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OVERBIDDING AND UNDERBIDDING IN PACKAGE ALLOCATION PROBLEMS

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Title: Overbidding and underbidding in package allocation problems

Abstract: We study the problem of allocating packages of different objects to a group of bidders. A rule is overbidding-proof if no bidder has incentives to bid above his actual valuations. We prove that if an efficient rule is overbidding-proof, then each winning bidder pays a price between his winning bid and what he would pay in a Vickrey auction for the same package. In counterpart, the set of rules that satisfy underbidding-proofness always charge a price below the corresponding Vickrey price. A new characterization of the Vickrey allocation rule is provided with a weak form of strategy-proofness. The Vickrey rule is the only rule that satisfies efficiency, individual rationality, overbidding-proofness and underbidding-proofness. Our results are also valid on the domains of monotonic valuations and of single-minded bidders. Finally, a rule is introduced that is overbidding proof and its payoffs are bidder-optimal in the core of the auction game according the reported valuations.

JEL Codes: D44, D47

Keywords: Strategy-proofness, overbidding, Vickrey allocation rule

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1 Introduction

This paper contributes to the study of package allocation problems from an auction-design perspective as described in Milgrom (2007). We study allocation problems in which a single seller auctions off multiple and different indivisible objects among a group of potential buyers each of whom might wish to acquire several objects. In this context, we analyze unilateral incentives for bidders to misreport their true valuations when asked to bid for packages. More specifically, we analyze under which circumstances potential buyers have incentives to manipulate allocation rules through the use of over- or underbids.

Package allocation problems are seemingly ubiquitous, present in numerous markets in both the public and private sectors. In small markets, the sale of packages of objects is certainly common practice. In large markets where determining the allocation of objects could become computationally impractical, the design and implementation of mechanisms to sell packages such as auctions have been clearly growing since the 1990's.[1] The design of rules whereby the Federal Communication Commission auctioned off the radio spectrum and the implementation of combinatorial auctions to assign bus routes in London are notable examples, which have been widely analyzed in the auction literature.[2]

In an auction, potential buyers, also referred to as bidders, typically compete with each other to buy or provide a package of goods or services. The rules of an auction determine the allocation procedure, which can be carried out in several stages or in a single shot as in the classic first- and second-price sealed-bid auctions (Milgrom, 2004). In the case of one-shot auctions, each bidder is typically asked to bid on packages. On that basis, the auctioneer treats reported values as truthful, calculates the allocation of objects and dictates the selling prices. In that context, a meaningful question is then whether bidders report willingness-to-pay truthfully or not.

The so-called Vickrey rule is a remarkable allocation mechanism.[3] Among other properties, the Vickrey rule satisfies strategy-proofness, i.e., the Vickrey rules never penalize a bidder for bidding truthfully.[4] In fact, the Vickrey rule is the only one that satisfies strategy-proofness among the family of efficient and individually rational rules (Holmström, 1979). Efficiency requires that once an allocation has been reached, there is no other alternative that makes one bidder better off without making another worse off. Individual rationality regards participation constraints, that is, each bidder is willing to participate when a Vickrey rule is in use. Thus, when implemented, the Vickrey rule promotes truthful participation across bidders.

[1] See for example (Milgrom, 2017) and Roth (2002).

[2] See Roth (2002) and Cantillon & Pesendorfer (2006), respectively.

[3] More precisely the Vickrey-Clarke-Groves mechanism, which generalizes some of the good properties of the second-price auction, see for example Cramton *et al.* (2006). However, what is known as generalized second-price auctions are not in general Vickrey auctions, see Edelman *et al.* (2007)

[4] In this paper, we apply strategy-proofness to allocation rules even though this is a property of social choice functions, Barberà *et al.* (2016).

Results from laboratory experiments show, however, that when second-price auctions are implemented, participants do not bid truthfully, on the contrary, they tend to overbid (Kagel *et al.* , 1987; Kagel & Levin, 1993). In an experiment on generalized second price auctions used for selling advertising positions in online search engines, Bae & Kagel (2019) also find that low value bidders tend to bid above their true values. Moreover, experiments on the Vickrey multi-unit demand auction reported in Kagel & Levin (2009) find a general pattern of bidding above true values, which resulted in substantially lower profits compared to those that would have been obtained with sincere bidding. A similar behavior has been also observed in laboratory experiments on combinatorial auctions with package bidding as reported by Kagel *et al.* (2014).

These references about experimental studies where overbid tendency exist hint two remarkable points. First, that bidders find it difficult to recognize that truthful bidding is a weakly dominant strategy when they bid under the rules of a Vickrey auction. This could be one of the reasons why, despite its prominent properties, the Vickrey allocation rule has not been much implemented in practice.[5] Second, that bidders tend to overbid, at least when they compete under the Vickrey rule.

Against this background, this paper investigates which allocation rules mute bidders' incentives to over- or underbid. To this end, we first introduce the notions of under- and overbidding as natural generalizations from single-object allocation situations to package allocation problems. Secondly, this paper examines bidders' incentives to over- or underbid and studies the relationship between overbidding- and underbidding-proof rules and the Vickrey rule.

To better illustrate our notions of under- and overbidding, first consider a single-object allocation situation. In that case, any bidder's misreport can be plainly characterized as an overbid or an underbid with respect to his/her willingness-to-pay. We generalize these notions to a framework with multiple units. A bidder overbids (underbids) when he reports a higher (lower) valuation for some packages compared to his actual willingness-to-pay for those packages. For example, a bidder's willingness-to-pay for packages A, B and C are \$10, \$15 and \$10, respectively. This bidder overbids when he reports \$10, \$20 and \$20, respectively. It is worth noting that, under our definition, the bidder could report some values truthfully, but he would be still overbidding. Taking these notions into account, we say that a rule satisfies overbidding-proofness (underbidding-proofness) when no agent has incentives to overbid (underbid).

Our first result shows that if a rule is efficient, individually rational and overbidding-proof, then each winning bidder pays a price between his winning bid and what he would pay for that package under a Vickrey allocation rule. Similarly, the rules that satisfy efficiency, individual rationality and underbidding-proofness charge a price lower than the corresponding Vickrey price. Our results contribute to the understanding of the non-manipulability properties of efficient and individually rational allocation rules. We

[5] Ausubel & Milgrom (2006) summarizes the virtues and weaknesses of the VCG mechanism.

demonstrate that the Vickrey rule is the only efficient and individually rational rule that satisfies these two non-manipulability properties. In other words, we provide a new characterization of the Vickrey rule with a clearly weaker form of strategy-proofness that consists in requiring only overbidding-proofness and underbidding-proofness but allowing succesfull manipulation by means of more sophisticated misreporting.

Additionally, we show that our results hold for two subclasses of bidders' valuations: monotonic valuations and single-minded bidders. A single-minded bidder only cares for one particular package and hence this bidder has a particular type of monotonic valuations that value at zero any package that does not contain the desired one (Mu'alem & Nisan, 2002) .

Besides strategy-proofness, another desired property for an allocation rule is stability or core-selection. In the context of package allocation problems, core-selection becomes a desired property as the set of competitive equilibria could be empty. An individually rational outcome is in the core of the so-called auction game if and only if there is no group of bidders who would strictly prefer an alternative assignment that is also strictly better for the seller. Hence, the failure to select a core allocation with respect to reported values implies that there is a group of bidders who have offered to pay more in total than the winning bidders, yet whose offer has been rejected.

