TRACES OF THE NEVANLINNA CLASS ON DISCRETE SEQUENCES

A. HARTMANN, X. MASSANEDA, A. NICOLAU

ABSTRACT. We show that a discrete sequence Λ of the unit disk is the union of n interpolating sequences for the Nevanlinna class $\mathcal{N}$ if and only if the trace of $\mathcal{N}$ on Λ coincides with the space of functions on Λ for which the divided differences of order $n-1$ are uniformly controlled by a positive harmonic function.

1. DEFINITIONS AND STATEMENT

This note deals with some properties of the classical *Nevanlinna class* consisting of the holomorphic functions in the unit disk $\mathbb{D}$ for which $\log_+|f|$ has a positive harmonic majorant. We denote by $\operatorname{Har}_+(\mathbb{D})$ the set of non-negative harmonic functions in $\mathbb{D}$. Equivalently,

$$\mathcal{N} = \left\{f \in \operatorname{Hol}(\mathbb{D}) : \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\, d\theta < \infty\right\}.$$

Definition. A discrete sequence of points Λ in $\mathbb{D}$ is called *interpolating for* N (denoted $\Lambda \in \operatorname{Int}\mathcal{N}$) if the trace space $N|\Lambda$ is ideal, or equivalently, if for every $v \in \ell^\infty$ there exists $f \in \mathcal{N}$ such that

$$f(\lambda_n) = v_n, \quad n \in \mathbb{N}.$$

Interpolating sequences for the Nevanlinna class were first investigated by Naftalevitch [6]. A rather complete study was carried out much later in [4]. Let B denote the Blaschke product associated to a Blaschke sequence Λ. Let

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar\lambda z} \quad \text{and} \quad B_\lambda(z) = \frac{B(z)}{b_\lambda(z)}\,.$$

Let's also consider the pseudohyperbolic distance in $\mathbb{D}$, defined as

$$\rho(z,w) = \left|\frac{z-w}{1-\bar z w}\right|\,,$$

and the corresponding pseudohyperbolic disks $D(z,r) = \{w \in \mathbb{D} : \rho(z,w) < r\}$.

According to [4, Theorem 1.2] $\Lambda \in \operatorname{Int}\mathcal{N}$ if and only if there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that

$$(1) \qquad |B_\lambda(\lambda)| = (1-|\lambda|)|B'(\lambda)| \geq e^{-H(\lambda)}, \quad \lambda \in \Lambda\,.$$

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Moreover in such case the trace space is

$$\mathcal{N}(\Lambda) = \Big\{\{\omega(\lambda)\}_{\lambda\in\Lambda} : \exists H \in \mathrm{Har}_+(\mathbb{D})\ ,\ \log_+|\omega(\lambda)| \leq H(\lambda),\ \lambda \in \Lambda\Big\}\ .$$

Other properties and characterizations of Nevanlinna interpolating sequences have been given recently in [3]. In these terms $\Lambda \in \mathrm{Int}\,\mathcal{N}$ when for every sequence $\omega(\Lambda) \in \mathcal{N}(\Lambda)$ there exists $f \in \mathcal{N}$ such that $f(\lambda) = \omega(\lambda)$, $\lambda \in \Lambda$. In terms of the restriction operator

$$\begin{aligned} \mathcal{R}_\Lambda : \mathcal{N} &\longrightarrow \mathcal{N}(\Lambda) \\ f &\mapsto \{f(\lambda)\}_{\lambda\in\Lambda}, \end{aligned}$$

Λ is interpolating when $\mathcal{R}_\Lambda(\mathcal{N}) = \mathcal{N}(\Lambda)$.

Definition 1.1. Let Λ be a discrete sequence in $\mathbb{D}$ and ω a function given on Λ. The *pseudohyperbolic divided differences of* ω are defined by induction as follows

$$\Delta^0\omega(\lambda_1) = \omega(\lambda_1)\ ,$$

$$\Delta^j\omega(\lambda_1,\dots,\lambda_{j+1}) = \frac{\Delta^{j-1}\omega(\lambda_2,\dots,\lambda_{j+1}) - \Delta^{j-1}\omega(\lambda_1,\dots,\lambda_j)}{b_{\lambda_1}(\lambda_{j+1})} \qquad j \geq 1.$$

For any $n \in \mathbb{N}$, denote

$$\Lambda^n = \{(\lambda_1,\dots,\lambda_n) \in \Lambda \times \overset{n}{\overbrace{\cdots}} \times \Lambda\ :\ \lambda_j \neq \lambda_k \text{ if } j \neq k\},$$

and consider the set $X^{n-1}(\Lambda)$ consisting of the functions defined in Λ with divided differences of order $n-1$ uniformly controlled by a positive harmonic function H i.e., such that for some $H \in \mathrm{Har}_+(\mathbb{D})$,

$$\sup_{(\lambda_1,\dots,\lambda_n)\in\Lambda^n} |\Delta^{n-1}\omega(\lambda_1,\dots,\lambda_n)| e^{-[H(\lambda_1)+\cdots+H(\lambda_n)]} < +\infty\ .$$

Lemma 1.2. *Let $n \in \mathbb{N}$. For any sequence $\Lambda \subset \mathbb{D}$, we have $X^n(\Lambda) \subset X^{n-1}(\Lambda) \subset \cdots \subset X^0(\Lambda) = \mathcal{N}(\Lambda)$.*

Proof. Assume that $\omega(\Lambda) \in X^n(\Lambda)$, that is,

$$\sup_{(\lambda_1,\dots,\lambda_{n+1})\in\Lambda^{n+1}} \left|\frac{\Delta^{n-1}\omega(\lambda_2,\dots,\lambda_{n+1}) - \Delta^{n-1}\omega(\lambda_1,\dots,\lambda_n)}{b_{\lambda_1}(\lambda_{n+1})}\right| e^{-[H(\lambda_1)+\cdots+H(\lambda_{n+1})]}\ <\ \infty\ .$$

