Axioms for the optimal stable rules and fair-division rules in a multiple-partners job market.⋆

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Abstract

In the multiple-partners job market, introduced in (Sotomayor, 1992), each firm can hire several workers and each worker can be hired by several firms, up to a given quota. We show that, in contrast to what happens in the simple assignment game, in this extension, the firms-optimal stable rules are neither valuation monotonic nor pairwise monotonic. However, we show that the firms-optimal stable rules satisfy a weaker property, what we call *firm-covariance*, and that this property characterizes these rules among all stable rules. This property allows us to shed some light on how firms can (and cannot) manipulate the firms-optimal stable rules. In particular, we show that firms cannot manipulate them by constantly over-reporting their valuations. Analogous results hold when focusing on the workers. Finally, we extend to the multiple-partners market a known characterization of the fair-division rules on the domain of simple assignment games.

Keywords: assignment game; stable rules; fair division.

1 Introduction

The aim of this paper is to study some allocation rules in a two-sided job market with firms on one side and workers on the other side. Each agent has a quota that determines in how many partnerships with agents of the opposite side this agent can enter. Each potential partnership has a value and a rule determines a matching and an allocation of

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the value of each partnership between the partners. This model is an extension of the well-known Shapley and Shubik assignment game.

The assignment game was introduced in Shapley & Shubik (1972) as a coalitional-game model for a two-sided market, formed by buyers and sellers or firms and workers, where each agent on one side is to be matched to at most one agent on the opposite side. The objective is to propose a matching and an allocation of the worth of each matched pair among the partners in such a way that no buyer-seller pair (or firm-worker pair) blocks the proposed matching because they can get a higher payoff by being matched together.

Shapley and Shubik prove that, for such markets, stable outcomes always exist and form a complete lattice, which guarantees the existence of an optimal stable outcome for each side of the market. They also prove the coincidence between the core, the set of stable payoff vectors and the set of competitive equilibria payoff vectors.

Many extensions of the Shapley and Shubik assignment game, that we will call the simple assignment game, have been studied since then. The first ones allow agents to be matched to more than one partner. Kaneko (1976) assumes that buyers can only buy one good from one seller while each seller can sell to more than one buyer. The core is also non-empty but (strictly) contains the set of competitive equilibrium payoff vectors. Thompson (1981) allows that both buyers and sellers can take part in multiple partnerships, up to a given quota exogenously determined for each agent. This extension was also studied in Sánchez-Soriano *et al.* (2001) with the name of transportation game and in Sotomayor (2002). It turns out that the core, that also contains the set of competitive equilibrium payoff vectors, is non-empty but has no longer a lattice structure. In this model, existence of optimal core allocations for each side of the market is still an open question.

A different extension of the simple assignment game was introduced in Sotomayor (1992) with the name of multiple-partners assignment game, and this is the model that better fits with our initial job market situation. In the multiple-partners assignment game, each agent can also take part in multiple partnerships, as many as the agent's quota allows, but can trade at most one unit with each possible partner. Utilities are assumed to be additively separable. Again, an outcome consists of a matching and an allocation of the worth of each partnership between the two partners. In this setting, a notion of (pairwise) stable outcome is similarly defined. Sotomayor (1992) shows that the set of stable payoffs is non-empty and a subset of the core, and that it can be strictly smaller than the core. Sotomayor (1999) adds that the set of stable payoffs is endowed with a complete lattice structure under two convenient partial order relations. Although these partial orders are not defined by the preferences of the agents, all agents on the same side of the market agree on the best stable payoff for them. The relationship with the set of competitive equilibrium payoffs is analysed in Sotomayor (2007) and a mechanism that yields the buyers-optimal competitive equilibrium payoff, which coincides with the buyers-optimal stable payoff, is obtained in Sotomayor (2009). Pérez-Castrillo & Sotomayor (2019) analyses how the optimal stable and competitive solutions react to the introduction of a new agent to the market, depending on whether it is a buyer or a seller.

The aim of our paper is to study stable allocation rules, that is, rules that given the values of all possible partnerships select a stable outcome. In particular we will focus on the two optimal stable rules. We first generalize some monotonicity properties: pairwise monotonicity and firm-valuation monotonicity (or worker-valuation monotonicity), that are satisfied by optimal stable rules in the simple assignment game. Firm-valuation monotonicity states that if the values of a firm weakly decrease but this does not modify the partners of this firm given by the rule, then this firm should not receive a higher payoff in any of its partnerships. In the simple assignment game this property characterizes the firms-optimal stable rules among all stable rules (van den Brink *et al.* , 2021).

We show that the optimal stable rules for the multiple-partners assignment game do not satisfy the aforementioned monotonicity properties: the firms-optimal stable rules are neither firm-valuation monotonic nor pairwise monotonic. However, the firms-optimal stable rules satisfy a weaker form of valuation monotonicity. We strengthen the conditions under which a decrease of the valuations of the firm should imply a decrease in that firm's payoffs: we only require monotonicity when all its valuations are decreased by the same amount. This weak firm-valuation monotonicity is a consequence of what we call firm-covariance. Roughly speaking, a rule is firm-covariant if when all valuations of a firm decrease in a constant amount and all optimal matchings of the initial market still remain optimal, then the payoff this firm obtains in each partnership decreases in exactly that constant amount. We prove that firm covariance characterizes the firms-optimal stable rules among all stable rules, and worker covariance characterizes the workers-optimal stable rules among all stable rules.

Secondly, we focus on how agents can misrepresent their preferences to manipulate a stable rule in the multiple-partners assignment game. Pérez-Castrillo & Sotomayor (2017) analyse the manipulability of competitive equilibrium rules for this market game (with buyers instead of firms and sellers instead of workers). They show that (i) any agent who does not receive her/his optimal competitive equilibrium payoff under a competitive rule can profitably misrepresent her/his valuations, assuming the others tell the truth; (ii) if the buyers-optimal (respectively, sellers-optimal) competitive equilibrium rule is used in a market with more than one vector of equilibrium prices, then there is a seller (respectively, buyer) who can profitably misrepresent his (respectively, her) valuations and (iii) an agent with a quota of one cannot manipulate a rule in a market if and only if the rule gives to this agent her/his most preferred equilibrium payoff.

Since in multiple-partners assignment games the payoff vector of the buyers-optimal stable rule coincides with that of the buyers-optimal competitive equilibrium rule (Sotomayor, 2007), only the buyers with capacity one cannot manipulate the rule. However, we show that these stable rules that are optimal for one side of the market have a weaker non-manipulability property: on the domain of multiple-partner job markets where all firm-worker pairs are acceptable, no firm can manipulate the firms-optimal stable rule by constantly over-reporting its valuations. Similarly, no worker can manipulate the workers-optimal stable rule by under-reporting his/her valuation.

There is some experimental evidence that bidders tend to over-report valuations in some auctions. See for instance Kagel & Levin (1993) for second price auctions, Kagel & Levin (2009) for the Vickrey multi-unit demand auction, or Kagel *et al.* (2014) for some combinatorial auctions with package bidding. We see that, although over-reporting may be profitable for firms (or buyers) if the firms-optimal stable rule is implemented in a multiple-partners job market, the least sophisticated form of over-reporting which consists in adding the same constant to all firm's valuations, does not bring any additional profit.

Finally, we consider the fair-division rules. The payoff vector of these rules is the midpoint between the firms-optimal and the workers-optimal payoff vectors. On the domain of simple assignment games, these rules have been characterized in van den Brink *et al.* (2021) by means of two properties: great valuation fairness and weak derived consistency. We adapt the definition of these two properties to the domain of multiple-partners job markets. Great valuation fairness requires that when the value of all firm-worker pair decreases by a constant amount (up to a given threshold that guarantees that all optimal matchings of the initial market remain optimal) then all players suffer the same reduction in the payoff they receive from the rule. Weak derived consistency only requires consistency of the payoffs when the market is reduced at a firm-worker pair that have the same payoff at any stable outcome. We show that these two properties individualize the fair-division rules among all stable rules.

The structure of the paper is as follows. In Section 2 we introduce the multiple-partners job market, Section 3 contains the characterizations of the two optimal stable rules, Section 4 discusses the manipulability of these rules and Section 5 characterizes the fair-division rules.

2 The multiple-partners assignment game

Let $F = \{f_1, f_2, \dots, f_m\}$ be a finite set of firms and $W = \{w_1, w_2, \dots, w_n\}$ a finite set of workers. Each firm f_i values in $h_{ij} \geq 0$ being matched to worker w_j. Also, each worker w_j has a reservation value $t_j \geq 0$, that can be interpreted as how much worker w_j values each one of his available slots. If firm f_i hires worker w_j, then a value $a_{ij} = \max\{h_{ij} - t_j, 0\} \geq 0$ is generated that has to be shared by both partners. Sometimes we will normalize reservation values of workers at zero and then $a_{ij} = h_{ij}$ for all $i = 1, 2 \dots, m$ and $j = 1, 2, \dots, n$. Each firm f_i can hire at most r_i workers and each worker w_j can work for at most s_j firms.

A *multiple-partners assignment market* or a *multiple-partners job market* is then defined by (F, W, a, r, s) where $a = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, $r = (r_i)_{i=1,\dots,m}$ and $s = (s_j)_{j=1,\dots,n}$. The set of all possible valuation profiles for a set F of firms and a set W of workers is denoted by $\mathcal{A}^{F \times W}$. We add a dummy agent on each side of the market, f_0 and w_0, such that $a_{00} = a_{i0} = a_{0j} = 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. As for the quotas, a dummy player may form as many partnerships as needed to fill up the quotas of the non-dummy players. We write $F_0 = F \cup \{f_0\}$ and $W_0 = W \cup \{w_0\}$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player. When all non-dummy agents have quota 1, this model coincides with the one in Shapley & Shubik (1972) and we will say it is a *simple assignment game*.

A *matching* μ is a subset of $F_0 \times W_0$, that does not violate the quotas of the players, that is, each $f_i \in F$ appears in exactly r_i pairs of μ and each $w_j \in W$ appears in exactly s_j pairs of μ, since a firm that does not fill some of its positions is assumed to be matched to the dummy worker w_0 (and similarly for workers with unfilled positions).

When necessary, we denote by μ_{f_i} the set of partners of firm $f_i \in F$ in matching μ, that is $\mu_{f_i} = \{j \in W_0 \mid (f_i, w_j) \in \mu\}$. Similarly, for all $w_j \in W$, $\mu_{w_j} = \{i \in F_0 \mid (f_i, w_j) \in \mu\}$.

The set of all matchings is $\mathcal{M}(F, W, r, s)$. A matching μ is *optimal* if, for any other $\mu' \in \mathcal{M}(F, W, r, s)$, $\sum_{(f_i,w_j)\in\mu} a_{ij} \geq \sum_{(f_i,w_j)\in\mu'} a_{ij}$. The set of optimal matchings is $\mathcal{M}_a(F, W, r, s)$.

From this market situation, a coalitional game $(F \cup W, w_a)$, the *multiple-partners assignment game* is defined with set of agents $F \cup W$ and coalitional function

$$w_a(T) = \max_{\mu\in\mathcal{M}(F\cap T, W\cap T, r, s)} \sum_{(f_i,w_j)\in\mu} a_{ij}$$

for all $T \subseteq F \cup W$ with $T \cap F \neq 0$ and $T \cap W \neq 0$, and $w_a(T) = 0$ otherwise.

