

VOLUME FLUCTUATIONS OF RANDOM ANALYTIC VARIETIES IN THE UNIT BALL

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ABSTRACT. Given a Gaussian analytic function f_L of intensity L in the unit ball of \mathbb{C}^n , $n \geq 2$, consider its (random) zero variety $Z(f_L)$. We study the variance of the $(n-1)$ -dimensional volume of $Z(f_L)$ inside a pseudo-hyperbolic ball of radius r . We first express this variance as an integral of a positive function in the unit disk. Then we study its asymptotic behaviour as $L \rightarrow \infty$ and as $r \rightarrow 1^-$. Both the results and the proofs generalise to the ball those given by Jeremiah Buckley for the unit disk.

1. DEFINITIONS AND STATEMENTS

Let \mathbb{B}_n denote the unit ball in \mathbb{C}^n and let ν denote the Lebesgue measure in \mathbb{C}^n normalised so that $\nu(\mathbb{B}_n) = 1$. Explicitly $\nu = \frac{n!}{\pi^n} dm = \beta^n$, where dm is the Lebesgue measure and $\beta = \frac{i}{2\pi} \partial \bar{\partial} |z|^2$ is the fundamental form of the Euclidean metric.

For $L > n$ consider the weighted Bergman space

$$B_L(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{n,L}^2 := c_{n,L} \int_{\mathbb{B}_n} |f(z)|^2 (1 - |z|^2)^L d\mu(z) < +\infty \right\},$$

where

$$(1) \quad d\mu(z) = \frac{d\nu(z)}{(1 - |z|^2)^{n+1}},$$

and $c_{n,L} = \frac{\Gamma(L)}{n! \Gamma(L-n)}$ is chosen so that $\|1\|_{n,L} = 1$.

Let

$$e_\alpha(z) = \left(\frac{\Gamma(L + |\alpha|)}{\alpha! \Gamma(L)} \right)^{1/2} z^\alpha$$

denote the normalisation of the monomial z^α in the norm $\|\cdot\|_{n,L}$, so that $\{e_\alpha\}_\alpha$ is an orthonormal basis of $B_L(\mathbb{B}_n)$. As usual, here we denote $z = (z_1, \dots, z_n)$ and use the multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

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The *hyperbolic Gaussian analytic function* (GAF) of *intensity* L is defined as

$$f_L(z) = \sum_{\alpha} a_{\alpha} \left(\frac{\Gamma(L + |\alpha|)}{\alpha! \Gamma(L)} \right)^{1/2} z^{\alpha} \quad z \in \mathbb{B}_n,$$

where a_{α} are i.i.d. complex Gaussians of mean 0 and variance 1 ($a_{\alpha} \sim N_{\mathbb{C}}(0, 1)$).

The sum defining f_L can be analytically continued to $L > 0$, which we assume henceforth.

The characteristics of the hyperbolic GAF are determined by its covariance kernel, which is given by (see [ST04, Section 1], [Sto94, p.17-18])

$$\begin{aligned} K_L(z, w) &= \mathbb{E}[f_L(z) \overline{f_L(w)}] = \sum_{\alpha} \frac{\Gamma(L + |\alpha|)}{\alpha! \Gamma(L)} z^{\alpha} \bar{w}^{\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(L + m)}{\Gamma(L)} \sum_{\alpha: |\alpha|=m} \frac{1}{\alpha!} z^{\alpha} \bar{w}^{\alpha} \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(L + m)}{m! \Gamma(L)} (z \cdot \bar{w})^m = \frac{1}{(1 - z \cdot \bar{w})^L}. \end{aligned}$$

A main feature of the hyperbolic GAF is that the distribution of the (random) integration current of its zero variety Z_{f_L} ,

$$[Z_{f_L}] = \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2,$$

is invariant under automorphisms of the unit ball (see [ST04, Section 1] or [BMP14]). Given $w \in \mathbb{B}_n$ there exists $\phi_w \in \text{Aut}(\mathbb{B}_n)$ such that $\phi_w(w) = 0$ and $\phi_w(0) = w$, and all automorphisms are essentially of this form: for all $\psi \in \text{Aut}(\mathbb{B}_n)$ there exist $w \in \mathbb{B}_n$ and \mathcal{U} in the unitary group such that $\psi = \mathcal{U} \phi_w$ (see [Rud08, 2.2.5]). Then the *pseudo-hyperbolic distance* ϱ in \mathbb{B}_n is defined as

$$\varrho(z, w) = |\phi_w(z)|, \quad z, w \in \mathbb{B}_n,$$

and the corresponding pseudo-hyperbolic balls as

$$E(w, r) = \{z \in \mathbb{B}_n : \varrho(z, w) < r\}, \quad r < 1.$$

The Edelman-Kostlan formula (see [HKPV09, Section 2.4] and [Sod00, Theorem 1]) gives the so-called *first intensity* of the GAF:

$$\mathbb{E}[Z_{f_L}] = \frac{i}{2\pi} \partial \bar{\partial} \log K_L(z, z) = L \omega(z),$$

where ω is the invariant form

$$\omega(z) = \frac{i}{2\pi} \partial \bar{\partial} \log \left(\frac{1}{1 - |z|^2} \right) = \frac{1}{(1 - |z|^2)^2} \sum_{j,k=1}^n [(1 - |z|^2) \delta_{j,k} + z_k \bar{z}_j] \frac{i}{2\pi} dz_j \wedge d\bar{z}_k.$$

In this paper we study the fluctuations of the $(n - 1)$ -dimensional volume of the random variety Z_{f_L} inside a pseudo-hyperbolic ball $E(z, r)$. By the invariance under automorphisms,

this is equivalent to measuring the $(n-1)$ -th volume of Z_{f_L} inside $B(0, r)$. This volume is given by the integral

$$I_{f_L}(r) = \int_{B(0,r) \cap Z_{f_L}} \omega_{n-1} = \int_{B(0,r)} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \omega_{n-1},$$

where for $p = 1, \dots, n$, $\omega_p = \omega^p/p!$.

Notice that now $\omega^n = d\mu$, so the Edelman-Kostlan formula yields trivially

$$\mathbb{E}[I_{f_L}(r)] = L \int_{B(0,r)} \frac{\omega^n}{(n-1)!} = \frac{L}{(n-1)!} \mu(B(0, r)) = \frac{L}{(n-1)!} \frac{r^{2n}}{(1-r^2)^n}.$$

Our main goal is to study the variance

$$\text{Var } I_{f_L}(r) = \mathbb{E}[(I_{f_L}(r) - \mathbb{E}(I_{f_L}(r)))^2],$$

and, particularly, to describe its asymptotic behaviour as $L \rightarrow \infty$ and as $r \rightarrow 1^-$.

The computations are much simpler if we consider the Euclidean volume instead of the invariant volume defined above. Let

$$E_{f_L}(r) = \int_{B(0,r) \cap Z_{f_L}} \beta_{n-1} = \int_{B(0,r)} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \beta_{n-1},$$

where, for $p = 1, \dots, n$, $\beta_p = \beta^p/p!$.

The key result is the following reduction of $\text{Var } E_{f_L}(r)$ to an integral of a positive function in the unit disk, together with the relation between $\text{Var } I_{f_L}(r)$ and $\text{Var } E_{f_L}(r)$.

Theorem 1. *Let $n \geq 2$. For $L > 0$ and $r \in (0, 1)$,*

$$\begin{aligned} \text{(a) } \text{Var } E_{f_L}(r) &= \frac{r^{4n} L^2 (1-r^2)^{2L-2}}{(n-1)!(n-2)!} \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{|1-r^2 w|^{2L} - (1-r^2)^{2L}} \frac{|1-w|^2}{|1-r^2 w|^2} \frac{dm(w)}{\pi}, \\ \text{(b) } \text{Var } I_{f_L}(r) &= \frac{\text{Var } E_{f_L}(r)}{(1-r^2)^{2n-2}}. \end{aligned}$$

From this we obtain the leading term in the asymptotics of $\text{Var } I_{f_L}(r)$ as $L \rightarrow \infty$.

Theorem 2. *Let $n \geq 2$ and fix $r \in (0, 1)$. Then, as $L \rightarrow \infty$,*

$$\text{Var } I_{f_L}(r) = \frac{1}{4\sqrt{\pi}} \frac{\zeta(n+1/2)}{(n-1)!} \frac{r^{2n-1}}{(1-r^2)^n} \frac{1}{L^{n-3/2}} [1 + o(1)].$$

Remarks. 1. Even though the proof of Theorem 2 is only valid for $n \geq 2$, the statement matches with the corresponding result for $n = 1$ (see [Buc13, Theorem 2(a)] and the references therein). Notice also that $\text{Var } E_{f_L}(r) = \text{Var } I_{f_L}(r)$ when $n = 1$.

2. As explained in [SZ06, Section 2.2], Theorem 2 can also be obtained with the methods used in the proof of the analogous result in the context of compact manifolds. Let p_N be a Gaussian holomorphic polynomial in $\mathbb{C}\mathbb{P}^n$ or, more generally, a section of a power L^N of a

positive Hermitian line bundle L over an n -dimensional Kähler manifold M . Given a domain $\mathcal{U} \subset M$ with sufficiently regular boundary, define

$$A_{p_N}(\mathcal{U}) = \int_{Z_{p_N} \cap \mathcal{U}} \frac{\omega^{n-1}}{(n-1)!},$$

where ω denotes the Kähler form of M . According to B. Shiffman and S. Zelditch [SZ08, Theorem 1.4.] (with $k = 1$)

$$\text{Var } A_{p_N}(\mathcal{U}) = \frac{1}{N^{n-3/2}} \left[\frac{\pi^{n-5/2}}{8} \zeta(n+1/2) \sigma(\partial\mathcal{U}) + O(N^{\epsilon-1/2}) \right],$$

where $\sigma(\partial\mathcal{U})$ denotes the $(2n-1)$ -volume of the boundary $\partial\mathcal{U}$. Notice that $\frac{r^{2n-1}}{(1-r^2)^n}$ is the $(2n-1)$ -volume (with respect to the invariant form) of $\partial B(0, r)$.

