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Moviment d'un satèl·lit artificial en prescència de frenat atmosfèric

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Contents

Abstract

In the modern world, artificial satellites orbiting around the Earth are used in a wide variety of areas. The tracking and modeling of their orbits is of utmost importance. In the following text, a mathematical framework for their study as well as a software implementation and the use of a software suite, GMAT, are presented.

Resum

En l'era moderna, satèl·lits artificial orbitant la Terra són utilitzats en una gran varietat d'aplicacions. El seguiment com la modelització de la seva orbita són d'una gran importància. En el següent text, es presenta un marc matemàtic pel seu estudi amb una implementació en software d'ell i el cas d'ús del paquet de software GMAT.

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Agraïments

Vull agrair a la meva familia i als meus amics per donar-me suport en els moments que no ho tenia del tot clar. Sobretot, agrair al meu tutor els seus coneixament com la seva paciència, una més infinita que l'altre.

Chapter 1

Introduction

In the modern world, artificial satellites orbiting around the Earth are used in a wide variety of areas: telecommunications, Earth observation, astronomical studies... Their use case, as well as their life-span, is greatly affected by our ability to track and modify their position with precision.

1.1 Objectives

Our main objective in this thesis is to show how a mathematical framework can be defined to accurately track and characterize artificial satellites orbiting the Earth. Furthermore, using the learned theory, the implementation of a software suite to simulate the orbit of a satellite has been done. And last, the study and use of a professional satellite orbit simulation software is also performed.

1.2 Structure

The content of this thesis is structured as follows. First, an introduction to the classical frame used to describe orbits, a intuitive frame of reference and the proof of Kepler's Laws. Second, a display of an analytical approach taking into account perturbations, in particular, atmospheric drag extending the classical framework, as well as a numerical reproduction of its solution. Third, a formal method of representing perturbations of a satellite's orbit. Fourth, a concrete use of this last method to characterize atmospheric drag, commonly use. Fifth, a comparison of the implementation in software using Julia of the previous mentioned method with a professionally graded simulation suite, named GMAT.

[Add references for each chapter]

1.3 Remarks

In order to simplify the text, some generalizations have been made:

1. Celestial bodies have been abstracted to particles. That is, they are spherical objects with uniform density, unless stated otherwise.
2. The study of satellite's movement will be relative to Earth, omitting the remaining bodies of the Solar System.
3. Newton's laws of mechanics and Newton's universal gravitational law are taken as fundamentals.

The interested reader can consider [Dan98] for a study of celestial motions without these abstractions.

Chapter 2

The Kepler Problem

Our aim is to study the motion of geocentric (artificial) satellites relative to Earth. The study of two bodies motion, product of their interaction in an isolated environment, is called the *Kepler problem*, or the *two-body problem*.

Formally, we consider two bodies M_1 and M_2 with masses m_1, m_2 respectively, moving in $\mathbb{R}^3$ under Newton's laws of mechanics and Newton's universal gravitational law:

$$\ddot{\boldsymbol{r}}_1 = Gm_2 \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{r_{12}^3}, \qquad \ddot{\boldsymbol{r}}_2 = Gm_1 \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{r_{12}^3}, \tag{2.1}$$

where $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are their position in a specific inertial reference system, G is the constant of universal gravitation and $r_{12} = \|r_1 - r_2\|$ is the distance between them.

The relative motion of the body M_2 with respect to M_1, if the reference system origin is set at M_1 and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, is described as:

Definition 2.1. *In $\mathbb{R}^3$, having fixed the coordinate system's origin at Earth, the Newtonian gravitational field F is a continuous function defined by*

$$\begin{aligned} F : \mathbb{R}^3 \setminus \{0\} &\longrightarrow \mathbb{R}^3 \\ F(\boldsymbol{r}) &\longmapsto -\frac{\mu}{r^3}\boldsymbol{r}, \end{aligned} \tag{2.2}$$

where $\mu = G(m_1 + m_2)$ and $r = |\boldsymbol{r}|$.

Remark 2.2. Note that F is not defined at the origin 0, which corresponds to the collision of M_1 and M_2. Collision orbits, as well as orbits with close approaches, can be studied using regularization techniques which are out of scope of this work.

An *orbit* (denoted by $\boldsymbol{r} = \boldsymbol{r}(t)$) is the position, as a function of time, of m_2 under the effects of a Newtonian gravitational field, that is, the solution of

$$\ddot{\boldsymbol{r}} = F(\boldsymbol{r}), \tag{2.3}$$

for a given initial condition, i.e. for a defined $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ at a specific instant. Note that this differential equation is autonomous.

From now on, the interval of definition of the orbit will be denoted by $r(t)$.

Remark 2.3. From now on, to be consistent with Definition (2.2), $r \neq 0$. Likewise, the maximal interval I of the solution $\boldsymbol{r}$ is the line $\mathbb{R}$.

2.1 The orbit and its elements

Equation (2.3) is equivalent to a system composed of three second-order differential equations (one for each coordinate), requiring six independent constants of integration for its complete solution . In this section, we present a study of Equation (2.3) in order to find the constants known as *elements of the orbit*.

Definition 2.4. *We define the angular momentum (per unit mass)* $\boldsymbol{c}$ *as*

$$\boldsymbol{c} = \boldsymbol{r} \times \dot{\boldsymbol{r}}.$$

Remark 2.5. We will omit the case $c = 0$, which specifies that the orbit is a straight line leading to (or starting from) collision.

Proposition 2.6. *The angular momentum* $\boldsymbol{c}$ *is a first integral of* (2.3)*.*

Proof. The only thing that must be seen is that c is constant along the solutions of Equation (2.8)

$$\frac{d}{dt}\boldsymbol{c} = \frac{d}{dt}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \boldsymbol{r} \times \ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} = \boldsymbol{r} \times \ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}(\boldsymbol{r} \times \boldsymbol{r}) = 0.$$

□

Corollary 2.7. *The orbit is planar. The director vector of the orbital plane OP is* $\boldsymbol{c}$*.*

Proof. Geometrically, the result from a cross product is a vector perpendicular to the two multiplying vectors and, as shown above, the vector $\boldsymbol{c}$ is constant. In particular, its direction is constant. Thus, the vectors $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$, and consequently the orbit, are contained in a plane. □

Definition 2.8. *We define the total energy per unit mass H as,*

$$H = V + P, \tag{2.4}$$

where V and P are the cinematic and potential energy respectively, given by

$$V = \frac{1}{2}\dot{r}^2, \qquad P = -\frac{\mu}{r}.$$

Proposition 2.9. *H is a first integral of* (2.3)*.*

Proof. To verify the statement the only thing necessary is to check that the Newtonian gravitational field is a conservative vector field. As if that is the case, the total energy is constant. Now, if we define

$$U : \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R},$$
$$U(\boldsymbol{r}) \longrightarrow \frac{\mu}{r}$$

U is C^1 and

$$grad(U) = -\frac{\mu}{r^3}\boldsymbol{r},$$

proving that the Newtonian gravitational force is conservative (is the gradient of a potential function), furthermore, the total energy is a first integral of (2.3).

□

Following a more constructive process, one last first integral (or three, one for each dimension of $\mathbb{R}^3$) can be obtained.

From the derivative of r, which is well defined as $\boldsymbol{r}(t) \in C^2$ and $r \neq 0$,

$$\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\dot{\boldsymbol{r}}r^2 - \langle \dot{\boldsymbol{r}}, \boldsymbol{r}\rangle \boldsymbol{r}}{r^3} = \frac{1}{r^3}\left[(\boldsymbol{r} \times \dot{\boldsymbol{r}}) \times \boldsymbol{r}\right],$$

where the last equality is derived from the identity

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = \langle \boldsymbol{u}, \boldsymbol{w}\rangle \boldsymbol{v} - \langle \boldsymbol{v}, \boldsymbol{w}\rangle \boldsymbol{u}.$$

Then, multiplying by μ and using Equation (2.3),

$$\mu\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = -\left[(\boldsymbol{r} \times \dot{\boldsymbol{r}}) \times -\frac{\mu \boldsymbol{r}}{r^3}\right] = -\boldsymbol{c} \times \ddot{\boldsymbol{r}} = \frac{d}{dt}(-\boldsymbol{c} \times \dot{\boldsymbol{r}}).$$

The last equality being true due the conservation of the angular momentum. From the equation above, we can derive that

$$\mu\left[\frac{\boldsymbol{r}(t)}{r(t)} + \boldsymbol{e}\right] = -\boldsymbol{c} \times \dot{\boldsymbol{r}}(t), \qquad t \in I,$$

where $\boldsymbol{e} \in \mathbb{R}^3$ is a vector introduced as a constant of integration. Multiplying by $\boldsymbol{r}(t)$,

$$\mu\left[\langle\frac{\boldsymbol{r}(t)}{r(t)}, \boldsymbol{r}(t)\rangle + \langle \boldsymbol{e}, \boldsymbol{r}(t)\rangle\right] = -\langle \boldsymbol{c} \times \dot{\boldsymbol{r}}(t), \boldsymbol{r}(t)\rangle, \qquad t \in I,$$

and

$$\mu\left[\langle\frac{\boldsymbol{r}(t)}{r(t)}, \boldsymbol{r}(t)\rangle + \langle \boldsymbol{e}, \boldsymbol{r}(t)\rangle\right] = -\langle \boldsymbol{c}, \dot{\boldsymbol{r}}(t) \times \boldsymbol{r}(t)\rangle = c^2, \qquad t \in I,$$

which follows from the identity

$$\langle \boldsymbol{u} \times \boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{u}, \boldsymbol{v} \times \boldsymbol{w}\rangle.$$

The above equation can be rewritten as

$$r(t) + \langle \boldsymbol{e}, \boldsymbol{r}(t)\rangle = \frac{c^2}{\mu}, \qquad t \in I. \tag{2.5}$$

Euqation (2.5) corresponds to the points on a conic with one focus at the origin[Ort10].

Definition 2.10. *Given an orbit $\boldsymbol{r}(t)$ in a Newtonian gravitational field in $\mathbb{R}^3$, we define the Laplace vector as*

$$\boldsymbol{e} = \frac{1}{\mu}\dot{\boldsymbol{r}}(t) \times \boldsymbol{c} - \frac{\boldsymbol{r}(t)}{r(t)}, \qquad t \in I,$$

and its module e is named eccentricity.

Proposition 2.11. *The Laplace vector $\boldsymbol{e}$ is a first integral.*

Proof. By definition, as $\boldsymbol{e}$ is a constant of integration. □

Remark 2.12. If $\boldsymbol{e} = 0$, then from Equation (2.5), we obtain that $r = c^2/\mu$ is constant; meaning that the orbit is circular.

Remark 2.13. From the definition of the Laplace vector, we can derive that $\boldsymbol{e}$ is on the orbital plane as both $\boldsymbol{r}$ and $\boldsymbol{c} \times \dot{\boldsymbol{r}}$ are on it.

Remark 2.14. Vector $\boldsymbol{e}$ and $\boldsymbol{c}$ are not independent.

It is of importance to note that we can identify the type of conic using the value of e (proof in [Ort10]),

- $e < 1$. The conic is an ellipse.
- $e > 1$. The conic is an hyperbola.
- $e = 1$. The conic is a parabola.

Similarly, as e and H are related by the expression[Ort10]

$$\mu^2(e^2 - 1) = 2H\,c^2, \tag{2.6}$$

the orbits can be classified as:

- $H < 0$. The conic is an ellipse.
- $H > 0$. The conic is an hyperbola.
- $H = 0$. The conic is a parabola.

Since the main interest of this text are the orbits corresponding to artificial satellites orbiting the Earth, focus is placed on the case of elliptical orbits, restricting e in the sequel to $0 \leq e < 1$, or equivalently, $H < 0$. The reader can refer to **[gerard]** for a development of the other cases.

When $\boldsymbol{e} \neq 0$, we can define polar coordinates on the orbital plane as

$$\boldsymbol{r}(t) = r(t)(\cos\theta(t), \sin\theta(t), 0), \qquad \forall t \in I,$$

where $r(t) = r(t)$ and $\theta(t)$ is the angle between the position vector $\boldsymbol{r}$ and the vector $\boldsymbol{e}$. With that, the following unitary reference system centered at the origin can be introduced: $\{\boldsymbol{q_p}, \boldsymbol{q_n}, \boldsymbol{q_c}\}$, where $\boldsymbol{q_p}$ goes along the vector $\boldsymbol{e}$, $\boldsymbol{q_c}$ along $\boldsymbol{c}$ and $\boldsymbol{q_n}$ is such that $\boldsymbol{q_p} \times \boldsymbol{q_n} = \boldsymbol{q_c}$.

Definition 2.15. *The angle $\theta(t)$ between the vector position $\boldsymbol{r}$ and the Laplace vector $\boldsymbol{e}$ is called the true anomaly.*

With this new notation, previous results can be revisited. The equation of the trajectories can be rewritten in polar coordinates as

$$r(t) = \frac{c^2/\mu}{1 + e\cos\theta(t)} = \frac{p}{1 + e\cos\theta(t)}, \tag{2.7}$$

where p is the *semi-lactus rectum* and denotes the distance of the satellite from the focus at the origin perpendicular to the Laplace vector. Equation (2.7) is known as *the Keplerian conics equation*.

In Equation (2.7) the existence of a minimum and a maximum value of $\boldsymbol{r}(t)$ dependent on the true anomaly can be easily seen. They are known respectively as the *peri-center* and *apo-center* and given by

$$r_{peri} = \frac{p}{1+e}$$

and

$$r_{apo} = \frac{p}{1-e}.$$

The line joining both is called the *line of apsides*. The mean value of the peri-center and the apo-center is the *semi-major axis* a,

$$a = \frac{1}{2}(r_{peri} + r_{apo}) = \frac{p}{1-e^2}. \tag{2.8}$$

Alternatively[**gerard**], using Equation (2.6), a can be expressed as

$$a = \frac{\mu}{2|H|}. \tag{2.9}$$

So far, the Keplerian orbits have been studied from a geometrical perspective, concluding that the trajectory of a satellite has the shape of a conic section. Still, no relation between the orbit and time has been established really. The objective now is to determine the satellite's position on its orbit at a particular time.

