Review Article Open Access

Xavier Massaneda and Pascal J. Thomas

From $\mathcal{H}^\infty$ to $\mathcal{N}$. Pointwise properties and algebraic structure in the Nevanlinna class

https://doi.org/10.1515/conop-2020-0007
Received November 4, 2019; accepted March 20, 2020

Abstract: This survey shows how, for the Nevanlinna class $\mathcal{N}$ of the unit disc, one can define and often characterize the analogues of well-known objects and properties related to the algebra of bounded analytic functions $\mathcal{H}^\infty$: interpolating sequences, Corona theorem, sets of determination, stable rank, as well as the more recent notions of Weak Embedding Property and threshold of invertibility for quotient algebras. The general rule we observe is that a given result for $\mathcal{H}^\infty$ can be transposed to $\mathcal{N}$ by replacing uniform bounds by a suitable control by positive harmonic functions. We show several instances where this rule applies, as well as some exceptions. We also briefly discuss the situation for the related Smirnov class.

Keywords: Nevanlinna class, interpolating sequences, Corona theorem, sampling sets, Smirnov class

MSC: 30H15 , 30H05, 30H80

1 Introduction

1.1 The Nevanlinna class

The class $\mathcal{H}^\infty$ of bounded holomorphic functions on the unit $\mathbb{D}$ disc enjoys a wealth of analytic and algebraic properties, which have been explored for a long time with no end in sight, see e.g. [11], [38]. These last few years, some similar properties have been explored for the Nevanlinna class, a much larger algebra which is in some respects a natural extension of $\mathcal{H}^\infty$. The goal of this paper is to survey those results.

For $\mathcal{H}^\infty$, and for a function algebra in general, under the heading "pointwise properties" we mean the characterization of zero sets, interpolating sequences, and sets of determination; while the (related) aspects of algebraic structure of $\mathcal{H}^\infty$ we are interested in concern its ideals: the Corona theorem, which says that $\mathbb{D}$ is dense in the maximal ideal space of $\mathcal{H}^\infty$, the computation of stable rank, and the determination of invertibility in a quotient algebra from the values of an equivalence class over the set where they coincide.

According the most common definition, the Nevanlinna class is the algebra of analytic functions

$$\mathcal{N} = \left\{ f \in \operatorname{Hol}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} \log_+ \left| f(re^{i\theta}) \right| \frac{d\theta}{2\pi} < \infty . \right\}.$$

We will take a different –but, of course, equivalent– perspective. One way to motivate the introduction of this algebra is to consider the quotients f/g when $f, g \in \mathcal{H}^\infty$ and f/g is holomorphic. Without loss of generality we may assume $\|f\|_\infty, \|g\|_\infty \le 1$. Once we factor out the common zeroes of f and g using a Blaschke product (see Section 2), we may assume that $g(z) \neq 0$ for any $z \in \mathbb{D}$. Then $|g| = \exp(-h)$, where h is a positive

Xavier Massaneda: Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08007-Barcelona, Catalonia, E-mail: xavier.massaneda@ub.edu
Pascal J. Thomas: Institut de Mathématiques de Toulouse, UMR5219, Université de Toulouse, CNRS, UPS, F-31062 Toulouse Cedex 9, France, E-mail: pascal.thomas@math.univ-toulouse.fr

harmonic function on the disc, and $\log |f/g| \leq h$. We thus make have the following definition, which will prove more useful in what follows.

Definition 1.1. A function f holomorphic on $\mathbb{D}$ is in the *Nevanlinna Class* $\mathcal{N}$, if and only if $\log |f|$ admits a positive harmonic majorant.

We denote by $\mathrm{Har}_+(\mathbb{D})$ the cone of positive (nonnegative) harmonic functions in the unit disc. For $z \in \mathbb{D}$ and $\theta \in [0, 2\pi)$ let the *Poisson kernel*

$$P_z(e^{i\theta}) := \frac{1}{2\pi} \frac{1 - |z|^2}{|1 - e^{i\theta}\bar{z}|^2}.$$

It is well-known that given μ a positive finite measure on $\partial\mathbb{D}$, its *Poisson integral*

$$\mathcal{P}[\mu](z) := \int_0^{2\pi} P_z(e^{i\theta}) d\mu(\theta)$$

belongs to $\mathrm{Har}_+(\mathbb{D})$, and reciprocally, any $h \in \mathrm{Har}_+(\mathbb{D})$ is the Poisson integral of a positive measure on the circle. Since any finite real measure is the difference of two positive measures, and since any harmonic function which admits a positive harmonic majorant is the difference of two positive harmonic functions, the set of Poisson integrals of finite real measures on the circle coincides with the set of differences of positive harmonic functions. This explains the equivalence of the definitions above, and shows that any function in $\mathcal{N}$ is a quotient f/g of bounded analytic functions.

The main goal of this survey is to provide, when possible, analogues for $\mathcal{N}$ of several well-known results on $\mathcal{H}^\infty$. We shall see that, in order to transfer the results we report on from $\mathcal{H}^\infty$ to $\mathcal{N}$, we must apply the following general principle. A function f belongs to $\mathcal{H}^\infty$ if and only if $\log_+ |f|$ is uniformly bounded above, while $f \in \mathcal{N}$ when there exists $h \in \mathrm{Har}_+(\mathbb{D})$ such that $\log_+ |f| \leq h$. Accordingly, in the hypotheses (and sometimes the conclusions) of the theorems regarding $\mathcal{H}^\infty$ we expect to replace uniform bounds by positive harmonic majorants. This turns out to yield very natural results and, sometimes, natural problems.

A function with a positive harmonic majorant is harder to grasp intuitively than a bounded function. Admitting a positive harmonic majorant is definitely a restriction; for instance, Harnack's inequalities (1) show that a positive harmonic function cannot grow faster that $C(1 - |z|)^{-1}$ as $|z| \to 1$. But there is no easy way to recognize whether a given nonnegative function on the disc has a harmonic majorant, although some conditions are given in [22], [4].

To give a very simple example, suppose that we are given a sequence $(z_k)_k \subset \mathbb{D}$ and positive numbers $(v_k)_k$. Define a function φ on $\mathbb{D}$ by $\varphi(z_k) = v_k$, $\varphi(z) = 0$ when $z \notin (z_k)_k$. It follows from Harnack's inequality that a necessary condition for φ to admit a harmonic majorant is $\sup_k (1 - |z_k|) v_k < \infty$. On the other hand, a sufficient condition is that $\sum_k (1 - |z_k|) v_k < \infty$, and the gap between the two conditions cannot be improved without additional information about the geometry of the sequence $(z_k)_k$ (see Corollary 3.8 in Section 3).

The paper is structured as follows. In the next Section we finish describing the setup and gather several well-known properties of Nevanlinna functions. In particular, we recall that from the point of view of the topology $\mathcal{N}$ is far worse than $\mathcal{H}^\infty$. Even though there is a well-defined distance d that makes $(\mathcal{N}, d)$ a metric space, this is not even a topological vector space. This is an important restriction, since all the nice theorems about Banach spaces available for $\mathcal{H}^\infty$ are no longer available. The canonical factorization of Nevanlinna functions, similar to that of bounded analytic functions, makes up for these shortcomings.

Section 3 studies the analogue in $\mathcal{N}$ of the interpolation problem in $\mathcal{H}^\infty$, which was completely solved in a famous theorem of Carleson (see e.g. [11]). The first issue is to give the appropriate definition of Nevanlinna interpolation. Once this is done we show that the general principle described above, applied to the various characterizations of $\mathcal{H}^\infty$-interpolating sequences, provides also characterizations of Nevanlinna interpolating sequences. The same principle works well when describing finite unions of interpolating sequences.

An old theorem of Brown, Shields and Zeller [6] shows that the $\mathcal{H}^\infty$-sampling sequences, i.e. sequences $(z_k)_k \subset \mathbb{D}$ such that

$$\|f\|_\infty = \sup_k |f(z_k)| \quad \text{for all } f \in \mathcal{H}^\infty,$$

are precisely those for which the non-tangential accumulation set of $(z_k)_k$ on $\partial\mathbb{D}$ has full measure. In Section 4 we set the analogous problem for $\mathcal{N}$, show the complete solution given by S. Gardiner, and provide some examples obtained by previous, more computable, conditions given by the authors.

R. Mortini showed that the corona problem in $\mathcal{N}$ can be solved analogously to Carleson's classical result for $\mathcal{H}^\infty$, provided that the substitution suggested by our guiding principle is performed. In Section 5 we show this and a result about finitely generated ideals, which is the natural counterpart of a theorem by Tolokonnikov [46]. We also show that, in contrast to the $\mathcal{H}^\infty$-case, the stable rank of the algebra $\mathcal{N}$ has to be strictly bigger that 1.

Section 6 studies the Corona problem in quotient algebras. Gorkin, Mortini and Nikolski [18] showed that $\mathcal{H}^\infty$ quotiented by an inner function has the Corona property if and only if it has the so-called Weak Embedding Property (essentially, the inner function is uniformly big away, by a fixed distance, from its zeroes; see Theorem 6.2(c)). The analogue is also true for $\mathcal{N}$, but unlike in the bounded case, this is equivalent to the inner function being a Blaschke product of a finite number of Nevanlinna interpolating sequences. We are also interested in the determination of invertibility in a quotient algebra from the values of an equivalence class over the set where they coincide.

The final Section 7 is devoted to describe the results for the Smirnov class $\mathcal{N}_+$, the subalgebra of $\mathcal{N}$ consisting of the functions f for which $\log_+|f|$ has a *quasi-bounded* harmonic majorant, i.e. a majorant of type $\mathcal{P}[w]$, where $w \in L^1(\partial\mathbb{D})$. A posteriori $\log|f(z)| \leq \mathcal{P}[\log|f^*|](z)$, $z \in \mathbb{D}$. The general rule, with few exceptions, is that the statements about $\mathcal{N}$ also hold for $\mathcal{N}_+$ as soon as the bounded harmonic majorants are replaced by quasi-bounded harmonic majorants.

A final word about notation. Throughout the paper $A \lesssim B$ will mean that there is an absolute constant C such that $A \leq CB$, and we write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

It has been known for a long time that functions in the Nevanlinna class have the same zeroes as bounded functions. These are the sequences $Z := (z_k)_{k\in\mathbb{N}} \subset \mathbb{D}$ satisfying the *Blaschke condition* $\sum_k (1-|z_k|) < \infty$. Note that when points in the sequence $(z_k)_k$ are repeated, we understand that f must vanish with the corresponding order.

That this is necessary is an easy application of Jensen's formula. To prove the reverse implication just notice that when $(z_k)_k$ is a Blaschke sequence the the associated *Blaschke product*

$$B_Z(z) := \prod_{k\in\mathbb{N}} \frac{\bar{z}_k}{|z_k|}\frac{z_k - z}{1 - z\bar{z}_k},$$

converges and yields a holomorphic function with non-tangential limits of modulus 1 almost everywhere on the unit circle $\partial\mathbb{D}$. We simply write $B(z)$ when no confusion can arise.

Theorem 2.1. *[11, Lemma 5.2, p. 69] Let $f \in \mathcal{N}$, $f \not\equiv 0$, $Z := f^{-1}\{0\}$. Then B_Z converges, and $g := f/B_Z \in \mathcal{N}$. Moreover,* $\log|g|$ *is the least harmonic majorant of* $\log|f|$.

Since $\mathcal{N}$ is the set of quotients f/g, where $f, g \in \mathcal{H}^\infty$ and g is zero free, an easy corollary is that $F \in \mathcal{N}$ admits an inverse in $\mathcal{N}$ if and only if it does not vanish on $\mathbb{D}$.

Any $f \in \mathcal{H}^\infty$ verifies $\log|f(z)| \leq \mathcal{P}[\log|f^*|](z)$, where we recall that f^* denotes the nontangential boundary values of f. Therefore if $F = f/g \in \mathcal{N}$, $F \not\equiv 0$, $F^*(e^{i\theta}) \neq 0$ almost everywhere, it admits finite nontangential boundary values almost everywhere, again denoted $F^* = (f/g)^*$. We will see more on this in Section 2.2.

Throughout the paper we will find it useful to consider the product with one factor removed and write

$$b_{z_k}(z) = \frac{\bar{z}_k}{|z_k|}\frac{z_k - z}{1 - z\bar{z}_k} \quad , \quad B_k(z) := B_{Z,k}(z) := \frac{B_Z(z)}{b_{z_k}(z)} = \prod_{j\in\mathbb{N}, j\neq k} \frac{\bar{z}_j}{|z_j|}\frac{z_j - z}{1 - z\bar{z}_j}.$$

We shall often need to use the *pseudohyperbolic* or *Gleason* distance between points of the disc given by

$$\rho(w,z) := \left|\frac{w-z}{1-z\bar{w}}\right|.$$

It is invariant under automorphisms (holomorphic bijections) of the disc, and closely related to the Poincaré distance. We shall denote by $D(a,r)$ the disk centered at $a \in \mathbb{D}$ and with radius $r \in (0,1)$, with respect to ρ, that is, $D(a,r) = \{z \in \mathbb{D} : \rho(z,a) < r\}$.

