Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

The differential geometry behind Maxwell's equations

Autor: Martín de Muniategui Climente

Director: Dr. Joana Cirici Núñez
Realitzat a: Departament de Matemàtiques i Informàtica
Barcelona, June 11, 2022

Abstract

Modern physics relies heavily on differential geometry in order to establish the mathematical formulation of its conceptual framework. This tendency started with Maxwell's equations in the XIX century and has since then only intensified. This work aims at establishing a more geometric approach to Maxwell's equations using differential forms in order to generalize them to other manifolds than $\mathbb{R}^3$, an imperative for any physical theory ever since Einstein laid the foundations of Special and General Relativity. We will therefore show a modern approach to physics delving into differential geometry to define the objects that we will deal with in Maxwell's equations which will give us deeper insight about the mathematical structure of these equations and their physical consequence.

Acknowledgements

First of all I want to specially thank to my advisor in this work, Dr. Joana Cirici, for guiding me in this process, for her helpful suggestions and pertinent corrections and for making the effort of squeezing in the time for all of this in her busy schedule. Next, I would like to thank my friends, specially those that have been accompanying me for all these years of study and who have been facing the same difficulties as I did. Of course, I also want to thank my family, and specially my mother, for all the unconditional support they have always given me. And finally, I want to thank both my cats for keeping me company and making me smile even when the situation was bleak.

Contents

Chapter 1

Introduction

Maxwell's equations in their classic form describe the behaviour of two vector fields, the **electric field** $\vec{E}$ and the **magnetic field** $\vec{B}$ over $\mathbb{R}^3$. These vector fields are defined over all space, taken as the space $\mathbb{R}^3$ with its usual metric and are also functions of time, a real-valued parameter $t \in \mathbb{R}$. These fields depend on the **electric charge density** ρ, which is a time-dependent function on space, and also on the **electric current density** $\vec{\jmath}$, which is a time-dependent vector field on space. Functions are assumed to be real-valued and both functions and vector fields are assumed to be C^∞.

Firstly written by Heaviside after independently developing vector calculus and in units where the speed of light is $c = 1$ - the natural choice of units for physicists -, the Maxwell's equations are the following set of four equations with given names:

$$\nabla \cdot \vec{E} = \rho \qquad \textbf{Gauss's Law}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{\jmath} \qquad \textbf{Ampère's circuital law}$$

$$\nabla \cdot \vec{B} = 0 \qquad \textbf{Gauss's Law for magnetism}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \textbf{Faraday's law of induction}$$

Another crucial formula, which will exhaust all needed formulas to explain electromagnetic phenomena, is **Lorentz's force law**, which is the following:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

where $\vec{F}$ denotes the force exerted over a charged particle of velocity $\vec{v}$ and charge q.

One should note that Lorentz's force law can not be derived from Maxwell's equations: since $\vec{F} = m\vec{a}$, Lorentz's force law involves mass, while Maxwell's equations remain indifferent to mass. One can also imagine a universe where both electric and magnetic field exist and are ruled by Maxwell's equations, but where both fields do not interact with matter. In this universe, even though one could not detect neither field, Maxwell's equations would still be true, but Lorentz's law would not hold.

There are some aspects of this equations worth mentioning, which will motivate their rewriting. Firstly, space and time are conceptually and explicitly separated in Maxwell's equations, as if we were stuck on the conceptual framework of Galilean spacetime, which is known to be deeply flawed. This problem is aggravated by the fact that a simple computation shows that Maxwell's equations are not invariant under Galilean transformations. In fact, this was the driving motivation to develop special relativity, which left behind pre-Einsteinian misconceptions, such as the non-existence of a universal speed limit - the speed of light in the vacuum -, and, as we mentioned before, the conceptual schism between space and time and Galilean relativity, i.e. the idea that the Galilean group was the fundamental group that left the laws of physics invariant.

Secondly, they hide a very non-physical fact. Using Lorentz's force law, one can easily determine the electric field with an experiment. To measure $\vec{E}$, we only need to measure by any means available the force exerted over a static particle, $\vec{F}$, and divide by the charge of that particle. To figure out $\vec{B}$, we can measure the force being exerted on charged particles with a variety of velocities. However, the definition of the cross product involves a completely arbitrary right-hand rule. If $\vec{v} = (v_x,\, v_y,\, v_z)$ and $\vec{B} = (B_x,\, B_y,\, B_z)$, define

$$\vec{v} \times \vec{B} = (v_y B_z - v_z B_y,\, v_z B_x - v_x B_z,\, v_x B_y - v_y B_x)$$

However, this convention is completely arbitrary, we could have as well set a left-hand rule,

$$\vec{v} \times \vec{B} = (v_z B_y - v_y B_z,\, v_x B_z - v_z B_x,\, v_y B_x - v_x B_y)$$

and the mathematics behind the cross product would work without a problem. However, when finding the magnetic field from the measurements on $\vec{F}$ using the left-hand convention, we would find a value of $\vec{B}$ with an opposite sign. Clearly, something like the magnetic field, a physical thing existing independently of us, should not depend on our conventions. There is something deeper to be said about how to mathematically understand the magnetic field, just as we did with the cross product of two vectors using differential forms.

Finally, Maxwell's equations come in two pairs, the homogeneous one

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

and the inhomogeneous one, which involves electric charge and electric current densities:

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{\jmath}$$

This two pairs look suspiciously alike, up to a minus sign. The symmetry is even clearer in the vacuum, where having zero electric charge density and current gives

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$$

Then, the transformation

$$(\vec{E}, \vec{B}) \mapsto (-\vec{B}, \vec{E})$$

leaves Maxwell's equations in the vacuum invariant. This internal symmetry is called duality, and is hinting to the fact that the electric field and the magnetic field are part of a bigger, unified whole: the electromagnetic field. Another clue towards this is the fact that Lorentz transformations do not just mix space and time, which are aspects of spacetime, they also mix the electric field and the magnetic field. If we introduce a complex-valued vector field

$$\vec{\mathcal{E}} = \vec{E} + i\vec{B}$$

duality can be expressed just by

$$\vec{\mathcal{E}} \mapsto -i\vec{\mathcal{E}}$$

and the Maxwell equations in the vacuum can succinctly be expressed as

$$\nabla \cdot \vec{\mathcal{E}} = 0 \qquad \nabla \times \vec{\mathcal{E}} = i\frac{\partial \vec{\mathcal{E}}}{\partial t}$$

This reformulation can be used to find solutions that correspond to plane waves moving at the speed of light in the vacuum, but the symmetry between $\vec{E}$ and $\vec{B}$ does not extend to the non-vacuum equations. We could consider the following:

$$\nabla \cdot \vec{\mathcal{E}} = \rho \qquad \nabla \times \vec{\mathcal{E}} = i(\frac{\partial \vec{\mathcal{E}}}{\partial t} + \vec{\jmath})$$

However, this introduces magnetic charge density and magnetic current density in the equations: since ρ and $\vec{\jmath}$ can be splitten into real and imaginary parts, we see that the imaginary part play a magnetic role:

$$\rho = \rho_e + i\rho_m \qquad \vec{\jmath} = \vec{\jmath}_e + i\vec{\jmath}_m$$

and we get the following Maxwell's equations:

$$\nabla \cdot \vec{B} = \rho_m \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{\jmath}_m$$

$$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{\jmath}_e$$

Even though these equations are much more charming, no magnetic charges - which are named magnetic monopoles - have ever been observed after many decades of scientific effort looking for them. We could just say, then, that $\rho_m = 0$ and $\vec{\jmath}_m = 0$ on the basis of experimental evidence, i.e. that ρ and $\vec{\jmath}$ are real-valued on this basis. However, finding a way to understand more deeply the Maxwell's equations is a much more interesting path to walk.

To do this, we will firstly generalize $\mathbb{R}^n$. Ever since Einstein developed the theories of Special Relativity and its generalization, General Relativity, it is known that space and time are not separate entities - they constitute space-time as a whole - and that this space-time is not flat. Our world is simply not $\mathbb{R}^3$ with a time parameter, so it is seems natural for our purpose that we model space-time as a whole using the concept of a manifold, and then specify the characteristics of space and time as components of this space-time, which we will do using a pseudo-Riemannian metric.

Then, we will find it convenient to specify what a vector is over a manifold. It is very clear what a vector in $\mathbb{R}^3$ is, but how can we speak about vectors on a mathematical object which is not flat? There is no thing such as a "curved vector". This will take us further to develop the concept of a vector bundle, which will allow us to define vector fields: we really do not want to talk about a single vector on a single point, but to define a field over the whole manifold.

However, as we noticed before, the magnetic field is a "weird" vector field. Apart from the fact mentioned before - its ambiguity -, when the vector field is reflected across a plane, it is not just reflected: it is reflected and reversed. How can a vector be a vector but behave differently than a vector? Physicists usually just stick "pseudo-" to the word vector, creating the ill-defined idea of a pseudo-vector: a vector which, sometimes, does not behave as such. This will take down to a path to formalize this using vector bundles, defining 1-forms, tensor fields and k-forms in the path, and their most important operator: the exterior derivative.

Later on, we will introduce what a pseudo-Riemannian metric is and some concepts which stem from it. Usually, in geometry, only Riemannian metrics are considered. However, in physics and motivated by Minkowksi space-time, we must generalize this concept to include a broader type of metrics which allow us to model space-time. Metrics will allow us to properly talk about distances, longitudes and volumes, concepts which are clearly relevant to physics. After all, we live in a geometrical reality, not just a topological one. After introducing pseudo-Riemannian manifolds, which are regular manifolds equipped with a pseudo-Riemannian metric, which will relate 1-forms with vectors via the musical isomorphisms, we will introduce the concept of a volume form in such a manifold. Then, we will introduce the Hodge star operator, a key operator in our formulation. This is a linear operator

$$\star : \Omega^*(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$$

acting on the differential forms of our manifold $\mathcal{M}$. Here n denotes the dimension of $\mathcal{M}$ and $\star$ is defined by the identity

$$\alpha \wedge \star\beta := \langle \alpha | \beta \rangle \, \omega_g,$$

where $\langle \alpha | \beta \rangle$ denotes the inner product of two forms, defined using the musical isomorphisms, and ω_g the volume form defined by the metric. Again, with the help of the musical isomorphisms, the Hodge star operator will let us redefine some old known operators in calculus, such as the gradient or the curl of a vector, but in the more general setting of a manifold.

After having developed all these mathematical concepts, we will finally be able to rewrite Maxwell's equations, which will turn to be only two equations, succinctly written as just:

$$dF = 0 \qquad \star\, d \star F = J$$

These two beautiful, compact equations describe the behaviour of the electromagnetic field: just like space-time (in fact, as a consequence of its indivisibility), we can not understand the electric and the magnetic fields separately. The electromagnetic field will turn to be a 2-form, and its behaviour will be described using the exterior derivative and the Hodge star operator.

We will then visit a notorious consequence of Maxwell's equations, which is much clearer in this setting: gauge freedom. On an intuitive level, it states that there are redundant degrees of freedom in the electromagnetic field variables, and we are free to impose further conditions which can help us do calculations. When generalized, this idea gives rise to the idea of gauge theories and Yang-Mills theories. We will not enter in detail in these because of their difficulty, but they are one of the most important ideas in modern physics.

Lastly, we will see how to find solutions for Maxwell's equations, which will relate again to the topology of the manifold, and will briefly discuss the Aharonov-Bohm effect as a consequence of the manifold in question not being simply connected. In the setting of this effect a part of the space will be inaccessible to an electron, so we can consider that space is not simply connected. This will give rise to a non-exact differential form which allows integrals along different paths to take different values. In the setting of the path-integral formulation of quantum mechanics, we have that if the electron starts at a state ψ at the point a and time $t = 0$, its state at a point b and time $t = T$ will be

$$\phi(b) = \int_{\mathcal{P}} e^{(\frac{i}{\hbar}s(\gamma) - q\int_\gamma A)} \psi(a)\mathcal{D}_\gamma$$

where $\mathcal{P}$ is the set of all possible paths from a at time $t = 0$ to b at time $t = T$. The presence of the non-exact differential form A will give a different phase to each possible path of the electron, which will then interfere with itself and will find it impossible to reach certain points which are classically allowed.

Chapter 2

Preliminaries on differential geometry

In order to rewrite Maxwell equations, we will need to lay the basis of the modern mathematical language for geometry: differential geometry. We will start by defining manifolds, and then move on to discuss some constructions over manifolds which we will need: vector bundles, vector fields, 1-forms, tensor fields and k-forms. This constructions are the basis of modern physics and will allow us to refer to the physical objects we are interested in - the magnetic and electric fields - without any use of coordinates, intrinsically. This section primarily based on [War83], [Lee09] and nLab.

2.1 Manifolds

Definition 2.1.1. Let X be a topological space. A **n-dimensional local chart on X** is a pair (U, φ) where $U \subseteq X$ is an open set and $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism. The functions $x^i = u^i \circ \varphi : U \to \mathbb{R}$ where $u^i : \mathbb{R}^n \to \mathbb{R}$ are the canonical coordinate functions on $\mathbb{R}^n$, $u^i(a^1, ..., a^n) = a^i$, are called the **coordinate functions of φ**.

Notice that given a chart (U, φ) on a topological space X, the coordinate functions of φ satisfy $\varphi = (x^1, ...x^n)$.

Given two charts (U, φ) and (V, ψ) of a topological space X such that $U \cap V \neq \emptyset$, the composition $\psi \circ \varphi^{-1}(U \cap V) : \varphi(U \cap V) \to \psi(U \cap V)$ which maps the coordinates of ψ to the coordinates of φ is called **the transition map**. It is a homeomorphism since the restriction of a homeomorphism and the composition of homeomorphisms are homeomorphisms.

Definition 2.1.2. Let X be a topological space. An **atlas $\mathcal{A}$** on X is a family of local charts on X, $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$, such that $X = \cup_{i \in I} U_i$. If all charts are n-dimensional, $\mathcal{A}$ is called an **n-dimensional atlas on X**.

Definition 2.1.3. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ on X is called a **smooth atlas on X** if for any non-disjoint charts the transition map $\varphi_j \circ \varphi_i^{-1}$ are C^∞-maps.

Definition 2.1.4. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two smooth atlases on a topological space X. We say that the two atlases are **compatible** if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas. Denote their compatibility by $\mathcal{A}_1 \sim \mathcal{A}_2$.

The compatibility of atlases, $\sim$, is an equivalence relation over the set of all atlases of a given topological space X. Equivalent atlases will give equivalent properties to the space. Therefore, we will be interested in working with their equivalence classes.

Definition 2.1.5. Let X be a topological space. A **smooth structure $[\mathcal{A}]$ on X** is an equivalence class on smooth manifolds on X given by the relationship of being compatible, $\sim$.

Definition 2.1.6. Let $\mathcal{M}$ be a topological space. We say that $\mathcal{M}$ is a **topological manifold** if it is Hausdorff, second countable and locally euclidean, i.e. there exists a number $n \in \mathbb{N}\backslash\{0\}$ such that for all $p \in \mathcal{M}$ there is a neighbourhood of p homeomorphic to $\mathbb{R}^n$.

Remark 2.1.7. The dimension n of a nonempty topological manifold is unique as a consequence of the topological invariance of the domain. This theorem implies that no non-empty open subset of $\mathbb{R}^n$ is homeomorphic to a subset of $\mathbb{R}^m$ if $n \neq m$.

Remark 2.1.8. The requirement for a topological manifold $\mathcal{M}$ to be locally euclidean implies the existence of an atlas over $\mathcal{M}$ of a certain dimension n. By the previous remark, a n-dimensional topological manifold only admits n-dimensional atlases.

While admitting a countable basis is a technical requirement that allows proving crucial theorems about manifolds, the Hausdorff condition is imposed in order to avoid pathological examples of manifolds that can be constructed using quotient spaces, such as the real line with two origins. This space verifies all imposed conditions except for being Hausdorff. However, it does not align with the idea underlying manifolds, i.e. a space locally resembling the m-dimensional real space.

Definition 2.1.9. A **n-dimensional smooth manifold** is a pair $(\mathcal{M}, [\mathcal{A}])$ where $\mathcal{M}$ is a topological manifold and $[\mathcal{A}]$ is a n-dimensional smooth structure on $\mathcal{M}$.

From now on, we will write $\mathcal{M}$ for a smooth manifold instead of $(\mathcal{M}, [\mathcal{A}]$ to ease notation and we will refer to smooth manifolds as just manifolds to ease verbosity.

Example 2.1.10. Let us see some examples of manifolds:

(i) For all $n \in \mathbb{N}\backslash\{0\}$, $\mathbb{R}^n$ with the standard structure $[\{(\mathbb{R}^n, id_{\mathbb{R}^n}\}]$ is a n-dimensional manifold. We will simply write it as $\mathbb{R}^n$.

