SOLVING $ax^p + by^p = cz^p$ WITH *abc* CONTAINING AN ARBITRARY NUMBER OF PRIME FACTORS

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ABSTRACT. In this paper we prove new cases of the asymptotic Fermat equation with coefficients. This is done by solving some remarkable *S*-unit equations and applying a method of Frey-Kraus-Mazur.

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INTRODUCTION

Let p be a rational prime and consider the degree p Fermat equation

$$x^p + y^p + z^p = 0. (1)$$

The group \mathbb{Q}^{\times} acts on the set of rational solutions of (1) by

$$\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z), \qquad \lambda \in \mathbb{Q}^{\times}.$$

That allows us to consider solutions in the rational projective plane

$$\mathbb{P}_2(\mathbb{Q}) = (\mathbb{Q}^3 \setminus \mathbf{0}) / \mathbb{Q}^{ imes}$$

That is, equation (1) defines a projective plane curve F_p in \mathbb{P}_2 .

By the genus-degree formula F_p has genus

$$g_p = (p-1)(p-2)/2.$$

Faltings' theorem [10] states that the set $F_p(\mathbb{Q})$ of \mathbb{Q} -rational points of F_p is finite if $g_p \geq 2$. Genus 0 and genus 1 curves, corresponding to p = 2 and p = 3 respectively, might have infinitely many rational points. The main goal in this paper is to prove a finiteness statement hence, we will avoid the case $p \leq 3$.

Fermat's last Theorem predicted that

$$F := \bigcup_{p \ge 5} F_p(\mathbb{Q}) = \{ [1:-1:0], [1:0:-1], [0:1:-1] \}.$$

In this paper we are interested in the finiteness of F and we shall generalize it to Fermat equations with coefficients. Let a, b, c be non-zero integers and let $F_p^{a,b,c}$ denote the projective curve given by

$$ax^p + by^p + cz^p = 0.$$

The Asymptotic Fermat Conjecture with coefficients a, b, c predicts that

Conjecture 1. The set

$$AF_{a,b,c} := \bigcup_{p \ge 5} F_p^{a,b,c}(\mathbb{Q})$$

is finite.

It is straightforward to see that the set of *trivial points* in $AF_{a,b,c}$, i.e. points [x:y:z] satisfying xyz = 0, is finite.

The very first non-trivial evidence of Conjecture 1 was established by Andrew Wiles when proving Taniyama-Shimura conjecture for the semistable case and hence proving the case a = b = c = 1.

Theorem (Wiles [32]).

$$AF_{1,1,1} = \{ [1:0:-1], [1:-1:0], [0:1:-1] \}$$

Remark. Case p = 3 of Fermat's last Theorem was proved by Leonhard Euler.

Jean-Pierre Serre, Barry Mazur and Gerhard Frey had previously established some cases of the conjecture, conditionally on Serre's conjecture or Taniyama-Shimura conjecture; both proved now.

Theorem (Serre [26]). Let n be a non-negative integer and let q be a prime in

 $\{3, 5, 7, 11, 13, 17, 19, 23, 29, 53, 59\}.$

Then

$$AF_{1,1,q^n} \subseteq F_5^{1,1,q^n}(\mathbb{Q}) \cup F_7^{1,1,q^n}(\mathbb{Q}) \cup F_q^{1,1q^n}(\mathbb{Q}) \cup \{trivial \ points\}$$

Theorem (Frey-Mazur [12]). Let q be an odd prime which is neither a Mersenne prime nor a Fermat prime, let n be a positive integer and m a non-negative integer. Then

 $AF_{1,q^n,2^m}$ is finite.

Kenneth Ribet, for the case $2 \le m < p$, and Henri Darmon, Loïc Merel, for the case m = 1 studied equation $X^p + Y^p + 2^m Z^p = 0$. In particular they proved that

Theorem (Ribet [24], Darmon-Merel [7]).

 $AF_{1,1,2^m} = \{ [1:-1:0] \} \cup \{ [2^r:2^r:-1] \mid m = rp+1 \text{ for } p \ge 5 \}.$

For a non-zero integer N let rad(N) denote the greatest square-free divisor of N. Let **P** denote the set of prime numbers. We can and will identify the image of $rad : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$ with the set of finite subsets of **P**. In particular the *radical* of ± 1 corresponds to the empty set under that identification. Similarly rad'(N) will denote the greatest odd divisor of rad(N).

Alain Kraus has given effective bounds related to the Asymptotic Fermat conjecture and proved the following.

Theorem (Kraus [14, Corollaire 1]). Let (a, b, c) be non-zero pairwise coprime integers such that rad(abc) = 2q for an odd prime q which is neither a Mersenne prime nor a Fermat prime. Then there is an explicit constant G = G(a, b, c) such that

$$AF_{a,b,c} = \{trivial \ points\} \cup \bigcup_{5 \le p < G} F_p^{a,b,c}(\mathbb{Q}).$$

Remark. Case $(a, b, c) = (1, 1, 2^{\alpha}q^{\beta})$ is not explicitly stated in Kraus' paper. Nevertheless, the same method as for the case $(1, 2^{\alpha}, q^{\beta})$ applies.

Related to this conjecture Nuno Freitas, Emmanuel Halberstadt and Alain Kraus, have recently developed the so-called *symplectic method* to solve Fermat equations for a positive density of exponents p, see [11] or [13]. Our approach follows similar strategies as in [14] and relies strongly on modularity; see [4] chapter 15 for an exposition of the modular method written by Samir Siksek and Theorem 15.5.3 therein for an improvement of Serre's Theorem, .

In this paper we exhibit non-trivial local obstructions¹ to some S-unit equations and we deduce results as the following. Let (a, b, c) be a primitive tern, i.e. gcd(a, b, c) = 1, of non-zero integers.

Theorem 2. Assume that rad(abc) is a product of primes all in $1+12\mathbb{Z}$ then $AF_{a,b,2^{r}c}$ is finite for every $r > 0, r \neq 1$.

Theorem 3. Assume that rad(abc) is a product of primes all in $1+3\mathbb{Z}$ then $AF_{a,b,16c}$ is finite.

We also consider some particular cases with $rad(abc) = q\ell$, for different odd primes q, ℓ . For example

Theorem 4. Let $q, \ell \geq 5$ be primes such that $q \equiv -\ell \equiv 5 \pmod{24}$. If rad(abc) = $q\ell$ then, $AF_{a,b,c}$ is finite.

See section 5 for the complete list of cases we consider.

Remark. We use Kraus method to deduce explicit bounds G(a, b, c) on p, see Section 6.

1. Fermat-type curves

Let $a, b, c \in \mathbb{Z}$, p prime, $abc \neq 0$. By a Fermat-type curve we mean a projective plane curve of the form

$$F_p^{a,b,c}: ax^p + by^p + cz^p = 0.$$

Notice that the Fermat-Type curve $F_p^{a,b,c}$ is a twist of the classical one $x^p + y^p + z^p = 0$ thus, they share some geometric properties as the genus. Also, the condition $abc \neq 0$ is equivalent to $F_p^{a,b,c}$ being non-singular.

Theorem 1.1 (Faltings, [10]). Let C/\mathbb{Q} be a projective curve of genus > 2. Then $C(\mathbb{Q})$ is finite.

By the genus-degree formula one has that $F_p^{a,b,c}/\mathbb{Q}$ has genus

$$(p-1)(p-2)/2.$$

This is a consequence of Hurwitz theorem, [27, II, 5.9].² Thus $F_p^{a,b,c}(\mathbb{Q})$ is finite for $p \geq 5$. The sets $F_2^{a,b,c}(\mathbb{Q})$, $F_3^{a,b,c}(\mathbb{Q})$ might be infinite³. Let a, b, c be non-zero integers and let us consider the set $AF_{a,b,c}$ defined in Con-

jecture 1. The following result is a direct consequence of [6, Proposition 1.1].

Proposition 1.2. Assume that there is a prime ℓ such that $v_{\ell}(a), v_{\ell}(b), v_{\ell}(c)$ are pairwise different. Then $AF_{a.b.c}$ is finite.

 $^{^{1}}$ We use the terminology *non-trivial local obstructions* to distinguish from the ones introduced in Proposition 1.2.

²Consider the degree p morphism $\phi: F_p^{a,b,c} \to \mathbb{P}_1, [x:y:z] \mapsto [x:y]$. It is ramified at p points with constant ramification index p.