Regarding basic properties of harmonic functions, we will use repeatedly the well-known *Harnack inequalities*: for $H \in \mathrm{Har}_+(\mathbb{D})$ and $z, w \in \mathbb{D}$,

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \le \frac{H(z)}{H(w)} \le \frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

In particular, taking $w = 0$,

$$H(0)\frac{1-|z|}{1+|z|} \le H(z) \le H(0)\frac{1+|z|}{1-|z|}. \tag{1}$$

We now collect some standard facts about the Nevanlinna class. We start with the natural metric and the topology it defines. Next we recap the canonical factorization of Nevanlinna functions.

2.1 Topological properties of $\mathcal{N}$

From the point of view of the topology, $\mathcal{N}$ is far worse than $\mathcal{H}^\infty$. One easily sees that, given $f \in \mathrm{Hol}(\mathbb{D})$, $\log(1+|f|)$ admits a harmonic majorant if and only if $\log_+|f|$ does. Thus we may define a distance between f and g in the Nevanlinna class by $d(f,g) = N(f-g)$, where

$$N(f) := \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\log(1+|f(re^{i\theta})|)\,d\theta\,. \tag{2}$$

The subharmonicity of $\log(1+|f|)$ yields the pointwise estimate

$$(1-|z|)\log(1+|f(z)|) \le 2N(f),$$

which shows that convergence in the distance d implies uniform convergence on compact sets [44, Proposition 1.1].

The value $N(f)$ can also be rewritten as an extremal solution to a harmonic majorant problem:

$$N(f) = \inf\left\{h(0) \,:\, \exists h \in \mathrm{Har}_+(\mathbb{D}) \text{ with } \log(1+|f|) \le h\right\}. \tag{3}$$

Although $(\mathcal{N}, d)$ is a metric space, it is not a topological vector space: multiplication by scalars fails to be continuous, and it contains many finite-dimensional subspaces on which the induced topology is discrete. In fact, the largest topological vector space contained in $\mathcal{N}$ is the Smirnov class $\mathcal{N}_+$ (see Section 8). These facts and many others are proved in [44].

The limitations imposed by this lack of structure are important, for example, when studying interpolating or sampling sequences (see Sections 3 and 4), since such basic tools as the Open Mapping or the Closed Graph theorems are not available.

2.2 Canonical factorization

To compensate for the lack of structure just mentioned, there is a canonical factorization of functions both in $\mathcal{H}^\infty$ and $\mathcal{N}$ (general references are e.g. [11], [38] or [42]). As a matter of fact this is at the core of our transit from $\mathcal{H}^\infty$ to $\mathcal{N}$, although quite often in the process the existing proofs for $\mathcal{H}^\infty$ have to be redone.

As mentioned in the previous section, given $f \in \mathcal{N}$ with zero set Z and associated Blaschke product B_Z, the function f/B_Z is zero-free and belongs to $\mathcal{N}$ as well (Theorem 2.1).

A function f is called *outer* if it can be written in the form

$$\mathcal{O}(z) = C \exp\left\{ \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log v(e^{i\theta}) d\sigma(\theta) \right\},$$

where $|C| = 1$, $v > 0$ a.e. on $\partial\mathbb{D}$ and $\log v \in L^1(\partial\mathbb{D})$. Such a function is the quotient $\mathcal{O} = \mathcal{O}_1/\mathcal{O}_2$ of two bounded outer functions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{H}^\infty$ with $\|\mathcal{O}_i\|_\infty \le 1$, $i = 1, 2$. In particular, the weight v is given by the boundary values of $|\mathcal{O}_1/\mathcal{O}_2|$. Setting $w = \log v$, we have

$$\log|\mathcal{O}(z)| = P[w](z) = \int_0^{2\pi} P_z(e^{i\theta}) w(e^{i\theta}) d\sigma(e^{i\theta}).$$

This formula allows us to freely switch between assertions about outer functions f and the associated measures $w d\sigma$.

Another important family in this context are *inner* functions: $I \in \mathcal{H}^\infty$ such that $|I| = 1$ almost everywhere on $\partial\mathbb{D}$. Any inner function I can be factorized into a Blaschke product B_Z carrying the zeros $Z = (z_k)_k$ of I, and a singular inner function S defined by

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(e^{i\theta}) \right\},$$

for some positive Borel measure μ singular with respect to the Lebesgue measure.

According to the Riesz-Smirnov factorization, any function $f \in \mathcal{N}$ is represented as

$$f = \alpha \frac{B S_1 \mathcal{O}_1}{S_2 \mathcal{O}_2}, \tag{4}$$

where $\mathcal{O}_1, \mathcal{O}_2$ are outer with $\|\mathcal{O}_1\|_\infty, \|\mathcal{O}_2\|_\infty \le 1$, S_1, S_2 are singular inner, B is a Blaschke product and $|\alpha| = 1$.

Similarly, any $f \in \mathcal{H}^\infty$ with zero set Z can be factored as $f = B_Z f_1$, with $f_1 \in \mathcal{H}^\infty$ and $\|f_1\|_\infty = \|f\|_\infty$. Actually f_1 takes the form $f_1 = \alpha S\mathcal{O}$, where $\mathcal{O}$ is outer and bounded, S is singular inner and $|\alpha| = 1$. In this sense $\mathcal{H}^\infty$ and $\mathcal{N}$ are comparable.

3 Nevanlinna Interpolation

The first difficulty when trying to study interpolating sequences for $\mathcal{N}$ is to find out what the precise problem should be. An interpolating sequence is one over which a certain natural set of sequences of values may be realized as the restriction of functions in the class being studied. That bounded functions should be required to interpolate all bounded values seems natural.

Definition 3.1. We say that $(z_k)_{k\in\mathbb{N}} \subset \mathbb{D}$ is an *interpolating sequence* for $\mathcal{H}^\infty$ if for any bounded complex sequence $v := (v_k)_k \in \ell^\infty$, there exists $f \in \mathcal{H}^\infty$ such that $f(z_k) = v_k$ for all $k \in \mathbb{N}$.

By an application of the Open Mapping theorem, one can show that when $(z_k)_k$ is $\mathcal{H}^\infty$-interpolating, there exists $M > 0$ such that f can be chosen with $\|f\|_\infty \le M\|v\|_\infty$. The minimum such M is called the *interpolation constant* of $(z_k)_k$.

The interpolating condition means that point evaluations at each z_k are independent of each other in a strong sense, and so the sequence has to be sparse. Actually, it is easy to deduce from the Schwarz-Pick Lemma that any interpolating sequence must be *separated*, i.e. $\inf_{j\neq k} \rho(z_j, z_k) > 0$.

Actually, a stronger uniform separation is necessary and sufficient.

Theorem 3.2 (Carleson, 1958, see e.g. [11]). *The sequence $(z_k)_k$ is interpolating for $\mathcal{H}^\infty$ if and only if*

$$\inf_{k\in\mathbb{N}} |B_k(z_k)| = \inf_{k\in\mathbb{N}} \prod_{j,j\neq k} \rho(z_j, z_k) > 0.$$

The $\mathcal{H}^\infty$-interpolating sequences, also known as Carleson sequences, have several equivalent characterizations, as we shall see soon.

Turning to the Nevanlinna class, we see that from the usual definition given at the beginning of the paper (in terms of limits of integrals over circles), it is not immediately obvious what the natural condition on pointwise values of f should be.

One way to circumvent this is to forego an explicit description of our target set of values, but to demand that it satisfies some property that guarantees that, in a sense, values can be chosen independently over the points of the sequence. Following N. K. Nikolski and his collaborators, we adopt the following definition.

Definition 3.3. Let X be a space of holomorphic functions in $\mathbb{D}$. A sequence $Z \subset \mathbb{D}$ is called *free interpolating* for X if the space of restrictions $X|Z$ of functions of X to Z, also called *trace space*, $X|Z$, is ideal. This means that if $(v_k)_k \in X|Z$ and $(c_k)_k \in \ell^\infty$, then necessarily $(c_k v_k)_k \in X|Z$.

In other words, Z being free interpolating means that whenever a sequence of values is a restriction of an element in X, any other sequence with pointwise values lesser or equal in moduli will also be a restriction. We also see that, when X is stable under multiplication by bounded functions, this immediately implies that Z is a zero set for X.

Observe also that for X a unitary algebra, $X|Z$ is ideal if and only if $\ell^\infty \subset X|Z$. Indeed, if $X|Z$ is ideal, then since the constant function 1 belongs to X, any bounded sequence must be the restriction of a function in X. Conversely, suppose that $|w_k| \le |v_k|$ for each k and that $v_k = f(z_k)$ for some $f \in X$. Let $h \in X$ satisfy $h(z_k) = w_k/v_k$ when $v_k \neq 0$; then $(hf)(z_k) = w_k$ and $hf \in X$.

As a consequence, we remark that any $\mathcal{H}^\infty$-interpolating sequence must be free interpolating for $\mathcal{N}$, since any bounded sequence of values will be the restriction of a bounded function, which is in the Nevanlinna class.

An alternative approach, when trying to define Nevanlinna interpolation, is to think in terms of harmonic majorants and consider, for a sequence $Z = (z_k)_k$, the subspace of values

$$\ell_{\mathcal{N}}(Z) := \left\{(v_k)_k : \exists\, h \in \mathrm{Har}_+(\mathbb{D}) \text{ such that } h(z_k) \ge \log_+ |v_k|, k \in \mathbb{N}\right\}.$$

Notice that it is immediate from the first property in Definition 1.1 that $\mathcal{N}|Z \subset \ell_{\mathcal{N}}(Z)$. We then want to claim that a sequence is interpolating for $\mathcal{N}$ when the reverse inclusion holds. Since $\ell_{\mathcal{N}}(Z)$ is clearly ideal, this implies in particular that the sequence is free interpolating.

Conversely, assume Z is free interpolating. Given values $(v_k)_k$ and h as above, we can construct a holomorphic function H such that $\mathrm{Re}\, H = h$; then $e^H \in \mathcal{N}$. The values $(e^{-H(z_k)} v_k)_k$ form a bounded sequence, so by the assumption there is $g \in \mathcal{N}$ such that $g(z_k) = e^{-H(z_k)} v_k$, and $e^H g \in \mathcal{N}$ will interpolate the given values. So the two conditions above turn out to be equivalent.

3.1 Main result and consequences.

In this section we report mostly on results from [22].

Nevanlinna interpolating sequences are characterized by the following Carleson type condition.

Theorem 3.4. *[22, Theorem 1.2] Let $Z = (z_k)_k$ be a sequence in $\mathbb{D}$. The following statements are equivalent:*
(a) Z is a free interpolating sequence for the Nevanlinna class $\mathcal{N}$, i.e. the trace space $\mathcal{N}|Z$ is ideal;
(b) The trace space $\mathcal{N}|Z$ contains $\ell_{\mathcal{N}}(Z)$, and therefore is equal to it;
(c) There exists $h \in \mathrm{Har}_+(\mathbb{D})$ such that

$$|B_k(z_k)| \ge e^{-h(z_k)} \quad k \in \mathbb{N}.$$

Notice that (c) could be rephrased as

(c) The function φ_Z defined by $\varphi_Z(z_k) = \log|B_k(z_k)|^{-1}$, $\varphi_Z(z) = 0$ when $z \in \mathbb{D} \setminus (z_k)_k$, admits a harmonic majorant.

In this form we see that this is the adaptation, according to our general principle, of Carleson's condition for $\mathcal{H}^\infty$: the uniform upper bound on $\log|B_k(z_k)|^{-1}$ is replaced by a harmonic majorant.

A first consequence of the theorem is that Nevanlinna-interpolating sequences must satisfy a weak separation condition.

Definition 3.5. A sequence $Z = (z_k)_k$ is *weakly separated* if there exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that the disks $D(z_k, e^{-H(z_k)})$, $k \in \mathbb{N}$, are pairwise disjoint.

Since for any $j \neq k$, $\rho(z_j, z_k) \geq |B_k(z_k)|$, we have $\rho(z_j, z_k) \geq e^{-h(z_k)}$, with h the harmonic majorant in condition (c). So any Nevanlinna-interpolating sequence is weakly separated. Another proof of this is given in [21, Corollary 2.5].

Although it is not easy to deduce simple geometric properties from condition (c), there is a vast and interesting class of sequences which turns out to be Nevanlinna-interpolating.

In order to see a first family of examples we point out that condition (c) is really about the "local" Blaschke product; it depends only on the behavior of the z_j, $j \neq k$, in a fixed pseudohyperbolic neighbourhood of z_k.