Strategy-proofness may be incompatible with core-selection. On those domains where the payoffs of the Vickrey rule belong to the core of the auction game defined by the reported valuations, it is the only core-selecting rule that is strategy-proof. Otherwise, there is no rule satisfying both properties. Day & Milgrom (2008) show that, among core-selecting auctions, the ones that maximize incentives for truthfull reporting are those that select a bidders-optimal core allocation (a core allocation that is Pareto-optimal for bidders). However, bidders-optimal core-selecting rules are not easy to define, specially if we are thinking of one-shot rule. Instead, if we allow for a sequential rule, the ascending proxy auction of Ausubel & Milgrom (2002) is a core-selecting rule and each bidders-optimal core allocation can be achieved in a Nash equilibrium.

Our approach is different, instead of minimizing the incentives of bidders to misrepresent their valuations, we look for rules that discard a particular sort of misrepresentation, which is overbidding, the one most observed in experiments. We consider the family of allocation rules that determine a price that is a given convex combination between the Vickrey price and the pay-as-bid price, and show that all rules in this family are overbidding-proof. There are always core-selecting rules in this family, in particular the pay-as-bid rule, which is seller-optimal. Moreover, we show that the core-selecting rule of this family with the maximum weight on the Vickrey price, is a bidders-optimal core selecting rule, and hence it rules out overbidding and at the same time maximizes incentives for truthfull reporting.

Our paper is organized as follows. Section 2 introduces preliminaries including classical axioms and three well-known allocation rules. Section 3 introduces two new axioms to capture bidders' over- and underbids in the context of allocation problems with multiple and different objects. Moreover, in this section we present our first results. Section 4 bridges between the notion of overbidding-proofness and core-selection. Section 5 shows that our previous results apply to the domain of monotonic valuations and to that of single-minded bidders. Finally, Section 6 concludes.

2 Preliminaries

Consider an allocation problem with a finite set of agents consisting of a single seller and a group of n bidders, let the seller by denoted by 0 and the set of bidders by $N = \{1, 2, ..., n\}$. The seller owns a finite set Q of indivisible objects that can be allocated among bidders.

Each bidder $i \in N$ has a quasi-linear preference with respect to money and a set of admissible packages $Q_i \subseteq Q$, *i.e.*, there is a valuation function $v_i : 2^{Q_i} \to \mathbb{R}_+$ such that $v_i(\emptyset) = 0$ and when he acquires the package $B \subseteq Q_i$ and pays $r \in \mathbb{R}_+$ for it, his utility is $v_i(B) - r$.[6] We interpret $v_i(B)$ as the willingness to pay of agent i for package B. A valuation profile consists of a valuation function v_i for each bidder $i \in N$, $v = (v_i)_{i \in N}$. We denote by V^n the set of all possible valuation profiles with n bidders. Hence, $v \in V^n$ and we usually use v_{-k} to refer to the valuations of all agents different from k, i.e., $(v_i)_{i \in N \setminus \{k\}}$.

An assignment of the objects is a list $z = (z_i)_{i \in N}$ such that for each $i \in N$, $z_i \subseteq Q_i$ and for each $i, k \in N$, $z_i \cap z_k = \emptyset$ with $i \neq k$, and we allow for $z_i = \emptyset$ for some or all $i \in N$. The set of all assignments is denoted by $\mathcal{Z}$.

An allocation consists of a pair $(z, p) \in \mathcal{Z} \times \mathbb{R}^n_+$ where $p = (p_i)_{i \in N} \in \mathbb{R}^n_+$ denotes the price that each agent $i \in N$ has to pay for his assigned package z_i. We denote by $\mathcal{A}$ the set of all possible allocations.

An *allocation rule* consists of a rule that assigns to each reported valuation profile, an allocation *i.e.*, $\varphi : V^n \to \mathcal{A}$. Hence, given a valuation profile v and allocation rule φ, we denote by $\varphi^o_i(v) \subseteq Q_i$ the package of objects assigned to agent i, by $\varphi^o(v) = (\varphi^o_i(v))_{i \in N}$ the corresponding assignment of objects and by $\varphi^m_i(v) \in \mathbb{R}_+$ the price that i has to pay. We say that a bidder is a winning bidder at v when φ is implemented and $\varphi^o_i(v) \neq \emptyset$.

Now, we introduce three classical properties for allocation rules, *efficiency*, *individual rationality* and *strategy-proofness*. In the literature with independent private values, efficiency and individual rationality are typically considered minimal requirements to be satisfied by any allocation. On the domain of quasi-linear preferences, efficiency captures the idea of maximizing the total sum of the valuations of the assigned objects, see, e.g., Chew & Serizawa (2007). Notice that such a maximizing allocation of objects may not be unique. Individual rationality assures unilateral incentives for the bidders to participate in the allocation problem.

[6] For each set B, we denote by $|B|$ the cardinality of B and by 2^B the power set of S.

Definition 2.1. *A rule φ is efficient at $v \in V^n$ if*

$$\varphi^o(v) \in argmax_{z \in \mathcal{Z}} \left\{ \sum_{i \in N} v_i(z_i) \right\}.$$

A rule φ satisfies efficiency if it is efficient at v for all $v \in V^n$.

The following property implies that every agent will pay for his assignment at most, his willingness to pay for it. This is understood as a participation constraint, since no agent will obtain a negative payoff after participation in the allocation problem.

Definition 2.2. *A rule φ satisfies individual rationality if for all $v \in V^n$ and all $i \in N$,*

$$v_i(\varphi_i^o(v)) - \varphi_i^m(v) \geq 0.$$

The next property, strategy-proofness requires that no agent will be better-off by individually claiming a false preference.

Definition 2.3. *A rule φ satisfies strategy-proofness if for all $v \in V^n$, all $i \in N$ and all $v'_i \in V$,*

$$v_i(\varphi_i^o(v)) - \varphi_i^m(v) \geq v_i(\varphi_i^o(v_{-i}, v'_i)) - \varphi_i^m(v_{-i}, v'_i).$$

Now, we consider three classical allocation rules. The first one is called the pay-as-bid or first-price menu auction (Bernheim & Whinston, 1986). Under a pay-as-bid scheme, each bidder simultaneously reports his/her willingness to pay for each package of objects on sale. Then, the seller allocates packages among bidders efficiently (maximizing the sum of the reported values of the allocated packages) and each bidder pays the announced offer for the package he/she receives.

Definition 2.4. *A rule φ is a pay-as-bid rule if for all $v \in V^n$, $\varphi^o(v) \in \mathcal{Z}$ is efficient at v and the price each agent $k \in N$ has to pay is $\varphi_k^m(v) = v_k(\varphi_k^o(v))$.*

In a pay-as-bid auction, each winning bidder pays according to his announced bids; therefore, individual rationality is trivially fulfilled.

The following definition introduces free-package rules which charge a price of zero to each bidder, winning or not. Free-package rules belong to a wider family of rules that charge a fix price to winning bidders irrespective of the bids. A free-package rule can be implemented, for example, by a food bank seeking an efficient allocation of food without expecting monetary compensations from its participants.

Definition 2.5. *A rule φ is a free-package rule if for all $v \in V^n$, $\varphi^o(v) \in \mathcal{Z}$ is efficient at v and the price each agent has to pay for his assigned package is zero.*

Certainly, among efficient and individually rational rules, free-package rules do not only provide incentives to participate but also make the highest profit to each bidder.

The following rule, the so-called Vickrey rule, has been widely studied in the literature of mechanism design and it plays a central role in the development of this paper.