The proof of this result is based on a (pluri)bipotent expression of $\text{Var } I_{f_L}(r)$ (see the beginning of Section 2) and on good estimates of the covariance kernel, something we certainly have for the hyperbolic GAF in the ball.

3. Theorem 2 shows a strong form of “self-averaging” of the volume $I_{f_L}(r)$, in the sense that the fluctuations are of smaller order than the expected value (see remarks also in [SZ08]). More precisely, as $L \rightarrow \infty$

$$\frac{\text{Var } I_{f_L}(r)}{(\mathbb{E}[I_{f_L}(r)])^2} = \mathcal{O}\left(\frac{1}{L^{n+1/2}}\right).$$

Notice also that the rate of self-averaging increases with the dimension.

We also study the behaviour of $\text{Var } I_{f_L}(r)$ as $r \rightarrow 1^-$.

Theorem 3. *Let $n \geq 2$ and fix $L > 0$. Then, as $r \rightarrow 1^-$:*

(a) *If $L < n/2$ then,*

$$\text{Var } I_{f_L}(r) = C(L, n) \frac{1}{(1-r^2)^{2(n-L)}} [1 + o(1)]$$

where

$$C(L, n) = \frac{L^2}{\sqrt{\pi}} \frac{2^{n-1}}{4^L (n-1)!} \frac{\Gamma(\frac{n}{2} - L) \Gamma(\frac{n+1}{2} - L)}{(\Gamma(n-L))^2}.$$

(b) *If $L = n/2$ then,*

$$\text{Var } I_{f_L}(r) = C(n/2, n) \frac{1}{(1-r^2)^n} \log\left(\frac{1}{1-r^2}\right) [1 + o(1)]$$

where

$$C(n/2, n) = \frac{(n/2)^2}{(n-1)! (\Gamma(\frac{n}{2}))^2}.$$

(c) If $L > n/2$ then,

$$\text{Var } I_{f_L}(r) = C(L, n) \frac{1}{(1-r^2)^n} [1 + o(1)]$$

where

$$C(L, n) = \frac{L^2}{4\sqrt{\pi}(n-1)!} \sum_{k=1}^{\infty} \frac{\Gamma(Lk - \frac{n}{2})\Gamma(Lk - \frac{n-1}{2})}{(\Gamma(Lk+1))^2} \left(Lk + \frac{n(n-1)}{2}\right).$$

Remarks. (a) Notice that for L fixed and $r \rightarrow 1^-$ there is also a strong self-averaging of the volume $I_{f_L}(r)$, which also increases with the dimension:

$$\frac{\text{Var } I_{f_L}(r)}{(\mathbb{E}[I_{f_L}(r)])^2} = \begin{cases} \mathcal{O}((1-r^2)^{2L}) & L < n/2 \\ \mathcal{O}((1-r^2)^n \log(\frac{1}{1-r^2})) & L = n/2 \\ \mathcal{O}((1-r^2)^n) & L > n/2. \end{cases}$$

(b) As before, the proofs we give are valid only for $n \geq 2$ but the values $C(n, L)$ above match those obtained by J. Buckley in the disk [Buc13, Theorem 1] (since $1-r^2 = 2(1-r) + o(1-r)$).

(c) As in dimension 1, there is a change of regime at $L = n/2$, which we don't know how to explain.

(d) Using the asymptotics $\lim_{k \rightarrow \infty} \frac{\Gamma(k+a)}{\Gamma(k)k^a} = 1$ we see that as $L \rightarrow \infty$

$$C(L, n) = \frac{L^2}{4\sqrt{\pi}\Gamma(n)} \left(\sum_{k=1}^{\infty} \frac{1}{(Lk)^{n+1/2}} \right) (1 + o(1)) = \frac{1}{4\sqrt{\pi}\Gamma(n)} \frac{\zeta(n+1/2)}{L^{n-3/2}} (1 + o(1))$$

and in particular, by Theorem 2, the limits in r and L can be interchanged:

$$\begin{aligned} \lim_{L \rightarrow \infty} \lim_{r \rightarrow 1^-} L^{n-3/2} (1-r^2)^n \text{Var } I_{f_L}(r) &= \lim_{r \rightarrow 1^-} \lim_{L \rightarrow \infty} L^{n-3/2} (1-r^2)^n \text{Var } I_{f_L}(r) \\ &= \frac{\zeta(n+1/2)}{4\sqrt{\pi}\Gamma(n)}. \end{aligned}$$

The scheme of the proofs of Theorems 1, 2, and 3 is the same as in the one dimensional case (see [Buc13]), although some of the computations are considerably more involved.

The paper is structured as follows. Section 2 contains the proof of Theorem 1, which is a long and sometimes tricky computation. In Section 3 we prove Theorem 2, while Section 4 is devoted to the proof of Theorem 3.

2. PROOF OF THE THEOREM 1

(a) As in dimension $n = 1$, or as in the context of compact manifolds, the variance we want to study can be expressed through the bipotential. Denote $S(0, r) = \{\zeta \in \mathbb{C}^n : |\zeta| = r\}$ and

$S = S(0, 1)$. Then (see [Buc13, Section 2], [SZ08, Theorem 3.11])

$$\text{Var } E_{f_L}(r) = \int_{S(0,r)} \int_{S(0,r)} \sum_{j,k=1}^n \frac{\partial^2 \rho_L}{\partial \bar{z}_j \partial \bar{w}_k}(z, w) \frac{i}{2\pi} d\bar{z}_j \wedge \beta_{n-1}(z) \wedge \frac{i}{2\pi} d\bar{w}_k \wedge \beta_{n-1}(w) .$$

Here $\rho_L(z, w) = \text{Li}((\theta(z, w))^L)$, where $\text{Li}(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^2}$ and

$$\theta(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2} .$$

Notice that $\theta^{L/2}(z, w)$ coincides with the modulus of the normalised kernel of the GAF.

For the sake of completeness we sketch the proof of this identity given in [SZ06, Proposition 3.5]. Let $U = B(0, r)$. By the Edelman-Kostlan formula

$$E_{f_L}(r) - \mathbb{E}[E_{f_L}(r)] = \int_U \frac{i}{2\pi} \partial \bar{\partial} \log |\hat{f}_L|^2 \wedge \beta_{n-1} = \int_{\partial U} \frac{i}{2\pi} \bar{\partial} \log |\hat{f}_L|^2 \wedge \beta_{n-1} ,$$

where

$$\hat{f}_L(z) = \frac{f_L(z)}{\sqrt{K_L(z, z)}}$$

is the normalised GAF. Then

$$\begin{aligned} \text{Var}(E_{f_L}(r)) &= \mathbb{E} [(E_{f_L}(r) - \mathbb{E}[E_{f_L}(r)])^2] \\ &= \mathbb{E} \left[\int_{\partial U} \frac{i}{2\pi} \bar{\partial} \log |\hat{f}_L(z)|^2 \wedge \beta_{n-1}(z) \int_{\partial U} \frac{i}{2\pi} \bar{\partial} \log |\hat{f}_L(w)|^2 \wedge \beta_{n-1}(w) \right] \\ &= \int_{\partial U} \int_{\partial U} \mathbb{E} \left[\frac{i}{2\pi} \bar{\partial}_z \log |\hat{f}_L(z)|^2 \wedge \frac{i}{2\pi} \bar{\partial}_w \log |\hat{f}_L(w)|^2 \right] \wedge \beta_{n-1}(z) \wedge \beta_{n-1}(w) \\ &= \int_{\partial U} \int_{\partial U} \left(\frac{i}{2\pi} \right)^2 \bar{\partial}_z \bar{\partial}_w \mathbb{E} [\log |\hat{f}_L(z)|^2 \log |\hat{f}_L(w)|^2] \wedge \beta_{n-1}(z) \wedge \beta_{n-1}(w) . \end{aligned}$$

The result follows from the fact that (see for instance [HKPV09, Lemma 3.5.2])

$$\mathbb{E} [\log |\hat{f}_L(z)|^2 \log |\hat{f}_L(w)|^2] = \rho_L(z, w) \boxtimes .$$

To compute the integrals above we use that $\beta_n = \bigwedge_{j=1}^n \frac{i}{2\pi} dz_j \wedge d\bar{z}_j$. For $j = 1, \dots, n$ define the $(n-1, n)$ -forms

$$(2) \quad \gamma_j(z) = \frac{i}{2\pi} d\bar{z}_j \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} dz_k \wedge d\bar{z}_k = \frac{i}{2\pi} d\bar{z}_j \wedge \beta_{n-1}(z) .$$

Denoting $S = S(0, 1)$ and letting $z = r\xi$, $w = r\eta$, where $\xi, \eta \in S$ we have, from the expression above,

$$(3) \quad \text{Var } E_{f_L}(r) = \int_S \int_S \sum_{j,k=1}^n \frac{\partial^2 \rho_L}{\partial \bar{z}_j \partial \bar{w}_k}(r\xi, r\eta) \gamma_j(r\xi) \gamma_k(r\eta) .$$