Proposition 3.6. *[22, Proposition 4.1] Let Z be a Blaschke sequence. For any $\delta \in (0,1)$, there exists a positive harmonic function h, such that*

$$\log \prod_{k:\rho(z_k,z)\geq\delta} |\rho(z_k, z)|^{-1} \leq h(z), \quad z \in \mathbb{D}.$$

The idea of the proof is that for any $a \in \mathbb{D}$ and $\delta \in (0,1)$, there is a constant C such that for $\rho(a,z) \geq \delta$,

$$\log \frac{1}{\rho(a,z)} \leq C\mathcal{P}[\chi_{I_a}](z),$$

where

$$I_a = \left\{e^{i\theta} \in \partial\mathbb{D} : \left|e^{i\theta} - \frac{a}{|a|}\right| < 1 - |a|\right\}$$

is the Privalov shadow of a (see e.g. [35, p. 124, lines 3 to 17]). One gets then the result by summing over the sequence Z, which gives the Poisson integral of an integrable function on the circle, because of the Blaschke condition.

An immediate consequence is the following.

Corollary 3.7. *Any separated Blaschke sequence is Nevanlinna-interpolating.*

There are other cases, when the geometry of Z is especially regular, where it is possible to characterize Nevanlinna interpolating sequences. This is the case for well concentrated sequences, in the sense that they are contained in a finite union of Stolz angles $\cup_i \Gamma(\theta_i)$, where

$$\Gamma(\theta) = \left\{z \in \mathbb{D} : |z - e^{i\theta}| \leq (1 - |z|)\right\}.$$

It is also the case for for well spread sequences, in the sense that the measure $\mu_Z = \sum_k (1 - |z_k|)\delta_{z_k}$ has bounded Poisson balayage, i.e.,

$$\sup_{\theta\in[0,2\pi)} \sum_k (1 - |z_k|) P_{z_k}(e^{i\theta}) \approx \sup_{\theta\in[0,2\pi)} \sum_k \frac{(1 - |z_k|)^2}{|z_k - e^{i\theta}|^2} < +\infty.$$

Corollary 3.8. *Assume $Z = (z_k)_k$ is a Blaschke sequence in $\mathbb{D}$.*

(a) *Let Z be contained in a finite number of Stolz angles. Then Z is Nevanlinna interpolating if and only if*

$$\sup_k (1-|z_k|)\log|B_k(z_k)|^{-1} < +\infty. \tag{5}$$

(b) *Let Z be such that $\mu_Z := \sum_k (1-|z_k|)\delta_{z_k}$ has bounded Poisson balayage. Then Z is Nevanlinna interpolating if and only if*

$$\sum_k (1-|z_k|)\log|B_k(z_k)|^{-1} < +\infty. \tag{6}$$

Condition (5) is always necessary for Nevanlinna interpolation. This is just a consequence of Theorem 3.4 (c) and Harnack's inequalities (see (1)).

In order to see the converse we can assume that the sequence is contained in just one Stolz angle: if $Z = \cup_{i=1}^n Z_i$, with $Z_i \subset \Gamma(\theta_i)$, $\theta_i \neq \theta_j$, then

$$\lim_{\substack{z\to e^{i\theta_i}\\ z\in\Gamma(\theta_i)}} |B_{Z_j}(z)| = 1\ ,$$

and therefore, for $z_k \in \Gamma(\theta_i)$, the value $\log|B_k(z_k)|^{-1}$ behaves asymptotically like $\log|B_{Z_i,k}(z_k)|^{-1}$.

In this situation the proof is immediate. Assume Z is contained in the Stolz angle of vertex $1 \in \partial\mathbb{D}$. Let C denote the supremum in (5) and define the positive harmonic (singular) function

$$h(z) := P[C\delta_1](z) = C\frac{1-|z|^2}{|1-z|^2},$$

where δ_1 indicates the Dirac mass on 1. From the hypothesis

$$\log|B_k(z_k)|^{-1} \leq \frac{C}{1-|z_k|} \leq h(z_k), \quad k\in\mathbb{N},$$

and the result follows from Theorem 3.4.

On the other hand, condition (6) is always sufficient, since then

$$w(\theta) = \sum_k (\log|B_k(z_k)|^{-1})\chi_{I_k}(e^{i\theta})$$

is in $L^1(\partial\mathbb{D})$ and clearly

$$\log|B_k(z_k)|^{-1} \leq P[w](z_k), \quad k\in\mathbb{N}.$$

In order to see that in case the Poisson balayage is finite (6) is also necessary, take $h \in \mathrm{Har}_+(\mathbb{D})$ satisfying Theorem 3.4(c) and let ν be a finite positive measure with $h = P[\nu]$. Then, by Fubini's theorem

$$\begin{aligned}\sum_k (1-|z_k|)h(z_k) &= \int_{\mathbb{D}} (1-|z|)h(z)d\mu_Z(z)\\ &= \int_0^{2\pi}\int_{\mathbb{D}} (1-|z|)P_z(e^{i\theta})d\mu_Z(z)d\nu(\theta) \approx \nu(\partial\mathbb{D}),\end{aligned}$$

and the result follows from the previous estimate.

3.2 Comparison with previous results.

There had been previous works on the question of interpolation in the Nevanlinna class. As early as 1956, Naftalevič [34] described the sequences Λ for which the trace $N|\Lambda$ coincides with the sequence space

$$\ell_{\mathrm{Na}}(Z) := \left\{(v_k)_k : \sup_k (1-|z_k|)\log_+|v_k| < \infty\right\}.$$

Theorem 3.9. *[34] $\mathcal{N}|Z = \ell_{Na}(Z)$ if and only if Z is contained in a finite union of Stolz angles and (5) holds.*

On the other hand, in a paper about the Smirnov class, Yanagihara [53] had introduced the sequence space

$$\ell_{\mathrm{Ya}}(Z) := \Big\{(z_k)_k : \sum_k (1 - |z_k|)\log_+ |v_k| < \infty\Big\}.$$

As we have just seen in the previous section, for any $Z \subset \mathbb{D}$, $\ell^\infty \subset \ell_{\mathrm{Ya}}(Z) \subset \ell_{\mathcal{N}}(Z) \subset \ell_{\mathrm{Na}}(Z)$.

The target space ℓ_{Na} seems "too big", since the growth condition it imposes forces the sequences to be confined in a finite union of Stolz angles. Consequently a big class of $\mathcal{H}^\infty$-interpolating sequences, namely those containing a subsequence tending tangentially to the boundary, cannot be interpolating in the sense of Naftalevič. This does not seem natural, for $\mathcal{H}^\infty$ is in the multiplier space of $\mathcal{N}$.

On the other hand, the target space $\ell_{\mathrm{Ya}}(Z)$ seems "too small": there are $\mathcal{H}^\infty$-interpolating sequences such that $\mathcal{N}|Z$ (or even $\mathcal{N}_+|Z$) does not embed into $l_{\mathrm{Ya}}(Z)$ [53, Theorem 3].

If one requires that $\mathcal{N}|Z \supset \ell_{\mathrm{Ya}}(Z)$, this implies that all bounded values can be interpolated, and the sequence is Nevanlinna-interpolating. But then the natural space of restrictions of functions in $\mathcal{N}$ is the a priori larger $\ell_{\mathcal{N}}(Z)$. As seen in Corollary 3.8, the target space $\ell_{\mathrm{Ya}}(Z)$ is only natural when considering sequences for which $\mu_Z := \sum_k (1 - |z_k|)\delta_{z_k}$ has bounded Poisson balayage.

3.3 Equivalent conditions for Nevanlinna interpolation

The following result collects several alternative descriptions of Nevanlinna interpolating sequences. All of them have their corresponding analogues in $\mathcal{H}^\infty$.

Given $H \in \mathrm{Har}_+(\mathbb{D})$, consider the disks $\mathcal{D}_k^H = D(z_k, e^{-H(z_k)})$ and the domain

$$\Omega_k^H = \mathbb{D} \setminus \bigcup_{\substack{j: j\neq k \\ \rho(z_j,z_k)\le 1/2}} \mathcal{D}_j^H .$$

The proof of Theorem 3.10 below shows clearly that the choice of the constant 1/2 in the definition of Ω_k^H is of no relevance; it can be replaced by any $c \in (0,1)$. Let $\omega(z, E, \Omega)$ denote the harmonic measure at $z \in \Omega$ of the set $E \subset \partial\Omega$ in the domain Ω.

Theorem 3.10. *([20, Theorem 1.2]) Let $Z = (z_k)_k$ be a Blaschke sequence of distinct points in $\mathbb{D}$ and let B be the Blaschke product with zero set Z. The following statements are equivalent:*

(a) Z is an interpolating sequence for $\mathcal{N}$, that is, there exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that

$$(1 - |z_k|^2)|B'(z_k)| = |B_k(z_k)| \ge e^{-H(z_k)}, \quad k \in \mathbb{N}.$$

(b) There exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that $|B(z)| \ge e^{-H(z)}\rho(z, Z)$, $z \in \mathbb{D}$,

(c) There exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that $|B(z)| + (1 - |z|^2)|B'(z)| \ge e^{-H(z)}$, $z \in \mathbb{D}$,

(d) There exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that the disks $\mathcal{D}_k^H$ are pairwise disjoint, and

$$\inf_{k\in\mathbb{N}} \omega(z_k, \partial\mathbb{D}, \Omega_k^H) > 0.$$

The proof of (d) shows that it can be replaced by an a priori stronger statement: for every $\epsilon \in (0,1)$ there exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that the disks $\mathcal{D}_k^H$ are pairwise disjoint, and

$$\inf_{k\in\mathbb{N}} \omega(z_k, \partial\mathbb{D}, \Omega_k^H) \ge 1 - \epsilon.$$

Vasyunin proved in [50] (see also [25]) that B_Z is an $\mathcal{H}^\infty$ interpolating Blaschke product if and only if there exists $\delta > 0$ such that

$$|B_Z(z)| \ge \delta\rho(z, Z), \quad z \in \mathbb{D}. \tag{7}$$

Therefore, condition (b) is, again, the natural counterpart of the existing condition for $\mathcal{H}^\infty$.

Similarly, statement (d) and its proof are modelled after the corresponding version for $\mathcal{H}^\infty$, proved by J.B. Garnett, F.W. Gehring and P.W. Jones in [12]. In that case the pseudohyperbolic discs $\mathcal{D}_k^H = D(z_k, e^{-H(z_k)})$ have to be replaced by uniform discs $D(z_k, \delta)$, $\delta > 0$.

A useful, and natural, consequence of Theorem 3.10(d) is that Nevanlinna interpolating sequences are stable under small pseudohyperbolic perturbations.

Corollary 3.11. *Let $Z = (z_k)_k$ be a Nevanlinna interpolating sequence and let $H \in \mathrm{Har}_+(\mathbb{D})$, satisfying Theorem 3.10(a). If $Z' = (z_k')_k \subset \mathbb{D}$ satisfies*

$$\rho(z_k, z_k') \le \frac{1}{4} e^{-H(z_k)}, \quad k \in \mathbb{N},$$

then Z' is also a Nevanlinna interpolating sequence.

3.4 Finite unions of interpolating sequences

Interpolation can be considered also with multiplicities, or more generally, with divided differences. We show next that a discrete sequence $Z = (z_k)_k$ of the unit disk is the union of n interpolating sequences for the Nevanlinna class $\mathcal{N}$ if and only if the trace $\mathcal{N}|Z$ coincides with the space of functions on Z for which the pseudohyperbolic divided differences of order $n-1$ are uniformly controlled by a positive harmonic function.

In Section 6 we will see other characterizations of finite unions of Nevanlinna interpolating sequences.

Definition 3.12. Let $Z = (z_k)_k$ be a discrete sequence in $\mathbb{D}$ and let ω be a function given on Z. The *pseudohyperbolic divided differences of ω* are defined by induction as follows

$$\Delta^0 \omega(z_{k_1}) = \omega(z_{k_1}),$$
$$\Delta^j \omega(z_{k_1}, \dots, z_{k_{j+1}}) = \frac{\Delta^{j-1}\omega(z_{k_2}, \dots, z_{k_{j+1}}) - \Delta^{j-1}\omega(z_{k_1}, \dots, z_{k_j})}{b_{z_{k_1}}(z_{k_{j+1}})} \qquad j \ge 1.$$

For any $n \in \mathbb{N}$, denote

$$Z^n = \{(z_{k_1}, \dots, z_{k_n}) \in Z \times \overset{n}{\cdots} \times Z \ : \ k_j \ne k_l \text{ if } j \ne l\},$$

and consider the set $X^{n-1}(Z)$ consisting of the functions defined in Z with divided differences of order $n-1$ uniformly controlled by a positive harmonic function H i.e., such that for some $H \in \mathrm{Har}_+(\mathbb{D})$,

$$\sup_{(z_{k_1}, \dots, z_{k_n}) \in Z^n} |\Delta^{n-1}\omega(z_{k_1}, \dots, z_{k_n})| e^{-[H(z_{k_1}) + \dots + H(z_{k_n})]} < +\infty .$$

It is not difficult to see that $X^n(Z) \subset X^{n-1}(Z) \subset \cdots \subset X^0(Z) = \mathcal{N}(Z)$. For example, if $\omega \in X^1(Z)$, we can take a fixed $z_{k_0} \in Z$ and write

$$\omega(z_k) = \frac{\omega(z_k) - \omega(z_{k_0})}{b_{z_k}(z_{k_0})} b_{z_k}(z_{k_0}) + \omega(z_{k_0}).$$

Using that there exists $H \in \mathrm{Har}_+(\mathbb{D})$ with

$$|\Delta^1(z_k, z_{k_0})| = \Big| \frac{\omega(z_k) - \omega(z_{k_0})}{b_{z_k}(z_{k_0})} \Big| \le e^{H(z_k) + H(z_{k_0})}$$

we readily see that there is $H_2 \in \mathrm{Har}_+(\mathbb{D})$, depending on H and z_{k_0}, such that

$$|\omega(z_k)| \le e^{H_2(z_k)} \qquad k \in \mathbb{N}.$$

The following result is the analogue of Vasyunin's description of the sequences Z in $\mathbb{D}$ such that the trace of the algebra $\mathcal{H}^\infty$ on Z equals the space of pseudohyperbolic divided differences of order n (see [51], [52]). Similar results hold also for Hardy spaces (see [7] and [19]) and the Hörmander algebras, both in $\mathbb{C}$ and in $\mathbb{D}$ [28].