³The set $F_2^{a,b,c}(\mathbb{Q})$ is infinite if and only if it is not empty. If $\mathcal{O} \in F_3^{a,b,c}(\mathbb{Q})$ then $(F_3^{a,b,c},\mathcal{O})$ is an elliptic curve over \mathbb{Q} and $F_3^{a,b,c}(\mathbb{Q})$ is a finitely generated group.

Proof. Let us see that

$$F_p^{a,b,c}(\mathbb{Q}) = \emptyset$$

for every $p > k := \max(v_\ell(a), v_\ell(b), v_\ell(c))$. Let $[x : y : z] \in F_p^{a,b,c}(\mathbb{Q})$ and
 $(A, B, C) = (ax^p, by^p, cz^p) \neq (0, 0, 0).$

Then $v_{\ell}(A) \leq v_{\ell}(B) \leq v_{\ell}(C) \leq \infty$ up to permutation of A, B, C and A+B+C=0. Notice that

$$v_{\ell}(A) = v_{\ell}(B) < \infty$$

since $v_{\ell}(A) = v_{\ell}(B) < \infty$ since $v_{\ell}(A) = v_{\ell}(B + C) \ge v_{\ell}(B)$ and $(A, B, C) \ne (0, 0, 0)$. Hence

$$v_{\ell}(a) \stackrel{(\text{mod } p)}{\equiv} v_{\ell}(A) = v_{\ell}(B) \stackrel{(\text{mod } p)}{\equiv} v_{\ell}(b)$$

Thus $v_{\ell}(a) = v_{\ell}(b)$ and

 $AF_{a,b,c} = \bigcup_{5 \le p \le k} F_p^{a,b,c}(\mathbb{Q})$

is finite.

We say that a tern a, b, c has a *trivial local obstruction* if there is a prime q such that $v_q(a), v_q(b), v_q(c)$ are pairwise different. Thus, we shall focus on terms with no trivial obstruction. We make the following hypothesis.

(F): The tern (a, b, c) has no trivial local obstruction and gcd(a, b, c) = 1.

Notice that a, b, c satisfies (F) if a, b, c are pairwise coprime.

Lemma 1.3. Let a, b, c be a tern satisfying (F) and let p be a prime such that

$$p > \max_{q \text{ prime}} \max(v_q(a), v_q(b), v_q(c)).$$

Then there are pairwise coprime integers α, β, γ such that $rad(\alpha\beta\gamma) = rad(abc)$ and

$$F_n^{a,b,c} \simeq F_n^{\alpha,\beta,\gamma}$$

as algebraic curves over \mathbb{Q} .

Proof. Let a, b, c be non-zero integers satisfying (F) and let

$$T_a = \gcd(b, c),$$

$$T_b = \gcd(a, c),$$

$$T_c = \gcd(a, b).$$

Then there are integers a', b', c' such that $a', b', c', T_a, T_b, T_c$ are pairwise coprime and

$$\begin{aligned} a &= a' \quad T_b \quad T_c, \\ b &= b' \quad T_a \quad T_c, \\ c &= c' \quad T_a \quad T_b. \end{aligned}$$

The Lemma follows by an induction on the number of prime divisors of $T_a T_b T_c$. Assume that $1 \leq e := v_q(T_a) < p$. The linear map $[x : y : z] \mapsto [qx : y : z]$ defines an isomorphism $F_p^{a_1,b_1,c_1} \to F_p^{a,b,c}$, where $q^e(a_1,b_1,c_1) = (q^p a, b, c)$. Hence

$$T_{a_1} = T_a/q^e,$$

 $T_{b_1} = T_b,$
 $T_{c_1} = T_c.$

and $\operatorname{rad}(a_1b_1c_1) = \operatorname{rad}(abc)$.

Remark 1.4. With the notation above one has that $v_q(\alpha\beta\gamma) = v_q(abc)$ if $q \nmid T_a T_b T_c$ and $v_q(\alpha\beta\gamma) = p - v_q(abc)/2 = p - v_q(T_a T_b T_c)$ otherwise.

2. S-unit equations

Let S be a finite set of primes. We identify S with its product in \mathbb{Z} . Let a, b, c be non-zero integers and consider the projective line L : aX + bY + cZ = 0 attached to it. The set

$$L(\mathbb{Q}) = \{ [x:y:z] \in \mathbb{P}^2(\mathbb{Q}) : L(x,y,z) = 0 \}$$

of \mathbb{Q} -rational points of L is infinite.

Definition 2.1. Let $P \in L(\mathbb{Q})$ and let $(x, y, z) \in \mathbb{Z}^3$ be a primitive representative of P, that is gcd(x, y, z) = 1. We say that P is an *S*-point of L if $xyz \neq 0$ and $rad(xyz) \mid S$.

Theorem 2.2 (Siegel-Mahler). Let a, b, c be non-zero integers and let S be a finite set of prime numbers. The set of S-points in the line

$$L:aX + bY + cZ = 0$$

is finite.

Proof (Lang [15, p. 28]). The S-points of L correspond to the points of the affine curve C: aX + bY + c = 0 with values in

$$\Gamma = \mathbb{Z}[1/S]^{\times} < \mathbb{Q}^{\times}.$$

The set S together with -1 generate the abelian group Γ , hence Γ/Γ^5 is finite. If C has infinitely many points with coefficients in Γ , then infinitely many points $\{(x_i, y_i)\}_{i\geq 1}$ coincide mod Γ^5 . Thus the curve $ax_1X^5 + by_1Y^5 + c = 0$ has infinitely many rational points and it has genus 6 since $ax_1by_1c \neq 0$. This contradicts Faltings' theorem.

Let us focus on the projective line

$$L_0: X + Y + Z = 0.$$

Frey-Kraus-Mazur (FKM) method on the Asymptotic Fermat Conjecture with coefficients (a, b, c) considers the set of primes S = rad(2abc) and seeks for S-points of L_0 . The 2-adic valuation of the S-points will play an important role.

Definition 2.3. Let S be a set of primes and let $P \in L_0(\mathbb{Q})$ be a S-point with primitive representative $(x, y, z) \in \mathbb{Z}^3$. We say that P is a proper S-point if $\operatorname{rad}(P) = S$. The height $h_2(P)$ is defined by $h_2(P) = v_2(xyz)$.

One has that

$$1 \le h_2(P) < \infty$$

for every $P \in L_0(\mathbb{Q})$. In particular L_0 has no S-points if $2 \notin S$.

Example 2.4. For S = 2 the S-point of L_0 is [2:-1:-1] up to permutation of coordinates.

Proposition 2.5. Let q be an odd prime and let S = 2q. Then the set of proper S-points of X + Y + Z = 0 is

- {[3:-2:-1], [3:-4:1], [9:-8:-1]}, if q = 3,
- { $[q:-2^n:-1]$ } with height a power of 2, if q is a Fermat prime ≥ 5 ,
- { $[q:-2^n:+1]$ } with prime height, if q is a Mersenne prime ≥ 7 ,
- \emptyset , otherwise,

up to permutation of coordinates.

Proof. Let (x, y, z) be a primitive representative of a proper S-point. Then x, y, z are non-zero pairwise coprime integers We may assume without loss of generality that $x = q^m, y = -2^n, z = \pm 1, m, n \ge 1$. By Theorem A.8 one has that either q = n = 3 and m = 2 or m = 1. We deduce that case q = 3 has 3 points. Case $m = 1, q \ge 5$ implies that either $q = 2^n + 1$ is a Fermat prime and n is a power of 2 or $q = 2^n - 1$ is a Mersenne prime and n is prime. Notice that 3 is the only prime being Fermat and Mersenne.

The FKM method to Conjecture 1 relies on finding pairs (S, H) such that

- S is a finite set of primes containing 2,
- *H* is a set of non-negative integers to be defined and
- there is no proper S-point of height $h_2 \in H$ in X + Y + Z = 0.

In the following subsections we exhibit infinite families of such pairs.

2.1. $S = 2q\ell$. Let q, ℓ be odd primes. In this subsection we deal with equations of the form

$$\begin{array}{rcl} 2^r q^s &=& \ell^t &\pm& 1\\ 2^r &=& q^s \ell^t &\pm& 1\\ 2^r &=& q^s &\pm& \ell^t \end{array}$$

For a non-zero integer k let $\sigma(k)$ denote the number of divisors of n and let $\omega(k)$ denote the number of prime divisors of k. Let Φ_k denote the kth cyclotomic polynomial. See Appendix A for further details on cyclotomic polynomials.