Lemma 4. *Let $r \in (0, 1)$ and $\xi, \eta \in S$. Then*

$$\begin{aligned} \frac{\partial^2 \rho_L}{\partial \bar{z}_j \partial \bar{w}_k}(r\xi, r\eta) &= \frac{L^2(1-r^2)^{2L-2}r^2}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}} \times \\ &\quad \times \frac{[(1-r^2)\eta_j - \xi_j(1-r^2\bar{\xi} \cdot \eta)][(1-r^2)\xi_k - \eta_k(1-\xi \cdot \bar{\eta})]}{|1-r^2\bar{\xi} \cdot \eta|^2}. \end{aligned}$$

Proof. Since $\text{Li}'(x) = \frac{1}{x} \log\left(\frac{1}{1-x}\right)$, we have

$$\begin{aligned} \frac{\partial \rho_L}{\partial \bar{z}_j} &= \text{Li}'(\theta^L) L \theta^{L-1} \frac{\partial \theta}{\partial \bar{z}_j} = \frac{1}{\theta^L} \log\left(\frac{1}{1-\theta^L}\right) L \theta^{L-1} \frac{\partial \theta}{\partial \bar{z}_j} \\ &= \frac{L}{\theta} \log\left(\frac{1}{1-\theta^L}\right) \frac{\partial \theta}{\partial \bar{z}_j} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \rho_L}{\partial \bar{z}_j \partial \bar{w}_k} &= -\frac{L}{\theta^2} \frac{\partial \theta}{\partial \bar{w}_k} \log\left(\frac{1}{1-\theta^L}\right) \frac{\partial \theta}{\partial \bar{z}_j} + \frac{L}{\theta} \frac{L \theta^{L-1}}{1-\theta^L} \frac{\partial \theta}{\partial \bar{w}_k} \frac{\partial \theta}{\partial \bar{z}_j} + \frac{L}{\theta} \log\left(\frac{1}{1-\theta^L}\right) \frac{\partial^2 \theta}{\partial \bar{z}_j \partial \bar{w}_k} \\ &= \frac{L}{\theta} \log\left(\frac{1}{1-\theta^L}\right) \left[\frac{\partial^2 \theta}{\partial \bar{z}_j \partial \bar{w}_k} - \frac{1}{\theta} \frac{\partial \theta}{\partial \bar{z}_j} \frac{\partial \theta}{\partial \bar{w}_k} \right] + \frac{L^2}{\theta^2} \frac{\theta^L}{1-\theta^L} \frac{\partial \theta}{\partial \bar{z}_j} \frac{\partial \theta}{\partial \bar{w}_k}. \end{aligned}$$

Using the definition of θ given above,

$$\frac{\partial \theta}{\partial \bar{z}_j} = \frac{-z_j(1-|w|^2)}{|1-\bar{z} \cdot w|^2} + \frac{(1-|z|^2)(1-|w|^2)}{(1-z \cdot \bar{w})(1-\bar{z} \cdot w)^2} w_j = \frac{1-|w|^2}{|1-\bar{z} \cdot w|^2} \left(\frac{1-|z|^2}{1-\bar{z} \cdot w} w_j - z_j \right).$$

We deduce that

$$\frac{\partial^2 \theta}{\partial \bar{z}_j \partial \bar{w}_k} - \frac{1}{\theta} \frac{\partial \theta}{\partial \bar{z}_j} \frac{\partial \theta}{\partial \bar{w}_k} = 0$$

and therefore,

$$\begin{aligned} \frac{\partial^2 \rho_L}{\partial \bar{z}_j \partial \bar{w}_k} &= \frac{L^2}{\theta^2} \frac{\theta^L}{1-\theta^L} \frac{\partial \theta}{\partial \bar{z}_j} \frac{\partial \theta}{\partial \bar{w}_k} \\ &= \frac{L^2}{\theta^2} \frac{\theta^L}{1-\theta^L} \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z} \cdot w|^4} \left(\frac{1-|z|^2}{1-\bar{z} \cdot w} w_j - z_j \right) \left(\frac{1-|w|^2}{1-z \cdot \bar{w}} z_k - w_k \right). \end{aligned}$$

Substituting $z = r\xi$, $w = r\eta$, and using the identity

$$\frac{\theta^L(r\xi, r\eta)}{1-\theta^L(r\xi, r\eta)} = \frac{(1-r^2)^{2L}}{|1-r^2\bar{\xi} \cdot \eta|^{2L}} \frac{1}{1-\frac{(1-r^2)^{2L}}{|1-r^2\bar{\xi} \cdot \eta|^{2L}}} = \frac{(1-r^2)^{2L}}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}}$$

we get the result. ■

Plugging this into (3) we finally have

$$(4) \quad \text{Var } I_{f_L}(r) = L^2(1-r^2)^{2(L-1)} r^2 \mathcal{I}(L, r),$$

where

$$\mathcal{I}(L, r) = \int_S \int_S \frac{1}{|1 - r^2 \bar{\xi} \cdot \eta|^{2L} - (1 - r^2)^{2L}} \frac{\Omega(r\xi, r\eta)}{|1 - r^2 \bar{\xi} \cdot \eta|^2}$$

and $\Omega(r\xi, r\eta)$ is the $(n-1, n-1)$ -form (in ξ and η) given by

$$\Omega(r\xi, r\eta) = \sum_{j,k=1}^n [(1-r^2)\eta_j - \xi_j(1-r^2\bar{\xi} \cdot \eta)][(1-r^2)\xi_k - \eta_k(1-r^2\xi \cdot \bar{\eta})]\gamma_j(r\xi)\gamma_k(r\eta).$$

We operate

$$\begin{aligned} & [(1-r^2)\eta_j - \xi_j(1-r^2\bar{\xi} \cdot \eta)][(1-r^2)\xi_k - \eta_k(1-r^2\xi \cdot \bar{\eta})] = \\ & = (1-r^2)^2 \xi_k \eta_j - (1-r^2)(1-r^2\bar{\xi} \cdot \eta)\xi_j \xi_k - (1-r^2)(1-r^2\xi \cdot \bar{\eta})\eta_j \eta_k + |1-r^2\bar{\xi} \cdot \eta|^2 \xi_j \eta_k \end{aligned}$$

and split $\mathcal{I}(L, r) = \sum_{m=1}^4 \mathcal{I}_m(L, r)$, where

$$\begin{aligned} \mathcal{I}_1(L, r) &= \int_S \int_S \frac{(1-r^2)^2}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}} \frac{1}{|1-r^2\bar{\xi} \cdot \eta|^2} \left(\sum_{j=1}^n \eta_j \gamma_j(r\xi) \right) \left(\sum_{k=1}^n \xi_k \gamma_k(r\eta) \right) \\ \mathcal{I}_2(L, r) &= - \int_S \int_S \frac{1-r^2}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}} \frac{1}{1-r^2\xi \cdot \bar{\eta}} \left(\sum_{j=1}^n \xi_j \gamma_j(r\xi) \right) \left(\sum_{k=1}^n \xi_k \gamma_k(r\eta) \right) \\ \mathcal{I}_3(L, r) &= - \int_S \int_S \frac{1-r^2}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}} \frac{1}{1-r^2\bar{\xi} \cdot \eta} \left(\sum_{j=1}^n \eta_j \gamma_j(r\xi) \right) \left(\sum_{k=1}^n \eta_k \gamma_k(r\eta) \right) \\ \mathcal{I}_4(L, r) &= \int_S \int_S \frac{1}{|1-r^2\bar{\xi} \cdot \eta|^{2L} - (1-r^2)^{2L}} \left(\sum_{j=1}^n \xi_j \gamma_j(r\xi) \right) \left(\sum_{k=1}^n \eta_k \gamma_k(r\eta) \right). \end{aligned}$$

In order to compute these integrals we first fix η and consider a unitary transformation \mathcal{U} taking $e_1 = (1, 0, \dots, 0)$ to η . Denote $v^1 = \eta$ and $v^j = \mathcal{U}(e_j)$, $j > 1$, where $e_j = (0, \dots, \overset{j}{1}, \dots, 0)$, so that η, v^2, \dots, v^n is an orthonormal system.

Consider the change of variables $\xi = \mathcal{U}(\alpha) = \sum_{j=1}^n \alpha_j v^j$. Then

$$\xi \cdot \bar{\eta} = \mathcal{U}(\alpha) \cdot \overline{\mathcal{U}(e_1)} = \alpha \cdot \bar{e}_1 = \alpha_1.$$

Also

$$\xi_k = \sum_{m=1}^n \alpha_m v_k^m, \quad \bar{\xi}_k = \sum_{j=1}^n \bar{\alpha}_m \bar{v}_k^m,$$

and therefore

$$d\xi_k = \sum_{m=1}^n v_k^m d\alpha_m, \quad d\bar{\xi}_k = \sum_{m=1}^n \bar{v}_k^m d\bar{\alpha}_m.$$

Since $\beta(\xi) = \frac{i}{2\pi} \partial \bar{\partial} |\xi|^2$ is invariant by unitary transformations

$$(5) \quad \gamma_j(r\xi) = \frac{i}{2\pi} r d\bar{\xi}_j \wedge \beta_{n-1}(r\xi) = \frac{i}{2\pi} \left(r \sum_{m=1}^n \bar{v}_j^m d\bar{\alpha}_j \right) \wedge \beta_{n-1}(r\alpha) = \sum_{m=1}^n \bar{v}_j^m \gamma_m(r\alpha).$$

Now parametrise $\alpha = \mathcal{U}^{-1}(\xi) \in S$ with coordinates $w = (w_1, \dots, w_{n-1}) \in \mathbb{B}_{n-1}$, $\psi \in [0, 2\pi)$ in the following way

$$\begin{cases} \alpha_j = w_j & j = 1, \dots, n-1 \\ \alpha_n = \sqrt{1 - |w|^2} e^{i\psi}. \end{cases}$$

Let us write the forms $\gamma_j(r\alpha)$ in this parametrisation.