Theorem 3.13 (Main Theorem [21]). *The trace $\mathcal{N}|Z$ of $\mathcal{N}$ on Z coincides with the set $X^{n-1}(Z)$ if and only if Z is the union of n interpolating sequences for $\mathcal{N}$.*

4 Sampling sets

4.1 Sets of determination for $\mathcal{H}^\infty$.

One may consider the dual problem to interpolation: which sequences are "thick" enough so that the norm of a function can be computed from its values on the sequence? Here there is no reason to restrict ourselves to sequences.

Definition 4.1. We say that $\Lambda \subset \mathbb{D}$ is a *set of determination* for $\mathcal{H}^\infty$ if for any $f \in \mathcal{H}^\infty$, $\|f\|_\infty = \sup_{z\in\Lambda} |f(z)|$.

Recall that we say that a sequence $(z_k)_k$ converges to $z^\star \in \partial\mathbb{D}$ *non-tangentially* if $\lim_{k\to\infty} z_k = z^\star$ and if there exists $A > 0$ such that for all k, $|z^\star - z_k| \le (1 + A)(1 - |z_k|)$. We write $NT \lim_{z\to e^{i\theta}} f(z) = \lambda$ if the limit is achieved over all sequences tending to $e^{i\theta}$ non-tangentially.

Also, as commented in the Preliminaries, for $f \in \mathcal{H}^\infty$ the non-tangential boundary value $f^\star(e^{i\theta}) = NT \lim_{z\to e^{i\theta}} f(z)$ exists a.e. $\theta \in [0, 2\pi)$ (see for instance [11, Theorem 3.1, p. 557]).

Determination sets for $\mathcal{H}^\infty$ are characterized by a simple geometric condition.

Theorem 4.2 (Brown, Shields and Zeller, 1960 [6]). *Λ is a set of determination for $\mathcal{H}^\infty$ if and only if the set $NT(\Lambda)$ consisting of the $\zeta \in \partial\mathbb{D}$ which are a non-tangential limit of a sequence of points in Λ has full measure, i.e. $|NT(\Lambda)| = 2\pi$.*

4.2 Defining the question for $\mathcal{N}$.

In general a sequence $Z = (z_k)_k$ is called "sampling" for a space of holomorphic functions X when any function $f \in X$ is determined by its restriction $f|Z$, with control of norms. For for the Nevanlinna class, which has nothing like a norm, the situation is not so obvious. We start from the notion of set of determination for $\mathcal{H}^\infty$. Instead of requiring that the least upper bound obtained from the values of $f|Z$ be the same as $\sup_{\mathbb{D}} |f|$, we will consider the set of harmonic majorants of $(\log_+ |f|)\,|Z$ compared to the set of harmonic majorants of $\log_+ |f|$.

Recall that the topology on $\mathcal{N}$ is defined with the help of the functional N, defined in (2) and (3). We give a variant.

$$N_+(f) = \lim_{r\to 1} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \inf\{h(0) : h \in \mathrm{Har}_+(\mathbb{D}) \text{ with } \log_+ |f| \le h\}.$$

Notice that the expression in (3) and the second expression in the equation above make sense for any measurable function on the disk, and we will apply them to $f|Z$.

Theorem 4.3. *[29, Theorem 2.2] The following properties of $Z = (z_k)_k \subset \mathbb{D}$ are equivalent:*
(a) There exists $C > 0$ such that for any $f \in \mathcal{N}$, $N(f) \le N(f|Z) + C$.
(b) For any $f \in \mathcal{N}$, $N_+(f) = N_+(f|Z)$.
(c) Z is a set of determination for $\mathcal{N}$, i.e. any $f \in \mathcal{N}$ with $\sup_Z |f| < \infty$ must be bounded on the whole unit disk.
(d) For any $f \in \mathcal{N}$ and $h \in \mathrm{Har}_+(\mathbb{D})$ such that $\log_+ |f(z_k)| \le h(z_k)$ for all k, then necessarily $\log_+ |f| \le h$ on the whole unit disk.

We say that Z is of determination (or sampling) for $\mathcal{N}$ if the above properties are satisfied.

4.3 Main result.

If property (b) above is satisfied, then in particular it must hold for zero-free functions in the Nevanlinna class, and passing to $\log|f|$, the set Z must be a set of determination for the class $\mathrm{Har}_\pm(\mathbb{D})$ of harmonic functions which are the difference of two positive harmonic functions.

To state subsequent results, we need a variant of the decomposition of the disc into Whitney squares, and points in them. Given $n \in \mathbb{N}$ and $k \in \{0, \dots, 2^{n+4} - 1\}$, let

$$S_{n,k} := \left\{ re^{i\theta} : 1 - 2^{-n} \le r \le 1 - 2^{-n-1}, \theta \in [2\pi k 2^{-n-4}, 2\pi(k+1)2^{-n-4}] \right\}; \tag{8}$$
$$z_{n,k} := (1 - 2^{-n}) \exp(2\pi i k 2^{-n-4}).$$

Theorem 4.4. *(Hayman-Lyons, [23]) Let $Z \subset \mathbb{D}$. The following properties are equivalent.*
(a) $\sup_Z h = \sup_{\mathbb{D}} h$ *for all* $h \in \mathrm{Har}_\pm(\mathbb{D})$.
(b) For every $\zeta \in \partial\mathbb{D}$, $\sum_{(n,k): Z \cap S_{n,k} \neq \emptyset} 2^{-n} P_{z_{n,k}}(\zeta) = \infty$.

Note that in contrast with the condition in Theorem 4.2, the condition of accumulation to the boundary must be met at every point with no exception. In particular, a set Z such that every boundary point is a non-tangential limit of points of Z will satisfy the above property.

The sets of determination for $\mathcal{N}$ have been characterized by S. Gardiner [10]. We need an auxiliary quantity depending on a set $A \subset \mathbb{D}$ and $t \ge 0$. If either $A = \emptyset$ or $t = 0$, we set $\mathcal{Q}(A, t) = 0$; otherwise,

$$\mathcal{Q}(A, t) := \min\left\{ k \in \mathbb{N} : \exists \xi_1, \dots, \xi_k \in \mathbb{C} \text{ s. t. } \sum_{1 \le j \le k} \log \frac{1}{|z - \xi_j|} \ge t, \forall z \in A \right\}.$$

Finally, for any set A and $\lambda > 0$, $\lambda A := \{\lambda z, z \in A\}$.

Theorem 4.5. *(Gardiner [10]) Let $Z \subset \mathbb{D}$. The following conditions are equivalent:*
(a) Z is a set of determination for $\mathcal{N}$;
(b) For every $\zeta \in \partial\mathbb{D}$, $\sum_{n,k} 2^{-n} \mathcal{Q}\left(2^n(Z \cap S_{n,k}), \lfloor P_{z_{n,k}}(\zeta) \rfloor\right) = \infty$.

One can show by an elementary computation [10] that

$$\mathcal{Q}\left(2^n(Z \cap S_{n,k}), \lfloor P_{z_{n,k}}(\zeta) \rfloor\right) \le 4 P_{z_{n,k}}(\zeta),$$

so that condition (b) in Theorem 4.5 implies condition (b) in Theorem 4.4, as could be expected.

4.4 Regular discrete sets.

In this section we give some computable conditions for regular sets.

Definition 4.6. Let $g : (0, 1] \longrightarrow (0, 1]$ be a non-decreasing continuous function with $g(0) = 0$. A sequence $Z = (z_k)_k$ is called a *g-net* if and only if
(i) The disks $D(z_k, g(1 - |z_k|))$, $k \in \mathbb{N}$, are mutually disjoint,
(ii) There exists $C > 0$ such that $\bigcup_{k \in \mathbb{N}} D(z_k, Cg(1 - |z_k|)) = \mathbb{D}$.

Another way to think of it is that a g-net is a maximal g-separated sequence. For n large enough, up to multiplicative constants, a g-net will have $g(2^{-n})^{-2}$ points in each domain $S_{n,k}$.

Theorem 4.7. *[29, Theorem 4.1] Let Z be a g-net. The following properties are equivalent:*
(a) Z is a set of determination for $\mathcal{N}$.
(b) $\displaystyle\int_0 \frac{dt}{t^{1/2} g(t)} = \infty$.

(c) $\sum_n 1/(2^{n/2} g(2^{-n})) = \infty$.

Taking regular sets with points which have a different distance in the radial and angular directions does not change things much. We describe a family of examples. Let $(r_m)_m \subset (0,1)$ be an increasing sequence of radii with $\lim_m r_m = 1$ and $\sup_m \frac{1-r_{m+1}}{1-r_m} < 1$. Let ϵ_m be a decreasing sequence of hyperbolic distances such that $\lim_m \epsilon_m = 0$. The *discretized rings associated to* $(r_m)_m$ *and* $(\epsilon_m)_m$ is the sequence $\Lambda = (\lambda_{m,j})_{m,j}$, where

$$\lambda_{m,j} = r_m \exp\left(2\pi i(1-r_m)\epsilon_m j\right) \qquad m \in \mathbb{N}, \quad 0 \le j < \left[\frac{1}{(1-r_m)\epsilon_m}\right].$$

Theorem 4.8. *[29, Theorem 4.4] Let $\Lambda = (\lambda_{m,j})_{m,j}$ be the sequence of discretized rings associated to $(r_m)_m$ and $(\epsilon_m)_m$. Then Z is a set of determination for $\mathcal{N}$ if and only if*

$$\sum_{m=0}^{\infty} \left(\frac{1-r_m}{\epsilon_m}\right)^{1/2} = \infty.$$

4.5 Uniformly dense disks

The sampling sets we consider here are no longer discrete. Recall a class of sequences considered by Ortega-Cerdà and Seip in [40].

Definition 4.9. A sequence $\Lambda = (\lambda_k)_k \subset \mathbb{D}$ is *uniformly dense* if
(i) Λ is separated, i.e. $\inf_{j\neq k} \rho(\lambda_j, \lambda_k) > 0$.
(ii) There exists $r < 1$ such that $\mathbb{D} = \bigcup_{k\in\mathbb{N}} D(\lambda_k, r)$.

In the terminology of the previous subsection, those are g-nets with a constant g.

Let φ be a non-decreasing continuous function, bounded by some constant less than 1. Given a uniformly dense Λ, define $D_k^\varphi = D(\lambda_k, \varphi(1-|\lambda_k|))$ and

$$\Lambda(\varphi) := \bigcup_{k\in\mathbb{N}} D_k^\varphi.$$

Theorem 4.10. *[29, Theorem 4.5] The set $\Lambda(\varphi)$ is sampling for $\mathcal{N}$ if and only if*

$$\int_0^1 \frac{dt}{t\log(1/\varphi(t))} = \infty\,. \tag{9}$$

This condition actually also characterizes determination sets for the space of subharmonic functions in the disk having the characteristic growth of the Nevanlinna class [29, Section 5].

Condition (9) is equivalent to the fact that the harmonic measure of the exterior boundary $\partial\mathbb{D}$ of $\mathbb{D}\setminus\overline{\Lambda(\varphi)}$ is zero, see [40, Theorem 1]. Notice also that for any fixed $K > 1$, condition (9) is equivalent to

$$\sum_n \frac{1}{\log(1/\varphi(K^{-n}))} = \infty\,. \tag{10}$$

The above family of examples allows us to see that there is no general relationship between $A^{-\alpha}$-sampling sets and Nevanlinna sampling sets. We recall this former, and better known, notion of sampling.

Definition 4.11. A set $\Lambda \subset \mathbb{D}$ is *sampling* for the space

$$A^{-\alpha} = \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_\alpha := \sup_{z\in\mathbb{D}} (1-|z|)^\alpha |f(z)| < \infty\} \qquad \alpha > 0,$$

when there exists $C > 0$ such that $\|f\|_\alpha \le C \sup_{\lambda\in\Lambda} (1-|\lambda|)^\alpha |f(z)|$ for all $f \in A^{-\alpha}$.