Definition 2.16. *Given an orbit $r(t)$ in a Newtonian gravitational field in R^3, with energy $H \neq 0$, the eccentricity anomaly E is defined by the following differential equation:*

$$\dot{E} = \frac{\mu^{1/2}}{a^{1/2} r}.$$

Then, if the notation $' = d/dE$ is introduced, by the chain rule,

$$r' = \frac{a^{1/2}}{\mu^{1/2}} \dot{r} r.$$

From the identity

$$r^2 \dot{r}^2 = (\boldsymbol{r}\dot{\boldsymbol{r}})^2 + (\boldsymbol{r} \times \dot{\boldsymbol{r}})^2 = (r\dot{r})^2 + c^2,$$

using the definition of H,

$$(r\dot{r})^2 + c^2 = 2(\mu r + Hr^2).$$

And by Equation (2.9),

$$r'^2 + \frac{ac^2}{\mu} = 2ar - r^2.$$

If $-a^2$ is added to both sides and Equation (2.8) is rewritten as

$$\frac{c^2}{\mu} = -a(e^2 - 1),$$

then

$$r'^2 - a^2e^2 = -(a - r)^2. \tag{2.10}$$

Defining the function $\rho(E)$ with the equation

$$e\,a\,\rho(E) = a - r,$$

replaced in Equation (2.10), gives

$$\rho'^2 + \rho^2 = 1,$$

from which, omitting the singular solution $p = 1$, has solution

$$\rho(E) = \cos(E + k);$$

obtaining

$$r(E) = a(1 - \cos(E + k)).$$

If we take $E = 0$ at the instant at the peri-center, t_p, then $k = 0$ and introducing this in the definition of E, we get

$$n(t - t_p) = E - e\sin E. \tag{2.11}$$

Equation (2.11) is known as *the Kepler equation*. The *mean motion* denoted by n has been introduced, defined as

$$n^2a^3 = \mu. \tag{2.12}$$

Definition 2.17. *Given an orbit $\mathbf{r}(t)$ in a Newtonian gravitational field in R^3, with energy $H \neq 0$, the mean anomaly M is defined by*

$$M = n(t - t_p),$$

where t_p is the instant at the peri-center.

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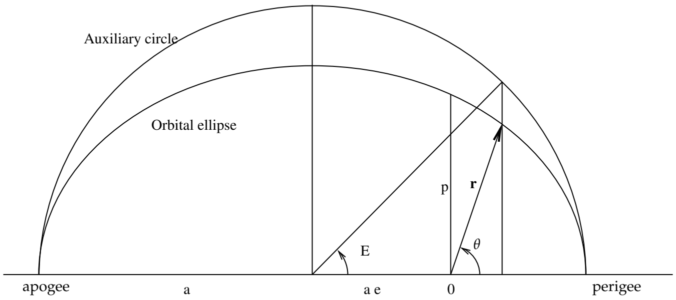


Figure 2.1: Orbital elements and their relation of an elliptic orbit.

2.2 The orbital plane in relation to space

Up to this point, the satellite motion has been described restricted to its orbital plane, that is, in $\mathbb{R}^2$. But it is essential to actively consider it in $\mathbb{R}^3$. Although the coordinate system $\{\boldsymbol{q_p}, \boldsymbol{q_n}, \boldsymbol{q_c}\}$ has already been introduced, the most common one for describing Earth-bound satellite orbits is the geocentric *equatorial coordinate system* or *inertial system*. Its origin is the Earth's center, the z-axis follows to the North Pole and the equatorial plane form the x, y-plane, with the x-axis being the intersection of the equatorial plane with the Earth's orbital plane and the y-axis being perpendicular to the y-axis.

The orientation of the orbital plane with respect to the equatorial plane allows for an intuitive coordinate system that can be described with:

- The *inclination* i indicates the angle of intersection between the orbital plane and the equator.
- The *right ascension of the ascending node* Ω is the (clock-wise) angle between the x-axis and the point where the satellite crosses the equator from south to north.
- The *argument of the perigee* ω is the angle between the direction of the ascending node and the direction of the peri-center.

Note, that since the orbit is restricted to the orbital plane, w, i and Ω are constants.

The conversion between the two already introduced reference frames, as well as a more indepth description of them and other helpful ones can be found in section 4.1.

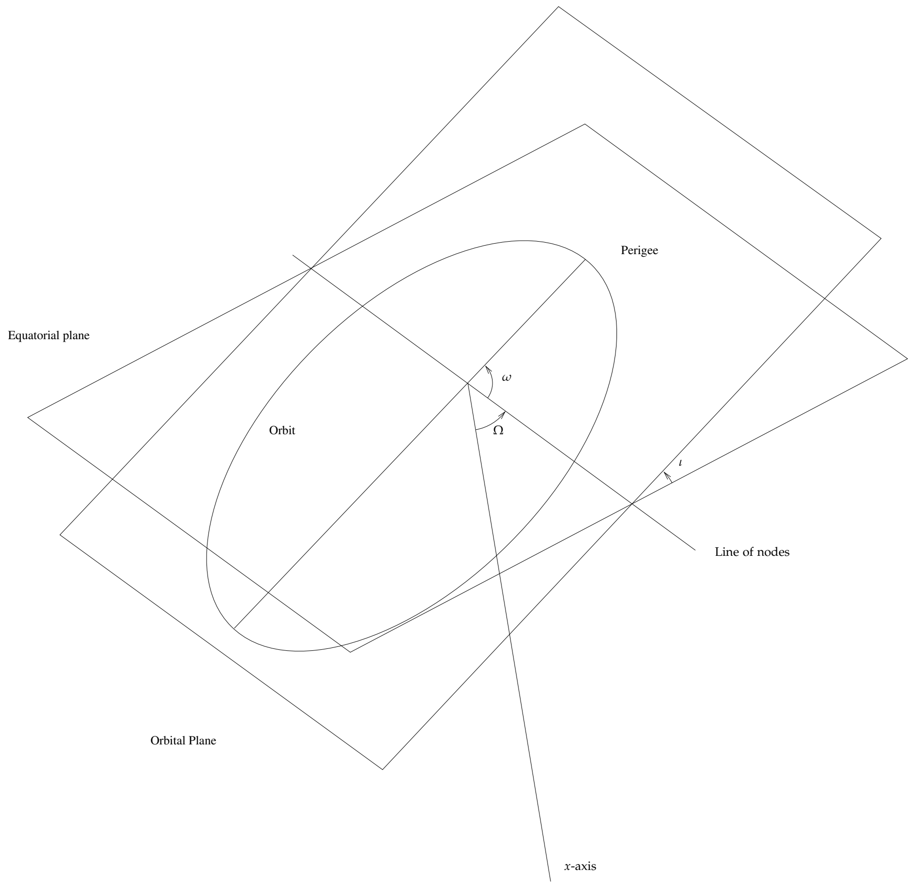


Figure 2.2: Relation between the Equatorial Plane and the Orbital Plane, being the *line of nodes* their intersection

2.3 Keplerian elements

As discussed, six constants are required to solve uniquely Equation (2.3). But not all the seven introduced above ($\{H, \boldsymbol{e}, \boldsymbol{c}\}$) are independent. Classically, the following constants have been used to determine uniquely the orbit of celestial bodies, refereed as the *Keplerian elements*:

$$(e, a, i, \Omega, w, \theta). \tag{2.13}$$

2.4 Kepler's Laws

Moreover, the mathematical framework presented above to describe a satellite orbit can be used to verify Kepler's Laws, established initially empirically.

Kepler's first law. Be $\boldsymbol{r}(t)$, $t \in I$, an orbit on a Newtonian gravitational field with angular momentum $c \neq 0$ and no restriction on H. Then, $\boldsymbol{r}$ moves on a conic (ellipse, parabola or hyperbola) with one focus at the origin.

From the law statement, following the same development as in section 2.1, Equation (2.5) is obtained. As mentioned, this equation corresponds to a conic with one focus at the origin. Note, that the case $e = 0$ is treated as a special case of an ellipse.

Kepler's second law. The area swept by the position vector $\boldsymbol{r}$ on an orbit $\boldsymbol{r}(t)$ is the same for equal intervals of time.

Fixed $t_0 < t_1 \in I$ with $\theta(t_1) - \theta(t_0) < 2\pi$, we define

$$D_{t_0 t_1} : = \{s\, \boldsymbol{r}(t) : t \in (t_0, t_1),\, s \in (0,1)\}.$$

Using the divergence theorem, the area of the region defined above in polar coordinates $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ is determined as

$$area(D_{t_0 t_1}) = \frac{1}{2} \int_{t_0}^{t_1} r(t)^2 \dot{\theta}(t) dt,$$

the proof can be found in [Ort10]. Now, the angular momentum can be rewritten in polar coordinates as

$$c = r(\cos\theta, \sin\theta, 0) \times [\dot{r}(\cos\theta, \sin\theta, 0) + r\dot{\theta}(-\sin\theta, \cos\theta, 0)] =$$
$$r^2\dot{\theta}(\cos\theta, \sin\theta, 0) \times (-\sin\theta, \cos\theta, 0) = r^2\dot{\theta}(0,0,1).$$

Then, by substitution,

$$area(D_{t_0 t_1}) = \frac{1}{2} \int_{t_0}^{t_1} c\, dt = \frac{1}{2} c\, (t_1 - t_0),$$

which only depends on the interval of time.

Kepler's third law. Be $\boldsymbol{r}(t)$ a movement in a Newtonian gravitational field in R^3, with specific angular momentum $c \neq 0$ and energy $H < 0$. Then $r(t)$ is periodic with minimum period

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}.$$

Equation (2.12) defines the mean motion of an orbit. This concept can also be understood as the angular speed a body would require to complete one orbit at constant angular speed (in an equivalent circular orbit). Thus, it can be understood as

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\mu}} a^{3/2}.$$

Interestingly, if Kepler's Laws are taken as fundamentals, Newton's laws of mechanics and Newton's universal gravitational law can be derived with a development of similar complexity to the one followed here [Góm22].

Chapter 3

An analytical solution of the Kepler problem with drag

Having introduced the Kepler problem in its classical form, we would like to modify it to more accurately portrait the conditions under which an artificial satellite in low Earth orbit moves. In particular, taking into account *the atmospheric drag*. A first analytical approach as described in [Mit81] follows.

3.1 Statement of the problem

Following the notation and concepts introduced in chapter 2, the Kepler problem with drag can be expressed by the differential equation

$$\ddot{\boldsymbol{r}} + \beta(\boldsymbol{r}, \dot{\boldsymbol{r}})\dot{\boldsymbol{r}} + \gamma(\boldsymbol{r})\boldsymbol{r} = 0, \tag{3.1}$$

where $\beta(\boldsymbol{r}, \dot{\boldsymbol{r}})$ and $\gamma(\boldsymbol{r})$ are arbitrary scalar coefficients.

Remark 3.1. With $\beta = 0$ and $\gamma = \mu/r^3$, Equation (3.1) is just the classic Kepler problem.

Equation (3.1) is transformed by introducing the true anomaly θ as the independent variable.

$$\dot{\boldsymbol{r}} = \boldsymbol{r}'\dot{\theta}, \qquad \ddot{\boldsymbol{r}} = \boldsymbol{r}''\dot{\theta}^2 + \boldsymbol{r}'\ddot{\theta},$$

where the prime denotes differentiation with respect to the angle θ. Equation (3.1) then becomes

$$\dot{\theta}^2\boldsymbol{r}'' + (\ddot{\theta} + \beta\dot{\theta})\boldsymbol{r}' + \gamma\boldsymbol{r} = 0. \tag{3.2}$$

Using the unit vector $\boldsymbol{\zeta} = \boldsymbol{r}/r$, Equation (3.2) becomes

$$r\dot{\theta}^2\boldsymbol{\zeta}'' + (2r'\dot{\theta}^2 + r\ddot{\theta} + \beta r\dot{\theta})\boldsymbol{\zeta}' + (r''\dot{\theta}^2 + r'\ddot{\theta} + \beta r'\dot{\theta} + r\gamma)\boldsymbol{\zeta} = 0. \tag{3.3}$$

Let

$$r = \frac{1}{u}, \qquad r' = -\frac{u'}{u^2}, \qquad r'' = -\frac{u''}{u^2} + 2\frac{(u')^2}{u^3}$$

then Equation (3.3) becomes:

$$\zeta'' + \left(\frac{\beta}{\dot{\theta}} + \frac{\ddot{\theta}}{\dot{\theta}^2}\right)\zeta' + \left(\frac{2(u')^2 - uu''}{u^2} - \frac{\ddot{\theta}u'}{\dot{\theta}^2 u} - \frac{\beta u'}{\dot{\theta}u} + \frac{\gamma}{\dot{\theta}^2}\right)\zeta = 0. \tag{3.4}$$

Equation (3.4) is written more simply if the variable ν is introduced

$$\nu = \frac{\beta}{\dot{\theta}} + \frac{\ddot{\theta}}{\dot{\theta}^2} - \frac{2u'}{u}, \tag{3.5}$$

$$\zeta'' + \nu\zeta' + \left(-\nu\frac{u'}{u} - \frac{u''}{u} + \frac{\gamma}{\dot{\theta}^2}\right)\zeta = 0. \tag{3.6}$$

3.2 Particular solution

To confirm the solution to the classical Keplerian problem, refraining from specifying β and γ, but exploring the possibility that $\nu = 0$, Equation (3.5) becomes

$$d\left(\frac{\theta}{u}\right)\dot{(}\frac{\dot{\theta}}{u^2}) + \frac{\beta\theta}{u^2} = 0$$

and recalling that $u = 1/r$,

$$d(r^2\dot{\theta}) + \beta r^2 d\theta = 0. \tag{3.7}$$

Note, that if $\beta = 0$, the angular momentum $c = r^2\dot{\theta}$ is constant, as is known from chapter 2.

Equation (3.7) readily admits another integral if $\beta = \alpha/r^2$ where α is a constant. This integral is

$$r^2\dot{\theta} + \alpha\theta = c_0, \tag{3.8}$$

where c_0 is the constant of integration.

With $\nu = 0$, Equation (3.6) reduces to

$$\zeta'' + \left(\frac{\gamma}{\dot{\theta}^2} - \frac{u''}{u}\right)\zeta = 0. \tag{3.9}$$

It is clear that ν is equal to zero in at least two cases, (1) when no drag is present and (2) when the drag is proportional to $1/r^2$. It should be noted that any perturbation having an effect directed along the position vector can be reflected in our choice of γ. Thus, $\nu = 0$ removes any coupling between the unit vector in the direction of $\boldsymbol{r}$ and its derivative with respect to θ.

