Highly Adiabatic Time-Optimal Quantum Driving at Low Energy Cost

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Time-efficient control schemes for manipulating quantum systems are of great importance in quantum technologies, where environmental forces rapidly degrade the quality of pure states over time. In this Letter, we formulate an approach to time-optimal control that circumvents the boundary-value problem that plagues the quantum brachistochrone equation at the expense of relaxing the form of the control Hamiltonian. In this setting, a coupled system of equations, one for the control Hamiltonian and another one for the duration of the protocol, realizes an ansatz-free approach to quantum control theory. We show how driven systems, in the form of a Landau-Zener type Hamiltonian, can be efficiently maneuvered to speed up a given state transformation in a highly adiabatic manner and with a low energy cost.

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Introduction.—Engineering a suitable Hamiltonian that evolves a given initial quantum state into a selected target state is essential in technologies such as quantum information processing [1,2], quantum simulation [3,4], and quantum sensing [5,6]. Inspired by Pontryagin's maximum principle [7,8], this problem has traditionally been addressed using optimal quantum control theory, where state transformations are implemented using either open-loop [9,10] or measurement-based [11–14] protocols. In most approaches to quantum control, however, the elapsed time over which the evolution is performed does not enter as a parameter to optimize and is typically "preset" before applying any optimization algorithm. Within this framework, the job reduces to parametrically optimizing a fixed form of the control Hamiltonian to reach maximum fidelity [15–18].

In a time-optimal version of quantum control theory, maximum fidelity transformations are sought in the least possible time to reduce the impact of decoherence, which rapidly degrades the quality of quantum states in quantum information processing [19,20]. Following a number of precursor works [21–25], a formal time-optimal version of quantum control theory was formulated by Carlini and coworkers [26–28] in analogy to Bernoulli's classical brachistochrone problem [29], and has since then been referred to as the quantum brachistochrone problem. If the time evolution has no constraints, the solution to the quantum brachistochrone problem reduces to finding the time-independent Hamiltonian that generates maximum speed of evolution along a geodesic [30–34]. In general, however, there can be a number of limitations that prohibits the implementation of such an elementary solution, specially for open quantum systems [35–37]. In closed systems, restrictions typically come from the available forms of the control Hamiltonian, which may yield a difficult-to-solve boundary-value problem [38]. Even if the form of the control Hamiltonian can be relaxed, limitations may still arise in situations where the system is immersed in an external field that cannot be altered by the controller. In analogy with the classical problem posed by Zermelo [29], this last situation has been often called the quantum Zermelo navigation problem [39–42].

The quantum Zermelo problem has been recently addressed for systems in a time-independent driving force and for initial and target states with at most a real overlap [43,44]. However, a general solution for transforming between arbitrary states in the presence of a time-dependent drift Hamiltonian has not been proven. Here, we show that this problem can be cast in the form of a system of two coupled equations, viz., one for the control Hamiltonian and another for the associated duration of the protocol, also known as the quantum speed limit formula. Importantly, we do not impose a preset form of the control Hamiltonian in

our derivation, and hence the resulting scheme lends itself as an "ansatz-free" approach to quantum control. This is illustrated for the case of a two-level system immersed in a Landau-Zener type drift Hamiltonian. In this context, the proposed control scheme embodies a low energy cost alternative to so-called transitionless driving [45–47].

Time-optimal control in a time-dependent drift Hamiltonian.—The quantum Zermelo navigation problem can be posed as follows. Given an initial state $|\psi_i\rangle$ and a time-dependent drift Hamiltonian $H_0(t)$, we aim to find the time-optimal control Hamiltonian $H_c(t)$ such that the total Hamiltonian $H(t) = H_0(t) + H_c(t)$ drives $|\psi_i\rangle$ to a desired final state $|\psi_f\rangle$ in the least possible time τ according to the time-dependent Schrödinger equation, $i(d/dt)|\psi(t)\rangle = H(t)|\psi(t)\rangle$ (atomic units are used throughout). The form of the drift Hamiltonian is not manipulable, while the control Hamiltonian is only constrained to fulfill two conditions: (i) it has to have a finite energy bandwidth, and (ii) it is restricted to a subspace of Hermitian operators.