(ii) For all $n \in \mathbb{N}\backslash\{0\}$, the n-dimensional sphere

$$\mathbb{S}^n := \{(x^1, ..., x^n) \in \mathbb{R}^{n+1} | (x^1)^2 + ... + (x^{n+1})^2 = 1\}$$

with the smooth structure given by the atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a n-dimensional manifold, where

$$U_N := \mathbb{S}^n \backslash p_N \qquad U_S := \mathbb{S}^n \backslash p_S$$

and $p_N = (1, 0, ..., 0)$, $p_S = (-1, 0, ..., 0)$, and $\varphi_N : U_N \to \mathbb{R}^n$ and $\varphi_S : U_S \to \mathbb{R}^n$ are the stereographic projections from points p_N and p_S respectively.

(iii) Given a manifold $\mathcal{M}$ with an atlas $\{(U_i, \varphi_i)\}_{i \in I}$, any open subset $V \subset \mathcal{M}$ is a manifold of the same dimension as $\mathcal{M}$, taking the induced atlas $\{(U_i \cap V, \varphi|_{U_i \cap V})\}_{i \in I}$. Specifically, any chart of $\mathcal{M}$ is a manifold.

(iv) Let $(\mathcal{M}, [\{(U_i, \varphi_i)\}_{i\in I}])$ and $(\mathcal{N}, [\{(V_j, \psi_j)\}_{j\in J}])$ be two manifolds of dimension n and m respectively. The **product manifold of $\mathcal{M}$ and $\mathcal{N}$** is the $(n+m)$-dimensional manifold $\mathcal{M} \times \mathcal{N}$ with the smooth structure represented by the atlas $(U_i \times V_j, \varphi_i \times \psi_j)_{i\in I, j\in J}$

Just as with any other mathematical structure, let us now define the correct concept for a structure-preserving map, i.e. a smooth map, and the concept of a map which expresses that two objects are essentially the same, i.e. a diffeomorphism.

Definition 2.1.11. Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. A **smooth map** $f : \mathcal{M} \to \mathcal{N}$ is a continuous map such that for any pair of charts (U, φ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is C^∞ as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Note that smooth maps are composable and that the identity is a smooth map by definition and the fact that transition functions are C^∞-maps.

Definition 2.1.12. A smooth map $f : \mathcal{M} \to \mathcal{N}$ is called a **diffeomorphism** if there exists a smooth map $g : \mathcal{N} \to \mathcal{M}$ such that $f \circ g = Id_{\mathcal{N}}$ and $g \circ f = Id_{\mathcal{M}}$.

2.2 The tangent vector space

We would now like to define the tangent space of a manifold $\mathcal{M}$ at a point p, generalizing the idea of the tangent directions of a point in $\mathcal{M}$. Since the notion of a manifold is independent of any kind of ambient space, we must find an intrinsic definition of the tangent vector. The key realization for this is that a vector can be thought as a direction we can derive in. Therefore, we will define the tangent space using the concept of derivations.

Given a manifold $\mathcal{M}$, denote by

$$\mathcal{F}(M) = \{f : \mathcal{M} \to \mathbb{R} \mid f_i := f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R} \text{ is } C^\infty \text{ for any chart } (U_i, \varphi_i)\}$$

the set of smooth functions on $\mathcal{M}$, which is the set of smooth maps from a given manifold $\mathcal{M}$ to the manifold $\mathbb{R}$. The set $\mathcal{F}(\mathcal{M})$ is real vector space with point-wise sum and multiplication:

$$(f + g)(p) := f(p) + g(p) \text{ and } (\lambda f)(p) := \lambda f(p)$$

It is also a ring, with the multiplication:

$$(fg)(p) := f(p)g(p)$$

Therefore, $\mathcal{F}(\mathcal{M})$ is an algebra: the **algebra of smooth functions on $\mathcal{M}$**.

Definition 2.2.1. Given a smooth map $f : \mathcal{M} \to \mathcal{N}$, we have the following induced operation:

$$\begin{aligned} f^* \colon \mathcal{F}(\mathcal{N}) &\to \mathcal{F}(\mathcal{M}) \\ h &\mapsto f^*(h) := h \circ f \end{aligned}$$

The map $f^*(h)$ is called the **pullback of h by f**.

Definition 2.2.2. A **derivation** at a point $p \in \mathcal{M}$ is a linear map $D : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g)$$

The **tangent space at a point $\boldsymbol{p}$ of $\boldsymbol{\mathcal{M}}$** is the set of all derivations at that point. We denote it using $T_p\mathcal{M}$.

Remark 2.2.3. Following the idea of generalizing derivatives in multivariable calculus, derivations are local too. Given two smooth function f, $g : \mathcal{M} \to \mathbb{R}$ which are equal at a neighbourhood of p, we have that $D(f) = D(g)$ for every $D \in T_p\mathcal{M}$.

Just as expected from generalizing the idea of the space of tangent directions of a point, $T_p\mathcal{M}$ is a vector space of dimension n, given the following operations:

$$(D_1 + D_2)(f) := D_1(f) + D_2(f) \text{ and } (\lambda D)(f) := \lambda D(f)$$

If the dimension of $\mathcal{M}$ is n and given a chart (U, φ) such that $p \in U$ and $\varphi = (x^1, \dots, x^n)$, we also have n canonical elements of $T_p\mathcal{M}$:

$$(\partial_1|_p, \dots, \partial_n|_p)$$

where each derivation acts on a smooth function $f : M \to \mathbb{R}$ as

$$\partial_i|_p(f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

Note that the notation $\partial_i|_p$ implicitly assumes a chart (U, φ). It is not shown as a sub-index in order to ease notation. Note also that, contrary to $\mathbb{R}^n$, we can not make sense out of the sum of two vectors from the tangent space of two different points. In order to sum two vectors, they must be elements of the tangent space of the same point.

Theorem 2.2.4. Let $p \in \mathcal{M}$ and let (U, φ) be a chart of $\mathcal{M}$ such that $p \in U$ and $\varphi(p) = (x^1, \dots, x^n)$. Then $\{\partial_i|_p\}_{i\{1,\dots,n\}}$ is a basis of $T_p\mathcal{M}$

The proof of this theorem is as well-known computation. Any standard differential geometry book, some of which the reader can find in this work's bibliography, will cover it.

There are two notorious consequences of this theorem. Firstly, it implies that every derivation $v \in T_p\mathcal{M}$ can be expressed as

$$v = \sum_{i=1}^{n} v(x^i)\partial_i|_p$$

Secondly, the dimension of $T_p\mathcal{M}$ is the same as $\mathcal{M}$. Since the manifold $\mathcal{M}$ and the manifolds U_i given by charts (U_i, φ_i) have the same dimension, so do $T_p\mathcal{M}$ and T_pU. Furthermore there is a canonical isomorphism between them given by $\tilde{v}(f) := v(f|_{U_i})$. This confirms the notion of locality of tangent spaces.

Definition 2.2.5. Given a smooth function $f : \mathcal{M} \to \mathbb{R}$ and a point $p \in \mathcal{M}$, the **differential of f at p** is defined as

$$\begin{aligned}(df)_p \colon T_p\mathcal{M} &\to \mathbb{R}\\ D &\mapsto (df)_p(D) := D(f)\end{aligned}$$

Theorem 2.2.6. Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map and let $p \in \mathcal{M}$. If (U, φ) is a chart of $\mathcal{M}$ and (V, ψ) a chart of $\mathcal{N}$ such that $p \in U$, $F(p) \in V$, $\varphi(p) = (x^1, \dots, x^n)$ and $\psi(q) = (y^1, \dots, x^m)$, then for all $j \in \{1, \dots, n\}$ we have the following equality:

$$d_pF\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{i=1}^{m} \frac{\partial F^i}{\partial x^j}(p)\frac{\partial}{\partial y^i}\Big|_{F(p)}$$

where we have $F^i := y^i \circ F \in \mathcal{F}(\mathcal{M})$.

Proof. We write $w := d_pF\left(\frac{\partial}{\partial x^j}\Big|_p\right) \in T_{F(P)}\mathcal{N}$. By Theorem 2.2.4 we can write

$$w = \sum_{i=1}^{m} w(y^i)\frac{\partial}{\partial y^i}\Big|_p$$

And by the definitions of both differential map and tangent vector,

$$w(y^i) = d_pF\left(\frac{\partial}{\partial x^j}\right)(y^i) = \frac{\partial}{\partial x^j}\Big|_p(y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p)$$

□

Definition 2.2.7. Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map and let $p \in \mathcal{M}$. The matrix J_pF associated to $d_pF : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ in basis $\{\partial/\partial x^j|_p\}_{j\in\{1,\dots,n\}}$ of $\mathcal{M}$ and $\{\partial/\partial y^i|_{F(p)}\}_{i\in\{1,\dots,m\}}$ in $\mathcal{N}$ is called **the Jacobian matrix** of F in p relative to (U, φ) and (V, ψ).

It is also possible to define the tangent space as a point $p \in \mathcal{M}$, $T_p\mathcal{M}$, using smooth curves.

Definition 2.2.8. Let $\mathcal{M}$ be a manifold. A **smooth curve on $\mathcal{M}$** is a smooth function $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$.

Any smooth curve defines a derivation at $\gamma(0) = p$ by

$$D_\gamma(f) = \left(\frac{d(f \circ \gamma)}{dt}\right)(0)$$

Then, an equivalent definition for $T_p\mathcal{M}$ is to define it as the space of equivalence classes of smooth curves $\gamma \colon (-\epsilon, \epsilon) \to \mathcal{M}$ with $\gamma(0) = p$, where $\gamma_1 \sim \gamma_2$ if the derivatives of $\varphi_i \circ \gamma_1$ and $\varphi_i \circ \gamma_2$ at 0 coincide for some chart (U_i, φ_i) with $p \in U_i$. This definition also captures the idea of the space of tangent directions of a point in a manifold, but it is less practical for calculations.

Definition 2.2.9. Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map and $h : \mathcal{N} \to \mathbb{R}$ a smooth function. Define the following induced map:

$$\begin{aligned}f_* \colon T_p\mathcal{M} &\to T_{f(p)}\mathcal{N}\\ D &\mapsto f_*(D)\end{aligned}$$

Where $f_*(D)(h) := D(h \circ f)$. We call the map $f_*(D)$ the **pushforward of D by f** .

2.3 Vector bundles

The following section will necessitate of further abstraction. In order to define vector fields, 1-forms, tensor fields and k-forms without any reference to their local components - i.e. how an observer is perceiving them - we will need to refer to vector bundles, a mathematical construction motivated by the tangent spaces. We want to avoid components so our equations are universal. For instance, when two observers see a vector in $\mathbb{R}^3$, they may define their own basis, so what they say about their vectors is subject to their point of view. However, the vector in itself is a mathematical entity in its own, so its properties should not depend on who is looking at it. For instance, it has a fixed length, direction, etc. These are properties that do not depend on the chosen observer or basis. This idea is crucial in the modern formulation of physics. To go on with this idea, we will use vector bundles: we will use two manifolds, one parametrizing the other. The basis manifold will be our space (or space-time) in the regular physical sense, and the bigger space (or total space) will be where our mathematical objects of interest live in. Firstly, we will define fiber bundles, and swiftly move on to vector bundles, which have the proper algebraic structure we need.

Definition 2.3.1. Let F, $\mathcal{M}$ and E be manifolds and let $\pi : E \to \mathcal{M}$ be a smooth onto map. The quadruple $(E, \pi, \mathcal{M}, F)$ is called a **fiber bundle** if it is locally trivial, i.e. if for each point $p \in \mathcal{M}$ there is an open set $U \subset \mathcal{M}$ and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the follow diagram commutes:

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}
$$

where pr_1 is the canonical projection of the first coordinate of the Cartesian product.

Definition 2.3.2. If $(E, \pi, \mathcal{M}, F)$ is a fiber bundle, E is called the **total space**, $\mathcal{M}$ is called the **base space**, π is called the **bundle projection** and F is called the **typical fiber**. The set $E_p := \pi^{-1}(p)$ is called **the fiber over p**.

For a given fiber bundle $(E, \pi, \mathcal{M}, F)$ we will usually write just E in order to ease notation. If this were to create ambiguity in a given situation, we will specify if by E we are referring to a fiber bundle or its total space or if we are using the different elements of the tuple. Note that, since π is onto, the total space is the union of the fibers over the points of the base space, i.e. $E = \cup_{p\in\mathcal{M}} E_p$.

Example 2.3.3. Let's us see some examples of fiber bundles:

(i) Given two smooth manifolds $\mathcal{M}$ and F, we have the smooth projection $pr_1 : \mathcal{M} \times F \to \mathcal{M}$. Then $(\mathcal{M} \times F, pr_1, \mathcal{M}, F)$ is a fiber bundle called **the trivial bundle**. In this case the fiber over a point $p \in \mathcal{M}$ is just $E_p = \{p\} \times F$ and there is a canonical diffeomorphism between each fiber and the typical fiber: for each $(p, f) \in E_p$, we have $(p, f) \mapsto f$.

(ii) The open Möbius strip is a fiber bundle with base space $\mathbb{S}^1$ and typical fiber $\mathbb{R}^1$.

The previous Möbius strip example shows that there exist nontrivial fiber bundles: fiber bundles are more general spaces than Cartesian products. However, the local triviality property ensures that any fiber bundle locally looks like a Cartesian product, even though the whole space may be more complicated, just like the Möbius strip has a twist.

Definition 2.3.4. Given a fiber bundle $(E, \pi, \mathcal{M}, F)$ and an open set $S \subset \mathcal{M}$, **the restriction of E to S** is the fiber bundle $(E|_S, \tilde{\pi}, S, F)$ where $E|_S := \{q \in E \mid \pi(q) \in S\}$ and $\tilde{\pi}$ is just the restriction of π to $E|_S$.

Let us now define the appropriate structure-preserving morphism between fiber bundles:

Definition 2.3.5. Let $(E, \pi, \mathcal{M}, F)$ and $(E', \pi', \mathcal{M}', F')$ be fiber bundles. A **fiber bundle morphism** from $(E, \pi, \mathcal{M}, F)$ to $(E', \pi', \mathcal{M}', F')$ is a pair of smooth maps $\psi : E \to E'$ and $\phi : \mathcal{M} \to \mathcal{M}'$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{M} & \xrightarrow{\phi} & \mathcal{M}' \end{array}$$

We write $(\psi, \phi) : (E, \pi, \mathcal{M}, F) \to (E', \pi', \mathcal{M}', F')$ and say that ψ is a bundle morphism along ϕ. If both ψ and ϕ are diffeomorphisms, then (ψ, ϕ) is called a **bundle isomorphism**. In this case, it is said that the bundles are isomorphic over ϕ.

Note that as a fiber-preserving map, i.e. $\psi(E_p) = E'_{\phi(p)}$, we have that ψ determines ϕ and it does not create any ambiguity to refer to the bundle morphism as just ψ. We could also now understand the local triviality property in the following terms: for any fiber bundle $(E, \pi, \mathcal{M}, F)$ and for any $p \in \mathcal{M}$, there exists an open subset $U \subset \mathcal{M}$ and a bundle isomorphism ϕ such that

$$\phi : E|_U \to U \times F$$

sending each fiber E_p to $\{p\} \times F$. ϕ is said to be a **local trivialization of E at p**.

Let's move on to a crucial definition, which will generalize physical fields:

Definition 2.3.6. A **section** of a fiber bundle $(E, \pi, \mathcal{M}, F)$ is a smooth map $s : \mathcal{M} \to E$ such that $\pi \circ s = Id_{\mathcal{M}}$. A **local section** over a set $U \subset \mathcal{M}$ is a smooth section over the restriction of E to U. The set of sections of a fiber bundle is noted by $\Gamma(E)$ and the set of local sections over a set U is noted by $\Gamma(E|_U)$.

Notice that both $\Gamma(E)$ and $\Gamma(E|_U)$ are C^∞-modules with the following operations:

$$(s + s') := s(p) + s'(p) \qquad (fs)(p) := f(p)s(p)$$

Example 2.3.7. If $E = M \times F$ is a trivial bundle with the standard fiber F, a section of E is a determined by a function from M to F: if we have a section $s : \mathcal{M} \to E$, there is a function $f : \mathcal{M} \to F$ such that

$$s(p) = (p, f(p)) \in E_p$$

Conversely, if we have a function $f : \mathcal{M} \to F$, the aforementioned formula gives a section.

We would now like to define vector bundles Informally speaking, a vector bundle is a fiber bundle where each fiber is a vector space. The idea behind it is to parameterize vector spaces using a manifold as the underlying parametrization.