The simplest solvable problem that includes a drag term is to choose $\gamma = \mu u^3$. For this choice, Equation (3.9) becomes, using Equation (3.8) to eliminate $\dot{\theta}$,

$$\zeta'' + \left(\frac{\mu}{(c_0 + \alpha\theta)^2 u} - \frac{u''}{u} \right) = 0. \tag{3.10}$$

Classically, when $\alpha = 0$, i.e. no drag present, the coefficient of ζ, when set equal to 1, produces the differential equation for the Keplerian conics. With $\alpha \neq 0$, setting the coefficient of ζ equal to 1, produces the differential equations

$$\zeta'' + \zeta = 0 \tag{3.11}$$

and

$$u'' + u = \frac{\mu}{\alpha^2 \left(\frac{c_0}{\alpha} - \theta\right)^2}. \tag{3.12}$$

Equation (3.12) is given by [Dan98].

3.3 Analytical solution of Equation (3.12)

Equation (3.12) can be solved applying the Laplace transform , which will convert the differential equation into an algebraic problem easily solvable, from which the solution of the original equation can be obtained.

For notation convenience, the independent variable is temporarily changed by letting $z = c_0/\alpha - \theta$. Equation (3.12) is now rewritten as

$$\frac{d^2u}{dz^2} + u = \frac{\mu}{\alpha^2 z^2}. \tag{3.13}$$

If u is the Laplace transform of U, then taking the inverse Laplace transform of Equation (3.13),

$$\tau^2 + U = \frac{\mu}{\alpha^2}\tau,$$

so that

$$U = \frac{\mu}{\alpha^2} \frac{\tau}{\tau^2 + 1}.$$

The Laplace transform of this equation, by definition, is

$$u(z) = \frac{\mu}{\alpha^2} \int_0^\infty \frac{\tau e^{-z\tau}}{\tau^2 + 1} d\tau.$$

Using the standard notation, $u(z) = \mu/\alpha^2 g(z)$ where

$$g(z) = \int_0^\infty \frac{t e^{-zt}}{t^2 + 1} dt. \tag{3.14}$$

Remark 3.2. An study of $g(z)$ can be found at [Mit81].

The general solution for $u(\theta)$ is:

$$u(\theta) = e_0 \cos(\theta - \theta_0) + \frac{\mu}{\alpha^2} g\left(\frac{c_0}{\alpha} - \theta\right), \tag{3.15}$$

where e_0 and θ_0 are constants of integration. Hence, the equation for the position r, is

$$r = \frac{p}{e \cos(\theta - \theta_0) + \left(\frac{c_0}{\alpha}\right)^2 g\left(\frac{c_0}{\alpha} - \theta\right)}, \tag{3.16}$$

where $e = e_0\, h_0^2/\mu$ and $p = c_0^2/\mu$. Notice that when

$$\lim_{\alpha \to 0^+} \left(\frac{c_0}{\alpha}\right)^2 g\left(\frac{c_0}{\alpha} - \theta\right) = 1,$$

Equation (3.16) is the solution to the classic two-body problem.

The solution for the position vector $\boldsymbol{r}$, as a function of *theta*, is obtained from Equation (3.11).

$$\boldsymbol{r} = r(\boldsymbol{q_p} \sin(\theta) + \boldsymbol{q_n} \cos(\theta)), \tag{3.17}$$

where $\boldsymbol{q_p}$, $\boldsymbol{q_n}$ were introduced in section 2.1 and r is given by Equation (3.16).

The solution, as given by Equation (3.16) and (3.17), involves nine constants of integration: c_0, e, θ_0 and the vectors $\boldsymbol{q_p}$ and $\boldsymbol{q_n}$. Since the original problem calls for the solution of a second-order vector differential equation, only six of these constants are independent.

3.4 Numerical Solution of Equation (3.12)

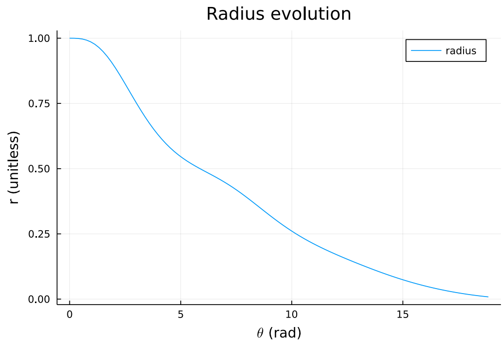
Alternatively, Equation (3.12) can be solved numerically using a numerical integrator. Transforming the equation into

$$u''(\theta) = \frac{\mu}{\alpha^2 \left(\frac{c_0}{\alpha} - \theta\right)^2} - u(\theta),$$

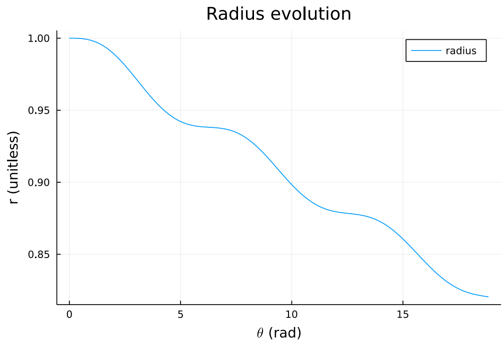
a second order ODE (ordinary differential equations) problem can be defined and solved using the package *DifferentialEquations.jl*. A longer explanation can be found at 7.1,

Similarly to [Mit81], the constant values $\mu = 1$ and $c_0 = 1$ and the parameter values $\alpha = \{0.05, 0.005\}$ have been used for the integration. The initial values for u and u' are 1 and 0 respectively.

From Figures 3.1 and 3.2, a great dependence between the magnitude of α, the atmospheric drag, and the rate of orbit altitude decrease can be observed. Therefore, is it of prime importance to accurately evaluate the atmospheric drag that affects the satellite. And in general, how to parameterize perturbations.

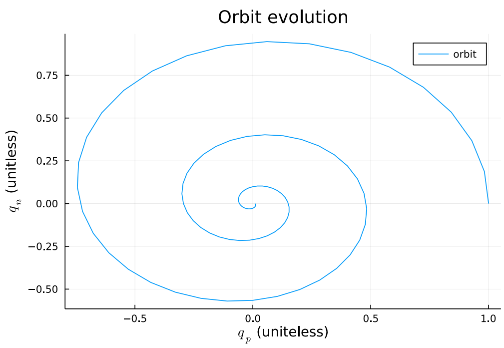


(a) $\alpha = 0.05$

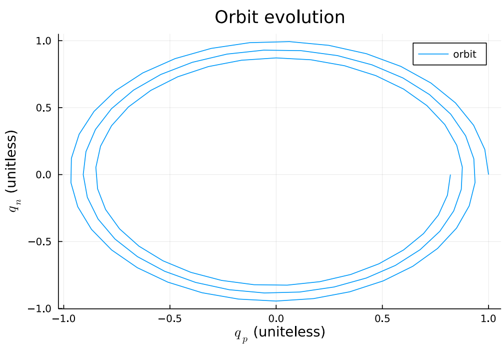


(b) $\alpha = 0.005$

Figure 3.1: Evolution of the module of the radius as a function of θ over three revolutions with parameter α



(a) $\alpha = 0.05$



(b) $\alpha = 0.005$

Figure 3.2: Evolution of the orbit over three revolutions with parameter α

Chapter 4

Perturbation theory and the Lagrange equations

Following the statement of the Kepler problem, an orbit of a satellite is the solution of Equation (2.3). Nevertheless, in reality the are more forces that interact with the satellite. These can be of gravitational type, such as other celestial bodies or the no completely spherical form of the Earth. Or they can be caused by other factors, for example, the atmospheric drag, the solar wind or the solar radiation pressure.

These forces *perturb* the orbit in varying degrees and in different manner, usually order of magnitudes lower than the principal force of Equation (2.2). Still, a framework to represent and study their effects is of interest.

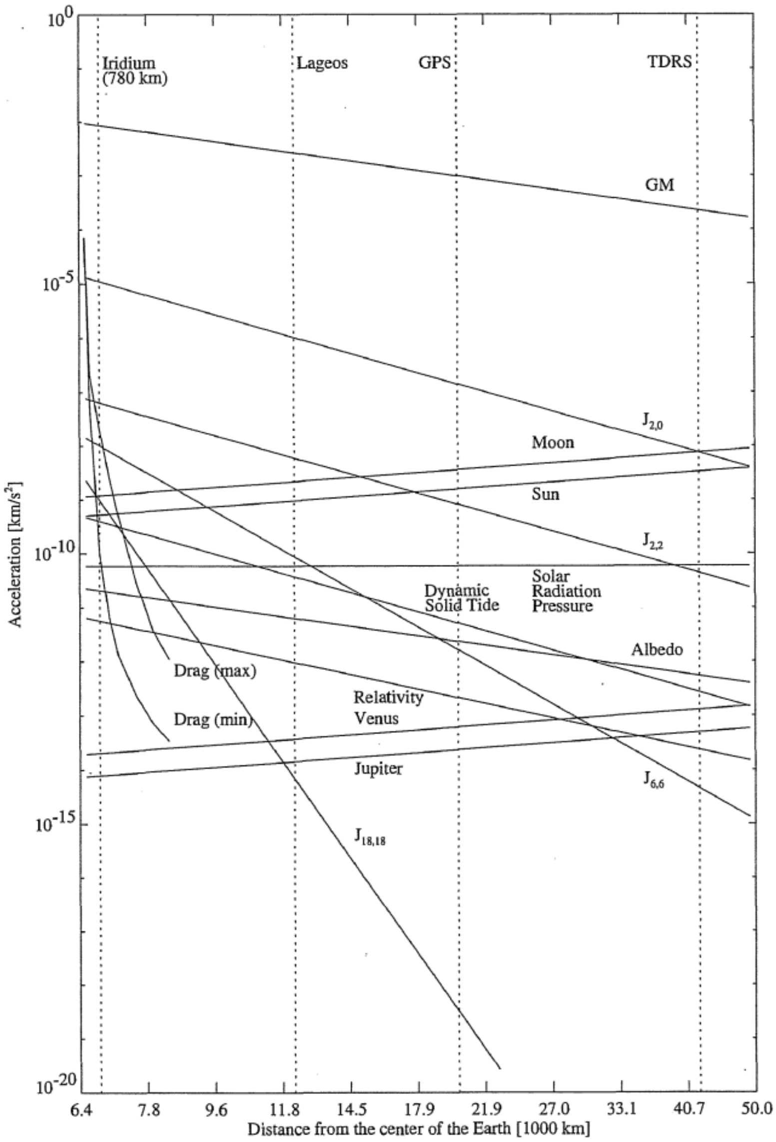


Figure 4.1: Order of magnitude of different perturbations, [Mon00] (p.55)

4.1 Reference systems

In astrodynamics, the use of different reference systems help to simplify some formulations. In our case, these reference systems have their origin at the primary focus of the Keplerian orbit, are orthogonal, unitary and directed. They are shown on the figure below, which is followed by its description.

- The reference system $\mathcal{I} = \{\boldsymbol{q}_x, \boldsymbol{q}_y, \boldsymbol{q}_z\}$. It is called the *inertial reference system* which the general equation of the two-body problem motion is based in. For the study of an artificial satellite's motion around the Earth, $\boldsymbol{q}_z$ is perpendicular to the equator plane, $\boldsymbol{q}_x$ is in the direction of the Aries point (the intersection between the Earth's orbital plane and the equatorial plane) and the axis $\boldsymbol{q}_y$ completes the trihedral.

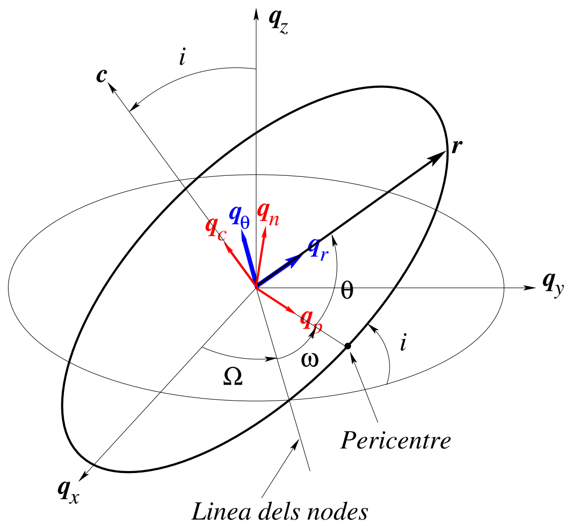


Figure 4.2: Representation of the unit vectors of the reference systems $\mathcal{I} = \{\boldsymbol{q}_x, \boldsymbol{q}_y, \boldsymbol{q}_z\}$, $\mathcal{P} = \{\boldsymbol{q}_p, \boldsymbol{q}_n, \boldsymbol{q}_c\}$ and $\mathcal{O} = \{\boldsymbol{q}_r, \boldsymbol{q}_\theta, \boldsymbol{q}_c\}$.

- The reference system $\mathcal{P} = \{\boldsymbol{q}_p, \boldsymbol{q}_n, \boldsymbol{q}_c\}$. In this reference system, the plane $(\boldsymbol{q}_p, \boldsymbol{q}_n)$ coincides with the satellite's orbital plane. The vector $\boldsymbol{q}_p$ follows the direction of the orbit's peri-center, the vector $\boldsymbol{q}_c$ the direction of the angular momentum (hence, is perpendicular to the orbital plane) and $\boldsymbol{q}_n$ is perpendicular to the previous two, such that $\boldsymbol{q}_c = \boldsymbol{q}_p \wedge \boldsymbol{q}_n$. This reference system is called the *perifocal reference system*.

- The reference system $\mathcal{O} = \{\boldsymbol{q}_r, \boldsymbol{q}_\theta, \boldsymbol{q}_c\}$. It is a rotating reference system in which the plane $(\boldsymbol{q}_r, \boldsymbol{q}_\theta)$ coincides with the orbital plane. The vector $\boldsymbol{q}_r$ follows the direction of the satellite's position vector $\boldsymbol{r}$, the vector $\boldsymbol{q}_c$ the direction of the angular momentum (hence, is perpendicular to the orbital plane) and $\boldsymbol{q}_\theta$ is perpendicular to the previous two, such that, $\boldsymbol{q}_c = \boldsymbol{q}_r \wedge \boldsymbol{q}_\theta$. This reference system is called the *orbital reference system*, or the *vertical-local-vertical-horizontal (LVLH)* reference frame.

- The reference system $\mathcal{V} = \{\boldsymbol{q}_m, \boldsymbol{q}_v, \boldsymbol{q}_c.\}$. It is a rotating reference system in which the plane $(\boldsymbol{q}_v, \boldsymbol{q}_m)$ coincides with the orbital plane. The vector $\boldsymbol{q}_v$ follows the direction of the velocity $\boldsymbol{v}$ of the satellite, the vector $\boldsymbol{q}_c$ the angular momentum (hence, is perpendicular to the orbital plane) and $\boldsymbol{q}_m$ is perpendicular to the previous two.