Then, given $(\lambda_1,\dots,\lambda_n) \in \Lambda^n$ and taking $\lambda_1^0,\dots,\lambda_n^0$ from a finite set (for instance the n first $\lambda_j^0 \in \Lambda$ different of all λ_j) we have

$$\begin{aligned} \Delta^{n-1}\omega(\lambda_1,\dots,\lambda_n) = &\frac{\Delta^{n-1}\omega(\lambda_1,\dots,\lambda_n) - \Delta^{n-1}\omega(\lambda_1^0,\lambda_1,\dots,\lambda_{n-1})}{b_{\lambda_1^0}(\lambda_n)} b_{\lambda_1^0}(\lambda_n) + \\ &+ \frac{\Delta^{n-1}\omega(\lambda_1^0,\lambda_1,\dots,\lambda_{n-1}) - \Delta^{n-1}\omega(\lambda_2^0,\lambda_1^0,\dots,\lambda_{n-2})}{b_{\lambda_2^0}(\lambda_{n-1})} b_{\lambda_2^0}(\lambda_{n-1}) + \cdots + \\ &\frac{\Delta^{n-1}\omega(\lambda_{n-1}^0,\dots,\lambda_1^0,\lambda_1) - \Delta^{n-1}\omega(\lambda_n^0,\dots,\lambda_1^0)}{b_{\lambda_n^0}(\lambda_1)} b_{\lambda_n^0}(\lambda_1) + \Delta^{n-1}\omega(\lambda_n^0,\dots,\lambda_1^0) \end{aligned}$$

Since $\omega \in X^{n-1}(\Lambda)$ there exists $H \in \operatorname{Har}_+(\mathbb{D})$ and a constant $K(\lambda_1^0,\dots,\lambda_n^0)$ such that

$$\begin{aligned}\left|\Delta^{n-1}\omega(\lambda_1,\dots,\lambda_n)\right| &\le e^{H(\lambda_1^0)+H(\lambda_1)\cdots+H(\lambda_n)}\rho(\lambda_1^0,\lambda_n)+e^{H(\lambda_1^0)+H(\lambda_2^0)\cdots+H(\lambda_{n-1})}\rho(\lambda_2^0,\lambda_{n-1})+\\ &+\cdots+e^{H(\lambda_1^0)+\cdots+H(\lambda_n^0)+H(\lambda_1)}\rho(\lambda_n^0,\lambda_1)+\Delta^{n-1}\omega(\lambda_n^0,\dots,\lambda_1^0)\\ &\le K(\lambda_1^0,\dots,\lambda_n^0)\,e^{H(\lambda_1)+\cdots+H(\lambda_n)},\end{aligned}$$

and the statement follows. ■

The main result of this note is modelled after Vasyunin's description of the sequences Λ in $\mathbb{D}$ such that the trace of the algebra of bounded holomorphic functions H^∞ on Λ equals the space of pseudohyperbolic divided differences of order n (see [7], [8]). Similar results hold also for Hardy spaces (see [1] and [2]) and the Hörmander algebras, both in $\mathbb{C}$ and in $\mathbb{D}$ [5]. The analogue in our context is the following.

Main Theorem. *The identity $\mathcal{N}|\Lambda = X^{n-1}(\Lambda)$ holds if and only if Λ is the union of n interpolating sequences for $\mathcal{N}$.*

2. General properties

Throughout the proofs we will use repeatedly the well-known *Harnack inequalities*: for $H \in \operatorname{Har}_+(\mathbb{D})$ and $z, w \in \mathbb{D}$,

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \le \frac{H(z)}{H(w)} \le \frac{1+\rho(z,w)}{1-\rho(z,w)}\,.$$

We shall always assume, without loss of generality, that $H \in \operatorname{Har}_+(\mathbb{D})$ is big enough so that for $z \in D(\lambda, e^{-H(\lambda)})$ the inequalities $1/2 \le H(z)/H(\lambda) \le 2$ hold. Actually it is sufficient to assume $\inf\{H(z) : z \in \mathbb{D}\} \ge \log 3$.

We begin by showing that one of the inclusions of the Main Theorem is inmediate.

Proposition 2.1. *For all $n \in \mathbb{N}$, the inclusion $\mathcal{N}|\Lambda \subset X^{n-1}(\Lambda)$ holds.*

Proof. Let $f \in \mathcal{N}$. Let us show by induction on $j \ge 1$ that there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that

$$|\Delta^{j-1} f(z_1,\dots,z_j)| \le e^{H(z_1)+\cdots+H(z_j)} \qquad \text{for all } (z_1,\dots,z_j) \in \mathbb{D}^j.$$

As $f \in \mathcal{N}$, there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that $|\Delta^0 f(z_1)| = |f(z_1)| \le e^{H(z_1)}$, $z_1 \in \mathbb{D}$.

Assume that the property is true for j and let $(z_1,\dots,z_{j+1}) \in \mathbb{D}^{j+1}$. Fix $z_1,\dots,z_j$ and consider z_{j+1} as the variable in the function

$$\Delta^j f(z_1,\dots,z_{j+1}) = \frac{\Delta^{j-1} f(z_2,\dots,z_{j+1}) - \Delta^{j-1} f(z_1,\dots,z_j)}{b_{z_1}(z_{j+1})}.$$

By the induction hypothesis, there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that

$$|\Delta^j f(z_1,\dots,z_{j+1})| \le \frac{1}{\rho(z_1,z_{j+1})}\big(e^{H(z_2)+\cdots+H(z_{j+1})} + e^{H(z_1)+\cdots+H(z_j)}\big).$$

If $\rho(z_1,z_{j+1}) \ge 1/2$ we get directly

$$|\Delta^j f(z_1,\dots,z_{j+1})| \le 4e^{H(z_1)+\cdots+H(z_{j+1})}\,,$$

and choosing for instance $\tilde{H} = H + \log 4$ we get the desired estimate.

