{\mathbf{M}} \restriction pred(x) \neq G \restriction pred(x)$. Then $\{x \in \mathbf{M} : G^{\mathbf{M}}(x) \neq G(x)\}$ has a minimal element by the Axiom of Foundation. Suppose y is one of these minimal elements. Then if $z \in y$, $G^{\mathbf{M}}(z) = G(z)$ and so $G^{\mathbf{M}} \restriction pred(y) = G \restriction pred(y)$. By this last equality and by the absoluteness of F, $F^{\mathbf{M}}(y, G^{\mathbf{M}} \restriction pred^{\mathbf{M}}(y)) = F(y, G \restriction pred(y))$ and so $G^{\mathbf{M}}(y) = G(y)$, which is a contradiction with $y \in \{x \in \mathbf{M} : G^{\mathbf{M}}(x) \neq G(x)\}$. Hence, G is absolute for $\mathbf{M}$. □

In fact, Proposition 2.17 says that notions defined by transfinite recursion from absolute functions are also absolute for transitive models of $ZF - P$.

Proposition 2.18. *Let* **M** *be a transitive model of* $ZF - P$. *Then the following are absolute for* **M**:

1. α^β *for* α, β *ordinals,*

2. x^+ *(the transitive closure of* x*).*

Proof. The ordinal exponentiation is defined by transfinite recursion, and so α^β is absolute for **M** by Proposition 2.17.
$rank(x)$ is defined by transfinite recursion, and so it is absolute for **M** by Proposition 2.17.
x^+ is defined from the $\bigcup^n x$'s, which are defined by recursion on n as

$$\bigcup^n x = \begin{cases} 0, & \text{if } n \notin \omega, \\ x, & \text{if } n = 0, \\ \bigcup(\bigcup^{n-1} x) & \text{if } 0 \in n \in \omega. \end{cases}$$

By Proposition 2.17, $\bigcup^n x$ is absolute for **M**, and so is $x^+ = \bigcup\{\bigcup^n x : n \in \omega\}$. $\square$

Let R^+ be the relation on **V** defined by $(x, y) \in R^+$ if and only if $x \in y^+$. Then, it is easy to check that R^+ is well-founded and set-like, and so the following result is a direct corollary of Proposition 2.17.

Corollary 2.2. *Let* $F : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ *be a function. We take* $pred(x) = pred(V, x, R^+)$.
Let $G : \mathbf{V} \to \mathbf{V}$ *be a function defined by transfinite induction:*
$\forall x(G(x) = F(x, G \restriction pred(x)))$.
Let **M** *be a transitive model of* $ZF - P$. *If* F *is absolute for* **M**, *then so is* G.

Note that, if **M** is a transitive model of ZFC, then we can use all the absolute notions and concepts freely within **M**. This will be helpful while studying the forcing method.

2.5 Relativization of the ZFC Axioms

Now, we are going to state the axioms of ZFC relativized to any class **M** and see some properties that **M** may have to satisfy these axioms. These criteria will be useful to prove that the model of $ZFC + \neg CH$ we want to produce by forcing is indeed a model of ZFC.

Set Existence Axiom
Relativized to **M**, this axiom states that **M** is non-empty. It is an obvious axiom.

Axiom of Extensionality
Relativized to $\mathbf{M}$, it is $\forall x, y \in \mathbf{M}(\forall z \in \mathbf{M}(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

Proposition 2.19. *If $\mathbf{M}$ is transitive, then the Axiom of Extensionality is true in $\mathbf{M}$.*

Proof. $\mathbf{M}$ being transitive means that if $x \in \mathbf{M}$ then $\forall z \in x(z \in \mathbf{M})$. So the condition $\forall z \in \mathbf{M}(z \in x \leftrightarrow z \in y)$ is equivalent to $\forall z(z \in x \leftrightarrow z \in y)$ since all the elements of x and y are in $\mathbf{M}$. Then, by the Axiom of Extensionality (not relativized), $x = y$. □

Axiom of Comprehension
The relativization of this axiom to $\mathbf{M}$ for each formula $\phi(x, z, w_1, \ldots, w_n)$ with all its free variables listed is
$\forall z, w_1, \ldots, w_n \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall x \in \mathbf{M}(x \in y \leftrightarrow x \in z \land \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n))$.

Lemma 2.8. *Let $\phi(x, z, w_1, \ldots, w_n)$ be a formula with no free variables except the already listed ones.*
If $\forall z, w_1, \ldots, w_n \in \mathbf{M}(\{x \in z : \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{M})$ then the Axiom of Comprehension is true in $\mathbf{M}$ for $\phi(x, z, w_1, \ldots, w_n)$.

Proof. For any $z, w_1, \ldots, w_n \in \mathbf{M}$, we take $y = \{x \in z : \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\}$. As $y \in \mathbf{M}$, then we have that $\forall x(x \in y \leftrightarrow \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n))$ and, in particular, $\forall x \in \mathbf{M}(x \in y \leftrightarrow \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n))$. As $y \subset z$, it is equivalent to $\forall x \in \mathbf{M}(x \in y \leftrightarrow x \in z \land \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n))$. Finally, as this is true for any $z, w_1, \ldots, w_n$, we have just shown that $\forall z, w_1, \ldots, w_n \in \mathbf{M}(x \in y \leftrightarrow x \in z \land \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n))$. As $y \in \mathbf{M}$, the Comprehension Axiom is true in $\mathbf{M}$. □

Proposition 2.20. *If $\forall z \in \mathbf{M}(\mathcal{P}(z) \subset \mathbf{M})$, then the Comprehension Axiom Scheme is true in $\mathbf{M}$.*

Proof. Let $z, w_1, \ldots, w_n \in \mathbf{M}$ and $\phi(x, z, w_1, \ldots, w_n)$ be a formula with no free variables except the already listed ones. Let $y = \{x \in z : \phi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\}$. Obviously, $y \subset z$, and so $y \in \mathcal{P}(z) \subset \mathbf{M}$. Then, $y \in \mathbf{M}$ and, by Lemma 2.8, the Comprehension Axiom is true in $\mathbf{M}$ for ϕ. □

Power Set Axiom
The Power Set Axiom relativized to $\mathbf{M}$ states that $\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}(z \cap \mathbf{M} \subset x \rightarrow z \in y)$.

Proposition 2.21. *If $\mathbf{M}$ is transitive, the Power Set Axiom is true in $\mathbf{M}$ if and only if $\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}(\mathcal{P}(x) \cap \mathbf{M} \subset y)$.*

Proof. If $\mathbf{M}$ is transitive, then for any $x \in \mathbf{M}$ $x \cap \mathbf{M} = x$. And, clearly, for $x, y \in \mathbf{M}$, $(x \subset y)^{\mathbf{M}}$ is equivalent to $x \subset y$. This means that the relativization of the Power Set Axiom for any transitive class $\mathbf{M}$ is $\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\ \forall z \in \mathbf{M}(z \subset x \to z \in y)$. We can re-write the condition $\forall z \in \mathbf{M}(z \subset x \to z \in y)$ as $\forall z \in \mathbf{M}(z \in \mathcal{P}(x) \to z \in y)$, which is equivalent to $\forall z(z \in \mathbf{M} \wedge z \in \mathcal{P}(x) \to z \in y)$ which is $\mathcal{P}(x) \cap \mathbf{M} \subset y$. □

Pairing Axiom
The relativization of this axiom to $\mathbf{M}$ is $\forall z \in \mathbf{M}\ \forall y \in \mathbf{M}\ \exists z \in \mathbf{M}(x \in z \wedge y \in z)$.

Union Axiom
The relativization of this axiom to $\mathbf{M}$ is $\forall \mathcal{F} \in \mathbf{M}\ \exists A \in \mathbf{M}\ \forall Y \in \mathbf{M}\ \forall x \in \mathbf{M}(x \in Y \wedge Y \in \mathcal{F} \to x \in A)$.
The formula $\forall x \in \mathbf{M}(x \in Y \wedge Y \in \mathcal{F} \to x \in A)$ is quivalent in $\mathbf{M}$ to $\bigcup \mathcal{F} \subset A$, and so the following holds.

Proposition 2.22. *The Union Axiom is true in* $\mathbf{M}$ *if* $\forall \mathcal{F} \in \mathbf{M}\ \exists A \in \mathbf{M}(\bigcup \mathcal{F} \subset A)$. □

Replacement Axiom
For each formula $\phi(x, y, A, w_1, \ldots, w_n)$ without free variables except the ones already listed, the Replacement Axiom relativized to $\mathbf{M}$ is
$\forall A \in \mathbf{M}\ \forall w_1, \ldots, w_n \in \mathbf{M}(\forall x \in \mathbf{M} \cap A\ \exists! y \in \mathbf{M} \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to \exists Y \in \mathbf{M}\ \forall x \in \mathbf{M} \cap A\ \exists y \in \mathbf{M} \cap Y \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n))$.

Proposition 2.23. *Let* $\phi(x, y, A, w_1, \ldots, w_n)$ *be a formula without free variables except the ones already listed. If* $(\forall x \in A\ \exists! y \in \mathbf{M}\ \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)) \to (\exists Y \in \mathbf{M}(\{y : \exists x \in A\ \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subset Y))$*, then the Replacement Axiom Scheme is true in* $\mathbf{M}$.

Proof. $\{y : \exists x \in A\ \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subset Y)$ implies that $\forall x \in \mathbf{M} \cap A\ \exists y \in \mathbf{M} \cap Y \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n))$. □

Axiom of Foundation
The relativization of this axiom to $\mathbf{M}$ is $\forall x \in \mathbf{M}(\exists y \in \mathbf{M}(y \in x) \to \exists y \in \mathbf{M}(y \in x \wedge \neg\exists z \in \mathbf{M}(z \in x \wedge z \in y)))$.

Proposition 2.24. *The Axiom of Foundation holds relativized to any class* $\mathbf{M} \subset WF$.

Proof. If $x \in \mathbf{M}$, then we take $y \in x$ such that $\forall z \in \mathbf{M}(rank(y) \leq rank(z))$ and the relativization of the Axiom of Foundation is true. □

Axiom of Infinity
The relativization of this axiom to **M** is $\exists x \in \mathbf{M}(0^{\mathbf{M}} \in x \wedge \forall y \in \mathbf{M}(y \in x \to S(y)^{\mathbf{M}} \in x))$.
Since ω satisfies this formula, the following holds.

Proposition 2.25. *Let* **M** *be a transitive model of at least* $ZF^{-} - P - Inf$*. If* $\omega \in \mathbf{M}$*, then the Axiom of Infinity is true in* **M**. □

Axiom of Choice
The relativization of this axiom to **M** is $\forall A \in \mathbf{M} \exists R \in \mathbf{M}(R\ well-orders\ A)$.

Proposition 2.26. *Let* $A \in WF$*. Then* A *can be well* $-$ *ordered* $\leftrightarrow$ $(A\ can\ be\ well-ordered)^{WF}$.

Proof. For the left to right implication, fix a relation $R \subset A \times A$ on $A \in WF$ such that R well-orders A. By Proposition 2.7, $A \times A \in WF$ and so $A \times A \subset WF$ by Proposition 2.5. Then, $R \subset WF$ and, again by Proposition 2.5, $R \in WF$. We can conclude that, by Theorem 2.7, $(R\ well-orders\ A)^{WF}$.
For the left to right implication, fix $R \in WF$ such that $(R\ well-orders\ A)^{WF}$. Then, by Theorem 2.7, $R\ well-orders\ A$, and so $A\ can\ be\ well-ordered$. □

Theorem 2.8. **V** *is a model of* ZFC.