A well-known result of K. Seip [43, Theorem 1.1] characterizes $A^{-\alpha}$-sampling sets as those Λ for which there exists a separated subsequence $\Lambda' = (\lambda_k)_k \subset \Lambda$ such that

$$D_-(\Lambda') := \liminf_{r\to 1^-} \inf_{z\in\mathbb{D}} \frac{\sum\limits_{k:1/2<\rho(\lambda_k,z)<r} \log \frac{1}{\rho(\lambda_k,z)}}{\log \frac{1}{1-r}} > \alpha .$$

Let Λ_g be a g-net associated to a function g with $\int_0 \frac{dt}{t^{1/2}g(t)} < \infty$, for instance $g(t) = t^{1/4}$. According to Theorem 4.7, Λ_g is not a Nevanlinna sampling set. On the other hand, for any given $\alpha > 0$, we can extract a maximal separated sequence Λ' with a separation constant small enough so that $D_-(\Lambda') > \alpha$, hence Λ_g is $A^{-\alpha}$-sampling for all $\alpha > 0$.

Also, given $\alpha > 0$, consider a uniformly dense sequence Λ with $D_-(\Lambda) < \alpha$ and take φ satisfying $\lim_{t\to 0}\varphi(t) = 0$ and (9). Then according to Theorem 4.10, $\Lambda(\varphi)$ is Nevanlinna sampling; but it is not $A^{-\alpha}$-sampling, since $D_-(\Lambda') < \alpha$ for any separated $\Lambda' \subset \Lambda(\varphi)$.

Alternatively, take a set Λ as in Theorem 4.8, sampling for $\mathcal{N}$, with $\lim\limits_{n\to\infty} \frac{1-r_{n+1}}{1-r_n} = 0$. Then $D_-(\Lambda) = 0$, so it cannot be $A^{-\alpha}$-sampling for any $\alpha > 0$.

5 Finitely Generated Ideals

Here we report mostly on [20].

In the study of the uniform algebra $\mathcal{H}^\infty$ it is important to know its maximal ideals, and thus to know whether an ideal is or is not the whole of the algebra. When considering the ideal generated by functions $f_1, \dots, f_n \in \mathcal{H}^\infty$, an easy necessary condition for it to be the whole of the algebra is $\inf_{z\in\mathbb{D}} \left(|f_1(z)| + \cdots + |f_n(z)|\right) > 0$. This turns out to be sufficient: this is the content of the Corona Theorem.

Theorem 5.1 (Carleson 1962 ; see [11] or [37])). *If $f_1, \dots, f_n \in \mathcal{H}^\infty$ and*

$$\inf_{z\in\mathbb{D}} \left(|f_1(z)| + \cdots + |f_n(z)|\right) > 0,$$

then there exist $g_1, \dots, g_n \in \mathcal{H}^\infty$ such that $f_1 g_1 + \cdots + f_n g_n \equiv 1$.

More generally, we denote by $I_{\mathcal{H}^\infty}(f_1, \dots, f_n)$ the ideal generated by the functions $f_1, \dots, f_n$ in $\mathcal{H}^\infty$. The general structure of these ideals is not well understood (see the references in [20]).

In certain situations the ideals can be characterized by growth conditions. In this context, the following ideals have been studied:

$$J_{\mathcal{H}^\infty}(f_1, \dots, f_n) = \left\{ f \in \mathcal{H}^\infty : \exists c = c(f) > 0 ,\ |f(z)| \le c \sum_{i=1}^n |f_i(z)| ,\ z \in \mathbb{D} \right\}.$$

It is obvious that $I_{\mathcal{H}^\infty}(f_1, \dots, f_n) \subset J_{\mathcal{H}^\infty}(f_1, \dots, f_n)$. This leads us to the type of results we are interested in here. Tolokonnikov [46] proved that the following conditions are equivalent:
(a) $J_{\mathcal{H}^\infty}(f_1, \dots, f_n)$ contains an interpolating Blaschke product for $\mathcal{H}^\infty$,
(b) $I_{\mathcal{H}^\infty}(f_1, \dots, f_n)$ contains an interpolating Blaschke product for $\mathcal{H}^\infty$,
(c) $\inf\limits_{z\in\mathbb{D}} \sum_{i=1}^n (|f_i(z)| + (1-|z|^2)|f_i'(z)|) > 0$.

As it turns out, in the special situation of two generators with no common zeros these conditions are equivalent to $I_{\mathcal{H}^\infty}(f_1, f_2) = J_{\mathcal{H}^\infty}(f_1, f_2)$. In the case of two generators f_1 and f_2 with common zeros, we have $I_{\mathcal{H}^\infty}(f_1, f_2) = J_{\mathcal{H}^\infty}(f_1, f_2)$ if and only if $I_{\mathcal{H}^\infty}(f_1, f_2)$ contains a function of the form $BB_{1,2}$ where B is a $\mathcal{H}^\infty$-interpolating Blaschke product and $B_{1,2}$ is the Blaschke product formed with the common zeros of f_1 and f_2 (see [18]).

For the Nevanlinna class R. Mortini observed that a well-known result of T. Wolff implies the following corona theorem (see [31] or [26]).

Theorem 5.2. *(R. Mortini, [31]) Let $I(f_1,\dots,f_n)$ denote the ideal generated in $\mathcal{N}$ by a given family of functions $f_1,\dots,f_m \in \mathcal{N}$. Then $I(f_1,\dots,f_n) = \mathcal{N}$ if and only if there exists $H \in \mathrm{Har}_+(\mathbb{D})$ such that*

$$\sum_{i=1}^{n} |f_i(z)| \geq e^{-H(z)}, \quad z \in \mathbb{D}.$$

The necessary condition is clear since the Bézout equation implies that $1 \leq \max_j(|f_j|)\sum_j |g_j|$. As observed in [31], the sufficient condition is a corollary of a theorem of Wolff [11, Theorem 8.2.3, p. 239].

Theorem 5.3. *If $g, g_1,\dots,g_N \in H^\infty(\mathbb{D})$ and for all $z \in \mathbb{D}$, $|g_1(z)| + \cdots + |g_N(z)| > |g(z)|$, then there exist $f_1,\dots,f_N \in H^\infty(\mathbb{D})$ such that $f_1g_1 + \cdots + f_Ng_N = g^3$.*

It will be enough to take a holomorphic g such that $|g| = e^{-H}$, and conclude using the fact that g^3 is invertible in $\mathcal{N}$.

The ideal corresponding to $J_{\mathcal{H}^\infty}$ in $\mathcal{N}$ is defined as:

$$J(f_1,\dots,f_n) = \left\{ f \in N : \exists H = H(f) \in \mathrm{Har}_+(\mathbb{D}),\ |f(z)| \leq e^{H(z)} \sum_{i=1}^{n} |f_i(z)|,\ z \in \mathbb{D} \right\}.$$

It is clear that $I(f_1,\dots,f_n) \subset J(f_1,\dots,f_n)$. Notice also that, by the previous corona theorem, in the case when $J(f_1,\dots,f_n) = \mathcal{N}$, then $I(f_1,\dots,f_n) = \mathcal{N}$.

The analogues for $\mathcal{N}$ of the results mentioned above in the context of $\mathcal{H}^\infty$ read as follows. We see, again, that the general principle of substituting $\mathcal{H}^\infty$ by $\mathcal{N}$ and boundedness by a control by a positive harmonic function remains valid.

Theorem 5.4. *Let $f_1,\dots,f_n$ be functions in $\mathcal{N}$. Then the following conditions are equivalent:*
(a) $I(f_1,\dots,f_n)$ contains a Nevanlinna interpolating Blaschke product,
(b) $J(f_1,\dots,f_n)$ contains a Nevanlinna interpolating Blaschke product,
(c) There exists a function $H \in \mathrm{Har}_+(\mathbb{D})$ such that

$$\sum_{i=1}^{n} (|f_i(z)| + (1-|z|^2)|f_i'(z)|) \geq e^{-H(z)},\ z \in \mathbb{D}.$$

In case $n = 2$, if f_1 and f_2 have no common zeros, the above conditions are equivalent to
(d) $I(f_1,f_2) = J(f_1,f_2)$.

As in $\mathcal{H}^\infty$, each of the conditions (a)-(c) implies $I(f_1,\dots,f_n) = J(f_1,\dots,f_n)$. On the other hand, an example similar to that given for the $\mathcal{H}^\infty$ case shows that when $n \geq 3$, the converse fails. This example is given by two Nevanlinna interpolating Blaschke products B_1, B_2 with respective zero-sets Z_1, Z_2. In this situation

$$I(B_1^2, B_2^2, B_1B_2) = J(B_1^2, B_2^2, B_1B_2).$$

But if Z_1 and Z_2 are too close, then condition (c) in Theorem 5.4 cannot hold.

Also, like in the $\mathcal{H}^\infty$-situation, if the two generators f_1 and f_2 have common zeros, then $I(f_1,f_2) = J(f_1,f_2)$ if and only if $I(f_1,f_2)$ contains a function of the form $BB_{1,2}$ where B is a Nevanlinna interpolating Blaschke product and $B_{1,2}$ is the Blaschke product formed with the common zeros of f_1 and f_2.

The proof Theorem 5.4 uses some of the ideas from the $\mathcal{H}^\infty$ case, but also some specific properties of the Nevanlinna class, in particular the description of Nevanlinna interpolating sequences in terms of harmonic measure seen in Theorem 3.10(d).

5.1 Stable rank and two open problems

The ideal generated by an n-tuple of functions $f_1,\dots,f_n \in \mathcal{H}^\infty$ satisfying the conclusion of the Corona Theorem 5.1 is the whole of $\mathcal{H}^\infty$. In general, in an algebra A (or even a unitary ring) an n-tuple which is not contained in any ideal smaller than A is called *unimodular*. An unimodular $n+1$-tuple $f_1,\dots,f_n,f_{n+1}\in A$ such that there exist $h_1,\dots,h_n\in A$ with $f_1+h_1f_{n+1},\dots,f_n+h_nf_{n+1}$ unimodular is called *reducible*.

Definition 5.5. The *stable rank* of an algebra A is the smallest integer n such that any $n+1$-tuple in A is reducible.

It is known, for example, that the disc algebra (consisting of the holomorphic functions countinuous up to the boundary) has stable rank 1 (see [8], [24]). The following result by Treil goes in the same direction.

Theorem 5.6 (Treil, 1992, [48]). *The stable rank of $\mathcal{H}^\infty$ is equal to 1; more explicitly, given any $f_1,f_2\in\mathcal{H}^\infty$ such that* $\inf_{z\in\mathbb{D}}\big(|f_1(z)|+|f_2(z)|\big)>0$, *there exists $h\in\mathcal{H}^\infty$ such that f_1+hf_2 is invertible in $\mathcal{H}^\infty$.*

As far as we know, the stable rank for the Nevanlinna class is unknown, but the following result shows that it is at least two.

Proposition 5.7. *([20, Proposition 5.1]) The stable rank of the Nevanlinna class is at least 2.*

Open problem: What is the stable rank of $\mathcal{N}$?
Our guess is that it is either two or infinity. In support of the first option is that any triple $(f_1,f_2,f_3)\in\mathcal{N}^3$ such that for some i the zeros of f_i form a Nevanlinna interpolating sequence, can be reduced.

In order to see that the stable rank of $\mathcal{N}$ cannot be 1 we construct a pair of Blaschke products B_1, B_2 such that (B_1,B_2) is unimodular but for which there are no $\phi\in\mathcal{N}$ and no H harmonic in $\mathbb{D}$ such that

$$\log|B_1(z)+\phi(z)B_2(z)|=H(z),\quad z\in\mathbb{D}. \tag{11}$$

This prevents the existence of $e^f\in\mathcal{N}$, invertible in $\mathcal{N}$, with

$$B_1+\phi B_2=e^f.$$

Let $Z_1=(z_k)_k:=(1-2^{-k})_k$ and B_1 its associated Blaschke product. Take now a point $\mu_k\in(0,1)$ close enough to z_k so that

$$|B_1(\mu_k)|=\begin{cases} e^{-\frac{1}{1-|\mu_k|^2}} & \text{if } k \text{ even}\\ e^{-\frac{2}{1-|\mu_k|^2}} & \text{if } k \text{ odd.}\end{cases}$$

Set $Z_2=(\mu_k)_k$ and B_2 its Blaschke product.

Since both Z_1 and Z_2 are $\mathcal{H}^\infty$-interpolating and on the segment $(0,1)$, is not difficult to see that for some $c>0$

$$|B_1(z)|+|B_2(z)|\ge e^{-\operatorname{Re}\left(c\frac{1+z}{1-z}\right)},\quad z\in\mathbb{D},$$

hence (B_1,B_2) is unimodular in $\mathcal{N}$.