To transform the inertial coordinates $\boldsymbol{x} = (x, y, z)$ in the reference system $\mathcal{I}$ into the coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ from the reference $\mathcal{P}$ it is necessary to first rotate an angle Ω around the axis $\boldsymbol{q}_z$ to transport the inertial axis $\boldsymbol{q}_x$ to the intersection of the orbital plane with the plane $z = 0$ (line of nodes). This rotation is denoted by $\boldsymbol{R}_3(\Omega)$. Then, a rotation of angle i is needed arount the line of nodes to transport the plane $z = 0$ to the orbital plane, that is, $\boldsymbol{R}_1(i)$. Finally, a rotation around the third axis, $\boldsymbol{R}_3(\omega)$, must be made, which transports the line of nodes to the direction of the peri-center. In summary, the transform between the inertial system $\mathcal{I}$ and the peri-focal system $\mathcal{P}$ is

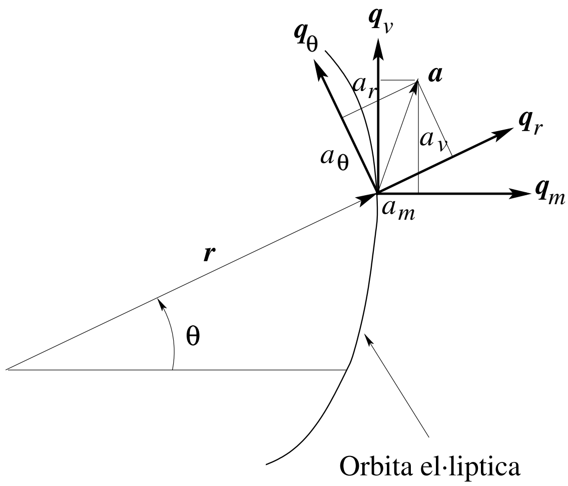


Figure 4.3: Representation of the orbital plane and the unitary vectors $\{q_r, q_\theta\}$ of the reference systems $\mathcal{O}$ and $\{q_v, q_m\}$ of the reference system $\mathcal{V}$, and the components of a vector $\boldsymbol{a}$ in both references.

given by $\boldsymbol{R_{qx}} = R_3(\omega)R_1(i)R_3(\Omega)$,

$$\boldsymbol{R_{qx}} = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega\cos\Omega + \sin\omega\sin\Omega\cos i & \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i & -\sin\omega\sin i \\ -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i & -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i & \cos\omega\sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{pmatrix}, \quad (4.1)$$

that is: $\boldsymbol{q} = \boldsymbol{R_{qx}}\boldsymbol{x}$.

The inverse transform is $\boldsymbol{R_{xq}} = \boldsymbol{R}_3(-\Omega)\boldsymbol{R}_1(-i)\boldsymbol{R}_3(-\omega)$ given by

$$\boldsymbol{R_{xq}} = \begin{pmatrix} \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i & -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i & \sin\Omega\sin i \\ \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i & -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix}. \quad (4.2)$$

If $\omega = 0$, (4.1) becomes

$$\boldsymbol{R_{\hat{q}x}} = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos i & \cos\Omega\cos i & \sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{pmatrix}, \quad (4.3)$$

(which will be useful in what follows), defines a new reference system $\mathcal{N}$ with

$$\begin{pmatrix} \boldsymbol{q}_s \\ \boldsymbol{q}_t \\ \boldsymbol{q}_c \end{pmatrix} = R_{\boldsymbol{\hat{q}x}} \begin{pmatrix} \boldsymbol{q}_x \\ \boldsymbol{q}_y \\ \boldsymbol{q}_z \end{pmatrix}, \quad (4.4)$$

where the vector $\boldsymbol{q}_s$ follows the direction of the line of nodes and $\boldsymbol{q}_c$ is perpendicular to the orbital plane

4.2 The method of variation of parameters

In the inertial reference frame, the equations of the perturbed relative movement of a mass m_2 with respect to another one m_1 are

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{a}_p, \tag{4.5}$$

where $\boldsymbol{a}_p$ is the perturbing acceleration product of an exterior force $\boldsymbol{f}$ action on the mass m_2 ($\boldsymbol{a}_p = \boldsymbol{f}/m_2$). As mentioned, the perturbing acceleration $\boldsymbol{a}_p$ has a considerable lower magnitude order than the gravitational force created by m_1.

Given initial conditions $(\boldsymbol{r}(t_0), \dot{\boldsymbol{r}}(t_0))$ for the perturbed problem (4.5), the six Keplerians elements (*osculators*) associated to the osculating orbit can be computed with these initial conditions: $a(t_0)$, $e(t_0)$, $i(t_0)$, $\Omega(t_0)$, $\omega(t_0)$, $M(t_0)$. On any instant after the initial one, $t > t_0$, the Keplerian elements computed from $(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))$, the solution to the differential equation of the disturbed orbit (4.5), will not coincide with the initial Keplerian elements, as they are no longer first integrals of the disturbed Equation (4.5). The goal of the method of variation of parameters is to determine which are the differential equations describing the variation of the Keplerian elements.

4.2.1 The general method

Given the position vector $\boldsymbol{r}(t) \in \mathbb{R}^3$ of the solution of the perturbed Kepler problem, which depends on 6 arbitrary constants of integration $\boldsymbol{s} = (s_1, ..., s_6)^T$. In absence of the perturbing acceleration ($\boldsymbol{a}_p = 0$), this problem can be expressed as

$$\boldsymbol{r}(t) = \boldsymbol{f}(t, \boldsymbol{s}), \quad \dot{\boldsymbol{r}}(t) = \frac{d\boldsymbol{f}(t, \boldsymbol{s})}{dt} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{s})}{\partial t}, \quad \ddot{\boldsymbol{r}}(t) = \frac{d^2\boldsymbol{f}(t, \boldsymbol{s})}{dt^2} = \frac{\partial^2 \boldsymbol{f}(t, \boldsymbol{s})}{\partial t^2}. \tag{4.6}$$

The method of variation of parameters tries to solve the perturbed motion with the same formal expression for $(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))$ than the non disturbed, but with $\boldsymbol{s}$ dependent on time, that is

$$(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t)) = \left(\boldsymbol{f}(t, \boldsymbol{s}(t)), \frac{d\boldsymbol{f}(t, \boldsymbol{s}(t))}{dt}\right), \quad \ddot{\boldsymbol{r}}(t) = \frac{d^2\boldsymbol{f}(t, \boldsymbol{s}(t))}{dt^2} + \boldsymbol{a}. \tag{4.7}$$

By the chain rule,

$$\dot{\boldsymbol{r}}(t) = \frac{\partial \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial t} + \frac{\partial \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt}, \quad \ddot{\boldsymbol{r}}(t) = \frac{\partial^2 \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial t^2} + \frac{\partial^2 \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial t \partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt}. \tag{4.8}$$

If the expression of $\dot{\boldsymbol{x}}(t)$ and $\ddot{\boldsymbol{x}}(t)$ are compared to (4.6), (4.7) i (4.8), it follows that

$$\begin{aligned} \frac{\partial \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} &= 0, \\ \frac{\partial^2 \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial t \partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} &= \boldsymbol{a}, \end{aligned}$$

which can be written in compact form as

$$\begin{pmatrix} \dfrac{\partial \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial \boldsymbol{s}} \\ \\ \dfrac{\partial^2 \boldsymbol{f}(t, \boldsymbol{s}(t))}{\partial t \partial \boldsymbol{s}} \end{pmatrix} \frac{d\boldsymbol{s}}{dt} \equiv L\dot{\boldsymbol{s}} = \begin{pmatrix} 0 \\ \boldsymbol{a} \end{pmatrix}.$$

With the inversion of matrix L, the differential equations of the variation of the "constants" $\boldsymbol{s}$ are derived:

$$\dot{\boldsymbol{s}} = L^{-1} \begin{pmatrix} 0 \\ \boldsymbol{a} \end{pmatrix}.$$

4.2.2 The Lagrange method

The Lagrange method constructs the matrix L, introduced above, in such a way that its inversion is easy to compute, obtaining for the perturbed Kepler problem an explicit expression for the differential equations that give the derivatives of the "constants" $\boldsymbol{s}$. It is assumed in this section that the perturbing force is the gradient of some potential function $R(\boldsymbol{r})$

$$\boldsymbol{a}_p = \left(\frac{\partial R}{\partial \boldsymbol{r}}\right)^T = \nabla R(\boldsymbol{r}),$$

for example, the gravity pull generated by other celestial bodies.

The potential energy per unit mass of the system is given by

$$V(\boldsymbol{r}) = -\frac{\mu}{r} - R(\boldsymbol{r}),$$

and the equations of motion are

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}, \tag{4.9}$$

$$\frac{d\boldsymbol{v}}{dt} = -\nabla V(\boldsymbol{r}) = -\frac{\mu}{r^3}\boldsymbol{r} + \nabla R(\boldsymbol{r}), \tag{4.10}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are functions depending on time t and on the six Keplerian elements $\boldsymbol{s} = (s_1, s_2, s_3, s_4, s_5, s_6)^T$: $\boldsymbol{r} = \boldsymbol{r}(t, \boldsymbol{s})$, $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{s})$. Hence,

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = \boldsymbol{v} + \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} \quad \Rightarrow \quad \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = 0. \tag{4.11}$$

Similarly, it can be computed that

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = \boldsymbol{a}_p.$$

Because $\partial \boldsymbol{v}/\partial t$ is the Keplerian component of the acceleration, $\partial \boldsymbol{v}/\partial t = -(\mu/r^3)\boldsymbol{r}$, obtaining that

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = \left(\frac{\partial R}{\partial \boldsymbol{r}}\right)^T. \tag{4.12}$$

Multiplying equation (4.12) by $(\partial \boldsymbol{r}/\partial \boldsymbol{s})^T$, (4.11) by $-(\partial \boldsymbol{v}/\partial \boldsymbol{s})^T$ and subtracting them, results in

$$\left[\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right)^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}} - \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}}\right)^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right] \frac{d\boldsymbol{s}}{dt} \equiv L \frac{d\boldsymbol{s}}{dt} = \left[\frac{\partial R}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right]^T = \left[\frac{\partial R}{\partial \boldsymbol{s}}\right]^T.$$

Thus, the Lagrange's equations of variation of constants are:

$$\frac{d\boldsymbol{s}}{dt} = L^{-1} \left[\frac{\partial R}{\partial \boldsymbol{s}}\right]^T. \tag{4.13}$$

The components L_{ij} of the matrix L are called Lagrange's parenthesis and if $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and $\boldsymbol{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)^T$, the Lagrange's parenthesis are defined as

$$L_{ij} = [s_i, s_j] = \left(\frac{\partial \boldsymbol{r}}{\partial s_i}\right)^T \frac{\partial \boldsymbol{v}}{\partial s_j} - \left(\frac{\partial \boldsymbol{v}}{\partial s_i}\right)^T \frac{\partial \boldsymbol{r}}{\partial s_j} = \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial s_i}\frac{\partial \dot{x}_k}{\partial s_j} - \frac{\partial \dot{x}_k}{\partial s_i}\frac{\partial x_k}{\partial s_j}\right).$$

The Lagrange Matrix L can be written in compact form introducing the state vector

$$\boldsymbol{X}(t, \boldsymbol{s}) = \begin{pmatrix} \boldsymbol{r}(t, \boldsymbol{s}) \\ \boldsymbol{v}(t, \boldsymbol{s}) \end{pmatrix},$$

then

$$L = \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right]^T \begin{bmatrix} 0 & I_{3\times 3} \\ -I_{3\times 3} & 0 \end{bmatrix} \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right] \equiv \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right]^T J \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right], \tag{4.14}$$

where J is a symmetric matrix such that $J^2 = -Id$.

4.3 The Lagrange planetary equations

To obtain Lagrange's planetary equations, the vector $\boldsymbol{s}$ is defined as

$$\boldsymbol{s} = (s_1, s_2, s_3, s_4, s_5, s_6)^T \equiv (a, e, i, \omega, \Omega, M_0)^T,$$

where $M_0 = n\, t_0$ is the initial mean anomaly , n is the mean motion and t_0 is the initial instant (for example, the epoch of pass at the peri-center).

In the reference system $\mathcal{P}$, following Figure 4.4, the coordinates of m_2, assuming it follows an elliptical orbit, can be written as a function of the eccentric anomaly E,

$$\boldsymbol{r} = \begin{pmatrix} \xi - ae \\ \eta \\ 0 \end{pmatrix} = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}, \tag{4.15}$$

where

$$r = a(1 - e\cos E),$$

$$\dot{\boldsymbol{r}} = \frac{na}{1 - e\cos E} \begin{pmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{pmatrix} = \frac{na}{\sqrt{1-e^2}} \begin{pmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{pmatrix}. \tag{4.16}$$

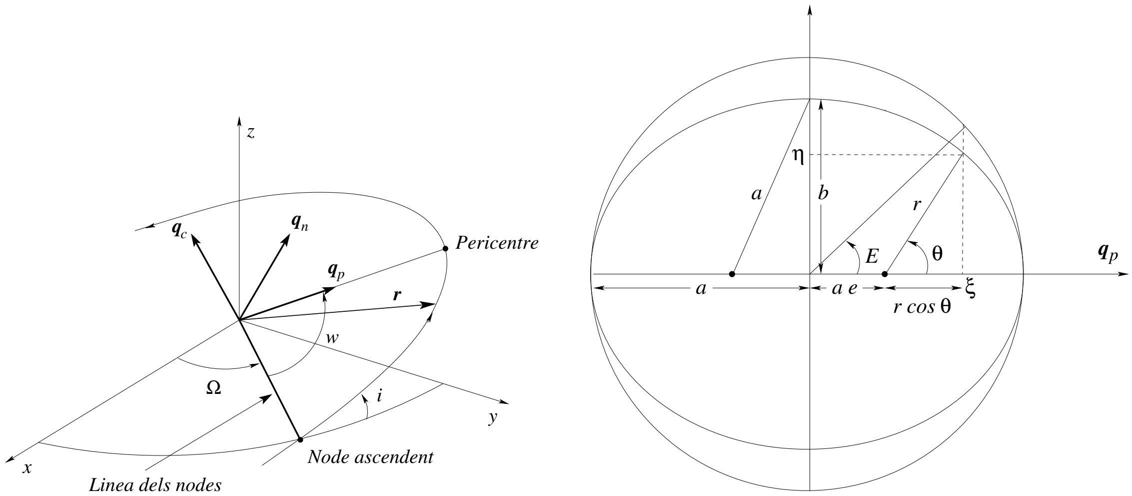


Figure 4.4: Relation between the Equatorial and the Orbital Plane as well as more details of the relation between the orbital elements.