An outcome for the market (F, W, a, r, s) consists of a matching and the payoffs that each agent obtains from each of the partnerships he/she establishes in this matching. That is, if firm f_i hires worker w_j at a salary v_{ij}, this firm receives $u_{ij} = a_{ij} - v_{ij}$.

Definition 2.1. *Let (F, W, a, r, s) be a multiple-partner job market. A feasible outcome is $(u, v; \mu)$ where $\mu \in \mathcal{M}(F, W, r, s)$ and for each $(f_i, w_j) \in \mu$,*

1. $u_{ij} + v_{ij} = a_{ij}$,
2. $u_{ij} \geq a_{i0}$ *and* $v_{ij} \geq a_{0j}$,
3. *if* $f_i = f_0$, *then* $v_{0j} = a_{0j}$,
4. *if* $w_j = w_0$, *then* $u_{i0} = a_{i0}$.

Notice that u_{ij} and v_{ij} are only defined if $(f_i, w_j) \in \mu$. Hence, u and v contain a list of dissagregated payoffs for each agent, one for each partnership established by μ. Also, as a consequence of the above definition, the payoff of the dummy agents is always zero.

For these markets, a notion of stability, sometimes called pairwise stability, is defined in Sotomayor (1992).

Definition 2.2. *Let (F, W, a, r, s) be a multiple-partner job market. A stable outcome is a feasible outcome $(u, v; \mu)$ such that for all $(f_i, w_j) \notin \mu$,*

$$u_{ik} + v_{lj} \geq a_{ij} \text{ for all } (f_i, w_k) \in \mu \text{ and } (f_l, w_j) \in \mu. \tag{1}$$

Notice that if there existed (f_i, w_k) and (f_l, w_j) in μ such that $u_{ik} + v_{lj} < a_{ij}$, then f_i and w_j might break their current partnerships with w_k and f_l, respectively, and form a new one together, because this could give to each of them a higher payoff.

It is shown in Sotomayor (1992) that if $(u, v; \mu)$ is a stable outcome, then μ is an optimal matching.

For the multiple-partners assignment game, stable outcomes always exist. This is proved in Sotomayor (1992) in two different ways: one of them uses linear programming[1]

[1]It is shown in Appendix 1 of Sotomayor (1992) the relationship between the set of stable outcomes of the multiple-partners assignment market and the set of dual solutions to the linear program that obtains an optimal matching for this market.

and the second one, that we comment on below, is based on a replication of the players and a convenient way of defining the valuation matrix.

Given any multiple partners-assignment game (F, W, a, r, s) we can define a *related simple assignment game* $(\tilde{F}, \tilde{W}, \tilde{a})$ in the following way. Each firm f_i with quota r_i is replicated r_i times and each worker w_j with quota s_j is replicated s_j times:

$$\tilde{F} = \{f_{ik} \mid i = 1, \dots, m;\ k = 1, \dots, r_i\} \text{ and } \tilde{W} = \{w_{kj} \mid j = 1, \dots, n;\ k = 1, \dots, s_j\}$$

with unitary quotas, $\tilde{r}_{ik} = 1$ for all $i = 1, \dots, m$ and $k = 1, \dots, r_i$, and $\tilde{s}_{kj} = 1$ for all $j = 1, \dots, n$ and $k = 1 \dots, s_j$. Moreover, given $\mu \in \mathcal{M}_a(F, W, r, s)$, we define a one-to-one matching $\tilde{\mu}$ between $\tilde{F}$ and $\tilde{W}$ in this way: (i) if $(f_{ik}, w_{lj}) \in \tilde{\mu}$, then $(f_i, w_j) \in \mu$ and (ii) if $(f_i, w_j) \in \mu$, there exist one and only one $k = 1, \dots, r_i$ and one and only one $l = 1 \dots, s_j$ such that $(f_{ik}, w_{lj}) \in \tilde{\mu}$. This means that if f_i hires w_j under μ, then one copy of f_i hires one copy of w_j under $\tilde{\mu}$ and that no other copies of them are matched. After defining $\tilde{a}$, it can be shown that $\tilde{\mu}$ is optimal for $(\tilde{F}, \tilde{W}, \tilde{a})$.

Then, given $\tilde{\mu}$ as defined above, the valuation matrix $\tilde{a}$ of this related simple assignment game $(\tilde{F}, \tilde{W}, \tilde{\alpha})$ is defined by

$$\tilde{a}_{ik,lj} = \begin{cases} 0 & \text{if } (f_i, w_j) \in \mu \text{ and } (f_{ik}, w_{lj}) \notin \tilde{\mu}, \\ a_{ij} & \text{otherwise.} \end{cases} \tag{2}$$

Now, if $(u', v'; \tilde{\mu})$ is a feasible outcome for the simple assignment game $(\tilde{F}, \tilde{W}, \tilde{a})$, we can built a feasible outcome $(u, v; \mu)$ for the multiple-partners assignment game (F, W, a, r, s) in the following way:

$$\begin{aligned} &\text{if } (f_{ik}, w_{lj}) \in \tilde{\mu}, \text{ then define } u_{ij} = u'_{ik},\ v_{ij} = v'_{lj}, \text{ and} \\ &u_{i0} = v_{0j} = 0, \text{ whenever } i \text{ or } j \text{ are matched to a dummy partner.} \end{aligned} \tag{3}$$

Proposition 2 in Sotomayor (1992) shows that $(u', v'; \tilde{\mu})$ is stable for $(\tilde{F}, \tilde{W}, \tilde{a})$ if and only if $(u, v; \mu)$ is stable for (F, W, a, r, s).

Since it is well-known that stable outcomes always exist for the simple assignment game, the above result guarantees also existence for the multiple-partners assignment game. Moreover, Sotomayor (1999) proves that the payoff vectors of the set of stable outcomes form a convex and compact lattice and, as a consequence, there exists a unique optimal stable payoff vector for each side of the market. To this end, the problem that u_{ij} and v_{ij} are indexed according to the current matching, that may differ from one stable matching to another, has to be solved. However, it is also proved in Theorem 1 in Sotomayor (1999) that in every stable outcome a player gets the same payoff in any nonessential partnership (those partnerships that occur in some but not all optimal matchings). Because of that, given a stable outcome $(u, v; \mu)$ and another optimal matching μ', we can reindex u_{ij} and v_{ij} according to μ' and still get a stable outcome compatible with μ'.

As a consequence of all that, to obtain the firms-optimal stable outcome in the multiple-partners assignment game we only need to obtain the firms-optimal stable payoff vector in the related simple assignment game. In the simple assignment game, the maximum stable payoff of a firm $f_{ik} \in \tilde{F}$ is its marginal contribution, $\overline{u}_{ik}(\tilde{a}) = w_{\tilde{a}}(\tilde{N}) - w_{\tilde{a}}(\tilde{N} \setminus \{f_{ik}\})$, and similarly for the workers, $\overline{v}_{lj}(\tilde{a}) = w_{\tilde{a}}(\tilde{N}) - w_{\tilde{a}}(\tilde{N} \setminus \{w_{lj}\})$

(see Demange (1982) and Leonard (1983)). Hence, $(\overline{u}(a), \underline{v}(a))$ defined from $(\overline{u}(\tilde{a}), \underline{v}(\tilde{a}))$ as in 3 are the optimal stable payoff vectors in the multiple-partners assignment game, and the maximum total stable payoff of an agent in the multiple-partners assignment game is

$$\overline{U}_i(a) = \sum_{(f_i, w_k) \in \mu} \overline{u}_{ik}(a) \text{ for all } f_i \in F \quad \text{and} \quad \overline{V}_j(a) = \sum_{(k,j) \in \mu} \overline{v}_{kj}(a) \text{ for all } w_j \in W,$$

given any optimal matching μ. Notice that, for all $f_i \in F$,

$$\begin{aligned} \overline{U}_i(a) = \sum_{(f_i, w_k) \in \mu} \tilde{w}_a(\tilde{N}) - \tilde{w}_a(\tilde{N} \setminus \{f_{ik}\}) \quad &\leq \quad \tilde{w}_a(\tilde{N}) - \tilde{w}_a(\tilde{N} \setminus \{f_{i1}, f_{i2}, \ldots, f_{ir_i}\}) \\ &= \quad w_a(N) - w_a(N \setminus \{f_i\}), \end{aligned}$$

where, in contrast to the simple assignment game, the inequality $\overline{U}_i(a) \leq w_a(N) - w_a(N \setminus \{f_i\})$ may be strict.

The set of total payoffs (U, V) to the agents in the multiple-partners assignment game has been studied in Fagebaume *et al.* (2010), where it is proved that the maximum of any pair of stable (total) payoffs for the firms is stable but the minimum need not be, even if we restrict the multiplicity of partnerships to one of the sides.

The aim of the present paper is to study the properties of stable allocation rules. An allocation rules selects a feasible outcome for each multiple-partners job market.

Definition 2.3. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule φ consists of maps $(u, v; \mu)$ from valuation profiles $a \in \mathcal{A}^{F \times W}$ to feasible outcomes $(u(a), v(a); \mu(a))$. That is, for each $a \in \mathcal{A}^{F \times W}$, $\varphi(a) \equiv (u(a), v(a); \mu(a))$ is a feasible outcome for (F, W, a, r, s).*

An allocation rule is a stable rule if it always selects a stable outcome.

Definition 2.4. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u, v; \mu)$ is a stable rule if for each valuation profile $a \in \mathcal{A}^{F \times W}$, $\varphi(a) \equiv (u(a), v(a); \mu(a))$ is a stable outcome for (F, W, a, r, s).*

In the next section we study some outstanding stable rules: the firms-optimal stable rules, that for each valuation profile select the firms-optimal stable payoffs together with a compatible matching, and the workers-optimal stable rules, that select the workers-optimal stable payoffs with a compatible matching. Notice from the above discussion of the literature, that for each of these two type of rules the associated payoff vector is uniquely determined, although the compatible matching may not be unique.

3 Valuation monotonicity properties

We begin by considering some monotonicity properties that are satisfied by the optimal stable rules in the simple assignment game and we see whether they are satisfied by the corresponding rules in the multiple-partners assignment game. The first one is *pairwise monotonicity*. A rule for the simple assignment game is pairwise monotonic if whenever a single valuation of the market weakly increases and the remaining ones do not change, then the rule decreases the payoff of neither the firm nor the worker related to that valuation. It turns out that both optimal stable rules, and also the fair division rules, are pairwise monotonic (Núñez & Rafels, 2002). If we want to discriminate between these stable rules, we need to consider different changes in the valuation profile. In van den Brink *et al.* (2021), a rule for the simple assignment game is said to be *firm-valuation monotonic*[2] if whenever all valuations of a single firm weakly decrease but this does not change which worker is hired by this firm, then the payoff to this firm cannot increase. It turns out that the firms-optimal stable rule is the only stable rule for the simple assignment game that is firm-valuation monotonic. Of course, parallel definitions and results follow for the workers-optimal stable rule.