Fix any valuation profile $v \in V^n$ and any efficient assignment $z \in \mathcal{Z}$ at v. Now, denote by $x_k(v, z)$ the Vickrey price, which can be seen as the social opportunity cost of bidder k's winning z_k at v. See for instance Ausubel & Milgrom (2006), *i.e.*,

$$x_k(v, z) = \max_{z' \in \mathcal{Z}} \left\{ \sum_{i \in N \setminus \{k\}} v_i(z'_i) \right\} - \sum_{i \in N \setminus \{k\}} v_i(z_i), \tag{1}$$

Definition 2.6. *A rule φ is a Vickrey rule if for all $v \in V^n$, $\varphi^o(v) \in \mathcal{Z}$ is efficient at v and the price each agent $k \in N$ has to pay is $\varphi^m_k(v) = x_k(v, \varphi^o(v))$.*

As it was already mentioned, Vickrey rules have been widely studied since Vickrey's seminal paper. One of the remarkable properties of Vickrey rules concerns its uniqueness since these rules are characterized by efficiency, individual rationality and strategy-proofness. Even more, the Vickrey rules satisfy a property related to the profit each bidder makes under two Vickrey rules. Fix a valuation profile v and consider two Vickrey rules φ and $\hat{\varphi}$. It is known that $v_i(\varphi^o_i(v)) - \varphi^m_i(v) = v_i(\hat{\varphi}^o_i(v)) - \hat{\varphi}^m_i(v)$ for all $i \in N$. Due to this indifference, it is usual the reference to the Vickrey rule as a unique rule.

3 Overbidding and underbidding proofness

In this section, we introduce new axioms to capture bidders possibility to unilaterally overbid or underbid their willingness-to-pay. Generally speaking, a bidder overbids (underbids) when he reports a higher (lower) value for some packages than his actual willingness-to-pay for them. These opposite bidding behaviors can be expressed by means of pairwise comparisons of valuation functions.

Definition 3.1. *Consider two different valuation functions $v_i, v'_i \in V$.*

1. *If* $v'_i(B) \geq v_i(B)$ *for all* $B \subseteq Q_i$, *and*
   $v'_i(B) > v_i(B)$ *for some* $B \subseteq Q_i$,
   then, v'_i is an overbid with respect to (wrt) v_i.
2. *If* $v'_i(B) \leq v_i(B)$ *for all* $B \subseteq Q_i$, *and*
   $v'_i(B) < v_i(B)$ *for some* $B \subseteq Q_i$,
   then, v'_i is an underbid wrt v_i.

The next two axioms make use of Definition 3.1 and concern particular misrepresentations of true preferences.

Definition 3.2. *A rule φ satisfies overbidding-proofness if for all $v \in V^n$, all $i \in N$ and all $v'_i \in V$,*

$$v'_i \text{ is an overbid wrt } v_i \Rightarrow v_i(\varphi^o_i(v)) - \varphi^m_i(v) \geq v_i(\varphi^o_i(v_{-i}, v'_i)) - \varphi^m_i(v_{-i}, v'_i).$$

Overbidding-proofness requires a weaker condition than strategy-proofness since it focuses only on misrepresentations that overrate the true value of objects. To illustrate the notion of overbidding-proofness, consider the simplest version of the pay-as-bid rule, a first-price sealed-bid auction with a single object. Under this scheme, no bidder benefits from reporting a valuation higher than his actual willingness-to-pay. Indeed, overbidding always leads the winning bidder to a negative profit. This rule satisfies overbidding-proofness.

Definition 3.3. *A rule φ satisfies underbidding-proofness if for all $v \in V^n$, all $i \in N$ and all $v_i' \in V$,*

$$v_i' \text{ is an underbid wrt } v_i \Rightarrow v_i(\varphi_i^o(v)) - \varphi_i^m(v) \geq v_i(\varphi_i^o(v_{-i}, v_i')) - \varphi_i^m(v_{-i}, v_i').$$

Consider again our example of the first-price auction rule. As remarked, this rule satisfies overbidding-proofness, but it does not survive to strategic manipulations via underbids. The winning bidder has incentives to underbid provided he outbids his competitors with a positive margin.

Along these lines, the natural question is then, which rules do survive to over or underbidding? Our next result offers a necessary condition for a rule to satisfy overbidding-proofness. Additionally, we show how a bidder can easily manipulate them. Roughly speaking, the set of rules that satisfy overbidding-proofness always price above the corresponding Vickrey price.

Proposition 3.4. *If an efficient and individually rational rule φ satisfies overbidding-proofness, then for each $v \in V^n$ the price is $\varphi_k^m(v) \in [x_k(v, \varphi^o(v)), v(\varphi_k^o(v))]$ for each $k \in N$.*

Proof. We proceed by contradiction. Assume that there is an efficient, individually rational and overbidding-proof rule, a valuation profile v and an agent k, such that $\varphi_k^m(v) \notin [x_k(v, \varphi^o(v)), v(\varphi_k^o(v))]$ where $x_k(v, \varphi^o(v))$ is the corresponding Vickrey price. By individual rationality, $v(\varphi_k^o(v)) < \varphi_k^m(v)$ cannot hold, therefore we have $x_k(v, \varphi^o(v)) > \varphi_k^m(v)$.

Define $\hat{v}_k$ such that $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi_k^o(v)$, and $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi_k^o(v)) > \varphi_k^m(v)$. Note that v_k is an overbid wrt $\hat{v}_k$.

By using point *1.* in Lemma 7.1, we have that $\varphi_k^o(v_{-k}, \hat{v}_k)$ is such that $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) = 0$. Then, individual rationality implies $\varphi_k^m(v_{-k}, \hat{v}_k) = 0$. Moreover, by definition of $\hat{v}_k$, we have $\hat{v}_k(\varphi_k^o(v)) > \varphi_k^m(v)$. Hence,

$$\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) = 0 < \hat{v}_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

which shows that agent k has incentives to overbid, i.e., bidder k claims v_k when his true preference is $\hat{v}_k$. This is a contradiction with the fact that φ satisfies overbidding-proofness. Hence, this shows that $\varphi_k^m(v) \in [x_k(v, \varphi^o(v)), v(\varphi_k^o(v))]$ as stated and completes the proof. □

The following result examines bidders incentives to misrepresent their valuations when they face an overbidding-proof rule.

Proposition 3.5. *If a rule φ satisfies efficiency, individual rationality, overbidding-proofnes and is not a Vickrey rule, then it can be manipulated by underbidding.*

Proof. By Proposition 3.4, we know that for any bidder k and any valuation profile v, $\varphi_k^m(v) \geq x_k(v, \varphi^o(v))$. Since φ is not the Vickrey rule, there is at least one profile v such that $\varphi_k^m(v) > x_k(v, \varphi^o(v))$ for some bidder k. Take this valuation profile v and this bidder.

Define $\hat{v}_k$ such that $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi_k^o(v)$, and $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v)) > x_k(v, \varphi^o(v))$. Therefore $\hat{v}_k$ is an underbid wrt v_k. By point *2.* in Lemma 7.1, $\varphi_k^o(v_{-k}, \hat{v}_k)$ is $\varphi_k^o(v)$. Then, by definition of $\hat{v}$, $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v)) = \hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k))$. By individual rationality, $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$. Then, $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$, hence $\varphi_k^m(v) > \varphi_k^m(v_{-k}, \hat{v}_k)$. As a consequence,

$$v_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) = v_k(\varphi_k^o(v)) - \varphi_k^m(v_{-k}, \hat{v}_k) > v_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

which shows that agent k has incentives to claim $\hat{v}_k$ when his true preference is v_k. □

The following results are the counterpart of Propositions 3.4 and 3.5 for underbidding. First we present a necessary condition on an efficient and individually rational rule to be underbidding-proof.