The satellite coordinates, velocity and acceleration are given by

$$\boldsymbol{x} = (x_1, x_2, x_3)^T, \quad \dot{\boldsymbol{x}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)^T, \quad \ddot{\boldsymbol{x}} = (\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)^T = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right)^T.$$

Using the rotation matrix (4.2), the inertial rectangular coordinates can be rewritten in terms of the Keplerian elements,

$$\boldsymbol{x} = \boldsymbol{R_{xq}}(\Omega, i, \omega)\, \boldsymbol{r}(a, e, M), \quad \dot{\boldsymbol{x}} = \boldsymbol{R_{xq}}(\Omega, i, \omega)\, \dot{\boldsymbol{r}}(a, e, M). \tag{4.17}$$

The derivatives of $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ can be expressed as a function of the Keplerian elements s_k derivatives, using

$$\dot{x}_i = \frac{dx_i}{dt} = \sum_{k=1}^{6} \frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt}, \qquad i = 1,2,3, \tag{4.18}$$

$$\frac{\partial V}{\partial x_i} = \frac{d\dot{x}_i}{dt} = \sum_{k=1}^{6} \frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt}, \qquad i = 1,2,3, \tag{4.19}$$

where the derivatives $\partial x_i/\partial s_k$ are obtained differentiating (4.15) and (4.17), and $\partial \dot{x}_i/\partial s_k$ differentiating (4.16) and (4.17).

Multiplying (4.18) by $-\partial \dot{x}_i/\partial s_l$ and adding up with respect to i,

$$-\sum_{i=1}^{3} \frac{\partial \dot{x}_i}{\partial s_l}\dot{x}_i = -\sum_{i=1}^{3}\sum_{k=1}^{6} \frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt}$$

results. Analogously, multiplying (4.19) by $\partial x_i/\partial s_l$ and adding up with respect to i,

$$\sum_{i=1}^{3} \frac{\partial x_i}{\partial s_l}\frac{\partial V}{\partial x_i} = \sum_{i=1}^{3}\sum_{k=1}^{6} \frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt}.$$

The sum from these last two equalities is

$$-\sum_{i=1}^{3}\frac{\partial \dot{x}_i}{\partial s_l}\dot{x}_i+\sum_{i=1}^{3}\frac{\partial x_i}{\partial s_l}\frac{\partial V}{\partial x_i}=-\sum_{i=1}^{3}\sum_{k=1}^{6}\frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt}+\sum_{i=1}^{3}\sum_{k=1}^{6}\frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt}, \tag{4.20}$$

which can be rewritten in compact form as

$$\sum_{k=1}^{6}[s_l,s_k]\frac{ds_k}{dt}=\frac{\partial F}{\partial s_l}, \tag{4.21}$$

where

$$[s_l,s_k]=\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k}-\frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}\right) \tag{4.22}$$

are the previously mentioned Lagrange parenthesis and $F = V - T$ is the total force, the difference between the potential energy (with reversed sign) V and the kinetic $T = (1/2)\sum_{i=1}^{3}\dot{x}_i^2$.

To have an explicit differential equation, it remains to: (1) compute the Lagrange parentheses, (2) write the potential function V as a function of the Keplerian elements.

Following the definition above,

$$[s_l,s_k]=-[s_k,s_l],\quad [s_k,s_k]=0.$$

Thus, only the computation of 15 Lagrange parenthesis is needed. Furthermore, their computation is independent of time. That is,

$$\begin{aligned}
\frac{\partial}{\partial t}[s_l,s_k] &= \sum_{i=1}^{3}\left(\frac{\partial^2 x_i}{\partial s_l\partial t}\frac{\partial \dot{x}_i}{\partial s_k}+\frac{\partial x_i}{\partial s_l}\frac{\partial^2 \dot{x}_i}{\partial s_k\partial t}-\frac{\partial^2 \dot{x}_i}{\partial s_l\partial t}\frac{\partial x_i}{\partial s_k}-\frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial^2 x_i}{\partial s_k\partial t}\right)\\
&= \sum_{i=1}^{3}\left(\frac{\partial}{\partial s_l}\left[\frac{\partial x_i}{\partial t}\frac{\partial \dot{x}_i}{\partial s_k}-\frac{\partial x_i}{\partial s_k}\frac{\partial \dot{x}_i}{\partial t}\right]-\frac{\partial}{\partial s_k}\left[\frac{\partial x_i}{\partial t}\frac{\partial \dot{x}_i}{\partial s_l}-\frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial t}\right]\right)\\
&= \sum_{i=1}^{3}\left(\frac{\partial}{\partial s_l}\left[\dot{x}_i\frac{\partial \dot{x}_i}{\partial s_k}-\frac{\partial x_i}{\partial s_k}\ddot{x}_i\right]-\frac{\partial}{\partial s_k}\left[\dot{x}_i\frac{\partial \dot{x}_i}{\partial s_l}-\frac{\partial x_i}{\partial s_l}\ddot{x}_i\right]\right)\\
&= \sum_{i=1}^{3}\left(\frac{\partial}{\partial s_l}\left[\frac{1}{2}\frac{\partial v^2}{\partial s_k}-\frac{\partial x_i}{\partial s_k}\frac{\partial(\mu/r)}{\partial x_i}\right]-\frac{\partial}{\partial s_k}\left[\frac{1}{2}\frac{\partial v^2}{\partial s_l}-\frac{\partial x_i}{\partial s_l}\frac{\partial(\mu/r)}{\partial x_i}\right]\right)\\
&= \frac{1}{2}\frac{\partial^2 v^2}{\partial s_l\partial s_k}-\frac{\partial^2(\mu/r)}{\partial s_l\partial s_k}-\frac{1}{2}\frac{\partial^2 v^2}{\partial s_l\partial s_k}-\frac{\partial^2(\mu/r)}{\partial s_l\partial s_k} \;=\; 0.
\end{aligned}$$

Consequently of this invariance, and from Equations (4.17), $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ can be determined at the point in which they are easier to compute, i.e. the peri-center, where the eccentric anomaly E is 0. In this manner, from Equations (4.15), (4.16) and (4.17), naming r_{ij} the components of the matrix $\boldsymbol{R_{xq}}$, it follows that

$$
\begin{aligned}
[s_l, s_k] &= \sum_{i=1}^{3} \left(\frac{\partial r_{i1}}{\partial s_l} \frac{\partial r_{i2}}{\partial s_k} - \frac{\partial r_{i2}}{\partial s_l} \frac{\partial r_{i1}}{\partial s_k} \right) na^2 \sqrt{1-e^2}, \\
& \quad \text{si} \quad s_l = \Omega,\ i,\ \omega, \quad s_k = \Omega,\ i,\ \omega. \\
[s_l, s_k] &= a(1-e) \sum_{i=1}^{3} \frac{\partial r_{i1}}{\partial s_l} \left(r_{i1} \frac{\partial \dot{q}_1}{\partial s_k} - r_{i2} \frac{\partial \dot{q}_2}{\partial s_k} \right) - \frac{\sqrt{1-e^2} na}{1-e} \sum_{i=1}^{3} \frac{\partial r_{12}}{\partial s_l} \left(r_{i1} \frac{\partial q_1}{\partial s_k} - r_{i2} \frac{\partial q_2}{\partial s_k} \right), \qquad (4.23) \\
& \quad \text{si} \quad s_l = \Omega,\ i,\ \omega, \quad s_k = \Omega,\ i,\ \omega. \\
[s_l, s_k] &= \sum_{i=1}^{3} r_{i1} r_{i1} \left(\frac{\partial q_1}{\partial s_l} \frac{\partial \dot{q}_1}{\partial s_k} - \frac{\partial q_1}{\partial s_k} \frac{\partial \dot{q}_1}{\partial s_l} \right) + \sum_{i=1}^{3} r_{i1} r_{i2} \left(\frac{\partial q_1}{\partial s_l} \frac{\partial \dot{q}_2}{\partial s_k} - \frac{\partial q_1}{\partial s_k} \frac{\partial \dot{q}_2}{\partial s_l} \right) \\
& \quad + \sum_{i=1}^{3} r_{i2} r_{i1} \left(\frac{\partial q_2}{\partial s_l} \frac{\partial \dot{q}_1}{\partial s_k} - \frac{\partial q_2}{\partial s_k} \frac{\partial \dot{q}_1}{\partial s_l} \right) + \sum_{i=1}^{3} r_{i2} r_{i2} \left(\frac{\partial q_2}{\partial s_l} \frac{\partial \dot{q}_2}{\partial s_k} - \frac{\partial q_2}{\partial s_k} \frac{\partial \dot{q}_2}{\partial s_l} \right) \\
& \quad \text{si} \quad s_l = a,\ e,\ M_0, \quad s_k = a,\ e,\ M_0.
\end{aligned}
$$

Therefore, using (4.17) and (4.23) it results that

$$
\begin{aligned}
[\Omega, i] &= \sum_{i=1}^{3} \left(\frac{\partial r_{i1}}{\partial \Omega} \frac{\partial r_{i2}}{\partial i} - \frac{\partial r_{i2}}{\partial \Omega} \frac{\partial r_{i1}}{\partial i} \right) na^2 \sqrt{1-e^2}, \\
&= [(-\sin\Omega\cos\omega - \cos\Omega\cos i\sin\omega)\sin\Omega\sin i\cos\omega \\
&\quad -(\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega)\cos\Omega\sin i\cos\omega \\
&\quad -(\sin\Omega\sin\omega - \cos\Omega\cos i\cos\omega)\sin\Omega\sin i\sin\omega \\
&\quad -(\cos\Omega\sin\omega + \sin\Omega\cos i\cos\omega)\cos\Omega\sin i\sin\omega] na^2\sqrt{1-e^2} \\
&= -na^2\sqrt{1-e^2}\sin i.
\end{aligned}
$$

The remaining non zero parenthesis are

$$
\begin{aligned}
[\Omega, i] &= -[i, \Omega,] &&= -na^2\sqrt{1-e^2}\sin i, \\
[\Omega, a] &= -[a, \Omega,] &&= \frac{1}{2} na\sqrt{1-e^2}\cos i, \\
[\Omega, e] &= -[e, \Omega,] &&= -na^2 \frac{e}{\sqrt{1-e^2}} \cos i, \qquad (4.24) \\
[\omega, a] &= -[a, \omega,] &&= \frac{1}{2} na\sqrt{1-e^2}, \\
[\omega, e] &= -[e, \omega,] &&= -na^2 \frac{e}{\sqrt{1-e^2}}, \\
[a, M_0] &= -[M_0, a,] &&= -\frac{1}{2} na.
\end{aligned}
$$

Lastly, introducing (4.24) in (4.21), the planetary Lagrange equations are obtained:

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial F}{\partial M_0}, \\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial F}{\partial M_0} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial F}{\partial \omega}, \\
\frac{di}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial F}{\partial \omega} - \frac{\partial F}{\partial \Omega}\right), \\
\frac{d\omega}{dt} &= -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial F}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial F}{\partial e}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial F}{\partial i}, \\
\frac{dM_0}{dt} &= -\frac{1-e^2}{na^2 e}\frac{\partial F}{\partial e} - \frac{2}{na}\frac{\partial F}{\partial a}.
\end{aligned}
\tag{4.25}
$$

Note that the denominators of equations (4.25) have terms e and $\sin i$, which could be close to 0, meaning that these equations are singular. It is possible to use other orbital elements when that is the case[Mon00](p29).

Chapter 5

Gauss variational equations

As mentioned above, the Lagrange equations are only well defined when the perturbing force is the gradient of a potential function. To remedy this restriction, a *Poisson matrix* is used.

5.1 Poisson parenthesis

Similarly to the expression (4.14) obtained for the Lagrange matrix, the Poisson P matrix is defined as

$$P = \left[\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right] J \left[\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right]^T. \tag{5.1}$$

The matrix P components are called Poisson parenthesis and are defined by

$$P_{ij} = \langle e_i, e_j \rangle = \frac{\partial s_i}{\partial \boldsymbol{r}} \left(\frac{\partial s_j}{\partial \boldsymbol{v}}\right)^T - \frac{\partial s_j}{\partial \boldsymbol{r}} \left(\frac{\partial s_i}{\partial \boldsymbol{v}}\right)^T.$$

A relation between L and P can be derived from their product

$$LP = \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right]^T J \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right] \left[\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right] J \left[\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right]^T = \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{s}}\right]^T J^2 \left[\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right]^T = -Id,$$

that is, $P = -L^{-1}$, considering that $P^T = -P$, it can be rewritten as $P^T = L^{-1}$.

Using this last identity, the Lagrange equations (4.13) can be rewritten as

$$\frac{d\boldsymbol{s}}{dt} = L^{-1} \left[\frac{\partial R}{\partial \boldsymbol{s}}\right]^T = P^T \left[\frac{\partial R}{\partial \boldsymbol{s}}\right]^T. \tag{5.2}$$

These equations will be used to obtain the equations for $d\boldsymbol{s}/dt$ when the perturbation force is not the gradient of a potential function , but it is caused by a general perturbing acceleration $\boldsymbol{a}_p$.

In this case, equation (4.10) is written as

$$\frac{d\boldsymbol{v}}{dt} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{a}_p,$$

and equations (4.11) and (4.12) as

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = 0, \quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}}\frac{d\boldsymbol{s}}{dt} = \boldsymbol{a}_p$$

and, as it was in the conservative case, the Lagrange equations are transformed into

$$\left[\left(\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right)^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}} - \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{s}}\right)^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right] \frac{d\boldsymbol{s}}{dt} \equiv L\frac{d\boldsymbol{s}}{dt} = \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right]^T \boldsymbol{a}_p,$$

from where it can be derived that

$$\frac{d\boldsymbol{s}}{dt} = P\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\right]^T \boldsymbol{a}_p = -\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}} J \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{s}}\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}}\right]^T \boldsymbol{a}_p = -\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{X}} J \left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{X}}\right]^T \boldsymbol{a}_p,$$

which, using the definitions of $\boldsymbol{X}$ and J, can be expressed as

$$\frac{d\boldsymbol{s}}{dt} = \left(\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}}\right]^T - \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{r}}\left[\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{v}}\right]^T\right)\boldsymbol{a}_p = \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{v}}\boldsymbol{a}_p, \tag{5.3}$$

considering that $\partial\boldsymbol{r}/\partial\boldsymbol{r} = Id$ and $\partial\boldsymbol{r}/\partial\boldsymbol{v} = 0$.