Proposition 2.6. Let ℓ, q be odd primes and assume that

$$2^r q^s = \ell^{2t} - 1$$

for some positive integers r, s, t. The solutions are given by the following equalities

 $2^{4} \cdot 5 = 3^{4} - 1$ $2^{3} \cdot 3 = 5^{2} - 1,$ $2^{4} \cdot 3 = 7^{2} - 1,$ $2^{5} \cdot 3^{2} = 17^{2} - 1.$

Proof. Let r, s, t, q, ℓ be a solution. Notice that $3 \mid \ell q$ since either $\ell \mid 3$ or $3 \mid \ell^{2t} - 1$. If $\ell = 3$ then $\sigma(2t) \leq 3$ by Corollary A.6. Hence $t \in \{1, 2\}$ with solutions $3^2 - 1 = 2^3, 3^4 - 1 = 2^4 \cdot 5$. The case $3^2 - 1 = 2^3$ is not allowed.

If q = 3 then t = 1 by Corollary A.6. The integers $\ell + 1$, $\ell - 1$ are consecutive even numbers and $gcd(\ell + 1, \ell - 1) = 2$. Case $\ell \pm 1 = 2^{r-1} \cdot 3^s$, $\ell \mp 1 = 2$ is not possible since $\ell \geq 5$. So assume

$$\ell + \varepsilon = 2 \cdot 3^s$$
$$\ell - \varepsilon = 2^{r-1}$$

for some unit ε . Then $3^s - 2^{r-2} = \varepsilon$ with solutions $3 - 2 = 1, 3 - 2^2 = -1, 3^2 - 2^3 = 1$ by Proposition 2.5. Hence $\ell \in \{5, 7, 17\}$.

Proposition 2.7. Let ℓ, q be odd primes and assume that

$$2^r q^s = \ell^t - 1$$

for some positive integers r, s, t such that t is odd and ≥ 3 . Then t is prime, $\Phi_1(\ell) = \ell - 1 = 2^r$ and $\Phi_t(\ell) = q^s$. Hence ℓ is a Fermat prime.

Proof. The odd integer t is prime since $2 \le \sigma(t) \le \omega(\ell^t - 1) = 2$ by Corollary A.6. Thus

$$2^r q^s = (\ell - 1)\Phi_t(\ell)$$

by the polynomial factorization in (3). Notice that $\Phi_t(\ell)$ is odd and has a prime divisor coprime to 2t by Theorem A.4, then $q \mid \Phi_t(\ell)$ and $q \neq t$. Notice that the greatest common divisor of $\ell - 1$ and $\Phi_t(\ell)$ divides t. Hence $\ell - 1$ and $\Phi_t(\ell)$ are coprime.

Proposition 2.8. Let ℓ, q be odd primes and assume that

$$2^r q^s = \ell^t + 1$$

for some positive integers r, s, t such that t is odd and ≥ 3 . Then t is prime, $\ell+1 = 2^r$ and $\Phi_{2t}(\ell) = q^s$. Hence ℓ is a Mersenne prime.

Proof. The integer t is prime since $2 \le \sigma(t) \le \omega(\ell^t + 1) = 2$ by Corollary A.7. Thus $2^r q^s = (\ell + 1)\Phi_{2t}(\ell)$

by the factorization of (4) in Appendix A. One proves as in Proposition 2.7 that $\Phi_{2t}(\ell)$ and $\ell + 1$ are coprime. Notice that $\Phi_{2t}(\ell)$ is odd.

Proposition 2.9. Let ℓ, q be odd primes and assume that

$$2^r q^s = \ell^{2t} + 1$$

for some positive integers r, s, t. Then r = 1 and $2t = 2^m$ for some $m \ge 1$ and $2^{m+1} | q - 1$.

Proof. Let t_2 be the largest odd divisor of 2t. Then $\ell^{2t} + 1$ has $\geq \sigma(t_2)$ odd prime divisors by Corollary A.7. Thus $t_2 = 1$. In particular ℓ has order 2^{m+1} in \mathbb{F}_q^{\times} . Notice that $\ell^{2^m} + 1 \equiv 2 \pmod{4}$ thus r = 1.

Let $n \ge 2$ be an integer. The equation $2X^n - 1 = Z^2$ was studied by Carl Störmer in [29, Section 3]. He proved that either n is a power of two or X = Z = 1 is the only solution in \mathbb{Z} . See also [22, A11.1].

Proposition 2.10. Assume that there are odd primes ℓ, q such that

$$2^r q^s = \ell^{2t} + 1$$

for some integers $r, s, t \ge 1$. Then r = 1 and either

• s = 1 and $q = \frac{\ell^{2t} + 1}{2}$, or • $(q, \ell) = (13, 239)$, or • s = 2, t = 1.

Proof. We have already seen in Proposition 2.9 that r is necessarily 1. Case s = 1 has many solutions. Assume $s \ge 2$. Then s, 2t are powers of two due to Störmer's result and Proposition 2.9. The curve $2x^4 = y^2 + 1$ has two positive integer solutions (1, 1) and (13, 239) due to [16] or [28]. For the study of $C : 2x^2 = y^4 + 1$, we consider the rational map $\varphi : C \dashrightarrow E$

$$(x,y)\mapsto \left(rac{4x}{x-y}-2,rac{4-4x^2}{(x-y)^2}
ight)$$

where E denotes the elliptic curve given by $y^2 = x^3 + 4x$ with Cremona Label 32a1. Notice that φ is well defined in the Zariski open $U = C \setminus \{x = y\}$ and that φ maps rational points of U to rational points of E. The Mordell-Weil group of E consists in 4 points, they are (0,0), (2,4), (2,-4) and the point at infinity, see [17]. The computation of $\varphi^{-1}E(\mathbb{Q})$ and $C \cap \{x = y\}$ provides the equality $C(\mathbb{Q}) = \{\pm(1,-1), \pm(1,1)\}.$

Thus equation $2q^{2^n} = \ell^{2^m} + 1$ has only solution $(q, \ell) = (13, 239)$ for $mn \ge 2$. \Box

Remark 2.11. Let q, ℓ be odd primes and assume that ℓ is neither a Fermat prime nor a Mersenne prime. One deduces from previous statements an algorithm to determine the solutions to the equation $2^r q^s = \ell^t + \varepsilon$. Indeed, the cases s = 1or t = 1 are easy to deal with. Let us assume that $s, t \geq 2$. The case $\varepsilon = -1$ is completely treated in Propositions 2.6 and 2.7. The case $\varepsilon = 1$ and t odd is solved in Proposition 2.8. For the case $\varepsilon = 1$ and t even one has by Proposition 2.10 that either $(q, \ell) = (13, 239)$ or s = t = 2 and r = 1.

One can use elementary algebraic number theory to attack the equation $\ell^2 - 2q^2 = -1$. Notice that an integer point of $x^2 - 2y^2 = -1$ corresponds to the unit $x + y\sqrt{2} \in \mathbb{Z}[\sqrt{2}]^{\times}$. Thus, all these points arise as powers of the fundamental unit $\eta = 1 + \sqrt{2}$, i.e. $\mathbb{Z}[\sqrt{2}]^{\times} = \{\pm 1\} \cdot \eta^{\mathbb{Z}}$. Four such pairs (ℓ, q) arise as coefficients of η^n , with $3 \le n \le 10^4$.

There are indeed solutions to equation $2^r q^s = \ell^t \pm 1$.

Examples 2.12. • $2 \cdot 5^2 = 7^2 + 1$ corresponding to η^3 ,

- $2 \cdot 29^2 = 41^2 + 1$ corresponding to η^5 ,
- η^{29} ,
- η^{59} .
- $2 \cdot 13^4 = 239^2 + 1$,
- $\Phi_5(3) = 11^2$ and $2\Phi_5(3) = 3^5 1$,
- $\Phi_7(5)$ is prime and $2^2 \cdot \Phi_7(5) = 5^7 1$,
- $\Phi_{34}(7)$ is prime and $2^3 \cdot \Phi_{34}(7) = 7^{17} + 1$.

We finish this subsection with a general statement about 2S-unit equations for |S| = 2.

Lemma 2.13. Let $q, \ell \geq 5$ be primes. Assume one of the following: (1) $(q, \ell) \equiv (-5, 5)$ or $(11, -11) \pmod{24}$. (2) $q \equiv 11 \pmod{24}, \ \ell \equiv 5 \pmod{24} \ and \ \left(\frac{q}{\ell}\right) = -1.$

(3) $q \equiv \pm 3 \pmod{8}, \ \ell \equiv -1 \pmod{24}, \ \ell \not\equiv -1 \pmod{q}.$

Then the $2\ell q$ -unit equation

$$X + Y + Z = 0$$

has no proper points of height ≥ 3 .