The above problem can be recast in the interaction picture of quantum mechanics by writing initial and final states respectively as $|\psi_i'\rangle = |\psi_i\rangle$ and $|\psi_f'\rangle = \mathcal{U}_0^\dagger(\tau)|\psi_f\rangle$, where $\mathcal{U}_0(t) = \mathcal{T}\exp(-i\int_0^t H_0(t_1)dt_1)$ and $\mathcal{T}$ defines the usual time ordering operator. For the particular case where initial and final states live in a two-dimensional Hilbert space $\mathcal{H}_2$, a simple derivation of the control Hamiltonian, $H_c'(t) = \mathcal{U}_0^\dagger(t)H_c(t)\mathcal{U}_0(t)$, can be obtained using geometric arguments on the Bloch sphere, as shown in Sec. I of the Supplemental Material [48]. For a general N-dimensional setting, we shall adopt an approach in the projective Hilbert space $\mathbb{P}(\mathcal{H}_N)$ [49]. Specifically, we use the Riemannian metric of Fubini and Study [51,52], which in the interaction picture reads as

$$d\mathcal{L}_{\mathrm{FS}}^2 = 4\left(\frac{\langle d\psi'|d\psi'\rangle}{\langle\psi'|\psi'\rangle} - \frac{|\langle\psi'|d\psi'\rangle|^2}{\langle\psi'|\psi'\rangle^2}\right), \tag{1}$$

with $|\psi'(t)\rangle = \mathcal{U}_0^\dagger(t)|\psi(t)\rangle$. Introducing the time-dependent Schrödinger equation into Eq. (1) allows one to set the length of the path followed by a normalized quantum state, i.e.,

$$\mathcal{L}_{\mathrm{FS}} = 2\int_0^\tau \Delta H_c'(t)dt, \tag{2}$$

where $\Delta H_c'(t)$ denotes the standard deviation of the control Hamiltonian $H_c'(t)$. Since the minimum distance path (or geodesic) between two states lies entirely in the subspace spanned by these two states [32,53], a state evolving along this path necessarily reads as

$$|\psi'(t)\rangle = \eta(t)|\psi_i\rangle + \zeta(t)|\bar{\psi}_f'\rangle, \tag{3}$$

where $|\bar{\psi}_f'\rangle = (1/\sqrt{1-s^2})(|\psi_f'\rangle - se^{i\beta}|\psi_i\rangle)$ (with $\langle\psi_i|\psi_f'\rangle = se^{i\beta}$) is the orthonormalized final state in the interaction picture. The Fubini-Study distance along a geodesic, $\mathcal{L}_{\mathrm{FS}}^{\min}$, is obtained by inserting Eq. (3) into Eq. (1), integrating $d\mathcal{L}_{\mathrm{FS}}$ to obtain $\mathcal{L}_{\mathrm{FS}}$ and then minimizing this length. This process yields (see, e.g., Appendix A of Ref. [54])

$$\mathcal{L}_{\mathrm{FS}}^{\min}(i,f') = 2\arccos(|\langle\psi_i|\psi_f'\rangle|). \tag{4}$$

Now, a state having evolved along a geodesic must obey $\mathcal{L}_{\mathrm{FS}} = \mathcal{L}_{\mathrm{FS}}^{\min}(i,f')$. Furthermore, if we impose the "transversality" condition

$$\langle\psi'(t)|H_c'(t)|\psi'(t)\rangle = i\langle\psi'(t)|\dot{\psi}'(t)\rangle = 0, \tag{5}$$

then the integrand of Eq. (2) takes its highest value (i.e., $\Delta H_c' = \sqrt{\langle\psi'(t)|H_c'^2(t)|\psi'(t)\rangle}$), thereby minimizing the upper limit of the integral τ. The above transversality condition allows us to infer a functional form of $H_c'(t)$, which is given by

$$H_c'(t) = i\left(\frac{d|\psi'(t)\rangle}{dt}\langle\psi'(t)| - |\psi'(t)\rangle\frac{d\langle\psi'(t)|}{dt}\right). \tag{6}$$

To express the above Hamiltonian in terms of initial and final states we shall first fix the specific form of the state in Eq. (3). For that, we here define the complex functions $\eta(t) = \cos\theta(t)$ and $\zeta(t) = e^{-i\beta}\sin\theta(t)$ [where $\theta(t) = \int_0^t \Delta H_c'(t')dt'$] and refer the interested reader to a more detailed derivation in Sec. II of the Supplemental Material [48]. Introducing Eq. (3) into Eq. (6) we then obtain

$$H_c'(t) = \frac{iv_z(t)}{\sqrt{1-s^2}}(e^{-i\beta}|\psi_f'\rangle\langle\psi_i| - e^{i\beta}|\psi_i\rangle\langle\psi_f'|), \tag{7}$$

where we have defined the "velocity" $v_z(t) = \sqrt{|\dot{\psi}'(t)|^2} = \Delta H_c'(t)$, with the second equality coming from the transversality condition.