If $\rho(z_1, z_{j+1}) \leq 1/2$ we apply the maximum principle and Harnack's inequalities

$$\begin{aligned}|\Delta^j f(z_1, \dots, z_{j+1})| &\leq \sup_{\xi:\rho(\xi,z_{j+1})=1/2} |\Delta^j f(z_1, \dots, z_j, \xi_{j+1})| \\ &\leq \sup_{\xi:\rho(\xi,z_{j+1})=1/2} 4e^{H(z_1)+\cdots+H(z_j)+H(\xi)} \\ &\leq 4e^{2[H(z_1)+\cdots+H(z_j)+H(z_{j+1})]}.\end{aligned}$$

Choosing here $\tilde{H} = 2H + \log 4$ we get the desired estimate. ■

Definition 2.2. A sequence Λ is *weakly separated* if there exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that the disks $D(\lambda, e^{-H(\lambda)})$, $\lambda \in \Lambda$, are pairwise disjoint.

Remark 2.3. If Λ is weakly separated then $X^0(\Lambda) = X^n(\Lambda)$, for all $n \in \mathbb{N}$.

By Lemma 1.2, to see this it is enough to prove (by induction) that $X^0(\Lambda) \subset X^n(\Lambda)$ for all $n \in \mathbb{N}$.

For $n = 0$ this is trivial.

Assume now that $X^0(\Lambda) \subset X^{n-1}(\Lambda)$ and take $\omega(\Lambda) \in X^0(\Lambda)$. Since $\rho(\lambda_1, \lambda_{n+1}) \geq e^{-H_0(\lambda_1)}$ for some $H_0 \in \mathrm{Har}_+(\mathbb{D})$ we have

$$\begin{aligned}|\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &= \left| \frac{\Delta^{n-1}\omega(\lambda_2, \dots, \lambda_{n+1}) - \Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)}{b_{\lambda_1}(\lambda_{n+1})} \right| \\ &\leq e^{H_0(\lambda_1)} \left(e^{H(\lambda_2)+\cdots+H(\lambda_{n+1})} + e^{H(\lambda_1)+\cdots+H(\lambda_n)} \right)\end{aligned}$$

for some $H \in \mathrm{Har}_+(\mathbb{D})$. Taking $\tilde{H} = H + H_0$ we are done.

Lemma 2.4. *Let $n \geq 1$. The following assertions are equivalent:*

(a) *Λ is the union of n weakly separated sequences,*
(b) *There exist $H \in \mathrm{Har}_+(\mathbb{D})$ such that*

$$\sup_{\lambda \in \Lambda} \#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] \leq n .$$

(c) $X^{n-1}(\Lambda) = X^n(\Lambda)$.

Proof. (a) ⇒(b). This is clear, by the weak separation.

(b) ⇒(a). We proceed by induction on $j = 1, \dots, n$. For $j = 1$, it is again clear by the definition of weak separation. Assume the property true for $j-1$. Let $H \in \mathrm{Har}_+(\mathbb{D})$, $\inf\{H(z) : z \in \mathbb{D}\} \geq \log 3$, be such that $\sup_{\lambda\in\Lambda} \#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] \leq j$. We split the sequence $\Lambda = \Lambda_a \cup \Lambda_b$ where

$$\begin{aligned}\Lambda_a &= \bigcup_{\{\lambda\in\Lambda:\#(\Lambda\cap D(\lambda,e^{-10H(\lambda)}))=j\}} (\Lambda \cap D(\lambda, e^{-10H(\lambda)})) \\ \Lambda_b &= \Lambda \setminus \Lambda_a\end{aligned}$$

Now, for every $\lambda \in \Lambda_b$, we have $\#(\Lambda \cap D(\lambda, e^{-10H(\lambda)})) \leq j-1$, and by the induction hypothesis, Λ_b splits into $j - 1$ separated sequences $\Lambda_1, \dots, \Lambda_{j-1}$.

In the case $\lambda \in \Lambda_a$, there is obviously no point in the annulus $D(\lambda, e^{-H(\lambda)}) \setminus D(\lambda, e^{-10H(\lambda)})$ which means that the j points in $D(\lambda, e^{-10H(\lambda)}))$ are far from the other points of Λ. So we can add each one of these j points in a weakly separated way to one of the sequences $\Lambda_1, \dots, \Lambda_{j-1}$, and the j-th point in a new sequence Λ_j (which is of course weakly separated since the groups $\Lambda \cap D(\lambda, e^{-10H(\lambda)})$ appearing in Λ_a are weakly separated).

(b)⇒(c). It remains to see that $X^{n-1}(\Lambda) \subset X^n(\Lambda)$. Given $\omega(\Lambda) \in X^{n-1}(\Lambda)$ and points $(\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}$, we have to estimate $\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})$. Under the assumption (b), at least one of these $n+1$ points is not in the disk $D(\lambda_1, e^{-H(\lambda_1)})$. Note that Λ^n is invariant by permutation of the $n+1$ points, thus we may assume that $\rho(\lambda_1, \lambda_{n+1}) \geq e^{-H(\lambda_1)}$. Using the fact that $\omega(\Lambda) \in X^{n-1}(\Lambda)$, there exists $H_0 \in \mathrm{Har}_+(\mathbb{D})$ such that

$$\begin{aligned} |\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &\leq \frac{|\Delta^{n-1}\omega(\lambda_2, \dots, \lambda_{n+1})| + |\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)|}{\rho(\lambda_1, \lambda_{n+1})} \\ &\leq e^{H(\lambda_1)} \left(e^{H_0(\lambda_2)+\cdots+H_0(\lambda_{n+1})} + e^{H_0(\lambda_1)+\cdots+H_0(\lambda_n)} \right) \\ &\leq 2 e^{H(\lambda_1)} e^{H_0(\lambda_1)+\cdots+H_0(\lambda_{n+1})} \, . \end{aligned}$$

Taking $\tilde{H} = H_0 + H + \log 2$ we get the desired estimate.