Lemma 5. *Let $d\beta_{n-1}(w) = \bigwedge_{k=1}^{n-1} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k$. Then*

$$\begin{cases} \gamma_j(\alpha) = \bar{w}_j \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) & j = 1, \dots, n-1, \\ \gamma_n(\alpha) = \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \end{cases}$$

Proof. Directly from the definition we have

$$\begin{aligned} d\alpha_n &= \sum_{l=1}^{n-1} \left(\frac{-\bar{w}_l e^{i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{i\psi}}{2\sqrt{1 - |w|^2}} d\bar{w}_l \right) + \sqrt{1 - |w|^2} i e^{i\psi} d\psi \\ d\bar{\alpha}_n &= \sum_{l=1}^{n-1} \left(\frac{-\bar{w}_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} d\bar{w}_l \right) - \sqrt{1 - |w|^2} i e^{-i\psi} d\psi. \end{aligned}$$

Assume first that $j < n$. Then $\bigwedge_{k \neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k$ contains the factor $\frac{i}{2\pi} d\alpha_n \wedge d\bar{\alpha}_n$, so

$$\begin{aligned} \bigwedge_{k \neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k &= \bigwedge_{k \neq j, n} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k \wedge \frac{i}{2\pi} d\alpha_n \wedge d\bar{\alpha}_n \\ &= \bigwedge_{k \neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k \wedge \frac{i}{2\pi} \left(\frac{-\bar{w}_j e^{i\psi}}{2\sqrt{1 - |w|^2}} dw_j + \frac{-w_j e^{i\psi}}{2\sqrt{1 - |w|^2}} d\bar{w}_j + \sqrt{1 - |w|^2} i e^{i\psi} d\psi \right) \wedge \\ &\quad \wedge \left(\frac{-\bar{w}_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} d\bar{w}_l - \sqrt{1 - |w|^2} i e^{-i\psi} d\psi \right) \\ &= \frac{1}{2\pi} d\psi \wedge (\bar{w}_j dw_j + w_j d\bar{w}_j) \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k. \end{aligned}$$

Therefore

$$\begin{aligned}
\gamma_j(\alpha) &= \frac{i}{2\pi} d\bar{\alpha}_j \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k \\
&= \frac{i}{2\pi} d\bar{w}_j \wedge \frac{1}{2\pi} d\psi \wedge (\bar{w}_j dw_j + w_j d\bar{w}_j) \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k \\
&= \bar{w}_j \frac{d\psi}{2\pi} \wedge \bigwedge_{k=1}^{n-1} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k
\end{aligned}$$

Assume now that $j = n$. Then

$$\gamma_n(\alpha) = \frac{i}{2\pi} d\bar{\alpha}_n \wedge \bigwedge_{k < n} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k = \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)$$

■

We finally use the parametrisation of Lemma 5 to compute the integrals $\mathcal{I}_j(L, r)$. We begin with \mathcal{I}_4 .

$\mathcal{I}_4(L, r)$. First make the change of variables $\xi = \mathcal{U}(\alpha)$. By invariance by unitary transformations,

$$\sum_{j=1}^n \xi_j \gamma_j(r\xi) = \frac{i}{2\pi} \bar{\partial}|\xi|^2 \wedge \beta_{n-1}(r\xi) = \sum_{j=1}^n \alpha_j \gamma_j(r\alpha),$$

and therefore

$$\begin{aligned}
\mathcal{I}_4(L, r) &= \int_S \int_S \frac{1}{|1 - r^2 \alpha_1|^{2L} - (1 - r^2)^{2L}} \left(\sum_{j=1}^n \alpha_j \gamma_j(r\alpha) \right) \left(\sum_{k=1}^n \eta_k \gamma_k(r\eta) \right) \\
&= A_n \int_S \frac{1}{|1 - r^2 \alpha_1|^{2L} - (1 - r^2)^{2L}} \left(\sum_{j=1}^n \alpha_j \gamma_j(r\alpha) \right),
\end{aligned}$$

where, by Stokes and the identity $\beta^n = d\nu$,

$$(6) \quad A_n := \int_S \sum_{k=1}^n \eta_k \gamma_k(r\eta) = r^{2n-1} \int_S \frac{i}{2\pi} \bar{\partial}|\eta|^2 \wedge \beta_{n-1}(\eta) = r^{2n-1} \int_{\mathbb{B}_n} \frac{\beta^n(z)}{(n-1)!} = \frac{r^{2n-1}}{(n-1)!}.$$

We compute the integral in α using the parametrisation of Lemma 5. Since $\gamma_j(r\alpha) = r^{2n-1} \gamma_j(\alpha)$ and

$$\begin{aligned}
\sum_{j=1}^n \alpha_j \gamma_j(\alpha) &= \sum_{j=1}^{n-1} w_j \bar{w}_j \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) + \sqrt{1 - |w|^2} e^{i\psi} \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\
(7) \quad &= (|w|^2 + 1 - |w|^2) \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) = \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w),
\end{aligned}$$

we have, after integrating $w_2, \dots, w_{n-1}, \psi$,

$$\begin{aligned} \mathcal{I}_4(L, r) &= A_n \int_{\mathbb{B}_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= A_n \int_{\mathbb{D}} \frac{r^{2n-1}}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} \frac{1}{(n-2)!} (1 - |w_1|^2)^{n-2} \frac{dm(w_1)}{\pi} \\ &= \frac{A_n r^{2n-1}}{(n-2)!} \int_{\mathbb{D}} \frac{(1 - |w_1|^2)^{n-2}}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} \frac{dm(w_1)}{\pi}. \end{aligned}$$

$\mathcal{I}_1(L, r)$. As in the previous case, first we change $\xi = \mathcal{U}(\alpha)$. By (5), and since $\eta = v^1$ and the system $\{v^l\}_{l=1}^n$ is orthonormal, the form to integrate is

$$\begin{aligned} \Gamma_1 &:= \sum_{j,k=1}^n \eta_j \xi_k \gamma_j(r\xi) \gamma_k(r\eta) = \sum_{j,k=1}^n \eta_j \left(\sum_{l=1}^n \alpha_l v_k^l \right) \left(\sum_{m=1}^n \bar{v}_j^m \gamma_m(r\alpha) \right) \gamma_k(r\eta) \\ &= \sum_{k,l,m=1}^n \left(\sum_{j=1}^n v_j^l \bar{v}_j^m \right) \alpha_l v_k^l \gamma_m(r\alpha) \gamma_k(r\eta) = \sum_{k=1}^n \left(\sum_{l=1}^n \alpha_l v_k^l \right) \gamma_1(r\alpha) \gamma_k(r\eta). \end{aligned}$$

Now we use the parametrisation given in Lemma 5. Then

$$\sum_{l=1}^n \alpha_l v_k^l = w_1 \eta_k + \sum_{l=2}^{n-1} w_l v_k^l + \sqrt{1 - |w|^2} e^{i\psi} v_k^n,$$

and therefore

$$\begin{aligned} \left(\sum_{l=1}^n \alpha_l v_k^l \right) \gamma_1(r\alpha) &= r^{2n-1} \left(w_1 \eta_k + \sum_{l=2}^{n-1} w_l v_k^l + \sqrt{1 - |w|^2} e^{i\psi} v_k^n \right) \bar{w}_1 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= r^{2n-1} \left(|w_1|^2 \eta_k + \sum_{l=2}^{n-1} \bar{w}_1 w_l v_k^l + \bar{w}_1 \sqrt{1 - |w|^2} e^{i\psi} v_k^n \right) \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w). \end{aligned}$$

This form will be integrated against a function which does not depend on $w_2, \dots, w_{n-1}, \psi$. The last term in the sum will vanish when integrating in ψ , while the second term will vanish when integrating in w_2, \dots, w_n . Thus, in terms of the integration we want to perform,

$$\begin{aligned} \Gamma_1 &= r^{2n-1} \sum_{k=1}^n |w_1|^2 \eta_k \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \gamma_k(\eta) + \text{vanishing terms} \\ &= r^{2n-1} \left(\sum_{k=1}^n \eta_k \gamma_k(\eta) \right) |w_1|^2 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) + \text{vanishing terms}. \end{aligned}$$

Letting A_n be the constant (6), and integrating $w_2, \dots, w_{n-1}, \psi$, we obtain

$$\begin{aligned} \mathcal{I}_1(L, r) &= A_n \int_{\mathbb{B}_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{(1-r^2)^2}{|1-r^2w_1|^2} |w_1|^2 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= \frac{r^{2n-1} A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1-|w_1|^2)^{n-2}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{(1-r^2)^2}{|1-r^2w_1|^2} |w_1|^2 \frac{dm(w_1)}{\pi} \end{aligned}$$