Let us now generalize the definition of the above monotonicity properties to the multiple-partners assignment game. Notice in the next definition that we can easily compare the payoffs a firm receives in different matchings since we requiere that the firm keeps the same partners after decreasing the valuations.

Definition 3.1. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u, v, ; \mu)$ satisfies* **firm-valuation monotonicity (FVM)** *if for all $a, a' \in \mathcal{A}^{F\times W}$ such that there is a firm $f_t \in F$ such that $a'_{ij} = a_{ij}$ for all $f_i \in F \setminus \{f_t\}$ and all $w_j \in W$ and $a'_{tj} \le a_{tj}$ for all $w_j \in W$, then*

$$\mu_{f_t}(a) = \mu_{f_t}(a') \Rightarrow u_{tk}(a') \le u_{tk}(a) \text{ for all } k \in \mu_{f_t}(a).$$

FVM means that if all valuations of a firm weakly decrease but this does not modify which workers it is assigned to, then the rule cannot give this firm a higher payoff in any of its partnerships.

When defining a pairwise monotonicity property for the multiple-partners assignment game, in order to be able to compare the payoffs before and after the change of a value, we also need to require that the two agents related to the value that has increased or decreased keep the same partners after this change. Hence, when applied to rules for the simple assignment game, this property is weaker than the usual pairwise monotonicity for these games.

Definition 3.2. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u, v, ; \mu)$ satisfies* **pairwise monotonicity (PM)** *if for all $a, a' \in \mathcal{A}^{F\times W}$ such that there is a firm-worker pair $(f_t, w_k) \in F \times W$ such that $a'_{ij} = a_{ij}$ if $(f_i, w_j) \neq (f_t, w_k)$ and $a'_{tk} \le a_{tk}$, then $\mu_{f_t}(a) = \mu_{f_t}(a')$ and $\mu_{w_k}(a) = \mu_{w_k}(a')$ imply*

$$u_{tj}(a') \le u_{tj}(a) \text{ for all } j \in \mu_{f_t}(a) \text{ and } v_{ik}(a') \le v_{ik}(a) \text{ for all } i \in \mu_{w_k}(a).$$

The next example shows that the firms-optimal stable rule does not satisfy any of the above monotonicity properties.

[2]The market considered in van den Brink *et al.* (2021) is formed by buyers and sellers and hence this property is called there *buyer-valuation monotonicity*

Example 3.3. *Let be a multiple partner assignment game with three firms, $F = \{f_1, f_2, f_3\}$ and three workers, $W = \{w_1, w_2, w_3\}$, all agents with quota 2, and valuation matrix*

$$a = \begin{pmatrix} 4.5 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

There is only one optimal matching $\mu = \{(f_1, w_2), (f_1, w_3), (f_2, w_1), (f_2, w_2), (f_3, w_1), (f_3, w_3)\}$, and hence any stable rule must select this matching μ. The worth of the grand coalition is $w_a(N) = 36$.

Assume the value a_{11} increases in 0.1, that is $a'_{11} = 4.6$ and the other values remain unchanged. Hence, the new valuation matrix is

$$a' = \begin{pmatrix} 4.6 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

and notice that μ is also the only optimal matching for (F, W, a', r, s).

To compute the payoffs in the firms-optimal stable rule, we obtain a related simple assignment game as in Sotomayor (1992): $\tilde{F} = \{f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32}\}$, $\tilde{W} = \{w_{11}, w_{21}, w_{12}, w_{22}, w_{13}, w_{23}\}$ and

$$\tilde{a} = \left(\begin{array}{cc|cc|cc} 4.5 & 4.5 & 20 & 0 & 0 & 0 \\ 4.5 & 4.5 & 0 & 0 & 4 & 0 \\ \hline 5 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ \hline 0 & 2 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{array}\right).$$

Here, $\overline{u}_{11}(\tilde{a}) = 36 - 17.5 = 18.5 = \overline{u}_{12}(a)$. If we replace in $\tilde{a}$ the 4.5 entries with 4.6, the resulting matrix $\tilde{a}'$ is a simple assignment game related to the valuation matrix a' and we can easily check that

$$\overline{u}_{11}(\tilde{a}') = 36 - 17.6 = 18.4 = \overline{u}_{12}(a') < \overline{u}_{12}(a).$$

As a consequence, the firms-optimal stable rule is not firm-valuation monotonic nor pairwise monotonic.

One may also ask about the behaviour of the total payoff of a firm in front of these changes. But in this example it is easy to check that $\overline{u}_{12}(\tilde{a}') = \overline{u}_{12}(\tilde{a}) = 36 - 32 = 4 = \overline{u}_{13}(a') = \overline{u}_{13}(a)$. Hence, $\overline{U}_1(a') = 22.4 < 22.5 = \overline{U}_1(a)$.

At the sight of the example above, we strengthen the requirement of firm-monotonicity by assuming that all valuations of a given firm are decreased by the same constant amount. Analogously, we will study a new worker-monotonicity property assuming the valuations of all firms with respect to a given worker decrease by the same constant amount. We will see how the two optimal stable rules react to these changes in the valuations.

Firm-covariance of the firms-optimal stable rules

We consider a multiple-partners assignment game (F, W, a, r, s) that is "balanced", in the sense that $\sum_{i\in F} r_i = \sum_{j\in W} s_j$. This assumption is without loss of generality since we could always add a fake dummy agent with the necessary quota. We analyse the behaviour of an allocation rule when the valuations of a firm f_{i_0} decrease by the same amount $c \geq 0$, under the assumption that values that become negative are truncated at zero: $a^c_{i_0 j} = \max\{0, a_{i_0 j} - c\}$ for all $w_j \in W$. These values are allowed to decrease in this way as long as no optimal matching of the initial problem becomes non-optimal. We then say that a rule for the multiple-partners assignment game is *firm-covariant* if the firm pays this cost c in each of its partnerships.

This property can be interpreted by saying that if a constant fee c is applied to some firm whenever it hires a worker, then this fee is completely paid by the firm, and not shared with the workers that it hires.

Definition 3.4. *A rule $\varphi \equiv (u, v; \mu)$ is* **firm-covariant (FC)** *if for all (F, W, a, r, s), all $f_{i_0} \in F$ and all $c \geq 0$ such that*

(i) $a^c_{i_0 j} = \max\{0, a_{i_0 j} - c\}$ *for all $w_j \in W$ and $a^c_{ij} = a_{ij}$ for all $f_i \in F \setminus \{f_{i_0}\}$,*

(ii) $c \leq a_{i_0 j}$ *for all $(f_{i_0}, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F, W)$ and*

(iii) $\mathcal{M}_a(F, W) \subseteq \mathcal{M}_{a^c}(F, W)$,

then,

$$\begin{aligned} u_{i_0 j}(a^c) &= u_{i_0 j}(a) - c \quad \text{for all } (f_{i_0}, w_j) \in \mu, \text{ and} \\ u_{ij}(a^c) &= u_{ij}(a), \qquad \text{for all } f_i \in F \setminus \{f_{i_0}\} \text{ and } (f_i, w_j) \in \mu. \end{aligned}$$

Notice that conditions (ii) and (iii) together imply that the worth of the grand coalition is still attained at the original optimal matchings.

As we remark after Definition A.1 in the Appendix, requiring that c satisfies conditions (ii) and (iii) in Definition 3.4 is equivalent to requiring $c \leq c^*$ where this threshold c^*, as defined in (5), is the minimum $c \geq 0$ such that there is an optimal matching of (F, W, a^c, r, s) with a zero entry.

When we analyse if the firms-optimal stable rules satisfy this property, we may consider the firms-optimal stable rules of the related simple assignment game and study there how the payoff of such a rule changes when all the copies of a given firm decrease their valuations by the same amount $c \geq 0$. To this end, in the Appendix we introduce the property of strong firm-covariance for stable rules of the simple assignment game, by requiring that several firms decrease their valuations in a given constant, and we provide an axiomatic characterization of their firms-optimal stable rules making use of this property. This strong firm-covariance can be defined analogously for the multiple-partners assignment game.

An analogous covariance property can be defined when all the valuations of a given worker are decreased by a constant amount.

Definition 3.5. *A rule $\varphi \equiv (u, v; \mu)$ is* **worker-covariant (WC)** *if for all (F, W, a, r, s), all $w_{j_0} \in F$ and all $c \geq 0$ such that*

(i) $a^c_{ij_0} = \max\{0, a_{ij_0} - c\}$ *for all* $f_i \in F$ *and* $a^c_{ij} = a_{ij}$ *for all* $w_j \in W \setminus \{f_{j_0}\}$*,*

(ii) $c \leq a_{ij_0}$ *for all* $(f_i, w_{j_0}) \in \mu$ *and* $\mu \in \mathcal{M}_a(F, W)$ *and*

(iii) $\mathcal{M}_a(F, W) \subseteq \mathcal{M}_{a^c}(F, W)$*,*

then,

$$\begin{array}{rcll} u_{ij_0}(a^c) & = & v_{ij_0}(a) - c & \text{for all } (f_i, w_{j_0}) \in \mu, \text{ and} \\ u_{ij}(a^c) & = & u_{ij}(a), & \text{for all } w_j \in W \setminus \{w_{j_0}\} \text{ and } (f_i, w_j) \in \mu. \end{array}$$

The next characterization of the firms-optimal stable rules of the multiple-partners assignment game follows from the results on the simple assignment game developed in the Appendix. We could also state this result replacing firm-covariance (worker-covariance) with strong firm-covariance (strong worker-covariance), since in any case it relies on the strong covariance of the optimal stable rules of the simple assignment game.

Theorem 3.6. *1. The firms-optimal stable rules are the only stable rules for the multiple-partners assignment game that are firm-covariant.*

2. The workers-optimal stable rules are the only stable rules for the multiple-partners assignment game that are worker-covariant.

Proof. Let (F, W, a, r, s) be a multiple-partners assignment game. Let $f_{i_0} \in F$ and $c \geq 0$ that satisfies the conditions in Definition 3.4. Take some $\mu \in \mathcal{M}_a(F, W, r, s)$ and let $(\tilde{F}, \tilde{W}, \tilde{a})$ be a related simple assignment game where firms and workers have been replicated according their capacity and the valuations are as described in (2), given that $\mu \in \mathcal{M}_a(F, W, r, s)$. Let (F, W, a^c, r, s) be the multiple-partners assignment game with a^c as in Definition 3.4. Notice that when we replicate this market we obtain $(\tilde{F}, \tilde{W}, \tilde{a^c})$ and the valuations satisfy $\tilde{a^c} = \tilde{a}^{c,I}$, as in Definition A.1 in the Appendix, where I consists of the r_{i_0} copies of firm f_{i_0}.