(c)⇒(b). We prove this by contraposition. Assume that for all $H \in \mathrm{Har}_+(\mathbb{D})$ there exists $\lambda \in \Lambda$ such that

$$\#[\Lambda \cap D(\lambda, e^{-H(\lambda)})] > n \, . \tag{2}$$

Consider the partition of $\mathbb{D}$ into the dyadic squares

$$Q_{k,j} = \left\{ z = re^{i\theta} \in \mathbb{D} : 1 - 2^{-k} \leq r < 1 - 2^{-k-1} \, , \, j\frac{2\pi}{k} \leq \theta < (j+1)\frac{2\pi}{k} \right\},$$

where $k \geq 0$ and $j = 0, \dots 2^k - 1$.

Let $\Lambda_{k,j} = \Lambda \cap Q_{k,j}$ and

$$r_{k,j} = \inf\{ r > 0 \ : \ \exists \lambda \in \Lambda_{k,j} \ : \ \#(\Lambda \cap \overline{D(\lambda, r)}) \geq n+1 \}.$$

Take $\alpha_{k,j} \in \Lambda_{k,j}$ such that $\#(\Lambda \cap \overline{D(\alpha_{k,j}, r_{k,j})}) \geq n+1$.

Claim: For all $H \in \mathrm{Har}_+(\mathbb{D})$,

$$\inf_{k,j} \frac{r_{k,j}}{e^{-H(\alpha_{k,j})}} = 0 \, .$$

To see this assume otherwise that there exist $H \in \mathrm{Har}_+(\mathbb{D})$ and $\eta > 0$ with

$$\frac{r_{k,j}}{e^{-H(\alpha_{k,j})}} \geq \eta \, .$$

In particular, by Harnack's inequalities,

$$\log \frac{1}{r_{k,j}} \leq 3H(z) + \log(\frac{1}{\eta}), \quad z \in Q_{k,j}. \tag{3}$$

Let $\tilde{H} := \log(2/\eta) + 4H \in \mathrm{Har}_+(\mathbb{D})$. By the hypothesis (2) there exist $k_0 \geq 0$, $j_0 \in \{0, \dots, 2^{k_0} - 1\}$, $\lambda_{k_0,j_0} \in \Lambda_{k_0,j_0}$ such that

$$\#\left[\Lambda \cap \overline{D(\lambda_{k_0,j_0}, e^{-\tilde{H}(\lambda_{k_0,j_0})})}\right] \geq n + 1 \; .$$

In particular, by definition of $r_{k,j}$, we have $r_{k_0,j_0} \leq e^{-\tilde{H}(\lambda_{k_0,j_0})}$, that is

$$\log \frac{1}{r_{k_0,j_0}} \geq \tilde{H}(\lambda_{k_0,j_0}) = \log(\frac{2}{\eta}) + 4H(\lambda_{k_0,j_0}),$$

which contradicts (3).

Now take a separated sequence $\mathcal{L} \subset \{\alpha_{k,j}\}_{k,j}$ for which the disks $D(\alpha, r_\alpha)$, $\alpha \in \mathcal{L}$, are disjoint, where for $\alpha = \alpha_{k,j} \in \mathcal{L}$ we denote $r_\alpha = r_{k,j}$. Given $\alpha \in \mathcal{L}$, let $\lambda_1^\alpha, \dots, \lambda_n^\alpha$ be its n nearest (not necessarily unique) points, arranged by increasing distance. Notice that $\rho(\alpha, \lambda_n^\alpha) = r_\alpha$.

In order to construct a sequence $\omega(\Lambda) \in X^{n-1}(\Lambda) \setminus X^n(\Lambda)$, put

$$\begin{cases} \omega(\alpha) = \prod_{j=1}^{n-1} b_\alpha(\lambda_j^\alpha), & \text{for all } \alpha \in \mathcal{L} \\ \omega(\lambda) = 0 & \text{if } \lambda \in \Lambda \setminus \mathcal{L}. \end{cases}$$

To see that $\omega(\Lambda) \in X^{n-1}(\Lambda)$ let us estimate $\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)$ for any given $(\lambda_1, \dots, \lambda_n) \in \Lambda^n$. By the separation conditions on $\mathcal{L}$, we know that none of the λ_j^α is in $\mathcal{L}$. Hence, we may assume that at most one of the points is in $\mathcal{L}$. On the other hand, it is clear that $\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n) = 0$ if all the points are in $\Lambda \setminus \mathcal{L}$. Thus, taking into account that Δ^{n-1} is invariant by permutations, we will only consider the case where λ_n is some $\alpha \in \mathcal{L}$ and $\lambda_1, \dots, \lambda_{n-1}$ are in $\Lambda \setminus \mathcal{L}$. In that case,

$$|\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_{n-1}, \alpha)| = |\omega(\alpha)| \prod_{j=1}^{n-1} \rho(\alpha, \lambda_j)^{-1} = \prod_{j=1}^{n-1} \frac{\rho(\alpha, \lambda_j^\alpha)}{\rho(\alpha, \lambda_j)} \leq 1,$$

as desired.

On the other hand, a similar computation yields

$$|\Delta^n \omega(\lambda_1^\alpha, \dots, \lambda_n^\alpha, \alpha)| = |\omega(\alpha)| \prod_{j=1}^{n} \rho(\alpha, \lambda_j^\alpha)^{-1} = \rho(\alpha, \lambda_n^\alpha)^{-1} = r_\alpha^{-1}.$$

The Claim above prevents the existence of $H \in \mathrm{Har}_+(\mathbb{D})$ such that

$$r_\alpha^{-1} = |\Delta^n \omega(\lambda_1^\alpha, \dots, \lambda_n^\alpha, \alpha)| e^{-(H(\lambda_1^\alpha) + \cdots + H(\lambda_n^\alpha) + H(\alpha))} \leq C \; ,$$

since otherwise, again by Harnack's inequalities, we would have

$$r_\alpha^{-1} \leq e^{3(n+1)H(\alpha)}, \quad \alpha \in \mathcal{L} \; .$$

■

It is clear from the characterization (1) of interpolating sequences for $\mathcal{N}$ that such sequences must be weakly separated. The previous result gives another way of showing it.