Proposition 3.6. *If an efficient and individually rational rule φ satisfies underbidding-proofness then, for each $v \in V^n$,*

$$\varphi_k^m(v) = [0, x_k(v, \varphi^o(v)] \text{ for each } k \in N. \tag{2}$$

Proof. We proceed by contradiction. Assume that there is an efficient, individually rational and underbidding-proof rule, a valuation profile v and an agent k, such that $\varphi_k^m(v) \notin [0, x_k(v, \varphi^o(v))]$. Therefore we have $x_k(v, \varphi^o(v)) < \varphi_k^m(v)$.

Define $\hat{v}_k$ such that $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi_k^o(v)$, and $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v)) > x_k(v, \varphi^o(v))$. Therefore $\hat{v}_k$ is an underbid wrt v_k.

By point *2.* in Lemma 7.1, $\varphi_k^o(v_{-k}, \hat{v}_k)$ is $\varphi_k^o(v)$, and then, by definition of $\hat{v}$, $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k))$. Moreover, by individual rationality, $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$. Hence, $\varphi_k^m(v) > \varphi_k^m(v_{-k}, \hat{v}_k)$ and as a consequence,

$$v_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) = v_k(\varphi_k^o(v)) - \varphi_k^m(v_{-k}, \hat{v}_k) > v_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

which shows that agent k has incentives to claim $\hat{v}_k$ when his true preference is v_k. This is a contradiction with the fact that φ satisfies underbidding-proofness, and completes the proof. □

The following result shows that underbidding rules can be manipulated by overbidding.

Proposition 3.7. *If a rule satisfies efficiency, individual rationality underbidding-proofness and is not a Vickrey rule, then it can be manipulated by overbidding.*

Proof. By Proposition 3.6, we know that for any bidder k and any valuation profile v, $x_k(v, \varphi^o(v)) \geq \varphi_k^m(v)$. Since φ is not the Vickrey rule, there is at least one valuation profile v such that $x_k(v, \varphi^o(v)) > \varphi_k^m(v)$ for some bidder k. Take this valuation profile v and this bidder.

Define $\hat{v}_k$ such that $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi^o_k(v)$, and $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$.

By using point *1.* in Lemma 7.1, we have that by reporting $\hat{v}_k$, $\varphi^o_k(v_{-k}, \hat{v}_k)$ is such that $\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) = 0$. By individual rationality, $\varphi^m_k(v_{-k}, \hat{v}_k) = 0$. Moreover, by definition of $\hat{v}_k$, we have $\hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$. Hence,

$$\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) - \varphi^m_k(v_{-k}, \hat{v}_k) = 0 < \hat{v}_k(\varphi^o_k(v)) - \varphi^m_k(v),$$

which shows that agent k has incentives to claim $\hat{v}_k$ when his true preference is v_k. □

The analysis in this section concludes with a new characterization of the Vickrey rule that replaces strategy-proofness with a weaker non-manipulability property that requires both over- and underbidding proofness.

Theorem 3.8. *The Vickrey rule is characterized by individual rationality, efficiency, underbidding-proofness and overbidding-proofness.*

Proof. If φ is the Vickrey rule, it satisfies individual rationality, efficiency and strategy-proofness, see e.g., Milgrom (2007). Hence, it also satisfies underbidding-proofness and overbidding-proofness. Now, we prove that if a rule satisfies individual rationality, efficiency, underbidding-proofness and overbidding-proofness, then it is the Vickrey rule. But this is directly implied by Propositions 3.4 and 3.6. □

4 Core-selecting overbidding-proof rules

To each package allocation problem we associate a coalitional game $(N \cup \{0\}, w_v)$ where the players are the bidders and the seller, and the worth $w_v(S)$ of a coalition $S \subseteq N \cup \{0\}$ is the maximum value that can be attained over all possible assignments to the bidders in S. Hence, if S does not contain the seller, then $w_v(S) = 0$, and otherwise

$$w_v(S) = \max_{z \in \mathcal{Z}} \sum_{i \in S \setminus \{0\}} v_i(z_i).$$

Besides strategy-proofness, another desired property for an allocation rule is stability or core-selection. An efficient and individually rational outcome is in the core of an auction game if and only if there is no group of bidders who would strictly prefer an alternative assignment that is also strictly better for the seller. More precisely, a payoff vector $x \in \mathbb{R}^{N \cup \{0\}}$ is in the core $C(w_v)$ of the game $(N \cup \{0\}, w_v)$ if $\sum_{k \in N \cup \{0\}} x_k = w_v(N \cup \{0\})$ and $\sum_{k \in S} x_k \geq w_v(S)$ for all $S \subseteq N \cup \{0\}$.

The core of the auction game, with respect to the reported valuations, is non-empty since the allocation that gives all the payoff $w_v(N \cup \{0\})$ to the seller is always in the core. This allocation is attained by any pay-as-bid rule. But there are other easy-to-compute core allocations. Give to a selected bidder $i^* \in N$ his/her marginal contribution $w_v(N \cup \{0\}) - w_v((N \setminus \{i^*\}) \cup \{0\})$, zero payoff to any other bidder and $w_v((N \setminus \{i^*\}) \cup \{0\})$ to the seller and, because of the monotonicity of the coalitional function w_v, this is also a core allocation. This allocation is attained by any efficient rule that assigns to bidder i^* the Vickrey price and to the remaining bidders their pay-as-bid price. Notice this rule is non-anonymous since the identity of bidder i^* plays a role in the determination of the prices.

Strategy-proofness may be incompatible with core-selection. On those domains where the payoffs of the Vickrey rule belong to the core of the auction game defined by the reported valuations, this is the only core-selecting rule that is strategy-proof. In that case, the payoffs of the Vickrey rule constitute the only bidder-optimal core allocation. But in general the Vickrey payoff vector might be outside the core and as a consequence, since all core allocations are efficient and individually rational, there is no rule satisfying both properties.

Day & Milgrom (2008) show that, among core-selecting auctions, the ones that maximize incentives for truthfull reporting are those that select a bidder-optimal core allocation (a core allocation that is Pareto-optimal for bidders). A well-known example of such a rule is the ascending proxy auction of Ausubel & Milgrom (2002). But we are considering in this paper direct mechanisms where bidders report their values (or willingness to pay) for each package and, based on that, the seller (or the auctioneer) determines the allocation of objects and dictates the price to be paid by each bidder. It is not clear in this setting how to define a bidder-optimal core allocation rule in a precise and clear way for the bidders to understand, since the bidder-optimal core payoff vector may not be unique.

We approach the problem in a different way. Instead of minimizing the profits from manipulation, we consider rules that discard one type of manipulation, the one that consists in overbidding, which is the most observed in some experiments. First we show that there exist rules other than the Vickrey rules that satisfy this weaker non-manipulability property that is overbidding-proofness. We show that this property holds for any rule that charges a price that is a fixed convex combination between the Vickrey price and the pay-as-bid price.

Proposition 4.1. *Consider an efficient and individually rational rule φ. If there exists $\alpha \in [0,1]$ such that for each $v \in V^n$ the price is*

$$\varphi_k^m(v) = \alpha x_k(v, \varphi^o(v)) + (1-\alpha) v_k(\varphi_k^o(v)) \text{ for each } k \in N, \tag{3}$$

then, the rule φ satisfies overbidding-proofness.