Proof. By Theorem 2.3, $V = WF$, and so we have to prove that WF is a model of ZFC.
By Proposition 2.3, WF is transitive, and so, by Proposition 2.19, the Axiom of Extensionality is true in WF.
Also because WF is transitive, if $A \in WF$ then $\mathcal{P}(A) \in WF$ by Proposition 2.7 and, by Proposition 2.20, the Comprehension Axiom is true in WF.
By Proposition 2.7, the pairing operation is closed in WF, and so it is clear that the Pairing Axiom is true in WF.
To see that the Union Axiom is true in WF, note that if $A \in WF$, then, by Proposition 2.7, $\bigcup A \in WF$ and so there is an ordinal α for which $\bigcup A \in V(\alpha)$ and then $\bigcup A \subset V(\alpha + 1)$. By Proposition 2.22, the Union Axiom is true in WF.
As WF is transitive, by Proposition 2.21 to show that the Power Set Axiom is true in WF we have to see that $\forall x \in WF\ \exists y \in WF(\mathcal{P}(x) \cap WF \subset y)$. But, if we fix $x \in WF$, then, by Proposition 2.7, $\mathcal{P}(x) \in WF$ and so $\mathcal{P}(x) \subset WF$ by Proposition 2.5. Thus, $\mathcal{P}(x) \cap WF = \mathcal{P}(x)$. As $\mathcal{P}(x) \in WF$, $\mathcal{P}(x) \subset V(\alpha) \in WF$ for some ordinal α.
In order to prove that the Replacement Axiom is true for WF, let $Y = \{y \in WF :$

$\exists x \in A\ \phi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\}$. It is obvious that $Y \subset WF$. By Proposition 2.5, $Y \in WF$. Then, there is an ordinal α for which $Y \in V(\alpha)$, which implies that $Y \subset V(\alpha+1)$. Hence, by Proposition 2.23, the Replacement Axiom is true for WF.
As $\omega \in WF$, by Proposition 2.25 the Axiom of Infinity is true in WF.
The Axiom of Foundation is true in WF by Proposition 2.24.
By Proposition 2.26, the Axiom of Choice is true in WF. □

2.6 The Reflection Theorems

We are going to state the Reflection Theorem and some related corollaries, which will be useful later. To do so, we first show the Mostowski Collapse theorem. For this, we need some preparation.

Definition 2.12. *R is an extensional relation on a class $\mathbf{A}$ if and only if* $\forall x \in \mathbf{A}\ \forall y \in \mathbf{A}(\forall z \in \mathbf{A}(zRx \leftrightarrow zRy) \to x = y)$.

We now define the *Mostowski collapsing function* and the *Mostowski collapse* of a class and a relation and give some of its basic properties.

Definition 2.13. *Let R be a well-founded and set-like relation on a class $\mathbf{A}$. We define G, the Mostowski collapsing function of $\mathbf{A}$, R, by $G(x) = \{G(y) : y \in \mathbf{A} \wedge yRx\}$.*
$\mathbf{M} = ran(G)$ is the Mostowski collapse of $\mathbf{A}$, R.

Lemma 2.9. *If G is the Mostowski collapsing function and $\mathbf{M}$ the Mostowski collapse of $\mathbf{A}$, R, then:*
a) $\forall x, y \in \mathbf{A}(xRy \to G(x) \in G(y))$,
b) $\mathbf{M}$ is transitive,
c) $\mathbf{M} \subset WF$.

Proof. a) is obvious by the definition of G.
b) If $\mathbf{M}$ is empty, it is obvious. Let $w \in \mathbf{M}$. So $\exists x \in \mathbf{A}(G(x) = w)$. If $z \in w$, then $z \in G(x)$ and so, by the definition of G, $\exists y \in \mathbf{A}(yRx \wedge z = G(y))$. Thus, $z \in \mathbf{M}$.
c) We will prove by induction on $rank(x)$ that $\forall x \in \mathbf{A}(G(x) \in WF)$. It will follow that, if $w \in \mathbf{M}$, then $\exists x \in \mathbf{A}(G(x) = w)$ and so $w \in WF$.
If $x \in \mathbf{A}$ is such that $\forall y \in \mathbf{A}(\neg\, yRx)$, then $G(x) = 0 \in WF$. Now, suppose that, for $x \in \mathbf{A}$, $\forall y \in \mathbf{A}(yRx \to G(y) \in WF)$. Then, $\forall w \in G(x)\ \exists y \in \mathbf{A}(yRx \wedge w = G(y))$, and so, by inductive hypothesis, $w \in WF$. Thus, $G(x) \subset WF$ and, by Proposition 2.5, $G(x) \in WF$. □

Proposition 2.27. *Let G be the Mostowski collapsing function of $\mathbf{A}$, R. If R is extensional on $\mathbf{A}$, then G is an isomorphism from $(\mathbf{A}, R)$ onto $(\mathbf{M}, \in)$.*

Proof. By the definition of G, it is enough to show that G is injective. Suppose that it is not. Then $X = \{x \in \mathbf{A} : \exists y \in \mathbf{A}(x \neq y \wedge G(x) = G(y)\}$ is non-empty. As R is well-founded on A, we can take $x \in \mathbf{A}$ as an $R-$minimal element of X. Let $y \in \mathbf{A}$ be such that $y \neq x$ and $G(y) = G(x)$. Since R is extensional on $\mathbf{A}$, $\exists z \in \mathbf{A}((zRx \wedge \neg\, zRy) \vee (\neg\, zRx \wedge zRy))$ (if this did not hold, we would have that $x = y$). So let $z \in \mathbf{A}$. If $zRx \wedge \neg\, zRy$, then by Lemma 2.9 we have that $G(z) \in G(x) = G(y)$, and so $\exists w \in \mathbf{A}(wRy \wedge G(z) = G(w))$. But $w \neq z$, which contradicts the minimality of x in X. On the other hand, if $\neg\, zRx \wedge zRy$, then by Lemma 2.9 we have that $G(z) \in G(y) = G(x)$ and so $\exists w \in \mathbf{A}(wRx \wedge G(w) = G(x))$, but $w \neq x$, which contradicts the minimality of x in X. □

Theorem 2.9 (Mostowski Collapsing Theorem)**.** *Let R be a well-founded, set-like and extensional relation on $\mathbf{A}$. Then there is a transitive class M and an isomorphism G from $(\mathbf{A},\ R)$ onto $(\mathbf{M},\ \in)$. Moreover, $\mathbf{M}$ and G are unique.*

Proof. By Proposition 2.27, if G and $\mathbf{M}$ are the Mostowski collapsing function and Mostowski collapse of $\mathbf{A}$, R, then they satisfy the Theorem. If suppose that $\mathbf{M}'$ and G' also satisfy the Theorem, it is easy to prove by induction on $rank(x)$ that if $x \in \mathbf{A}$ then $G'(x) = G(x)$ and so $\mathbf{M} = \mathbf{M}'$. □

Taking $R = \in$ in Theorem 2.9, we obtain the following corollary.

Corollary 2.3. *If $\in$ is extensional on $\mathbf{A}$, then there is a transitive class $\mathbf{M}$ and an isomorphism G from $\mathbf{A}$ onto $\mathbf{M}$.* □.

Before stating the Reflection Theorem, we introduce the following notion.

Definition 2.14. *Let $\phi_1, \ldots, \phi_n$ be a list of formulas. We say that this list is subformula closed if and only if all the subformulas of any ϕ_i on the list are also on the list.*

Lemma 2.10. *Let $\mathbf{M}$ and $\mathbf{N}$ be classes with $\mathbf{M} \subset \mathbf{N}$ and $\phi_1, \ldots, \phi_n$ be a subformula closed list of formulas. Then these two statements are equivalent:*
a) $\phi_1, \ldots, \phi_n$ are absolute for $\mathbf{M}, \mathbf{N}$.
b) For any ϕ_i on the list, if ϕ_i is of the form $\exists x\ \phi_j(x, y_1, \ldots, y_l)$, with all the free variables of ϕ_j listed, then
$\forall y_1, \ldots, y_n \in \mathbf{M}(\exists x \in \mathbf{N}\ \phi_j^{\mathbf{N}}(x, y_1, \ldots, y_l) \rightarrow \exists x \in \mathbf{M}\ \phi_j^{\mathbf{N}}(x, y_1, \ldots, y_l))$.

Proof. For a) $\rightarrow$ b), assume that $\phi_1, \ldots, \phi_n$ are absolute for $\mathbf{M}, \mathbf{N}$. Let $y_1, \ldots, y_l \in \mathbf{M}$ and $\phi_i = \exists x\ \phi_j(x, y_1 \ldots, y_l)$. Assume that $\phi_i^{\mathbf{N}}$ holds. By the absoluteness of ϕ_i, we have that $\phi_i^{\mathbf{M}}$ also holds, and so $\exists x \in \mathbf{M}\ \phi_j^{\mathbf{M}}(x, y_1, \ldots, y_l)$. By the absoluteness of ϕ_j, we finally have that $\exists x \in \mathbf{M}\ \phi_j^{\mathbf{N}}(x, y_1, \ldots, y_l)$.
For b) $\rightarrow$ a), we prove by induction on the length of ϕ_i that ϕ_i is absolute for $\mathbf{M}, \mathbf{N}$. If ϕ_i is an atomic formula, then ϕ_i is absolute for $\mathbf{M}, \mathbf{N}$ by Lemma 2.5.

If ϕ_i is $\neg\phi_j$ for some absolute ϕ_j for $\mathbf{M}, \mathbf{N}$ or ϕ_i is $\phi_j \wedge \phi_k$ for some absolute ϕ_j and ϕ_k for $\mathbf{M}, \mathbf{N}$, then ϕ_i is obviously also absolute for $\mathbf{M}, \mathbf{N}$.
If ϕ_i is $\exists x\ \phi_j(x, y_1, \ldots, y_l)$ with ϕ_j absolute for $\mathbf{M}, \mathbf{N}$, we fix $y_1, \ldots, y_l \in \mathbf{M}$ and then $\phi_i^{\mathbf{M}}(y_1, \ldots, y_l) \leftrightarrow \exists x \in \mathbf{M}\ \phi_j^{\mathbf{M}}(x, y_1, \ldots, y_l) \leftrightarrow \exists x \in \mathbf{M}\ \phi_j^{\mathbf{N}}(x, y_1, \ldots, y_l) \leftrightarrow \exists x \in \mathbf{N}\ \phi_j^{\mathbf{N}}(x, y_1, \ldots, y_l) \leftrightarrow \phi_i^{\mathbf{N}}(y_1, \ldots, y_n)$. The first and the last equivalence are by definition. The second one follows from the absoluteness of ϕ_j. The third one follows from the hypothesis b). □

The next theorem is the general case of the Reflection Theorem.

Theorem 2.10 (Reflection Theorem)**.** *Let* $\mathbf{Z}$ *be a class and for each ordinal* α *let* $Z(\alpha)$ *be a set such that:*
a) For any $\beta \in \alpha$, $Z(\beta) \subset Z(\alpha)$,
b) $Z(\alpha) = \bigcup_{\beta<\alpha} Z(\beta)$ *if* α *is a limit ordinal,*
c) $\mathbf{Z} = \bigcup_{\alpha\in\mathbf{ON}} Z(\alpha)$.
Then, for any formulas $\phi_1, \ldots, \phi_n$,
$\forall\alpha\exists\beta > \alpha(\phi_1, \ldots, \phi_n \textit{ are absolute for } Z(\beta), \mathbf{Z})$.

Proof. We assume that $\phi_1, \ldots, \phi_n$ is a subformula closed list (if not, we expand the list by adding the required subformulas). We define a function $F_i : \mathbf{ON} \to \mathbf{ON}$ for each $i = 1, \ldots, n$ as follows.
If ϕ_i is not of the form $\exists x\ \phi_j(x, y_1, \ldots, y_n)$, then we put $F_i(\xi) = 0$ for any ordinal ξ.
If ϕ_i is $\exists x\ \phi_j(x, y_1, \ldots, y_n)$, we define the function $G_i(y_1, \ldots, y_l)$ to be the least η such that $\exists x \in Z(\eta)\ \phi_j^{\mathbf{Z}}(x, y_1, \ldots, y_l)$. If $\neg\exists x \in \mathbf{Z}\ \phi^{\mathbf{Z}}(x, y_1, \ldots, y_l)$, then $G_i(y_1, \ldots, y_l) = 0$. We set $F_i(\xi) = sup\{G_i(y_1, \ldots, y_l) : y_1, \ldots, y_l \in Z(\xi)\}$. This supreme exists due to the Replacement Axiom.
If β is a limit ordinal and, for each i, $\forall\xi < \beta(F_i(\xi) < \beta$, then $\phi_1, \ldots, \phi_n$ will be absolute by Lemma 2.10. Indeed, if $\exists x \in \mathbf{Z}\ \phi_j^{\mathbf{Z}}(x, y_1, \ldots, y_l)$, then this x will also be in β and thus in $Z(\beta)$ since, for any ordinal ξ, $F_i(\xi) < \beta$. We fix an ordinal α. Let's construct this limit ordinal β.
We define by recursion $\{\beta_k\}_{k\in\omega}$ as follows. $\beta_0 = \alpha$ and, for each $k \in \omega$, $\beta_{k+1} = max\{\beta_k + 1, F_1(\beta_k), \ldots, F_n(\beta_k)\}$. Let $\beta = sup\{\beta_k : k \in \omega\}$. We have that $\beta_{k+1} \geq \beta_k + 1 > \beta_k$ and so $\alpha = \beta_0 < \beta_1 < \beta_2 \cdots$. Thus, β is a limit ordinal with $\beta > \alpha$.
Moreover, $\forall\xi_1 \in \mathbf{ON}\ \forall\xi_2 \in \mathbf{ON}(\xi_1 < \xi_2 \to F_i(\xi_1) \leq F_i(\xi_2))$. So, if $\xi < \beta$, then $\xi < \beta_k$ for some ordinal k, and then $F_i(\xi) \leq F_i(\beta_k) \leq \beta_{k+1} < \beta$, giving the desired inequality. □

The following result is a direct corollary of Theorem 2.10, taking $\mathbf{Z} = \mathbf{V}$ and $Z(\alpha) = V(\alpha)$.