As a consequence, if $\overline{u}_{ik}(\tilde{a})$ and $\overline{u}_{ik}(\tilde{a^c})$ are the maximum stable payoffs of the k copy of firm f_i in $(\tilde{F}, \tilde{W}, \tilde{a})$ and $(\tilde{F}, \tilde{W}, \tilde{a^c})$, respectively, then from Proposition A.3 in the Appendix,

$$\overline{u}_{i_0k}(\tilde{a^c}) = \overline{u}_{i_0k}(\tilde{a}) - c \quad \text{ and } \quad \overline{u}_{ik}(\tilde{a^c}) = \overline{u}_{ik}(\tilde{a}) \text{ if } i \neq i_0.$$

Hence, if $(f_{i_0}, w_j) \in \mu$ and $(f_{i_0k}, w_{lj}) \in \tilde{\mu}$,

$$\overline{u}_{i_0j}(a^c) = \overline{u}_{i_0k}(\tilde{a^c}) = \overline{u}_{i_0k}(\tilde{a}) - c = \overline{u}_{i_0j}(a) - c.$$

Similarly, if $f_i \in F \setminus \{i_0\}$, $(f_i, w_j) \in \mu$ and $(f_{ik}, w_{lj}) \in \tilde{\mu}$, then

$$\overline{u}_{ij}(a^c) = \overline{u}_{ik}(\tilde{a^c}) = \overline{u}_{ik}(\tilde{a}) = \overline{u}_{i_0j}(a),$$

which shows that the firms-optimal stable rules of the multiple-partners assignment game are firm-covariant.

The converse implication is straightforward since any stable rule for the multiple-partners assignment game that is FC induces a stable rule for the simple assignment game that is strong firm-covariant, and by Theorem A.4 in the Appendix we know this can only be a firms-optimal stable rule. □

If we are interested in the maximum total payoff of the firms in a stable outcome, then we have

$$\overline{U}_{i_0}(a^c) = \sum_{k=1}^{r_{i_0}} \overline{u}_{i_0k}(\tilde{a^c}) = \sum_{k=1}^{r_{i_0}} (\overline{u}_{i_0k}(\tilde{a}) - c) = \overline{U}_{i_0}(a) - r_{i_0}c$$

and for all $f_i \in F \setminus \{f_{i_0}\}$

$$\overline{U}_i(a^c) = \sum_{k=1}^{r_i} \overline{u}_{ik}(\tilde{a^c}) = \sum_{k=1}^{r_i} \overline{u}_{ik}(\tilde{a}) = \overline{U}_i(a),$$

Notice now that, as a consequence of Theorem 3.6, we deduce that the firms-optimal stable rules of the multiple partners assignment game satisfy a weaker form of valuation monotonicity. We strengthen the conditions under which a decrease of the valuations of the firm should imply a decrease in that firm's payoffs: we only require monotonicity when all valuations are decreased by the same amount.

Definition 3.7. *A rule $\varphi \equiv (U, V; \mu)$ is* **weak firm-valuation monotonic (WFVM)** *if for all (F, W, a, r, s), all $f_{i_0} \in F$ and all $c \geq 0$ such that*

(i) $a^c_{i_0j} = \max\{0, a_{i_0j} - c\}$ *for all $w_j \in W$ and $a^c_{ij} = a_{ij}$ for all $f_i \in F \setminus \{f_{i_0}\}$,*

(ii) $c \leq a_{i_0j}$ *for all $(f_{i_0}, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F, W)$ and*

(iii) $\mathcal{M}_a(F, W) \subseteq \mathcal{M}_{a^c}(F, W)$,

then,

$$u_{i_0j}(a^c) \quad \leq \quad u_{i_0j}(a) \qquad \text{for all } (f_{i_0}, w_j) \in \mu.$$

Since the firms-optimal stable rules are firm-covariant, they trivially satisfy weak firm-valuation monotonicity.

Corollary 3.8. *On the domain of multiple-partners assignment game, the firms-optimal stable rules satisfy weak firm-valuation monotonicity.*

We can analogously define weak worker-valuation monotonicity (WWVM). A rule for the multiple-partners assignment game is weak worker-valuation monotonic if whenever the values all firms obtain with a given worker decrease by the same amount (with truncation to avoid negative valuations), in such a way that all optimal matchings of the initial market still remain optimal, then the payoff this worker obtains in each partnership does not increase. It is then obtained that the workers-optimal stable rules are weak worker-valuation monotonic.

4 Non-manipulability properties

Given a multiple-partners job market, when an allocation rule is to be adopted, then firms and workers are required to report their valuations and this induces a strategic game. Recall that each firm f_i values in $h_{ij} \geq 0$ the possibility of hiring worker w_j and each worker w_j has a reservation value $t_j \geq 0$ and will not accept being hired with a salary below his/her reservation value. Once agents report their valuations, an allocation rule $\varphi(h,t)$ selects a matching $\mu \in \mathcal{M}(F,W,r,s)$ and determines how to split the net profit $a_{ij} = \max\{h_{ij} - t_j, 0\}$ of each partnership $(f_i, w_j) \in \mu$. We may assume the rule simply determines the salary m_{ij} that firm f_i pays to worker w_j if they are matched. Then, in the partnership (f_i, w_j), the payoff of the firm is $u_{ij} = h_{ij} - m_{ij}$ and the payoff of the worker is $v_{ij} = m_{ij} - t_j$.

The question is whether a firm (or a worker) has incentives not to report its true valuations, once known which allocation rule will be applied. In particular, we want to study whether firms (workers) have incentives to manipulate the firms-optimal (workers-optimal) stable rule, since it is something they cannot do in the simple assignment game.

For this strategic analysis, and since population will not change, we may consider the sets of firms and workers, F and W, and their capacities fixed. Then, for any reported valuations (h,t), the firms-optimal stable rule selects an optimal matching μ and for all $(f_i, w_j) \in \mu$ determines a salary $\underline{m}_{ij}$ such that $\underline{v}(h,t)_{ij} = \underline{m}_{ij} - t_j$, where $(\overline{u}(h,t), \underline{v}(h,t))$ is the firms-optimal stable payoff vector. Similarly, the workers-optimal stable rule selects an optimal matching μ and for all $(f_i, w_j) \in \mu$ determines a salary $\overline{m}_{ij}$ such that $\overline{v}(a)_{ij} = \overline{m}_{ij} - t_j$, where $(\underline{u}(h,t), \overline{v}(h,t))$ is the workers-optimal stable payoff vector according the reported valuations.

From Pérez-Castrillo & Sotomayor (2017), that studies the manipulability of the optimal competitive equilibrium rules of the multiple-partners assignment game, and taking into account that every firms-optimal stable rule coincides with a firms-optimal competitive equilibrium rule, we deduce that these rules are manipulable by any firm with capacity greater than one. However, in the example provided in Pérez-Castrillo & Sotomayor (2017), the firm that manipulates the firms-optimal stable rule increases its valuations in a non-constant way, that is, it increases some valuations but not all of them by the same amount.

We may think that "naive" firms, when trying to manipulate an allocation rule, only consider whether to increase or decrease all its valuations by the same constant amount. This idea leads to a weaker non-manipulability property.

Definition 4.1. *Let F be a set of firms with capacities $r = (r_i)_{i \in F}$ and W a set of workers with capacities $s = (s_j)_{j \in W}$. A firm $f_{i_0} \in F$* **manipulates** *a rule $\varphi \equiv (m; \mu)$ in a multiple-partners job market (F, W, h, t, r, s)* **by constantly over-reporting its valuations** *if there exists $c > 0$ such that f_{i_0} gets a higher payoff at $(v(h',t); \mu(h',t))$ than at $(v(h,t); \mu(h,t))$, where $h'_{i_0 j} = h_{i_0 j} + c$ for all $w_j \in W$ and $h'_{ij} = h_{ij}$ for all $f_i \in F \setminus \{f_{i_0}\}$ and all $w_j \in W$.*

We intend to make use of the firm-covariance property that we introduced in the previous section. But notice that the fact that $h'_{i_0 j} = h_{i_0 j} + c$ for all $w_j \in W$ does not imply $a'_{i_0 j} = \max\{h'_{i_0 j} - t_j, 0\} = a_{i_0 j} + c$, since for some $w_j \in W$ it may happen that $h_{i_0j} - t_j < 0$. Because of that, we will restrict the study to the domain of multiple-partners job market where all firm-worker pairs are *mutually acceptable*, that is, $h_{ij} - t_j \geq 0$ for all $(f_i, w_j) \in F \times W$.

Proposition 4.2. *On the domain of multiple-partners job market where all firm-worker pairs are mutually acceptable, no firm can manipulate the firms-optimal stable rule by constantly over-reporting its valuations.*

Proof. Let (F, W, h, t, r, s) be a multiple-partners job market such that $h_{ij} - t_j \geq 0$ for all $(f_i, w_j) \in F \times W$. If firm f_{i_0} reports $h'_{i_0j} = h_{i_0j} + c$ for some $c > 0$, then $a'_{i_0j} = \max\{h'_{i_0j} - t_j, 0\} = a_{i_0j} + c$ and both markets have the same set of optimal matchings. From Theorem 3.6 and the proof of Proposition A.3 the salaries m'_{ij} determined by $\varphi(h', t)$, where φ is the firms-optimal stable rule, are the same as the salaries m_{ij} determined by $\varphi(h, t)$, since $\underline{v}_{ij}(a') = \underline{v}_{ij}(a)$, for each (f_i, w_j) in an optimal matching $\mu \in \mathcal{M}_a(F, W, r, s)$. Then,

$$h_{i_0j} - m'_{i_0j} = h_{i_0j} - (\underline{v}_{i_0j}(a') + t_j) = h_{i_0j} - (\underline{v}_{i_0j}(a) + t_j) = h_{i_0j} - m_{i_0j}, \text{ for all } (f_{i_0}, w_j) \in \mu.$$

Hence, the total payoff of firm f_{i_0} does not improve when reporting h'_{i_0}:

$$\overline{U}_{i_0}(a') = \sum_{(i_0, j) \in \mu} h_{i_0j} - m'_{i_0j} = \sum_{(i_0, j) \in \mu} h_{i_0j} - m_{i_0j} = \overline{U}_{i_0}(a).$$

□

Notice that, because each firm may value differently each worker in the market, firms may have more sophisticated strategies than the constant over-reporting of Definition 4.1. Take for instance Example 4.2 in Pérez-Castrillo & Sotomayor (2017) that consists in a market with three workers with capacity one and null reservation value and two firms, the first of them with capacity two, with valuations $h_1 = (7, 6, 4)$ and $h_2 = (8, 6, 3)$. Notice that all firm-worker pairs are acceptable. Since there is only one optimal matching, this is the matching selected by any stable rule, $\mu = \{(f_1, w_2), (f_1, w_3), (f_2, w_1)\}$. In the firms-optimal stable rule, f_1 pays salaries $\underline{v}_{12}(a) = 1$ and $\underline{v}_{13}(a) = 0$, with a net profit of $\overline{U}_1(a) = 9$. If f_1 reports $h'_1 = (8, 7, 7)$, which is a non-constant over-report of its valuations, then the optimal matching does not change but now the salaries paid by f_1 in the firms-optimal stable rule are $\underline{v}_{12}(a') = \underline{v}_{13}(a') = 0$ and the payoff of f_1, taking into account its true valuations, is 10.

Instead, the reservation value of a worker does not depend on which firm he/she is matched to. Hence, when a worker under-reports his reservation value, his/her net valuations with all firms increase by the same amount, provided all firm-worker pairs are acceptable.