Proof. Take any bidder k and consider any v_k and v_k' such that v_k' is an overbid wrt to v_k. We denote by $v' \in V^n$ the valuation profile where agent k bids v_k' and $v_i' = v_i$ for all

$i \in N \setminus \{k\}$. Then we have

$$\begin{aligned}
v_k(\varphi_k^o(v)) - \varphi_k^m(v) &= v_k(\varphi_k^o(v)) - \alpha x_k(v, \varphi^o(v)) - (1-\alpha)v_k(\varphi_k^o(v)) \\
&= \alpha v_k(\varphi_k^o(v)) - \alpha x_k(v, \varphi^o(v)) = \alpha\Big(v_k(\varphi_k^o(v)) - x_k(v, \varphi^o(v)\Big) \\
&\geq \alpha\Big(v_k(\varphi_k^o(v')) - x_k(v', \varphi^o(v'))\Big) \\
&= v_k(\varphi_k^o(v')) - \alpha x_k(v', \varphi^o(v')) - (1-\alpha)v_k(\varphi_k^o(v')) \\
&\geq v_k(\varphi_k^o(v')) - \alpha x_k(v', \varphi^o(v')) - (1-\alpha)v'_k(\varphi_k^o(v')) \\
&= v_k(\varphi_k^o(v')) - \Big(\alpha x_k(v', \varphi^o(v')) + (1-\alpha)v'_k(\varphi_k^o(v'))\Big) \\
&= v_k(\varphi_k^o(v')) - \varphi_k^m(v'),
\end{aligned}$$

where the first inequality follows from the strategy-proofness of the Vickrey rule and the second one because v'_k is an overbid wrt v_k. Hence, the desired inequality has been shown. □

The above proposition offers a simple formula for the auctioneer to identify some overbidding-proof rules using the Vickrey rule as a benchmark. It is enough to announce that the packages will be allocated efficiently and each bidder will pay a mixture between his/her Vickrey price for the allocated package and the pay-as-bid price, that is the bidder's reported valuation for that package. Among the rules in this family (3) there is at least one core-selecting rule that is the pay-as-bid rule that corresponds to $\alpha = 0$. Hence, overbidding-proofness is compatible with core-selection.

In the family of rules defined by (3), the ratio $\alpha \geq 0$ is fixed, but we could consider it depends on the valuation profile, $\alpha(v) \in [0, 1]$ for all $v \in V^n$. Then, if we require some monotonicity such as $\alpha(v) \geq \alpha(v')$ whenever v' is an overbid of v, then the corresponding rule would also be overbidding-proof.

Let us denote by $\mathcal{R}_V^\alpha$ the family of rules defined by (3). It contains an appealing rule that approaches as much as possible the Vickrey rule while preserving core-selection.

Definition 4.2. *Let φ^{α^*} be any rule in $\mathcal{R}_V^\alpha$ such that $\alpha^* \in [0, 1]$ is maximum such that the payoff vector where each bidder gets $v_i(\varphi_i^{\alpha^* o}(v)) - \varphi_i^{\alpha^* m}(v)$ and the seller gets $\sum_{i \in N} \varphi_i^{\alpha^* m}(v)$ is in the core with respect to the reported valuations v.*

Notice first that the rule φ^{α^*} coincides with the Vickrey rule whenever this is in the core, and then $\alpha^* = 1$. By the definition of α^*, φ^{α^*} is weak bidder Pareto-optimal in the core. But we prove next that it is in fact bidder Pareto-optimal, that is, it selects a bidder-optimal core allocation.

Theorem 4.3. *The rule φ^{α^*} that for any valuation profile v selects an efficient assignment $\varphi^{\alpha^* o}(v)$ and the prices*

$$\varphi_k^{\alpha^* m}(v) = \alpha^* x_k(v, \varphi^{\alpha^* o}(v)) + (1-\alpha^*)v_k(\varphi_k^{\alpha^* o}(v)) \text{ for each } k \in N, \tag{4}$$

with α^* maximum $\alpha \in [0, 1]$ such that φ^α is a core-selecting rule, is bidder-optimal in the core with respect to the reported valuations v.

Proof. Let us consider any valuation profile v and denote by $y^i \in \mathbb{R}^{N\cup\{0\}}$ the core payoff where each bidder in $N \setminus \{i\}$ gets 0 except for bidder i that gets his/her marginal contribution $M_i^v = w_v(N \cup \{0\}) - w_v((N \setminus \{i\}) \cup \{0\})$. Recall that by efficiency the seller gets $w_v((N \setminus \{i\}) \cup \{0\})$. Denote by x^{α^*} the payoff vector that results when the rule φ^{α^*} is applied to profile v, and by x^V the payoff vector that results when applying the Vickrey rule, where every bidder gets his/her marginal contribution M_i^v.

By efficiency, the seller's payoff is uniquely determined by the bidders' payoffs. Hence we will focus on the bidders' payoffs but keeping for simplicity the notation of the complete payoff vectors. If x^{α^*} is not bidder Pareto-optimal, there exists $i' \in N$ and

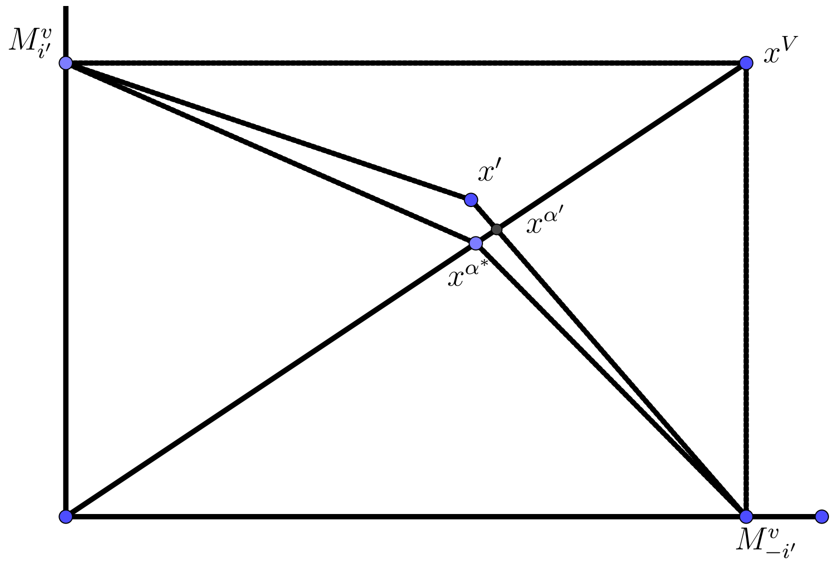

Figure 1: An illustration of the rule φ^{α^*}

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$x' \in C(w_v)$ such that $x'_{i'} > x_{i'}^{\alpha^*}$ and $x'_i = x_i^{\alpha^*}$ for all $i \in N \setminus \{i'\}$. Let $px = q$, with $p \in \mathbb{R}^N$ and $q \in \mathbb{R}$, be the hyperplane that contains the polytope with extreme points $x^{i'}$ and y^i for all $i \in N \setminus \{i'\}$, see Figure 1. Note that, from convexity of the core, this polytope is contained in $C(w_v)$. Moreover, $px' = q$, $px^V > q$ and $px^{\alpha^*} < q$.

Then, there is $\alpha' > \alpha^*$ such that $x_i^{\alpha'} = v_i(\varphi_i^{\alpha' o}(v)) - \varphi_i^{\alpha' m}(v)$ where $\varphi_i^{\alpha' m}(v) = \alpha x_i(v, \varphi^{\alpha' o}(v)) + (1-\alpha) v_i(\varphi_k^{\alpha' o}(v))$ for each $i \in N$.The payoff vector $x^{\alpha'}$ belongs to the core $C(w_v)$ and then contradicts that α^* is the maximum $\alpha \in [0,1]$ such that φ^α is a core-selecting rule. □

The rule φ^{α^*} can be interpreted as a one-shot auction rule since it is anonymous. The seller announces that the objects will be assigned efficiently according to the reported valuations and the price every bidder will pay for the assigned package will be a fixed proportion between the Vickrey price and the pay-as-bid price. The weight of the Vickrey price will be the maximum such that the final payoff is in the core of the auction game according the reported valuations.