Theorem 2.11 (Reflection Theorem)**.** *For any list of formulas* $\phi_1, \ldots, \phi_n$,
$ZF \vdash \forall\alpha\ \exists\beta > \alpha(\phi_1, \ldots, \phi_n \textit{ are absolute for } V(\beta))$. □

Corollary 2.4. *Let S be any set of axioms extending ZF and $\phi_1, \ldots, \phi_n$ any axioms of S. Then, $S \vdash \forall\alpha\ \exists\beta > \alpha\ \forall i \leq n(\phi_i^{V(\beta)})$.*

Proof. By Theorem 2.11, $ZF \vdash \forall\alpha\ \exists\beta > \alpha\ \forall i \leq n(\phi_i \leftrightarrow \phi_i^{V(\beta)})$. Since S extends ZF, we also have that $S \vdash \forall\alpha\ \exists\beta > \alpha\ \forall i \leq n(\phi_i \leftrightarrow \phi_i^{V(\beta)})$. As, for each i, $S \vdash \phi_i$ since ϕ_i is an axiom of S, then $S \vdash \forall\alpha\ \exists\beta > \alpha\ \forall i \leq n(\phi_i^{V(\beta)})$. □

A consequence of Corollary 2.4 is that, if S is a collection of axioms extending ZFC, then there is no finite sublist of axioms of S from which it is possible to prove all the axioms of S. It follows that no finite list of axioms is equivalent to ZFC. To end this section, we are going to see an important result for the forcing theory. It will grant us the existence of a countable transitive model of any finite list of axioms of ZFC from which we will construct the desired model of $ZFC + \neg CH$. But first, let's see some preliminary lemmas.

Lemma 2.11. *Let $\mathbf{Z}$ be a class and $\phi_1, \ldots, \phi_n$ be a list of formulas. Then,*
$\forall X \subset \mathbf{Z}\ \exists A(X \subset A \subset \mathbf{Z}\ \wedge (\phi_1, \ldots, \phi_n \text{ are absolute for } A, \mathbf{Z}) \wedge |A| \leq max(\omega, |X|))$.

Proof. We can assume that the list of formulas is subformula closed. If not, we add the needed formulas to the list. We define $Z(\alpha) = V(\alpha) \cap \mathbf{Z}$ for any ordinal α. $\mathbf{Z}$ and $Z(\alpha)$ satisfy the hypothesis of Theorem 2.10. Indeed, if $\alpha < \beta$, then $V(\alpha) \subset V(\beta)$ and so $Z(\alpha) = V(\alpha) \cap \mathbf{Z} \subset V(\beta) \cap \mathbf{Z} = Z(\beta)$; if α is a limit ordinal, then
$Z(\alpha) = V(\alpha) \cap \mathbf{Z} = (\bigcup_{\beta<\alpha} V(\beta)) \cap \mathbf{Z} = \bigcup_{\beta<\alpha}(V(\beta) \cap \mathbf{Z}) = \bigcup_{\beta<\alpha} Z(\beta)$; $\bigcup_{\alpha\in\mathbf{ON}} Z(\alpha) = \bigcup_{\alpha\in\mathbf{ON}}(V(\alpha) \cap \mathbf{Z}) = (\bigcup_{\alpha\in\mathbf{ON}} V(\alpha)) \cap \mathbf{Z} = \mathbf{V} \cap \mathbf{Z} = \mathbf{Z}$.
Let α be an ordinal such that $X \subset Z(\alpha)$. By Theorem 2.10, we can pick an ordinal $\beta > \alpha$ such that $\phi_1, \ldots, \phi_n$ are absolute for $Z(\beta), \mathbf{Z}$. By the Axiom of Choice, there is a well-order of $Z(\beta)$, that we call $\prec$. Let β_0 be the $\prec$-first element of $Z(\beta)$.
For each $i = 1, \ldots, n$, assuming that ϕ_i has l_i free variables, we define a function $H_i : Z(\beta)^{l_i} \to Z(\beta)$ as follows. If $l_i = 0$, then we fix a $b \in Z(\beta)$ and $H_i(0) = b$. If ϕ_i is $\exists x(\phi_j(x, y_1, \ldots, y_{l_i})$ and $\exists x \in Z(\beta)\phi_j^{Z(\beta)}(x, y_1, \ldots, y_{l_i}$, then let x_0 be the first such x and let $H_i(y_1, \ldots, y_{l_i}) = x_0$. If $\neg\exists x(\phi_j(x, y_1, \ldots, y_{l_i})$ or ϕ_i is not an existential quantification, then $H_i(y_1, \ldots, y_{l_i}) = \beta_0$. Let A be the closure of X under $H_1, \ldots, H_i$, that is the least Y such that $X \subset Y$ and $ran(H_j \restriction Y^{l_j}) \subset B$ (if $l_j = 0$, $H_j \in Y$). Then, by Proposition 1.2, as A is closed under each H_i, we have that $|A| \leq max(\omega, |X|)$ which certifies that $\forall y_1, \ldots, y_{l_i} \in A(\exists x \in Z(\beta)\ \phi_j^{Z(\beta)}(x, y_1, \ldots, y_{l_i}) \leftrightarrow \exists x \in A\ \phi_j^{Z(\beta)}(x, y_1, \ldots, y_{l_i}))$. By Lemma 2.10, $\phi_1, \ldots, \phi_n$ are absolute for $A, Z(\beta)$ and thus for $A, \mathbf{Z}$. □

The following Lemma is easy to prove by induction on the lenght of ϕ.

Lemma 2.12. *Let $G : A \to M$ be an isomorphism. Then for each formula $\phi(x_1, \dots, x_n)$ with all its variables listed, we have that*
$\forall x_1, \dots, x_n \in A(\phi(x_1, \dots, x_n)^A \leftrightarrow \phi(G(x_1), \dots, G(x_n))^M)$. $\square$

Lemma 2.13. *Let* $\mathbf{Z}$ *be a transitive class and $\phi_1, \dots, \phi_n$ be a list of sentences. Then* $\forall X \subset \mathbf{Z}(X \text{ is transitive} \to \exists M(X \subset M \land \forall i \leq n(\phi_i^M \leftrightarrow \phi_i^{\mathbf{Z}})) \land M \text{ is transitive} \land |M| \leq max(\omega, |X|)))$.

Proof. We assume that the Axiom of Extensionality is on the list. If not, add it. Fix $X \subset \mathbf{Z}$. By Lemma 2.11, we can take an A such that $X \subset A \subset \mathbf{Z}$, $\phi_i^A \leftrightarrow \phi_i^{\mathbf{Z}}$ for each $i \leq n$ and $|A| \leq max\{w, |X|\}$. As $\mathbf{Z}$ is transitive, by Proposition 2.19 the Axiom of Extensionality holds in $\mathbf{Z}$ and so it does on A. By Corollary 2.3, there is a transitive set M and an isomorphism $G : A \to M$. By Lemma 2.12, $\phi_i^A \leftrightarrow \phi_i^M$, and so $\phi_i^{\mathbf{Z}} \leftrightarrow \phi_i^M$. Finally, let's show that $X \subset M$. If $x \in X$, then $G(x) = \{G(y) : y \in A \land y \in x\} = \{G(y) : y \in x\}$ since X is transitive (and so the condition $y \in x$ is equivalent to $y \in A \land y \in x$ when $x \in A$). So, by induction on $rank(x)$, $G(x) = x$ for all $x \in X$. Indeed, if $x = 0$, it is obvious. If $\forall y \in x(G(y) = y)$, then $G(x) = \{y : y \in x\} = x$. Hence, $X = ran(G \restriction X) \subset M$. $\square$

Finally, taking $\mathbf{Z} = \mathbf{V}$ and $X = \omega$, we obtain the following immediate consequence of Lemma 2.13.

Proposition 2.28. *Let S be a set of axioms extending ZFC and $\phi_1, \dots, \phi_n$ be axioms of S. Then*
$S \vdash \exists M(|M| = \omega \land M \text{ is transitive} \land \forall i \leq n(\phi_i^M))$. $\square$

So, Proposition 2.28 states that, assuming that ZFC is consistent, there is a countable transitive model of any finite set of axioms extending ZFC. As we will discuss later, it is from this model that we will be able to find a model of $ZFC + \neg CH$ that will confirm the consistency of the negation of the Continuum Hypothesis with ZFC. To end this chapter, we briefly present the model of $ZFC + CH$ that shows the relative consistency of the Continuum Hypothesis with ZFC. In addition with our work, we will conclude that CH is independent of ZFC.

2.7 The Constructible Universe

In this section we are going to define the class of constructible sets, $\mathbf{L}$. This class is defined from the concept of $definable\ set$. Given a set A, we wish to define the set $Df(A, n)$ of the subsets of A^n that contain all the ordered n-tuples that satisfy some formula with n free variables relativized to A. These subsets would be defined from A by a formula. Nonetheless, formalizing this idea is not immediate since there are infinite formulas. The approach will be to consider sets constructed from basic relations (as $x \in y$ or $x = y$, that would give, for example, due to the Axiom of Comprehension, the sets $\{(x,y) \in A^2 : xRy\}$, for $R \in \{\in, =\}$) and sets obtained by operating with already defined sets recursively, using complementation, intersection and projection. Each of these set operations will have a correspondence with the logical connectors and the existential quantifier $\neg$, $\wedge$ and $\exists$ respectively. Thereby, $Df(A,n)$ will be the least union of those sets closed under the aforementioned operations.

Definition 2.15. *If $n \in \omega$ and $i, j < n$, we define:*
a) $Proj(A,R,n) = \{s \in A^n : \exists t \in R(t \restriction n = s)\}$,
b) $Diag_{\in}(A,n,i,j) = \{s \in A^n : s(i) \in s(j)\}$,
c) $Diag_{=}(A,n,i,j) = \{s \in A^n : s(i) = s(j)\}$,
d) $Df'(0,A,n) = \{Diag_{\in}(A,n,i,j) : i,j < n\} \cup \{Diag_{=}(A,n,i,j) : i,j < n\}$ *and* $Df'(k+1,A,n) = Df'(k,A,n) \cup \{A^n \setminus R : R \in Df'(k,A,n)\} \cup \{R \cap S : R, S \in Df'(k,A,n)\} \cup \{Proj(A,R,n) : R \in Df'(k,A,n+1)\}$ *for* $k \in \omega$.

Definition 2.16. *We define* $Df(A,n) = \bigcup\{Df'(k,A,n) : k \in \omega\}$ *and* $D(A) = \{X \subset A : \exists n \in \omega\ \exists s \in A^n\ \exists R \in Df(A,n+1)(X = \{x \in A : s^\frown\langle x\rangle \in R\})\}$.

The following Lemma is easy to show by induction on the length of ϕ.

Lemma 2.14. *Let $\phi(v_0, \ldots, v_{n-1}, x)$ be any formula with all its free variables listed. Then,* $\forall A\ \forall v_0, \ldots, v_{n-1} \in A(\{x \in A : \phi^A(v_0, \ldots, v_{n-1}, x)\} \in D(A))$. □

We will use the definable sets to define the constructible universe **L**.