The role of the $v_z(t)$ can be elucidated by noting that, due to the structure of the Hamiltonian in Eq. (6), $2[\Delta H_c'(t)]^2 = \mathrm{tr}[H_c'^2(t)] = \mathrm{tr}[H_c^2(t)]$, where the last equality follows from the cyclic property of the trace. This property allows us to write $v_z(t) = \sqrt{\mathrm{tr}[H_c^2(t)]/2} = \|H_c(t)\|$, which establishes a clear-cut relation between the velocity $v_z(t)$ and the Hilbert-Schmidt (or Frobenius) norm of the control Hamiltonian, also known as the energy resource of the control [55,56]. In this respect, by equating Eqs. (2) and (4) we obtain $\int_0^\tau v_z(t)dt = \arccos|\langle\psi_i|\psi_f'\rangle|$.

To minimize the protocol time τ, we can further impose the "full throttle" condition that the energy disposal of the control is held constant at the maximum attainable value, i.e., $v_z(t) = v_z$. This last condition allows us to find an equation for the protocol time τ. By simply equating Eqs. (2) and (4) we find

$$\tau = \frac{1}{v_z}\arccos\sqrt{\mathcal{F}_0}, \tag{8}$$

where $\mathcal{F}_0 \equiv |\langle\psi_i|\psi_f'\rangle|^2$ is the fidelity of the process dictated solely by the drift Hamiltonian. Equation (8) represents an alternative expression for Bhattacharyya's quantum speed limit written in the interaction picture [31]. In particular, in the absence of a drift Hamiltonian $\mathcal{F}_0 = |\langle\psi_i|\psi_f\rangle|^2$, and then Eq. (8) coincides with the well-known Mandelstam-Tamm (MT) bound [30], i.e.,

$$\tau_{\mathrm{MT}} = \frac{1}{v_z}\arccos|\langle\psi_i|\psi_f\rangle|. \tag{9}$$

In general, the protocol time τ can be either smaller or larger than τ_{MT} depending on the fidelity of the process dictated solely by the drift Hamiltonian. Specifically, comparing Eqs. (8) and (9) yields

$$\tau \lesseqgtr \tau_{\mathrm{MT}} \quad \text{if } |\langle\psi_i|\psi_f\rangle|^2 \lesseqgtr \mathcal{F}_0. \tag{10}$$

If the evolution dictated by the drift Hamiltonian brings the initial state closer to the final state in a time τ, then it represents a "favorable wind" and $\tau < \tau_{\mathrm{MT}}$. Contrarily, if the drift Hamiltonian takes the initial state away from the final state in a time τ, then it represents an "unfavorable wind" and $\tau > \tau_{\mathrm{MT}}$. Note, however, that Eq. (8) is in general a nontrivial function of the energy disposal of the control v_z. That is, whether a drift Hamiltonian embodies a favorable or unfavorable wind ultimately depends on the duration of the protocol τ, which in turn is a function of the energy resource of the control v_z. Therefore, a given drift Hamiltonian $H_0(t)$ may represent either a favorable or unfavorable wind depending on the energy resource of the control v_z.

Time-optimal control of a Landau-Zener model system.—To assess the performance of the above protocol, we investigate the time-optimal control for the "simplest nontrivial quantum problem," that is, the evolution of a two-level system under a Landau-Zener (LZ) drift Hamiltonian as shown in Sec. III of the Supplemental Material [48]. Here, two states $|0(t)\rangle$ and $|1(t)\rangle$, the diabatic levels, are coupled through a LZ Hamiltonian [57,58]:

$$H_{\mathrm{LZ}}(t) = \Gamma(t)\sigma_z + \omega\sigma_x, \tag{11}$$

($\sigma_{x,z}$ being the Pauli operators with $\sigma_x|0(t)\rangle = |1(t)\rangle$) characterized by the instantaneous adiabatic levels of the system $|g(t)\rangle$ and $|e(t)\rangle$. In Eq. (11), ω represents the coupling between the diabatic levels and is kept constant, and $\Gamma(t)$ is a piecewise-defined function which is chosen to be either a linear or polynomial function of time:

$$\Gamma_{\mathrm{linear}}(t) = 4\left(\frac{t}{\tau} - 1/2\right), \quad \text{for } 0 \le t \le \tau, \tag{12a}$$

$$\Gamma_{\mathrm{poly}}(t) = a\left(\frac{t}{\tau}\right)^2 + b\left(\frac{t}{\tau}\right) - 2, \quad \text{for } 0 \le t \le \tau, \tag{12b}$$

with $\Gamma(t<0) = -2$ and $\Gamma(t>\tau) = 2$ in both cases. In Eq. (12b) we picked $a = b^2/32$ and $b = -16 - \sqrt{1536}/2$ to find a situation with a "nonfavorable wind" (as will be clarified later on).

Our goal is to design a control protocol that drives the system in the least time through the anticrossing point in such a way that at the end of the evolution the final state is the adiabatic ground state, i.e., $|\psi(\tau)\rangle = |g(\tau)\rangle$. The system is initially prepared in the adiabatic ground state $|\psi(0)\rangle = |g(0)\rangle$ and, in the absence of a control Hamiltonian, undergoes tunneling to the excited state $|e(t)\rangle$ with a finite probability [57,58]. Alternatively, under the action of the control Hamiltonian obtained from Eqs. (7) and (8), the evolution of the initial adiabatic ground state will reach the target state in the least time and with unit fidelity. Notably, we do not impose any restriction on the form of the control Hamiltonian and hence our job is in contrast with previous works, where a "preset" form of the control Hamiltonian has been optimized at the price of degrading either fidelity or speed [18,59–61]. The interested reader can find more details on the numerical implementation of Eqs. (7) and (8) in Sec. IV of the Supplemental Material [48].

We begin by illustrating the influence of the drift Hamiltonian in the duration of the control protocol. Our aim is to show that a given drift Hamiltonian $H_{\mathrm{LZ}}(t)$ may represent both a favorable and an unfavorable wind depending on the energy resource of the control v_z. Figure 1(a) shows $\mathcal{F}_0$ as a function of τ for $\omega = 2$, either for Γ_{linear} or Γ_{poly}. In the figure, the regimes where the top and bottom inequalities of Eq. (10) are fulfilled can be readily identified. This can be checked in Figure 1(b), where we draw the protocol time τ as a function of the energy resource of the control v_z. For large values of v_z, the drift Hamiltonian (no matter whether it is favorable or unfavorable) is irrelevant, and thus $\tau \sim \tau_{\mathrm{MT}}$. For lower values of v_z, however, Γ_{poly} may represent a "nonfavorable wind" and, hence, delay the arrival of the initial state to the target state as compared to cases with a "favorable wind" (Γ_{linear}) or no wind, as shown in Sec. V of the Supplemental Material [48]. For slow enough processes, the drift Hamiltonian prevails over the control field and therefore the protocol time τ becomes always smaller than the MT bound, τ_{MT}, which diverges as $v_z \to 0$ as shown in Sec. V of the Supplemental Material [48].

Next, we assess the adiabaticity of the dynamics dictated by the control protocol. Specifically, we evaluate the mean adiabaticity of the overall control process:

$$\bar{\mathcal{A}} = \frac{1}{\tau}\int_0^\tau \mathcal{A}(t)dt, \tag{13}$$

where $\mathcal{A}(t) = |\langle g(t)|\psi(t)\rangle|^2$ quantifies how close the evolved state of the system is to the instantaneous ground state of $H_{\mathrm{LZ}}(t)$. Figure 2(a) shows $\bar{\mathcal{A}}$ as a function of τ for different values of the coupling constant ω. The minimum