It is well known that not all core allocation of the auction game can be priced by means of the usual competitive prices, where the price of a package is linear on the price of the objects and does not depend on the buyer of the package. However, Bikhchandani & Ostroy (2002) show that the core coincides with the set of competitive allocations if we allow for prices of packages that are not linear and not anonymous. As a consequence, the price of the rule φ^{α^*} is competitive in that sense.

Day & Milgrom (2008) also prove that in a core-selecting auction no bidder can earn more than its Vickrey payoff by disaggregating and bidding with shills. Hence, this is another kind of misrepresentation that the φ^{α^*} rules out.

5 Monotonicity and single-minded bidder domains

This section is devoted to establish the results of the previous sections on two different domains: monotonic valuations and single-minded bidders.

Monotonicity states that for any bidder, the more objects in a package, the better. For each buyer $i \in N$, his valuation v_i is monotonic if it satisfies

$$v_i(B') \leq v_i(B) \text{ for all } B' \subseteq B \subseteq Q_i$$

Denote by $V_{\mathcal{M}}$ the set of all monotonic valuations.

Theorem 5.1. *On the domain of monotonic valuations, $V^n_{\mathcal{M}}$, the following assertions hold:*

1. *If an efficient and individually rational rule φ satisfies overbidding-proofness (underbidding-proofnes), then prices satisfy*

   $$\varphi^m_k(v) \in [x_k(v, \varphi^o(v)),\ v_k(\varphi^o_k(v))] \text{ for each profile } v \text{ and for each } k \in N.$$
   $$(\varphi^m_k(v) \in [0, x_k(v, \varphi^o(v))] \text{ for each profile } v \text{ and for each } k \in N, \text{ respectively.})$$

2. *If a rule φ satisfies efficiency, individual rationality, overbidding-proofnes (underbidding-proofness) and is not a Vickrey rule, then it can be manipulated by underbidding (overbidding, respectively).*

3. *The Vickrey rule is characterized by individual rationality, efficiency, underbidding-proofness and overbidding-proofness.*

Proof. We first prove statement *1.* for overbidding-proofness. By contradiction, assume that there is an efficient, individually rational and overbidding-proof rule φ, a valuation profile v and an agent k, such that $\varphi^m_k(v) \notin [x_k(v, \varphi^o(v)), v(\varphi^o_k(v))]$. By individual rationality, $v(\varphi^o_k(v)) \geq \varphi^m_k(v)$. Therefore, $x_k(v, \varphi^o(v)) > \varphi^m_k(v)$.

Define $\hat{v}_k$ for every $B \subseteq Q_k$ in the following way:

(i) $\hat{v}_k(B) = 0$ for all $B \not\supseteq \varphi^o_k(v)$,

(ii) $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$, and

(iii) $\hat{v}_k(B) = \hat{v}_k(\varphi^o_k(v))$ for all $B \supseteq \varphi^o_k(v)$.

Note that $\hat{v}_k \in V_{\mathcal{M}}$. It is straightforward to see that v_k is an overbid wrt $\hat{v}_k$. Take any $B \subseteq Q_k$ and notice that if $B \not\supseteq \varphi_k^o(v)$, then $v_k(B) \geq 0 = \hat{v}_k(B)$, and if $B \supseteq \varphi_k^o(v)$, then

$$v_k(B) \geq v_k(\varphi_k^o(v)) \geq x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi_k^o(v)) = \hat{v}_k(B),$$

where the first inequality holds because of monotonicity of v_k, the second inequality by definition of the Vickrey price, the strict inequality and the equality, by definition of $\hat{v}_k$.

Now, by statement *3.* in Lemma 7.1, we have that $\varphi_k^o(v_{-k}, \hat{v}_k)$ is such that $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) = 0$. By individual rationality, $\varphi_k^m(v_{-k}, \hat{v}_k) = 0$. Moreover, by definition of $\hat{v}_k$, we have $\hat{v}_k(\varphi_k^o(v)) > \varphi_k^m(v)$ and hence,

$$\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) = 0 < \hat{v}_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

which shows that agent k has incentives to overbid, i.e., bidder k claims v_k when his true preference is $\hat{v}_k$. This is a contradiction with the fact that φ satisfies overbidding-proofness. Hence, this shows that $\varphi_k^m(v) \in [x_k(v, \varphi^o(v)), v(\varphi_k^o(v))]$ as stated.

We continue proving statement *1.* for underbidding-proofness. By contradiction, assume that there is an efficient, individually rational and underbidding-proof rule, a valuation profile v and an agent k, such that $\varphi_k^m(v) \notin [0, x_k(v, \varphi^o(v))]$. Therefore we have $x_k(v, \varphi^o(v)) < \varphi_k^m(v)$.

Define $\hat{v}_k$ for each $B \subseteq Q_k$ such that:

(i) $\hat{v}_k(B) = 0$ for all $B \not\supseteq \varphi_k^o(v)$,

(ii) $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v)) > x_k(v, \varphi^o(v))$, and

(iii) $\hat{v}_k(B) = \hat{v}_k(\varphi_k^o(v))$ for all $B \supseteq \varphi_k^o(v)$.

Note that $\hat{v}_k \in V_{\mathcal{M}}$. Now, we see that $\hat{v}_k$ is an underbid wrt v_k. Take any $B \subseteq Q_k$ and notice that if $B \nsubseteq \varphi_k^o(v)$, then $v_k(B) \geq 0 = \hat{v}_k(B)$, and if $B \subseteq \varphi_k^o(v)$, then $v_k(B) \geq v_k(\varphi_k^o(v)) \geq x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi_k^o(v)) = \hat{v}_k(B)$, where the first inequality holds because of monotonicity of v_k, the second inequality by definition of the Vickrey price, the strict inequality and the equality, by definition of $\hat{v}_k$.

By statement *4.* in Lemma 7.1, we have $\varphi_k^o(v_{-k}, \hat{v}_k) \supseteq \varphi_k^o(v)$. Moreover, by definition of $\hat{v}$ and by individual rationality, $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$ and $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$, hence $\varphi_k^m(v) > \varphi_k^m(v_{-k}, \hat{v}_k)$. As a consequence,

$$v_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) \geq v_k(\varphi_k^o(v)) - \varphi_k^m(v_{-k}, \hat{v}_k) > v_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

where the first inequality comes from monotonicity of v_k. The expression above shows that agent k has incentives to claim $\hat{v}_k$ when his true preference is v_k. This is a contradiction with the fact that φ satisfies underbidding-proofness.

Let us now prove statement *2.* for overbidding-proofness. We know from statement *1.* in this Theorem, that for any bidder k and any valuation profile v, $\varphi_k^m(v) \geq x_k(v, \varphi^o(v))$. Since φ is not the Vickrey rule, there is at least one v such that $\varphi_k^m(v) > x_k(v, \varphi^o(v))$ for some bidder k. Take this valuation profile v and this bidder.

Define $\hat{v}_k$ for each $B \subseteq Q_k$ such that $\hat{v}_k(B) = 0$ for all $B \not\supseteq \varphi_k^o(v)$, $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v)) > x_k(v, \varphi^o(v))$, and $\hat{v}_k(B) = \hat{v}_k(\varphi_k^o(v))$ for all $B \supseteq \varphi_k^o(v)$. Note that $\hat{v}_k \in V_{\mathcal{M}}$ and $\hat{v}_k$ is an underbid wrt v_k.