Definition 2.17. *We define $L(\alpha)$ for any ordinal α as follows:*
a) $L(0) = 0$,
b) $L(\alpha+1) = D(L(\alpha))$,
c) $L(\alpha) = \bigcup_{\beta<\alpha} L(\beta)$ *if α is a limit ordinal.*

Definition 2.18. $\mathbf{L} = \bigcup\{L(\alpha) : \alpha \in \mathbf{ON}\}$.

Now, we can state the fundamental result for **L**.

Theorem 2.12. **L** *is a model of* $ZFC + GCH$. □

Since **L** is a model of $ZFC + GCH$, and so in particular of $ZFC + CH$, constructed from ZFC, then, by Proposition 2.1, we have that $Con(ZFC) \rightarrow Con(ZFC + GCH)$ and so $Con(ZFC) \rightarrow Con(ZFC + CH)$.
This result, proved by Kurt Gödel in 1939, states that the Continuum Hypothesis is consistent with the ZFC axioms. Along with the proof of the consistency of the negation of the Continuum Hypothesis, which is the aim of this work, we will conclude that CH is independent of ZFC. In the next chapter we will study the Forcing technique that is used to find a model of $ZFC + \neg CH$.

Chapter 3

The Forcing Method

In this chapter, we are going to present the general method of forcing to obtain, from a model of ZFC, M, called a *ground model*, another model of ZFC extending the ground one and satisfying, additionally, some other statement, ϕ. Thus, it will follow that the consistency of ZFC implies the consistency of $ZFC + \phi$.
Formally, we could not be able to suppose that a countable transitive model M of ZFC exists because ZFC is infinite. By Proposition 2.28, we can only prove from ZFC that there is a countable transitive model of any finite list of axioms of ZFC. The proof of the relative consistency $Con(ZFC) \to Con(ZFC + \neg CH)$ will be made by contraposition: if we assume the inconsistency of $ZFC + \neg CH$ we will deduce the inconsistency of ZFC.
So, suppose that $ZFC + \neg CH$ is not consistent. Let $\phi_1, \ldots, \phi_n$ be a finite list of axioms of $ZFC + \neg CH$ such that they lead to a contradiction, i.e., for some formula ψ, $\phi_1 \cdots \phi_n \vdash \psi \wedge \neg\psi$. As we will see in the next chapter, ZFC implies that there is a model N such that $\phi_1, \ldots, \phi_n$ are true in N, and so $\psi \wedge \neg\psi$ holds in N. Hence ZFC is inconsistent.
Instead of doing all this reasoning, we can just assume that there is a countable transitive model of ZFC to produce another countable transitive model of $ZFC + \neg CH$ (for all the arguments that we will see involved in forcing, we need a countable transitive model of some axioms, and so, instead of considering a model of enough axioms to follow each argument, we simply suppose that there is a countable transitive model of all ZFC). To force the desired model, we will work with *partial orders* in M. The features of those partial orders will give the extending model some properties that will allow us to prove that some other statement is true in it. In our case, as we will see in Chapter 5, we can use this technique with a suitable partial order to find a model of $ZFC + \neg CH$.

3.1 Partial Orders

Definition 3.1. *Let $\leq$ be a relation on a non-empty set A. We say that $\leq$ partially orders A if and only if $\leq$ is reflexive ($\forall a \in A(a \leq a)$) and transitive ($\forall a, b, c \in A((a \leq b \wedge b \leq c) \rightarrow a \leq c)$) on A.*
A partial order is an ordered triplet $\mathbb{P} = (P, \leq, \mathbb{1})$ where $\mathbb{P} \neq 0$, $\leq$ partially orders P and $\mathbb{1}$ is the largest element of P ($\forall p \in P(p \leq \mathbb{1})$).

The elements of P are called *conditions*, and, for two conditions p, q, we say that p *extends* q (or p is an *extension* of q) if and only if $p \leq q$. Note that saying that $\mathbb{P} \in M$ means that $P \in M$, $\leq \in M$ and $\mathbb{1} \in M$.

Definition 3.2. *Let $p, q \in P$. We say that p and q are compatible if and only if $\exists r \in P(r \leq p \wedge r \leq q)$. We say that p and q are incompatible, in symbols $p \perp q$, if and only if there is no such $r \in P$.*

Definition 3.3. *Let $\mathbb{P}$ be a partial order and $D, G \subset \mathbb{P}$. We say that*

1. *D is dense in $\mathbb{P}$ if and only if $\forall p \in P\ \exists q \in D(q \leq p)$.*
2. *G is a filter in $\mathbb{P}$ if and only if*
   (a) *$\forall p, q \in G\ (\neg p \perp q)$ and*
   (b) *$\forall p \in G\ \forall q \in P(p \leq q \rightarrow q \in G)$.*

For the rest of this section, we will consider M to be a countable transitive model of ZFC and $\mathbb{P}$ a partial order such that $\mathbb{P} \in M$.

Proposition 3.1. *The following notions are absolute:*
a) $\mathbb{P}$ is a partial order,
b) D is dense.
c) G is a filter.

Proof. It is clear that the formulas that defines these notions are equivalent to some Δ_0 formulas since all the quantifiers that appear in those formulas are bounded. Then, they are absolute. □

Later, we will also need the following concept.

Definition 3.4. *If $E \subset \mathbb{P}$ and $p \in P$, then we say that E is dense below p is and only if $\forall q \leq p\ \exists r \in E(r \leq q)$.*

Since, if $q \leq p$, then $\forall r \leq q(r \leq p)$, it is clear that if E is dense below p and $q \leq p$, then E is dense below q.

Lemma 3.1. *If $\{q : E \text{ is dense below } q\}$ is dense below p, then E is dense below p.*

Proof. If $\{q : E \text{ is dense below } q\}$ is dense below p, then $\forall q \leq p \, \exists r \leq q(E \text{ is dense below } r)$ and so, if we fix $q \leq p$, we can take $r \leq q$ such that $\forall x \leq r \, \exists y \leq x(y \in E)$. In particular, taking $x = r$, we can fix $y \leq r$ such that $(y \in E)$. So, we have that $y \leq r \leq q$ and $y \in E$ and hence E is dense below p. □

3.2 Generic Extensions

In this section, we will see how to construct a *generic extension*, $M[G]$, of our ground model M relative to a special kind of filter G called $\mathbb{P}$-generic. Let's first see the definition of this notion and two immediate results.

Definition 3.5. *Let $G \subset \mathbb{P}$ be a filter in $\mathbb{P}$. G is $\mathbb{P}$-generic over M if and only if for all dense $D \subset \mathbb{P}$, $D \in M \to G \cap D \neq 0$.*

Henceforth, we assume that the filters that we consider are non-empty. The next proposition says that any condition of $\mathbb{P}$ is in some $\mathbb{P}$-generic filter over M.

Proposition 3.2. *Let $p \in P$. Then, there is a $G \subset \mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M.*

Proof. We fix a $p \in P$. Since M is countable, the number of dense subsets of $\mathbb{P}$ in M must also be countable. So, let D_n for $n \in \omega$ be all the dense subsets of $\mathbb{P}$ in M. For each $n \in \omega$, as D_n is dense, we can pick a q_n such that $q_{n+1} \in D_n$ and $q_{n+1} \leq q_n$, with $q_0 = p$. Let $G = \{q \in P : \exists n \in \omega(q_n \leq q)\}$. Then, it is easy to check that G is a filter in $\mathbb{P}$ with $p \in G$. Since, for any $n \in \omega$, $q_{n+2} \leq q_{n+1}$, then $q_n \in G$ and thus $G \cap D_n \neq 0$ for all the dense subsets of $\mathbb{P}$ in M. Hence, G is $\mathbb{P}$-generic over M. □

Proposition 3.3. *If $\mathbb{P}$ is such that $\forall p \in P \, \exists q, r \in \mathbb{P}(q \leq p \wedge r \leq p \wedge q \perp r)$ and G is a $\mathbb{P}$-generic filter over M, then $G \notin M$.*

Proof. Suppose $G \in M$. Then, as $\mathbb{P} \in M$ and the set difference operation is absolute, $D = \mathbb{P} \setminus G \in M$. Moreover, if $p \in P$ and $q, r \in P$ are such that $q \leq p \wedge r \leq p \wedge q \perp r$, then, since G is filter, q and r cannot both be in G (if they were, they would be compatible, which contradicts our assumption). Without loss of generality, we can suppose that $q \notin G$, and so $q \in D$ with $q \leq p$, which means that D is dense. But, by how we defined D, we have that $D \cap G = 0$, which is a contradiction with G being $\mathbb{P}$-generic. □

Once we already saw the preliminary notions and results, we can finally show how to construct a model extending M by means of a $\mathbb{P}$-generic filter. Given G

$\mathbb{P}$-generic over M, Proposition 3.3 says that in general $G \notin M$. So, we will make up $M[G]$ to be the least transitive model of ZFC such that $M \subset M[G]$, $G \in M$ and the ordinals in $M[G]$ are just the ordinals in M. In fact, $M[G]$ will be the set that contains all the sets constructed from G following certain procedures definable in M. The elements of $M[G]$ are constructed from their *names* in M, defined as follows.

Definition 3.6. *Let $\tau \in M$. We say that τ is a $\mathbb{P}$-name if and only if τ is a relation such that $\forall(\sigma, p) \in \tau(\sigma$ is a $\mathbb{P} - name \wedge p \in P)$. Furthermore, if τ is a $\mathbb{P}$-name, we define the domain of τ as $dom(\tau) = \{\sigma : \exists p((\sigma, p) \in \tau)\}$.*
If M is a countable transitive model of ZFC and $\mathbb{P} \in M$, $\mathbf{V}^{\mathbb{P}}$ is the class of $\mathbb{P}$-names and $M^{\mathbb{P}} = \mathbf{V}^{\mathbb{P}} \cap M$.

Given a filter G on $\mathbb{P}$, each name in $\mathbf{V}^{\mathbb{P}}$ will have a *value* associated with G. We will define the elements of $M[G]$ as the values of the names in $M^{\mathbb{P}}$.

Definition 3.7. *Let G be a filter on $\mathbb{P}$. For $\tau \in \mathbf{V}^{\mathbb{P}}$, we define the value of τ with respect to G, in symbols $\tau[G]$, by transfinite recursion: $\tau[G] = \{\sigma[G] : \exists p \in G((\sigma, p) \in \tau)\}$.*

Definition 3.8. *If M is a countable transitive model of ZFC, $\mathbb{P} \in M$ and $G \subset \mathbb{P}$ is a filter, we define $M[G] = \{\tau[G] : \tau \in M^{\mathbb{P}}\}$. $M[G]$ is called a generic extension of M.*

The following definition and lemma show that the elements of M are in fact the values of their own names in $M^{\mathbb{P}}$ and so are in $M[G]$.

Definition 3.9. *For $x \in M$, we define $\check{x}$ recursively: $\check{x} = \{(\check{y}, \mathbb{1}) : y \in x\}$.*

The following Lemma is easy to show by induction on $rank(x)$.

Lemma 3.2. *If G is a filter on $\mathbb{P}$ then $\forall x \in M(\check{x} \in M^{\mathbb{P}} \wedge \check{x}[G] = x)$.* □

We now show that $M[G]$ is the least transitive extension of M containing G.

Definition 3.10. *Let M be a countable transitive model of ZFC. Let $\mathbb{P} \in M$ be a partial order. We define $\Gamma = \{(p, \check{p}) : p \in P\}$.*

Proposition 3.4. *If M is a countable transitive model of ZFC, $\mathbb{P} \in M$ and G is a filter on $\mathbb{P}$, then:*
a) $M \subset M[G]$,
b) $G \in M[G]$,
c) $M[G]$ is transitive,
d) If N is a transitive model of ZFC with $M \subset N$ and $G \in N$, then $M[G] \subset N$.