Corollary 2.5. *If Λ is an interpolating sequence, then it is weakly separated.*

Proof. If Λ is an interpolating sequence, then $\mathcal{N}|\Lambda = X^0(\Lambda)$. On the other hand, by Proposition 2.1, $\mathcal{N}|\Lambda \subset X^1(\Lambda)$. Thus $X^0(\Lambda) = X^1(\Lambda)$. We conclude by the preceding lemma applied to the particular case $n = 1$. ■

The covering provided by the following result will be useful.

Lemma 2.6. *Let $\Lambda_1, \dots, \Lambda_n$ be weakly separated sequences. There exist $H \in \mathrm{Har}_+(\mathbb{D})$, positive constants α, β, a subsequence $\mathcal{L} \subset \Lambda_1 \cup \dots \cup \Lambda_n$ and disks $D_\lambda = D(\lambda, r_\lambda)$, $\lambda \in \mathcal{L}$, such that*

(i) $\Lambda_1 \cup \dots \cup \Lambda_n \subset \cup_{\lambda \in \mathcal{L}} D_\lambda$,
(ii) $e^{-\beta H(\lambda)} \le r_\lambda \le e^{-\alpha H(\lambda)}$ *for all* $\lambda \in \mathcal{L}$,
(iii) $\rho(D_\lambda, D_{\lambda'}) \ge e^{-\beta H(\lambda)}$ *for all* $\lambda, \lambda' \in \mathcal{L}$, $\lambda \ne \lambda'$.
(iv) $\#(\Lambda_j \cap D_\lambda) \le 1$ *for all* $j = 1, \dots, n$ *and* $\lambda \in \mathcal{L}$.

Proof. Let $H \in \mathrm{Har}_+(\mathbb{D})$ be such that

$$\rho(\lambda, \lambda') \ge e^{-H(\lambda)}, \qquad \forall \lambda, \lambda' \in \Lambda_j,\ \lambda \ne \lambda',\ \forall j = 1, \dots, n\,. \tag{4}$$

We will proceed by induction on $k = 1, \dots, n$ to show the existence of a subsequence $\mathcal{L}_k \subset \Lambda_1 \cup \dots \cup \Lambda_k$ such that:

$$(i)_k \quad \Lambda_1 \cup \dots \cup \Lambda_k \subset \cup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k),$$
$$(ii)_k \quad e^{-\beta_k H(\lambda)} \le R_\lambda^k \le e^{-\alpha_k H(\lambda)},$$
$$(iii)_k \quad \rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)) \ge e^{-\beta_k H(\lambda)} \text{ for any } \lambda, \lambda' \in \mathcal{L}_k,\ \lambda \ne \lambda'.$$

Then it suffices to chose $\mathcal{L} = \mathcal{L}_n$, $\alpha = \alpha_n$, $\beta = \beta_n$, $r_\lambda = R_\lambda^n$. The weak separation and the fact that $r_\lambda < e^{-H(\lambda)}/3$ implies that $\#\Lambda_j \cap D(\lambda, r_\lambda) \le 1$, $j = 1, \dots, k$, hence the lemma follows.

For $k = 1$, the property is clearly verified with $\mathcal{L}_1 = \Lambda_1$ and $R_\lambda^1 = e^{-CH(\lambda)}$, with C big enough so that $(iii)_1$ holds ($C = 3$, for instance). Properties $(i)_1$, $(ii)_1$ follow immediately.

Assume the property true for k and split $\mathcal{L}_k = \mathcal{M}_1 \cup \mathcal{M}_2$ and $\Lambda_{k+1} = \mathcal{N}_1 \cup \mathcal{N}_2$, where

$$\mathcal{M}_1 = \{\lambda \in \mathcal{L}_k \ :\ D(\lambda, R_\lambda^k + 1/4\, e^{-\beta_k H(\lambda)}) \cap \Lambda_{k+1} \ne \emptyset\},$$
$$\mathcal{N}_1 = \Lambda_{k+1} \cap \bigcup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k + 1/4\, e^{-\beta_k H(\lambda)}),$$
$$\mathcal{M}_2 = \mathcal{L}_k \setminus \mathcal{M}_1,$$
$$\mathcal{N}_2 = \Lambda_{k+1} \setminus \mathcal{N}_1.$$

Now, we put $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \mathcal{N}_2$ and define the radii R_λ^{k+1} as follows:

$$R_\lambda^{k+1} = \begin{cases} R_\lambda^k + 1/4\, e^{-\beta_k H(\lambda)} & \text{if } \lambda \in \mathcal{M}_1, \\ R_\lambda^k & \text{if } \lambda \in \mathcal{M}_2, \\ 1/8\, e^{-\beta_k H(\lambda)} & \text{if } \lambda \in \mathcal{N}_2. \end{cases}$$

It is clear that $(i)_{k+1}$ holds:

$$\Lambda_1 \cup \dots \cup \Lambda_{k+1} \subset \bigcup_{\lambda \in \mathcal{L}_{k+1}} D(\lambda, R_\lambda^{k+1})\,.$$

Also, by the induction hypothesis,

$$\frac{1}{8}e^{-\beta_k H(\lambda)} \leq R_\lambda^{k+1} \leq e^{-\alpha_k H(\lambda)} + \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

Thus, to see $(ii)_{k+1}$ there is enough to choose $\alpha_{k+1}, \beta_{k+1}$ such that

$$e^{-\alpha_k H(\lambda)} + 1/4\, e^{-\beta_k H(\lambda)} \leq e^{-\alpha_{k+1} H(\lambda)},$$

for instance $\alpha_{k+1} = \alpha_k - 1$, and

$$1/8\, e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}\ , \tag{5}$$

that is $\beta_{k+1} H(\lambda) \geq \beta_k H(\lambda) + \log 8$. Assuming without loss of generality that $H(\lambda) \geq \log 8$, there is enough choosing $\beta_{k+1} \geq \beta_k + 1$.