From statement *4.* in Lemma 7.1, we have $\varphi_k^o(v_{-k}, \hat{v}_k) \supseteq \varphi_k^o(v)$. Then, by definition of $\hat{v}$ and by individual rationality, $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$ and $\varphi_k^m(v) > \hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) \geq \varphi_k^m(v_{-k}, \hat{v}_k)$, hence $\varphi_k^m(v) > \varphi_k^m(v_{-k}, \hat{v}_k)$. As a consequence,

$$v_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) \geq v_k(\varphi_k^o(v)) - \varphi_k^m(v_{-k}, \hat{v}_k) > v_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

where the first inequality comes from monotonicity of v_k. The expression above shows that agent k has incentives to claim $\hat{v}_k$ when his true preference is v_k.

We continue with the proof of statement *2.* for underbidding-proofness. From statement *1.* in this Theorem, we know that for any bidder k and any valuation profile v, $x_k(v, \varphi^o(v)) \geq \varphi_k^m(v)$. Since φ is not the Vickrey rule, there is at least one v such that $x_k(v, \varphi^o(v)) > \varphi_k^m(v)$ for some bidder k. Take this valuation profile v and this bidder.

Define $\hat{v}_k$ for every $B \subseteq Q_k$ such that $\hat{v}_k(B) = 0$ for all $B \not\supseteq \varphi_k^o(v)$, $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi_k^o(v)) > \varphi_k^m(v)$, and $\hat{v}_k(B) = \hat{v}_k(\varphi_k^o(v))$ for all $B \supseteq \varphi_k^o(v)$. Note that $\hat{v}_k \in V_{\mathcal{M}}$ and v_k is an overbid wrt $\hat{v}_k$.

From statement *3.* in Lemma 7.1, we have that by reporting $\hat{v}_k$, $\varphi_k^o(v_{-k}, \hat{v}_k)$ is such that $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) = 0$. Then, by individual rationality, $\varphi_k^m(v_{-k}, \hat{v}_k) = 0$. Moreover, by definition of $\hat{v}_k$, we have $\hat{v}_k(\varphi_k^o(v)) > \varphi_k^m(v)$. Hence,

$$\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) - \varphi_k^m(v_{-k}, \hat{v}_k) = 0 < \hat{v}_k(\varphi_k^o(v)) - \varphi_k^m(v),$$

which shows that agent k has incentives to overbid, i.e., bidder k claims v_k when his true preference is $\hat{v}_k$.

Finally, statement *3.* is a direct consequence of statements *1.* and *2.* □

Now, let us consider the domain of single-minded bidders. Let $r \in \mathbb{R}_+$ be a non-negative number and fix a package $B_i \subseteq Q_i$ for agent i, the referring package for i. Bidder i is single-minded if

(i) $v_i(B') = r$ for all $B' \subseteq Q_i$ such that $B_i \subseteq B'$, and

(ii) $v_i(B') = 0$ for all $B' \subseteq Q_i$ such that $B_i \nsubseteq B'$.

When a bidder is single-minded, the seller and every other bidder knows it, that is, the referring package B_i is known by the other participants in the allocation problem.[7] Another characteristic of the single-minded condition is that it belongs to the family of monotonic valuations. A single-minded bidder i is interested in a particular package B_i and he is not interested in packages that do not contain B_i. Denote by $V_{\mathcal{SM}}$ the set of all single-minded valuations and notice that $V_{\mathcal{SM}} \subseteq V_{\mathcal{M}}$.

The following result extends our previous Theorem to the class of single-minded bidders. We state the result without proof since it is analogous to the proof of Theorem 5.1. The reason is that whenever in the proof of the above theorem we select a misrepresentation of the valuations of a bidder, the reported valuations are not only monotonic but also single-minded.

[7]We consider the case of known single-minded bidders, i.e., the referring package B_i of bidder i is common knowledge, see e.g., Mu'alem & Nisan (2002).

Theorem 5.2. *On the domain of single-minded valuations, $V^n_{\mathcal{SM}}$, the following assertions hold:*

1. *If an efficient and individually rational rule φ satisfies overbidding-proofness (underbidding-proofnes), then prices satisfy*

$$\varphi^m_k(v) \in [x_k(v, \varphi^o(v)),\ v_k(\varphi^o_k(v))] \text{ for each profile } v \text{ and for each } k \in N.$$
$$(\varphi^m_k(v) \in [0, x_k(v, \varphi^o(v))] \text{ for each profile } v \text{ and for each } k \in N,\ \text{respectively.})$$

2. *If a rule φ satisfies efficiency, individual rationality, overbidding-proofnes (underbidding-proofness) and is not a Vickrey rule, then it can be manipulated by underbidding (overbidding, respectively).*

3. *The Vickrey rule is characterized by individual rationality, efficiency, underbidding-proofness and overbidding-proofness.*

6 Concluding remarks

This paper analyzes weaker forms of strategy-proofness for package allocation rules, such as overbidding-proofness and underbidding-proofness, and shows that these two properties together with efficiency and individually rationality already characterize the Vickrey rule. And this holds even if we restrict to monotonic valuations or single-minded valuations.

Moreover, overbidding-proofness is compatible with core-selection. We introduce a new rule that is efficient and makes each bidder pay a fixed combination between his/her Vickrey price and his/her pay-as-bid price. This rule is easy to understand since the Vickrey auction and the pay-as-bid auction are two well-known mechanisms. In the rule we introduce, the seller will announce that the weight of the Vickrey price will be the highest such that the resulting payoff vector is in the core of the auction game.

In this way, this rule overcomes some drawbacks of the Vickrey rule: the prices the bidders pay are (non-linear and non-anonymous) competitive prices, the payoff to the seller is not too low and the seller cannot disqualify bidders to get higher revenues, as it may happen in the Vickrey rule.

7 Appendix

We include here a lemma that puts together some facts that have been used while proving Propositions 3.4, 3.5, 3.6, 3.7 and Theorems 5.1 and 5.2.

Lemma 7.1. *Fix a bidder k, consider any efficient rule φ, two valuation profiles v and $\hat{v} = (v_{-k}, \hat{v}_k)$ where $v_k \neq \hat{v}_k$. Then the following statements hold.*

1. *If $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi^o_k(v)$ and $x_k(\varphi^o(v), v) > \hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$, then $\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) = 0$.*

2. *If $\hat{v}_k(B) = 0$ for all $B \subseteq Q_k$, $B \neq \varphi^o_k(v)$ and $\varphi^m_k(v) > \hat{v}_k(\varphi^o_k(v)) > x_k(v, \varphi^o(v))$, then $\varphi^o_k(\hat{v}) = \varphi^o_k(v)$.*

3. *If for all $B \subseteq Q_k$, we have $\hat{v}_k(B) = 0$ for $B \not\supseteq \varphi^o_k(v)$, $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$, and $\hat{v}_k(B) = \hat{v}_k(\varphi^o_k(v))$ for all $B \supseteq \varphi^o_k(v)$, then $\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) = 0$.*

4. *If for all $B \subseteq Q_k$, we have $\hat{v}_k(B) = 0$ for all $B \not\supseteq \varphi^o_k(v)$, $\varphi^m_k(v) > \hat{v}_k(\varphi^o_k(v)) > x_k(v, \varphi^o(v))$, and $\hat{v}_k(B) = \hat{v}_k(\varphi^o_k(v))$ for all $B \supseteq \varphi^o_k(v)$, then $\varphi^o_k(v_{-k}, \hat{v}_k) \supseteq \varphi^o_k(v)$.*