Proof. a) If $x \in M$, then, by Lemma 3.2, $x = \check{x}[G] \in M[G]$.

b) If $p \in G$, then $p \in M$ and so, by Lemma 3.2, $p = \check{p}[G]$. With that in mind, let's find the $\mathbb{P}$-name for G.
$G = \{p : p \in G\} = \{\check{p}[G] : p \in G\} = \{\sigma[G] : \exists p \in G(\sigma = \check{p})\} = \{\sigma[G] : \exists p \in G((\sigma, p) \in \{(\check{p}, p) : p \in P\})\} = \{(\check{p}, p) : p \in P\}[G] = \Gamma[G] \in M[G]$.

c) Let $x \in M[G]$. Then $x = \tau[G]$ for some $\tau \in M^{\mathbb{P}}$. If $\sigma[G] \in \tau[G]$, then $\sigma \in M^{\mathbb{P}}$ and so $\sigma[G] \in M[G]$.

d) Let $\tau[G] \in M[G]$. Then $\tau \in M^{\mathbb{P}}$ and, since $M^{\mathbb{P}} \subset M \subset N$, we have that $\tau \in N$. As $G \in N$ and N is transitive, then $G \subset N$ and so $G \cap N = N$. Thus, $(\tau[G])^N = \{\sigma[G] : \exists p \in G \cap N((\sigma, p) \in \tau)\} = \{\sigma[G] : \exists p \in G((\sigma, p) \in \tau)\} = \tau[G] \in N$. □

Moreover, the ordinals in M are just the ordinals in $M[G]$.

Definition 3.11. $o(M) = M \cap \mathbf{ON}$.

Proposition 3.5. *If G is a filter on $\mathbb{P}$, then:*
a) $\forall \tau \in M^{\mathbb{P}}(rank(\tau[G]) \leq rank(\tau))$.
b) $o(M) = o(M[G])$.

Proof. a) Let's prove it by induction on τ. If $\forall \sigma \in dom(\tau)(rank(\sigma[G]) \leq rank(\sigma))$, then $rank(\tau[G]) = sup\{rank(\sigma[G]) + 1 : (\sigma, p) \in \tau\} \leq sup\{rank(\sigma) + 1 : (\sigma, p) \in \tau\} = rank(\tau)$ by the inductive hypothesis.

b) Since $M \subset M[G]$, we trivially have that $o(M) \subset o(M[G])$. On the other hand, let $\alpha \in o(M[G])$. Then $\alpha = \tau[G]$ for some $\tau \in M^{\mathbb{P}}$, and so, as $\tau[G]$ is an ordinal and by clause *a*), $\tau[G] = rank(\tau[G]) \leq rank(\tau) \in M \cap \mathbf{ON} = o(M)$ by the absoluteness of the notion of rank. □

Proposition 3.6. *The following notions are absolute for M:*
a) τ is a $\mathbb{P}$-name,
b) $\tau[G]$,
c) $\check{x}$.

Proof. a) We define the function $F(\mathbb{P}, \tau) = 1$ if and only if τ is a relation and $\forall(\sigma, p) \in \tau(F(\mathbb{P}, \tau) = 1 \wedge p \in P)$ and $F(\mathbb{P}, \tau) = 0$ otherwise.
It is clear that τ is a $\mathbb{P}$-name if and only if $F(\mathbb{P}, \tau) = 1$.
Since $F(\mathbb{P}, \tau)$ is defined by absolute concepts for M (which is a transitive model of, in particular, $ZFC - P$) from all the $F(\mathbb{P}, \sigma)$ with $(\sigma, p) \in \tau$ and $p \in P$, and $F(\mathbb{P}, \sigma)$

is defined from all the $F(\mathbb{P}, \pi)$ with $(\pi, q) \in \sigma$ and $q \in P$, it follows recursively that $F(\mathbb{P}, \tau)$ is defined from $F \upharpoonright \tau^+$. By Proposition 3.29, F is absolute, and so is "τ is a $\mathbb{P}$-name".

b)/c) $\tau[G]$ and $\check{x}$ are defined from absolute notions and are then absolute for M. □

Since these notions are absolute, we can use them within M freely.

3.3 Forcing formulas in the generic extension

We now wish to establish some relationship between the truth of a formula in $M[G]$ and some property in M. This relationship will come from the notion of *forcing* a formula.
Let $\phi(x_1, \ldots, x_n)$ be a formula with all its free variables listed, M be a countable transitive model of ZFC, $\mathbb{P} \in M$ a partial order, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.

Definition 3.12. *Let $p \in P$. We say that p forces $\phi(\tau_1, \ldots, \tau_n)$, in symbols $p \Vdash_{\mathbb{P},M} \phi(\tau_1, \ldots, \tau_n)$, if and only if*

$$\forall G((G \text{ is } \mathbb{P} - \text{generic over } M \wedge p \in G) \rightarrow \phi^{M[G]}(\tau_1[G], \ldots, \tau_n[G])).$$

We will use $\Vdash$ instead of $\Vdash_{\mathbb{P},M}$ if no confusion can occur. This definition says that a condition forces a formula if this formula is true in any generic extension of M that is relative to a $\mathbb{P}$-generic filter over M that contains this condition. After seeing two obvious properties about $\Vdash$, we will show that this notion can be represented by a formula in M and is so definable within M.

Lemma 3.3. *Let $p, q \in P$. Then:*
a) $(p \Vdash \phi(\tau_1, \ldots, \tau_n) \wedge q \leq p) \rightarrow q \Vdash \phi(\tau_1, \ldots, \tau_n)$.
b) $(p \Vdash \phi(\tau_1, \ldots, \tau_n)) \wedge (p \Vdash \psi(\tau_1, \ldots, \tau_n))$ if and only if $p \Vdash (\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n))$. □

Now, we will define $\Vdash^*$, a concept related to $\Vdash$. This $\Vdash^*$ relativized to M is equivalent to $\Vdash$, and it will allow us to decide in M whether a condition forces a formula or not.

Definition 3.13. *Let $\mathbb{P}$ be a partial order. Let $\phi(x_1, \ldots, x_n)$ be a formula with all its free variables listed. Let $\tau_1, \ldots, \tau_n \in \mathbf{V}^{\mathbb{P}}$ and $p \in P$. We define $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$ by recursion as follows.*

1. $p \Vdash^* \tau_1 = \tau_2$ *if and only if*
   a) $\forall(\pi_1, s_1) \in \tau_1$
   $\{q \leq p : q \leq s_1 \rightarrow \exists(\pi_2, s_2) \in \tau_2(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$
   is dense below p, and

   b) $\forall(\pi_2, s_2) \in \tau_2$
   $\{q \leq p : q \leq s_2 \rightarrow \exists(\pi_1, s_1) \in \tau_1 \, (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$
   is dense below p.

2. $p \Vdash^* \tau_1 \in \tau_2$ *if and only if*
   $\{q : \exists(\pi, s) \in \tau_2 \, (q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$ *is dense below p.*

3. $p \Vdash^* (\phi\,(\tau_1, \ldots, \tau_n) \wedge \psi\,(\tau_1, \ldots, \tau_n))$ *if and only if*
   $p \Vdash^* \phi\,(\tau_1, \ldots, \tau_n)$ *and* $p \Vdash^* \psi\,(\tau_1, \ldots, \tau_n)$.

4. $p \Vdash^* \neg\phi\,(\tau_1, \ldots, \tau_n)$ *if and only if there is no* $q \leq p$ *such that*
   $q \Vdash^* \phi\,(\tau_1, \ldots, \tau_n)$.

5. $p \Vdash^* \exists x \; \phi\,(x, \tau_1, \ldots, \tau_n)$ *if and only if*
   $\left\{q : \exists\tau \in V^{\mathbb{P}} \, (q \Vdash^* \phi\,(\tau, \tau_1, \ldots, \tau_n))\right\}$ *is dense below p.*

Lemma 3.4. *Let* $\mathbb{P}$ *be a partial order. Let* $\phi(x_1, \ldots, x_n)$ *be a formula with all its free variables listed. Let* $\tau_1, \ldots, \tau_n \in \mathbf{V}^{\mathbb{P}}$ *and* $p \in P$*. Then the following are equivalent:*
a) $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$,
b) $\forall q \leq p(q \Vdash^* \phi(\tau_1, \ldots, \tau_n))$,
c) $\{q : q \Vdash^* \phi(\tau_1, \ldots, \tau_n)\}$ *is dense below p.*

Proof. For $a) \rightarrow b)$, we proceed by induction on the length of ϕ. If it is an atomic formula, then if $q \leq p$ and the sets in clause 1. and 2. of Definition 3.13 are dense below p then they are dense below q. The same argument holds if ϕ is of the form $\exists x \; \psi$ using clause 5. of this same Definition. The rest of the cases are easy to show using induction on the length of ϕ.
The implication $b) \rightarrow c)$ is obvious.
For $c) \rightarrow a)$, we proceed by induction on the length of ϕ. Suppose that it is $\tau_1 = \tau_2$. If $\{q : q \Vdash^* \tau_1 = \tau_2\}$ is dense below p, then
$\{q : clause\ a)\ from\ Definition\ 3.13\ holds\ for\ q\}$ is dense below p. But these is a set of conditions for which some sets (the ones of the Definition) are dense below. By Lemma 3.1, these sets are also dense below p, and so $p \Vdash^* \tau_1 = \tau_2$. The same argument holds if ϕ is $\tau_1 \in \tau_2$ or if ϕ is $\exists x \; \psi(\tau_1, \ldots, \tau_n)$. The rest of the cases are easy to show using the inductive hypothesis. □

Proceeding by induction on the length of ϕ, we can prove the following theorem. It shows the relation between $\Vdash^*$ in M and the truth of a formula in $M[G]$. After this result, we will be able to state the equivalence between $\Vdash$ and $\Vdash^*$ in M.

Theorem 3.1. *Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a partial order. Let $G \subset \mathbb{P}$ be $\mathbb{P}$-generic over M. Let $\phi(x_1, \dots, x_n)$ be a formula with all its free variables listed. Let $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$. Then,*
a) If $p \in G$ and $(p \Vdash^* \phi(\tau_1, \dots, \tau_n))^M$, then $(\phi(\tau_1[G], \dots, \tau_n[G]))^{M[G]}$ and
b) if $\phi(\tau_1[G], \dots, \tau_n[G])^{M[G]}$, then $\exists p \in G((p \Vdash^* \phi(\tau_1 \dots, \tau_n))^M)$. $\square$

Theorem 3.2. *Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a partial order. Let $\phi(x_1, \dots, x_n)$ be a formula with all its free variables listed. Let $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$. Then,*
a) $\forall p \in P$

$$(p \Vdash \phi(\tau_1, \dots, \tau_n) \leftrightarrow (p \Vdash^* \phi(\tau_1, \dots, \tau_n))^M.$$

b) If G is $\mathbb{P}$-generic over M, then

$$\phi(\tau_1[G], \dots, \tau_n[G])^{M[G]} \leftrightarrow \exists p \in G(p \Vdash \phi(\tau_1, \dots, \tau_n)).$$

Proof. a) For the implication from right to left, fix $p \in P$ and let G be $\mathbb{P}$-generic over M such that $p \in G$. Assume that $(p \Vdash^* \phi(\tau_1, \dots, \tau_n))^M$. Then, by Theorem 3.1 a), we have that $(\phi(\tau_1[G], \dots, \tau_n[G]))^{M[G]}$. Hence, by the definition of $\Vdash$, $p \Vdash \phi(\tau_1, \dots, \tau_n)$.
For the left to right implication, assume that $p \Vdash \phi(\tau_1, \dots, \tau_n)$. Let $D = \{q : (q \Vdash^* \phi(\tau_1, \dots, \tau_n))^M\}$. By Lemma 3.4, if we manage to show that D is dense below p, then $(p \Vdash^* \phi(\tau_1, \dots, \tau_n))^M$. Suppose D is not dense below p. Then, let $q \leq p$ such that $\neg\exists r \in D(r \leq q)$. By Definition 3.13 5., this means that $(q \Vdash^* \neg\phi(\tau_1, \dots, \tau_n))^M$ and so, by the right to left implication of the clause a) of this Theorem, $q \Vdash \neg\phi(\tau_1, \dots, \tau_n)$. Now, fix G $\mathbb{P}$-generic over M with $q \in G$. Since G is a filter and $q \leq p$, we also have that $p \in G$. By the definition of $\Vdash$, the hypothesis that $p \Vdash \phi(\tau_1, \dots, \tau_n)$ implies that $(\phi(\tau_1[G], \dots, \tau_n[G]))^{M[G]}$ holds, while $q \Vdash \neg\phi(\tau_1, \dots, \tau_n)$ implies that $(\neg\phi(\tau_1[G], \dots, \tau_n[G]))^{M[G]}$, which is a contradiction.

b) For the left to right implication, the following holds.

$$\phi(\tau_1[G], \dots, \tau_n[G])^{M[G]} \rightarrow \exists p \in G((p \Vdash^* \phi(\tau_1, \dots, \tau_n))^M) \rightarrow \\ \exists p \in G((p \Vdash \phi(\tau_1, \dots, \tau_n))^M).$$

The first implication is Theorem 3.1 b). The second one follows from the clause a) that we just showed.
The other implication is immediate from the definition of $\Vdash$. $\square$

Clause $a)$ of Theorem 3.2 says that forcing is definable within M since the fact that a condition may force a formula is equivalent to something ($p \Vdash^* \phi$) happening in M. Clause $b)$ states the equivalence between the truth of a formula in a generic extension of M and the existence of a condition in the $\mathbb{P}$-generic filter over M relative to this generic extension that forces this formula. Mixing up the two clauses, we finally see that the truth of a formula ϕ in $M[G]$ is equivalent to the existence of some p for whom a formula in M, $(p \Vdash^* \phi)^M$, is true. To close this section, let's see some final results about $\Vdash$.