In order to prove $(iii)_k$ take now $\lambda, \lambda' \in \mathcal{L}_{k+1}$, $\lambda \neq \lambda'$. Notice that

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^{k+1} - R_{\lambda'}^{k+1}.$$

Split into four different cases:

1. $\underline{\lambda, \lambda' \in \mathcal{L}_k}$. Assume without loss of generality that $H(\lambda) \leq H(\lambda')$. Then, by the definition of R_λ^{k+1}, we see that

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^k - R_{\lambda'}^k - \frac{1}{4}e^{-\beta_k H(\lambda)} - \frac{1}{4}e^{-\beta_k H(\lambda')}.$$

By inductive hypothesis

$$\rho(\lambda, \lambda') - R_\lambda^k - R_{\lambda'}^k = \rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)) \geq e^{-\beta_k H(\lambda)}\ .$$

Thus, by (5),

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-\beta_k H(\lambda)} - \frac{1}{2}e^{-\beta_k H(\lambda)} = \frac{1}{2}e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}.$$

2. $\underline{\lambda, \lambda' \in \mathcal{N}_2}$. Assume also $H(\lambda) \leq H(\lambda')$. Condition (4) implies $\rho(\lambda, \lambda') \geq e^{-H(\lambda)}$, hence

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-H(\lambda)} - \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

If $\beta_k \geq 2$, by (5) we have

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq e^{-2H(\lambda)} \geq e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}.$$

3. $\underline{\lambda \in \mathcal{M}_1, \lambda' \in \mathcal{N}_2}$ By definition of $\mathcal{M}_1$ there exists $\beta \in \mathcal{N}_1$ such that

$$\rho(\lambda, \beta) \leq R_\lambda^k + \frac{1}{4}e^{-\beta_k H(\lambda)}.$$

Then, using (4) on $\beta, \lambda' \in \Lambda_{k+1}$, we have, by Harnack's inequalities (if $\beta_k \geq 4$),

$$\begin{aligned}\rho(\lambda, \lambda') &\geq \rho(\beta, \lambda') - \rho(\lambda, \beta) \geq e^{-H(\beta)} - R_\lambda^k - \frac{1}{4}e^{-\beta_k H(\lambda)} \geq e^{-2H(\lambda)} - \frac{5}{4}e^{-\beta_k H(\lambda)} \\ &\geq e^{-4H(\lambda)} \geq e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)}\ .\end{aligned}$$

4. $\underline{\lambda \in \mathcal{M}_2, \lambda' \in \mathcal{N}_2}$. Taking into account the definition of $R_\lambda^{k+1}, R_{\lambda'}^{k+1}$ we have

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = \rho(\lambda, \lambda') - R_\lambda^k - \frac{1}{8} e^{-\beta_k H(\lambda)}$$

Since

$$\rho(\lambda, \lambda') - R_\lambda^k \geq \rho(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)),$$

by inductive hypothesis and by (5)

$$\rho(D(\lambda, R_\lambda^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq \frac{1}{4} e^{-\beta_k H(\lambda)} - \frac{1}{8} e^{-\beta_k H(\lambda)} \geq e^{-\beta_{k+1} H(\lambda)} .$$

All together, it is enough to start with $C > n$, define $\alpha_1 = \beta_1 = C$, and then define α_k, β_k inductively by

$$\alpha_{k+1} = \alpha_k - 1 = \cdots = C - k , \qquad \beta_{k+1} = \beta_k + 1 = \cdots = C + k .$$

■

3. Proof of Main Theorem. Necessity

Assume $\mathcal{N}|\Lambda = X^{n-1}(\Lambda)$, $n \geq 2$. Using Proposition 2.1, we have $X^{n-1}(\Lambda) = X^n(\Lambda)$, and by Lemma 2.4 we deduce that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$, where $\Lambda_1, \ldots, \Lambda_n$ are weakly separated sequences. We want to show that each Λ_j is an interpolating sequence.

Let $\omega(\Lambda_j) \in \mathcal{N}(\Lambda_j) = X^0(\Lambda_j)$. Let $\cup_{\lambda \in \mathcal{L}} D_\lambda$ be the covering of Λ given by Lemma 2.6. We extend $\omega(\Lambda_j)$ to a sequence $\omega(\Lambda)$ which is constant on each $D_\lambda \cap \Lambda_j$ in the following way:

$$\omega_{|D_\lambda \cap \Lambda} = \begin{cases} 0 & \text{if } D_\lambda \cap \Lambda_j = \emptyset \\ \omega(\alpha) & \text{if } D_\lambda \cap \Lambda_j = \{\alpha\} . \end{cases}$$

We verify by induction that the extended sequence is in $X^{k-1}(\Lambda)$ for all $k \leq n$. It is clear that it belongs to $X^0(\Lambda)$.

Assume that $\omega \in X^{k-2}(\Lambda)$, $k \geq 2$, and consider $(\alpha_1, \ldots, \alpha_k) \in \Lambda^k$. If all the points are in the same D_λ then $\Delta^{k-1}\omega(\alpha_1, \ldots, \alpha_k) = 0$, so we may assume that $\alpha_1 \in D_\lambda$ and $\alpha_k \in D_{\lambda'}$ with $\lambda \neq \lambda'$. Then we have, for some $H_0 \in \operatorname{Har}_+(\mathbb{D})$,

$$\rho(\alpha_1, \alpha_k) \geq e^{-\beta H_0(\alpha_1)}, \qquad k \neq 1.$$

With this and the induction hypothesis it is clear that for some $H \in \operatorname{Har}_+(\mathbb{D})$,

$$\begin{aligned} |\Delta^{k-1}\omega(\alpha_1, \ldots, \alpha_k)| &= \left| \frac{\Delta^{k-2}\omega(\alpha_2, \ldots, \alpha_k) - \Delta^{k-2}\omega(\alpha_1, \ldots, \alpha_{k-1})}{b_{\alpha_1}(\alpha_k)} \right| \\ &\leq e^{\beta H_0(\alpha_1)} \left(e^{H(\alpha_2) + \cdots + H(\alpha_k)} + e^{H(\alpha_1) + \cdots + H(\alpha_{k-1})} \right) . \end{aligned}$$