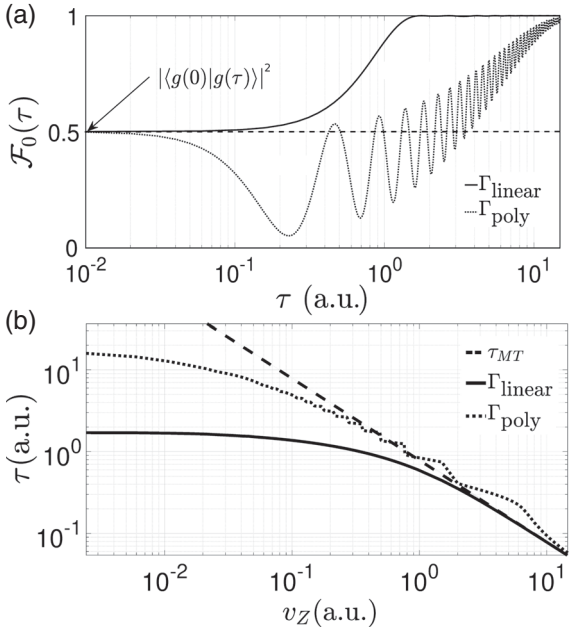

FIG. 1. (a) Fidelity $\mathcal{F}_0$ as a function of τ, Eq. (8), for $\omega = 2$ and two different LZ Hamiltonians with Γ_{linear} (solid line) and Γ_{poly} (dotted line). The dasehd line corresponds to $|\langle g(0)|g(\tau)\rangle|^2 = |\langle\psi_i|\psi_f\rangle|^2$. (b) Duration of the protocol time τ as a function of the energy resource of the control v_z for the same settings (where a.u. refers to atomic units). Both regimes in Eq. (10) can be identified depending on the energy resource of the control for Γ_{poly}. Contrarily, for Γ_{linear} the LZ Hamiltonian always acts as a favorable wind.

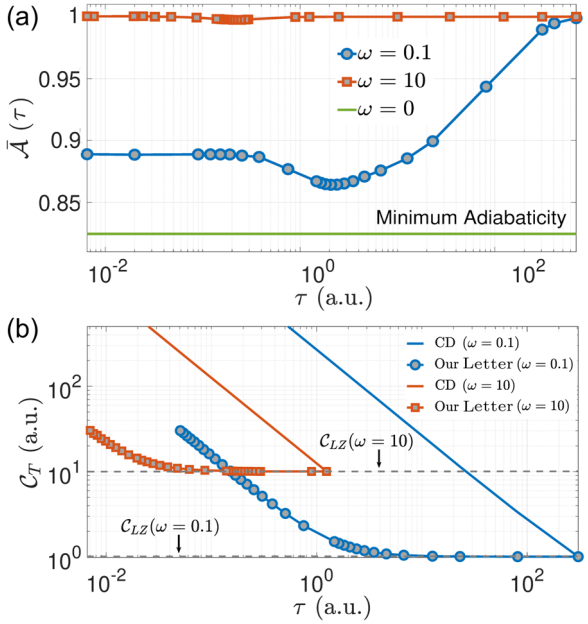

FIG. 2. (a) Time-averaged adiabaticity $\bar{\mathcal{A}}$, Eq. (13), as a function of τ for Γ_{linear} and different values of the coupling constant ω. Note that the minimum adiabaticity (green solid line) occurs at $\omega = 0$, where the level crossing induces an exchange of roles between ground and excited states. (b) Cost (in atomic units, a.u.) of implementing the proposed (solid lines with markers) and the counterdiabatic (solid lines without markers) protocols calculated through Eq. (14). $\mathcal{C}_{\text{LZ}}$ (dashed lines) denotes the cost of implementing solely the LZ Hamiltonian. It is clear from the above panels that imposing full adiabaticity at high speeds may result in being orders of magnitude more expensive than assuming a slightly nonadiabatic evolution.

adiabaticity of our method is expected at $\omega = 0$, where the level crossing induces an exchange of roles between ground and excited states. In this limit, Eqs. (7) and (8) can be solved analytically and yield $\bar{\mathcal{A}}_{\text{min}} \approx 0.82$ independently of the energy resource of the control, as shown in Sec. VI of the Supplemental Material [48]. By increasing the value of the coupling constant, the resulting dynamics are more and more adiabatic, with $\bar{\mathcal{A}}$ showing an asymptotic restoring of full adiabaticity for slow processes. We found that for $\omega \geq 10$, the resulting dynamics are adiabatic within an error less than 0.1%.