Definition 2.3.8. Let V be a n-dimensional real vector space. A **n-dimensional vector fiber bundle with typical fiber V** is a fiber bundle $(E, \pi, \mathcal{M}, V)$ such that each fiber E_p is a vector space over $\mathbb{R}$ isomorphic to V and such for each point $p \in \mathcal{M}$ there exists a neighbourhood U of p and a fiberwise linear local trivialization, i.e. there exists a local trivialization

$$\phi : E|_U \to U \times \mathbb{R}^n$$

that maps each fiber E_p to the fiber $\{p\} \times \mathbb{R}^n$ linearly. The number n is called the **rank** of the vector bundle.

Definition 2.3.9. Let $(E, \pi, \mathcal{M}, V)$ and $(E', \pi', \mathcal{M}', V')$ be vector bundles. A **vector bundle morphism** $(\psi, \phi) : (E, \pi, \mathcal{M}, V) \to (E', \pi', \mathcal{M}', V')$ is a fiber bundle morphism such that its restriction to each fiber is linear, i.e. $\psi|_{E_p} \colon E_p \to E_{\phi(p)}$ is a $\mathbb{R}$linear map. A **vector bundle isomorphism** is a vector bundle morphism such that it is a bundle isomorphism and the restriction to each fiber is a vector space isomorphism.

We will now briefly define a specific type of vector bundle which will be crucially important in the mathematical foundation of physics:

Definition 2.3.10. Let $(E, \pi, \mathcal{M}, V)$ be a vector bundle. Its **dual vector bundle** is the vector bundle $(E^*, \pi^*, \mathcal{M}, V^*)$ which is obtained by passing each fiber E_p to its dual vector space, such that $E^* = \cup_{p \in \mathcal{M}} (E_p)^*$ and by defining $\pi^* : E^* \to \mathcal{M}$ as $(p, v^*) \mapsto p$.

Notice that, clearly, the rank of a vector bundle and its dual vector bundle is the same.

2.4 Vector fields

In our next step we will define vector fields. In order to do so, we will need to previously define the tangent bundle of a manifold $\mathcal{M}$. The idea behind this is to consider a new manifold built up from the tangent spaces of all points $p \in \mathcal{M}$, $T\mathcal{M}$. Then, we will consider the vector bundle where the total space is $T\mathcal{M}$ and the base space is $\mathcal{M}$, and we will define vector fields intrinsically using the concepts we build up in the previous section.

Firstly, fix a manifold $\mathcal{M}$ and define the following set:

$$T\mathcal{M} := \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M} = \{(p, v) \,|\, p \in \mathcal{M} \text{ and } v \in T_p\mathcal{M}\}$$

and the following map:

$$\begin{aligned} \pi \colon T\mathcal{M} &\to \mathcal{M} \\ (p, v) &\mapsto p \end{aligned}$$

We can equip the set $T\mathcal{M}$ with the smooth structure in the following way:

Fix a n-dimensional manifold $\mathcal{M}$. Given a chart (U, φ) of $\mathcal{M}$ with $\varphi = (x^1, \dots, x^n)$, let's consider

$$\begin{aligned} \psi_U \colon \pi^{-1}(U) &\to \mathbb{R}^{2n} \\ (p, v) &\mapsto (p_1, ..., p_n, v_1, ..., v_n) \end{aligned}$$

where $\varphi(p) = (p_1, ..., p_n) = (x^1(p), ..., x^n(p))$ and $v := \sum v_i \partial_i|_p$. Then, the topology of $T\mathcal{M}$ is generated by the preimages of ψ_U of all open sets of $\mathbb{R}^{2n}$ and all charts U of $\mathcal{M}$. In addition, if $\{(U_i, \varphi_i)\}_{i\in I}$ represents the smooth structure on $\mathcal{M}$, $\{(\pi^{-1}(U_i), \psi_{U_i})\}_{i\in I}$ represents the smooth structure on $T\mathcal{M}$. Clearly, $T\mathcal{M}$ is a $2n$-dimensional manifold and it is now just a calculation to check that the map $\pi : T\mathcal{M} \to \mathcal{M}$ is smooth.

Definition 2.4.1. Let $\mathcal{M}$ be a n-dimensional manifold. The **tangent bundle of $\mathcal{M}$** is the vector bundle $(T\mathcal{M}, \pi, \mathcal{M}, \mathbb{R}^n)$.

Remark 2.4.2. In order to ensure that $(T\mathcal{M}, \pi, \mathcal{M}, \mathbb{R}^n)$ is a vector bundle we should prove that it is locally trivial. To see why, notice that all charts are homeomorphic to $\mathbb{R}^n$ by definition. Taking any point p, we can find a chart U that contains it and so the neighbourhood of p given by U with the tangent spaces attached will look like $\mathbb{R}^{2n}$.

Notice that the fiber of $T\mathcal{M}$ over p, $T\mathcal{M}_p := \pi^{-1}(p)$ is canonically identified with $T_p\mathcal{M}$ by the mapping $(p, v) \to v$.

Notice that with the definition of the tangent bundle $T\mathcal{M}$ we can now combine the push-forwards of a tangent vector by a smooth map $f : \mathcal{M} \to \mathcal{N}$, which was defined in Definition 2.2.9, to define a neew map $Tf : T\mathcal{M} \to T\mathcal{N}$ on the tangent bundles of the manifolds which is linear in each fiber. This map is called **the tangent lift of f**. For smooth maps $f : \mathcal{M}_1 \to \mathcal{M}_2$, $g : \mathcal{M}_2 \to \mathcal{M}_3$, we have the following simple-looking version of the chain rule: $T(g \circ f) = T(g) \circ T(f)$.

Let's now finally define vector fields using the tangent bundle of $\mathcal{M}$:

Definition 2.4.3. A **vector field over $\mathcal{M}$** is a section of the tangent bundle, i.e. it is a smooth map $X : \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ X = Id_{\mathcal{M}}$. Therefore, a vector field over $\mathcal{M}$ is given by $X = (p, X_P)$, where $X_p \in T_p\mathcal{M}$. Denote by $\mathcal{X}(\mathcal{M}) := \Gamma(T\mathcal{M})$ the set of all vector fields.

Since $\mathcal{X}(\mathcal{M})$ is the set of all sections of a vector bundle, it has a natural real vector-space structure and a $\mathcal{F}(\mathcal{M})$-module structure, defined by the operations:

$$(X + Y)(p) := (p, X_p + Y_P),\ (\lambda X)(p) := (p, \lambda X_p),\ \text{and}\ \ (fX)(p) := (p, f(p)X_p)$$

Moreover, $\mathcal{X}(\mathcal{M})$ acts over $\mathcal{F}(\mathcal{M})$ as $X(f)(p) := X_p(f)$, where $X \in \mathcal{X}(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$.

We will now see some important cases of vector fields:

Definition 2.4.4. Given a manifold $\mathcal{M}$, a point $p \in \mathcal{M}$ and a chart (U, φ) such that $p \in U$ and $\varphi = (x^1, \dots, x^m)$, the map

$$\begin{aligned} \partial^i \colon U &\to T_pU \\ p &\mapsto (p, \partial^i|_p) \end{aligned}$$

is a vector field, since the functions $\partial_i(f) : \mathcal{M} \to \mathbb{R}$ given by $\partial_i(f)(p) = \partial_i|_p(f)$ are smooth for all $f \in \mathcal{F}(\mathcal{M})$. This map is called **the coordinate vector field of (U, φ) in the x^i direction**.

Definition 2.4.5. The **Lie bracket of two vector fields X and Y** is the unique vector field $[X, Y]$ such that

$$[X, Y] := X(Y(f)) - Y(X(f))$$

The proof of its existence and uniqueness can be found in [Lee09].
On a chart (U, φ) such that $\varphi(p) = (x^1, \dots, x^m)$ we may locally write a vector field as:

$$X = \sum a_i \partial_i$$

where a_i are smooth functions. In the case of the Lie bracket $[X, Y]$, with $X = \sum a_i \partial_i$ and $Y = \sum b_i \partial_i$ given a chart (U, φ), we may write:

$$[X, Y] = \sum_{i,j} \left(a_j \frac{\partial b_i}{\partial x^j} - b_j \frac{\partial a_i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

We will briefly refer to the crucial properties of the Lie bracket:

Theorem 2.4.6. The map $[\cdot, \cdot] : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$ such that $(X, Y) \mapsto [X, Y]$ is bilinear over $\mathbb{R}$ and for $X, Y, Z \in \mathcal{X}(\mathcal{M})$ the following properties hold:

(i) It is anti-symmetric: $[X, Y] = -[Y, X]$

(ii) The Jacobi identity holds: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

(iii) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for all $f, g \in C^\infty(\mathcal{M})$

We will no further talk about the Lie bracket and move on to 1-forms, not because their lack of interest but the other way around. Lie brackets are a far-reaching concept about which rivers of ink have flown, so in order to keep this work on track, we will have to move on.

2.5 1-forms

We will now define 1-forms intrinsically on a manifold, without referencing components or coordinates in our definition. 1-forms (or more generally, k-forms, as we will see later on) are a founding rock of our understanding of modern geometry and physics, even though most physicists are not aware of such fact. Instead, they think of forms as vectors which behave in weird ways. For instance, physicists think about the magnetic field as a vector field. However, under a reflection, the magnetic field is not reflected, but reflected and reversed. It is impossible to explain this in terms of vectors, so physicists stack the prefix pseudo in front of the term, naming the magnetic field as a "pseudo-vector". In reality, the magnetic field is what is called a 2-form. We will now start to formally cement a theory that can explain this behaviour using dual vector spaces.

Definition 2.5.1. Given a manifold $\mathcal{M}$ and $p \in \mathcal{M}$, the **cotangent space of $\mathcal{M}$ at p** is the dual space of $T_p\mathcal{M}$, and we write $T_p^*\mathcal{M} := (T_p\mathcal{M})^*$. Its elements are called **linear forms** or **covectors**.

Now, define the following set:

$$T^*\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} T^*_p\mathcal{M} = \{(p, v^*) \,|\, p \in \mathcal{M} \text{ and } v^* \in T^*_p\mathcal{M}\}$$

We will use the previous Definition 2.3.10 to to define a new vector bundle:

Definition 2.5.2. Given a manifold $\mathcal{M}$, the **cotangent bundle of $\mathcal{M}$** is the dual vector bundle of the tangent bundle of $\mathcal{M}$.

Unpacking this definition a little bit, we get that the cotangent bundle of $\mathcal{M}$ is the vector bundle $(T^*\mathcal{M}, \pi^*, \mathcal{M}, \mathbb{R}^n)$ where $\pi^*(p, v^*) = p$ and n is the dimension of $\mathcal{M}$. We have that $T^*\mathcal{M}$ is a $(2n)$-dimensional manifold and that the fiber over a point p is just the cotangent space over that point, i.e. $T^*\mathcal{M}_p = T^*_p\mathcal{M}$.

Definition 2.5.3. A **(differential) 1-form on $\mathcal{M}$** is a section of the cotangent bundle, i.e. a smooth map $\omega : \mathcal{M} \to T^*\mathcal{M}$ such that $\pi^* \circ \omega = Id_{\mathcal{M}}$.

Denote the set of all 1-forms as $\mathcal{X}^*(\mathcal{M}) := \Gamma(T^*\mathcal{M})$. It has a real vector-space structure and a $\mathcal{F}(\mathcal{M})$-module structure with analogous operations to the ones defined point-wise in $\mathcal{X}(\mathcal{M})$. Given a vector field $X \in \mathcal{X}(\mathcal{M})$ we can define different maps $\omega(X) : \mathcal{M} \to \mathbb{R}$ by $\omega(X)(p) := \omega_p(X_p)$ and identify 1-forms with the maps $\omega : \mathcal{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$, which let us think of the space 1-forms, $\mathcal{X}^*(\mathcal{M})$ as the dual space of vector fields, $\mathcal{X}(\mathcal{M})$ - hence the notation.

Definition 2.5.4. Given a smooth function $f \in \mathcal{F}(M)$, the **differential of f** is the 1-form $df \in \mathcal{X}^*(M)$ defined by $df(p) := (p, d_pf)$, where d_pf is the differential of f at p as defined in Definition 2.2.2.

We will now express 1-forms locally so we can do computations with them. Given a manifold $\mathcal{M}$, a point $p \in \mathcal{M}$ and a chart (U, φ) such that $p \in U$ and $\varphi(p) = (x^1, \dots, x^m)$, we have that $x^i \in \mathcal{F}(\mathcal{M})$ and that the form $d_px^i \in T^*_p\mathcal{M}$ satisfies that

$$d_px^j(\partial_i|_p) = \frac{\partial x^j}{\partial x^i}(p) = \delta^j_i$$

where δ^j_i is the Kronecker delta. Therefore $\{d_px^i\}_{i\in\{1,\dots,n\}}$ is the dual basis of $\{\partial^i|_p\}_{i\in\{1,\dots,n\}}$. We can then consider the **coordinate 1-forms** $dx^1, \dots, dx^n$ on U which satisfy $dx^j(\partial|_i) = \delta^j_i$. Since any vector field can be locally expressed as $X = \sum a_i\partial_i$, by applying to both sides the coordinate 1-forms it follows that any 1-form $\omega \in \mathcal{X}^*(M)$ can be locally expressed in U as

$$\omega|_U = \omega_i dx^i$$

where $\omega_i := \omega(\partial_i)$ are smooth functions. In the particular case of the differential of a function, we have the following expression:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

2.6 Tensor fields

We would now like to generalize the concept of a 1-form. In fact, by doing so, we will acquire a point of view which regards functions, vector fields and 1-forms as instances of the same mathematical object: a tensor field. It will also provide the basis to define the semi-Riemannian metric, which will allow us to generalize the inner-product of $\mathbb{R}^n$ to an arbitrary manifold.
We will first need to introduce the concept of a tensor over a module. Let X be a module over a ring R and X^* its dual R-module. For $r, s \in \mathbb{N}$, we can consider the following R-modules:

$$(X^*)^r := X^* \times \overset{(r)}{\cdots} \times X^* \text{ and } X^s := X \times \overset{(s)}{\cdots} \times X$$

with component-wise induced operations.

Definition 2.6.1. Given $r, s \in \mathbb{N}$, a **(r, s)-tensor** over a module X is an R-multilinear map

$$A\colon (X^*)^r \times X^s \to R$$

i.e. it is linear in each component of the Cartesian product.

Example 2.6.2. A $(0, 0)$-tensor is an element of R, a $(0, 1)$-tensor is a linear form, i.e. an element of X^* and a $(1, 0)$-tensor is a linear form on X^*, i.e. an element of X^{**}, which can be canonically identified with an element of X. A $(0, 2)$-tensor is a bilinear form and a $(1, 1)$-tensor is a linear transformation.

Denote by $T^r_s(X)$ the set of all (r, s)-tensors over X, and we write $\mathcal{T}(X) := \bigoplus_{r,s\in\mathbb{N}} T^r_s(X)$. This set is called the **tensor algebra of $\boldsymbol{X}$**. We will justify that $\mathcal{T}(X)$ is an algebra right away. $T^r_s(X)$ is an R-module with the usual sum of functions and multiplication by a scalar.

Definition 2.6.3. The **tensor product** of two tensors $A \in T^r_s(X)$ and $B \in T^t_u(X)$ is the tensor $A \otimes B \in T^{r+t}_{s+u}(X)$ defined by

$$(A\otimes B)(\alpha^1, \dots, \alpha^{r+t}, v_1, \dots, v_{s+u}) := A(\alpha^1, \dots, \alpha^r, v_1, \dots, v_s)\cdot B(\alpha^{r+1}, \dots, \alpha^{r+t}, v_{s+1}, \dots, v_{s+u})$$

The tensor product defines an associative operation on the set $\mathcal{T}(X)$ which is not commutative. It is compatible with the other operations over $\mathcal{T}(X)$ induced by the operations over the R-modules $T^r_s(X)$. Therefore, the tensor algebra $\mathcal{T}(X)$ is a (graded) R-algebra, and hence its name is indeed justified.

Theorem 2.6.4. Let V be a vector space, V^* its dual vector space, $\{e_i\}_{i\in I}$ a basis of V and $\{e^{*j}\}_{j\in I}$ its dual basis. The set $\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{*j_1} \otimes \dots \otimes e^{*j_s}\}_{i_k, j_l \in I}$ is well-defined and is a basis of $T^r_s(V)$.

Proof. The result follows from linear algebra: one must check on the linear independence of the set and then use multilinearity and use the expressions of coordinates relative to V and V^* on the arguments of A. One can find the full proof in [Spi79]. □

The previous theorem ensures that all (r, s)-tensors can be expressed as

$$A = A^{i_1 \dots i_r}_{j_1 \dots j_s}\, e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{*j_1} \otimes \dots \otimes e^{*j_s}$$

where $A^{i_1\dots i_r}_{j_1\dots j_s} := A(e^{*i_1}, \dots, e^{*i_r}, e_{j_1}, \dots, e_{j_s})$ are called the **components** of A in that basis.