Taking for instance $\tilde{H} = H + \beta H_0 + \log 2$ we get

$$|\Delta^{k-1}\omega(\alpha_1, \ldots, \alpha_k)| \leq e^{\tilde{H}(\alpha_1) + \cdots + \tilde{H}(\alpha_k)} ,$$

thus $\omega(\Lambda) \in X^{k-1}(\Lambda)$. By assumption there exist $f \in \mathcal{N}$ interpolating the values $\omega(\Lambda)$. In particular f interpolates $\omega(\Lambda_j)$.

4. Proof of the Main Theorem. Sufficiency

Assume $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$, where $\Lambda_j \in \operatorname{Int} \mathcal{N}$, $j = 1, \ldots, n$, and denote $\Lambda_j = \{\lambda_k^{(j)}\}_{k \in \mathbb{N}}$. Denote also by B_j the Blaschke product with zeros on Λ_j. We will use the following property of the Nevanlinna interpolating sequences (see Theorem 1.2 in [3]).

Lemma 4.1. *Let $\Lambda \in \operatorname{Int} N$ and let B the Blaschke product associated to Λ. There exists $H_1 \in \operatorname{Har}_+(\mathbb{D})$ such that*

$$|B(z)| \geq e^{-H_1(z)} \rho(z, \Lambda) \qquad z \in \mathbb{D} .$$

According to Proposition 2.1 we only need to see that $X^{n-1}(\Lambda) \subset \mathcal{N}|\Lambda$. Let then $\omega(\Lambda) \in X^{n-1}(\Lambda)$ and split it

$$\{\omega(\lambda)\}_{\lambda \in \Lambda} = \{\omega_k^{(1)}\}_{k \in \mathbb{N}} \cup \cdots \cup \{\omega_k^{(n)}\}_{k \in \mathbb{N}} ,$$

where $\omega_k^{(j)} = \omega(\lambda_k^{(j)})$, $j = 1, \ldots, n$, $k \in \mathbb{N}$. By Lemma 1.2 and the hypothesis $\{\omega_k^{(1)}\}_{k \in \mathbb{N}} \in X^0(\Lambda_1)$, hence there exists $f_1 \in \mathcal{N}$ such that

$$f_1(\lambda_k^{(1)}) = \omega_k^{(1)} , \qquad k \in \mathbb{N} .$$

In order to interpolate also the values $\{\omega_k^{(2)}\}_k$ consider functions of the form

$$f_2(z) = f_1(z) + B_1(z) g_2(z) .$$

Immediately $f_2(\lambda_k^{(1)}) = f_1(\lambda_k^{(1)}) = \omega_k^{(1)}$, $k \in \mathbb{N}$, and we will have $f_2(\lambda_k^{(2)}) = \omega_k^{(2)}$ as soon as we find $g_2 \in \mathcal{N}$ such that

$$g_2(\lambda_k^{(2)}) = \frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} , k \in \mathcal{N} .$$

Since $\Lambda_2 \in \operatorname{Int} \mathcal{N}$ such g_2 will exist as soon as the sequence in the right hand side is majorized by a sequence of the form $\{e^{H(\lambda_k^{(2)})}\}_k$.

Given $\lambda_k^{(2)} \in \Lambda_2$ pick $\lambda_k^{(1)}$ such that $\rho(\lambda_k^{(2)}, \Lambda_1) = \rho(\lambda_k^{(2)}, \lambda_k^{(1)})$. There is no restriction in assuming that $\rho(\lambda_k^{(2)}, \lambda_k^{(1)}) \leq 1/2$. Then, by Lemma 4.1 there exists $H_1 \in \operatorname{Har}_+(\mathbb{D})$ such that

$$|B_1(\lambda_k^{(2)})| \geq e^{-H_1(\lambda_k^{(2)})} \rho(\lambda_k^{(1)}, \lambda_k^{(2)}) \qquad k \in \mathbb{N}.$$

Now, since $f_1(\lambda_k^{(1)}) = \omega_k^{(1)}$ we have

$$\begin{aligned} \left| \frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| &\leq \left| \frac{\omega_k^{(2)} - \omega_k^{(1)}}{B_1(\lambda_k^{(2)})} \right| + \left| \frac{f_1(\lambda_k^{(1)}) - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})} \right| \\ &\leq \Big(\Delta^1(\omega_k^{(1)}, \omega_k^{(2)}) + \Delta^1(f_1(\lambda_k^{(1)}), f_1(\lambda_k^{(2)})) \Big) e^{H_1(\lambda_k^{(2)})} . \end{aligned}$$

By hypothesis, and since $f_1 \in \mathcal{N}$, there exists $H_2 \in \operatorname{Har}_+(\mathbb{D})$ such that

$$\Delta^1(\omega_k^{(1)}, \omega_k^{(2)}) + \Delta^1(f_1(\lambda_k^{(1)}), f_1(\lambda_k^{(2)})) \leq e^{H_2(\lambda_k^{(1)}) + H_2(\lambda_k^{(2)})},$$

and therefore, by Harnack's inequalities,

$$\left|\frac{\omega_k^{(2)} - f_1(\lambda_k^{(2)})}{B_1(\lambda_k^{(2)})}\right| \leq e^{H_2(\lambda_k^{(1)})+H_2(\lambda_k^{(2)})} e^{H_1(\lambda_k^{(2)})} \leq e^{3(H_1+H_2)(\lambda_k^{(2)})},$$

In general, assume that we have $f_{n-1} \in \mathcal{N}$ such that

$$f_{n-1}(\lambda_k^{(j)}) = \omega_k^{(j)} \qquad k \in \mathbb{N},\ j = 1, \ldots, n-1\ .$$

We look for a function $f_n \in \mathcal{N}$ interpolating the whole Λ of the form

$$f_n = f_{n-1} + B_1 \cdots B_{n-1} g_n\ .$$

We need then $g_n \in \mathcal{N}$ with

$$g_n(\lambda_k^{(n)}) = \frac{\omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)})}{B_1(\lambda_k^{(n)}) \cdots B_{n-1}(\lambda_k^{(n)})}, \qquad k \in \mathbb{N}\ .$$

Let us see that the sequence of values in the right hand side of this identity have a majorant of the form $\{e^{H(\lambda_k^{(n)})}\}_k$.