On the other hand, for any $\phi\in\mathcal{N}$,

$$(1-|\mu_k|^2)\log|B_1(\mu_k)+\phi(\mu_k)B_2(\mu_k)|=(1-|\mu_k|^2)\log|B_1(\mu_k)|=\begin{cases}-1 & \text{if } k \text{ even}\\ -2 & \text{if } k \text{ odd.}\end{cases}$$

This prevents (11) from holding for any $H=P[\nu]$ harmonic, with ν finite measure on $\partial\mathbb{D}$, since

$$\lim_{k\to\infty}(1-|\mu_k|^2)P[\nu](\mu_k)=\nu(\{1\}).$$

5.2 The f^2 problem

In the late seventies T. Wolff presented a problem on ideals of $\mathcal{H}^\infty$, known now as the f^2 problem, which was finally solved by S. Treil in [49]. The analogue for the Nevanlinna class is the following: let $f_1,\dots,f_n$ be functions in the Nevanlinna class, and let $f\in\mathcal{N}$ be such that there exists $H\in\operatorname{Har}_+(\mathbb{D})$ with

$$|f(z)|\le e^{H(z)}(|f_1(z)|+\cdots|f_n(z)|)^p,\quad z\in\mathbb{D}, \tag{12}$$

for some $p \geq 1$. Does it follow that $f \in I(f_1, \dots, f_n)$?

As in the $\mathcal{H}^\infty$ case, when $p > 2$, the $\overline{\partial}$ estimates by T. Wolff show that the answer is affirmative. When $p < 2$ the answer is in general negative, as the following example shows. Let N be an integer such that $(N+1)p > 2N$, $f = B_1^N B_2^N$, $f_1 = B_1^{N+1}$ and $f_2 = B_2^{N+1}$. Then (12) holds but $f \notin I(f_1, f_2)$ if (B_1, B_2) is not unimodular in $\mathcal{N}$.

Open problem: What happens in the case $p = 2$?

6 Weak embedding property and invertibility threshold

6.1 Weak embedding property in $\mathcal{H}^\infty$.

The quotient algebras of $\mathcal{H}^\infty$ arise naturally in several questions, notably problems of invertibility of operators. The paper [17] gives more detail about this, in the more general framework of uniform algebras; we shall concentrate on the question of invertibility within a quotient algebra of $\mathcal{H}^\infty$.

Recall that a function $I \in \mathcal{H}^\infty$ is called *inner* if $\lim_{r\to 1} |I(r\xi)| = 1$ for almost every $\xi \in \partial\mathbb{D}$. It is a consequence of Beurling's theorem about cyclic functions in the Hardy classes that any closed principal ideal of $\mathcal{H}^\infty$ is of the form $I\mathcal{H}^\infty$. Given two functions f, g in the same equivalence class $[f] \in \mathcal{H}^\infty/I\mathcal{H}^\infty$, they always coincide on $I^{-1}\{0\} =: Z$. If $[f]$ is invertible in $\mathcal{H}^\infty/I\mathcal{H}^\infty$, then $\inf_Z |f| > 0$. One may ask whether the converse holds. This is the $n = 1$ case of the following property (see [18]).

Definition 6.1. Let I be an inner function with zero set Z. We say that $\mathcal{H}^\infty/I\mathcal{H}^\infty$ has the Corona Property if: given $f_1, \dots, f_n \in \mathcal{H}^\infty$ such that

$$|f_1(z)| + \cdots + |f_n(z)| \geq \delta \qquad z \in Z,$$

for some $\delta > 0$, then there exist $g_1, \dots, g_n, h \in \mathcal{H}^\infty$ such that $f_1 g_1 + \cdots + f_n g_n = 1 + hI$.

Theorem 6.2 (Gorkin, Mortini, Nikolski, [18]). *Given an inner function I with $Z = I^{-1}\{0\}$, the following are equivalent:*

(a) Any $[f] \in \mathcal{H}^\infty/I\mathcal{H}^\infty$ such that $\inf_Z |f| > 0$ is invertible;

(b) $\mathcal{H}^\infty/I\mathcal{H}^\infty$ has the Corona Property;

(c) I satisfies the Weak Embedding Property *(WEP):*
For any $\epsilon > 0$, there exists $\eta > 0$ such that $|I(w)| \geq \eta$ when $\inf_{z\in Z} \rho(w, z) > \epsilon$.

Among inner functions with the same zero set, Blaschke products are the largest, and thus most likely to enjoy the WEP. Vasyunin's condition (7) (see Section 3.3) shows that $\mathcal{H}^\infty$-interpolating Blaschke products satisfy the weak embedding property with $\eta = \epsilon/C$.

Thus for any Carleson-Newman Blaschke product, i.e. a finite product of interpolating Blaschke products, we will have the WEP with $\eta = \epsilon^N/C$, and if it is satisfied for I with that value of η, then I is a Carleson-Newman Blaschke product [14], [2]. But there are other functions with the WEP [18]. No geometric characterization is known for the WEP, but further examples and results can be found in [2], [4].

6.2 Weak embedding property in the Nevanlinna Class.

In the algebra $\mathcal{N}$, any nonvanishing function is invertible and so any principal ideal is generated by some Blaschke product B with zero set Z. The elements of the quotient algebra $\mathcal{N}_B = \mathcal{N}/B\mathcal{N}$ are in one-to-one correspondence with their traces over Z.

Definition 6.3. Let B be a Blaschke product with zero set Z. We say that the Corona Property holds for $\mathcal{N}_B$ if for any positive integer n and any $f_1, \dots, f_n \in \mathcal{N}$ for which there exists $H \in \operatorname{Har}_+(\mathbb{D})$ such that

$$|f_1(z)| + \cdots + |f_n(z)| \geq e^{-H(z)} \qquad z \in Z, \tag{13}$$

there exist $g_1, \dots, g_n, h \in \mathcal{N}$ such that $f_1 g_1 + \cdots + f_n g_n = 1 + Bh$, that is, there exist $g_1, \dots, g_n \in \mathcal{N}$ such that

$$f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) = 1, \quad z \in Z.$$

Observe that condition (13) is necessary, and that the case $n = 1$ simply expresses invertibility in $\mathcal{N}_B$. Observe also that when Z is an interpolating sequence and $n = 1$, condition (13) means that $\log\left(1/|f(z_k)|\right) \leq H(z_k)$, and so the values $1/f(z_k)$ can be interpolated by $g \in \mathcal{N}$. So in the Nevanlinna case we immediately see that interpolating sequences are related to the Corona property for the quotient algebra.

As in the case of $\mathcal{H}^\infty$, we can see that the Corona property for $\mathcal{N}_B$ can be reduced to bounding $|B|$ from below, except that in this case both the distance condition and the bounds have to be expressed in terms of positive harmonic functions rather than constants.

Theorem 6.4. *[27] Let B be a Blaschke product and let Z be its zero sequence. The following conditions are equivalent:*
(a) The Corona Property holds for $\mathcal{N}_B$.
(b) For any $H_1 \in \mathrm{Har}_+(\mathbb{D})$, there exists $H_2 \in \mathrm{Har}_+(\mathbb{D})$ such that $|B(z)| \geq e^{-H_2(z)}$ for any $z \in \mathbb{D}$ such that $\rho(z, Z) \geq e^{-H_1(z)}$.

Blaschke products satisfying those equivalent properties are said to have the *Nevalinna WEP*. Once this is known, it becomes clear that a finite product of Nevanlinna-interpolating Blaschke products will have the Nevalinna WEP. Assume that $Z = \cup_{j=1}^N Z_j$, with Z_j Nevanlinna interpolating, and let $H_j \in \mathrm{Har}_+(\mathbb{D})$ be such that the estimate in Theorem 3.10(b) holds for Z_j and its associated Blaschke product B_j. Now, given $z \in \mathbb{D}$ with $\rho(z, Z) \geq e^{-H_0(z)}$,

$$|B(z)| = \prod_{j=1}^N |B_j(z)| \geq \prod_{j=1}^N e^{-H_j(z)} \rho(z, Z_j) \geq \exp\left(-\left(\sum_{j=1}^N H_j(z)\right) - NH_0(z)\right),$$

so Theorem 3.10(b) holds for the whole Z.

In contrast with the $\mathcal{H}^\infty$ case, the converse to this holds.

Theorem 6.5. *[27] The Corona Property holds for $\mathcal{N}_B$ if and only if B is a finite product of Nevanlinna interpolating Blaschke products.*

6.3 Some elements of proof.

The class of Nevanlinna WEP sequences is thus quite different, and much simpler, than that of WEP sequences. For instance, in the setting of $\mathcal{H}^\infty$, any WEP Blaschke product which is not a finite union of interpolating Blaschke products admits a subproduct which fails to be WEP [27, Lemma 1.3]. The following result follows immediately from Theorem 6.5, but is actually a step in its proof.

Lemma 6.6. *Any subproduct of a Nevanlinna WEP Blaschke product is also a Nevanlinna WEP Blaschke product.*

To give a flavor of how the Nevanlinna version of the WEP differs from the $\mathcal{H}^\infty$ version, we will sketch the proof of this. First we observe that points which are "far away" from the zero set, in the sense of distances measured with positive harmonic functions, can be found in a neighborhood of any point.

We will need an auxiliary function.

Definition 6.7. Given a Blaschke sequence $Z = (z_k)_k$, let H_Z denote the positive harmonic function defined by

$$H_Z(z) = \sum_k \int_{I_k} \frac{1 - |z|^2}{|\xi - z|^2} |d\,\xi|, \qquad z \in \mathbb{D}, \tag{14}$$

where $I_k := \{\xi \in \partial\mathbb{D} : |\xi - z_k/|z_k|| \le 1 - |z_k|\}$ denotes the Privalov shadow of z_k.

Lemma 6.8. *[27, Lemma 1.1] There exists $c_0 > 0$ such that for all $H \in \mathrm{Har}_+(\mathbb{D})$ with $H \ge c_0 H_Z$ the following property holds: for all $z \in \mathbb{D}$ there exists $\tilde{z}$ such that $\rho(\tilde{z}, z) \le e^{-H(z)}$ and $\rho(\tilde{z}, Z) \ge e^{-10H(z)}$.*

This is achieved through a simple counting argument. Note that $e^{-10H(z)} << e^{-H(z)}$, although the respective harmonic functions differ only by a constant.

Proof of Lemma 6.6. Assume $B = B_1 B_2$ is a Nevanlinna WEP Blaschke product. Denote by Z_i the zero sequences of B_i, $i = 1, 2$. Let $z \in \mathbb{D}$, and $H_1 \in \mathrm{Har}_+(\mathbb{D})$ be such that $\rho(z, Z_1) \ge e^{-H_1(z)}$. If needed, make H_1 larger so that Lemma 6.8 applies. If z verifies $\rho(z, \Lambda_2) \ge e^{-10H_1(z)}$ as well, since B is Nevanlinna WEP, there exists $H_2 \in \mathrm{Har}_+(\mathbb{D})$ such that

$$|B_1(z)| \ge |B(z)| \ge e^{-H_2(z)}.$$

If on the other hand $\rho(z, \Lambda_2) \le e^{-10H_1(z)}$, by Lemma 6.8 we can pick $\tilde{z} \in \mathbb{D}$ with $\rho(\tilde{z}, z) \le e^{-10H_1(z)}$ and $\rho(\tilde{z}, Z_2) \ge e^{-100H_1(z)}$. Hence $\rho(\tilde{z}, Z) \ge e^{-100H_1(z)}$, so there exists $H_3 \in \mathrm{Har}_+(\mathbb{D})$ such that

$$|B_1(\tilde{z})| \ge |B(\tilde{z})| \ge e^{-H_3(\tilde{z})}.$$

Since B_1 has no zeros in $D(z, e^{-5H_1(z)})$, Harnack's inequalities applied in that disc give $|B_1(z)| \ge e^{-2H_3(z)}$.

In the $\mathcal{H}^\infty$ framework, one easily sees from Carleson's Theorem 3.2 that any Blaschke product with separated zero set enjoying the WEP is actually an interpolating Blaschke product. The Nevanlinna analogue holds.

Lemma 6.9. *A Nevanlinna WEP Blaschke product with weakly separated zero set (as in Definition 3.5) is a Nevanlinna interpolating Blaschke product.*

Proof. Let $H_1 \in \mathrm{Har}_+(\mathbb{D})$ giving the separation. Since B is Nevanlinna WEP there exists $H_2 \in \mathrm{Har}_+(\mathbb{D})$ such that

$$|B(z)| \ge e^{-H_2(z)} \quad \text{for} \quad z \in \cup_{\lambda \in Z} \partial D(\lambda, e^{-H_1(\lambda)})$$

In particular,

$$|B_\lambda(z)| \ge e^{-H_2(z)} \quad \text{for} \quad z \in \partial D(\lambda, e^{-H_1(\lambda)}).$$

Since B_λ has no zeros in $D(\lambda, e^{-H_1(\lambda)})$ we can apply the maximum principle to the harmonic function $\log |B_\lambda|^{-1}$ to deduce that

$$|B_\lambda(\lambda)| \ge \min_{z \in \partial D(\lambda, e^{-H_1(\lambda)})} e^{-H_2(z)} \ge e^{-2H_2(\lambda)}.$$

∎

The bulk of the proof of Theorem 6.5 is spent on showing that any Nevanlinna WEP Blaschke product must have a zero set which is a finite union of weakly separated subsequences [27, Theorem B and Section 3]. Then Lemma 6.6 shows that each of those subsequences generates a Nevanlinna WEP Blaschke product, which then must be Nevanlinna interpolating by Lemma 6.9.