Proof. This is a mod 24 exercise. See Appendix C.

2.2. Large |S|.

Lemma 2.14 $(h_2 = 4)$. Let S be a finite set of primes in $1 + 3\mathbb{Z}$. Then L_0 has no 2S-points of height 4.

Proof. Let (A, B, C) be a (primitive representative of a) 2S-point of height 4 and let $\varepsilon_A, \varepsilon_B, \varepsilon_C$ the sign of A, B, C, respectively. Then

$$0 = A + B + C \equiv \varepsilon_A + \varepsilon_B + \varepsilon_C \pmod{3}.$$

Hence $(\varepsilon_A, \varepsilon_B, \varepsilon_C) = \pm (1, 1, 1)$ and A + B + C is either strictly positive or strictly negative.

Notice that the same proof applies to every even height case. In particular, L_0 has no 2S-point of even height with the notation of Lemma 2.14.

Lemma 2.15 $(h_2 = 4)$. Let n be a positive integer not dividing 14, 16 nor 18 and let S be a finite set of primes in $\pm 1 + n\mathbb{Z}$. Then L_0 has no 2S-points of height 4.

Proof. Let (A, B, C) be a 2S-point of height 4. Say $A = 2^4 A'$, then $A', B, C \equiv \pm 1 \pmod{n}$. Thus

$$0 = A + B + C \equiv \pm 16 \pm 1 \pm 1 \pmod{n}.$$

Hence $n \mid 14, 16 \text{ or } 18$.

Lemma 2.16 $(h_2 \ge 2)$. Let p be an odd prime. Let S be a finite set of primes in $1 + 4p\mathbb{Z}$. Then L_0 has no 2S-points of height ≥ 2 .

Proof. Let (A, B, C) be a proper point. Say

$$\begin{array}{rcl}
A &=& \varepsilon_A & A' & 2^r \\
B &=& \varepsilon_B & B' \\
C &=& \varepsilon_C & C'
\end{array}$$

for $r \ge 2$ and $\varepsilon_x = \operatorname{sign} x$. Then $A' \equiv B' \equiv C' \equiv 1 \pmod{4p}$ and

$$0 = A + B + C \equiv 2^r \varepsilon_A + \varepsilon_B + \varepsilon_C \pmod{4p}$$

Thus $\varepsilon_B \equiv -\varepsilon_C \pmod{4}$ and $2^r \varepsilon_A + \varepsilon_B + \varepsilon_C \equiv 0 \pmod{p}$. Then $\varepsilon_B = -\varepsilon_C$ and $p \mid 2^r$.

3. Frey-Kraus-Mazur Method

In this section we recall the FKM method. The standard references are Frey's [12] and Kraus' [14] papers. Let a, b, c be non-zero pairwise coprime integers and let

$$p > \max\left(4, \max_{q \text{ prime}} v_q(abc)\right) \tag{2}$$

be a prime. Assume that $F_p^{a,b,c}(\mathbb{Q})$ has a non-trivial point P and let (x, y, z) be a primitive tern of non-zero integers such that [x : y : z] = P. That is, $xyz \neq 0$, gcd(x, y, z) = 1 and

$$ax^p + by^p + cz^p = 0.$$

Notice that $(A, B, C) = (ax^p, by^p, cz^p)$ are pairwise coprime integers.⁴

3.1. The Frey curve. Following the notation above consider the elliptic curve

$$E = E_{A,B,C} : Y^2 = X(X - A)(X + B)$$

over \mathbb{Q} . The definition of $E_{A,B,C}$ is sensible to the order of (A, B, C). More precisely, the curve $E_{A,B,C}$ is a twist of $E_{B,A,C}$ by the quadratic twist of $\mathbb{Q}(i)/\mathbb{Q}$ while even permutations of (A, B, C) define \mathbb{Q} -isomorphic elliptic curves. Hence $E_{A,B,C}$, $E_{B,A,C}$ have common prime-to-2 conductor. Let us reorder (A, B, C) so that E has minimal conductor exponent over \mathbb{Q}_2 .⁵

Proposition 3.1. *E* has conductor $2^r \operatorname{rad}'(abcxyz)$ where

$$r = \begin{cases} 1 & \text{if } xyz \text{ is } even \text{ or } v_2(abc) \geq 5, \\ 0 & \text{if } xyz \text{ is } odd \text{ and } v_2(abc) = 4, \\ 3 & \text{if } xyz \text{ is } odd \text{ and } v_2(abc) \in \{2,3\}, \\ 5 & \text{if } xyz \text{ is } odd \text{ and } v_2(abc) = 1. \end{cases}$$

Proof. The elliptic curve E has semi-stable reduction at every odd prime since A, B, C are pairwise coprime. Let ℓ be an odd prime, then E has bad reduction over \mathbb{Q}_{ℓ} if and only if $\ell \mid ABC$. Thus, E has prime-to-2 conductor rad'(ABC) =rad'(xyzabc) by Neron-Ogg-Shafarevich. The conductor exponent of E over \mathbb{Q}_2 has been computed in [8, Lemma 2]. If xyz is even then $v_2(ABC) \ge pv_2(xyz) \ge p > 4$ by hypothesis, thus r = 1. Notice that $v_2(abc) = 0$ implies xyz even.

Lemma 3.2. The Galois representation

$$\bar{\rho}_{E,p} : \operatorname{Gal}(\mathbb{Q}/\mathbb{Q}) \longrightarrow \operatorname{Aut}(E[p]) \simeq \operatorname{GL}_2(\mathbb{F}_p)$$

is irreducible.

Proof. Recall that $p \ge 5$ by assumption (2). The irreducibility condition is proved in Serre's paper [26, Proposition 6] for the semi-stable case, i.e. $r \le 1$. Serre's proof relies on Mazur's theorem [19, Theorem 2].

Let us prove irreducibility for $r \in \{3, 5\}$. Consider the local Galois representation

$$\rho_{E,p}|_{G_2} : \operatorname{Gal}(\mathbb{Q}_2/\mathbb{Q}_2) \to \operatorname{Aut}(\mathcal{T}_p(E))$$

⁴Indeed, if $q \mid x, y$ then $q^p \mid c$ and $p \leq v_q(c) \leq v_q(abc)$.

⁵For example one can take B even and $A \equiv -1 \pmod{4}$.

and the residual representation

$$\bar{\rho}_{E,p}|_{G_2} : \operatorname{Gal}(\mathbb{Q}_2/\mathbb{Q}_2) \to \operatorname{Aut}(E[p]),$$

where $\mathcal{T}_p(E)$ denotes the *p*-adic Tate module of E and G_2 denotes a decomposition subgroup of $G_{\mathbb{Q}}$ over 2. The conductor of $\rho_{E,p}|_{G_2}$ is larger than or equal to the conductor of $\bar{\rho}_{E,p}|_{G_2}$. Henri Carayol computed in [3] the cases where the inequality is strict. See the discussion in page 789 and Proposition 2 therein. Since $\rho_{E,p}|_{G_2}$ has unramified determinant and $r \geq 3$ one deduces that $\rho_{E,p}|_{G_2}, \bar{\rho}_{E,p}|_{G_2}$ have common conductor 2^r . Assume that $\bar{\rho}_{E,p}$ is reducible then

$$\bar{\rho}_{E,p}|_{G_2} \simeq \left(\begin{array}{cc} \chi_1 & * \\ & \chi_2 \end{array} \right)$$

with $\chi_1\chi_2$ being the (unramified) mod p cyclotomic character. Thus χ_1, χ_2 have common conductor. The Swan conductor is invariant under semisimplification. Thus, the Swan conductor of $\bar{\rho}_{E,p}|_{G_2}$ coincides with the Swan conductor of $\chi_1 \oplus \chi_2$. That is, either χ_1, χ_2 are unramified and $\bar{\rho}_{E,p}|_{G_2}$ has conductor ≤ 1 or χ_1, χ_2 are ramified with common Swan conductor m. In the last case one has that $\bar{\rho}_{E,p}|_{G_2}$ has even conductor exponent $r = \dim_{\mathbb{F}_p} E[p] - \dim_{\mathbb{F}_p} E[p]^{I_2} + 2m = 2 + 2m$.

3.2. Lowering the level. We shall lower the level of E via E[p]. The standard reference here is Ribet's level lowering theorem, [23]. Let us recall some notation therein. Let $\bar{\rho} := E[p]$ be the mod p irreducible representation attached to E. Then $\bar{\rho}$ is modular of level $N = 2^r \operatorname{rad}'(abcxyz)$ by Wiles [32], see also [8]. Let ℓ be a prime divisor of N with $\ell \parallel N$, that is $\ell \mid N$ and $\ell^2 \nmid N$. The representation $\bar{\rho}$ is finite at ℓ if by definition some geometric condition is satisfied.⁶ For the case of modular elliptic curves that condition has a pleasant equivalence.