5.2 Variation of the Keplerian elements

5.2.1 Variation of the semi-major axis a

From equation (2.9) and the definition (2.4) of energy H and its conservation property, it follows that

$$v^2 = \boldsymbol{v}^T\boldsymbol{v} = \frac{2\mu}{r} - \frac{\mu}{a},$$

from which

$$\frac{\partial a}{\partial \boldsymbol{v}} = \frac{2a^2}{\mu}\boldsymbol{v}^T.$$

Introducing this expression into (5.3) produces

$$\frac{da}{dt} = \frac{\partial a}{\partial \boldsymbol{v}}\boldsymbol{a}_p = \frac{2a^2}{\mu}\boldsymbol{v}^T\boldsymbol{a}_p. \tag{5.4}$$

This equation is independent of the chosen reference frame in which vector $\boldsymbol{v}$ and $\boldsymbol{a}_p$ are defined.

5.2.2 Variation of the eccentricity e

From the definition of angular momentum $\boldsymbol{c} = \boldsymbol{r} \wedge \boldsymbol{v}$,

$$c^2 = r^2\boldsymbol{v}^T\boldsymbol{v} - (\boldsymbol{r}^T\boldsymbol{v})^2,$$

and differentiating with respect to the velocity,

$$\frac{\partial c}{\partial \boldsymbol{v}} = \frac{1}{h}\left(r^2\boldsymbol{v}^T - (\boldsymbol{r}^T\boldsymbol{v})\boldsymbol{r}^T\right). \tag{5.5}$$

Now, from the definition $p = c^2/\mu = a(1-e^2)$, $c^2 = \mu a(1-e^2)$ is obtained, which once differentiated gives

$$2c\frac{\partial c}{\partial \boldsymbol{v}} = \mu\frac{\partial a}{\partial \boldsymbol{v}}(1-e^2) - 2\mu a e\frac{\partial e}{\partial \boldsymbol{v}}.$$

If (5.4) and (5.5) are introduced in this last expression, then

$$\frac{\partial e}{\partial \boldsymbol{v}} = \frac{1}{\mu a e}\left((pa - r^2)\boldsymbol{v}^T + (\boldsymbol{r}^T\boldsymbol{v})\boldsymbol{r}^T\right).$$

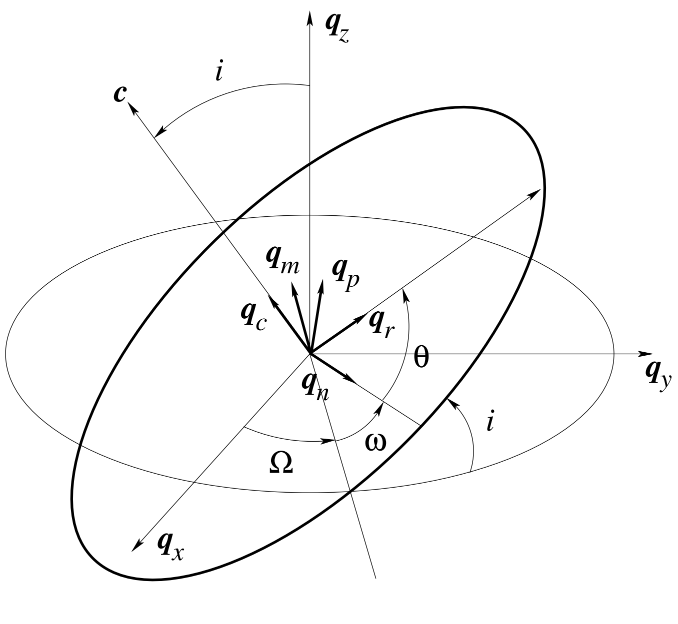
The eccentricity variation equation is

$$\frac{de}{dt} = \frac{\partial e}{\partial \boldsymbol{v}}\boldsymbol{a}_p = \frac{1}{\mu a e}\left((pa - r^2)\boldsymbol{v}^T\boldsymbol{a}_p + (\boldsymbol{r}^T\boldsymbol{v})\boldsymbol{r}^T\boldsymbol{a}_p)\right). \tag{5.6}$$

5.2.3 Variation of the right ascension of the ascending node Ω and the inclination i

To compute $d\Omega/dt$ and di/dt the inertial reference system $\mathcal{N}$, defined by the vectors $\{\boldsymbol{q}_s, \boldsymbol{q}_t, \boldsymbol{q}_c\}$ using the rotation (4.4), will be used, that is,

$$\begin{pmatrix} \boldsymbol{q}_s \\ \boldsymbol{q}_t \\ \boldsymbol{q}_c \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos i & \cos\Omega\cos i & \sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_x \\ \boldsymbol{q}_y \\ \boldsymbol{q}_z \end{pmatrix}. \tag{5.7}$$



The angular momentum can also be written as

$$\boldsymbol{c} = c\,(\sin\Omega\sin i\,\boldsymbol{q}_x - \cos\Omega\sin i\,\boldsymbol{q}_y + \cos i\,\boldsymbol{q}_z),$$

from which, using (5.7),

$$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{v}} = c \sin i \, \boldsymbol{q}_s \frac{\partial \Omega}{\partial \boldsymbol{v}} - c \, \boldsymbol{q}_t \frac{\partial i}{\partial \boldsymbol{v}} + \boldsymbol{q}_c \frac{\partial c}{\partial \boldsymbol{v}}$$

is obtained. By the definition of $\boldsymbol{c}$,

$$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{v}} = \frac{\partial \boldsymbol{r} \wedge \boldsymbol{v}}{\partial \boldsymbol{v}} = \boldsymbol{r} \wedge$$

and since,

$$-\boldsymbol{r} \wedge = c \sin i \left(\frac{\partial \Omega}{\partial \boldsymbol{v}} \right)^T \boldsymbol{q}_s^T - c \left(\frac{\partial i}{\partial \boldsymbol{v}} \right)^T \boldsymbol{q}_t^T + \left(\frac{\partial c}{\partial \boldsymbol{v}} \right)^T \boldsymbol{q}_c^T. \tag{5.8}$$

The product of this last equation with $\boldsymbol{i}_n$ is

$$\frac{\partial \Omega}{\partial \boldsymbol{v}} = \frac{1}{c \sin i} (\boldsymbol{q}_s \wedge \boldsymbol{r})^T.$$

And because $\boldsymbol{r}$ can be expressed as

$$\boldsymbol{r} = r \left(\cos(\omega + \theta) \boldsymbol{q}_s + \sin(\omega + \theta) \boldsymbol{q}_t \right),$$

it follows that

$$\frac{\partial \Omega}{\partial \boldsymbol{v}} = \frac{r \sin(\omega + \theta)}{c \sin i} \boldsymbol{i}_c^T.$$

Using the previous expression for $\boldsymbol{r}$, and the product of equation (5.8) and $\boldsymbol{q}_t$,

$$\frac{\partial i}{\partial \boldsymbol{v}} = \frac{r \cos(\omega + \theta)}{c} \boldsymbol{i}_c^T$$

is derived. Introducing these last two derivatives into (5.3), the following system is obtained,

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial \boldsymbol{v}} \boldsymbol{a}_p = \frac{r \sin(\omega + \theta)}{c \sin i} \boldsymbol{i}_c^T \boldsymbol{a}_p, \tag{5.9}$$

$$\frac{di}{dt} = \frac{\partial i}{\partial \boldsymbol{v}} \boldsymbol{a}_p = \frac{r \cos(\omega + \theta)}{c} \boldsymbol{i}_c^T \boldsymbol{a}_p, . \tag{5.10}$$

5.2.4 Variation of the anomalies

First, the derivative of the true anomaly θ with respect to time will be derived, followed by the eccentric E and mean M anomalies.

From equation (2.7),

$$r = \frac{c^2/\mu}{1 + e \cos \theta},$$

it follows that

$$r e \sin \theta \frac{\partial \theta}{\partial \boldsymbol{v}} = r \cos \theta \frac{\partial e}{\partial \boldsymbol{v}} - \frac{2c}{\mu} \frac{\partial c}{\partial \boldsymbol{v}}. \tag{5.11}$$

In the reference system $\mathcal{O} = \{\boldsymbol{q}_r, \boldsymbol{q}_\theta, \boldsymbol{q}_c\}$, the vector position is $\boldsymbol{r} = (r, 0, 0)^T$, and in the inertial reference system $\mathcal{I} = \{\boldsymbol{q}_x, \boldsymbol{q}_y, \boldsymbol{q}_z\}$,

$$\boldsymbol{r} = r \begin{pmatrix} \cos\Omega\cos(\omega+\theta) - \sin\Omega\sin(\omega+\theta)\cos i \\ \sin\Omega\cos(\omega+\theta) + \cos\Omega\cos(\omega+\theta)\cos i \\ \sin(\omega+\theta)\cos i \end{pmatrix},$$

$$\dot{\boldsymbol{r}} = -\frac{\mu}{c} \begin{pmatrix} \cos\Omega(\sin(\omega+\theta) + e\sin\omega) + \sin\Omega(\cos(\omega+\theta) + e\cos\omega)\cos i \\ \sin\Omega(\sin(\omega+\theta) + e\sin\omega) - \cos\Omega(\cos(\omega+\theta) + e\cos\omega)\cos i \\ -(\cos(\omega+\theta) + e\cos\omega)\sin i \end{pmatrix}.$$

Following from these two expressions,

$$\frac{\mu}{c} re\sin\theta = \boldsymbol{r}^T\boldsymbol{v}.$$

By differentiation,

$$re\cos\theta\frac{\partial\theta}{\partial\boldsymbol{v}} = -r\sin\theta\frac{\partial e}{\partial\boldsymbol{v}} + \frac{\boldsymbol{r}^T\boldsymbol{v}}{\mu}\frac{\partial c}{\partial\boldsymbol{v}} + \frac{c}{\mu}\boldsymbol{r}^T. \tag{5.12}$$

The sum of (5.11) multiplied by $\sin\theta$ and (5.12) multiplied by $\cos\theta$ is

$$r\,e\,c\,\frac{\partial\theta}{\partial\boldsymbol{v}} = \frac{c^2}{\mu}\cos\theta\,\boldsymbol{r}^T - \left(\frac{c^2}{\mu} + r\right)\sin\theta\frac{\partial c}{\partial\boldsymbol{v}}. \tag{5.13}$$

Using (5.5),

$$\frac{\partial\theta}{\partial\boldsymbol{v}} = \frac{1}{ce}\left(\frac{c^2/\mu}{r}\cos\theta + \frac{c^2/\mu + r}{c^2/\mu}e\sin^2\theta\right)\boldsymbol{r}^T - \frac{r}{c^2e}\left(\frac{c^2}{\mu} + r\right)\sin\theta\boldsymbol{v}^T. \tag{5.14}$$

The parenthesis term may be simplified multiplying by $\boldsymbol{r}^T$,

$$\begin{aligned} \frac{c^2/\mu}{r}\cos\theta + \left(1 + \frac{r}{c^2/\mu}\right)e\sin^2\theta &= \cos\theta + e\cos^2\theta + e + \frac{re}{c^2/\mu} - \frac{re}{c^2/\mu}\cos^2\theta - e\cos^2\theta \\ = e + \cos\theta\left(1 + \frac{e\cos\theta}{1+e\cos\theta}\right) + \frac{re}{c^2/\mu} &= e + \frac{r}{c^2/\mu}(\cos\theta + e). \end{aligned}$$

Thus,

$$\frac{\partial\theta}{\partial\boldsymbol{v}} = \frac{1}{ce}\left(e + \frac{r}{c^2/\mu}(\cos\theta + e)\right)\boldsymbol{r}^T - \frac{r}{c^2e}\left(\frac{c^2}{\mu} + r\right)\sin\theta\boldsymbol{v}^T. \tag{5.15}$$

When calculating the derivatives of the anomalies, it has to be taken into consideration that they have a non perturbed anomaly, that is

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial\boldsymbol{v}}\boldsymbol{a}_p = \frac{c}{r^2} + \frac{1}{ce}\left(e + \frac{r}{c^2/\mu}(\cos\theta + e)\right)\boldsymbol{r}^T\boldsymbol{a}_p - \frac{r}{c^2e}\left(\frac{c^2}{\mu} + r\right)\sin\theta\boldsymbol{v}^T\boldsymbol{a}_p. \tag{5.16}$$

Now, from Figure 4.4, the orbital equation can be rewritten as

$$x = r\cos\theta = a(\cos E - e), \quad y = r\sin\theta = a\sqrt{1-e^2}\sin E,$$

from where it can be derived that

$$\cos E = \frac{\cos\theta + e}{1 + e\cos\theta}, \quad \sin E = \frac{\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta}.$$

The partial derivative of $\cos E$ is

$$-\sin E(1+e\cos\theta)\frac{\partial E}{\partial \boldsymbol{v}} = (e\cos E-1)\sin\theta\frac{\partial \theta}{\partial \boldsymbol{v}} - (1-\cos E\cos\theta)\frac{\partial e}{\partial \boldsymbol{v}},$$

replacing $\sin E$ and $\cos E$ by their definition gives, after some computation, into

$$\frac{\partial E}{\partial \boldsymbol{v}} = \frac{r}{a\sqrt{1-e^2}}\frac{\partial \theta}{\partial \boldsymbol{v}} - \frac{r}{(c^2/\mu)\sqrt{1-e^2}}\sin\theta\frac{\partial e}{\partial \boldsymbol{v}} = \frac{r}{\mu a\sqrt{1-e^2}e}\left(\frac{c}{c^2/\mu}(\cos\theta+e)\boldsymbol{r}^T - (r+a)\sin\theta\boldsymbol{v}^T\right). \tag{5.17}$$

Then, using equation (5.2), and keeping in mind that $dE/dt = \sqrt{\mu/a}/r$, the derivative of the eccentric anomaly is

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \frac{\partial E}{\partial \boldsymbol{v}}\boldsymbol{a}_p = \frac{na}{r} + \frac{r}{\mu a\sqrt{1-e^2}e}\left(\frac{\mu}{c}(\cos\theta+e)\boldsymbol{r}^T\boldsymbol{a}_p - (r+a)\sin\theta\boldsymbol{v}^T\boldsymbol{a}_p\right). \tag{5.18}$$