Pick $\lambda_k^{(j)} \in \Lambda_j$, $j = 1, \ldots, n-1$ such that $\rho(\lambda_k^{(n)}, \Lambda_j) = \rho(\lambda_k^{(n)}, \lambda_k^{(j)})$. There is no restriction in assuming that $\rho(\lambda_k^{(n)}, \lambda_k^{(j)}) \leq 1/2$. Since $f_{n-1}(\lambda_k^{(j)}) = \omega_k^{(j)}$, $j = 1, \ldots, n-1$, an immediate computation shows that

$$\omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)}) = \Big[\Delta^{n-1}(\omega_k^{(1)}, \ldots, \omega_k^{(n-1)}, \omega_k^{(n)}) - \\ -\Delta^{n-1}(f_{n-1}(\lambda_k^{(1)}), \ldots, f_{n-1}(\lambda_k^{(n-1)}), f_{n-1}(\lambda_k^{(n)}))\Big]\ b_{\lambda_k^{(1)}}(\lambda_k^{(n)}) \cdots b_{\lambda_k^{(n-1)}}(\lambda_k^{(n)})\ .$$

Again by Lemma 4.1, there exists $H_1 \in \operatorname{Har}_+(\mathbb{D})$ such that

$$|B_j(\lambda_k^{(n)})| \geq e^{-H_1(\lambda_k^{(n)})} \rho(\lambda_k^{(j)}, \lambda_k^{(n)})\ , k \in \mathbb{N},\ j = 1, \ldots, n-1.$$

Hence, by hypothesis and the fact that $f_{n-1} \in \mathcal{N}$ there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that

$$\left|\frac{\omega_k^{(n)} - f_{n-1}(\lambda_k^{(n)})}{B_1(\lambda_k^{(n)}) \cdots B_{n-1}(\lambda_k^{(n)})}\right| \leq [|\Delta^{n-1}(\omega_k^{(1)}, \ldots, \omega_k^{(n)})| + |\Delta^{n-1}(f_{n-1}(\lambda_k^{(1)}), \ldots, f_{n-1}(\lambda_k^{(n)}))|]\ e^{(n-1)H_1(\lambda_k^{(n)})} \\ \leq e^{H(\lambda_k^{(1)}) + \cdots + H(\lambda_k^{(n-1)}) + H(\lambda_k^{(n)}) + (n-1)H_1(\lambda_k^{(n)})}\ .$$

Finally, by Harnack's inequalities, this is bounded by $e^{2n(H(\lambda_k^{(n)}) + H_1(\lambda_k^{(n)}))}$.

REFERENCES

[1] Bruna, J.; Nicolau, A.; Øyma, K. *A note on interpolation in the Hardy spaces of the unit disc*. Proc. Amer. Math. Soc. 124 (1996), no. 4, 1197–1204.

[2] Hartmann, A. *Une approche de l'interpolation libre gnralise par la thorie des oprateurs et caractrisation des traces $H^p|\Lambda$. (French) [An approach to generalized free interpolation using operator theory and characterization of the traces $H^p|\Lambda$.]* J. Operator Theory 35 (1996), no. 2, 281-316.

[3] Hartmann, A., Massaneda, X., Nicolau, A. *Finitely generated Ideals in the Nevanlinna class*. arXiv:1605.08160.

[4] Hartmann, A., Massaneda, X., Nicolau, A., Thomas, P. *Interpolation in the Nevanlinna and Smirnov classes and harmonic majorants*. J. Funct. Anal. 217 (2004), no. 1, 1-37.

[5] Massaneda, X. Ortega-Cerdà, J., Ounaïes, M. *Traces of Hörmander algebras on discrete sequences*. Analysis and Mathematical Physics. Birkhäuser Verlag (2009) 397–408.

[6] Naftalevič, A.G., *On interpolation by functions of bounded characteristic (Russian)*, Vilniaus Valst. Univ. Mokslų Darbai. Mat. Fiz. Chem. Mokslų **Ser. 5** (1956), 5–27.

[7] Vasyunin, V. I. *Traces of bounded analytic functions on finite unions of Carleson sets (Russian)*. Investigations on linear operators and the theory of functions, XII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 31–34.

[8] Vasyunin, V. I. *Characterization of finite unions of Carleson sets in terms of solvability of interpolation problems (Russian)*. Investigations on linear operators and the theory of functions, XIII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **135** (1984), 31–35.

A.HARTMANN: UNIVERSITÉ DE BORDEAUX, IMB, 351 COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE

E-mail address: Andreas.Hartmann@math.u-bordeaux1.fr

X.MASSANEDA: UNIVERSITAT DE BARCELONA, DEPARTAMENT DE MATEMÀTIQUES I INFORMÀTICA AND BGSMATH, GRAN VIA 585, 08007-BARCELONA, SPAIN

E-mail address: xavier.massaneda@ub.edu

A.NICOLAU: UNIVERSITAT AUTÒNOMA DE BARCELONA, DEPARTAMENT DE MATEMÀTIQUES, EDIFICI C, 08193-BELLATERRA, SPAIN

E-mail address: artur@mat.uab.cat