$\mathcal{I}_2(L, r)$. Once more, changing $\xi = \mathcal{U}(\alpha)$ and parametrising as in Lemma 5,

$$\begin{aligned} \Gamma_2 &:= \sum_{j,k=1}^n \xi_j \xi_k \gamma_j(r\xi) \gamma_k(r\eta) = \sum_{j,k=1}^n \left(\sum_{l=1}^n \alpha_l v_j^l \right) \left(\sum_{m=1}^n \alpha_m v_k^m \right) \left(\sum_{t=1}^n \bar{v}_j^t \gamma_t(r\alpha) \right) \gamma_k(r\eta) \\ &= \sum_{k,l,m,t=1}^n \alpha_l \alpha_m v_k^m \left(\sum_{j=1}^n v_j^l \bar{v}_j^t \right) \gamma_t(r\alpha) \gamma_k(r\eta) = \left(\sum_{l=1}^n \alpha_l \gamma_l(r\alpha) \right) \left(\sum_{k,m=1}^n \alpha_m v_k^m \gamma_k(r\eta) \right). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k,m=1}^n \alpha_m v_k^m \gamma_k(r\eta) &= \sum_{m=1}^{n-1} w_m \left(\sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \sqrt{1-|w|^2} e^{i\psi} \left(\sum_{k=1}^n v_k \gamma_k(r\eta) \right) \\ &= w_1 \sum_{k=1}^n \eta_k \gamma_k(r\eta) + \sum_{m=2}^{n-1} w_m \left(\sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \sqrt{1-|w|^2} e^{i\psi} \left(\sum_{k=1}^n v_k \gamma_k(r\eta) \right) \end{aligned}$$

and therefore, using (7),

$$\begin{aligned} \Gamma_2 &= r^{2n-1} \left[w_1 \sum_{k=1}^n \eta_k \gamma_k(r\eta) + \sum_{m=2}^{n-1} w_m \left(\sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \right. \\ &\quad \left. + \sqrt{1-|w|^2} e^{i\psi} \left(\sum_{k=1}^n v_k \gamma_k(r\eta) \right) \right] \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w). \end{aligned}$$

As before, the third term in the bracket will vanish when integrating in ψ , while the second term will vanish when integrating in w_2, \dots, w_{n-1} . Thus, finally,

$$\begin{aligned} \mathcal{I}_2(L, r) &= -A_n \int_{\mathbb{B}_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{1-r^2}{1-r^2w_1} w_1 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= -\frac{r^{2n-1} A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{1-r^2}{1-r^2w_1} w_1 \frac{dm(w_1)}{\pi}. \end{aligned}$$

$\mathcal{I}_3(L, r)$. Here the form to integrate is, in the terms of the parametrisation,

$$\begin{aligned} \Gamma_3 &= \sum_{j,k=1}^n \eta_j \eta_k \gamma_j(r\xi) \gamma_k(r\eta) = \sum_{k=1}^n \eta_k \sum_{j=1}^n v_j^1 \left(\sum_{m=1}^n \bar{v}_j^m \gamma_m(r\alpha) \right) \gamma_k(r\eta) \\ &= \sum_{k=1}^n \eta_k \left[\sum_{m=1}^n \left(\sum_{j=1}^n v_j^1 \bar{v}_j^m \right) \gamma_m(r\alpha) \right] \gamma_k(r\eta) = r^{2n-1} \left(\sum_{k=1}^n \eta_k \gamma_k(r\eta) \right) \bar{w}_1 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{I}_3(L, r) &= -A_n \int_{\mathbb{B}_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{1-r^2}{1-r^2\bar{w}_1} \bar{w}_1 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= -\frac{r^{2n-1}A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{1-r^2}{1-r^2\bar{w}_1} \bar{w}_1 \frac{dm(w_1)}{\pi}. \end{aligned}$$

Finally, adding up the expressions of $\mathcal{I}_j(L, r)$ and using that

$$1 + \frac{(1-r^2)^2}{|1-r^2w_1|^2} - \frac{1-r^2}{1-r^2w_1}w_1 - \frac{1-r^2}{1-r^2\bar{w}_1}\bar{w}_1 = \left| 1 - \frac{(1-r^2)w_1}{1-r^2w_1} \right|^2 = \frac{|1-w_1|^2}{|1-r^2w_1|^2},$$

we obtain

$$\mathcal{I}(L, r) = \sum_{m=1}^4 \mathcal{I}_m(L, r) = \frac{r^{2n-1}A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{|1-r^2w_1|^{2L} - (1-r^2)^{2L}} \frac{|1-w_1|^2}{|1-r^2w_1|^2} \frac{dm(w_1)}{\pi}.$$

Plugging this into (4) and writing down the value A_n (see (6)) we get Theorem 1(a).

(b) We just need to repeat the proof of (a) replacing β by ω . The corresponding bipotential formula for the variance is now

$$\text{Var } E_{f_L}(r) = \int_{S(0,r)} \int_{S(0,r)} \sum_{j,k=1}^n \frac{\partial^2 \rho_l}{\partial \bar{z}_j \partial \bar{w}_k}(z, w) \frac{i}{2\pi} d\bar{z}_j \wedge \omega_{n-1}(z) \wedge \frac{i}{2\pi} d\bar{w}_k \wedge \omega_{n-1}(w).$$

Replacing $\gamma_j(z)$ by

$$\Gamma_j(z) = \frac{i}{2\pi} d\bar{z}_j \wedge \omega_{n-1}(z)$$

in the calculations above we get the ‘‘invariant’’ version of (4):

$$\text{Var } I_{f_L}(r) = L^2(1-r^2)^{2(L-1)}r^2 \sum_{m=1}^4 \mathcal{I}_m^I(L, r),$$

where the integrals $\mathcal{I}_m^I(L, r)$ are obtained from $\mathcal{I}_m(L, r)$ replacing the γ_j by Γ_j .

As before, change now $\alpha = \mathcal{U}^{-1}(\xi) \in S$ and parametrise α with the same coordinates coordinates $w = (w_1, \dots, w_{n-1}) \in \mathbb{B}_{n-1}$, $\psi \in [0, 2\pi)$. We will be done as soon as we prove the following lemma.

Lemma 6. For $r < 1$ and $j = 1, \dots, n$

$$\Gamma_j(r\alpha) = \frac{\gamma_j(r\alpha)}{(1-r^2)^{n-1}}.$$

Proof. From the definition we have

$$\begin{aligned} \omega_{n-1}(z) = \frac{1}{(1-|z|^2)^n} & \left\{ \sum_{k=1}^n (1-|z_k|^2) \bigwedge_{j \neq k} \frac{i}{2\pi} dz_j \wedge d\bar{z}_j + \right. \\ & \left. + \sum_{\substack{j,k=1 \\ j \neq k}}^n \bar{z}_j z_k \frac{i}{2\pi} dz_j \wedge d\bar{z}_k \wedge \bigwedge_{l \neq j,k} \frac{i}{2\pi} dz_l \wedge d\bar{z}_l \right\} \end{aligned}$$

Therefore

$$\Gamma_m(z) = \frac{i}{2\pi} d\bar{z}_k \wedge \omega_{n-1}(z) = \frac{1}{(1-|z|^2)^n} \left[(1-|z_m|^2) \gamma_m(z) - \sum_{j \neq m} \bar{z}_m z_j \gamma_j(z) \right],$$

and in particular

$$\Gamma_m(r\alpha) = \frac{1}{(1-r^2)^n} \left[(1-r^2|\alpha_m|^2) \gamma_m(r\alpha) - r^2 \sum_{j \neq m} \bar{\alpha}_m \alpha_j \gamma_j(r\alpha) \right].$$

For $m < n$ we have then

$$\begin{aligned} \Gamma_m(r\alpha) &= \frac{1}{(1-r^2)^n} \left[(1-r^2|w_m|^2) \gamma_m(r\alpha) - r^2 \sum_{j \neq m} \bar{w}_m w_j \gamma_j(r\alpha) - r^2 \sqrt{1-|w|^2} e^{i\psi} \bar{w}_m \gamma_n(r\alpha) \right] \\ &= \frac{r^{2n-1}}{(1-r^2)^n} \left[(1-r^2|w_m|^2) \bar{w}_m \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) - r^2 \sum_{j \neq m} \bar{w}_m |w_j|^2 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) - \right. \\ & \quad \left. - r^2 (1-|w|^2) \bar{w}_m \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \right] \\ &= \frac{r^{2n-1}}{(1-r^2)^n} \left[1-r^2|w_m|^2 - r^2 \sum_{j \neq m} |w_j|^2 - r^2 (1-|w|^2) \right] \bar{w}_m \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \\ &= \frac{r^{2n-1}}{(1-r^2)^{n-1}} \bar{w}_m \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \end{aligned}$$

By Lemma 5 we have thus $\Gamma_m(r\alpha) = (1-r^2)^{1-n} \Gamma_m(r\alpha)$, $m < n$.

If $m = n$ we have similarly

$$\begin{aligned} \Gamma_m(r\alpha) &= \frac{r^{2n-1}}{(1-r^2)^n} \left[(1-r^2(1-|w|^2)) \sqrt{1-|w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) - \right. \\ &\quad \left. - r^2 \sum_{j=1}^{n-1} w_j \sqrt{1-|w|^2} e^{-i\psi} \bar{w}_j \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \right] \\ &= \frac{r^{2n-1}}{(1-r^2)^{n-1}} \sqrt{1-|w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w). \end{aligned}$$

Again by Lemma 5 we have $\Gamma_n(r\alpha) = (1-r^2)^{1-n} \Gamma_n(r\alpha)$. ■

3. PROOF OF THE THEOREM 2

We start with the following consequence of Theorem 1(a).

Lemma 7. *Let $n \geq 2$. For $L > 0$ and $r \in (0, 1)$,*

$$\text{Var } E_{f_L}(r) = \frac{2L^2(1-r^2)^{n-2}}{\pi(n-1)!(n-2)!} K(L, r),$$

where

$$K(L, r) = \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} \int_0^{\alpha(s,r)} (2\cos\theta - 2\cos\alpha(s,r))^{n-2} \left(s + \frac{1}{s} - 2\cos\theta\right) d\theta ds$$

and $\alpha(s, r) = \arccos\left[\frac{1}{2}\left((1+r^2)s + \frac{1-r^2}{s}\right)\right]$.