In the case of the derivative of the mean anomaly, differentiating its definition $M = M_0 + n(t-t_0) = E - e\sin E$,

$$\frac{\partial M}{\partial \boldsymbol{v}} = \frac{ra\sqrt{1-e^2}}{ca^2e}\left(\cos\theta\boldsymbol{r}^T + \frac{a}{c}\left((r+\frac{c^2}{\mu}\right)\sin\theta\boldsymbol{v}^T\right).$$

From which it simply follows that

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial \boldsymbol{v}}\boldsymbol{a}_p = n + \frac{r\sqrt{1-e^2}}{cae}\left(\cos\theta\boldsymbol{r}^T\boldsymbol{a}_p + \frac{a}{c}\left(r+\frac{c^2}{\mu}\right)\sin\theta\boldsymbol{v}^T\boldsymbol{a}_p\right). \tag{5.19}$$

5.2.5 Variation of the argument of the peri-center

The variation of the argument of the peri-center can be derived from a new definition, the *argument of the latitude*, $\phi = \omega + \theta$, which is the angle between the position vector $\boldsymbol{q}_r$ and the direction of the ascending node $\boldsymbol{q}_a$, thus

$$\cos\phi = \boldsymbol{q}_a^T\boldsymbol{q}_r,$$

which, using transformation (5.7), can be written as

$$\cos\phi = \cos\Omega(\boldsymbol{q}_x^T\boldsymbol{q}_r) + \sin\Omega(\boldsymbol{q}_y^T\boldsymbol{q}_x).$$

Differentiating the previous equation, and taking into account that the unit vectors $\boldsymbol{q}_x$, $\boldsymbol{q}_y$ and $\boldsymbol{q}_r$ are independent from $\boldsymbol{v}$, it follows that

$$-\sin\phi\frac{\partial \phi}{\partial \boldsymbol{v}} = \left(-\sin\Omega(\boldsymbol{q}_x^T\boldsymbol{q}_r) + \cos\Omega(\boldsymbol{q}_y^T\boldsymbol{q}_x)\right)\frac{\partial \Omega}{\partial \boldsymbol{v}}.$$

Using the definition of $\boldsymbol{q}_r$,

$$\frac{\partial \phi}{\partial \boldsymbol{v}} = -\cos i\frac{\partial \Omega}{\partial \boldsymbol{v}},$$

and using the definition $\phi = \omega + \theta$,

$$\frac{\partial \omega}{\partial \boldsymbol{v}} = -\frac{\partial \theta}{\partial \boldsymbol{v}} - \cos i\frac{\partial \Omega}{\partial \boldsymbol{v}}$$

is derived. Replacing the derivatives of θ and Ω,

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial \boldsymbol{v}}\boldsymbol{a}_p = -\frac{1}{ce}\left(\frac{r}{c^2/\mu}(\cos\theta+e)+e\right)\boldsymbol{r}^T\boldsymbol{a}_p - \frac{r}{c^2e}\left(\frac{c^2}{\mu}+r\right)\sin\theta\boldsymbol{v}^T\boldsymbol{a}_p - \frac{r\sin\phi}{c\tan i}\boldsymbol{q}_c^T\boldsymbol{a}_p. \tag{5.20}$$

5.3 Gaussian variational equations

The Gaussian variation equations describe the variation of the Keplerian elements and are well defined when the perturbing force is the gradient of a potential function, as well as when that is not the case. To derive them, the reference system *LVLH* is used, associated to the Keplerian orbit and defined by the unit vectors

$$\boldsymbol{q}_r = \frac{\boldsymbol{r}}{r},\ \boldsymbol{q}_\theta,\ \boldsymbol{q}_c.$$

The position vector $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and perturbing acceleration $\boldsymbol{a}_p$ are expressed in the *LVLH* frame.

With respect to the inertial reference system $\mathcal{I}$, the reference system *LVLH* rotates around the axis $\boldsymbol{q}_c$ with angular velocity $\dot{\theta}$, in such a way that from the derivative of $\boldsymbol{r} = r\boldsymbol{q}_r$ and applying the chain rule, it can be derived that

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{q}_r + r\dot{\theta}\boldsymbol{q}_\theta, \tag{5.21}$$

where $\dot{r}$ and $r\dot{\theta}$ are, respectively, the radial and tangencial components of the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$.

From the orbital equation

$$r = \frac{p}{1 + e\cos\theta} = \frac{c^2/\mu}{1 + e\cos\theta},$$

recalling that $c = r^2\dot{\theta}$, it follows that

$$\dot{r} = \frac{r}{1 + e\cos\theta}\, e\sin\theta\,\dot{\theta} = \frac{r^2}{p}\, e\sin\theta\,\dot{\theta} = \frac{\mu}{c}\, e\sin\theta.$$

The tangential component of the velocity can be written as

$$r\,\dot{\theta} = \frac{c}{r} = \frac{\mu\, p}{c\, r},$$

and

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{\mu}{c}\left(e\sin\theta\,\boldsymbol{q}_r + \frac{p}{r}\,\boldsymbol{q}_\theta\right). \tag{5.22}$$

The variational equation of the semi-major axis a as a function of the components a_r and a_θ of the acceleration $\boldsymbol{a}_p$ in the reference system *LVLH*, is obtained replacing equation (5.22) into (5.4),

$$\frac{da}{dt} = \frac{2a^2}{c}\left(e\sin\theta a_r + \frac{p}{r}a_\theta\right).$$

Similarly, using Equation (5.22) in (5.6),

$$\frac{de}{dt} = \frac{1}{c}\left(p\sin\theta a_r + \frac{(pa - r^2)(1 + e\cos\theta)}{ae}a_\theta\right),$$

taking into consideration that $r = p/(1 + e\cos\theta) = a(1 - e^2)/(1 + e\cos\theta)$, this last equation can be written as

$$\frac{de}{dt} = \frac{1}{c}\left(p\sin\theta a_r + \left((p + r)\cos\theta + re\right)a_\theta\right).$$

The variational equations of the right ascension of the ascending node Ω and the inclination i are obtained using $\boldsymbol{q}_c^T \boldsymbol{a}_p = a_c$ in equations (5.9) and (5.10)

$$\frac{d\Omega}{dt} = \frac{r\sin\theta}{c\sin i}a_c,$$

$$\frac{di}{dt} = \frac{r\cos\theta}{c}a_c.$$

The variational equations of the anomalies and the argument of the peri-center are derived from the substitution of equation (5.22) into equations (5.20), (5.16), (5.18) and (5.19) respectively simplifying,

$$\begin{aligned}
\frac{d\omega}{dt} &= -\frac{1}{c\,e}\cos\theta, p\,a_r + \frac{1}{c\,e}(p+r)\sin\theta\,a_\theta - \frac{r\sin\theta\cos i}{c\sin i}\,a_c, \\
\frac{d\theta}{dt} &= \frac{c}{r^2} + \frac{1}{c\,e}\left(p\cos\theta\,a_r - (p+r)\sin\theta\,a_\theta\right), \\
\frac{dE}{dt} &= \frac{n\,a}{r} + \frac{p}{a\sqrt{1-e^2}c\,e}\left((a(\cos\theta - e)\,a_r + (r+a)\sin\theta\,a_\theta\right), \\
\frac{dM}{dt} &= n + \frac{a\sqrt{1-e^2}}{ace}\left((p\cos\theta - 2r\,e)\,a_r - (p+r)\sin\theta\,a_\theta\right).
\end{aligned}$$

As mentioned, Gaussian variational equations are useful, in particular, when the perturbing acceleration is not conservative, as other methods may not be usable. If the perturbation is caused by a control maneuver, the Gaussian variational equations reflect the effect of the maneuver on the Keplerian elements accordingly. Inversely, useful information for dynamical control can be obtained from them. For example, from the equation of the right ascension of the ascending node Ω and the inclination i, the most efficient instant to correct the node is when $\sin\theta$ is maximum, and the most efficient instant to adjust the orbit's inclination is when $\cos\theta$ is maximum. Furthermore, writing the acceleration vector in the reference system *LVLH*, the variation of the orbital elements can be integrated with ease.

In summary, the Gaussian variational equations are:

$$\begin{aligned}
\frac{da}{dt} &= \frac{2a^2}{c}\left(e\sin\theta\,a_r + \frac{p}{r}\,a_\theta\right), \\
\frac{de}{dt} &= \frac{1}{c}\left(p\sin\theta\,a_r + ((p+r)\cos\theta + r\,e)\,a_\theta\right), \\
\frac{di}{dt} &= \frac{r\cos\theta}{c}\,a_c, \\
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{c\sin i}\,a_c, \\
\frac{d\omega}{dt} &= -\frac{1}{c\,e}\cos\theta\,p\,a_r + \frac{1}{c\,e}(p+r)\sin\theta\,a_\theta, \\
\frac{dM}{dt} &= n + \frac{\sqrt{1-e^2}}{e}\left(8p\cos\theta - 2r\,e\right)a_r - (p+r)\sin\theta\,a_\theta).
\end{aligned} \tag{5.23}$$

Instead of using the reference system $\mathcal{O} = \{\boldsymbol{q}_r,\ \boldsymbol{q}_\theta,\ \boldsymbol{q}_c\}$ or *LVLH*, the Gaussian equations can be rewritten using the reference system $\mathcal{V} = \{\boldsymbol{q}_m,\ \boldsymbol{q}_v,\ \boldsymbol{q}_c\}$ in which $\boldsymbol{q}_v$ is a unitary vector with the direction of the velocity. Note that, if the orbit is circular, then $\boldsymbol{q}_r = \boldsymbol{q}_m$ i $\boldsymbol{q}_\theta = \boldsymbol{q}_v$.

Using equation (5.22), $\boldsymbol{q}_v$ can be expressed as

$$\boldsymbol{q}_v = \frac{\boldsymbol{v}}{v} = \frac{c}{pv}\left(e\sin\theta\boldsymbol{q}_r + \frac{p}{r}\boldsymbol{q}_\theta\right). \tag{5.24}$$

Because $\boldsymbol{q}_m$ s perpendicular to $\boldsymbol{q}_v$ and $\boldsymbol{q}_c$ by definition, it follows that

$$\boldsymbol{q}_m = \frac{c}{pv}\left(\frac{p}{r}\boldsymbol{q}_r - e\sin\theta\boldsymbol{q}_\theta\right). \tag{5.25}$$

Expanding the definition of acceleration

$$\boldsymbol{a} = a_r\boldsymbol{q}_r + a_\theta\boldsymbol{q}_\theta + a_c\boldsymbol{q}_c = a_m\boldsymbol{q}_m + a_v\boldsymbol{q}_v + a_c\boldsymbol{q}_c,$$

and applying equations (5.24) and (5.25), the system

$$\begin{aligned}
\begin{pmatrix} a_r \\ a_\theta \end{pmatrix} &= \frac{c}{pv}\begin{pmatrix} p/r & e\sin\theta \\ -e\sin\theta & p/r \end{pmatrix}\begin{pmatrix} a_m \\ a_v \end{pmatrix}, \\
\begin{pmatrix} a_m \\ a_v \end{pmatrix} &= \frac{c}{pv}\begin{pmatrix} p/r & -e\sin\theta \\ e\sin\theta & p/r \end{pmatrix}\begin{pmatrix} a_r \\ a_\theta \end{pmatrix}
\end{aligned}$$

is obtained. Using the expression for the module of the velocity as a function of θ,

$$v(\theta) = \frac{c}{p}\sqrt{1+e^2+2e\cos\theta},$$

the previous transformation becomes

$$\begin{aligned}
\begin{pmatrix} a_r \\ a_\theta \end{pmatrix} &= \frac{1}{\sqrt{1+e^2+2e\cos\theta}}\begin{pmatrix} 1+e\cos\theta & e\sin\theta \\ -e\sin\theta & 1+e\cos\theta \end{pmatrix}\begin{pmatrix} a_m \\ a_v \end{pmatrix}, \\
\begin{pmatrix} a_m \\ a_v \end{pmatrix} &= \frac{1}{\sqrt{1+e^2+2e\cos\theta}}\begin{pmatrix} 1+e\cos\theta & -e\sin\theta \\ e\sin\theta & 1+e\cos\theta \end{pmatrix}\begin{pmatrix} a_r \\ a_\theta \end{pmatrix}.
\end{aligned}$$

Applying these transformations on the Gaussian variational equations (5.23), produces the following alternative expressions,

$$\begin{aligned}
\frac{da}{dt} &= \frac{2a^2 v}{\mu}a_v, \\
\frac{de}{dt} &= \frac{1}{v}\left(\frac{r}{a}\sin\theta\, a_n + 2(e+\cos\theta)\, a_v\right), \\
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{c\sin\theta}a_c, \\
\frac{di}{dt} &= \frac{r\cos\theta}{c}a_c, \\
\frac{d\omega}{dt} &= -\frac{1}{e\,v}\left(-\left(2e+\frac{r}{a}\right)\cos\theta\, a_n + 2\sin\theta\, a_v\right) - \frac{r\sin\theta\cos i}{c\sin i}a_c, \\
\frac{dM}{dt} &= n + \frac{\sqrt{1-e^2}}{e\,v}\left(\frac{r}{a}\cos\theta\, a_n - 2\left(1+e^2\frac{r}{p}\right)\sin\theta\, a_v\right).
\end{aligned} \tag{5.26}$$

Chapter 6

Atmospheric Drag

Although a first approach to characterize the effects of the atmospheric drag on satellites has been presented in section 3, there is a need for more detailed control on the parameters to more accurately portrait the effects of the atmosphere. Currently, the atmospheric drag is defined as proportional to the product of the atmospheric density ρ with the square of the relative velocity module v^2. As the drag force is opposite to the motion direction, the equation of perturbed motion can be written as

$$\frac{d^2\boldsymbol{r}}{dt^2} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{a}_f = -\frac{AC\rho}{m}\frac{v^2}{2}\boldsymbol{q}_v, \tag{6.1}$$

where A is the are of the satellite's section transversal to the velocity vector, m is the satellite mass, C is the drag coefficient (a unit-less constants), ρ the air density, v the inertial velocity module of the satellite and $q_v = \boldsymbol{v}/v$ an unit vector with $\boldsymbol{v}$ direction.

The density of the atmosphere depends on the altitude, decreasing exponentially as higher altitudes are reached. Moreover, other factors, such as temperature also influence it. There exists a number of models with varying degrees of accuracy, complexity and computational load trade-offs. A very rough estimate is the exponential model, defining density as a function of altitude using

$$\rho(r) = \rho_0\, e^{-\dfrac{r - R_E}{H_0}},$$

where ρ_0 is the reference density at the nearest height h_0 to the desired altitude, H_0 is the corresponding density scale height and R_E is the Earth radius. The USSA76, *US Standard Atmosphere 1976*, is one model of this type, see Figure 7.1.