Definition 4.3. *Let F be a set of firms with capacities $r = (r_i)_{i \in F}$ and W a set of workers with capacities $s = (s_j)_{j \in W}$. A worker $w_{j_0} \in W$* **manipulates** *a rule $\varphi \equiv (m; \mu)$ in a multiple-partners job market (F, W, h, t, r, s)* **by under-reporting his/her reservation value** *if there exists $0 \leq c \leq t_{j_0}$ such that w_{j_0} gets a higher payoff at $(v(h, t'); \mu(h, t'))$ than at $(v(h, t); \mu(h, t))$, where $t'_{j_0} = t_{j_0} - c$ and $t'_j = t_j$ for all $w_j \in W \setminus \{w_{j_0}\}$.*

Notice that given a multiple-partners job market where all firm-worker pairs are acceptable, then all firm-worker pair in the market that results when some worker under-reports his/her reservation value are also acceptable.

Proposition 4.4. *On the domain of multiple-partners job market where all firm-worker pairs are mutually acceptable, no worker can manipulate the workers-optimal stable rule by under-reporting his/her reservation value.*

Proof. Let (F, W, h, t, r, s) be a multiple-partners job market such that $h_{ij} - t_j \geq 0$ for all $(f_i, w_j) \in F \times W$. If worker w_{j_0} reports $t'_{j_0} = t_{j_0} - c$ for some $c > 0$, then $a'_{ij_0} = \max\{h'_{ij_0} - t_{j_0}, 0\} = a_{ij_0} + c$ and both markets have the same set of optimal matchings. From Theorem 3.6 and the proof of Proposition A.3 the salary m'_{ij_0} determined by $\varphi(h, t')$, where φ is the workers-optimal stable rule is the same as the salary m_{ij_0} determined by $\varphi(h, t)$:

$$m'_{ij_0} = \overline{v}_{ij_0}(h, t') + t'_{j_0} = \overline{v}_{ij_0}(h, t) + c + t_{j_0} - c = m_{ij_0}.$$

Hence, given any $\mu \in \mathcal{M}_a(F, W, r, s)$, the payoff to worker w_{j_0} in each partnership $(f_i, w_{j_0}) \in \mu$ is $m'_{ij_0} - t_{j_0} = m_{ij_0} - t_{j_0}$ and w_{j_0} has no incentives to report t'_{j_0} instead of t_j. □

This may not be the case when a worker over-reports his/her reservation value. Example 4.1 in Pérez-Castrillo & Sotomayor (2017) shows that in that case such a worker may manipulate the workers-optimal stable rule.

However, the above non-manipulability properties do not characterize neither the firms-optimal stable rules nor the workers-optimal stable rules on the domain where all firm-worker pairs are acceptable. Notice for instance that the workers-optimal stable rule is also non-manipulable by constant over-reporting of one firm's valuations. Take (F, W, a, r, s) where all pairs are acceptable, a firm $f_{i_0} \in F$ and an optimal matching $\mu \in \mathcal{M}_a(F, W)$. Assume $h'_{i_0 j} = h_{i_0 j} + c$ for some $c > 0$, while $h'_{ij} = h_{ij}$ for $i \in F \setminus \{f_{i_0}\}$ and $j \in W$. This implies $a'_{i_0 j} = a_{i_0 j} + c$ for all $j \in W$, while $a'_{ij} = a_{ij}$ otherwise. Consider the related simple assignment game $(\tilde{F}, \tilde{W}, \tilde{a})$ and the corresponding optimal matching $\tilde{\mu}$. If $w_{j_0} \in W$ is such that $(f_{i_0 k}, w_{l j_0}) \in \tilde{\mu}$, where $f_{i_0 k}$ and $w_{l j_0}$ are copies of f_{i_0} and w_{j_0} respectively, then $\overline{v}_{i_0 j_0}(a') = \overline{v}_{l j_0}(\tilde{a}') = w_{\tilde{a}'}(\tilde{F} \cup \tilde{W}) - w_{\tilde{a}'}(\tilde{F} \cup (\tilde{W} \setminus \{w_{l j_0}\}))$ is either $\overline{v}_{i_0 j_0}(a) + c$ or $\overline{v}_{i_0 j_0}(a)$, since $\mu' \in \mathcal{M}_{\tilde{a}'}(\tilde{F}, \tilde{W} \setminus \{w_{l j_0}\})$ if and only if $\mu' \in \mathcal{M}_{\tilde{a}}(\tilde{F}, \tilde{W} \setminus \{w_{l j_0}\})$. This means $\overline{v}_{i_0 j_0}(a') \geq \overline{v}_{i_0 j_0}(a)$ and hence $h_{i_0 j_0} - \overline{v}_{i_0 j_0}(a') \leq h_{i_0 j_0} - \overline{v}_{i_0 j_0}(a)$ and i_0 has no incentives to constantly over-report its valuations.

5 The fair division rules

In some situations, specially in two-sided markets without money, it is usual to implement an allocation rule that favours one side of the market. Take for instance the allocation of students to colleges or resident doctors to hospitals. But in a job market, and also in a market with buyers and sellers, it makes sense to assume that matched agents enter a negotiation and agree on a price or salary that is in between those that are optimal for each side.

We extend to the multiple-partners job market the notion of fair division payoff vector that was introduce by Thompson (1981) for the simple assignment game as the midpoint between the two optimal stable payoff vectors. That is, given a set F of firms with quotas r and a set of workers W with quotas s, and a valuation profile $a \in \mathcal{A}^{F_0 \times W_0}$, a fair division rule is $\varphi^\tau \equiv (u^\tau(a), v^\tau(a); \mu)$ where

$$u^\tau_{ij}(a) = \frac{1}{2}\overline{u}_{ij}(a) + \frac{1}{2}\underline{u}_{ij}(a) \text{ and } v^\tau_{ij}(a) = \frac{1}{2}\overline{v}_{ij}(a) + \frac{1}{2}\underline{v}_{ij}(a) \text{ for all } (f_i, w_j) \in \mu$$

and μ is a compatible matching. Notice that there may be several compatible matchings but the payoff vector is uniquely defined.

In van den Brink *et al.* (2021), and for the simple assignment game, the fair division rule is characterized as the only stable rule that satisfies grand valuation fairness and weak derived consistency. Our aim is to extend these two properties to the multiple-partners job market and see whether they also individualize the fair division rules. To extend the notion of (derived) consistency, that reflects how a solution behaves when some agents leave the market, we will allow in this section for positive values a_{i0} and a_{0j}, for all $f_i \in F$ and $w_j \in W$. Hence, now a valuation profile is $(a_{ij})_{\substack{i \in F_0 \\ j \in W_0}}$ with $a_{00} = 0$, and we denote by $\mathcal{A}^{F_0 \times W_0}$ the set of all valuation profiles. We assume again that all firm-worker pairs are mutually acceptable, which in the notation just introduced translates to saying that for all $(f_i, w_j) \in F^0 \times W^0$, $a_{ij} \geq a_{i0} + a_{0j}$.

Roughly speaking, grand valuation fairness requires that if all firm-worker valuations decrease by a same amount $c \geq 0$, as long as all optimal matchings of the initial market remain optimal, the payoff all agents receive from each partnership decreases equally.

Definition 5.1. *A rule $\varphi \equiv (u, v; \mu)$ satisfies great valuation fairness (GVF) if for all (F, W, a, r, s) and all $c \geq 0$ such that*

(i) $a^c_{ij} = \max\{0, a_{ij} - c\}$ *for all $f_i \in F$ and $w_j \in W$,*

(ii) $c \leq a_{ij}$ *for all $(f_i, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F, W, r, s)$ and*

(iii) $\mathcal{M}_a(F, W) \subseteq \mathcal{M}_{a^c}(F, W, r, s)$,

then,

$$u_{ij}(a^c) - u_{ij}(a) = v_{ij}(a^c) - v_{ij}(a) \text{ for all } (f_i, w_j) \in \mu. \tag{4}$$

From firm-covariance and worker-covariance of the two optimal stable rules, it follows quite straightforwardly that the fair division rules satisfy GVF.

Proposition 5.2. *On the domain of multiple-partners assignment markets, the fair division rules satisfy GVF.*

Proof. Recall that the minimum c satisfying (i), (ii) and (iii) is the c^* defined in (5). As a consequence, if a multiple-partners job market (F, W, a, r, s) is "unbalanced", in the sense that $\sum_{f_i \in F} r_i \neq \sum_{w_j \in W} s_j$, then $c^* = 0$ and GVF does not impose any restriction. Hence, we may focus on markets where the sum of capacities of firms equals those of workers.

Let (F, W, a, r, s) be a multiple-parters job market with $\sum_{f_i \in F} r_i = \sum_{w_j \in W} s_j$, μ an optimal matching and $c \geq 0$ satisfying (i), (ii), and (iii) in Definition 5.1. By strong firm-covariance of the firms-optimal stable rules, taking $I = F$, that is, assuming all firms decrease their valuations in c, we have that $\overline{u}_{ij}(a^c) = \overline{u}_{ij}(a) - c$ and $\underline{v}_{ij}(a^c) = \underline{u}_{ij}(a)$ for all $(f_i, w_j) \in \mu$. Similarly, from worker-covariance of the workers-optimal stable rule, taking $I = W$, we have $\overline{v}_{ij}(a^c) = \overline{v}_{ij}(a) - c$ and $\underline{u}_{ij}(a^c) = \underline{v}_{ij}(a)$ for all $(f_i, w_j) \in \mu$. As a consequence,

$$u^{\tau}_{ij}(a^c) = u^{\tau}_{ij}(a) - \frac{c}{2} \text{ and } v^{\tau}_{ij}(a^c) = v^{\tau}_{ij}(a) - \frac{c}{2} \text{ for all } (f_i, w_j) \in \mu$$

which implies $u^{\tau}_{ij}(a^c) - u^{\tau}_{ij}(a) = \frac{c}{2} = v^{\tau}_{ij}(a^c) - v^{\tau}_{ij}(a)$ and GVF holds. □

The idea now is to proceed as in the simple assignment game (van den Brink *et al.* , 2021). In that case, we decrease all firm-worker values until reaching the threshold c^*; at this point there is a firm-worker pair (f_i, w_j) whose payoff is fixed, and equal to their individual values a_{i0} and a_{0j}, at any stable outcome. Then, these two agents leave the market with their fixed payoff and we must define the reduced game in such a way that the fair division rule is consistent with respect to this reduction.

For the simple assignment game, a notion of reduced game is introduced in Owen (1992) with the name of derived game. Several solution concepts, such as the core, the optimal stable rules for any side of the market or the nucleolus (Llerena *et al.* , 2015), are derived consistent, that meaning that when we restrict the solution payoff vector to the agents that remain in the derived market game, we get a solution payoff vector of the derived market game. The fair-division rules are not derived consistent, unless the agents that leave the market have a unique stable payoff.

We now propose how to reduce a multiple-partners job market when an individual or a firm-worker pair have a unique stable payoff. Since agents may have capacities that allow for multiple partnership, the firm and worker in that pair may not leave the market but simply reduce their capacities in one unit.