Lemma 3.3. Let p be a prime, let E' be an elliptic curve over \mathbb{Q} of conductor N' and let $\ell \parallel N'$ be a prime. Then E'[p] is finite at ℓ if and only if $p \mid v_{\ell}(j_{E'})$. If $p \neq \ell$ then E'[p] is finite at ℓ if and only if E'[p] is unramified at ℓ .

Proof. The lemma is a consequence of Tate's uniformization for multiplicative reduction elliptic curves over \mathbb{Q}_{ℓ} . See [5, 2.12] and [9, 8.2].

Let s be the conductor exponent of E[p] at 2, $s \le r$. If $r \in \{0, 3, 5\}$ then s = r. If r = 1 then s is ruled by Tate's uniformization. That is, s = 0 if and only if $v_2(abc) = 4$.

Theorem 3.4. Following the notation above, let

$$\bar{\rho} : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{F}_p)$$

be the Galois representation attached to the p-torsion of $E = E_{A,B,C}$. There is a newform $f \in S_2(2^s \operatorname{rad}'(abc))$ whose mod p Galois representation is isomorphic to $\bar{\rho}$.

⁶More precisely, $\bar{\rho}$ is finite at ℓ if there is a finite flat \mathbb{F}_p -vector space scheme H over \mathbb{Z}_ℓ such that $H(\bar{\mathbb{Q}}_\ell)$ is isomorphic to $\bar{\rho}|_{G_\ell}$ as $\mathbb{F}_p[G_\ell]$ -modules.

$$j_E = \frac{2^8 (C^2 - AB)^3}{A^2 B^2 C^2}$$

and $v_{\ell}(j_E) = -2v_{\ell}(abc) - 2pv_{\ell}(xyz)$. Thus $p \mid v_{\ell}(j_E)$ if and only if

 $p \mid v_{\ell}(abc).$

Recall that $p > v_{\ell}(abc)$ by assumption (2). Thus E[p] is finite at an odd prime ℓ if and only if $\ell \nmid abc$.

Ribet's level lowering Theorem states that $\bar{\rho}$ is modular of level $2^s \operatorname{rad}'(abc)$. I.e., there is a newform in $S_2(M)$, for some $M \mid 2^s \operatorname{rad}'(abc)$, and a prime $\mathfrak{p} \ni p$ such that $\bar{\rho}_{f,\mathfrak{p}}$ and $\bar{\rho}$ are isomorphic. See B for a proof of the equality $M = 2^s \operatorname{rad}'(abc)$.

The final step is to connect f with S-unit equations. Kraus method allows us to impose the following conditions.

- [Q_f: Q] = 1 so that f corresponds to an elliptic curve E'/Q.
 E' has full rational 2-torsion.

Theorem 3.5. There is a constant H = H(rad(abc), s) such that if p > H then the newform described in Theorem 3.4 corresponds to an elliptic curve over \mathbb{Q} with full rational 2-torsion (up to isogeny).

Proof. See Théorème 3 and Théorème 4 in [14].

Notice that H depends on [x: y: z] since r and s may vary from point to point. Nevertheless, one can still give a uniform bound depending only on a, b, c by taking $\max_{s}(H(\operatorname{rad}(abc), s)).$

Proposition 3.6. Let N be a square-free odd integer and let $r \in \{0, 1, 3, 5\}$. The existence of a Frey Curve of conductor $2^r N$ is equivalent to the existence of a proper 2N-point of height

$$\begin{array}{ll} \geq 5 & if \ r = 1, \\ 4 & if \ r = 0, \\ 2 \ or \ 3 & if \ r = 3, \\ 1 & if \ r = 5. \end{array}$$

Proof. The existence of a Frey curve attached to a 2N-point follows as in Proposition 3.1. For the other implication let

$$E: Y^2 = X(X - A)(X + B)$$

be a Frey curve of conductor $2^r N$, $A, B \in \mathbb{Z}$. There is a Frey curve

$$E': Y^2 = X(X - a)(X + b)$$

twist of E such that a, b are coprime, $a \equiv -1 \mod 4$ and b is even. Mainly, the term $(a, b, -a-b) \in \mathbb{Z}^3$ is a primitive representative of [A:B:-A-B] up to permutation of coordinates. Let us see that (a, b, c) is a proper 2N point with the corresponding

constrains on the height. The curve E' has conductor $2^{r'} \operatorname{rad}'(ab(a+b))$ where

$$r' = \begin{cases} 0 & \text{if } v_2(b) = 4, \\ 1 & \text{if } v_2(b) \ge 5, \\ 3 & \text{if } v_2(b) = 2, 3, \\ 5 & \text{if } v_2(b) = 1 \end{cases}$$

as described in [8]. Thus, it is enough to prove that r' = r and $N = \operatorname{rad}'(ab(a+b))$. Let $g \in \mathbb{Z}$ square-free such that E is a twist of E' by the quadratic character χ attached to $\mathbb{Q}(\sqrt{g})$. Equivalently,

$$E \simeq E'': Y^2 = X(X - ag)(X + bg).$$

This model of E'' is minimal over \mathbb{Z}_p for every odd prime p. Thus E has additive reduction at every odd prime divisor of g. Since N is square-free one deduces that $g \in \{\pm 1, \pm 2\}$ and $N = \operatorname{rad}'(ab(a + b))$. The conductor of E' over \mathbb{Q}_2 needs special consideration. Assume $g \neq 1$, otherwise E and E' have common conductor. The character χ has conductor $2^{|g|+1}$. We check r = r' via modularity. Let f be the weight 2, level $2^{r'}N$, trivial character newform attached to E', by Wiles. Then $f \otimes \chi$ is the newform attached to E and has level 2^rN . If $r' \neq 5$ or $g \neq -1$, then the level of $f \otimes \chi$ is $2^{2|g|+2}N$ by [1, Theorem 3.1]. This contradicts that $r \in \{0, 1, 3, 5\}$. If r' = 5 and g = -1 then $f \otimes \chi$ has level 2^5N . Indeed, if $g = f \otimes \chi$ has level 2^sN with $s \leq 3$ then $g \otimes \chi = f$ has level 2^4N , loc. cit. This contradicts $r' \in \{0, 1, 3, 5\}$. Hence, either g = 1 or g = -1 and r = r' = 5.

One can also use Tate's algorithm [30] to compute those conductors. See also [31, Proposition 1] for the case where $\mathbb{Q}(\sqrt{g})$ is more deeply ramified than $\rho_{E',\ell}$, i. e. $r' \leq 1$ and $g \neq 1$.

4. Kraus Theorem

The following is a slight generalization of Kraus' Theorem [14, Théorème 1]. That is, the assumption a, b, c being pairwise coprime has been relaxed to (\mathbf{F}) .

Let a, b, c be non-zero integers satisfying condition (F) defined in Section 1. We assume without loss of generality that a is odd.

Theorem 4.1 (2-good). Assume that b is odd and $v_2(c) = 4$. Let S = rad(abc) and assume that there are no proper S-points of L_0 of height 4. Then there is a constant G(a, b, c) such that

$$AF_{a,b,c} \subseteq \bigcup_{5 \le p \le G(a,b,c)} F_p^{a,b,c} \cup \{trivial \ points\}.$$

Theorem 4.2 (2-node). Assume one of the following

- (1) bc is odd,
- (2) b is odd and $v_2(c) \ge 5$, or
- (3) $v_2(b) = v_2(c) \ge 1$.

Let S = rad(2abc) and assume that there are no proper S-points of height ≥ 5 . Then there is a constant G(a, b, c) such that

$$AF_{a,b,c} \subseteq \bigcup_{5 \le p \le G(a,b,c)} F_p^{a,b,c} \cup \{trivial \ points\}$$

Theorem 4.3. Assume that b is odd and $v_2(c) \in \{2,3\}$. Assume that there are no proper S-points of height 2, 3 or ≥ 5 for S = rad(abc). Then there is a constant G(a, b, c) such that

$$AF_{a,b,c} \subseteq \bigcup_{5 \le p \le G(a,b,c)} F_p^{a,b,c} \cup \{trivial \ points\}.$$

Theorem 4.4. Assume that b odd, $v_2(c) = 1$ and let S = rad(abc). Assume that there are no proper S-points of height 1 or ≥ 5 . Then there is a constant G(a, b, c) such that

$$AF_{a,b,c} \subseteq \bigcup_{5 \le p \le G(a,b,c)} F_p^{a,b,c} \cup \{trivial \ points\}.$$

See 6 for a description of G(a, b, c).