The above results indicate that quantum systems, in the form of a LZ type Hamiltonian, can be maneuvered at the quantum speed limit in a quasiadiabatic manner without imposing a preset form of the control Hamiltonian. A relevant question may then be how the proposed control scheme compares to transitionless driving, where the so-called counterdiabatic (CD) field, $H_{\text{CD}}(t)$, is designed in such a way that the system is driven precisely through the adiabatic manifold of the drift Hamiltonian (i.e., $\bar{\mathcal{A}}_{\text{CD}} = 1$) [45–47,62]. We evaluate the energetic cost of implementing our control scheme and the CD driving field. The notion of cost has been somewhat loosely employed and hence different quantifiers probe different aspects of the system's energy [63]. Therefore, we are free to choose any meaningful quantifier that provides a sound basis for drawing a comparison. Simply determining the average energy of the control is insufficient as both our protocol and CD drivings obey the transversality condition in Eq. (5) and would appear thermodynamically for free. Instead, here we choose the cost defined by Campbell and coworkers [55,56,68] and use the Frobenius norm of the total Hamiltonian to define the cost of the control protocol as

$$\mathcal{C}_T = \frac{1}{\tau}\int_0^\tau \|H(t)\|dt, \tag{14}$$

where $\|H(t)\| = \sqrt{\text{tr}[H^2(t)]/2}$ and $H(t)$ is the total Hamiltonian including the drift and control fields. As shown in Fig. 2(b), for very low speed processes the cost of implementing both protocols converges to the cost of implementing solely the LZ Hamiltonian $\mathcal{C}_{\text{LZ}} = (1/\tau)\int_0^\tau \|H_{\text{LZ}}(t)\|dt$. Imposing full adiabaticity at higher speeds, however, may result in being orders of magnitude more expensive than assuming a slightly nonadiabatic evolution. For example, at $\omega = 10$, implementing a counterdiabatic driving field can be as costly as $\times 100$ the

cost of implementing our control protocol, which is already 99.9% adiabatic. At lower coupling constants and high speeds, the cost difference between the two protocols increases even further while the adiabaticity error in the proposed protocol is always kept below 20%. Remarkably, for $\omega = 0$, the CD driving field does not bring the system to the target state. This is in contrast to the proposed protocol, which reaches the final state through a highly adiabatic path.

For the above example, the interested reader can find a detailed comparison of the matrix elements of $H_c(t)$ and $H_{\mathrm{CD}}(t)$ in Sec. VII of the Supplemental Material [48]. Let us stress, however, that the practical implementation of $H_c(t)$ and $H_{\mathrm{CD}}(t)$ ultimately depends on the physical system under consideration. In this respect, in Sec. VII of the Supplemental Material [48] we also provide some hints on the practical implementation of $H_c(t)$ for a spin $1/2$ in a time-varying magnetic field and a Bose Einstein condensate in an optical conveyor belt [69,70], both immersed in a LZ type Hamiltonian.

Conclusions.—To summarize, based on a pure geometric derivation in a projective Hilbert space, we have presented an "ansatz-free" approach to time-optimal quantum control. The analysis of this scheme as applied to the Landau-Zener model yielded two important conclusions for maximum speed transformations. First, no "guessed" form of the control Hamiltonian is required for designing a time-optimal control protocol that, along with the action of a time-dependent drift Hamiltonian, drives an initial state to a target state in the least time and with unit fidelity. The solution to the system of Eqs. (7) and (8) can be thus exploited to conceive new, unforeseen, time-optimal control protocols. Second, quasiadiabatic dynamics with less than 0.1% deviation from the full adiabatic path can be attained at the quantum speed limit with an energetic cost that is orders of magnitude lower than the cost of implementing a counterdiabatic field. Therefore, the proposed control method lends itself as a "low-cost" alternative to transitionless driving. Overall, the quantum control approach that we establish opens a new avenue in the search for more time- and energy-efficient control protocols [71].

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[49] For a general N-dimensional setting, it may be natural to expand the density matrix in the interaction picture using the generators of the $SU(N)$ group. One could then argue that a minimum angular distance would be traveled by a Bloch vector restricted to move in the plane defined by the generalized Bloch vectors associated to $|\psi_i\rangle$ and $|\psi'_f\rangle$. Unfortunately, while for $N = 2$ all Bloch vectors result in positive semi-definite density operators, this is not true for $N \geq 3$, and no general parametrisation has been found to avoid regions of the hypersphere that do not correspond to positive semi-definite density matrices [50].
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