Our last result collects several different descriptions of products of exactly N Nevanlinna interpolating Blaschke products, which are then equivalent to the property about the trace space on the zero set given in Theorem 3.13. Analogous results for interpolating Blaschke products were proved by Kerr-Lawson [25], Gorkin and Mortini [14] and Borichev [2].

Given a Blaschke product B and $z \in \mathbb{D}$, let $|B(N)(z)|$ denote the value at the point $z \in \mathbb{D}$ of the modulus of the Blaschke product obtained from B after deleting the N zeros of B closest to z (in the pseudo-hyperbolic metric).

Theorem 6.10. *[27, Theorem C] Let B be a Blaschke product with zero set Z and let N be a positive integer. The following conditions are equivalent:*

(a) B is a product of N Nevanlinna interpolating Blaschke products.

(b) *There exists* $H_1 \in \mathrm{Har}_+(\mathbb{D})$ *such that*

$$|B(z)| \geq e^{-H_1(z)}\rho^N(z, Z), \qquad z \in \mathbb{D}.$$

(c) *There exists* $H_2 \in \mathrm{Har}_+(\mathbb{D})$ *such that* $|B(N)(z)| \geq e^{-H_2(z)}$, $z \in \mathbb{D}$.
(d) *There exists* $H_3 \in \mathrm{Har}_+(\mathbb{D})$ *such that*

$$D_N(B)(z) = \sum_{j=0}^{N}(1-|z|)^j|B^{(j)}(z)| \geq e^{-H_3(z)}, \qquad z \in \mathbb{D}.$$

The equivalence between (a), (b) and (d) for $N = 1$ is stated in Theorem 3.10 (see [20, Theorem 1.2]).

A consequence of these characterizations is a kind of stability of the property of being a finite product of Nevanlinna Interpolating Blaschke products.

Corollary 6.11. *Let* B *be a finite product of Nevanlinna interpolating Blaschke products. Then, there exists* $H_0 = H_0(B) \in \mathrm{Har}_+(\mathbb{D})$ *such that for any* $g \in H^\infty$ *with* $|g(z)| \leq e^{-H_0(z)}$, $z \in \mathbb{D}$, *the function* $B - g$ *factors as* $B - g = B_1 G$, *where* B_1 *is a finite product of Nevanlinna interpolating Blaschke products and* $G \in H^\infty$ *is such that* $1/G \in H^\infty$.

6.4 Invertibility threshold in the Nevanlinna Class.

Along with the definition of the WEP, the even more subtle question of the invertibility threshold was raised in [18]: is there $c \in [0, 1)$ such that given any $[f] \in \mathcal{H}^\infty/I\mathcal{H}^\infty$ with $\|[f]\| := \inf\{\|g\| : g \in [f]\} = 1$ and $\inf_{I^{-1}\{0\}} |f| > c$, then f is invertible in $\mathcal{H}^\infty/I\mathcal{H}^\infty$? The case of the WEP corresponds to $c = 0$. It was shown in [39] that for any value of $c \in (0, 1)$, there is some Blaschke product with zero set Z so that under the condition $\inf_Z |f| > c$, then f is invertible in $\mathcal{H}^\infty/B\mathcal{H}^\infty$, but no $c' < c$ will work.

In the case of the Nevanlinna Class, $|f|$ being bounded from below by a small constant will be replaced by $|f|$ being bounded from below by e^{-H}, where H is a large positive harmonic function. To study the question, we need a quantitative version of Theorem 6.4. Since any $f \in \mathcal{N}$ can be written $f = g_1/g_2$ with $g_1, g_2 \in \mathcal{H}^\infty$, invertibility of f is equivalent to that of g_1. We thus state the result for bounded functions, which is a way of normalizing the functions we are considering.

Observe that a restriction arises which does not occur in the $\mathcal{H}^\infty$ case: the two properties below are equivalent only when considering positive harmonic functions which are larger than H_Z, the function in Definition 6.7.

Theorem 6.12. *[36, Theorem 1] Let* B *be a Blaschke product with zero set* $Z = (z_k)_k$.
(a) *There exists a universal constant* $C > 0$ *such that the following statement holds. Let* $H \in \mathrm{Har}_+(\mathbb{D})$ *and assume that the function* $-\log|B|$ *has a harmonic majorant on the set* $\{z \in \mathbb{D} : \rho(z, Z) \geq e^{-H(z)}\}$. *Then for any* $f \in \mathcal{H}^\infty$, $\|f\|_\infty \leq 1$ *such that*

$$|f(z_k)| > e^{-CH(z_k)}, \quad k = 1, 2, \ldots, \tag{15}$$

there exist $g, h \in \mathcal{N}$ *such that* $fg = 1 + Bh$.
(b) *There exist universal constants* $C_0 > 0$ *and* $C > 0$ *such that the following statement holds. Let* $H \in \mathrm{Har}_+(\mathbb{D})$ *with* $H \geq C_0 H_Z$. *Assume that for any* $f \in H^\infty$, $\|f\|_\infty \leq 1$ *such that* (15) *holds, there exist* $g, h \in \mathcal{N}$ *such that* $fg = 1 + Bh$. *Then, the function* $-\log|B|$ *has a harmonic majorant on the set* $\{z \in \mathbb{D} : \rho(z, Z) \geq e^{-H(z)}\}$.

The result can be extended to Bézout equations with any number of generators [36, Corollary 2].

So we get a sufficient condition (a) for a solution to the invertibility problem in the quotient algebra $\mathcal{N}/B\mathcal{N}$, and a necessary condition when the harmonic function under consideration is large enough. This condition is in terms of the following class.

Definition 6.13. Given a Blaschke product B, let $\mathcal{H}(B)$ be the set of functions $H \in \mathrm{Har}_+(\mathbb{D})$ such that $-\log|B|$ has a harmonic majorant on the set $\{z \in \mathbb{D} : \rho(z, Z) \geq e^{-H(z)}\}$.

It is easy to see that constant functions are always in $\mathcal{H}(B)$ (see Proposition 4.1 of [22]), and that if $H_1 \in \mathcal{H}(B)$ and $H_2 \in \mathrm{Har}_+(\mathbb{D})$, $H_2 \leq H_1$, then $H_2 \in \mathcal{H}(B)$. In this language Theorem 6.5 reads as follows: $\mathcal{H}(B) = \mathrm{Har}_+(\mathbb{D})$ if and only if Z is a finite union of interpolating sequences for $\mathcal{N}$.

For any Blaschke product B, $\mathcal{H}(B)$ does contain unbounded functions [36, Theorem 2]. We have the following analogue of the Nikolski-Vasyunin result, showing that given any positive harmonic function H, there exist Blaschke products so that H is in some sense on the boundary of the class $\mathcal{H}(B)$.

Theorem 6.14. *[36, Theorem 3]*
(a) Let $H_1, H_2 \in \mathrm{Har}_+(\mathbb{D})$ such that

$$\limsup_{|z|\to 1} \frac{H_1(z)}{H_2(z)} = +\infty.$$

Then there exists a Blaschke product B with zero set Z such that $H_2 \in \mathcal{H}(B)$ but $H_1 \notin \mathcal{H}(B)$.
(b) For any $\eta_0 > 0$, and any unbounded positive harmonic function H, there exists a Blaschke product B such that $H \in \mathcal{H}(B)$ but $(1 + \eta_0)H \notin \mathcal{H}(B)$.

The Blaschke products which we exhibit in Theorem 6.14 have a distribution of zeroes tailored to the variation of the positive harmonic functions involved. When the harmonic functions we consider are “too big” compared to the density of zeroes of B, the delicate phenomenon involved in Theorem 6.14(b) disappears, and multiplying by a constant does not affect membership in $\mathcal{H}(B)$.

Recall that the disc is partitioned in the Whitney squares $S_{n,k}$ defined in (8).

Theorem 6.15. *[36, Theorem 4] Let B be a Blaschke product with zero set Z. Let $H \in \mathrm{Har}_+(\mathbb{D})$ such that*

$$\inf\{e^{H(z)} : z \in Q\} \geq \#(Z \cap Q), \tag{16}$$

for any dyadic Whitney square Q. Assume that $H \in \mathcal{H}(B)$; then, for any $C > 0$, $CH \in \mathcal{H}(B)$.

Of course, the result is non-trivial for $C > 1$ only. Note that the harmonic functions that verify the hypothesis of Theorem 6.12(b) will verify (16).

It would be nice, given any Blaschke product, to determine the class $\mathcal{H}(B)$. Although this seems out of reach in general, a sufficient condition is known. Given a dyadic Whitney square Q, let $z(Q)$ denote its center.

Theorem 6.16. *Let B be a Blaschke product with zero set Z. Let $\mathcal{A}$ be the collection of dyadic Whitney squares Q such that $\#(Z \cap Q) > 0$. Let $H \in \mathrm{Har}_+(\mathbb{D})$. Assume that the map $z_Q \mapsto \#(Z \cap Q) \cdot H(z(Q))$ for any $Q \in \mathcal{A}$ (and 0 elsewhere) admits a harmonic majorant. Then $H \in \mathcal{H}(B)$.*

Notice that we impose no direct restriction on the values of H in the dyadic squares where no zero of B is present. Moreover, there is a class of Blaschke products for which this sufficient condition is also necessary [36, Section 2].

7 The Smirnov class

The usual definition of the Smirnov class is

$$\mathcal{N}_+ = \Big\{f \in \mathcal{N} : \lim_{r\nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} \log_+ |f(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log_+ |f^*(re^{i\theta})|\, d\theta\Big\}.$$

However, as everywhere else, we take the equivalent definition in terms of harmonic majorants.

Definition 7.1. A harmonic function h is called *quasi-bounded* if it is the Poisson integral of a measure absolutely continuous with respect to the Lebesgue measure on the circle, i. e. $h = P[w]$ for some $w \in L^1(\partial\mathbb{D})$. Let $QB(\mathbb{D})$ denote the set of quasi-bounded harmonic functions, and $QB_+(\mathbb{D})$ the cone of those which are nonnegative.

Definition 7.2. The *Smirnov class* $\mathcal{N}_+$ is the set of $f \in \mathcal{N}$ such that $\log|f|$ has a quasi bounded harmonic majorant. Equivalently, it is the set of $f \in \mathcal{N}$ such that $\log|f(z)| \leq \mathcal{P}(\log|f^*|)(z)$.

In terms of the factorization (4), Smirnov functions are the Nevanlinna functions with singular factor $S_2 \equiv 1$. Then, any $f \in \mathcal{N}_+$ has a factorization of the form

$$f = \alpha \frac{BS_1\mathcal{O}_1}{\mathcal{O}_2},$$

where $\mathcal{O}_1, \mathcal{O}_2$ are outer with $\|\mathcal{O}_1\|_\infty, \|\mathcal{O}_2\|_\infty \leq 1$, S_1 is singular inner, B is a Blaschke product and $|\alpha| = 1$.

In particular, unlike Nevanlinna functions, the inverse of a nonvanishing Smirnov function is not necessarily a Smirnov function. Actually, the class $\mathcal{N}^+$ is the subalgebra of $\mathcal{N}$ where the invertible functions are exactly the outer functions.

Other aspects of $\mathcal{N}_+$ are better than in $\mathcal{N}$. For instance, $(\mathcal{N}_+, d)$ is a topological vector space (see [44]). It can also be represented as the union of certain weighted Hardy spaces $\mathcal{H}^2(w)$, and hence given the inductive limit topology [30].

According to the above definition, we expect the results explained in the previous sections to hold as well for $\mathcal{N}_+$ as soon as $H \in \mathrm{Har}_+(\mathbb{D})$ is replaced by $H \in QB_+(\mathbb{D})$. This is indeed the case most of the times, but not always.

Let us briefly comment how the previous results are modified or adapted to $\mathcal{N}_+$.

7.1 Interpolation in $\mathcal{N}_+$

Theorem 3.4 holds as expected, replacing $\mathcal{N}$ by $\mathcal{N}_+$ and $h \in \mathrm{Har}_+(\mathbb{D})$ is replaced by $h \in QB_+(\mathbb{D})$. As for Corollary 3.8(a), condition (5) has to be replaced by

$$\lim_{k\to\infty} (1 - |z_k|) \log|B_k(z_k)|^{-1} = 0.$$

This is in accordance with the fact that for $h \in QB(\mathbb{D})$

$$NT \lim_{z\to e^{i\theta}} (1 - |z|)h(z) = 0, \quad \text{for all } \theta \in [0, 2\pi).$$

As for (b), when $\mu_Z = \sum_k (1 - |z_k|)\delta_{z_k}$ has bounded Poisson balayage, the same argument as before shows that Z is Smirnov interpolating if and only if (6) holds. In particular, in this situation Nevanlinna and Smirnov interpolating sequences are the same.