Proof. Assume that there is a prime p > G(a, b, c) such that $F_p^{a,b,c}(\mathbb{Q})$ has nontrivial points. Let α, β, γ be pairwise coprime integers such that $F_p^{a,b,c} \simeq F_p^{\alpha,\beta,\gamma}$ and $\operatorname{rad}(abc) = \operatorname{rad}(\alpha\beta\gamma)$ by Lemma 1.3 and let (x, y, z) be a primitive representative of a non trivial point P in $F_p^{\alpha,\beta,\gamma}(\mathbb{Q})$. Let $(A, B, C) = (\alpha x^p, \beta y^p, \gamma z^p)$ and reorder (A, B, C) so that $E = E_{A,B,C}$ has minimal conductor. If b, c are both even then the choice of G(a, b, c) ensures that $v_2(\alpha) \geq 5$, see Remark 1.4. Otherwise, $v_2(\alpha\beta\gamma) =$ $v_2(abc)$. The level lowering trick combined with the fact that p is large, see Theorem 3.5, implies that there is an elliptic curve E' over \mathbb{Q} with full rational 2-torsion such that E[p] = E'[p]. Moreover E' has conductor $2^s \operatorname{rad}'(abc)$ where

$$s = \begin{cases} 0 & \text{if } b \text{ is odd and } v_2(c) = 4, \\ 1 & \text{if } b, c \text{ have same parity,} \\ 1 & \text{if } b \text{ is odd, } v_2(c) \in \{1, 2, 3\} \text{ and } xyz \text{ is even,} \\ 3 & \text{if } b \text{ is odd, } v_2(c) \in \{2, 3\} \text{ and } xyz \text{ is odd,} \\ 5 & \text{if } b \text{ is odd, } v_2(c) = 1 \text{ and } xyz \text{ is odd.} \end{cases}$$

Thus $E' = E_{R,S,T}$ is a Frey curve, with rad(RST) = rad(2abc). That is, there is a proper rad(2abc)-point of height

$$\begin{cases} 1 & \text{if } s = 5, \\ 2 \text{ or } 3 & \text{if } s = 3, \\ 4 & \text{if } s = 0, \\ \ge 5 & \text{if } s = 1, \end{cases}$$

by Proposition 3.6.

5. Statements

In this section we translate Lemmas 2.13, 2.14, 2.15 and 2.16 to new cases of Asymptotic Fermat Conjecture with coefficients. Let (a, b, c) be a tern of non-zero integers satisfying (F), a odd.

Theorem 5.1. Let S be a set of primes all in $1+3\mathbb{Z}$. and assume that rad(abc) = S. Then the Fermat equation

$$ax^p + by^p + 16cz^p = 0$$

has no solutions other than xyz = 0 for p larger than G(a, b, 16c).

Proof. There are no proper 2S-points of height 4 by Lemma 2.14. The theorem follows due to Theorem 4.1. \Box

Theorem 5.2. Let n be a positive integer not dividing 14, 16, 18 and let S be a finite set of primes all in $(1 + n\mathbb{Z}) \cup (-1 + n\mathbb{Z})$. Assume $\operatorname{rad}(abc) = S$. Then the Fermat equation

$$ax^p + by^p + 16cz^p = 0$$

has no solutions other than xyz = 0 for p larger than G(a, b, 16c).

Proof. There are no proper 2S-points of height 4 by Lemma 2.15. The theorem follows due to Theorem 4.1. \Box

Theorem 5.3. Let q be an odd prime and let S be a finite set of primes all in $1 + 4q\mathbb{Z}$. Assume that either rad(abc) = S or rad(abc) = 2S and $v_2(bc) \ge 2$. Then the Fermat equation

$$ax^p + by^p + cz^p = 0$$

has no solutions other than xyz = 0 for p larger than G(a, b, c).

Proof. Case rad(abc) = S follows from Theorem 4.2 and Lemma 2.16. If $v_2(bc) \ge 2$ then either b and c are even or b is odd and $m = v_2(c) \ge 2$. The first case follows by Theorem 4.2 and Lemma 2.16. The second case follows by Theorem 4.2 for $m \ge 5$, by Theorem 4.1 for m = 4 and by Theorem 4.3 for $2 \le m \le 3$.

Theorem 5.4. Let $q, \ell \geq 5$ be primes. Assume one of the following:

(1) $(q, \ell) \equiv (-5, 5)$ or $(11, -11) \pmod{24}$.

(2) $q \equiv 11 \pmod{24}, \ \ell \equiv 5 \pmod{24} \ and \ \left(\frac{q}{\ell}\right) = -1.$

(3) $q \equiv \pm 3 \pmod{8}, \ \ell \equiv -1 \pmod{24}, \ \ell \not\equiv -1 \pmod{q}.$

Assume that $rad(abc) = \ell q$.

Let n = 0 or ≥ 4 then the Fermat equation

$$ax^p + by^p + 2^n cz^p = 0$$

has no solutions other than xyz = 0 for p larger than $G(a, b, 2^n c)$. Let r > 1 then the Fermat equation

$$ax^p + 2^r by^p + 2^r cz^p = 0,$$

has no solutions other than xyz = 0 for p larger than $G(a, 2^rb, 2^rc)$.

Proof. If either n = 0 or $n \ge 5$ or $r \ge 1$ then this is Theorem 4.2 with Lemma 2.13. If n = 4 then this is Theorem 4.1 with Lemma 2.13.

6. Bounds

Let us recall the explicit bound G(a, b, c) as in Kraus' paper. In the following presentation we relax the bound so that statements are shorter. For example, G(a, b, c)is taken so that a, b, c are *p*th-power-free for every p > G(a, b, c). Let us describe the bound G(a, b, c). Let N be a positive integer and let

$$\mu(N) = [\operatorname{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod_{\ell \mid N \text{ prime}} \left(1 + \frac{1}{\ell}\right)$$
$$g(N) = \dim_{\mathbb{C}} S_2^{new}(N),$$
$$F(N) = \left(\sqrt{\frac{\mu(N)}{6}} + 1\right)^{2g(N)}.$$

where $S_2^{new}(N)$ denotes the space of weight 2 newforms of level N. Let a, b, c be non-zero integers satisfying $(\mathbf{F}), 0 = v_2(a) \leq v_2(b) \leq v_2(c)$. Let

$$N = \begin{cases} \operatorname{rad}'(abc) & \text{if } b \text{ is odd and } v_2(c) = 4, \\ 2^3 \operatorname{rad}'(abc) & \text{if } b \text{ is odd and } v_2(c) = 2, 3, \\ 2^5 \operatorname{rad}'(abc) & \text{if } b \text{ is odd and } v_2(c) = 1, \\ 2 \operatorname{rad}'(abc) & \text{otherwise.} \end{cases}$$

If b is odd then G is defined by

$$G(a, b, c) := \max(F(N), \max_{q \text{ prime}} v_q(a), \max_{q \text{ prime}} v_q(b), \max_{q \text{ prime}} v_q(c)).$$

If b is even, that is $v_2(b) = v_2(c) \ge 1$ then G is defined by

$$G(a, b, c) := \max(F(N), \max_{q \text{ prime}} v_q(a), \max_{q \text{ prime}} v_q(b), \max_{q \text{ prime}} v_q(c), v_2(c) + 4).$$

Example 6.1. Let $S \neq \emptyset$ be a finite set of primes in $1 + 12\mathbb{Z}$ and let a, b, c be non-zero, square-free, pairwise coprime integers such that rad(abc) = S. Then

$$N = 2 \operatorname{rad}(abc) = 2S,$$

$$g(N) = \frac{\varphi(S)}{12} + (-1)^{\omega(2S)},$$

$$\mu(N) = 3 \prod_{\ell \in S} (\ell + 1).$$

Here φ denotes the Euler's totient function and $\omega(2S)$ the number of prime divisors of 2S. The dimension g(N) of $S_2^{new}(N)$ has been computed in [18].

APPENDIX A. PRIME DIVISORS OF CYCLOTOMIC POLYNOMIALS

In this appendix we give some lower bounds for the number of prime divisors of $\ell^n \pm 1$ for integers $\ell \geq 3$ and $n \geq 1$.