Given a vector space V and a basis $\{e_i\}_{i\in I}$, $A \in T^r_s(V)$ and $B \in T^t_u(V)$ with components $A^{i_1\dots i_r}_{j_1\dots j_s}$ and $B^{i_1\dots i_t}_{j_1\dots j_u}$, the components of the tensor $C = A \otimes B$ in the same basis are

$$C^{i_1,\dots,i_{r+t}}_{j_1,\dots,j_{s+i}} = A^{i_1\dots i_r}_{j_1\dots j_s} \cdot B^{i_{r+1}\dots i_{r+t}}_{j_{s+1}\dots j_{s+u}}$$

Now, given a vector space V and two bases of this space, $\{e_i\}_{i\in I}$ and $\{e'_j\}_{j\in I}$, we can consider the change of basis matrix $\Lambda := \Lambda^i_j$ defined by $e_j = \Lambda^i_j e'_i$ and its inverse matrix Λ'. Then, if the components of a tensor A in the first basis are $A^{i_1\dots i_r}_{j_1\dots j_s}$, the components of A in the second basis are

$$A'^{\,i_1\dots i_r}_{\,j_1\dots j_s} = \Lambda^{i_1}_{k_1} \cdots \Lambda^{i_r}_{k_r} \Lambda'^{\,l_1}_{\,j_1} \cdots \Lambda'^{\,l_s}_{\,j_s} A^{k_1\dots k_r}_{l_1\dots l_s}$$

The upper indices of the tensor components, which change according to Λ, are said to be **contravariant**, and the lower indices of the tensor components, which change according to Λ', are said to be **covariant**.

The previous discussion justifies the following definition:

Definition 2.6.5. A $(r, 0)$-tensors is named a **contravariant tensor** and a $(0, s)$-tensors is named **covariant tensor**.

Definition 2.6.6. Let A be a covariant or a contravariant tensor of at least rank 2. A is said to be a **symmetric tensor** if transposing any two of its arguments leaves its image fixed. A is said to be a **skew-symmetric tensor** if transposing two any two of its arguments changes the sign of the image. By convention, we say that $(0, 0)$, $(1, 0)$ and $(0, 1)$ tensors are both symmetric and skew-symmetric.

Lets now define a tensor bundle over a manifold. Fix a manifold $\mathcal{M}$ and define the following set:

$$T^{r,s}(\mathcal{M}) := \bigsqcup_{p\in\mathcal{M}} T^r_s(T_p\mathcal{M})$$

Definition 2.6.7. Given a manifold $\mathcal{M}$, the **(r, s)-tensor bundle over $\mathcal{M}$** is the vector bundle $(T^{r,s}(\mathcal{M}), \tilde{\pi}, \mathcal{M}, \mathbb{R}^{n^{(r+s)}})$ where $\tilde{\pi}(p, T) = p$.

The details of the topology of the set $T^{r,s}(\mathcal{M})$ can be found in [Gol98]. Finally, we can get to define tensor fields via:

Definition 2.6.8. Given a manifold $\mathcal{M}$ and $r, s \in \mathbb{N}$ a **(r, s)-tensor field over $\mathcal{M}$** is a section over the (r, s)-tensor bundle over $\mathcal{M}$. Denote the $\mathcal{F}(\mathcal{M})$-module of all (r, s)-tensor fields over $\mathcal{M}$ as $T^r_s(\mathcal{M}) := \Gamma(T^{r,s}(\mathcal{M}))$

Example 2.6.9. Smooth functions are $(0, 0)$-tensor fields by convention, vector fields are $(1, 0)$-tensor fields and 1-forms are $(0, 1)$-tensor fields.

Remark 2.6.10. Given a manifold $\mathcal{M}$, and $r, s \in \mathbb{N}$, we can also understand a **(r, s)**-tensor field over **$\mathcal{M}$** as a (r, s)-tensor over the $\mathcal{F}(\mathcal{M})$-module $\mathcal{X}(\mathcal{M})$, that is a $\mathcal{F}(\mathcal{M})$-multilinear map

$$A\colon (\mathcal{X}^*(\mathcal{M}))^r \times (\mathcal{X}(\mathcal{M}))^s \to \mathcal{F}(\mathcal{M})$$

We write $T(\mathcal{M})$ instead of $\mathcal{T}(\mathcal{X}(\mathcal{M}))$ to signify the tensor algebra over $\mathcal{X}(\mathcal{M})$, which is $T(\mathcal{M}) = \bigoplus r, s \in \mathbb{N} T^{r,s}(\mathcal{M})$.

Given a manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$, a tensor over the tangent space $T_p\mathcal{M}$ is called a **tensor at *p***. A fiber of the (r,s)-tensor bundle over a point $p \in \mathcal{M}$ is the just the set of (r,s)-tensors over $T_p\mathcal{M}$, i.e. $T^r_s(T_p\mathcal{M})$, the set of all tensors at p. A (r,s)-tensor field A is equivalent to a field on $\mathcal{M}$ smoothly assigning to each $p \in \mathcal{M}$ a tensor at that point, $A_p : (T^*_p\mathcal{M})^r \times (T_p\mathcal{M})^s \to \mathbb{R}$.

Since $A(\omega^1, \dots, \omega^r, X_1, \dots, X_s)(p)$ only depends on the local values of the vector fields and one-forms and not on the whole vector fields and one-forms, we define:

$$A_p(\alpha^1, \dots, \alpha^r, v_1, \dots, v_s) := A(\omega^1, \dots, \omega^r, X_1, \dots, X_s)(p)$$

with ω^i is any one-form such that $\omega^i_p = \alpha^i \in T^*_p\mathcal{M}$ and X_j is any vector field such that $X_{ip} = v_i \in T_p\mathcal{M}$. Conversely, a choice of A_p determines uniquely a tensor field A. In an analogous way to vector fields and one-forms, the smoothness can be defined in terms of the smoothness of the map $A(\omega^1, \dots, \omega^r, X_1, \dots, X_s) : \mathcal{M} \to \mathbb{R}$ for all $\omega^i \in T^*_p\mathcal{M}$ and for all $X_j \in T_p\mathcal{M}$.

Theorem 2.6.11. Let (U, φ) be a chart on a manifold $\mathcal{M}$ such that $\varphi(p) = (x^1, ..., x^n)$. Any tensor field $A \in \mathcal{T}^r_s(\mathcal{M})$ can be written on U as

$$A = A^{i_1 \dots i_r}_{j_1 \dots j_s}\, \partial_{i_1} \otimes \dots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

where $i_k,\ j_l \in \{1, ..., n\}$ for all $k \in \{1, ..., s\}$ and $l \in \{1, ...r\}$. The smooth functions $A^{i_1 \dots i_r}_{j_1 \dots j_s} := A(dx^{i_1}, ..., dx^{i_r}, \partial_{j_1}, ..., \partial_{j_s}) \in \mathcal{F}(U)$ are called **the components of A relative to (U, φ)**.

Proof. It is a direct proof considering the basis $\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{*j_1} \otimes \dots \otimes e^{*j_s}\}_{i_k, j_l \in \{1,\dots,n\}}$ of $\mathcal{T}^r_s(T_p\mathcal{M})$ given by Theorem 2.6.4 and the local expressions for vector fields and one-forms. $\square$

Remark 2.6.12. Let (V, ψ) be another chart of $\mathcal{M}$ such that $\psi(p) = (y^1, ..., y^n)$ and $U \cap V \neq \emptyset$. In the overlap $U \cap V$ we have

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

Therefore the components of a tensor A relative to (V, ψ) on $U \cap V$ can be expressed relative to (U, φ) using the Jacobian matrix $Q^i_j = \frac{\partial y^i}{\partial x^j} = J(\psi \circ \varphi^{-1})^i_j$ in the expression obtained after Theorem 2.2.4.

2.7 Differential forms

Differential forms are a particular type of tensor fields which constitute the right mathematical frame that allows us to generalize integration, curves and surfaces from $\mathbb{R}^n$ to arbitrary (smooth) manifolds. In an intuitive manner, if dx^i is an infinitesimal variation in the x_i direction, a covariant tensor such as $dx^i \otimes dx^j$ can be interpreted as a 2-dimensional infinitesimal variation or a surface which locally approximates the plane $\partial_i - \partial_j$. This intuitions works in general for k dimensions. However, since the tensor product of two covariant tensors is commutative, we can not distinguish between the different coordinate orders, i.e. $dx^i \otimes dx^j = dx^j \otimes dx^i$. To naturally keep track of the coordinate order (and therefore a notion of orientation), we will define k-forms as skew-symmetric covariant tensors. Let's start with some linear algebra.

Let V be a $\mathbb{R}$-vector space for all this section. We will denote the space of all skew-symmetric covariant tensors over V of range k as $\bigwedge^k(V)$. This space is clearly a real vector space. Notice that, by previous convention, we have that $\bigwedge^0(V) = T_0^0(V)$ and $\bigwedge^1(V) = T_1^0(V)$. Let's now define a map which allows us to generate elements of $\bigwedge^k(V)$ from elements of $T_k^0(V)$.

Definition 2.7.1. The **antisymmetrization map** $Alt^k : T_k^0(V) \to \bigwedge^k(V)$ is defined by

$$Alt^k(\omega)(v_1, ..., v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma(n)})$$

where S_k is the k-symmetric group and $\varepsilon(\sigma)$ is the sign of the permutation $\sigma \in S_k$.

The following immediate properties can easily be derived from the definition:

Theorem 2.7.2. For $\alpha \in T_{k_1}^0(V)$ and $\beta \in T_{k_2}^0(V)$, we have

$$Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes \beta) = Alt^{k_1+k_2}(\alpha \otimes \beta)$$

and

$$Alt^{k_1+k_2}(\alpha \otimes Alt^{k_2}\beta) = Alt^{k_1+k_2}(\alpha \otimes \beta)$$

Definition 2.7.3. Given $\omega \in \bigwedge^{k_1}(V)$ and $\eta \in \bigwedge^{k_2}$, we can define their **exterior product** or **wedge product**, $\omega \wedge \eta \in \bigwedge^{k_1+k_2}(V)$, as

$$\omega \wedge \eta := \frac{1}{k_1!\, k_2!} Alt^{k_1+k_2}(\omega \otimes \eta)$$

Written out, the previous definition is

$$\omega \wedge \eta = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \varepsilon(\sigma)\sigma(\omega \otimes \eta) = \frac{1}{k_1!\, k_2!} \sum_{\sigma \in S_{k_1+k_2}} \epsilon(\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma(k_1)})\eta(v_{\sigma(k_1+1)}, ..., v_{\sigma(k_2)})$$

Readers should be careful with this definition. There is another convention in use which defines the wedge product the same as we did but multiplied by a factor of $(k_1 + k_2)!$, which may affect computations. Let's now discuss some properties of this product:

Theorem 2.7.4. Let $\alpha \in \bigwedge^{k_1}(V)$, $\beta \in \bigwedge^{k_2}(V)$ and $\gamma \in \bigwedge^{k_3}(V)$. We have the following properties:

(i) $\wedge : \bigwedge^{k_1}(V) \times \bigwedge^{k_2}(V) \to \bigwedge^{k_1+k_2}(V)$ is $\mathbb{R}$-linear.

(ii) $\alpha \wedge \beta = (-1)^{k1k_2} \beta \wedge \alpha$

(iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

Theorem 2.7.5. Let $\{e^1, ..., e^n\}$ be a basis of V^*. The set of elements

$$\{e^{i_1} \wedge ... \wedge e^{i_k} \mid 1 \leq i_1 < i_2 < ... < i_k \leq n\}$$

is a basis for $\bigwedge^k(V)$.

A direct consequence of this theorem is that the dimension of $\bigwedge^k(V)$ is $dim \bigwedge^k(V) = \binom{n}{k}$, and in particular if $k > n$ we have that $dim \bigwedge^k(V) = 0$. Now, since $\bigwedge^0(V) = \mathbb{R}$ and $\bigwedge^1(V) = V^*$ and the wedge product is compatible with the sum and the scalar multiplication, the direct sum $\bigwedge(V) = \bigoplus_{k=0}^n \bigwedge^k(V)$ is an alternating $\mathbb{R}$-graded algebra with the operations $(+, \cdot, \wedge)$. It has dimension $dim \bigwedge(V) = \sum_{k=0}^n \binom{n}{k} = 2^n$. $\bigwedge(V)$ is called the **exterior algebra of $\boldsymbol{V}$**.

Now that we have finished our linear-algebra digression, we can finally get to define k-forms. Given a manifold $\mathcal{M}$, define the following set:

$$\bigwedge^k(T\mathcal{M}) := \bigsqcup_{p \in \mathcal{M}} \bigwedge^k(T_p\mathcal{M})$$

Definition 2.7.6. The vector bundle $(\bigwedge^k(T\mathcal{M}), \pi, \mathcal{M}, \mathbb{R}^{\binom{n}{k}})$ is called the **bundle of differential $\boldsymbol{k}$-forms**.

The smooth structure of $\bigwedge^k(T\mathcal{M})$ is defined in a similar way as the previous bundle examples that we have seen. One can find the detailed description in the reference used for this section. We will denote the set of sections over this vector bundle as $\Omega^k(\mathcal{M}) = \Gamma(\bigwedge^k(T\mathcal{M}))$.

Definition 2.7.7. A **differential $\boldsymbol{k}$-form over $\boldsymbol{\mathcal{M}}$** or simply a **$\boldsymbol{k}$-form over $\boldsymbol{\mathcal{M}}$** is an element of $\Omega^k(\mathcal{M})$, i.e. a section of the bundle of k-forms over $\mathcal{M}$.

Given a $k \in \mathbb{N}$, an equivalent definition for a differential k-form over $\mathcal{M}$ is that it is a skew-symmetric $(0, k)$-tensor field over $\mathcal{M}$.

Notice that we have overlapping notations: we have that $\Omega^0(\mathcal{M}) = \mathcal{F}(\mathcal{M})$ and that $\Omega^1(\mathcal{M}) = \mathcal{X}^*(\mathcal{M})$. In general, the set of all k-forms $\Omega^k(\mathcal{M})$ can be regarded as a subset of $\mathcal{T}_k^0(\mathcal{M})$ specifically as a $\mathcal{F}(\mathcal{M})$-submodule of $\mathcal{T}_k^0(\mathcal{M})$. The tensor product of two skew-symmetric tensors is not, in general, skew-symmetric. Therefore, the tensor product of two k-forms is not in general a k-form. However we can define a new binary operation, $\wedge$ over k-forms which gives a k-form via skew-symmetrizing their tensor product. This will allow us to consider new differential forms such as $dx^i \wedge dx^j$ which represent infinitesimal oriented surfaces. The orientation comes from the skew-symmetric property, $dx^i \wedge dx^j = dx^j \wedge dx^i$, which keeps track of the order of the coordinates, and therefore the orientation of the surface.

Definition 2.7.8. Given two forms $\omega \in \Omega^k(\mathcal{M})$ and $\eta \in \Omega^l(\mathcal{M})$, their **exterior product** $\wedge : \Omega^k(\mathcal{M}) \times \Omega^l(\mathcal{M}) \to \Omega^{k+l}(\mathcal{M})$ is defined as

$$\omega \wedge \eta(p) := \omega(p) \wedge \eta(p)$$

This definition can also be seen as

$$\omega \wedge \eta = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \varepsilon(\sigma)\sigma(\omega \otimes \eta)$$

where S_n is the n-symmetric group, $\varepsilon(\sigma)$ is the sign of the permutation $\sigma \in S_n$ and $\sigma(\omega\otimes\eta)(X_1, \dots, X_{k+l}) := (\omega\otimes\eta)(X_{\sigma(1)}, \dots, X_{\sigma(k+l)})$, and where X_i are vector fields over $\mathcal{M}$.

By definition it is clear that the exterior product is associative and skew-symmetric: for $\omega \in \Omega^k(\mathcal{M})$ and $\eta \in \Omega^k(\mathcal{M})$, we have

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$$

Therefore for any $\omega \in \Omega(\mathcal{M})$ we have

$$\omega \wedge \omega = 0$$

Remark 2.7.9. Since the exterior product defines a binary operation $\wedge : \Omega^k(\mathcal{M})\times\Omega^l(\mathcal{M}) \to \Omega^{k+l}(\mathcal{M})$ which is compatible with the other operations defined over $\Omega^k(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$, we have a graded alternating $\mathcal{F}(\mathcal{M})$-algebra $(\Omega(\mathcal{M}), +, \cdot, \wedge)$ where $\Omega(\mathcal{M}) := \bigoplus_{k\geq 0}\Omega^k(\mathcal{M})$. It is called **algebra of differential forms** of the manifold $\mathcal{M}$.