Proof. Starting from the expression given in Theorem 1(a), write

$$(8) \quad \text{Var } E_{f_L}(r) = \frac{r^{4n} L^2 (1-r^2)^{-4}}{\pi(n-1)!(n-2)!} J(L, r),$$

where

$$J(L, r) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{1 - \left(\frac{1-r^2}{|1-r^2w|}\right)^{2L}} \left(\frac{1-r^2}{|1-r^2w|}\right)^{2L+2} |1-w|^2 dm(w)$$

We write this integral in (a sort of) polar coordinates s, θ : let

$$w = \frac{1}{r^2} + \frac{1-r^2}{r^2s} e^{i(\pi-\theta)},$$

so that

$$\frac{1-r^2}{|1-r^2w|} = s.$$

Let $p(s, r)$ denote the intersection point of the unit circle and the circle $|w - 1/r^2| = \frac{1-r^2}{r^2s}$. In these coordinates, $w \in \mathbb{D}$ if and only if $s \in (\frac{1-r^2}{1+r^2}, 1)$ and $\theta \in (-\alpha(s, r), \alpha(s, r))$, where $\alpha(s, r)$ is the angle between the complex numbers $-1/r^2$ and $p(s, r) - 1/r^2$.

In these coordinates

$$(9) \quad 1 - |w|^2 = 1 - \left| \frac{1}{r^2} + \frac{1-r^2}{r^2s} e^{i(\pi-\theta)} \right|^2 = \frac{1-r^2}{r^4s} [2 \cos \theta - 2 \cos \alpha(s, r)] ,$$

since

$$(10) \quad 2 \cos \alpha(s, r) = (1+r^2)s + \frac{1-r^2}{s} .$$

Also

$$|1-w|^2 = \left| 1 - \frac{1}{r^2} + \frac{1-r^2}{r^2s} e^{i(\pi-\theta)} \right|^2 = \frac{(1-r^2)^2}{r^4s} \left(s + \frac{1}{s} - 2 \cos \theta \right) .$$

Since $dm(w) = \frac{(1-r^2)^2}{r^4s^3} d\theta ds$, and letting $\mathbb{D}_+ = \{z \in \mathbb{D} : \text{Im } z > 0\}$,

$$J(L, r) = 2 \int_{\mathbb{D}_+} \frac{(1-|w|^2)^{n-2}}{1 - \left(\frac{1-r^2}{|1-r^2w|} \right)^{2L}} \left(\frac{1-r^2}{|1-r^2w|} \right)^{2L+2} |1-w|^2 dm(w) = \frac{2(1-r^2)^{n+2}}{r^{4n}} K(L, r) ,$$

where

$$K(L, r) = \int_{\frac{1-r^2}{1+r^2}}^1 \int_0^{\alpha(s,r)} \frac{s^{2L-n}}{1-s^{2L}} (2 \cos \theta - 2 \cos \alpha(s, r))^{n-2} \left(s + \frac{1}{s} - 2 \cos \theta \right) d\theta ds .$$

Going back to (8) we obtain Lemma 7. ■

Our starting point in the proof of Theorem 2 is the expression of $\text{Var } E_{f_L}(r)$ given by Lemma 7.

Fix $r < 1$ and, in order to simplify the notation, let $\alpha(s) = \alpha(s, r)$.

1. In the first place we observe that for the asymptotics of $\text{Var } E_{f_L}(r)$ as $L \rightarrow \infty$, only the part of the integral $K(L, r)$ corresponding to s close to 1 is relevant. Fix $\epsilon > 0$ and let us see that the part of the integral up to $s = 1 - \epsilon$ decays exponentially in L ; we will see later that the part corresponding to $s \in (1 - \epsilon, 1)$ decays polynomially. From (9) we have

$$2 \cos \theta - 2 \cos \alpha(s, r) \leq \frac{r^4s}{1-r^2}$$

and therefore, for L big enough,

$$\begin{aligned}
& \int_{\frac{1-r^2}{1+r^2}}^{1-\epsilon} \frac{s^{2L-n}}{1-s^{2L}} \int_0^{\alpha(s)} (2 \cos \theta - 2 \cos \alpha(s, r))^{n-2} \left(s + \frac{1}{s} - 2 \cos \theta \right) d\theta ds \\
& \leq \frac{(1-\epsilon)^{2L-n}}{1-(1-\epsilon)^{2L}} \int_{\frac{1-r^2}{1+r^2}}^{1-\epsilon} \left(\frac{r^4 s}{1-r^2} \right)^{n-2} \left[\left(s + \frac{1}{s} \right) \alpha(s) - 2 \sin \alpha(s) \right] d\theta ds \\
& \leq \frac{(1-\epsilon)^{2L-2} r^{4n-2}}{(1-r^2)^{n-2}} \int_{\frac{1-r^2}{1+r^2}}^1 \left[\left(s + \frac{1}{s} - 2 \right) \alpha(s) + 2\alpha(s) \right] ds \\
& \leq \frac{(1-\epsilon)^{2L-2} r^{4n-2}}{(1-r^2)^{n-2}} \int_{\frac{1-r^2}{1+r^2}}^1 \left[\left(s + \frac{1}{s} - 2 \right) \frac{\pi}{2} + \pi \right] ds = C_{n,r} (1-\epsilon)^{2L-2}.
\end{aligned}$$

2. For s near 1 we can assume that $\alpha(s)$ is small. More precisely, given $\delta > 0$, there exists $\epsilon = \epsilon(\delta) > 0$ such that $\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0$ and

$$s \in (1-\epsilon, 1) \implies \alpha(s) \leq \arccos(1-\delta) = O(\sqrt{\delta}).$$

To see this notice that, by (10), $\alpha(s) \leq \arccos(1-\delta)$ precisely when $(1+r^2)s + \frac{1-r^2}{s} \geq 2(1-\delta)$, which is equivalent to $s^2 - \frac{2(1-\delta)}{1+r^2} s + \frac{1-r^2}{1+r^2} \geq 0$, and to

$$s \notin \left(\frac{1}{(1+r^2)} (1-\delta - \sqrt{(1-\delta)^2 - (1-r^4)}), \frac{1}{(1+r^2)} (1-\delta + \sqrt{(1-\delta)^2 - (1-r^4)}) \right).$$

Hence it is enough to take

$$\epsilon = 1 - \frac{1}{(1+r^2)} (1-\delta + \sqrt{(1-\delta)^2 - (1-r^4)}).$$

3. For $\alpha(s)$ close to 0,

$$\alpha^2(s) = (1+r^2) \frac{1-s}{s} \left(s - \frac{1-r^2}{1+r^2} \right) + \dots$$

(here and in the remaining of this section, $+\dots$ will indicate terms of lower order as $s \rightarrow 1^-$.)

To see this notice that, from (10),

$$(11) \quad 2 - 2 \cos \alpha(s) = -\frac{(1-s)^2}{s} + \frac{r^2}{s} - sr^2 = (1+r^2) \frac{1-s}{s} \left(s - \frac{1-r^2}{1+r^2} \right),$$

which together with the approximation $2 - 2 \cos \alpha(s) = \alpha^2(s) + o(\alpha^2(s))$ gives the statement.

4. We will see next that the whole integral, with s from $\frac{1-r^2}{1+r^2}$ up to 1, has polynomial decay. As seen in the previous point, for s close to 1 we have $\alpha^2(s) = 2r^2(1-s) + \dots$. On the other

hand, by (10)

$$\begin{aligned} 2 \cos \theta - 2 \cos \alpha(s) &= -2(1 - \cos \theta) + 2 - s - \frac{1}{s} + r^2 \left(\frac{1}{s} - s \right) \\ &= 2r^2(1 - s) - 2(1 - \cos \theta) - (1 - r^2) \sum_{j=2}^{\infty} (1 - s)^j, \end{aligned}$$

and

$$s + \frac{1}{s} - \cos \theta = s + \frac{1}{s} - 2 + 2(1 - \cos \theta) = 2(1 - \cos \theta) + \sum_{j=2}^{\infty} (1 - s)^j.$$

Together with the approximation $2(1 - \cos \theta) = \theta^2 + \dots$, and writing only the leading term around $s = 1$, this yields

$$\begin{aligned} A(s) &:= \int_0^{\alpha(s)} (2 \cos \theta - 2 \cos \alpha(s))^{n-2} \left(s + \frac{1}{s} - 2 \cos \theta \right) d\theta \\ &= \int_0^{\sqrt{2r^2(1-t)}} (2r^2(1-s) - \theta^2)^{n-2} \theta^2 d\theta + \dots \\ &= \sum_{j=0}^{n-2} \binom{n-2}{j} (2r^2(1-s))^{n-2-j} (-1)^j \int_0^{\sqrt{2r^2(1-t)}} \theta^{2j+2} d\theta + \dots \\ &= (2r^2(1-s))^{n-1/2} \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(-1)^j}{2j+3} + \dots \end{aligned}$$

A straightforward computation shows that for $m \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0, 1, \dots, m\}$,

$$(12) \quad \sum_{j=0}^m \binom{m}{j} \frac{(-1)^j}{z+j} = \frac{m!}{z(z+1)\cdots(z+m)} = \frac{m!\Gamma(z)}{\Gamma(z+m+1)}.$$