Proposition 3.7. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a partial order. Let $\sigma, \tau_1, \dots, \tau_n \in M^{\mathbb{P}}$. Then,*
a) $\{p \in P : (p \Vdash \phi(\tau_1, \dots, \tau_n)) \vee (p \Vdash \neg\phi(\tau_1, \dots, \tau_n))\}$ is dense,
b) $p \Vdash \neg\phi(\tau_1, \dots, \tau_n)$ if and only if $\neg\exists q \leq p(q \Vdash \phi(\tau_1, \dots, \tau_n))$,
c) $p \Vdash \exists x\phi(x, \tau_1, \dots, \tau_n)$ if and only if $\{q \leq p : \exists \tau \in M^{\mathbb{P}}(r \Vdash \phi(\tau, \tau_1, \dots, \tau_n))\}$ is dense below p,
d) if $p \Vdash \exists x(x \in \sigma \wedge \phi(x, \tau_1, \dots, \tau_n))$, then $\exists q \leq p\, \exists \pi \in dom(\sigma)(q \Vdash \phi(\pi, \tau_1, \dots, \tau_n))$.

Proof. Note that clauses $a)$, $b)$ and $c)$ are true for $\Vdash^*$ instead of $\Vdash$ and the formulas relativized to M. Hence, by Theorem 3.2, $a)$, $b)$ and $c)$ hold.

d) Fix G $\mathbb{P}$-generic such that $p \in G$. $p \Vdash \exists x(x \in \sigma \wedge \phi(x, \tau_1, \dots, \tau_n))$ implies that $(\exists x(x \in \sigma[G] \wedge \phi(x, \tau_1[G], \dots, \tau_n[G])))^{M[G]}$ holds, and so there is an $x \in \sigma[G]$ such that $(\phi(x, \tau_1[G], \dots, \tau_n[G]))^{M[G]}$ ($\sigma[G] \in M[G]$ and so $\sigma[G] \cap M[G] = \sigma[G]$. Moreover, as x must be a value of a name such that this name is in the domain of σ, we have that, for some $\pi \in dom(\sigma)$, $x = \pi[G]$. Now, due to Theorem 3.2 $b)$, we can fix $r \in G$ such that $r \Vdash \phi(\pi, \tau_1, \dots, \tau_n)$. Since G is a filter, p and r share a common extension, q. Then, $q \Vdash \phi(\pi, \tau_1, \dots, \tau_n)$ by Lemma 3.3, and $q \leq p$. □

3.4 Every generic extension satisfies ZFC

In this section, we are going to show that, given a countable transitive model M for ZFC, a partial order $\mathbb{P} \in M$ and a $\mathbb{P}$-generic filter over M, G, then the generic extension $M[G]$ is indeed a model of ZFC. By Proposition 3.4, we will conclude that $M[G]$ is the least transitive model of ZFC extending M with $G \in M[G]$.
First, note that, since $M[G]$ is transitive by Proposition 3.4, then $M[G]$ satisfies the Axiom of Extensionality by Proposition 2.19. Also, by Proposition 2.24, the Axiom of Foundation is true in $M[G]$ since $M[G] \subset WF$. The following lemma holds.

Lemma 3.5. *The Axioms of Extensionality and Foundation are true in $M[G]$.* □

Lemma 3.6. *The Axiom of Comprehension is true in $M[G]$.*

Proof. By Lemma 2.8, we have to show that, for any formula $\phi(x, v, y_1, \ldots, y_n)$ with all its free variables listed and for any $\sigma, \tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ we have that $A = \{a \in \sigma[G] : (\phi(a, \sigma[G], \tau_1[G], \ldots, \tau_n[G]))^{M[G]}\}$ is in $M[G]$. We are going to see that A is indeed an element of $M[G]$, whose associated name is

$$\rho = \{(\pi, p) \in dom(\sigma) \times \mathbb{P} : p \Vdash (\pi \in \sigma \wedge \phi(\pi, \sigma, \tau_1, \ldots, \tau_n))\}.$$

But, by Theorem 3.2 a), ρ is definable in M , and so $\rho \in M^{\mathbb{P}}$. Now, we prove that $\rho[G] = A$. In order to simplify the exposition, we will not mention $\tau_1[G], \ldots, \tau_n[G]$. To see the left to right inclusion, fix $\pi[G] \in \rho[G]$ such that $(\pi, p) \in \rho$ for some $p \in G$. We have that $p \Vdash (\pi \in \sigma \wedge \phi(\pi))$. Thus, $\pi[G] \in \rho[G]$ and $\phi(\pi[G])^{M[G]}$ holds by the definition of $\Vdash$. Hence, $\pi[G] \in A$.
For the right to left inclusion, let $a \in A$. Then, $a \in \sigma[G]$ and $\phi(a)^{M[G]}$. Also, $a = \pi[G]$ with $(\pi, q) \in \sigma$ for some $q \in G$. So we have that $(\pi[G] \in \sigma[G] \wedge \phi(\pi[G]))^{M[G]}$. By Theorem 3.2 b), there is a $p \in P$ such that $p \Vdash (\pi \in \sigma \wedge \phi(\pi)$. This means that $(\pi, p) \in \rho$ and so $\pi[G] = a \in \rho[G]$. □

Lemma 3.7. *The Union Axiom is true in $M[G]$.*

Proof. Let $a \in M[G]$. By Proposition 2.22, we have to show that $\exists b \in M[G](\bigcup a \subset b)$. Since $a \in M[G]$, $a = \tau[G]$ for some $\tau \in M^{\mathbb{P}}$. Let $\pi = \bigcup \tau$. Since, if $\sigma \subset \tau$ then $\sigma \in M^{\mathbb{P}}$, we have that $\pi \in^{\mathbb{P}}$, and so $\pi[G] \in M[G]$. Now, let $c \in a = \tau[G]$. Then $c = \rho[G]$ for some $\rho \in dom(\tau)$. So $\rho \subset dom(\tau) = \pi$ which implies that $c = \rho[G] \subset \pi[G]$. Hence, $\bigcup a \subset \pi[G]$. □

To show that the Pairing Axiom holds in $M[G]$, we are going to define, given two names τ and σ in $M^{\mathbb{P}}$, a further name whose value is $\{\tau[G], \sigma[G]\}$.

Definition 3.14. *Let $\tau, \sigma \in M^{\mathbb{P}}$. We define a) $up(\tau, \sigma) = \{(\tau, \mathbb{1}), (\sigma, \mathbb{1})\}$,
b) $op(\tau, \sigma) = up(up(\tau, \tau), up(\tau, \sigma))$.*

Proposition 3.8. *Let $\tau, \sigma \in M^{\mathbb{P}}$. Then, $up(\tau, \sigma) \in M^{\mathbb{P}}$ and $up(\tau, \sigma)[G] = \{\tau[G], \sigma[G]\}$. Also, $op(\tau, \sigma) \in M^{\mathbb{P}}$ and $op(\tau, \sigma)[G] = (\tau[G], \sigma[G])$.* □

The next Lemma is immediate from the preceding Proposition.

Lemma 3.8. *The Axiom of Pairing is true in $M[G]$.* □

Lemma 3.9. *The Axiom of Replacement is true in $M[G]$.*

Proof. Fix a formula $\phi(x, u, v, z_1, \ldots, z_n)$ with all its free variables listed and let $\tau[G], \tau_1[G], \ldots, \tau_n[G] \in M[G]$ such that

$$(\forall x \in \tau[G]\ \exists! y\ \phi(x, y, \tau[G], \tau_1[G], \ldots, \tau_n[G]))^{M[G]}.$$

If we show that there is a $\rho[G] \in M[G]$ such that

$$\forall x \in \tau[G]\ \exists y \in \rho[G](\phi(x, y, \tau[G], \tau_1[G], \ldots, \tau_n[G]))^{M[G]},$$

then it will follow that the Replacement Axiom is true in $M[G]$ by Proposition 2.23. In order to simplify the exposition, we will not mention $\tau_1, \ldots, \tau_n$.
We consider $\psi(\sigma, A)$ to be the formula with free variables σ and A:

$$\forall \pi \in dom(\sigma)\ \forall p \in P\ (\exists \mu \in M^{\mathbb{P}}(p \Vdash \phi(\pi, \mu)) \to \exists \mu \in A(p \Vdash \phi(\pi, \mu))).$$

Since, by Theorem 3.2*a*), $p \Vdash \phi(\pi, \tau)$ is equivalent to a formula relativized to M, so is $\psi(\sigma, A)$. The Reflection Theorem 2.11 ensures the existence of an ordinal α for which $\psi(\sigma, A)$ is absolute for $V(\alpha)$. So, taking $B = V(\alpha) \cap M^{\mathbb{P}}$, $\psi(\tau, B)$ holds. Let $\rho = B \times \{\mathbb{1}\}$. By definition, $\rho[G] = \{\sigma[G] : \exists p \in G((\sigma, p) \in B \times \{\mathbb{1}\})\} = \{\sigma[G] : \sigma \in B\}$ because $\mathbb{1} \in G$. We assert that $\rho[G]$ satisfies the desired property. Indeed, let $x \in \tau[G]$. Then, $x = \pi[G]$ for some $\pi \in dom(\tau)$. We have that $(\exists y\ \phi(\pi[G], y))^{M[G]}$ holds, so, for some $\mu \in M^{\mathbb{P}}$, $\phi(\pi[G], \mu[G])^{M[G]}$ holds. By Theorem 3.2*b*), there exists a $p \in G$ such that $p \Vdash \phi(\pi, \mu)$ and, since, as discussed above, $\psi(\tau, B)$ is true, there exists a $\sigma \in B$ such that $p \Vdash \phi(\pi, \sigma)$. By the definition of $\Vdash$, $\phi(\pi[G], \sigma[G])^{M[G]}$ is true with $\sigma[G] \in \rho[G]$. □

Note that $\omega = \check{\omega} \in M[G]$ (by Proposition 3.4 and because $\omega \in M$), and so, as we have already proved that $M[G]$ is a model of $ZF^- - P - Inf$ the following Lemma is immediate by Proposition 2.25.