Remark 2.7.10. Since a k-form ω is a tensor field, given a chart (U, φ) of a manifold $\mathcal{M}$ such that $\varphi(p) = (x^1, ..., x^n)$ we can locally express the k-form as $\omega|_U = \omega_{i_1\dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_l}$ with $\omega_{i_1\dots i_k} = \omega(\partial_{i_1}, ..., \partial_{i_k})$. Since skew-symmetry implies that if there is a repeated index the coefficient is zero, we can rewrite this expression as:

$$\omega|_U = \sum_{i_1<\dots<i_k}\sum_{\sigma\in S_k} \omega_{i_{\sigma(1)}\cdots i_{\sigma(k)}} dx^{i_{\sigma(1)}} \otimes \dots \otimes dx^{i_{\sigma(k)}} = \sum_{i_1<\dots<i_k} \omega_{j_1\dots j_k} dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

which characterizes local expressions of differential k-forms

The following theorem will characterize the exterior derivative, an operator over differential forms that will be crucial in our rewriting of Maxwell equations.

Theorem 2.7.11. *Let $\mathcal{M}$ be a manifold. There exists a unique map $d : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ such that the following properties hold:*

(i) *If $\omega \in \Omega^k(\mathcal{M})$, then $d\omega \in \Omega^{k+1}(\mathcal{M})$.*

(ii) *d is $\mathbb{R}$-linear.*

(iii) *If $\omega \in \Omega^p(\mathcal{M})$ and $\eta \in \Omega^q(\mathcal{M})$, $d(\omega\wedge\eta) = d\omega \wedge \eta + (-1)^p\omega \wedge d\eta$.*

(iv) *For any $\omega \in \Omega(\mathcal{M})$, $d^2(\omega) := d(d\omega) = 0$.*

One can find the detailed proof of this theorem in [Spi79]. The restriction $d : \Omega^0(\mathcal{M}) \to \Omega^1(\mathcal{M})$ agrees with the differential of a function as previously defined. The previous unique operator $d : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ is called **the exterior derivative**.

This operator is far more interesting than it seems at first sight. A differential form α is called **closed** if $d\alpha = 0$ and **exact** if $\alpha = d\beta$ for some differential form β. The last property, $d^2 = 0$, is called the **exactness** property of the exterior derivative: all exact differential forms are closed. The converse is generally not true, and the extent to which it fails to be true gives a lot of information on the topological properties of the manifold.

We will not enter in much detail, because the study of the failure of closed differential forms to be exact is a field of study in its own: de Rahm cohomology. However we shall mention that, by the theorem of Poincaré duality, the study of de Rahm cohomology groups, i.e. of differential forms, give the singular homology groups of the manifold, whose importance can not be overemphasized in topology.

Furthermore, the study of de Rahm cohomology can help us find solutions to Maxwell differential equations, and are the basis of study for Gauge invariance and scalar electric potential and vector magnetic potential which are intimately related to the electric and magnetic fields. This potentials and the de Rahm cohomology can even explain the Aharonov-Bohm effect in quantum electromagnetism, an effect by which an electrically charged particle is affected by the electromagnetic potential despite being confined to a region in which both the electric and magnetic fields are zero. We will treat this effect briefly in the last section of this work.

Chapter 3

Constructions in Pseudo-Riemannian geometry

Even though we have discussed many topics on differential geometry so far, we have been constrained to a domain where we didn't need any metric. However, we will now need to touch on some pseudo-Riemannian geometric ideas. Firstly, we will discuss pseudo-Riemannian metrics and manifolds and the natural isomorphism between vector fields and 1-forms which a metric allows. Then, we will move on to volume forms and the metric form, and finally we will define the Hodge star operator, which will allow us to rewrite Maxwell equations in a general setting. This section follows [Lee09], [BM94] and [Spi79].

3.1 Pseudo-Riemannian manifolds

Pseudo-Riemannian metrics generalize the concept of an inner-product from Euclidean vector-spaces, such as $\mathbb{R}^n$, to manifolds. Lorentzian manifolds, which are the right mathematical setup for modelling space-time, will emerge from this notion as a particular case.

Definition 3.1.1. Let V be a finite-dimensional real vector space and let $g \in T_2^0(V)$ be a symmetric bilinear form on V. We say that g is an **inner product** on V if it is non-degenerate, i.e. if $g(v, w) = 0$ for all w, then $v = 0$.

Definition 3.1.2. An inner product g on V is said to be **positive-definite** if $g(v, v) > 0$ for all $v \neq 0$ in V. Equivalently, it is said to be **negative-definite** if $g(v, v) < 0$ for all $v \neq 0$ in V.

Remark 3.1.3. We will not include positive-(semi)definiteness in the definition of an inner product. This will allow us to distinguish between two kind of subspaces, which will keep model the differences between space and time in our lorentzian manifold.

Definition 3.1.4. Let g be an inner product on V. The **index** ν of the inner product is the highest dimension possible of the subspaces $F \subset V$ over which $g|_F$ is negative-definite.

Remark 3.1.5. It is a well-known fact that given an inner product g over an n-dimensional vector space V, there exists an orthonormal basis $\{e_1, \ \dots \ , e_n\}$ on V for g, and that the number of basis vectors such that $g(e_i, e_i) = -1$ is equal to the index of g for any chosen basis.

Remark 3.1.6. Notice that an orthonormal basis on V for g diagonalizes the matrix of g, with components defined as $g_{ij} = g(e_i, e_j) = \pm\delta^i_j$. We will consider orthonormal basis in an order such that the n-tuple $(\varepsilon_1, ..., \varepsilon_n) := (g_{11},\ \dots\ , g_{nn})$ - the **signature** of g - satisfies that $g_{ii} = -1$ for all $i \leq \nu$ and $g_{ii} = +1$ for all $i > \nu$. This is often denoted as $(\nu, n-\nu)$. For instance, if $dimV = 4$ and the index of g is $\nu = 1$, we say that its signature is $(1, 3)$. This example will be important later on, since it is the one that models space-time. Now, if $v = v^i e_i$ and $w = w^i e_i$, we have that

$$g(v, w) = -\sum_{i=1}^{\nu} v^i w^i + \sum_{i=\nu+1}^{n} v^i w^i$$

We will now generalize this notion to an arbitrary manifold using tensor fields:

Definition 3.1.7. A **metric** over a manifold $\mathcal{M}$ is a symmetric non-degenerate $(0, 2)$-tensor field on $\mathcal{M}$ of constant index ν. The non-degeneracy and the constant index can be understood regarding $g \in \mathcal{T}^0_2(\mathcal{M})$ as a smooth assignation - as we explained before - of an inner product $g_p \in T_p\mathcal{M}$ to each point $p \in \mathcal{M}$, with a constant index ν for all points.

Definition 3.1.8. A **pseudo-Riemannian manifold** is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a (smooth) manifold and g a metric on $\mathcal{M}$. We say that it is a **Lorentzian manifold** if $dim\mathcal{M} > 2$ and $\nu_g = 1$. We will usually denote pseudo-Riemannian manifolds using just $\mathcal{M}$.

Remark 3.1.9. Given a local chart (U, φ) of a pseudo-Riemannian manifold $(\mathcal{M}, g)$, its metric tensor can be locally expressed on U as

$$g = g_{ij} dx^i \otimes dx^j$$

where $g_{ij} = g(\partial_i, \partial_j)$ are the local components of the tensor with respect to the chart U. The symmetry implies that $g_{ij} = g_{ji}$ and the non-degeneracy implies that $(g_{ij})_{i,j}$ is a non-singular matrix - i.e. its kernel as a linear application is trivial. If $x, Y \in \mathcal{X}(\mathcal{M})$ such that $X = X^i\partial_i$ and $Y = Y^j\partial_j$ locally over U, then we have

$$g(X, Y) = g_{ij} X^i Y^j$$

Definition 3.1.10. Given a pseudo-Riemannian manifold $(\mathcal{M}, g)$, we have the following natural $\mathcal{F}(\mathcal{M})$-linear isomorphisms:

$$\begin{aligned} \flat \colon \mathcal{X}(\mathcal{M}) &\to \mathcal{X}^*(\mathcal{M}) \\ X &\mapsto X^\flat : \mathcal{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) \\ &\qquad\qquad Y \mapsto X^\flat(Y) = g(X, Y) \end{aligned}$$

which is named the **flat operator**, and its inverse:

$$\begin{aligned} \sharp \colon \mathcal{X}(\mathcal{M}) &\to \mathcal{X}^*(\mathcal{M}) \\ \omega &\mapsto \omega^\sharp \end{aligned}$$

given by the equality $g(\omega^\sharp, X) = \omega(X)$ for $X \in \mathcal{X}(\mathcal{M})$ is named the **sharp operator**. The pair of isomorphisms is called the **musical isomorphisms**.

Remark 3.1.11. In musical notation, the symbol for flat, $\flat$, is used to signify a lower pitch and the symbol sharp, $\sharp$, is used to signify a higher pitch. This motivates the choice for the notation: in a chart (U, φ) of $\mathcal{M}$ with $\varphi(p) = (x^1, ..., x^n)$, if we have $X \in \mathcal{X}(\mathcal{M})$ locally written as $X = X^i \partial_i$ in U, then $X^\flat = X_i dx^i$ with $X_i = g_{ij} X^j$, so given another vector field locally written as $Y = Y^k \partial_k$, we have

$$X^\flat(Y) = g_{ij} X^j dx^i(Y^k \partial_k) = g_{ij} X^j Y^k \delta^i_k = g_{ij} X^j Y^i = g(X, Y)$$

Therefore we can say that the $\flat$ operator is lowering the index by the metric. The situation is analogous for the $\sharp$ operator, and we say that it is raising the index by the metric, just like in musical notation. The situation is analogous for $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$ at any $p \in \mathcal{M}$ with g_p.

Remark 3.1.12. The metric g over a manifold $\mathcal{M}$ induces a symmetric $C^\infty(\mathcal{M})$-bilinear map $\langle \cdot, \cdot \rangle : \Omega^k(\mathcal{M}) \times \Omega^k(\mathcal{M}) \to C^\infty(\mathcal{M})$ defined by $\langle \omega, \eta \rangle(p) := g_p(\omega^\sharp, \eta^\sharp) = \langle \omega^\sharp, \eta^\sharp \rangle_p$ In fact, we can now give an inner product for any tensor at a point $p \in \mathcal{M}$ by assigning

$$\langle \alpha^1 \otimes ... \otimes \alpha^r \otimes v_1 \otimes ... \otimes v_s, \beta^1 \otimes ... \otimes \beta^r \otimes u_1 \otimes ... \otimes u_s \rangle_p = g_p(\alpha^{1\sharp}, \beta^{1\sharp}) \cdots g_p(\alpha^{r\sharp}, \beta^{r\sharp}) g_p(v_1, u_1) \cdots g_p(v_s, u_s)$$

and generalizing to any k-form using linearity

3.2 Volume forms

Definition 3.2.1. Given a manifold $\mathcal{M}$, two of its charts (U, φ), (V, ψ) are said to be **positively compatible** if $U \cap V = \emptyset$ or if for all $p \in U \cap V$ we have $|J_p(\varphi \circ \psi^{-1})| > 0$. A manifold is said to be **orientable** if their charts are positively compatible by pairs.

Even though all points of any manifold have an orientable neighbourhoods (also known as being locally orientable), not all manifolds are orientable. The easiest example of a non-orientable manifold is the Möbius strip. The proof of this can be found in [Spi79].

Definition 3.2.2. Let $\mathcal{M}$ be an orientable manifold and $\mathcal{P}(\mathcal{M})$ the set of possible atlases of $\mathcal{M}$ formed by positively compatible charts. Two atlases A_1 and A_2 of $\mathcal{P}(\mathcal{M})$ are said to be positively compatible if $(\mathcal{M}, A_1 \cup A_2)$ is orientable. This defines an equivalence relation with classes named **orientations of** $\mathcal{M}$. We say that $(\mathcal{M}, \mathcal{O})$ is an **oriented manifold**, where $\mathcal{M}$ is a manifold and $\mathcal{O}$ a fixed orientation on $\mathcal{M}$. We usually just write $\mathcal{M}$ for an oriented manifold.

Definition 3.2.3. A **volume-form** ω on a n-dimensional manifold $\mathcal{M}$ is a non-vanishing n-form on $\mathcal{M}$, i.e. a n-form ω such that for all vector fields $X_1, ..., X_n \in \mathcal{X}(\mathcal{M})$ the functions $\omega(X_1, ..., X_n)(p)$ are non-zero for all $p \in \mathcal{M}$. Equivalently, for all $p \in \mathcal{M}$ we have that $\omega_p(v_1, ..., v_n)$ is non-zero for all $v_1, ..., v_n \in T_p\mathcal{M}$.

From [Spi79] we know the following proposition:

Theorem 3.2.4. A manifold is orientable if and only if there exists a volume form.

Given a volume form ω on a manifold $\mathcal{M}$, an ordered basis of $T_p\mathcal{M}$ is said to be **positively oriented** by ω if $\omega_p(v_1, ..., v_n) > 0$. Equivalently, it is said to be **negatively oriented** by ω if $\omega_p(v_1, ..., v_n) < 0$. Therefore orientations on $\mathcal{M}$ can be regarded as an equivalence class

over the set of volume forms quotiented by the relation of defining the same set of positively oriented tangent vectors.
The last missing piece in order to define the Hodge star operator will be metric volume forms. In order to define them, we must however introduce local (co)frame fields and the metric volume form.

Definition 3.2.5. Let $(\mathcal{M}, g)$ be a n-dimensional pseudo-Riemannian manifold. A **local frame field** on $U \subset \mathcal{M}$ is a set $\{E_1, ..., E_n\}$ of orthonormal vector fields on U, i.e. for every $p \in \mathcal{M}$ $\{E_1(p), ..., E_n(p)\}$ is an orthonormal basis on $T_p\mathcal{M}$. Its **local coframe field** is the set of dual one-forms $\{\omega^1, ..., \omega^n\}$, i.e. $\omega^i(E_j) = \delta^i_j$.

Definition 3.2.6. Let $(\mathcal{M}, g)$ be an oriented pseudo-Riemannian manifold of dimension n and let U and V be two non-disjoint charts of $\mathcal{M}$. Let $\{\omega^1, ..., \omega^n\}$ and $\{\eta^1, ..., \eta^n\}$ be two local coframe fields on U and V respectively. Given that the change of basis matrix Q will be orthogonal at each point, some calculations give the result:

$$\omega^1 \wedge ... \wedge \omega^n = \frac{1}{det(Q)}\eta^1 \wedge ... \wedge \eta^n = \pm\eta^1 \wedge ... \wedge \eta^n$$

on the overlap $U \cap V$. Taking a local coframe field compatible with the orientation of the manifold implies that $\omega^1 \wedge ... \wedge \omega^n = +\eta^1 \wedge ... \wedge \eta^n$. Therefore, the n-forms will agree on all overlaps, defining a unique volume form for the whole manifold. We name this new global volume form the **metric volume form** and note it by ω_g.

Remark 3.2.7. The calculation needed to proof that the metric volume form is not ill-defined is the following one:

Let (U, φ) and (V, ψ) be charts of $\mathcal{M}$ such that $\varphi(p) = (x^1, ..., x^n)$ and $\psi(p) = (y^1, ..., y^n)$. Let ω be a n-form with coordinate $\omega_{1...n} = \omega(\partial_1, ..., \partial_n)$ in (U, φ). Let the two charts be positively compatible and non-disjoint. Then, on the intersection $U \cap V$, using the expression for a vector obtained after Theorem 2.2.4 we have the following:

$$\begin{aligned}\omega_{1...n} = \frac{\partial^{l_1}}{\partial x^1} \cdots \frac{\partial y^{l_n}}{\partial x^n}\omega'_{l_1...l_n} &= \sum_{l_1<...<l_n}\sum_{\sigma\in S_n}\varepsilon(\sigma)\frac{\partial y^{l_{\sigma(1)}}}{\partial x^1} \cdots \frac{\partial y^{l_{\sigma(n)}}}{\partial x^n}\omega'_{l_1...l_n} \\ &= \sum_{\sigma\in S_n}\varepsilon(\sigma)\frac{\partial y^{\sigma(1)}}{\partial x^1} \cdots \frac{\partial y^{\sigma(n)}}{\partial x^n}\omega'_{1...n} = det\ J(\psi\circ\varphi^{-1})\omega'_{1...n}\end{aligned}$$

In our case we have that $J(\psi \circ \varphi^{-1}) = Q$ and that Q is orthogonal gives $detQ = \pm 1$, which gives the desired result.