Applying this to $z = 3/2$, $m = n - 2$, and using that $\Gamma(1/2) = \sqrt{\pi}$ we have

$$A(s) = \frac{(n-2)! \sqrt{\pi}}{4\Gamma(n+1/2)} (2r^2(1-s))^{n-1/2}.$$

Since for L big the integral for $s \in (0, \frac{1-r^2}{1+r^2})$ tends to 0, this implies that

$$\begin{aligned} K(L, r) &= \frac{(n-2)! \sqrt{\pi}}{4\Gamma(n+1/2)} \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} (2r^2(1-s))^{n-1/2} ds + \dots \\ &= \sqrt{\frac{\pi}{2}} \frac{(n-2)! 2^{n-2}}{\Gamma(n+1/2)} r^{2n-1} \sum_{k=0}^{\infty} \int_0^1 s^{2L+2Lk-n} (1-s)^{n-1/2} ds + \dots \\ &= \sqrt{\frac{\pi}{2}} \frac{(n-2)! 2^{n-2}}{\Gamma(n+1/2)} r^{2n-1} \sum_{k=0}^{\infty} \frac{\Gamma(2L+2Lk-n+1) \Gamma(n+1/2)}{\Gamma(2L+2Lk+3/2)} + \dots \end{aligned}$$

Using the asymptotics $\lim_{k \rightarrow \infty} \frac{\Gamma(k+a)}{\Gamma(k)k^a} = 1$ we have then

$$\begin{aligned} K(L, r) &= \sqrt{\frac{\pi}{2}} \frac{(n-2)! 2^{n-2}}{\Gamma(n+1/2)} r^{2n-1} \Gamma(n+1/2) \sum_{k=0}^{\infty} \frac{1}{(2L+2kL)^{n+1/2}} + \dots \\ &= \frac{\sqrt{\pi}}{8} (n-2)! \zeta(n+1/2) r^{2n-1} \frac{1}{L^{n+1/2}} + \dots \end{aligned}$$

Finally, by Lemma 7 we get

$$\begin{aligned} \text{Var } E_{f_L}(r) &= \frac{2L^2(1-r^2)^{n-2}}{\pi(n-1)!(n-2)!} \frac{\sqrt{\pi}}{8} (n-2)! \zeta(n+1/2) r^{2n-1} \frac{1}{L^{n+1/2}} + \dots \\ &= \frac{1}{4\sqrt{\pi}} \frac{\zeta(n+1/2)}{(n-1)!} r^{2n-1} (1-r^2)^{n-2} L^{3/2-n} + \dots \end{aligned}$$

This and Theorem 1(b) finish the proof.

4. PROOF OF THE THEOREM 3

Let us see first that the order of growth as $r \rightarrow 1^-$ is as stated. Later on we will see how the constants $C(L, n)$ can be determined. The notation \simeq indicates that there exists a constant C independent of r such that $C^{-1}A \leq B \leq CA$.

According to Lemma 7 it is enough to study the asymptotics of $K(L, r)$ as $r \rightarrow 1^-$.

Since $2 \cos x - 2 \cos a \simeq (\sin a)(a - x)$ for $0 \leq x \leq a \leq \pi/2$, we see that in the range of integration of θ in $K(L, r)$

$$2 \cos \theta - 2 \cos \alpha(s, r) \simeq (\sin \alpha(s, r))(\alpha(s, r) - \theta) \simeq \alpha(s, r)(\alpha(s, r) - \theta).$$

Therefore

$$K(L, r) \simeq \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} (\alpha(s, r))^{n-2} \int_0^{\alpha(s, r)} (\alpha(s, r) - \theta)^{n-2} \left[\frac{(1-s)^2}{s} + 2(1 - \cos \theta) \right] d\theta ds$$

Denote temporarily $\alpha = \alpha(s, r)$. Using now that $1 - \cos \theta \simeq \theta^2$ for $\theta \in [0, \pi/2]$ we can estimate the integral in θ :

$$\begin{aligned} \int_0^\alpha (\alpha - \theta)^{n-2} \left[\frac{(1-s)^2}{s} + 2(1 - \cos \theta) \right] d\theta &\simeq \\ &\simeq \frac{(1-s)^2}{s} \int_0^\alpha (\alpha - \theta)^{n-2} d\theta + \int_0^\alpha (\alpha - \theta)^{n-2} \theta^2 d\theta \\ &\simeq \frac{(1-s)^2}{s} \int_0^\alpha (\alpha - \theta)^{n-2} d\theta + \int_0^\alpha (\alpha - \theta)^n d\theta + 2\alpha \int_0^\alpha (\alpha - \theta)^{n-1} d\theta + \alpha^2 \int_0^\alpha (\alpha - \theta)^{n-2} d\theta \\ &\simeq \frac{(1-s)^2}{s} \alpha^{n-1} + \alpha^{n+1} = \alpha^{n-1} \left[\frac{(1-s)^2}{s} + \alpha^2 \right]. \end{aligned}$$

Hence

$$K(L, r) \simeq \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} (\alpha(s, r))^{2n-3} \left[\frac{(1-s)^2}{s} + \alpha^2(s, r) \right] ds.$$

Using (11) we have also,

$$\alpha(s, r)^2 \simeq 2 - 2 \cos \alpha(s, r) = (1+r^2) \frac{1-s}{s} \left(s - \frac{1-r^2}{1+r^2} \right),$$

and therefore

$$\begin{aligned} K(L, r) &\simeq \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} \left(\frac{1-s}{s} \left(s - \frac{1-r^2}{1+r^2} \right) \right)^{n-3/2} \left[\frac{(1-s)^2}{s} + \frac{1-s}{s} \left(s - \frac{1-r^2}{1+r^2} \right) \right] ds \\ &\simeq \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} \left(s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} \left((1-s) + \left(s - \frac{1-r^2}{1+r^2} \right) \right) ds \\ &\simeq \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} \left(s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} ds. \end{aligned}$$

Let us see now that for the asymptotics as $r \rightarrow 1^-$ it is enough to take care of s small. Notice, for instance, that the portion of the integral where $s \in (1/2, 1)$ tends to the constant

$$\int_{1/2}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} s^{n-3/2} ds = \int_{1/2}^1 \frac{s^{2L-n-1}}{1-s^{2L}} (1-s)^{n-1/2} ds.$$

On the other hand, for $s \leq 1/2$ we have $1-s \simeq 1-s^{2L} \simeq 1$ and therefore,

$$\begin{aligned} \int_{\frac{1-r^2}{1+r^2}}^{1/2} \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} \left(s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} ds &\simeq \int_{\frac{1-r^2}{1+r^2}}^{1/2} s^{2L-2n+1/2} \left(s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} ds \\ &\simeq \begin{cases} (1-r^2)^{2L-n} + \dots & \text{if } 2L-n < 0 \\ \log \frac{1}{1-r^2} + \dots & \text{if } 2L-n = 0 \\ 1 & \text{if } 2L-n > 0. \end{cases} \end{aligned}$$

This gives the order of growth of $\text{Var } E_{f_L}(r)$ as stated in Theorem 3. Once we know this we can determine the values $C(L, n)$, $L > 0$.

Case $L \leq n/2$. Here $K(L, r)$ tends to ∞ as $r \rightarrow 1^-$, at speed $(1 - r^2)^{2L-n}$, so it is enough to consider the terms giving this maximal order of growth. Since

$$s + \frac{1}{s} - 2 \cos \theta = \frac{(1-s)^2}{s} + 2(1 - \cos \theta) = \frac{1}{s} + \dots$$

and

$$\frac{1}{1-s^{2L}} = 1 + \dots$$

(where the dots indicate lower order terms) we have

$$\begin{aligned} K(L, r) &= \int_{\frac{1-r^2}{1+r^2}}^1 s^{2L-n-1} \int_0^{\alpha(s,r)} (2 \cos \theta - 2 \cos \alpha(s, r))^{n-2} d\theta ds + \dots \\ &= \int_{\frac{1-r^2}{1+r^2}}^1 \int_0^{\alpha(s,r)} s^{2L-n-1} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j (2 \cos \alpha(s, r))^j (2 \cos \theta)^{n-2-j} d\theta ds + \dots \end{aligned}$$

In order to apply Fubini's theorem –and to simplify the notation– denote $\epsilon(r) = \frac{1-r^2}{1+r^2}$. The domain of the double integral above is thus given by the conditions

$$\begin{cases} \epsilon(r) \leq s \leq 1 \\ 0 \leq \theta \leq \alpha(s, r) . \end{cases}$$

To determine the global range of θ , notice that the function

$$h(s) := 2 \cos \alpha(s, r) = (1+r^2)s + \frac{1-r^2}{s}$$

has a minimum at $s = \sqrt{\epsilon(r)}$ and that $h(\sqrt{\epsilon(r)}) = 2\sqrt{1-r^4}$. Therefore

$$0 \leq \theta \leq \alpha_0(r) := \arccos(\sqrt{1-r^2}) .$$

Once θ is fixed, the inequalities above determine the range of s :

$$s \in A(\theta) := \left\{ s : \epsilon(r) \leq s \leq 1, 2 \cos \theta \geq (1+r^2)s + \frac{1-r^2}{s} \right\} .$$

Therefore

$$(13) \quad K(L, r) = \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\alpha_0(r)} (2 \cos \theta)^{n-2-j} \left[\int_{A(\theta)} (2 \cos \alpha(s, r))^j ds \right] d\theta + \dots$$

Here we wish to determine the leading term of

$$J := \int_{A(\theta)} (2 \cos \alpha(s, r))^j ds = \int_{A(\theta)} \left((1+r^2)s + \frac{1-r^2}{s} \right)^j ds .$$

After the change of variable $s = \sqrt{\epsilon(r)}x$ the restrictions on s imposed by $A(\theta)$ are equivalent to the conditions

$$\sqrt{\epsilon(r)} \leq x \leq 1/\sqrt{\epsilon(r)} \quad , \quad (x + 1/x)\sqrt{1-r^4} \leq 2 \cos \theta \quad ,$$

hence

$$(14) \quad J = (\epsilon(r))^{L-\frac{n}{2}}(1-r^4)^{j/2} \int_{\substack{\sqrt{\epsilon(r)} \leq x \leq 1/\sqrt{\epsilon(r)} \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} x^{2L-n-1} \left(x + \frac{1}{x}\right)^j dx + \dots$$

$$= \frac{(1-r^2)^{L-\frac{n}{2}+\frac{j}{2}}}{2^{L-\frac{n}{2}-\frac{j}{2}}} \int_{\substack{\sqrt{\epsilon(r)} \leq x \leq 1/\sqrt{\epsilon(r)} \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} x^{2L-n-1} \left(x + \frac{1}{x}\right)^j dx + \dots$$

We split the integral above into two parts, depending on whether or not $x \leq 1$. For both parts we perform the same change of variables $y = x + 1/x$ and extract the terms of maximal order.