Definition 5.3. *Let (F, W, a, r, s) be a multiple-partner assignment market, μ an optimal matching, $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ such that $a_{ij} = a_{i0} + a_{0j}$ and $z = (u, v; \mu)$ a stable outcome.*

The **derived assignment market** *relative to T at z is $(F^T, W^T, a^{T,z}, r^T, s^T)$ where*

$$F^T = \begin{cases} F \setminus \{f_i\} & \text{if } f_i \in F,\ r_i = 1, \\ F & \text{otherwise} \end{cases} \quad \text{and} \quad W^T = \begin{cases} W \setminus \{w_j\} & \text{if } w_j \in W,\ s_j = 1, \\ W & \text{otherwise} \end{cases},$$

$a^{T,z}_{kl} = a_{kl}$ *for all* $f_k \in F^T,\ w_l \in W^T$,

$$\begin{aligned} (i)\ a^{T,z}_{k0} &= \max_{(f_k,w_j)\notin\mu}\{a_{k0}, a_{kj} - v_{ij}\},\ \text{for all } f_k \in F^T, \\ (ii)\ a^{T,z}_{0k} &= \max_{(f_i,w_k)\notin\mu}\{a_{0k}, a_{ik} - u_{ij}\},\ \text{for all } w_k \in W^T, \end{aligned}$$

and $r^T_k = r_k - 1$ *if* $f_k \in T \cap F^T$, $r^T_k = r_k$ *if* $f_k \in F^T \setminus T$,
$s^T_k = s_k - 1$ *if* $w_k \in T \cap W^T$, $s^T_k = s_k$ *if* $w_k \in W^T \setminus T$.

In the derived assignment market relative to a coalition $T = \{f_i, w_j\}$ such that $a_{ij} = a_{i0} + a_{0j}$, (i) values for the firm-worker pairs 'that are still in the market' are the same as in the original market, (ii) the individual values are modified taking into account the possibilities to trade with agents outside the derived market and (iii) the capacity of each agent in T decreases in one unit.

Notice that the above definition allows for T to contain a dummy agent, like $T = \{f_i, w_0\}$ if $(f_i, w_0) \in \mu$.

The next example illustrates the definition of derived market.

Example 5.4. *Let us consider a market with two firms, $F = \{f_1, f_2\}$, the first with capacity 2 and the second with capacity 1, and three workers, $W = \{w_1, w_2, w_3\}$, each of them with unitary capacity. The valuation matrix is*

$$a = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 5 & 1 & 0 \\ 5 & 4 & 2 \end{pmatrix} \end{array},$$

with individual reservation values $a_{i0} = a_{0j} = 0$ for all $f_i \in F$ and $w_j \in W$.

Notice there is only one optimal matching $\mu = \{(f_1, w_1), (f_1, w_3), (f_2, w_3)\}$ for a total value of 9. Any stable payoff vector (u, v) must be compatible with this matching and hence $u_{13} = v_{13} = 0$.

We now consider the derived gamed at coalition $T = \{f_1, w_3\}$ and at any stable payoff, since the stables payoffs of f_1 and w_3 at this partnership are fixed at $u_{13} = v_{13} = 0$.

In this derived game, these agents establish the partnership (f_1, w_3) and are paid according the only possible stable payoff, in that case each of them gets 0. Then, w_3 has exhausted his capacity and leaves the market, while f_1 remains with its capacity reduced by one unit.

The individual reservation values are now modified to take into account the possibility of reaching an agreement with the worker that has left the market. That is,

$$\begin{array}{ll} a'_{10} = \max\{a_{10}\} = 0, & a'_{20} = \max\{a_{20}, a_{23} - v_{13}\} = \max\{0, 2-0\} = 2, \ \textit{since } (f_2, w_3) \notin \mu, \\ a'_{01} = \max\{a_{01}\} = 0, & a'_{02} = \max\{a_{02}, a_{12} - v_{13}\} = \max\{0, 1-0\} = 1, \ \textit{since } (f_1, w_2) \notin \mu. \end{array}$$

This means that whenever it is unmatched in the derived game, f_2 has a value of 2, and hence any stable payoff vector must allocate f_2 at least this amount. Similarly, w_2 has now a reservation value of 1. The valuations of the derived game can be represented in a table, with the reservation values of firms in a first column, and the reservation values of workers in a first row:

$$a' = \begin{array}{c} \\ \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} & w_1 & w_2 \end{array} \\ \left(\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 5 & 1 \\ 2 & 5 & 4 \end{array}\right) \end{array},$$

Weak derived consistency means that in a derived market at a coalition $T = \{f_i, w_j\}$ such that $a_{ij} = a_{i0} + a_{0j}$, the payoffs for the firms and workers that remain in the market do not change. For instance, when computing the firms-optimal stable payoff vector in the above example, we get $(\overline{u}(a), \underline{v}(a)) = (\overline{u}_{11}(a), \overline{u}_{13}(a), \overline{u}_{22}(a); \underline{u}_{11}(a), \underline{u}_{13}(a), \underline{u}_{22}(a)) = (3, 0, 3; 2, 0, 1)$, and when we compute it in the derived market we get $(\overline{u}(a'), \underline{v}(a')) = (\overline{u}_{11}(a'), \overline{u}_{22}(a'); \underline{u}_{11}(a'), \underline{u}_{22}(a')) = (3, 3; 2, 1)$. The formal definition of weak derived consistency follows below.

Definition 5.5. *On the domain of multiple-partners job markets, a stable allocation rule φ is **weak derived consistent (WDC)** if for all market (F, W, a, r, s) and all coalition $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ and $a_{ij} = a_{i0} + a_{0j}$, it holds*

$$\begin{array}{lll} (i) & \mu' = \mu \setminus \{(f_i, w_j)\} \text{ is optimal for } (F^T, W^T, a^{T,(u,v)}, r^T, s^T), & \\ (ii) & u_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = u_{kl} & \text{for all } (f_k, w_l) \in \mu' \text{ and} \\ (iii) & v_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = v_{kl} & \text{for all } (f_k, w_l) \in \mu' \end{array}$$

where $\varphi(F, W, a, r, s) = (u, v; \mu)$.

Let us first argue in the next lemma that condition (i) in the above definition always holds, since it is necessary to guarantee that u_{kl} and v_{kl} are well-defined.

Lemma 5.6. *Let $(F^T, W^T, a^{T,z}, r^T, s^T)$ be the derived game at $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ such that $a_{ij} = a_{i0} + a_{0j}$ and $z = (u, v; \mu)$ is a stable outcome. Then,*

1. *$a_{k0}^{T,(u,v)} = a_{k0}$ if $(f_k, w_0) \in \mu$ (and $a_{0l}^{T,(u,v)} = a_{0l}$ if $(f_0, w_l) \in \mu$),*
2. *$(u', v'; \mu')$ is a stable outcome for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$, where $\mu' = \mu \setminus \{(f_i, w_j)\}$, $u'_{kl} = u_{kl}$ and $v'_{kl} = v_{kl}$ for all $(f_k, w_l) \in \mu'$.*
3. *$\mu' = \mu \setminus \{(f_i, w_j)\}$ is optimal for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$.*

Proof. Notice that $(f_k, w_0) \in \mu$, implies $a_{k0} = u_{k0} \geq a_{kj} - v_{ij}$ for all $(f_k, w_j) \notin \mu$ because of the stability of (u, v). In the same way, $(f_0, w_l) \in \mu$ implies $a_{0l} = v_{0l} \geq a_{il} - u_{ij}$ for all $(f_i, w_l) \notin \mu$, and statement 1) is proved.

To prove 2) notice that for all $(f_k, w_l) \in \mu'$, it holds $(f_k, w_l) \in \mu$ and hence $u'_{kl} + v'_{kl} = u_{kl} + v_{kl} = a_{kl} = a_{kl}^{T,(u,v)}$. This includes the case where $(f_k, w_0) \in \mu'$ and then $u'_{k0} = a_{k0} = a_{k0}^{T,(u,v)}$, and similarly for $(f_0, w_l) \in \mu$. This means that (u', v') is feasible with respect to μ'. Now, if $(f_k, w_l) \in F^T \times W^T$ with $(f_k, w_l) \notin \mu'$ then either $(f_k, w_l) \notin \mu$ and in this case $u'_{kq} + v'_{pl} = u_{kq} + v_{pl} \geq a_{kl} = a_{kl}^{T,(u,v)}$ for $(f_k, w_q), (f_p, w_l) \in \mu'$, or $(f_k, w_l) = (f_i, w_j)$. In this second case, if $(f_i, w_q), (f_p, w_l) \in \mu'$, we have $u'_{iq} + v'pj = u_{iq} + v_{pj} \geq a_{i0} + v_{0j} = a_{ij}$, where the last equality follows from the assumption. Finally, if $(f_k, w_l) \in \mu'$, then $u_{kl} \geq \max\{a_{k0}, a_{kj} - v_{ij}\} = a_{k0}^{T,(u,v)}$, and similarly $v_{lk} \geq a_{0k}^{T,(u,v)}$ for all $(f_l, w_k) \in \mu'$. As a consequence, $(u', v'; \mu')$ satisfies all stability constraints.

Once known that $(u', v'; \mu')$ is a stable outcome for the market $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$, it follows from Sotomayor (1992) that μ' is optimal for this market. $\square$

Notice that statement 2) in Lemma 5.6 crucially depends on the fact that $a_{ij} = a_{i0} + a_{0j}$. This condition is also important to notice that for all stable payoff vectors it holds $u_{ij} = a_{i0}$ and $v_{ij} = a_{0j}$. Then, a sort of converse of statement 2) holds: under the assumptions of Lemma 5.6, if (u'', v'') is a stable payoff vector for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$, then, by completing it with the payoffs u_{ij} and v_{ij} we obtain a payoff vector of the initial market (F, W, a, r, s). As an immediate consequence, if (u, v) is a stable payoff vector of (F, W, a, r, s), then the set of stable payoff vectors of $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$ is precisely the restriction of the set of stable payoff vectors of (F, W, a, r, s) to $F^T \cup W^T$. In particular, the restrictions of $(\overline{u}(a), \underline{v}(a))$, $(\underline{u}(a), \overline{v}(a))$ and $(u^\tau(a), v^\tau(a))$ to $F^T \cup W^T$ are, respectively, the firms-optimal stable payoff vector, the workers-optimal stable payoff vector and the fair division payoff vector of the derived game $(F^T, W^T, a^T, r^T, s^T)$. This proves the next proposition.

Proposition 5.7. *On the domain of multiple-partners job markets, the firms-optimal stable rules, the workers-optimal stable rules and the fair division rules are weak derived consistent.*

We have seen until now that the fair division rules satisfy GVF and WDC. It only remains to see that these two properties characterize these rules among all stable rules. We only sketch the proof, since it is very similar to the one for the simple assignment game in van den Brink *et al.* (2021).