Let Φ_n be the *n*th cyclotomic polynomial. A usual description of Φ_n is given by the formula

$$\Phi_n(X) = \prod_k (X - \zeta_n^k)$$

where $\zeta_n = e^{2\pi i/n}$ is a primitive *n*th root of unity and *k* ranges over the units of $\mathbb{Z}/n\mathbb{Z}$. Gauss proved that Φ_n is irreducible in $\mathbb{Z}[X]$, hence $\mathbb{Z}[X]/\Phi_n \simeq \mathbb{Z}[\zeta_n] \subseteq \mathbb{C}$ is a domain. In particular

$$X^n - 1 = \prod_{d|n} \Phi_d(X) \tag{3}$$

,

is the factorization of $X^n - 1$ in irreducible factors over $\mathbb{Z}[X]$. Similarly, write $n = 2^m n_2$ where n_2 is the largest odd divisor of n. Then

$$X^{n} + 1 = \prod_{d|n_{2}} \Phi_{2^{m+1}d}(X)$$
(4)

since $X^{2^{m+1}n_2} - 1 = (X^n - 1)(X^n + 1).$

Let k be a positive integer. The map $\mathbb{Z}[X] \to \mathbb{Z}/k\mathbb{Z}, X \mapsto \ell$ factors through $\mathbb{Z}[\zeta_n] \to \mathbb{Z}/k\mathbb{Z}, \zeta_n \mapsto \ell$ if, and only if, $k \mid \Phi_n(\ell)$.

Lemma A.1. Let $p \nmid n$ be a prime and assume that there is a ring homomorphism $\theta : \mathbb{Z}[\zeta_n] \to \mathbb{F}_p$. Then $\theta(\zeta_n)$ has order n in \mathbb{F}_p^{\times} and $n \mid p-1$.

Proof. Let $\alpha = \theta(\zeta_n)$. Then $\alpha^n - 1 = \prod_d \Phi_d(\alpha) = 0$. Notice that $X^n - 1$ is separable over \mathbb{F}_p since $nX^{n-1} \neq 0$ in $\mathbb{F}_p[X]$. Hence α has order n in \mathbb{F}_p^{\times} and the lemma follows.

Lemma A.2. Let p be an odd prime. There is no ring homomorphism $\mathbb{Z}[\zeta_p] \to \mathbb{Z}/p^2\mathbb{Z}$. There is no ring homomorphism $\mathbb{Z}[\zeta_4] \to \mathbb{Z}/4\mathbb{Z}$.

Proof. It is enough to prove that $\Phi_p(X) = \sum_{i=0}^{p-1} X^i$ has no roots in $\mathbb{Z}/p^2\mathbb{Z}$. The following proof is standard. Assume that there is a root a of Φ_p in $\mathbb{Z}/p^2\mathbb{Z}$. Then $a = 1 \mod p$, since $\Phi_p = (X-1)^{p-1}$ in \mathbb{F}_p . Notice that $\Phi_p(1+pb) = \sum_{i=0}^{p-1} 1+ipb = p$ in $\mathbb{Z}/p^2\mathbb{Z}$ for every b. Hence $\Phi_p(a) = p$ for every $a \equiv 1 \pmod{p}$. Notice that $\Phi_4(X) = X^2 + 1$ has no roots in $\mathbb{Z}/4\mathbb{Z}$.

Lemma A.3. Let $\ell \geq 3$, $n \geq 2$ be integers and let p be the largest prime divisor of n, then $|\Phi_n(\ell)| > p$.

Proof. The Euler's totient function φ , satisfies that

$$p-1 \mid \varphi(n).$$

Hence

$$|\Phi_n(\ell)| = \prod_k |\ell - \zeta_n^k| \ge \prod_k 2 \ge 2^{p-1}$$

and case $p \geq 3$ follows.

If p = 2 then n is a power of 2, $n = 2^m$, and

$$\Phi_n(\ell) = \ell^{2^{m-1}} + 1 > 2.$$

The polynomial Φ_n has no real roots for $n \ge 3$, hence $|\Phi_n(\ell)| = \Phi_n(\ell)$.

Theorem A.4. Let $\ell \geq 3, n \geq 3$ be integers. There is a prime divisor p of $\Phi_n(\ell)$ not dividing 2n. Hence, ℓ has order n in \mathbb{F}_p^{\times} .

Proof. Case $n = 2^m \ge 4$.

One has that $\Phi_{2^m}(X) = X^{2^{m-1}} + 1$ and $\Phi_n(\ell) \ge 10$. If $4 \mid \Phi_n(\ell)$ then $\mathbb{Z}[\zeta_n] \to \mathbb{Z}/4\mathbb{Z}$, $\zeta_n \mapsto \ell$ defines a ring homomorphism that restricts to $\mathbb{Z}[\zeta_4] \subseteq \mathbb{Z}[\zeta_n]$. This contradicts Lemma A.2. Hence either $\Phi_n(\ell)$ is odd or $\Phi_n(\ell)/2 \ge 5$ is odd. Case $p \mid n, p \text{ odd.}$

Notice that $\Phi_n(\ell)$ is odd. Indeed, if $2 \mid \Phi_n(\ell)$ then there exists a ring homomorphism

 $\mathbb{Z}[\zeta_n] \to \mathbb{F}_2$

which induces by restriction a map

$$\mathbb{Z}[\zeta_p] \to \mathbb{F}_2$$

hence $p \mid 2 - 1$ by Lemma A.1.

Let us see that either $\Phi_n(\ell)$ and n are coprime or there is a prime p such that $\Phi_n(\ell)/p, n$ are coprime. Assume that p < q are prime divisors of $\Phi_n(\ell)$ and n. Then there is a ring homomorphism

$$\mathbb{Z}[\zeta_q] \subseteq \mathbb{Z}[\zeta_n] \to \mathbb{F}_p$$

and $q \mid p-1$ by Lemma A.1 which contradicts p < q. Hence the greatest common divisor of $\Phi_n(\ell)$ and n is a possibly trivial power of an odd prime p. If $p \mid n$ then $p^2 \nmid \Phi_n(\ell)$ by Lemma A.2. Hence either $\Phi_n(\ell), 2n$ are coprime or there is an odd prime divisor p of $\Phi_n(\ell)$ such that $\Phi_n(\ell)/p$ and 2n are coprime. In the second case $\Phi_n(\ell)/p$ is an odd integer > 1 by Lemma A.3 and the first part of the theorem follows. The order of ℓ is computed in Lemma A.1.

Corollary A.5. Let $\ell \geq 3$. Assume that n_1, \ldots, n_r are pairwise different integers ≥ 3 . Then

$$\prod_i \Phi_{n_i}(\ell)$$

has at least r odd prime divisors.

Proof. Let p_i be a prime divisor of $\Phi_{n_i}(\ell)$ coprime to $2n_i$ as in Theorem A.4. Then ℓ has order n_i in $\mathbb{F}_{p_i}^{\times}$, thus $p_i \neq p_j$ for different i, j.

For an integer k let $\omega(k)$ denote the number of prime divisors of k and let $\sigma(k)$ denote the number of divisors of k.

Corollary A.6. Let $\ell \ge 3$, $n \ge 1$ be integers. If $(\ell, n) \ne (3, even)$ then $\omega(\ell^n - 1) \ge \sigma(n).$

$$\omega(3^{2t} - 1) \ge \sigma(2t) - 1.$$

Proof. Let $i \in \{1, 2\}$ such that $n \equiv i \pmod{2}$. Then

$$A := \prod_{\substack{d \mid n \\ d \ge 3}} \Phi_d(\ell) = \frac{\ell^n - 1}{\ell^i - 1}$$

has at least $\sigma(n) - i$ odd prime divisors $S = \{p_d\}_{d|n,d\geq 3}$ as in Theorem A.4. Notice that $p_d \nmid \ell^i - 1$ for every $p_d \in S$. Indeed, if an odd prime p divides $\ell^i - 1$ then ℓ has order $\leq i$ in \mathbb{F}_p^{\times} by Lemma A.1. Thus

$$\omega(\ell^n - 1) \ge \sigma(n) - i + \omega(\ell^i - 1).$$

It is enough to prove that $\omega(\ell^i - 1) \ge i$ if and only if $(\ell, i) \ne (3, 2)$. If i = 1 then $\ell - 1 \ge 2$ and $\omega(\ell - 1) \ge 1$. If i = 2 then $gcd(\ell - 1, \ell + 1) \le 2$. Assume $\omega(\ell^2 - 1) < 2$ then $\ell - 1, \ell + 1$ are powers of two. Hence $\ell = 3$.