Theorem 3.10 and Corollary 3.11 work according to the general substitution explained above. Same thing with Theorem 3.13.

7.2 Sampling sets for $\mathcal{N}_+$

Sampling (or determination) sets for the Smirnov class can be defined as in Section 4, with the standard replacement of $\mathcal{N}$ by $\mathcal{N}_+$ and $\mathrm{Har}_+(\mathbb{D})$ by $QB_+(\mathbb{D})$. Then the analogue of Theorem 4.3 holds, but much more can be said.

By Theorem 4.2, for $\Lambda \subset \mathbb{D}$ to be sampling for $\mathcal{N}_+$ it is necessary that $|NT(\Lambda)| = 2\pi$. But Smirnov functions are determined by their boundary values, so this condition is also sufficient (see [29, Theorem 2.3]).

7.3 Finitely generated ideals

Mortini's solution to the corona problem given in Theorem 5.2 works also for $\mathcal{N}_+$, with the usual substitution explained above. The same thing happens with Theorem 5.4.

The situation changes for the stable rank. Notice that the example showing that the stable rank of $\mathcal{N}$ has to be at least two does not work for $\mathcal{N}_+$, because the function $\mathrm{Re}(\frac{1+z}{1-z})$ is not quasi-bounded (it is the inverse of the Poisson integral of a delta measure on 1). As far as we know the study of the stable rank of $\mathcal{N}_+$ is wide open.

7.4 Corona problem in quotient algebras

As mentioned before, the class $\mathcal{N}_+$ is an algebra where the invertible functions are exactly the outer functions. So any quotient of the Smirnov class by a principal ideal (which are the only closed ideals [41, Theorem 2]) can be represented by $\mathcal{N}_I^+ := \mathcal{N}_+/I\mathcal{N}_+$, where I is an inner function.

Carefully following the proof of Theorems 6.4 and 6.5 above, we see that the natural analogues hold for $\mathcal{N}_+$, with the usual replacement of $\mathrm{Har}_+(\mathbb{D})$ by $QB(\mathbb{D})$.

This explains the situation for quotients of type $\mathcal{N}_B^+$, with B Blaschke product, but not the general situation of quotients by inner functions. Here we say that an inner function I with zero set Z satisfies the Smirnov WEP if and only if for any $H_1 \in QB_+(\mathbb{D})$ there exists $H_2 \in QB_+(\mathbb{D})$ such that $|B(z)| \geq e^{-H_2(z)}$ for any $z \in \mathbb{D}$ such that $\rho(z, Z) \geq e^{-H_1(z)}$.

Notice that if it does and $I = BS$, where B is a Blaschke product and S a singular inner function, then since $|I| \leq |B|$ and $I^{-1}\{0\} = B^{-1}\{0\}$, it is clear that B must be Smirnov WEP as well. One may wonder which singular inner functions are admissible as divisors of Smirnov WEP functions, in analogy to the study begun in [3], [4]. It turns out that there aren't any besides the constants.

Theorem 7.3. *Let $I = BS$ be an inner function, where B is a Blaschke product and S is singular inner. Then I satisfies the Smirnov WEP if and only if $S = e^{i\theta}$ (a unimodular constant) and $B^{-1}\{0\}$ is a finite union of Smirnov interpolating sequences.*

As a consequence, if I is an inner function, the Corona Property holds for $\mathcal{N}_I^+$ if and only if I is a Blaschke product with its zero set being a finite union of Smirnov interpolating sequences.

To see this one has to check that the WEP condition forces $-\log|S|$ to have a quasi-bounded harmonic majorant, which is only possible when S is a constant.

References

[1] H. Bass, *K-theory and stable algebra,* Inst. Hautes Etudes Sci. Publ. Math. 22 (1964), 5–60.

[2] A. Borichev, *Generalized Carleson–Newman inner functions*, Math. Z. 275 (2013), 1197–1206.

[3] A. Borichev, A. Nicolau, P.J. Thomas, *Harmonic and superharmonic majorants on the disk*. Bull. London Math. Soc. 38 (2006), 250–260.

[4] A. Borichev, A. Nicolau, P.J. Thomas, *Weak embedding property, inner functions and entropy*. Math. Ann. 368 (2017), no. 3–4, 987–1015.

[5] J. Bourgain, *On finitely generated closed ideals in $H^\infty(\mathbb{D})$*. Ann. Inst. Fourier (35), n° 4 (1985), 163–174.

[6] L. Brown, A. Shields, K. Zeller, *On absolutely convergent exponential sums*, Trans. Amer. Math. Soc. 96 (1960), 162–183.

[7] J. Bruna, A. Nicolau, K. Øyma, *A note on interpolation in the Hardy spaces of the unit disc*. Proc. Amer. Math. Soc. 124 (1996), no. 4, 1197–1204.

[8] G. Corach, D. Suárez, *On the stable range of uniform algebras and H^∞*. Proc. Amer. Math. Soc. 98 (1986), no. 4, 607–610.

[9] U. Daepp, P. Gorkin, R. Mortini, *Finitely generated radical ideals in H^∞*. Proc. Amer. Math. Soc. 116 (1992), no. 2, 483–488.

[10] S. J. Gardiner, *Sets of determination for the Nevanlinna class*, Bull. London Math. Soc. 42 (2010), 1115–1120.

[11] J. B. Garnett, *Bounded analytic functions*. Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007. xiv+459 pp.

[12] J. B. Garnett, F. W. Gehring, P. W. Jones, *Conformally invariant length sums*. Indiana Univ. Math. J. 32 (1983), no. 6, 809–829.
[13] P. Gorkin, K. Izuchi, R. Mortini, *Higher order hulls in H^∞ II*. J. Funct. Anal. 177 (2000), no. 1, 107–129.
[14] P. Gorkin, R. Mortini, *Two new characterizations of Carleson-Newman Blaschke products*. Israel J. Math. 177 (2010), 267–284.
[15] P. Gorkin, R. Mortini, *Hulls of closed prime ideals in H^∞*. Illinois J. Math. 46 (2002), no. 2, 519–532.
[16] P. Gorkin, R. Mortini, *Alling's conjecture on closed prime ideals in H^∞*. J. Funct. Anal. 148 (1997), no. 1, 185–190.
[17] P. Gorkin, R. Mortini, A. Nicolau, *The generalized corona theorem*. Math. Ann. 301 (1995), no. 1, 135–154.
[18] P. Gorkin, R. Mortini, N. Nikolski, *Norm controlled inversions and a corona theorem for H^∞-quotient algebras*. J. Funct. Anal. 255 (2008), 854–876.
[19] A. Hartmann, *Une approche de l'interpolation libre généralisée par la théorie des opérateurs et caractérisation des traces $H^p|\Lambda$. (French) [An approach to generalized free interpolation using operator theory and characterization of the traces $H^p|\Lambda$.]* J. Operator Theory 35 (1996), no. 2, 281–316.
[20] A. Hartmann, X. Massaneda, A. Nicolau, *Finitely generated ideals in the Nevanlinna class*. Israel J. Math. 231 (2019), no. 1, 139–179.
[21] A. Hartmann, X. Massaneda, A. Nicolau, *Traces of the Nevanlinna class on discrete sequences*. Complex Anal. and Oper. Theory 12 (2018), no. 8, 1945–1958.
[22] A. Hartmann, X. Massaneda, A. Nicolau, P. J. Thomas, *Interpolation in the Nevanlinna class and harmonic majorants*, J. Funct. Anal. 217 (2004), 1–37.
[23] W. K. Hayman and T. J. Lyons, *Bases for positive continuous functions*, J. London Math. Soc. 42 (1990), 292–308.
[24] P. W. Jones, D. Marshall, T. Wolff, *Stable rank of the disc algebra*. Proc. Amer. Math. Soc. 96 (1986), no. 4, 603–604.
[25] A. Kerr-Lawson, *Some lemmas on interpolating Blaschke products and a correction*. Canad. J. Math. 21 (1969), 531–534.
[26] R. Martin, *On the ideal structure of the Nevanlinna class*. Proc. Amer. Math. Soc. 114 (1992), no. 1, 135–143.
[27] X. Massaneda, A. Nicolau, P.J. Thomas, *The Corona property in Nevanlinna quotient algebras and interpolating sequences*. J. Funct. Anal. 276 (2019), no. 8, 2636–2661.
[28] X. Massaneda, J. Ortega-Cerdà, M. Ounaïes, *Traces of Hörmander algebras on discrete sequences*. Analysis and Mathematical Physics. Birkhäuser Verlag (2009), 397–408.
[29] X. Massaneda, P.J. Thomas, *Sampling sets for the Nevanlinna class*, Rev. Mat. Iberoamericana 24 (2008), no. 1, 353–385.
[30] John E. McCarthy, *Topologies on the Smirnov class*. J. Funct. Anal. 104 (1992), no. 1, 229–241.
[31] R. Mortini, *Zur Idealstruktur von Unterringen der Nevanlinna-Klasse N. (German) [On the ideal structure of subrings of Nevanlinna class N]*. Publ. du CUL, Séminaire de Math. de Luxembourg, Travaux mathématiques I (1989), 81–91.
[32] R. Mortini, *Generating sets for ideals of finite type in H^∞*. Bull. Sci. Math. 136 (2012), no. 6, 687–708.
[33] R. Mortini, *Finitely generated closed ideals in H^∞*. Arch. Math. (Basel) 45 (1985), no. 6, 546–548.
[34] A.G. Naftalevič, *On interpolation by functions of bounded characteristic* (Russian), Vilniaus Valst. Univ. Mokslų Darbai. Mat. Fiz. Chem. Mokslų Ser. 5 (1956), 5–27.
[35] A. Nicolau, J. Pau, P.J. Thomas, *Smallness sets for bounded holomorphic functions,* J. Anal. Math. 82 (2000), 119–148.
[36] A. Nicolau, P.J. Thomas, *Invertibility Threshold for Nevanlinna Quotient Algebras,* arXiv:1904.06908
[37] N.K. Nikolski, *Treatise on the shift operator. Spectral function theory. With an appendix by S.V. Hruščev [S.V. Khrushchev] and V.V. Peller.* Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 273. Springer-Verlag, Berlin, 1986.
[38] N.K. Nikolski [Nikol'skiĭ], *Operators, functions, and systems: an easy reading. Vol. 1, Hardy, Hankel, and Toeplitz; Vol.2, Model Operators and Systems*, Mathematical Surveys and Monographs, 92 and 93. American Mathematical Society, Providence, RI, 2002.
[39] N. Nikolski, V. Vasyunin, *Invertibility threshold for the H^∞-trace algebra and the efficient inversion of matrices,* Algebra i Analiz 23 (1) (2011), 87–110.
[40] J. Ortega-Cerdà, K. Seip, *Harmonic measure and Uniform densities*, Indiana Univ. Math. J. 53 (2004), no. 3, 905–923.
[41] J.V. Roberts, M. Stoll, *Prime and principal ideals in the algebra N^+*. Arch. Math. (Basel) 27 (1976), 387-393.
[42] M. Rosenblum, J. Rovnyak, *Hardy classes and operator theory*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.
[43] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. 113 (1993), 21–39.
[44] J.H. Shapiro, A.L. Shields, *Unusual topological properties of the Nevanlinna class*. Amer. J. Math. 97 (1975), no. 4, 915–936.
[45] P.J. Thomas, *Sampling sets for Hardy spaces of the disk*, Proc. Amer. Math.Soc. 126 (1998), 2927–2932.
[46] V.A. Tolokonnikov, *Interpolating Blaschke products and ideals of the algebra H^∞*. (Russian) Investigations on linear operators and the theory of functions, XII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 126 (1983), 196–201.
[47] V.A. Tolokonnikov, *Blaschke products with the Carleson-Newman condition, and ideals of the algebra H^∞*. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149 (1986), Issled. Linein. Teor. Funktsii. XV, 93–102, 188; translation in J. Soviet Math. 42 (1988), no. 2, 1603–1610.
[48] S. Treil, *The stable rank of the algebra H^∞ equals 1*, J. Funct. Anal. 109 (1992), no. 1, 130–154.
[49] S. Treil, *Estimates in the corona theorem and ideals of H^∞: a problem of T. Wolff*. Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. 87 (2002), 481–495.
[50] V.I. Vasyunin, *Unconditionally convergent spectral decompositions and interpolation problems (Russian)*. Spectral theory of functions and operators. Trudy Mat. Inst. Steklov. 130 (1978), 5–49, 223.

[51] V.I. Vasyunin, *Traces of bounded analytic functions on finite unions of Carleson sets (Russian)*. Investigations on linear operators and the theory of functions, XII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 126 (1983), 31–34.
[52] V.I. Vasyunin, *Characterization of finite unions of Carleson sets in terms of solvability of interpolation problems (Russian)*. Investigations on linear operators and the theory of functions, XIII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 135 (1984), 31–35.
[53] N. Yanagihara, *Interpolation theorems for the class N^+*, Illinois J. Math. 18 (1974), 427–435.