Supposing that the atmospheric drag is the only perturbation force in the reference frame

$$\mathcal{V} = \{\boldsymbol{q}_m,\ \boldsymbol{q}_v,\ \boldsymbol{q}_c\},$$

and denoting $C_f(r) = \frac{A}{m}C\rho(r)$, then

$$\boldsymbol{a}_f = \left(0, -C_f(r)\frac{v^2}{2}, 0\right).$$

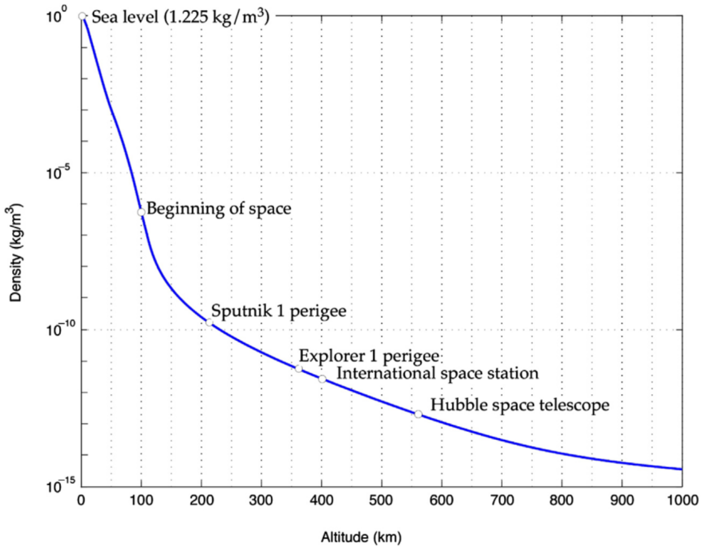


Figure 6.1: US Standard Atmosphere 1976: density versus altitude, [Cur14] (p.657)

Replacing these values into the Gaussian variational equations (5.26), the variation of the Keplerian elements under atmospheric drag perturbations is described by the next differential equation system:

$$\frac{da}{dt} = -C_f(r)\frac{v^3}{an^2}, \tag{6.2}$$

$$\frac{de}{dt} = -C_f(r)(e+\cos\theta)v, \tag{6.3}$$

$$\frac{d\omega}{dt} = -C_f(r)\frac{\sin\theta}{e}v, \tag{6.4}$$

$$\frac{di}{dt} = 0, \tag{6.5}$$

$$\frac{d\Omega}{dt} = 0, \tag{6.6}$$

$$\frac{dM}{dt} = n + \frac{\sqrt{1-e^2}}{e}C_f(r)\left(1+e^2\frac{r}{p}\right)\sin\theta\, v. \tag{6.7}$$

With these equations, the behavior of the variation can be studied qualitatively. From the first equation, the semi-major axis a is observed to be monotonically decreasing. Indeed, the atmospheric drag decreases the Keplerian energy of the satellite ($|H| = \mu/2a$). The rate at which a decreases is an important factor to determine the satellite's life-cycle on a low Earth orbit (LEO), lower than 2000 kilometers of altitude.

To reliably predict a satellite's lifetime, a precise estimation of the atmospheric properties at great altitudes for long periods of time (years or decades) is needed. However, aleatory variations of the solar radiation (i.e. solar storms) and Earth's geomagnetic field usually make impossible this task.

The atmospheric drag doesn't affect the longitude (another word for right of ascension of the ascending node) Ω nor the inclination i, which define the orbital plane.

Therefore, the orbital plane is invariant under atmospheric drag (and no other perturbations are present).

Considering the exponential increase in atmospheric density by height decrease, an elliptical orbit experiments the greater atmospheric drag at its peri-center, ρ diminishing when moving towards the apo-center. Thus, eccentricity e will have its bigger variation at $\cos\theta \approx 1$. Note that, on average, $\dot{e} < 0$, meaning that over time the orbit will be becoming circular.

The atmosphere of Earth or Mars is negligible over 150 km height. In the case of a satellite with an orbit with an initial large eccentricity and peri-center height inferior to 150km orbiting Earth or Mars, it decreases its velocity each time it passes trough the orbit's peri-center, which stays "fixed" in space (although hard to prove analytically, it can be shown with numerically). Hence, if an interplanetary hyperbola orbit is transformed to an elliptical orbit with large eccentricity, with the use of retro grade rockets (a maneuver called retro grade engine burn) or by an initial pass through the atmosphere (named aerocapture), the orbit's semi-major axis and its eccentricity can be reduced by successive passes through the atmosphere. This method is known as aeroassisted orbital transfer, and its has been used in various missions, as the Margellan Venus, the Mars Global Surveyor (MGS) and Mars Odyssey.

Chapter 7

Numerical simulations

Two satellites will be used in order to analyze the effects of atmospheric drag at different altitudes. The first one is named *Satellite1* and it is a fictitious satellite with meaningful orbital values, obtained from [Cur14](p.659); it's spherical with diameter of 1 meter and 100 kg of mass. The second one is the *Enxaneta*, a satellite developed by the IEEC (*Insistut d'Estudis Especials de Catalunya*), with 10 kg of mass and assumed spherical shape and 1 meter of diameter.

Orbit characteristics	Satellite1	Enxaneta
Pericenter altitude	215 km	541.6 km
Apocenter altitude	939 km	571.6 km
Right ascension of the ascending node	$340°$	?
Inclination	$65.1°$	$97.6°$
Argument of pericenter	$58°$	30
True anomaly	$332°$	?

Table 7.1: Initial orbit characteristics of Satellite1 and Enxaneta.

7.1 Orbits using a numerical integrator

For the computation of the solutions of ordinary differential equations (as well as double-checking some algebraic results), the scientific programming language *Julia* has been used. It is very similar to *Matlab*, but has a more consistent logic and it's free software (as in free beer). In particular, the package *DifferentialEquations.jl* has been extensively used, which provides a very similar form to the one used in mathematics to define initial value problems. More details can be found at the code itself and its comments.

Using the software mentioned above, Equations (6.2) have been resolved, as well as Equation (6.1) and are shown below.

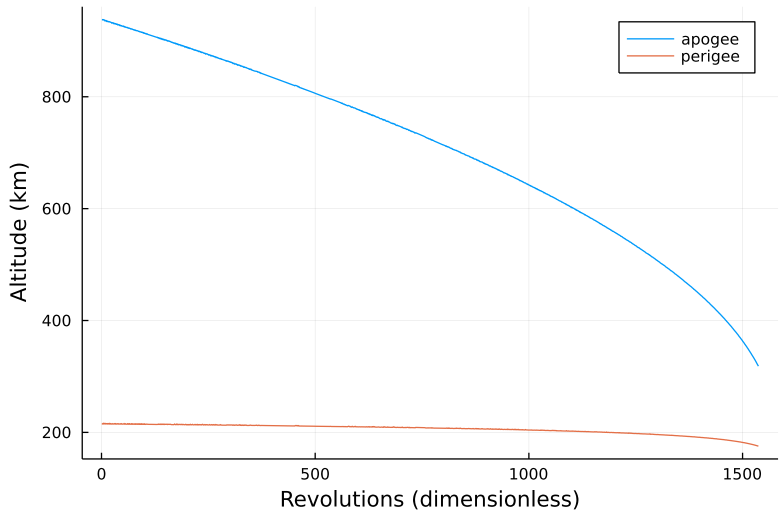


Figure 7.1: Progression of the apogee and the perigee of Satellite1 over 80 days.

Although the progression of the apocenter in Figure 7.1 and the ones from the semi-axis major a and the eccentricity e will be verified with the study done in the next section, the behavior of w and M doesn't seem to make sense qualitatively nor numerically and its probably due to a magnitude error or an error propagation, so they will not be added to this text.

7.2 Orbits using GMAT

GMAT is a simulation tool developed by NASA that can be used with ease following online tutorials while being professional graded. For the use of GMAT, the variation of the orbital elements $\{a, e, \omega, M\}$ and the satellite's altitude have been recorded in the course of 48 hours with atmospheric drag in both proposed satellites and in the case of "Enxaneta", also with all available perturbations.

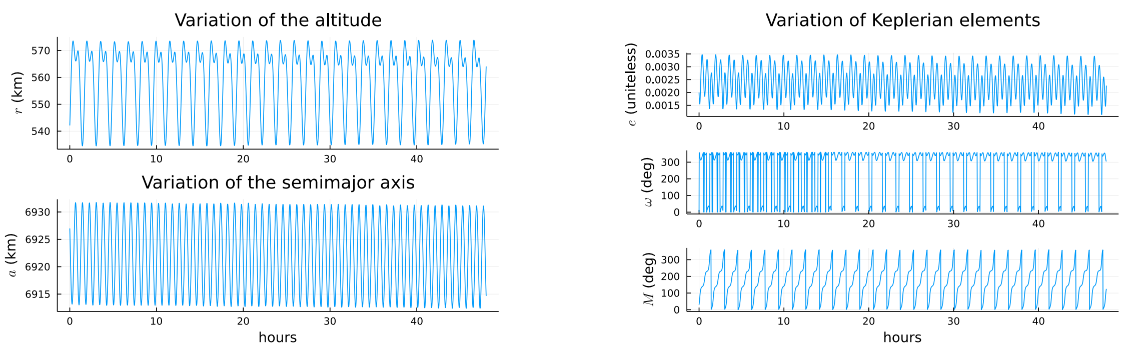


Figure 7.2: Variation of the orbit of satellite "Enxaneta" with atmospheric drag perturbation during 48 hours.

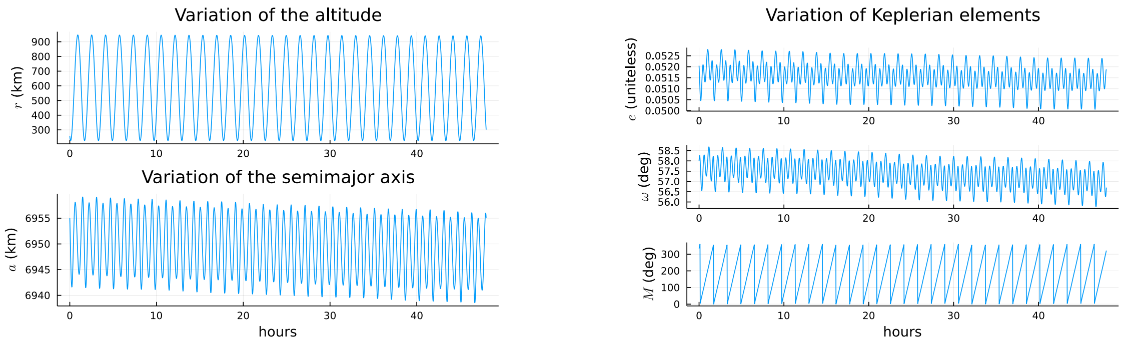


Figure 7.3: Variation of the orbit of satellite “Satellite1” with atmospheric drag perturbation during 48 hours.

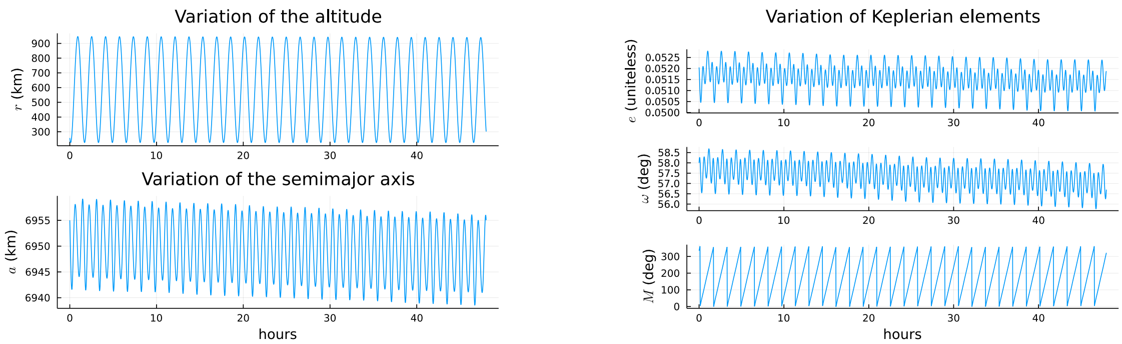


Figure 7.4: Variation of the orbit of satellite “Satellite1” with all perturbations during 48 hours.

To reach a compromise between local and global behavior, a 48 hour period of observation was chosen. Nevertheless, because the range of values of r is large, the difference in it along 48 hours is hard to observe and its better to analyze by proxy using a. Moreover, variables with degrees for units, have the same value at their extremes (0°, 360 °) which explains the “jumps” in the graphics.

Figure 7.2 and 7.3 confirm the qualitative study of Section 6, under atmospheric drag the orbit leans to a circular shape, given the decrease of both e and a. Furthermore, the decrease of a is more “sharp” for more eccentric orbits. From these figures it can also be confirmed that orbit rotates inside the orbital plane, with the Laplace vector becoming parallel to the line of nodes, ω tends to 0.

By the lack of differences between Figure 7.3 and 7.4, atmospheric drag stands at a higher order of magnitude than the other perturbations for LEO orbits such as the two simulated, as the Figure 4.1 shows.

Chapter 8

Conclusions and future work

8.1 Conclusions

After some assumptions and review, a consistent framework to study and characterize artificial satellite's orbits has been defined with enough expressiveness to confirm the Kepler's Laws. Furthermore, it has been extended in a constructive manner to add previously omitted details, such as the perturbation of the atmospheric drag, which has led us to be able to extract qualitative details of LEO orbits.

It has been possible to use the framework and its extension in a practical manner with the aid of a numerical integrator, although with varying degree of success and further detail study of their operation is needed. Lastly, we have learn and used GMAT, which has help us to acquire a "ground truth" of how the study of orbits is done in a professional setting and mark a positive goal for our implementation.

8.2 Future work

Taking as reference GMAT, a complete application for the modeling and study of satellite's orbit, there are various directions to which expand the work of this thesis:

- **Improvement of the orbital model.** Other perturbations could be added to the orbital model for a more complete solution. Otherwise, the development done in this text could be added to an already existing open source modeling package.
- **Transition from development to production.** Using the Object Oriented Programming paradigm, the study realized in this text could be translated into an Orbit Propagator, an entity which given some initial values can solve an orbit following a model. Moreover, this object can be integrated with a GUI for ease of use and a 3D plotting library for better visualization. In a sense, it will be to extend our initial study into a tool that can be used in the real world, as we have used GMAT.

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