Lemma 3.10. *The Axiom of Infinity is true in $M[G]$.* □

Lemma 3.11. *The Power Set Axiom is true in $M[G]$.*

Proof. Let $\tau[G] \in M[G]$. We have to show that there is a $\sigma[G] \in M[G]$ such that $\forall \pi[G] \in M[G](\pi[G] \subset \tau[G] \to \pi[G] \in \sigma[G])$. Let $A = \{\rho \in M^{\mathbb{P}} : dom(\rho) \subset dom(\tau)\}$ and $\sigma = A \times \{\mathbb{1}\}$. We fix $\pi[G] \in M[G]$ such that $\pi[G] \subset \tau[G]$ and we will show that $\pi[G] \in \sigma[G]$.
For this, let

$$\mu = \{(\nu, p) : \nu \in dom(\tau) \land p \Vdash \nu \in \pi\}.$$

Clearly $dom(\mu) \subset dom(\tau)$, so $\mu \in A$, which implies that $\mu[G] \in \sigma[G]$. If we show that $\mu[G] = \pi[G]$, the proof will be done.
First, let's see that $\pi[G] \subset \mu[G]$. Let $x \in \pi[G]$. Since $\pi[G] \subset \tau[G]$, $x \in \tau[G]$ and so $x = \nu[G]$ for some $\nu \in dom(\tau)$. But $\nu[G] \in \pi[G]$, which means that $(\nu[G] \in \pi[G])^{M[G]}$. By Theorem 3.2 *b*), there is a $p \in G$ such that $p \Vdash \nu \in \pi$. Hence, by the definition of μ, $(\nu, p) \in \mu$ and so $\nu[G] = x \in \mu[G]$.
Now, let's see that $\mu[G] \subset \pi[G]$. Let $x \in \mu[G]$. Then, $x = \nu[G]$ for some $\nu \in dom(\mu)$. In particular, $(\nu, p) \in \mu$ for some $p \in G$. By the definition of μ, $p \Vdash \nu \in \pi$ and so $\nu[G] = x \in \pi[G]$. □

Regarding this proof note that, $\tau[G] \in M[G]$ being fixed, $\{\sigma \in M^{\mathbb{P}} : \sigma[G] \subset \tau[G]\}$ is in general not contained in any subset of M. Nonetheless, for any $\sigma[G] \subset \tau[G]$ there is a subset of names of M where all the names of $\sigma[G]$ are contained. Now, let's see an equivalent condition of the Axiom of Choice.

Proposition 3.9. *The following condition is equivalent to the Axiom of Choice.*

$$\forall x\ \exists \alpha \in ON(f\ is\ a\ function \wedge dom(f) = \alpha \wedge x \subset ran(f)). \tag{3.1}$$

Proof. In order to see that this condition implies the Axiom of Chocie, fix x and α as in (3.1). Let $g : x \to \alpha$ be the function defined as $g(y) = min(f^{-1}\{y\})$ for $y \in x$. Note that, if $y, z \in x$, then $g(y) = g(z)$ implies that $min(f^{-1}\{y\}) = min(f^{-1}\{z\})$ and so $y = z$. Thus, g is injective. Let R be the relation on x defined by $yRz \leftrightarrow g(y) < g(z)$. Then, R well-orders x and so the Axiom of Choice is true.
The other implication is clear. □

Lemma 3.12. *The Axiom of Choice is true in $M[G]$.*

Proof. Let $x \in M[G]$. Then $x = \tau[G]$ for some $\tau \in M^{\mathbb{P}}$. In particular, $\tau \in M$, and so, since the Axiom of Choice is true in M, by Proposition 3.9, there is a $\alpha \in o(M)$ and there is a function $f \in M$ such that $dom(f) = \alpha$ and $dom(\tau) \subset ran(f)$. Since the Axiom of Choice holds in M, $dom(\tau)$ can be enumerated. Then, there is a $\beta < \omega$ such that $dom(\tau) = \{f(\gamma) : \gamma < \beta\}$.
Now let $\sigma = \{op(\check{\gamma}, f(\gamma)) : \gamma < \beta\} \times \{\mathbb{1}\}$. By Proposition 3.8, for $\gamma < \beta$ we have that $op(\check{\gamma}, f(\gamma)) \in M^{\mathbb{P}}$ and $op(\check{\gamma}, f(\gamma))[G] = (\gamma, f(\gamma))[G]$. Thus, $\sigma \in M$ and $\sigma[G] = \{(\gamma, f(\gamma)[G] : \gamma < \beta)\}$. We can conclude that $\sigma[G]$ is a function in $M[G]$, $dom(\sigma[G]) = \beta$ and $\tau[G] \subset ran(\sigma[G])$. By Proposition 3.9, the Axiom of Choice is true in $M[G]$. □

So we have proved the following theorem.

Theorem 3.3. *Let M be a countable transitive model of ZFC. Let $\mathbb{P} \in M$ be a partial order. Let G be a $\mathbb{P}$-generic filter over M. Then, the generic extension $M[G]$ is a transitive model of ZFC.* □

Now, let's find a generic extension that satisfies, additionally, $\neg CH$.

Chapter 4

Forcing ZFC+¬CH

In the previous chapter, we have shown that any generic extension relative to a $\mathbb{P}$-generic filter over a countable transitive model M of ZFC is also a model of ZFC (and is in fact the minimum one containing G). We wish to find a generic extension M[G] that also satisfies $\neg CH$ in order to show the consistency of $ZFC + \neg CH$. To do this, we will use a particular partial order, as we will see now.

4.1 Forcing with finite partial functions

We are now going to construct a generic extension satisfying $\neg CH$. In fact, we will find a generic extension $M[G]$ where $(2^\omega \geq \omega_2)^{M[G]}$. The partial order that will allow us to do this is the following one.

Definition 4.1. *Let I and J be sets. We define the set of finite partial functions from I into J as $Fn(I,J) = \{p : |p| < \omega \wedge p \text{ is a function } \wedge \ dom(p) \subset I \ \wedge \ ran(p) \subset J\}$.*

Let M be a countable transitive model of ZFC. For any sets I and J, define $\mathbb{F}n(I,J) = (Fn(I,J), \leq, 0)$, with $p \leq q$ if and only if $q \subset p$. It is easy to show that $\mathbb{F}n(I,J)$ is a partial order. Moreover, all the notions involved in the definition of $\mathbb{F}n(I,J)$ are absolute, and so is $\mathbb{F}n(I,J)$. It follows that, if $I,J \in M$, $\mathbb{F}n(I,J) = \mathbb{F}n(I,J)^M \in M$.

Lemma 4.1. *Let $I,J \in M$. Let $G \in Fn(I,J)$ be a filter. Then $\bigcup G$ is a function with $dom(\bigcup G) \subset I$ and $ran(\bigcup G) \subset J$.*
Furthermore, if I is infinite, $J \neq 0$ and G is $\mathbb{F}n(I,J)$-generic over M, then $\bigcup G$ is a surjective function from I onto J.

Proof. Since G is a filter, $\forall p,q \in G \ \exists r \in G(p \subset r \ \wedge q \subset r)$ and so it is clear that $\bigcup G$ is a function with $dom(\bigcup G) \subset I$ and $ran(\bigcup G) \subset J$.
Now, suppose that $J \neq 0$. For any $i \in I$, let $D_i = \{p \in Fn(I,J) : i \in dom(p)\}$.

Then, by absoluteness, $D_i \in M$ for any $i \in I$. Moreover, D_i is dense for any $i \in I$ since any condition can be extended to another containing i in its domain. So, if G is $\mathbb{F}n(I, J)$-generic, $\forall i \in I(G \cap D_i \neq 0)$. Hence, $dom(\bigcup G) = I$.
As I is infinite, $E_j = \{p \in Fn(I, J) : j \in ran(p)\}$ is dense and in M by the same arguments and so $ran(\bigcup G) = J$. □

Recall that the cardinal κ^λ corresponds to the cardinality of the set of functions f with $dom(f) = \lambda$ and $ran(f) \subset \kappa$. Let κ be an uncountable cardinal in M. We consider $\mathbb{F}n(\kappa \times \omega, 2)$. Let G be $\mathbb{F}n(\kappa \times \omega, 2)$-generic over M. By Lemma 4.1, $\bigcup G$ is a function from $\kappa \times \omega$ onto 2. For any $\alpha < \kappa$, we define the function $f_\alpha(n) = \bigcup G(\alpha, n)$ from ω into 2 and we consider the family of functions $\{f_\alpha\}_{\alpha<\kappa}$. By absoluteness, this family is in $M[G]$. The next proposition says that $f_\alpha \neq f_\beta$ for $\alpha \neq \beta$ and so there are at least $|\kappa|$ many functions from ω into 2 in $M[G]$.

Proposition 4.1. *If $\kappa \in M$ is an uncountable cardinal and G is a $\mathbb{F}n(\kappa \times \omega, 2)$-generic filter over M, then the f_α described above are all distinct in $M[G]$ and so we have that $(2^\omega \geq |\kappa|)^{M[G]}$.*

Proof. Let $\alpha \neq \beta$. To see that $f_\alpha \neq f_\beta$, let

$$D_{\alpha\beta} = \{p \in Fn(\kappa \times \omega, 2) : \exists n \in \omega((\alpha, n) \in dom(p) \wedge (\beta, n) \in dom(p) \wedge \\ \text{p}(\alpha, n) \neq p(\beta, n))\}.$$

Clearly $D_{\alpha\beta}$ is in M since it is defined from concepts defined in M. Also, $D_{\alpha\beta}$ is dense. Indeed, if $p \in P$ such that $p \notin D_{\alpha\beta}$, then $\forall n \in \omega((\alpha, n), (\beta, n) \in dom(p) \rightarrow p(\alpha, n) = p(\beta, n))$. Since $|p| < \omega$, there is some $N \in \omega$ such that $(\alpha, N) \notin dom(p)$ and we can extend p to another condition q with $(\alpha, N) \in dom(q)$ and $q(\alpha, N) \neq q(\beta, N)$, which implies that $q \in D_{\alpha\beta}$.
Thus, $G \cap D_{\alpha\beta} \neq 0$. This implies that there is a $p \in G$ and a $n \in \omega$ such that $p(\alpha, n) \neq p(\beta, n)$ and, since $p \subset \bigcup G$, we have that $f_\alpha \neq f_\beta$.
Hence, there are at least $|\kappa|$ many functions from ω into 2 in $M[G]$, which means that $(2^\omega \geq |\kappa|)^{M[G]}$. □

Unfortunately, we do not have control over this κ. We wish to have $\kappa = \omega_2$ in $M[G]$ to obtain that $(2^\omega \geq \omega_2)^{M[G]}$ but, whereas the ordinals in $M[G]$ are just the ordinals in M, we do not know whether a cardinal in M remains the same in $M[G]$ because the notion of cardinal is not absolute: an uncountable cardinal in M can collapse to a countable ordinal in $M[G]$. Thereby, $\kappa = \omega_2$ in M generally does not imply that $\kappa = \omega_2$ in $M[G]$. So, we need the following notion.

Definition 4.2. *If $\mathbb{P} \in M$ is a partial order, we say that $\mathbb{P}$ preserves cardinals if and only if, for all $\mathbb{P}$-generic G over M,*

$$\forall \kappa \in o(M)((\kappa \text{ is a cardinal})^M \leftrightarrow (\kappa \text{ is a cardinal})^{M[G]}).$$

To assure $\mathbb{F}n(\omega_2 \times \omega, 2)$ preserves cardinals, we will need to use the concept of *countable chain condition*. We need some preparation.

4.2 The basic Δ-system lemma

In this section, we are going to see a combinatoric result called the Δ-system Lemma that we will use later to show that $\mathbb{F}n(\omega_2 \times \omega, 2)$ preserves cardinals.

Definition 4.3. *A family F of sets is called a Δ-system if and only if there is a set x^* such that, $\forall x, y \in F$, if $x \neq y$ then $x \cap y = x^*$. We say that x^* is the root of F.*

Now, let's see a preliminary result.