Remark 3.2.8. Let (U, φ) be a chart on a pseudo-Riemannian manifold $(\mathcal{M}, g)$ such that $\varphi(p) = (x^1, ..., x^n)$, $\{E_1, ..., E_n\}$ a frame field on U with $\{\omega^1, ..., \omega^n\}$ its coframe field and $\omega_g = \omega^1 \wedge ... \wedge \omega^n$ the metric volume form on $(\mathcal{M}, g)$. Let Q be the change of basis matrix $E_i = Q^j_i\partial_j$. Just as before, we have:

$$\omega^1 \wedge ... \wedge \omega^n = \frac{1}{det(Q)}dx^1 \wedge ... \wedge dx^n$$

Since $\{E_1, ..., E_n\}$ is an orthonormal frame, we have $g(E_i, E_j) = \varepsilon_i \delta_{ij}$, with $(\varepsilon_1, ...\varepsilon_n)$ the signature of g. We have

$$\varepsilon_i \delta_{ij} = g(E_i, E_j) = Q_i^k Q_j^l g(\partial_k, \partial_l) = Q_i^k Q_j^l g_{kl}$$

and taking determinants at both sides we get

$$\pm 1 = det(Q^2) det(g)$$

Therefore, for a positively oriented coordinate system we obtain

$$\omega_g = \sqrt{|detg|}\; dx^1 \wedge ... \wedge dx^n$$

Remark 3.2.9. If $\alpha, \beta \in \Omega^k(\mathcal{M})$ such that $\alpha = \alpha^1 \wedge ... \wedge \alpha^n$ and $\beta = \beta^1 \wedge ... \wedge \beta^n$, we will write $\langle \alpha | \beta \rangle = [det\langle \alpha^i, \beta^k \rangle]$ for $k > 0$ and $\langle a | b \rangle = ab$ for $k = 0$. Notice the difference in notation between $\langle \cdot, \cdot \rangle$ and $\langle \cdot | \cdot \rangle$. This map is an inner product which makes the basis $\{e^{i_1}, ..., e^{i_k}\}_{i_1 < ... < i_k}$ orthonormal if $\{e_i\}_{i \in \{1,...,n\}}$ is orthonormal. $\langle \cdot | \cdot \rangle$ defines an inner product over the space of k-forms for any k. It is well defined using universal properties, as one can check in [Lee09]. One will also be able to see that

$$\langle \alpha | \beta \rangle = \frac{1}{k!} \langle \alpha, \beta \rangle$$

so the two inner products are the same up to a constant factor.

3.3 The Hodge star operator

Finally, we can define the the Hodge star operator, $\star$. The intuitive idea behind it is that given a pseudo-Riemannian manifold, the Hodge star operator over a form gives the "dual" of that form which completes the original form up to the metric volume form. For instance, in $\mathbb{R}^3$, we could visualize this as taking the normal direction of a given oriented surface $dx \wedge dy$, which would be dz, so that the wedge product between the original form representing the plane and its dual gives the volume form: $(dx \wedge dy) \wedge \star(dx \wedge dy) = (dx \wedge dy) \wedge (dz) = dx \wedge dy \wedge dz = vol_{\mathbb{R}_{euc}}$.

Theorem 3.3.1. Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold with a metric volume form ω_g. There exists a unique linear function $\star : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ satisfying $\star : \Omega^k(\mathcal{M}) \to \Omega^{n-k}(\mathcal{M})$ defined by the identity

$$\alpha \wedge \star\beta := \langle \alpha | \beta \rangle\, \omega_g$$

The proof of its existence and uniqueness can be found in [Lee09]. The previous operator $\star : \omega(\mathcal{M}) \to \Omega(\mathcal{M})$ is **the Hodge star operator**. Let us now name some of its basic properties:

Theorem 3.3.2. Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold with ν its metric index and ω_g its metric volume form. The following identities hold:

(i) $\star 1 = \omega_g$, where 1 is the constant 1 function.

(ii) $\star\omega_g = (-1)^\nu$

(iii) $\alpha \wedge \star\alpha = 0$ if and only if $\alpha = 0$

(iv) For all $\alpha, \beta \in \Omega^k(\mathcal{M})$, $\alpha \wedge \star\beta = \beta \wedge \star\alpha$

(v) If (U, φ) is a chart of $\mathcal{M}$ with $\varphi(p) = (x^1, ...x^n)$ and $\sigma \in S_n$, we have

$$\star(dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}) = \varepsilon_{\sigma(1)} \cdots \varepsilon_{\sigma(k)}\varepsilon(\sigma)dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}$$

(vi) For all $k \leq n$, $\alpha \in \Omega^k(\mathcal{M})$, $\star\star\,\alpha = (-1)^\nu(-1)^{k(n-k)}\alpha$

(vii) For all $\alpha, \beta \in \Omega^k(\mathcal{M})$ we have $\langle\star\alpha|\star\beta\rangle = (-1)^\nu\langle\alpha|\beta\rangle$

Proof. Properties (i), (ii), (iii) and (iv) follow immediately from the definition of the Hodge star operator:

(i) $\star 1 = 1 \wedge \star 1 = \langle 1|1\rangle\,\omega_g = 1 \cdot \omega_g = \omega_g$

(ii) $\omega_g \wedge \star\omega_g = \langle\omega_g|\omega_g\rangle\,\omega_g = det[\langle\omega^i, \omega^j\rangle]\,\omega_g = |detg|\,det[\langle dx^i, dx^j\rangle]\,\omega_g = |detg|\,\omega_g = (-1)^\nu\omega_g \iff \star\omega_g = (-1)^\nu$

(iii) $\alpha \wedge \star\alpha = \langle\alpha|\alpha\rangle\,\omega_g = 0 \iff \langle\alpha|\alpha\rangle = 0 \iff \alpha = 0$, since $\langle\cdot|\cdot\rangle$ is an inner product.

(iv) $\alpha \wedge \star\beta = \langle\alpha|\beta\rangle\,\omega_g = \langle\beta|\alpha\rangle\,\omega_g = \beta \wedge \star\alpha$ since $\langle\cdot|\cdot\rangle$ is an inner product and therefore symmetric.

To prove (v) it suffices to check that

$$\begin{aligned}(dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}) &\wedge \varepsilon_{\sigma(1)} \cdots \varepsilon_{\sigma(k)}\varepsilon(\sigma)dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)} \\ &= \langle dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}|\varepsilon_{\sigma(1)} \cdots \varepsilon_{\sigma(k)}\varepsilon(\sigma)dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}\rangle\,\omega_g\end{aligned}$$

and by uniqueness of the Hodge star operator, we will have the desired result. A detailed calculation can be found in [Lee09].

We will now proceed to prove (vi). It suffices to consider $\alpha = dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}$ for some permutation $\sigma \in S_n$. We first compute $\star(dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)})$. We must have $\star(dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}) = c\,dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}$ for some constant c by (v). On the other hand,

$$\begin{aligned}\varepsilon_{\sigma(k+1)}...\varepsilon_{\sigma(n)}\,\omega_g &= \langle dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}|dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}\rangle\,\omega_g \\ &= (dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}) \wedge \star(dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}) \\ &= (dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)}) \wedge c\,dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)} = (-1)^{k(n-k)}c\varepsilon(\sigma)\omega_g\end{aligned}$$

so that $c = \varepsilon_{\sigma(k+1)} \cdots \varepsilon_{\sigma(n)}(-1)^{k(n-k)}\varepsilon(\sigma)$. Using this and (v), we have that

$$\begin{aligned}\star\star(dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}) &= \star(\varepsilon_{\sigma(1)} \cdots \varepsilon_{\sigma(k)}\varepsilon(\sigma)dx^{\sigma(k+1)} \wedge ... \wedge dx^{\sigma(n)} \\ &= \varepsilon_{\sigma(1)} \cdots \varepsilon_{\sigma(k)}\varepsilon_{\sigma(k+1)} \cdots \varepsilon_{\sigma(n)}\varepsilon(\sigma)^2(-1)^{k(n-k)}dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)} \\ &= (-1)^\nu(-1)^{k(n-k)}dx^{\sigma(1)} \wedge ... \wedge dx^{\sigma(k)}\end{aligned}$$

Finally, we will prove (vii):

$$\langle \star\alpha | \star\beta \rangle \, \omega_g = \star\alpha \wedge \star\star\,\beta = (-1)^{\nu}(-1)^{k(n-k)} \star \alpha \wedge \beta = (-1)^{\nu}\beta \wedge \star\alpha = (-1)^{\nu}\langle \alpha|\beta \rangle \, \omega_g$$

which implies the equality which we wanted to prove. □

Now, using this newly defined operator and the musical isomorphisms, we can write a more general version of some old known operators:

Definition 3.3.3. Let $\mathcal{M}$ be a manifold, $f \in \mathcal{F}(\mathcal{M})$ and $X, Y \in \mathcal{X}(\mathcal{M})$. Define the following operators:

(i) The **gradient of a function *f*** is $grad\, f := (df)^\sharp$

(ii) The **divergence of a vector field *X*** is $div\, X := \star d \star X^\flat$

And over a 3-manifold we can define the following operators:

(i) The **cross product of two vector fields *X* and *Y*** is $X \times Y := (\star(X^\flat \wedge Y^\flat))^\sharp$

(ii) The **curl of a vector field *X*** is $curl\, X := (\star d X^\flat)^\sharp$

This generalized operators keep the properties of the old definitions intact. For instance, $div\,(curl\, X) = 0$, $curl\,(grad\,)f = 0$ and $curl\, X = 0 \Longrightarrow X|_U = grad\, f$.

We will now see a simple example of a computation using the Hodge star operator to get a basic idea of how it works:

Example 3.3.4. Consider the pseudo-Riemannian manifold $\mathbb{R}^3$ with its usual euclidean metric and orientation. Consider dx, dy and dz as a basis of 1-forms on $\mathbb{R}^3$. We then have:

$$\begin{aligned} \star\colon \Omega^1(\mathcal{M}) &\to \Omega^2(\mathcal{M}) \\ dx &\mapsto \star\, dx = dy \wedge dz \\ dy &\mapsto \star\, dy = dz \wedge dx = -dx \wedge dz \\ dz &\mapsto \star\, dz = dx \wedge dy \end{aligned}$$

and also

$$\begin{aligned} \star\colon \Omega^2(\mathcal{M}) &\to \Omega^1(\mathcal{M}) \\ dx \wedge dy &\mapsto \star\,(dx \wedge dy) = dz \\ dy \wedge dz &\mapsto \star\,(dy \wedge dz) = dx \\ dz \wedge dx &\mapsto \star\,(dz \wedge dx) = dy \end{aligned}$$

This example will be important when we proceed to rewrite Maxwell's equations.

Chapter 4

Rewriting Maxwell's equations

Finally we will be able to rewrite Maxwell's equations in an explicitly covariant way using differential forms. This process will uncover many interesting things about both the equations and the mathematical objects that intervene in them. This section will mainly follow [BM94]

4.1 The first equation

We will firstly generalize the homogeneous Maxwell's equations, namely, Gauss's Law for magnetism and Faraday's law of induction, to any manifold. This generalization will also unify both equations in a single one. We have the following equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Let us first consider the static case. We have just the following two equations:

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = 0$$

We notice that we are using the divergence and curl of vector fields in $\mathbb{R}^3$. In the language of differential geometry, as we saw in Definition 3.3.3, divergence becomes the exterior derivative on 2-forms on $\mathbb{R}^3$ and curl becomes the exterior derivative on 1-forms on $\mathbb{R}^3$. Thus, instead of treating the magnetic field as a vector field $\vec{B} = (B_x, B_y, B_z)$ and the electric field as a vector field $\vec{E} = (E_x, E_y, E_z)$, we will treat them as:

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

and

$$E = E_x dx + E_y dy + E_z dz$$

We define B this way because we can identify dx, dy and dz with $dy \wedge dz$, $dx \wedge dz$ and $dx \wedge dy$ in $\mathbb{R}^3$ as shown in Example 3.3.4. However, to do this, we must use a metric and an orientation over $\mathbb{R}^3$. It is this identification which introduces the right-hand rule in the cross product. Over $\mathbb{R}^3$, we have that both the spaces of 1-forms and 2-forms are 3-dimensional. The convention of the right-hand rule in the cross product arises from trying to identify this two different spaces using the Hodge star operator. However, this operator presupposes a metric and an orientation, which are the ones that will establish the right hand rule. Had we chosen another metric and orientation, different from the usual ones, we could as well have a left-hand rule for the cross product. This is why, when considering the magnetic field as a vector field, there exist two opposite consistent choices possible. The solution is to this apparent problem is realizing that the magnetic field is not a vector field, but a 2-form.

With this new expressions, the static case can be expressed just as:

$$dB = 0 \qquad dE = 0$$

We will now consider the general, time-dependant case. We must think about the magnetic and electric fields as living on spacetime. We will begin by working on the Minkowski spacetime, the manifold $\mathbb{R}^4$, using the standard coordinates (x^0, x^1, x^2, x^3). We will often write t for the time coordinate x^0 and (x, y, z) for the space coordinates (x^1, x^2, x^3). We say that index $\mu = 0$ is time-like and that indices $\mu = 1, 2, 3$ are space-like. We can combine both the electric field form E and the magnetic field form B in a 2-form in the following way:

$$F = B + E \wedge dt$$

We name this 2-form the **electromagnetic field**. Looking at the components, we have $F = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ or using the Einstein summation convention, $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. We can write the components in a matrix as:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

This way, the homogeneous Maxwell's equations become just

$$dF = 0$$

To prove this, let's remember that any k-form on can be locally written as

$$\omega|_U = \sum_{i_1<...<i_k} \omega_{j_1...j_k} dx^{j_1} \wedge ... \wedge dx^{j_k}$$

by Remark 2.7.10. Therefore, taking the global chart of $\mathbb{R}^n$, any form on $\mathbb{R}^n$ can be written as:

$$\omega = \omega_I dx^I$$

where $I = (i_1, ..., i_k)$ is a multi-index and $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_k}$, and where we are using the Einstein summation convention. We therefore have

$$d\omega = d\omega_I \wedge dx^I$$

since $d(dx^I) = 0$, and we get

$$d\omega = \partial_\mu \omega^I dx^\mu \wedge dx^I$$

Since the manifold is $\mathbb{R}^4$, we get

$$d\omega = \partial_\mu \omega_I dx^\mu \wedge dx^I = \partial_i \omega_I dx^i \wedge dx^I + \partial_0 \omega_I dx^0 \wedge dx^I = \partial_i \omega_I dx^i \wedge dx^I + dt \wedge \partial_t \omega$$

where $i \in \{1, 2, 3\}$, and writing $d_S\omega := \partial_i \omega_I dx^i \wedge dx^I$ for the "space-like part" of the form and $d_t\omega := dt \wedge \partial_t \omega$ for the "time-like part" of the form, we get

$$d\omega = d_S\omega + d_t\omega$$

Therefore, for the electromagnetic field F, the equation $dF = 0$ implies that

$$\begin{aligned} dF &= d(B + E \wedge dt) = dB + dE \wedge dt \\ &= d_S B + dt \wedge \partial_t B + (d_S E + dt \wedge \partial_t E) \wedge dt = d_S B + (\partial_t B + d_S E) \wedge dt = 0 \end{aligned}$$

Where we have used that $dt \wedge \partial_t B =$ The first term has no dt while the second one does. Also, the second term will vanish if and only if the expression inside the parenthesis is 0. Therefore, it follows that $dF = 0$ is equivalent to the following pair of equations,

$$d_S B = 0 \qquad \partial_t B + d_S E = 0$$

which are just the original Maxwell's equations in a different notation in the case of $\mathcal{M} = \mathbb{R}^4$.

We could now take any manifold $\mathcal{M}$ of any dimension n modelling spacetime, define the electromagnetic field to be a 2-form F on $\mathcal{M}$. The homogeneous Maxwell's equations would then be just $dF = 0$. Sometimes, but not always, we can split spacetime up in space and time, and we would have $\mathcal{M} = \mathbb{R} \times S$ for some $(n-1)$- manifold S representing space and $\mathbb{R}$ representing time. If this is possible, we can then write t for the usual coordinate on $\mathbb{R}$ and split electric and magnetic fields, and we can also split the exterior derivative in a space-like part and a time-like part, and we get back the original Maxwell's equations. We have arrived, then, at a more general way of expressing the homogeneous Maxwell's equations for any manifold, which in the Minkowskian case will reduce to our old equations.