Theorem 5.8. *On the domain of multiple-partners job markets, the fair division rules are the only stable rules that satisfy GVF and WDC.*

Proof. Let φ be a stable rule that satisfies GVF and WDC, and take a multiple-partners assignment game (F, W, a, r, s). Let $c_1 \geq 0$ be the maximum $c \geq 0$ such that $\mathcal{M}_a(F, W, r, s) \subseteq \mathcal{M}_{a^c}(F, W, r, s)$, and $c \leq a_{ij}$ (and thus $a^c_{ij} = a_{ij} - c$) for all $(f_i, w_j) \in \mu$ for some $\mu \in \mathcal{M}_a(F, W, r, s)$. Then, there is some $f_{i_1}, w_{j_1})$ in an optimal matching of (F, W, a^{c_1}, r, s) such that $a^{c_1}_{i_1 j_1} = a_{i_1 0} + a_{0 j_1}$, which means that $u_{i_1 j_1} = a_{i_1 0}$ and $v_{i_1 j_1} = a_{0 j_1}$ for each stable payoff vector (u, v) of (F, W, a^{c_1}, r, s). Define $T_1 = \{f_{i_1}, w_{j_1}\}$.

Let $z^\varphi(a) = (u^\varphi(a), v^\varphi(a))$ and $z^\varphi(a^{c_1}) = (u^\varphi(a^{c_1}), v^\varphi(a^{c_1}))$ be the payoff vectors selected by the rule φ when applied to (F, W, a, r, s) and (F, W, a^{c_1}, r, s) respectively, and $z^\tau(a) = (u^\tau(a), v^\tau(a))$ and $z^\tau(a^{c_1}) = (u^\tau(a^{c_1}), v^\tau(a^{c_1}))$ the payoff vectors selected by any fair division rule in these markets. Trivially, because of the selection of T_1, $u^\varphi_{i_1 j_1}(a^{c_1}) = u^\tau_{i_1 j_1}(a^{c_1})$ and $v^\varphi_{i_1 j_1}(a^{c_1}) = v^\tau_{i_1 j_1}(a^{c_1})$. And by GVF of both φ and τ,

$$
\begin{aligned}
u^\varphi_{i_1 j_1}(a) &= u^\varphi_{i_1 j_1}(a^{c_1}) + \frac{c_1}{2} = u^\tau_{i_1 j_1}(a^{c_1}) + \frac{c_1}{2} = u^\tau_{i_1 j_1}(a) \\
v^\varphi_{i_1 j_1}(a) &= v^\varphi_{i_1 j_1}(a^{c_1}) + \frac{c_1}{2} = v^\tau_{i_1 j_1}(a^{c_1}) + \frac{c_1}{2} = v^\tau_{i_1 j_1}(a)
\end{aligned}
$$

Consider now

$$
F_1 = \begin{cases} F \setminus \{f_{i_1}\} & \text{if } f_{i_1} \in F \text{ and } r_{i_1} = 1, \\ F & \text{otherwise} \end{cases} \quad \text{and ;}
$$

$$
W_1 = \begin{cases} W \setminus \{w_{j_1}\} & \text{if } w_{j_1} \in W \text{ and } s_{j_1} = 1, \\ W & \text{otherwise} \end{cases} ,
$$

and the derived market at T_1 and $z^\varphi(a)$. That is, $a_1 = a^{T_1, z^\varphi(a)}$, $r^1 = r^{T_1}$ and $s^1 = s^{T_1}$, as in Definition 5.3. By WDC of φ, $u^\varphi_{ij}(a_1) = u^\varphi_{ij}(a^{c_1})$ and $v^\varphi_{ij}(a_1) = v^\varphi_{ij}(a^{c_1})$ for all $(f_i, w_j) \in \mu_1 = \mu \setminus \{(f_{i_1}, w_{j_1})\}$.

We now repeat the procedure, that is, given $(F_1, W_1, a_1, r^1, s^1)$ we take $c_2 \geq 0$ the maximum $c \geq 0$ such that $\mathcal{M}_{a_1}(F^1, W^1, r^1, s^1) \subseteq \mathcal{M}_{a_1^c}(F^1, W^1, r^1, s^1)$, and $c \leq (a_1)_{ij}$ (and thus $(a_1)^c_{ij} = (a_1)_{ij} - c$) for all $(f_i, w_j) \in \mu$ for some $\mu \in \mathcal{M}_{a_1}(F^1, W^1, r^1, s^1)$. Then, there is some (f_{i_2}, w_{j_2}) in an optimal matching μ_1 of $(F^1, W^1, a_1^{c_2}, r^1, s^1)$ such that $(a_1)^{c_2}_{i_2 j_2} = (a_1)_{i_2 0} + (a_1)_{0 j_2}$, which means that $u_{i_2 j_2} = a_{i_2 0}$ and $v_{i_2 j_2} = a_{0 j_2}$ for each stable payoff vector (u, v) of $(F^1, W^1, (a_1)^{c_2}, r^1, s^1)$. And we define $T_2 = \{f_{i_2}, w_{j_2}\}$. Notice at this point that, from Sotomayor (1999), the components of any stable payoff vector can be reindexed according to the new optimal matching μ_1.

Since at each step the aggregated capacity strictly decreases, we can guarantee that the procedure is finite. Moreover, at each step, some payoffs u_{ij} and v_{ij}, for (f_i, w_j) optimally matched, are proved to coincide in φ and in τ, and by GVF and WDC they also coincide in the initial market. $\square$

The above theorem shows that the known axiomatic characterization of the fair division rules in the simple assignment game extends to the multiple-partners assignment games. As a consequence, it also follows the logical independence of the two axioms.

6 Concluding remarks

The axiomatic characterizations given in this paper for the two-optimal stable rules and the fair-division rules, on the domain of multiple-partners job markets, have in common that all of them rely on the behaviour of the rules when some firm-worker valuations decrease in a constant amount. This provides a unifying approach to all these stable rules.

Furthermore, from the discussion at the end of Section 4, it follows that on the domain where all firm-worker pairs are acceptable, the fair-division rules are also non-manipulable by constant over-reporting of one firm's valuations. Although this is a weak non-manipulability property, we find it interesting since it rules out a kind of manipulation that is frequently observed in experiments.

A Strong firm-covariance for stable rules in the simple assignment game

We analyse the behaviour of an allocation rule for the simple assignment game when the valuations of an arbitrary set I of firms decrease by the same amount $c \geq 0$, under the assumption that values that become negative are truncated at zero: $a^c_{ij} = \max\{0, a_{ij} - c\}$ for all $(f_i, w_j) \in I \times W$. These row values are allowed to decrease in this way as long as no optimal matching of the initial problem becomes non-optimal. A rule is covariant with respect this change if all firms that have seen their values decreased in c, also see their payoff decreased in c.

Definition A.1. *A rule $\varphi \equiv (u, v; \mu)$ for the simple assignment game is* **strong firm-covariant (SFC)** *if for all (F, W, a), all $I \subseteq F$ and all $c \geq 0$ such that*

(i) $a^{c,I}_{ij} = \max\{0, a_{ij} - c\}$ *for all* $(f_i, w_j) \in I \times W$ *and* $a^c_{ij} = a_{ij}$ *for all* $(f_i, w_j) \in (F \setminus I) \times W$,

(ii) $c \leq a_{ij}$ *for all* $f_i \in I$, $(f_i, w_j) \in \mu$ *and* $\mu \in \mathcal{M}_a(F, W)$ *and*

(iii) $\mathcal{M}_a(F, W) \subseteq \mathcal{M}_{a^{c,I}}(F, W)$,

then,

$$\begin{aligned} u_i(a^{c,I}) &= u_i(a) - c, \quad \textit{for all } f_i \in I \textit{ and} \\ u_i(a^{c,I}) &= u_i(a), \quad\quad \textit{for all } f_i \in F \setminus I. \end{aligned}$$

We can give a threshold for those $c \geq 0$ on the conditions of the above definition.

Lemma A.2. *Conditions* (*ii*) *and* (*iii*) *in Definition A.1 are equivalent to considering* $c \leq c^*$, *where*

$$c^* = \min\{c \geq 0 \mid \text{ there exist } \mu \in \mathcal{M}_{a^{c,I}}(F,W) \text{ and } (f_i, w_j) \in \mu \text{ with } f_i \in I \text{ and } a_{ij} = 0\}. \tag{5}$$

Proof. Let us define $m_a^I = \min\{a_{ij} \mid (f_i, w_j) \in \mu$ for some $\mu \in \mathcal{M}_a(F,W)$ and $f_i \in I\}$. It is quite clear that $c^* \leq m_a^I$. Otherwise, if $m_a^I < c^*$, taking $c = m_a^I$, by definition of c^*, we have that for any $\mu \in \mathcal{M}_{a^{c,I}}(F,W)$ it holds $a_{ij}^{c,I} > 0$ for all $(f_i, w_j) \in \mu$ and $f_i \in I$. This implies that there is an optimal matching μ' of the initial market that is not optimal in $(F, W, a^{c,I})$, since by definition of m_a^I, μ' will have a null entry. But $\sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I}$ implies $\sum_{(f_i,w_j)\in\mu} a_{ij} > \sum_{(f_i,w_j)\in\mu'} a_{ij}$, and contradicts $\mu' \in \mathcal{M}_a(F,W)$.

We now show that if $0 \leq c \leq c^*$, then c satisfies (*ii*) and (*iii*) in Definition A.1. First, since $c \leq c^* \leq m_a^I$, $a_{ij}^{c,I} = a_{ij} - c \geq 0$ for all $(f_i, w_j) \in \mu$, for all $\mu \in \mathcal{M}_a(F,W)$, and (*ii*) is satisfied. Moreover, since $c \leq c^*$, by definition of c^*, we have $a_{ij}^{c,I} = a_{ij} - c \geq 0$ for all $(f_i, w_j) \in \mu \in \mathcal{M}_{a^{c,I}}(F,W)$. This implies that all $\mu \in \mathcal{M}_a(F,W)$ is also optimal for $(F, W, a^{c,I})$. Otherwise, if there exists $\mu' \in \mathcal{M}_{a^{c,I}}(F,W)$ such that $\sum_{(f_i,w_j)\in\mu'} a_{ij}^{c} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I}$, then

$$\sum_{(f_i,w_j)\in\mu'} a_{ij} - |I|c = \sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I} = \sum_{(f_i,w_j)\in\mu} a_{ij} - |I|c,$$

in contradiction with $\mu \in \mathcal{M}_a(F,W)$.

Conversely, if c satisfies (*ii*) and (*iii*), we show that $c \leq c^*$. Indeed, (*ii*) implies that $c \leq m_a^I$. To see that $c \leq c^*$, if we assume on the contrary that $c^* < c \leq m_a^I$, we know that none of the matchings $\mu \in \mathcal{M}_a(F,W)$ has a null entry neither in $(F, W, a^{c^*,I})$ nor in $(F, W, a^{c,I})$. Instead, by definition of c^* there is $\mu' \in \mathcal{M}_{a^{c^*,I}}(F,W)$ with $(f_{i_0}, w_{j_0}) \in \mu'$ and $a_{i_0 j_0}^{c^*,I} = 0 \geq a_{i_0 j_0} - c^*$. Then, since $c^* < c \leq m_a^I$, $\sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I}$ for all $\mu \in \mathcal{M}_a(F,W)$, in contradiction with (*iii*). □

If a rule φ satisfies Definition A.1 for $|I| = 1$, we will say φ is **firm-covariant (FC)**. And notice that this definition coincides with Definition 3.4 when applied to a rule for the simple assignment game. We first prove that the firms-optimal stable rules are strong firm-covariant.

We could similarly define when an allocation rule is strong worker-covariant and we would obtain, in an analogous way, that the workers-optimal stable rules are strong worker-covariant.