Lemma 4.2. *Let $n > 1$. Let F be an uncountable family of finite sets such that $\forall x \in F(|x| = n)$. Then, F has an uncountable subfamily that is a Δ-system.*

Proof. We proceed by induction on n.
If $n = 1$, then F forms a Δ-system with root $\emptyset$.
Suppose that $n > 1$. Let F' be a maximal disjoint subfamily of F.
If F' is uncountable, then F' is itself a Δ-system with root $\emptyset$.
So, suppose that F' is countable. Then, since $|x| = n$ for all $x \in F'$, we have that $\bigcup F'$ is also countable, and, by the Axiom of Choice, we can enumerate it. So, we put $\bigcup F' = \{f_n : n \in \omega\}$. Let, for all $n \in \omega$, $F_n = \{X \in F : f_n \in X\}$.
Now, we will show that there is an $f \in \bigcup F'$ such that f belongs to ω_1 many elements of F. Suppose that this f does not exist. Then, $\forall n \in \omega(|F_n| \leq \omega)$. Since F is uncountable and $\bigcup\{F_n : n \in \omega\}$ is countable, there is an $Y \in F \setminus \bigcup\{F_n : n \in \omega\}$. Then, $Y \cap \{f_n : n \in \omega\} = \emptyset$. Thus, $F' \cup \{Y\}$ is disjoint, which contradicts the fact that F' is a maximal disjoint subfamily in F.
So there is an $f \in \bigcup F'$ and there is a subfamily $F'' \subset F$ such that $|F''| = \omega_1$ and $\forall X \in F''(f \in X)$.
Let $Z = \{X \setminus \{f\} : X \in F''\}$. By the inductive hypothesis, there is an uncountable subfamily $G \subset Z$ such that G is a Δ-system. Hence, $\{Y \cup \{f\} : Y \in G\}$ is an uncountable subfamily of F and forms a Δ-system. □

Lemma 4.3 (Δ-system Lemma)**.** *Let F be an uncountable family of finite sets. Then there is an uncountable subfamily of F that is a Δ-system.*

Proof. Let F be an uncountable family of finite sets. Let $F_n = \{X \in F : |X| = n\}$. Then $F = \bigcup\{F_n : n \in \omega\}$. Since F is uncountable, there must be a F_n that is uncountable for some $n \in \omega$. By Lemma 4.2, F_n has an uncountable subfamily that is a Δ-system. □

4.3 Preservation of cardinals in $\mathbb{F}n(\omega_2 \times \omega, 2)$

In this section, we are going to prove that cardinals are preserved in $\mathbb{F}n(\omega_2 \times \omega, 2)$. It will follow that, if M is a countable transitive model of ZFC with $\mathbb{F}n(\omega_2 \times \omega, 2) \in M$ and $(\kappa = \omega_2)^M$ then we also have that $(\kappa = \omega_2)^{M[G]}$. For this, we need the following combinatoric concept.

Definition 4.4. *$A \subset \mathbb{P}$ is an antichain in $\mathbb{P}$ if and only if $\forall p, q \in A(p \neq q \rightarrow p \perp q)$. We say that $\mathbb{P}$ satisfies the countable chain condition if and only if every antichain in $\mathbb{P}$ is countable.*

Now, our aim is to prove that $\mathbb{F}n(\kappa \times \omega, 2)$ satisfies the countable chain condition for any uncountable cardinal κ. For this, we show the following more general result.

Proposition 4.2. *Let I, J be sets with J countable. Then $\mathbb{F}n(I, J)$ satisfies the countable chain condition.*

Proof. For each $\alpha < \omega_1$, let $p_\alpha \in Fn(I, J)$ with $\mathbb{P}_{\alpha\beta}$ if $\alpha \neq \beta$ and let $a_\alpha = dom(p_\alpha)$. We will show that $\{p_\alpha : \alpha < \omega_1\}$ is not an antichain. By the Δ-system Lemma 4.3, let $X \subset \omega_1$ such that X is uncountable and $\{a_\alpha : \alpha \in X\}$ forms a Δ-system. Let r be its root. Then, J^r is countable (because J is countable) and so $\{p_\alpha \restriction r : \alpha < \omega_1\}$ is countable since each p_α is finite. This implies that there is an uncountable subset Y of X such that the $p_\alpha \restriction r$ are all the same for all $\alpha \in Y$ (if there was not, there would be uncountably many $p_\alpha \restriction r$). But this implies that, if $\alpha, \beta \in Y$, then p_α and p_β are compatible. □

So, in particular, $\mathbb{F}n(\kappa \times \omega, 2)$ satisfies the countable chain condition for any uncountable cardinal κ. Now, we will show that if $\mathbb{P}$ is a partial order with the countable chain condition, then $\mathbb{P}$ preserves cardinals. We will use the following notion.

Definition 4.5. *If $\mathbb{P} \in M$, we say that $\mathbb{P}$ preserves cofinalities if and only if, for any $\mathbb{P}$-generic G over M and for any limit ordinal γ in M, $cf(\gamma)^M = cf(\gamma)^{M[G]}$.*

Proposition 4.3. *If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.*

Proof. We suppose that $\mathbb{P}$ preserve cofinalities. The fact that finite cardinals are preserved is obvious by absoluteness. We will show that regular cardinals in M are regular cardinals in $M[G]$ and limit cardinals in M are limit cardinals in $M[G]$. Since any infinite cardinal is either regular or a limit cardinal, this will imply that cardinals are preserved.
Let $\kappa \in M$ be an infinite cardinal. If κ is a regular cardinal in M, then $cf(\kappa)^M = \kappa$

and so $cf(\kappa)^{M[G]} = cf(\kappa)^M = \kappa$; it follows that κ is also a regular cardinal in $M[G]$. If κ is an uncountable limit cardinal in M, then the set of successor cardinals in M smaller than κ is unbounded in κ. As these successor cardinals are regular, they are also regular in $M[G]$, which implies that the set of successor cardinals smaller than κ in $M[G]$ are unbounded in κ, and so κ is an uncountable limit cardinal in $M[G]$. □

The following two lemmas are essential to show that any partial order with the countable chain condition preserves cardinals.

Lemma 4.4. *Let $\mathbb{P} \in M$ such that $(\mathbb{P}\ satisfies\ the\ countable\ chain\ condition)^M$. Let G be $\mathbb{P}$-generic over M. Let $A, B \in M$ and $f : A \to B$ such that $f \in M[G]$. Then, there is an $F : A \to \mathcal{B}$ such that $F \in M$ and, for all $a \in A$, $f(a) \in F(a)$ and the cardinality of $F(a)$ is countable.*

Proof. Since $f \in M[G]$, $f = \tau[G]$ for some $\tau \in M^{\mathbb{P}}$.
We have that $(\tau[G]\ is\ a\ function\ from\ A\ into\ B)^{M[G]}$ holds and so, by Theorem 3.2 a), there is a $p \in G$ such that $p \Vdash \tau\ is\ a\ function\ from\ \check{A}\ into\ \check{B}$. We define the function F from A into B such that

$$F(a) = \{b \in B : \exists q \leq p(q \Vdash \tau(\check{a}) = \check{b})\}.$$

Since $\Vdash$ can be defined within M by Theorem 3.2 a), we have that $F \in M$. We now show that this function satisfies the desired properties: let $a \in A$ and let's prove that $f(a) \in F(a)$ and $(|F(a)| \leq \omega)^M$.
Put $b = f(a)$. So, there is a $q \in G$ such that $q \Vdash \tau(\check{a}) = b$. Since p and q share a common extension, r, because G is a filter, then $q \Vdash \tau(\check{a}) = \check{b}$ and so $f(a) \in F(a)$.
Now, for any $b \in F(a)$, let $X_b = \{q \in P : q \leq p \wedge q \Vdash \tau(\check{a}) = \check{b}\}$. Since $b \in F(a)$, X_b is non-empty. By the Axiom of Choice in M, let R_b be a well-ordering of X_b. Then, we define the function $Q : F(a) \to P$ such that, for any $b \in F(a)$, $Q(b)$ is the R_b-least element of X_b. By the definition of X_b, $Q(b) \leq p$ and $Q(b) \Vdash \tau(\check{a}) = \check{b}$. Moreover, all the $Q(b)$'s are incompatible. Indeed, if there were $b, c \in F(a)$ such that $Q(b)$ and $Q(c)$ were compatible, as they both extend p there would be a $\mathbb{P}$-generic filter H over M containing both of them and so, in the generic extension $M[H]$, $\tau[H] : A \to B$ would be a function such that $\tau[H](a) = b$ and $\tau[H](a) = c$, which is a impossible. Also, $Q \in M$ by absoluteness.
So the set $Y = \{Q(b) : b \in F(a)\}$ is an antichain in P and so, as $\mathbb{P}$ satisfies the countable chain condition in M, Y must be countable in M, which clearly implies that $F(a)$ is countable in M. □

Lemma 4.5. *Let $\mathbb{P} \in M$. If, for any regular uncountable cardinal κ in M and any $\mathbb{P}$-generic G over M we have that $(\kappa\ is\ regular)^{M[G]}$, then $\mathbb{P}$ preserves cofinalities.*

Proof. Let γ be a limit ordinal in M such that $(\kappa = cf(\gamma))^M$. By Proposition 1.1*a*), let f be a strictly increasing function mapping κ into γ cofinally. Also, we have that $(\kappa\ \textit{is a regular cardinal})^M$. If $(\kappa = \omega)^M$, then, by absoluteness of ω, $(\kappa = \omega)^{M[G]}$, and if $(\kappa > \omega)^M$ then $(\kappa\ \textit{is regular})^{M[G]}$. Now, as $f \in M[G]$, $(\kappa = cf(\gamma))^{M[G]}$ by Proposition 1.1*b*). □

Proposition 4.4. *If* $\mathbb{P} \in M$ *and* $(\mathbb{P}\ \textit{satisfies the countable chain condition})^M$*, then* $\mathbb{P}$ *preserves cofinalities and hence cardinals.*

Proof. Suppose that $\mathbb{P}$ does not preserve cofinalities. By Lemma 4.5, there must be an uncountable cardinal $\kappa \in M$ such that $(\kappa\ \textit{is regular})^M$ but $(\kappa\ \textit{is not regular})^{M[G]}$. It follows that for some $\alpha < \kappa$ there is a function $f \in M[G]$ such that f maps α cofinally into κ. Now, by Lemma 4.4, there is a function $F \in M$, $F : \alpha \to \mathcal{P}(\kappa)$, such that, for any $\beta < \alpha$, $f(\beta) \in F(\beta)$ and the cardinality of $F(\beta)$ is countable in M. Let $A = \bigcup_{\beta<\alpha} F(\beta)$. By absoluteness of the union, we have that $A \in M$ and clearly A is an unbounded subset of κ. Moreover, $|A| = |\alpha|$ since each $F(\beta)$ has countable cardinality and there are as many as $|\alpha|$. Thus, $(|A| < \kappa)^M$, which contradicts the fact that $(\kappa\ \textit{is regular})^M$. □

We now have all the elements to find a generic extension where CH is false.

Theorem 4.1. *Let M be a countable transitive model of ZFC such that* $\mathbb{F}n(\omega_2^M \times \omega, 2) \in M$. *Let G be a* $\mathbb{F}n(\omega_2^M \times \omega, 2)$*-generic filter over M. Then,* $(2^\omega \geq \omega_2)^{M[G]}$ *and hence the Continuum Hypothesis fails in* $M[G]$.

Proof. By Proposition 4.2, $\mathbb{F}n(\omega_2^M \times \omega, 2)$ satisfies the countable chain condition in M. Then, by Proposition 4.4, it preserves cofinalities and so it preserves cardinals. Thus, $\omega_2^{M[G]} = \omega_2^M$, and so $(2^\omega \geq \omega_2)^{M[G]}$ by Proposition 4.1. Hence, $(2^\omega \geq \omega_1)^{M[G]}$, and so the Continuum Hypothesis fails in $M[G]$. □

Also, by using a class of special names, the so-called *nice names*, we can prove that if κ is an uncountable cardinal of M such that $(\kappa^\omega = \kappa)^M$, $\mathbb{P} = \mathbb{F}n(\kappa \times \omega, 2) \in M$ and G is $\mathbb{P}$-generic over M, then $(2^\omega = \kappa)^{M[G]}$. It follows that it is consistent that 2^ω is any cardinal whose cofinality is uncountable.

Conclusions

In this work we have seen the main properties of the forcing method introduced by Cohen in 1966 and we have proved one of the main applications of forcing: the consistency of the negation of CH with ZFC. Previously, the consistency of CH with ZFC was shown by Gödel. So, CH is independent of ZFC, and this means that neither CH nor $\neg CH$ can be proved by ordinary mathematical means.

In order to introduce the method of forcing, we have studied the central notions of relativization and absoluteness and we have proved the Reflection Theorems, which permit us to construct countable transitive models of any set of axioms that extends ZFC and, so, allow us to consider a ground countable transitive model M of ZFC as a starting point. Then, by using the method of forcing, we can construct from M, a partial order $\mathbb{P}$ in M and a $\mathbb{P}$-generic filter G over M, a generic extension $M[G]$ of the model M satisfying additional properties. In particular, by using the partial order of the set of finite partial functions from $\omega_2 \times \omega$ into 2, we have proved that the generic extension $M[G]$ contains ω_2 many functions from ω into 2, and so the Continuum Hypothesis fails in $M[G]$.

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Index