Note, however, that electric and magnetic fields can only be defined after we choose how to split spacetime $\mathcal{M}$ up in space and time. $\mathcal{M}$ could be diffeomorphic to $\mathbb{R} \times S$ in a different number of ways for different S, or in no way at all. It is a lesson from special relativity that different inertial frames give different splittings of spacetime into $\mathbb{R} \times \mathbb{R}^3$, which are related by the Lorentz group of transformations. This means that electric and magnetic fields will be mixed up after a Lorentz transformation, recovering a fact that we mentioned before as a motivation for our rewriting of Maxwell's equations.

4.2 The second equation

We will now rewrite the inhomogeneous Maxwell's equations using differential geometry. It is now where finally the Hodge star operator will play a key role. Remember that all Maxwell's equations are:

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times \vec{B} - \frac{\partial E}{\partial t} = \vec{\jmath}$$

In order to rewrite the first pair of equations, the homogeneous one, we treated B as a 2-form and E as a 1-form. However, the second pair seems to treat E and B the other way around (up to a minus sign); taking the divergence of $\vec{E}$ would imply that it should be treated as a 2-form, and taking the curl of $\vec{B}$ would imply that it should be treated as a 1-form. We should therefore use what we have been building up to, the Hodge star operator, which in a 3-dimensional manifold $\mathcal{M}$ maps 1-forms to 2-forms and vice-versa. However, it comes at the price of choosing a metric and an orientation for the manifold $\mathcal{M}$.

We will first understand the effect of taking the Hodge dual of the electromagnetic field F, $F \mapsto \star F$. We will consider the manifold $(\mathbb{R}^4, \eta)$ - the Minkowski spacetime - whith its usual coordinates x^μ, $\mu \in \{0,1,2,3\}$ as our manifold for the following computations. The Minkowski metric η with this choice of coordinates (the global chart for all $\mathbb{R}^4$) is the following one:

$$\eta(v,w) = -v^0w^0 + v^1w^1 + v^2w^2 + v^3w^3$$

Since we can split the manifold as $\mathcal{M} = \mathbb{R}\times S = \mathbb{R}\times\mathbb{R}^3$ with $t := x^0$ the coordinate of time for $\mathbb{R}$ and $(x,y,z) := (x^1,x^2,x^3)$, we can write the electromagnetic field as $F = B + E\wedge dt$, where B is a time-dependant 2-form and E is a time-dependant 1-form. We have $F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$, and therefore

$$\star(F) = \star(\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu) = \frac{1}{2}F_{\mu\nu} \star (dx^\mu \wedge dx^\nu)$$

Taking into account that the signature of η is $\sigma(\eta) = (-+++)$ and the definition of the Hodge star operator, an easy but tedious computation shows that

$$\star(dx^0 \wedge dx^1) = -dx^2 \wedge dx^3 \quad \star(dx^0 \wedge dx^2) = dx^1 \wedge dx^3 \quad \star(dx^0 \wedge dx^3) = -dx^1 \wedge dx^2$$

$$\star(dx^1 \wedge dx^2) = dx^0 \wedge dx^3 \quad \star(dx^1 \wedge dx^3) = -dx^0 \wedge dx^2 \quad \star(dx^2 \wedge dx^3) = dx^0 \wedge dx^1$$

and therefore we have that the components of $\star F$ in the chosen basis are

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

In other words, taking the dual by the Hodge star operator of the electromagnetic field F amounts to doing the replacement $(E_i, B_i) \mapsto (-B_i, E)$, which is the symmetry which we named duality, and was one of the driving motivations to rewrite Maxwell equations. We have then found that the origin of the symmetry between the electric and magnetic field is rooted in the Hodge star operator. This duality is the main difference between the first pair of Maxwell's equations

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

and the second pair

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times \vec{E} - \frac{\partial \vec{E}}{\partial t} = \vec{\jmath}$$

The second difference between the two pairs is that the second one is not homogeneous: it contains ρ and $\vec{\jmath}$, which are non-zero in general. To get a differential form, we can transform the current density

$$\vec{\jmath} = j^1\partial_1 + j^2\partial_2 + j^3\partial_3$$

to a 1-form using the flat isomorphism:

$$j := \vec{\jmath}^{\,\flat} = j^1dx^1 + j^2dx^2 + j^3dx^3$$

Similarly we can combine the current density $\vec{\jmath}$ and the electric charge density ρ in a single vector field in the Minkowski spacetime:

$$\vec{J} = \rho\partial_0 + j^1\partial_1 + j^2\partial_2 + j^3\partial_3$$

and using the flat isomorphism given by the Minkowski metric we get

$$J := \vec{J}^{\,\flat} = -\rho dx^0 + j^1dx^1 + j^2dx^2 + j^3dx^3 = j - \rho dt$$

which we call the **current** in the Minkowski spacetime.

We claim that, in the same way that the homogeneous Maxwell's equations could be reduced to the equation $dF = 0$, the inhomogeneous Maxwell's equations can be generalized to the single equation:

$$\star d \star F = J$$

We will prove this for a more general space than the Minkowski space, which we were considering before. Consider any $\mathcal{M}$ manifold as the spacetime. Then the electromagnetic field over $\mathcal{M}$ is a 2-form F and the current J is a 1-form over $\mathcal{M}$. The first Maxwell equation is $dF = 0$. We must assume that $\mathcal{M}$ is a pseudo-Riemannian, oriented manifold in order to write the second Maxwell equation, $\star d\star F = J$. To introduce magnetic and electric fields we must assume that $\mathcal{M} = \mathbb{R}\times S$ where S is named the space. We can then write $F = B + E\wedge dt$ and $J = j - \rho dt$. Suppose that the space is 3-dimensional and that the metric over M is a static one of the form $\eta = -dt^2 +^3 g$, where ^3g is a Riemannian metric over S. Using the notation presented in the last section, we would then have that the $\mathcal{M}$ is a Lorentzian manifold.

Now, to prove what we want, we will split the Hodge star operator as we did with the exterior derivative. We will consider the operator $\star_S$, which is the Hodge star operator viewing the forms on $\mathcal{M}$ as time-dependant forms over S, where the operator is acting with the metric ^3g over S. We will firstly compute $\star F$ in terms of $\star_S$. Using linearity we get

$$\star F = \star B + \star(E \wedge dt)$$

The key realization here is that since every component of $E\wedge dt$ contains a dt, when we apply the $\star$ operator no components will contain it. The converse is true for B: since no terms contain dt, all terms of $\star B$ will. Doing a little computation using the Hodge star operator and factoring the common dt, we get

$$\star F = -\star_S B \wedge dt + \star_S E$$

which reformulates the electromagnetic duality as $(B, E) \mapsto (\star_S E, -\star_S B)$.

Now

$$d \star F = d(\star_S E - \star_S B \wedge dt) = d_S \star_S E + d_t \star_S E - d_S(\star_S B \wedge dt) - d_t(\star_S B \wedge dt)$$

and since

$$d_t(\star_S B \wedge dt) = dt \wedge \partial_t(\star_S B \wedge dt) = 0$$

we will get

$$\begin{aligned} d \star F &= d_S \star_S E + d_t \star_S E - d_S(\star_S B \wedge dt) \\ &= d_S \star_S E + \star_S \partial_t E \wedge dt - d_S \star_S B \wedge dt = d_S \star_S E + (\star_S \partial_t E - d_S \star_S B) \wedge dt \end{aligned}$$

Taking the Hodge star operator once again on both sides, the terms with dt will lose it and the term without it will gain a dt, and so we get

$$\star d \star F = - \star_S d_S \star_S E \wedge dt + (-\partial_t E + \star_S d_S \star_S B)$$

Finally, setting $\star d \star F = J$ and equating the terms we get

$$\star_S d_S \star_S E = \rho \qquad - \partial_t E + \star_S d_S \star_S B = j$$

which in the case of the Minkowski spacetime, are just the old acquaintances

$$\nabla \cdot \vec{E} = \rho \qquad \nabla \times B - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

In conclusion, for any pseudo-Riemannian, orientable manifold $(\mathcal{M}, \eta)$ where $\mathcal{M} = \mathbb{R} \times S$ and $\eta = -dt^2 +^3 g$ with S a 3-dimensional manifold and ^3g is a Riemannian metric over S, and where F is a 2-form over $\mathcal{M}$ and J is a 1-form over $\mathcal{M}$, the Maxwell's equations over $\mathcal{M}$ are

$$dF = 0$$

$$\star d \star F = J$$

Let us now discuss three interesting consequence of our newly formulated Maxwell's equations. Firstly, it is interesting to note that in the static case, where F is independent of t - or, equivalently, B and E are independent of t -, Maxwell's equations can be written as a pair involving only B

$$dB = 0 \qquad \star_S d \star_S B = j$$

and a pair involving only E

$$dE = 0 \qquad \star_S d \star_S E = \rho$$

This shows that only when the electric and magnetic field are time-dependant can they affect each other. It was Faraday who in 1831 first discovered that a changing magnetic field causes a nonzero curl in the electric field, being thus responsible for the $\frac{\partial \vec{B}}{\partial t}$ term in the Maxwell's equations.

In 1861 Maxwell hypothesized that a changing electric field should induce a nonzero curl on the magnetic field, guessing that there should be a $\frac{\partial \vec{E}}{\partial t}$ term too. It is only when both these effects are taken into account that we can explain electromagnetic radiation, in which changes in E produce changes in B and vice-versa, creating a wave which propagates through space: light.

Secondly, let's derive a crucial law from our equations. Let's consider the second Maxwell's equation, $\star d \star F = J$. Let's take the Hodge star operator on both sides to get $d \star F = \pm \star J$, where the sign depends on the value of $\star^2$ over 1-forms, i.e. on the metric of the manifold. Taking now the exterior derivative again, since $d^2 = 0$, we get

$$d \star J = 0$$

This equation is named the **continuity equation** and mathematically expresses the local conservation law for charge: not only is the total charge of the universe constant, but in order to move charge from one point to another, it must go through intermediate regions. In terms of components, this equation is written just as

$$\partial^\mu J_\mu = 0$$

which in the Minkowski space time can be written as

$$\frac{d\rho}{dt} + \nabla \cdot \vec{\jmath} = 0$$

This was the original formulation of the law, which motivated Maxwell to introduce the $\frac{\partial \vec{E}}{\partial t}$ term in the old Maxwell's equations in order to derive the local conservation of charge from the equations for electromagnetism. To derive it, we just have to take the divergence on both sides of Ampère's circuital law:

$$\nabla \cdot (\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t}) = \nabla \cdot \vec{\jmath}$$

Since $\nabla \cdot (\nabla \times \vec{B}) = 0$ and $\nabla \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = \frac{\partial \rho}{\partial t}$ by Schwarz's theorem and Gauss's Law for the electric field, we get the continuity law in its original formulation.

Thirdly, the first Maxwell equation, namely $dF = 0$ is specially charming: it is generally covariant. This means that it is independent of any fixed choice of metric or other geometrical structure on spacetime. More specifically, it implies that the equation is conserved by any diffeomorphism. Therefore, the equation $dF = 0$ is not only invariant under the Lorentz group of transformations, but under any kind of coordinate transformation.

Finally, it is worth noting that one can also express Lorentz's Force Law easily using differential forms as

$$f_\mu = q F^\mu_\nu u^\nu$$

where f_μ are the components of the electromagnetic force acting over a particle of charge q, F^μ_ν are the components of the electromagnetic 2-form and u^ν are the components of the quadri-velocity vector, given by $u = (mc, v^1, v^2, v^3)$ where c is the speed of light in vacuum and m the mass of that particle.

4.3 Potentials and gauge freedom

Let's now see the most notorious consequence of Maxwell's equations, which is immediate in this form, and which is usually unnecessarily mystified: gauge freedom. As we will see later on, this concept is key in modern physics.

This newly found equations are a more general, compact, tidy and elegant formulation of Maxwell's equations. An immediate consequence of the couple in question,

$$dF = 0 \qquad \star d \star F = J$$

is that if the electromagnetic field form F is exact - $F = dA$ for some 1-form A -, the first equation is automatically true by the exactness property of the exterior derivative, $dF = d(dA) = d^2 A = 0$, and the second equation is reduced to $\star d \star dA = J$. A 1-form A such that $F = dA$ is a **electromagnetic potential**. Notice that we say it is "a" electromagnetic potential and not "the" electromagnetic potential because A is not uniquely determined: indeed, if we have an electromagnetic potential which just differs from A by an exact 1-form, i.e. $A' = A + df$, it also satisfies that $dA' = F$, since $dA' = d(A + df) = dA + d^2 f = dA = F$. This way of changing A is called a **gauge transformation** and our freedom for choosing A is called **gauge freedom**.

It can be very convenient to use gauge freedom to make the vector potential satisfy handy extra conditions. Ending with these redundant degrees of freedom via fixing one of these conditions is called **choosing a gauge**.

The simplest gauge one can imagine is **temporal gauge**. If we suppose that we are on a Lorentzian spacetime such that $\mathcal{M} = \mathbb{R} \times S$, where S is the space, with a given metric $dt^2 - ^3g$, where ^3g is a Riemannian metric over S, and time t is a coordinate over $\mathbb{R}$. Differentiating with respect to t can be thought as the vector field ∂_t on $\mathcal{M}$. If the 1-form A on $\mathcal{M} = \mathbb{R} \times S$ satisfies

$$A(\partial_t) = 0$$

we say that A is in temporal gauge. For instance, in Minkowski spacetime $\mathbb{R}^4$, any 1-form can be written as

$$A = A_0 dt + A_1 dx + A_2 dy + A_3 dz$$

and temporal gauge is simply the condition that $A_0 = 0$. To keep notation simple we will define $A_0 = A(\partial_t)$ for any space-time of the form $\mathbb{R} \times S$, so that A is in temporal gauge if $A_0 = 0$.

Let's now show that given any exact 2-form F on $\mathcal{M} = \mathbb{R} \times S$, we can find some A in temporal gauge. Let's start with A, not necessarily in temporal gauge, such that $dA = F$. Let f be a function on $\mathcal{M} = \mathbb{R} \times S$ such that for any point $(t, p) \in \mathbb{R} \times S$,

$$f(t, p) = \int_0^t A_0(s, p) ds$$

and let

$$A' = A - df$$

We have that $dA' = f$ and that A' is in temporal gauge:

$$\begin{aligned} A'_0(t,p) = A'(\partial_t)(t,p) = A_0(\partial_t)(t,p) - (df(\partial_t))(t,p) = \\ = A_0(t,p) - (df(\partial_t))(t,p) = A_0(t,p) - (\partial_t f)(t,p) = \\ = A_0(t,p) - \partial \int_0^t A_0(s,p)ds = 0 \end{aligned}$$

Let us see how Maxwell's equations on $\mathcal{M} = \mathbb{R} \times S$ look like when the electromagnetic potential A is in temporal gauge. Since $A_0 = 0$ we can think of A as a 1-form on S that is a function of time. Since $F = B + E \wedge dt$ and

$$F = da = d_S A + d_t A = d_S A + dt \wedge \partial_t A$$

we have that

$$E = -\partial_t A \qquad B = d_S A$$

We will now rewrite Maxwell's equation in terms of the **Cauchy data** (A, E) on a space-like surface, a surface of the form $\{t\} \times S$. The first pair of Maxwell's equations

$$d_S B = 0 \qquad \partial_t B + d_S E = 0$$

become thus tautological in terms of A:

$$d_S^2 A = 0 \qquad \partial_t d_S A - d_S \partial_t A = 0$$

while the second pair becomes

$$\star_S d_S \star_S E = \rho \qquad -\partial_E + \star_S d_S \partial_S B = j$$

which describe the behaviour of the Cauchy data: the first equation - Gauss's Law - becomes a constraint that the Cauchy data must satisfy at any given time and the second equation together with the fact that $\partial_t A = -E$ describe the evolution of the Cauchy data with time:

$$\partial_t(A, E) = (-E, \star_S d_S \star_S d_S A - j)$$

With this, we can compute the Cauchy data any later or earlier time provided that we know it at time t. It is worth noting that as long as the continuity equation

$$\partial_t \rho + \star_S d_S \star_S j = 0$$

holds, Gauss's Law together with the evolutionary equation imply that Gauss's Law holds at later times, i.e. Gauss's Law is preserved in by time evolution. Lets proof this with a simple computation. Take Gauss's Law and derive it with respect to t:

$$\partial_t(\star_S d_S \star_S E - \rho) = 0$$

and now using the continuity equation $\partial_t \rho = -\star_S d_S \star_S j$ and the evolutionary equation $\partial_t E = \star_s d_S \star_S d_S A - j$ we get that

$$\star_S d_S \star_S (\star_S d_S \star_S d_S A - j) - \star_S d_S \star_S j = \pm \star_S d_S^2(\star_S d_S A) = 0$$

One should also note that an exact 2-form $F =$ on $\mathbb{R} \times S$, the 1-form A in temporal gauge such that $F = dA$ is not unique. In other words, there is still some gauge freedom. The reason is that if ω is any fixed, closed 1-form on space, $A' = A + \omega$ will again be a 1-form on $\mathbb{R} \times S$ that is in temporal gauge and has $dA' = F$. In particular, we can take $\omega = df$ fore some function f on space. Getting rid of the remaining gauge, for any reason we would want to do so, freedom would suppose more work.