Proposition A.3. *The firms-optimal stable rules of the simple assignment game are strong firm-covariant.*

Proof. We can assume without loss of generality that there are as many firms as workers (otherwise we only need to add dummy agents with null valuations in the short side of the market). If $a_{ij} = 0$ for some $f_i \in I$ such that $(f_i, w_j) \in \mu$ and $\mu \in \mathcal{M}_a(F,W)$, then only $c = 0$ satisfies the conditions on Definition A.1, and SFC is trivially satisfied in that case. So, assume $a_{ij} > 0$ for all $(f_i, w_j) \in \mu$ such that $f_i \in I$ and $\mu \in \mathcal{M}_a(F, W)$.

Let $c \geq 0$ be a constant under the conditions of Definition A.1, that is, $c \leq a_{ij}$ for all $(f_i, w_j) \in \mu$ with $f_i \in I$ and $\mu \in \mathcal{M}_a(F, W)$ (and thus $a_{ij}^{c,I} = a_{ij} - c$), and moreover any matching μ that is optimal for (F, W, a) is also optimal for $(F, W, a^{c,I})$.

From now one, to simplify notation, we will write just a^c instead of $a^{c,I}$.

Consider the two optimal stable payoff vectors, $(\overline{u}(a), \underline{v}(a))$ and $(\underline{u}(a), \overline{v}(a))$, for (F, W, a). Let $(\overline{u}^c(a), \underline{v}(a))$ be given by

$$\overline{u}^c(a) = \overline{u}_i(a) - c \text{ for all } f_i \in I;\ \overline{u}^c(a) = \overline{u}_i(a) \text{ for all } f_i \in F \setminus I.$$

We show that $(\overline{u}^c(a), \underline{v}(a))$ is stable for (F, W, a^c), that is, we show individual rationality for each firm and worker, and the stability requirements for each firm-worker pair.

(i) Individual rationality for the workers (i.e. $\underline{v}_j \geq 0$ for all $w_j \in W$) follows trivially from the stability of $(\overline{u}(a), \underline{v}(a))$.

(ii) The stability requirements for every firm-worker pair (i.e. $\overline{u}^c(a)_i + \underline{v}(a)_j \geq a_{ij}^c$ for all $f_i \in F$ and $w_j \in W$) follows trivially from the stability of $(\overline{u}(a), \underline{v}(a))$ (under the assumption that $\overline{u}_i^c(a) \geq 0$, which we show next under (iii)).

(iii) It only remains to show individual rationality for the firms. It is obvious that $\overline{u}_i^c(a) = \overline{u}_i(a) \geq 0$ for all $f_i \in F \setminus I$, so we only need to prove that $\overline{u}_i^c(a) = \overline{u}_i(a) - c \geq 0$ for all $f_i \in I$. This implies to show that any c on the conditions of Definition A.1 satisfies $c \leq \min_{i \in I} \overline{u}_i(a)$. Let us denote by $f_{i_1} \in I$ the firm such that $\overline{u}_{i_1}(a) = \min_{i \in I} \overline{u}_i(a)$.

Notice first that trivially if $c' = \overline{u}_{i_1}(a)$, then $(\overline{u}^{c'}(a), \underline{v}(a))$ is stable for $(F, W, a^{c'})$. Let k be the cardinality of I, $\mu \in \mathcal{M}_a(F, W)$ and $\mu' \in \mathcal{M}_a(F \setminus \{f_{i_1}\}, W)$. Then,

$$\overline{u}_{i_1}(a) = \sum_{(f_i, w_j) \in \mu} a_{ij} - \sum_{(f_i, w_j) \in \mu'} a_{ij}. \tag{6}$$

Furthermore,

$$\sum_{(f_i, w_j) \in \mu'} a_{ij}^{c'} \geq \sum_{(f_i, w_j) \in \mu'} a_{ij} - (k-1)c'.$$

On the other hand, $\sum_{(f_i, w_j) \in \mu} a_{ij} - \sum_{(f_i, w_j) \in \mu} a_{ij}^{c'} = kc'$, and then

$$\sum_{(f_i, w_j) \in \mu} a_{ij}^{c'} = \sum_{(f_i, w_j) \in \mu} a_{ij} - kc' = \sum_{(f_i, w_j) \in \mu'} a_{ij} + c' - kc' = \sum_{(f_i, w_j) \in \mu'} a_{ij} - (k-1)c' \leq \sum_{(f_i, w_j) \in \mu'} a_{ij}^{c'},$$

where the second equality follows from (6). This implies that μ' is also optimal for $(F, W, a^{c'})$ and, as a consequence, if $w_{j_2} \in W$ is the worker unmatched by μ', then $a_{i_1 j_2}^{c'} = 0$. Otherwise, $a_{i_1 j_2}^{c'} > 0$ would contradict the optimality of μ in $(F, W, a^{c'})$.

We finally show that $c \leq \overline{u}_{i_1}(a)$. On the contrary, suppose that $c > c' = \overline{u}_{i_1}(a)$. Since $a_{i_1 j_2}^{c'} = 0$, we have $a_{i_1 j_2} - c < 0$. Then, $\{f_{i_1}, w_{j_2}\}$ belonging to an optimal matching of $(F, W, a^{c'})$ and $c > c'$ implies that the optimal matchings of (F, W, a), which entries have not been truncated, are no longer optimal in (F, W, a^c). This contradicts that c satisfies the conditions of Definition A.1. So, we have proved that $c \leq \overline{u}_{i_1}(a)$, and as a consequence individual rationality for the firms is satisfied.

Since we showed individual rationality for the firms and the workers ((i) and (iii) above), and the stability requirements for all firm-worker pairs ((ii) above), we have that $(\overline{u}^c(a), \underline{v}(a))$ is a stable payoff vector for (F, W, a^c), for all c under the conditions of Definition A.1. Analogously, it can be shown that $(\underline{u}(a), \overline{v}^c(a))$ is a stable payoff vector for (F, W, a^c).

Notice that, by $(\overline{u}^c(a), \underline{v}(a))$ being a stable payoff vector of (F, W, a^c), it is the optimal stable payoff vector of (F, W, a^c). Otherwise, one can derive a contradiction with $(\overline{u}(a), \underline{v}(a))$ being the optimal stable payoff vector of (F, W, a). This completes the proof of SFC for the firms-optimal stable rule. $\square$

The converse implication also holds. In fact, it is even stronger. Any stable rule that satisfies Definition A.1 for $|I| = 1$ (any single row) must be the firms-optimal stable rule. We state the result for both optimal stable rules but only prove it for the firms-optimal one.

Theorem A.4. *1. The firms-optimal stable rules are the only stable rules for the simple assignment game that are firm-covariant.*

2. The workers-optimal stable rules are the only stable rules for the simple assignment game that are worker-covariant.

Proof. It has already been proved in Proposition A.3 that any firms-optimal stable rule is SFC. We need to prove the converse implication. Let $\varphi \equiv (u^\varphi, v^\varphi; \mu)$ be a stable rule that satisfies FC. If φ is not the firms-optimal stable rule, there exists $f_{i_0} \in F$ and a simple assignment game (F, W, a) such that $0 \leq u^\varphi_{i_0}(a) < \overline{u}_{i_0}(a)$. Take then $I = \{f_{i_0}\} \subseteq F$ and $c^* = \overline{u}_{i_0}(a)$, where c^* as defined in (5) satisfies the requirements of Definition A.1.

Then, by firm-covariance of φ, we get $u^\varphi_{i_0}(a^{c^*}) = u^\varphi_{i_0}(a) - c^* < \overline{u}_{i_0}(a) - c^* = 0$ which contradicts the stability of φ. $\square$

The combination of the above results leads to the next straightforward characterization.

Corollary A.5. *1. The firms-optimal stable rules are the only stable rules for the simple assignment game that are strong firm-covariant.*

2. The workers-optimal stable rules are the only stable rules for the simple assignment game that are strong worker-covariant.

References

Demange, G. 1982. Strategyproofness in the assignment market game. *Laboratoire d'Econometrie de l'Ecole Polytechnique.*

Fagebaume, A., Gale, D., & Sotomayor, M. 2010. A note on the multiple-partners assignment game. *Journal of Mathematica Economics*, **46**, 388–392.

Kagel, J., & Levin, D. 1993. Independent Private Value Auctions: Bidder Behaviour in First-, Second- and Third-Price Auctions with Varying Numbers of Bidders. *Economic Journal*, **103(409)**, 868–879.

Kagel, J., & Levin, D. 2009. Implementing efficient multi-object auction institutions: An experimental study of the performance of boundedly rational agents. *Games and Economic Behavior*, **66**, 221–237.

Kagel, J., Lien, Y., & Milgrom, P. 2014. Ascending prices and package bidding: Further experimental analysis. *Games and Economic Behavior*, **85**, 210–231.

Kaneko, M. 1976. On the core and competitive equilibria of a market with indivisible goods. *Naval Research Logistics Quarterly*, **23**, 321–337.

Leonard, H.B. 1983. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, **91**, 461–479.

Llerena, F., Núñez, M., & Rafels, C. 2015. An axiomatization of the nucleolus of the assignment game. *International Journal of Game Theory*, **44**, 1–15.

Núñez, M., & Rafels, C. 2002. The assignment game: the τ-value. *International Journal of Game Theory*, **31**, 411–422.

Owen, G. 1992. The assignment game: the reduced game. *Annales d'Économie et de Statistique*, 71–79.

Pérez-Castrillo, D., & Sotomayor, M. 2017. On the manipulability of competitive equilibrium rules in many-to-many buyer-seller markets. *International Journal of Game Theory*, **46**, 1137–1161.

Pérez-Castrillo, D., & Sotomayor, M. 2019. Comparative statics in the multiple-partners assignment game. *Games and Economic Behavior*, **114**, 177–192.

Sánchez-Soriano, J., López, M.A., & García-Jurado, I. 2001. On the core of transportation games. *Mathematical Social Sciences*, **41**, 215–225.

Shapley, L. S., & Shubik, M. 1972. The assignment game I: the core. *International Journal of Game Theory*, **1**, 111–130.

Sotomayor, M. 1992. The multiple partners game. *Pages 269–283 of:* Majumdar, M. (ed), *Equilibrium and Dynamics: Essays in Honor to David Gale*, vol. 31.

Sotomayor, M. 1999. The lattice structure of the set of stable outcomes of the multiple partners game. *International Journal of Game Theory*, **28**, 567–583.

Sotomayor, M. 2002. A labour market with heterogeneous firms and workers. *International Journal of Game Theory*, **31**, 269–283.

Sotomayor, M. 2007. Connecting the cooperative and the competitive structures of the multiple-patrtners assignment game. *Journal of Economic Theory*, **134**, 155–174.

Sotomayor, M. 2009. Adjusting prices in the multiple-partners assignment game. *International Journal of Game Theory*, **38**, 575–600.

Thompson, G. L. 1981. Auctions and market games. *In:* Aumann, R. (ed), *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern.*

van den Brink, R., Núñez, M., & Robles, F. 2021. Valuation monotonicity, fairness and stability in assignment problems. *Journal of Economic Theory*, **195**, 105277.