5. *If B_k is the referring package for k, as defined in Section 5, and for all $B \subseteq Q_k$ we have $\hat{v}_k(B) = 0$ for all $B \not\supseteq B_k$, $x_k(v, \varphi^o(v)) > \hat{v}_k(\varphi^o_k(v)) > \varphi^m_k(v)$, and $\hat{v}_k(B) = \hat{v}_k(\varphi^o_k(v))$ for all $B \supseteq B_k$, then $\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) = 0$.*

6. *If B_k is the referring package for k, as defined in Section 5, and for all $B \subseteq Q_k$ we have $\hat{v}_k(B) = 0$ for all $B \not\supseteq B_k$, $x_k(v, \varphi^o(v)) < \hat{v}_k(\varphi^o_k(v)) < \varphi^m_k(v)$, and $\hat{v}_k(B) = \hat{v}_k(\varphi^o_k(v))$ for all $B \supseteq B_k$, then $\varphi^o_k(v_{-k}, \hat{v}_k) \supseteq B_k$.*

Proof. We first prove statement 1. Notice that

$$\begin{aligned}\max_{z \in \mathcal{Z}: z_k = \varphi^o_k(v)} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\} &= \sum_{i \in N} \hat{v}_i(\varphi^o_i(v)) = \sum_{i \in N \setminus \{k\}} v_i(\varphi^o_i(v)) + \hat{v}_k(\varphi^o_k(v)) \\ &< \sum_{i \in N \setminus \{k\}} v_i(\varphi^o_i(v)) + x_k(\varphi^o(v), v) \\ &= \sum_{i \in N \setminus \{k\}} v_i(\varphi^o_i(v)) + \max_{z \in \mathcal{Z}_{-k}} \left\{ \sum_{i \in N \setminus \{k\}} v_i(z_i) \right\} \\ &- \sum_{i \in N \setminus \{k\}} v_i(\varphi^o_i(v)) = \max_{z \in \mathcal{Z}: \hat{v}_k(z_k) = 0} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\}.\end{aligned}$$

where the first equation follows from efficiency of φ, the second equation because $\hat{v}_i = v_i$ for all $i \in N \setminus \{k\}$, the strict inequality by the assumption $x_k(\varphi^o(v), v) > \hat{v}_k(\varphi^o_k(v))$ and the last but one equation from (1). Hence, we have

$$\max_{z \in \mathcal{Z}: z_k = \varphi^o_k(v)} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\} < \max_{z \in \mathcal{Z}: \hat{v}_k(z_k) = 0} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\}. \quad (5)$$

Now, since

$$\max_{z \in \mathcal{Z}_{-k}} \left\{ \sum_{i \in N \setminus \{k\}} v_i(z_i) \right\} = \max_{z \in \mathcal{Z}: \hat{v}_k(z_k) = 0} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\},$$

by substitution in (5), we have

$$\max_{z \in \mathcal{Z}: z_k = \varphi^o_k(v)} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\} < \max_{z \in \mathcal{Z}: \hat{v}_k(z_k) = 0} \left\{ \sum_{i \in N} \hat{v}_i(z_i) \right\},$$

and, from the assumption $\hat{v}(B) = 0$ for all $B \neq \varphi^o(v)$ and the efficiency of φ, this implies that $\hat{v}_k(\varphi^o_k(v_{-k}, \hat{v}_k)) = 0$.

To prove statement 2., notice first that

$$\begin{aligned}\sum_{i\in N}\hat{v}_i(\varphi_i^o(v)) &= \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \hat{v}_k(\varphi_k^o(v)) > \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + x_k(v,\varphi^o(v))\\
&= \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \max_{z\in\mathcal{Z}_{-k}}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)\right\} - \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v))\\
&= \max_{z\in\mathcal{Z}_{-k}}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)\right\} \geq \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)+\hat{v}_k(z_k)\right\}\\
&= \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\},\end{aligned}$$

where the strict inequality follows from the assumption that $\hat{v}_k(\varphi_k^o(v)) > x_k(v,\varphi^o(v))$, the second equation from (1), and the weak inequality follows since each $z\in\mathcal{Z}$ with $\hat{v}(z_k)=0$ corresponds to an assignment in $\mathcal{Z}_{-k}$ with the same total value.

Moreover, since $\hat{v}_k(\varphi_k^o(v)) > 0$, we have

$$\max_{z\in\mathcal{Z}:\hat{v}_k(z_k)\neq 0}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\} \geq \sum_{i\in N}\hat{v}_i(\varphi_i^o(v)) > \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\},$$

and efficiency, together with the assumption $\hat{v}(B)=0$ for all $B\neq\varphi^o(v)$, implies that $\varphi_k^o(v_{-k},\hat{v}_k)$ is $\varphi_k^o(v)$.

Let us now prove statement 3. By definition of $\hat{v}$, we have,

$$\begin{aligned}\max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=\hat{v}_k(\varphi_k^o(v))}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\} &= \sum_{i\in N}\hat{v}_i(\varphi_i^o(v)) = \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \hat{v}_k(\varphi_k^o(v))\\
&< \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + x_k(v,\varphi^o(v))\\
&= \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \max_{z\in\mathcal{Z}_{-k}}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)\right\}\\
&- \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) = \max_{z\in\mathcal{Z}_{-k}}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)\right\}.\end{aligned}$$

Now, taking into account

$$\max_{z\in\mathcal{Z}_{-k}}\left\{\sum_{i\in N\setminus\{k\}} v_i(z_i)\right\} = \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\}.$$

we obtain

$$\max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=\hat{v}_k(\varphi_k^o(v))}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\} < \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0}\left\{\sum_{i\in N}\hat{v}_i(z_i)\right\}.$$

This shows that $\varphi_k^o(v_{-k}, \hat{v}_k)$ is such that $\hat{v}_k(\varphi_k^o(v_{-k}, \hat{v}_k)) = 0$

Under the assumptions of statement 5., this same argument is valid to prove that $\hat{v}(\varphi_k^o(v_{-k}, \hat{v}_k)) = 0$.

To prove statement 4., notice first that

$$\begin{aligned}\sum_{i\in N} \hat{v}_i(\varphi_i^o(v)) &= \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \hat{v}_k(\varphi_k^o(v)) > \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + x_k(v, \varphi^o(v)) \\ &= \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) + \max_{z\in\mathcal{Z}_{-k}} \left\{ \sum_{i\in N\setminus\{k\}} v_i(z_i) \right\} - \sum_{i\in N\setminus\{k\}} v_i(\varphi_i^o(v)) \\ &= \max_{z\in\mathcal{Z}_{-k}} \left\{ \sum_{i\in N\setminus\{k\}} v_i(z_i) \right\} \geq \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0} \left\{ \sum_{i\in N\setminus\{k\}} v_i(z_i) + \hat{v}_k(z_k) \right\} \\ &= \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0} \left\{ \sum_{i\in N} \hat{v}_i(z_i) \right\}.\end{aligned}$$

Moreover, since $\hat{v}_k(B) > 0$, for every $B \supsetneq \varphi_k^o(v)$ we have

$$\max_{z\in\mathcal{Z}:\hat{v}_k(z_k)\neq 0} \left\{ \sum_{i\in N} \hat{v}_i(z_i) \right\} \geq \sum_{i\in N} \hat{v}_i(\varphi_i^o(v)) > \max_{z\in\mathcal{Z}:\hat{v}_k(z_k)=0} \left\{ \sum_{i\in N} \hat{v}_i(z_i) \right\},$$

and then efficiency implies that $\varphi_k^o(v_{-k}, \hat{v}_k) \supsetneq \varphi_k^o(v)$.

Under the assumptions of statement 6., this same argument is valid to prove that $\varphi_k^o(v_{-k}, \hat{v}_k)) \supsetneq B_k$.

□

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