Corollary A.7. Let $\ell \geq 3$, $n \geq 1$ be integers and let n_2 be the largest odd divisor of n. Then

$$\omega(\ell^n + 1) \ge \sigma(n_2).$$

Proof. Let $n = 2^m n_2$ then $\ell^n + 1 = \prod_{d \mid n_2} \Phi_{2^{m+1}d}(\ell)$ by the polynomial factorization of (4). For every d such that $2^{m+1}d \neq 2$ consider a prime $p_d \mid \Phi_{2^{m+1}d}(\ell)$ as in Theorem A.4. If m = 0 let p_1 be an arbitrary prime divisor of $\Phi_2(\ell) = \ell + 1$. Then $\prod_{d \mid n_2} p_d$ is a squarefree divisor of $\ell^n + 1$.

A.1. Catalan Conjecture. One deduces a case of Catalan's Conjecture.

Theorem A.8 (Partial Catalan's Conjecture). Let $\ell \geq 3$ be an integer and assume that

 $2^m - \ell^n \in \{\pm 1\}$

for some integers $m, n \ge 2$. Then $m = \ell = 3, n = 2$.

Proof. Assume that $2^m = \ell^n + 1$, $n \ge 2$ and let n_2 be the largest odd divisor of n. Then ℓ is odd and $\ell^n + 1 \ge 4$. By Corollary A.7 we have that $1 = \omega(\ell^n + 1) \ge \sigma(n_2)$, hence $n_2 = 1$ and $n = 2^r$ for some positive r. Since $2^m = \ell^{2^r} + 1 \equiv 2 \pmod{4}$ one has that m = 1 and $2 = \ell^{2^r} + 1$.

Assume that $2^m = \ell^n - 1$, $n \ge 2$. If $(\ell, n) = (3, 2t)$ with t an integer then $1 = \omega(3^{2t} - 1) \ge \sigma(2t) - 1$ by Corollary A.6. Hence t = 1.

If $(\ell, n) \neq (3, even)$, by Corollary A.6 one has that $1 = \omega(\ell^n - 1) \geq \sigma(n)$. Hence n = 1.

This partial result is well known to experts, see [22, B3.3]. See *ibid* for a complete treatment of Catalan's conjecture written before Preda Mihăilescu's proof [20]. See also Bilu - Bugeaud - Mignotte's book [2] for a *minimalistic* approach of the proof or Schoof's book [25] based on two sets of lecture notes by Yuri Bilu.

Appendix B. The conductor of E[p]

The j-invariant of a Frey curve is given by the formula

$$j_E = \frac{2^8 (C^2 - AB)^3}{A^2 B^2 C^2}.$$

Thus one has for the case $(A, B, C) = (ax^p, by^p, cz^p)$ being pairwise coprime that $C^2 - AB$ and ABC are coprime. Let ℓ be a prime divisor of ABC. Then

$$v_{\ell}(j_E) = 8v_{\ell}(2) - 2v_{\ell}(ABC) \equiv 8v_{\ell}(2) - 2v_{\ell}(abc) \pmod{p}.$$

Thus $p \mid v_{\ell}(j_E)$ if and only if

- ℓ is odd and $p \mid v_{\ell}(abc)$, or
- $\ell = 2$ and $v_2(abc) \equiv 4 \pmod{p}$.

Proposition B.1. Let $E = E_{A,B,C}$ be the Frey curve as in Theorem 3.4. Let f be a newform in $S_2(M)$ for some divisor M of $2^s \operatorname{rad}'(abc)$ and let \mathfrak{p} be a prime ideal such that

$$E[p] \simeq \bar{\rho}_{f,\mathfrak{p}}$$

as $\mathbb{F}_p[G_{\mathbb{Q}}]$ -modules. Then $M = 2^s \operatorname{rad}'(abc)$.

Let $\ell = 2$. If $s \in \{3, 5\}$ then Carayol [3] predicts that the lifting $\rho_{f,\mathfrak{p}}$ of $\bar{\rho}_{f,\mathfrak{p}}$ has conductor exponent s. Thus $2^s \mid M$. If s = 0 then M is odd and so is R. If s = 1then E[p] is ramified at 2 and so is $\bar{\rho}_{f,\mathfrak{p}}$. Hence $2 \mid M$.

One could just avoid case $p \mid M$ since we will consider big primes p with respect to rad(*abc*). Still, if $p \mid \text{rad}'(abc)$ then E[p] is not finite at p. That is, $E[p]|_{G_p}$ is reducible and not *peu ramifié* by [9, Proposition 8.2]. If $p \nmid M$ then $\bar{\rho}_{f,\mathfrak{p}}|_{G_p}$ is either irreducible or reducible and *peu ramifié*. Thus $E[p]|_{G_p} \not\simeq \bar{\rho}_{f,\mathfrak{p}}|_{G_p}$. This completes the proof.

Appendix C. Mod 24 exercises

Proof of Lemma 2.13: Let (A, B, C) be a primitive S-unit point of height ≥ 3 . Assume $A = 2^r$, $r \geq 3$. Then $B + C \equiv 0 \pmod{8}$ and $B + C \not\equiv 0 \pmod{3}$. Hence,

$$BC \equiv -1 \pmod{8}$$

since $C^{-1} \equiv C \pmod{8}$ and

$$BC \equiv 1 \pmod{3}$$

since $B, C \in \{\pm 1\} \mod 3$. Thus

$$\pm q^s \ell^t = BC \equiv 7 \pmod{24}.$$

(1) By hypothesis $(q, \ell) \equiv (-5, 5)$ or $(11, -11) \pmod{24}$. Notice that

 $q^{s}\ell^{t} \equiv \pm q^{s+t} \not\equiv \pm 7 \pmod{24},$

hence A is not a power of two.

Assume that

$$0 \equiv 2^r q^s = \ell^t + \varepsilon \equiv (-3)^t + \varepsilon \pmod{8}$$

for some $\varepsilon \in \{\pm 1\}$. Then $\varepsilon = -1$ and t is even. Proposition 2.6 implies

$$(q, \ell) \in \{(3, 5), (5, 3), (3, 7), (3, 17)\}.$$

Condition $q \equiv -\ell \pmod{24}$ leads to a contradiction. Similarly, $2^r \ell^t = q^r + \varepsilon$ has no solution.

(2) Assume that $(2^r, -q^s \ell^t, \varepsilon)$ is an S-point for some unit ε . Then $-\varepsilon q^s \ell^t \equiv 7 \pmod{24}$. Thus s, t are odd and $\varepsilon = -1$. That is

$$2^r = q^s \ell^t + 1 \equiv -1 \pmod{3},$$

hence r is odd, r = 2f + 1. Thus, 2 is a square in \mathbb{F}_q , i.e. $q \equiv \pm 1 \pmod{8}$. Indeed

$$\left(\frac{1}{q}\right) = \left(\frac{2}{q}\right)' = \left(\frac{2}{q}\right)$$

Assume that $2^r + (-1)^a q^s + (-1)^b \ell^t = 0$. Then

$$(-1)^{a+b}q^{s}\ell^{t} \equiv 7 \pmod{24}.$$

Hence a, b have same parity and s, t are odd. Thus

$$2^r = q^s + \ell^t \equiv 1 \pmod{3}$$

and r is even. Thus q is a square in \mathbb{F}_{ℓ} .

Assume that $(2^r q^s, -\ell^t, \varepsilon)$ is an S-point. Then $\ell^t \equiv \varepsilon \pmod{8}$ and hence t is even and $\varepsilon = 1$.

Assume that $(2^r \ell^t, -q^s, \varepsilon)$ is an S-point. Then $\varepsilon = 1$ and s is even. By Proposition 2.6 $q \in \{3, 5, 7, 17\}$, hence

$$q \not\equiv 11 \pmod{24}.$$

(3) By hypothesis

 $\ell \equiv -1 \pmod{24}$

and $q \equiv \pm 5$ or $\pm 11 \pmod{24}$ since $q \ge 5$. Thus $q^s \ell^t \not\equiv \pm 7 \pmod{24}$ and A is not a power of two.

Assume that $2^r q^s = \ell^t + 1$. Then t is either 1 or an odd prime by Lemma 2.8. Case t = 1 implies $\ell \equiv -1 \pmod{q}$. Case t odd prime implies ℓ Mersenne hence

$$\ell \equiv 0, 1 \pmod{3}.$$

Assume that $2^r q^s = \ell^t - 1$. Hence t is even and Proposition 2.6 implies $\ell \in \{3, 5, 7, 17\}$, then $\ell \not\equiv -1 \pmod{24}$. Similarly, case $2^r \ell^s = q^t \pm 1$ is not allowed by Lemma 2.6.

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