Other well-known gauges in the Minkowski space-time are **Couloumb gauge**, which is equal to fixing

$$A(\partial_x + \partial_y + \partial_z) = \partial_i(A_i)^\sharp = A_1 + A_2 + A_3 = 0$$

and **Lorentz gauge** which is

$$A(-\partial_0 + \partial_x + \partial_y + \partial_z) = \partial_\mu(A_\mu)^\sharp = -A_0 + A_1 + A_2 + A_3 + A_4 = 0$$

using Einstein notation and the typical notation that Latin letters exclude the time-index and Greek letters range over all indices, i.e. $i \in \{1, 2, 3\}$ and $\mu \in \{0, 1, 2, 3\}$.

Gauge freedom may not seem that important at first sight, but the realization that Maxwell's equation possess this symmetry was one of the key advancements in the history of theoretical physics. This realization and a consequent generalization - gauge theory - gave rise to Yang-Mills equations (or more generally, Yang-Mills theory). Gauge theory made it possible to establish the standard model of particles using similar equations to Maxwell's equation, Yang-Mills equations. They can be very easily stated as just:

$$d_D F = 0 \qquad \star\, d_D \star F = J$$

which look just like Maxwell's equations with a sub-index. However, making sense of that sub-index requires constructions in differential geometry related to G-bundles, connections, curvature, holonomy, generalizing the exterior derivative and the Hodge operator, etc. The case of electromagnetism is special because it turns out that it is a specially easy case, an Abelian gauge theory using a relatively simple group, $U(1)$.

The main take-away we want to express is that a modern approach to Maxwell's equations provides the foundation of modern physics. With the exception of general relativity, all important theories of modern physics are quantized versions of Yang-Mills theory. These include quantum electrodynamics, the electroweak theory by Salam and Weinberg General and the standard model of particle physics. The most important of these theories is the standard model, which is what is called a Yang-Mills theory with an $U(1) \times SU(2) \times SU(3)$ gauge symmetry. General relativity also satisfies a what is called a gauge symmetry, even though it is not known if it can be cast as a Yang-Mills theory, being that a crucial problem in today's theoretical physics which, if solved, would be a great advance towards a Grand Unified Theory of physics.

4.4 Solutions to Maxwell's equations and the Aharonov-Bohm effect

As we have seen, we can find solutions for Maxwell's equations using the electromagnetic potential. However, can we get all solutions using this procedure? In our previous discussion, we were speaking about exact electromagnetic fields and we know that any exact form is closed. As we will see, the converse is not always true, and it gives rise, as we explained in the differential's form section, to de Rahm cohomology. Lets first introduce some terms.

A 0-form ϕ - also known as a function - such that $E = -d\phi$ is known as a **scalar potential** for E and a 1-form such that $B = dA$ is called a **vector potential** for B. Notice that the minus sign is just a convention.

We will now study when a 1-form is exact. Let S, space, be a manifold with a 1-form E such that $dE = 0$. Can we obtain a function ϕ such that $E = -d\phi$? We will attempt to do so integrating the 1-form E along a smooth path γ in S, i.e. a smooth map $\gamma : [0, T] \to S$. Technically we should have previously defined a manifold with a boundary to properly speak about a smooth map from $[0, T]$. However, since we will be integrating and the boundary of that manifold with boundary would be $0, T$, which is of measure 0, it will not affect our computations. We will denote

$$\int_\gamma E = \int_0^T E_{\gamma(t)}(\gamma'(t))dt$$

where $E_{\gamma(t)} = E(\gamma(t))$ is the 1-form evaluated at the point $\gamma(t)$ and $\gamma'(t) = \partial_t\gamma(t)$ is the tangent vector at the point $\gamma(t)$. We will define ϕ as follows: fix any point $p \in S$ and for any $q \in S$ let γ be some path from p to q and

$$\phi(q) = -\int_\gamma E$$

In the case of $S = \mathbb{R}^3$, this is how one would write a curl-free vector field as the gradient of a function.

Notice that in order to define a path from p to q, it must be that p and q are in the same arc-connected component of S. Since we are talking about space, we will further assume that S is arc-connected. Were it to be that there are different arc-connected components of space, we could just apply the following procedure to each arc-component independently.

Now, a watchful reader would have noticed that the details of that integral will in general depend on the details of the path γ and not just its endpoints $\gamma(0) = p$ and $\gamma(T) = q$. We will analyze this further: suppose that we have another path γ' from p to q and a smooth path homotopy between them, i.e. a smooth function $H : [0, 1] \times [0, T] \to S$ such that $H(s, 0) = p$ and $H(s, T) = q$ for any $s \in [0, T]$ and that $H(0, t) = \gamma$ and $H(1, t) = \gamma'$. Let's now see how

$$I_s = \int_0^T E_{\gamma(s,t)}\gamma'(s, t))dt = \int_s^T E_\mu(\gamma(s, t))\partial_t\gamma^\mu(s, t)dt$$

depends on s by differentiating:

$$\begin{aligned}\partial_s I_s = \int_0^T \partial_s[E_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t)]dt = \\ = \int_0^T [\partial_s E_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) + E_\mu(\gamma(s,t))\partial_s\partial_t\gamma^\mu(s,t)]dt = \\ = \int_0^T [\partial_s E_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) - \partial_t E_\mu(\gamma(s,t))\partial_s\gamma^\mu(s,t)]dt = \\ = \int_0^T \partial_\nu E_\mu(\gamma(s,t))[\partial_s\gamma^\nu\partial_t\gamma^\mu - \partial_t\gamma^\nu\partial_s\gamma^\mu]dt\end{aligned}$$

where we have used the product rule, integration by parts and the chain rule. Now, since $dE = (\partial_\mu E_\nu - \partial_\nu E_\mu)dx^\mu dx^\nu$ we obtain that

$$\partial_s I_s = \int_0^T (dE)_\mu\nu\partial_s\gamma^\mu\partial_t\gamma^\nu dt = 0$$

since $dE = 0$, and thus I_s is independent of s. We have proven that a closed 1-form has the same integral along any two homotopic paths. Therefore, given a manifold S which is arc-connected and simply connected - i.e. we can speak about the fundamental group without fixing a base-point and it has the trivial fundamental group, $\pi(X) = 0$ - we can define ϕ unambiguously for E. A reader which is interested in treating homotopy, the fundamental group of a topological space and simple connectedness more thoroughly should refer to [Hat02].

However, if S is not simply connected, we can find counter-examples for this. For instance, given $S = \mathbb{R}^2 - \{0\}$, the form $E = \frac{xdy - ydx}{x^2+y^2}$ integrated along the non-homotopic paths $\gamma_1 : [0,\pi] \to S$ defined by $\gamma_1(t) = (\cos t, \sin t)$ and $\gamma_2 : [0,\pi] \to S$ defined by $\gamma_2(t) = (\cos t, -\sin t)$ gives $\int_{\gamma_1} E = \pi$ and $\int_{\gamma_2} E = -\pi$.

Now, let's show that indeed $E = -d\phi$. To show that they agree at a point p, we have to show that they agree when applied to any tangent vector $v \in T_p\mathcal{M}$. Therefore we need to show that $E(v) = d\phi(v)$. To do this, lets pick a path $\gamma : [0,2] \to S$ with $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(1) = v$. We then have:

$$E(v) = E(\gamma'(1)) = \frac{d}{ds}\int_0^s E(\gamma'(t))dt\bigg|_{s=1} = -\frac{d}{ds}\phi(\gamma(s))\bigg|_{s=1} = -v(\phi)$$

using that the derivative of $\phi(\gamma(s))$ with respect to s is the same as the derivative of ϕ in the direction $\gamma'(s) = v$.

We could think of using the same procedure for finding solutions for the vector potential of the magnetic field. However, a quick reader will notice a crucial difference: we do not know how to integrate it, since it is not a function.

An omission we have been making is that k-forms are something that can be integrated over. We will not analyze this much further, since it would take many pages: defining manifolds with boundaries, how to integrate k-forms and stating and proving Stokes' theorem. We will take all of these for granted in order to briefly speak about the Aharonov-Bohm effect, which shows the importance of the vector potential in electromagnetism, specially on a quantum setting. An interested reader - and they should indeed be - should check the reference [Fra12] for a thorough mathematical understanding of the topic.

The setting behind the Aharonov-Bohm is how it follows. We will first consider cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$. We notice that z is a smooth function over $\mathbb{R}^3$, so dz is a form defined over all $\mathbb{R}^3$, but r is only smooth away from the z-axis, i.e. $r = 0$, so dr is only defined away from this axis. Moreover, *theta* is not well-defined over the z-axis and only up to modulo 2π. Nonetheless it is tradition to define the 1-form

$$d\theta = \frac{xdy - ydx}{x^2 + y^2}$$

even if it is not an exact form. It is, however, a closed form. Let's consider a solenoid by winding a wire around a cylinder in a tight spiral, with the cylinder centered at the z-axis. If current j flows through the wire, one obtains a constant magnetic field inside the solenoid and zero magnetic field outside, i.e.

$$B = f(r) r dr \wedge d\theta$$

where f is constant for $r < R$ and zero for $r > R$. We also obtain

$$\star B = f(r) dz$$

and the vector potential

$$A = g(r) d\theta$$

with

$$g(r) = \int_0^r s f(s) ds$$

that fulfills $dA = B$. One can find the proof of this computation in [BM94]. In particular, outside the solenoid we have

$$A = \frac{\Phi}{2\pi} d\theta$$

where Φ is the magnetic flux through the solenoid, i.e. the integral of the B 2-form over the disc $r \leq R$ in any plane of constant z:

$$\Phi = \int_D B = \int_D f(r) r dr \wedge d\theta = 2\pi \int_0^R f(r) r dr$$

Remember that we haven't learnt how to integrate 2-forms, but the basic idea is that the part $dr \wedge d\theta$ becomes the $drd\theta$ inside the integral and that we integrate the "function part" of the form, $f(r)r$ over the area and the differentials $drd\theta$. Remember also that A is not the unique vector potential such that $dA = B$ by gauge freedom, but we only want one vector potential for this discussion.

The Aharonov-Bohm effect occurs when a charged particle passes around the solenoid. It is a purely quantum effect, so we will briefly explain the ideas we will need.

In quantum mechanics, physical systems are described by wave-functions, unit vectors in some Hilbert space $\mathcal{H}$. The inner product of $\mathcal{H}$ is related to the probabilistic nature of quantum mechanics as it follows. Suppose that there is a system represented by the state $\psi \in \mathcal{H}$ and that an experiment is performed to check if the system is in a state $\phi \in \mathcal{H}$. The probability to find that the system is in the state ϕ in that experiment is

$$|\langle\phi|\psi\rangle|^2$$

which is called the transition probability. Also, while we represent states as unit vectors, it is important to note that if two states ψ and ψ' differ by a phase, i.e. if $\psi' = e^{i\theta}\psi$ for $\theta \in \mathbb{R}$, they describe the same state, because transition probabilities are not affected by the change: for all $\phi \in \mathcal{H}$, we have

$$|\langle\phi|\psi'\rangle|^2 = \langle\phi|\psi\rangle|^2$$

Now, suppose we have a particle in $\mathbb{R}^3$ with electric charge q. In classical mechanics, a particle moves along some path γ in $\mathbb{R}^3$ and there is a function called the Lagrangian, $\mathcal{L} = \mathcal{L}(\gamma(t), \gamma'(t))$, describing the particle which depends on the particle's position and velocity. If we consider the particle's path from time 0 to T and integrate the Lagrangian over this interval of time we get a quantity called the action, S,

$$S = \int_0^T \mathcal{L}dt$$

In classical mechanics, a particle going from a point p at time 0 to a point q at time T will always follow a path which is a critical point of the action! It often implies that the path minimizes the action, but it is not always the case.

In quantum mechanics, the Lagrangian also plays an important role. However, particles in quantum mechanics do not move through a path. Now suppose that a particle starts at a state ψ at $t = 0$ and we wish to compute its state ϕ at some other time $t = T$. Suppose also at first that there is no magnetic field, Let

$$\mathcal{P} = \{\gamma : [0, T] \to \mathbb{R}^3 : \gamma(0) = a,\ \gamma(T) = b\}$$

denote the space of all paths that start at a at time 0 and end at b at time T. Then, as stated by Feynmann's path-integral quantum-mechanics formulation

$$\phi(b) = \int_{\mathcal{P}} e^{\frac{i}{\hbar}S(\gamma)}\psi(a)\mathcal{D}_\gamma$$

where $\mathcal{D}_\gamma$ is a kind of measure on the space $\mathcal{P}$ (of which we will not enter in detail, which would require another 50 pages of mathematics) and $\hbar$ is Planck's constant. In words, we can think that the particle takes all possible paths weighted by a factor of $e^{\frac{i}{\hbar}S(\gamma)}$. One can show that as $\hbar \to 0$, this phase factor oscillates rapidly near the paths that are critical points, cancelling out in such a way that only the classical path contributes.

Now, funnily enough, suppose that there is a magnetic field B on $\mathbb{R}^3$ with a vector potential A, and they are independent of time. The new path integral should be

$$\phi(b) = \int_{\mathcal{P}} e^{(\frac{i}{\hbar}s(\gamma) - q\int_\gamma A)}\psi(a)\mathcal{D}_\gamma$$

In other words, a new phase factor appears:

$$e^{-\frac{i}{\hbar}q\int_\gamma A}$$

In the case which $a = b$, the extra phase factor is just

$$e^{-\frac{i}{\hbar}q\oint_\gamma A}$$

or if γ bounds the disk D, by Stokes' theorem,

$$e^{-\frac{i}{\hbar}q\int_D B}$$

This phase does indeed have physical effects: it can differ for different loops producing constructive or destructive interference in the path integral. Now, in the previous solenoid setting, consider an electron completely excluded by the solenoid, which is taken to be of radius $\frac{1}{2}$. Since the electron is excluded from the solenoid, we may as well take space to be $S = \mathbb{R}^3 - \{r \leq \frac{1}{2}\}$, which is not simply connected. Now, the magnetic field vanishes in S, but A does not. Suppose that we send an electron from $a = (-1, 0, 0)$ to $b = (1, 0, 0)$ in S. Since it is quantum-mechanical, the electron will take any path in S from a to b.

However, due to the vector potential, the electron can pick different phases depending on the path it takes from a to b. This gives rise to interference, which is the Aharonov-Bohm effect. In short, in quantum mechanics the vector potential can affect the wavefunction in a significant way even when the magnetic field is 0 in the region!

To see this precisely, notice that since the vector potential (up to gauge freedom) is $A = \frac{\Phi}{2\pi}d\theta$, the phase factor $e^{-\frac{i}{\hbar}q\int_\gamma A}$ equals $e^{\frac{-iq\Phi}{2\hbar}}$ on the path of module 1 at one side of the solenoid and $e^{\frac{iq\Phi}{2\hbar}}$ at the other. By adjusting a proper A, the factors are i and $-i$. By symmetry, every path from a to b has its reflected path which cancels its factor, and thus the path-integral vanishes.

In other words, for the right value of Φ it is impossible for an electron to go from point a to point b because it interferes with itself, and all due to the vector potential A, while $B = 0$. This effect is commonly observed and the basis for the superconducting quantum interference devices, which measure magnetic flux accurately.

The crucial hidden property here is that the space S is not simply connected. This is what allows the integral along different paths to take different values. While real space is indeed connected, the space accessible by the electron is not, so Aharonov-Bohm effect can be easily understood using a model of space which is not simply connected, which gives rise to a non-exact differential form which makes the electron interfere with itself so it can never go from one point to another one.

Bibliography

[BM94] John Baez and Javier P. Muniain, *Gauge fields, knots and gravity*, Series on Knots and Everything, vol. 4, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.

[Fra12] Theodore Frankel, *The geometry of physics*, third ed., Cambridge University Press, Cambridge, 2012, An introduction.

[Gol98] Samuel I. Goldberg, *Curvature and homology*, Dover Publications, Inc., Mineola, NY, 1998, Revised reprint of the 1970 edition.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Lee09] Jeffrey M. Lee, *Manifolds and differential geometry*, Graduate Studies in Mathematics, vol. 107, American Mathematical Society, Providence, RI, 2009.

[Spi79] M. Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish, Inc., Wilmington, Del., 1979.

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983, Corrected reprint of the 1971 edition.