(i) Assume $x \leq 1$. Here $y = 1/x + \dots$ and therefore $x^{2L-n-1} = y^{n+1-2L} + \dots$. Similarly

$$dy = \left(1 - \frac{1}{x^2}\right)dx = -\frac{dx}{x^2} + \dots = -y^2 dx + \dots$$

As for the limits of integration, notice that if $x = \sqrt{\epsilon(r)}$ then $y = \sqrt{\epsilon(r)} + 1/\sqrt{\epsilon(r)} = \frac{2}{\sqrt{1-r^4}}$. This yields

$$(15) \quad A_1 := \int_{\substack{\sqrt{\epsilon(r)} \leq x \leq 1 \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} x^{2L-n-1} \left(x + \frac{1}{x}\right)^j dx = \int_2^{\frac{2 \cos \theta}{\sqrt{1-r^4}}} y^{n+1-2L} y^j \frac{dy}{y^2} + \dots$$

$$= \frac{1}{n-2L+j} \left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right)^{n-2L+j} + \dots = \frac{2^{\frac{n}{2}-L+\frac{j}{2}} (\cos \theta)^{n-2L+j}}{(n-2L+j)(1-r^2)^{\frac{n}{2}-L+\frac{j}{2}}} + \dots$$

(ii) Let us see now that the integral corresponding to $x \geq 1$ is of smaller order. Here $y = x + \dots$ and $dy = dx + \dots$, thus

$$A_2 := \int_{\substack{1 \leq x \leq 1/\sqrt{\epsilon(r)} \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} x^{2L-n-1} \left(x + \frac{1}{x}\right)^j dx = \int_2^{\frac{2 \cos \theta}{\sqrt{1-r^4}}} y^{2L-n-1+j} dy + \dots$$

If $2L-n+j < 0$ these integrals tend to a constant. If $2L-n+j = 0$ then this is $O(\log(\frac{1}{1-r^2}))$, which is obviously of smaller order than $(1-r^2)^{L-\frac{n}{2}-\frac{j}{2}} = (1-r^2)^{-j}$. Finally, if $2L-n+j > 0$ then

$$A_2 = \frac{1}{2L-n+j} \left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right)^{2L-n+j} = o\left(\frac{1}{(1-r^2)^{\frac{n}{2}-L+\frac{j}{2}}}\right).$$

Therefore the leading term of the integral in (14) is A_1 , and by (15) we get

$$\begin{aligned} J &= \frac{(1-r^2)^{L-\frac{n}{2}+\frac{j}{2}}}{2^{L-\frac{n}{2}-\frac{j}{2}}} \frac{2^{\frac{n}{2}-L+\frac{j}{2}} (\cos \theta)^{n-2L+j}}{(n-2L+j)(1-r^2)^{\frac{n}{2}-L+\frac{j}{2}}} + \dots \\ &= \frac{2^{n-2L+j}}{n-2L+j} (\cos \theta)^{n-2L+j} (1-r^2)^{2L-n} + \dots \end{aligned}$$

Plugging this into (13) we obtain

$$K(L, r) = 2^{2n-2L-2} \left[\sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(-1)^j}{n-2L+j} \right] \left[\int_0^{\pi/2} (\cos \theta)^{2n-2L-2} d\theta \right] (1-r^2)^{2L-n} + \dots$$

The sum in j is computed using (12) with $z = n - 2L$ and $m = n - 2$. The integrals in θ are taken care of by the following identity, which is a simple computation: for $m \in \mathbb{N}$

$$(16) \quad \int_0^{\pi/2} (\cos \theta)^m d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2} + 1)}.$$

We have thus

$$K(L, r) = 2^{2n-2L-2} \frac{(n-2)! \Gamma(n-2L)}{\Gamma(2n-2L-1)} \frac{\sqrt{\pi}}{2} \frac{\Gamma(n-L-1/2)}{\Gamma(n-L)} (1-r^2)^{2L-n} + \dots$$

and therefore

$$\text{Var } E_{f_L}(r) = \frac{L^2}{\sqrt{\pi}} \frac{2^{2n-2L-2}}{(n-1)!} \frac{\Gamma(n-L-1/2)}{\Gamma(2n-2L-1)} \frac{\Gamma(n-2L)}{\Gamma(n-L)} (1-r^2)^{2L-n} + \dots$$

This expression can be simplified by means of the duplication formula for the Γ -function:

$$(17) \quad \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z+1).$$

Taking $z = n - L - 1/2$ we see that

$$\frac{2^{2n-2L-2}}{\sqrt{\pi}} \frac{\Gamma(n-L-1/2)}{\Gamma(2n-2L-1)} = \frac{1}{\Gamma(n-L)}$$

and therefore

$$\text{Var } E_{f_L}(r) = \frac{L^2}{(n-1)!} \frac{\Gamma(n-2L)}{(\Gamma(n-L))^2} (1-r^2)^{2L-n} + \dots$$

Applying once more the duplication formula, now with $z = n/2 - L$, we finally get

$$\text{Var } E_{f_L}(r) = \frac{L^2}{(n-1)!} \frac{2^{n-2L-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-L) \Gamma(\frac{n+1}{2}-L)}{(\Gamma(n-L))^2} (1-r^2)^{2L-n} + \dots$$

Case $L = n/2$. Proceeding as in the previous case we arrive at

$$K(L, r) = \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\alpha_0(r)} (2 \cos \theta)^{n-2-j} \left[\int_{A(\theta)} \frac{1}{s} (2 \cos \alpha(s, r))^j ds \right] d\theta + \dots$$

where

$$J := \int_{A(\theta)} \frac{1}{s} (2 \cos \alpha(s, r))^j ds = 2^{j/2} (1 - r^2)^{j/2} \int_{\substack{\sqrt{\epsilon(r)} \leq x \leq 1/\sqrt{\epsilon(r)} \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} \frac{1}{x} \left(x + \frac{1}{x}\right)^j dx + \dots$$

As before we split the integral into two parts, depending on whether $x \leq 1$ or not. Unlike the previous case here both parts have the same order.

(i) Assume $x \leq 1$. Then

$$\begin{aligned} A_1 &:= \int_{\substack{\sqrt{\epsilon(r)} \leq x \leq 1 \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} \frac{1}{x} \left(x + \frac{1}{x}\right)^j dx = \int_2^{\frac{2 \cos \theta}{\sqrt{1-r^4}}} y^{j-1} dy + \dots \\ &= \begin{cases} \log\left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right) + \dots = \frac{1}{2} \log\left(\frac{1}{1-r^2}\right) + \log(\cos \theta) \dots & \text{if } j = 0 \\ \frac{1}{j} \left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right)^j + \dots = \frac{2^{j/2} (\cos \theta)^j}{j(1-r^2)^{j/2}} + \dots & \text{if } j \geq 1. \end{cases} \end{aligned}$$

(ii) For $x > 1$ we have the same values as in the previous case:

$$A_2 := \int_{\substack{1 < x \leq 1/\sqrt{\epsilon(r)} \\ x+1/x \leq \frac{2 \cos \theta}{\sqrt{1-r^4}}}} \frac{1}{x} \left(x + \frac{1}{x}\right)^j dx = \int_2^{\frac{2 \cos \theta}{\sqrt{1-r^4}}} y^{j-1} dy + \dots$$

Then

$$J = \begin{cases} \log\left(\frac{1}{1-r^2}\right) + 2 \log(\cos \theta) + \dots & \text{if } j = 0 \\ \frac{2^{j+1}}{j} (\cos \theta)^j + \dots & \text{if } j \geq 1. \end{cases}$$

With this we get

$$\begin{aligned} K(L, r) &= \int_0^{\alpha_0(r)} (2 \cos \theta)^{n-2} \left(\log\left(\frac{1}{1-r^2}\right) + 2 \log(\cos \theta) \right) d\theta + \\ &\quad + \sum_{j=1}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\alpha_0(r)} (2 \cos \theta)^{n-2-j} \frac{2^{j+1}}{j} (\cos \theta)^j d\theta. \end{aligned}$$

Since $(\cos \theta)^{n-2} \log(\cos \theta)$ is integrable in $[0, \pi/2]$ and the sum in $j \geq 1$ is bounded independently of r , the leading term of $K(L, r)$ is given by the factor $\log(1/(1-r^2))$ above. Then (16)

yields

$$\begin{aligned} K(L, r) &= 2^{n-2} \int_0^{\alpha_0(r)} (\cos \theta)^{n-2} \log\left(\frac{1}{1-r^2}\right) + \dots = 2^{n-2} \int_0^{\pi/2} (\cos \theta)^{n-2} \log\left(\frac{1}{1-r^2}\right) + \dots \\ &= 2^{n-3} \sqrt{\pi} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \log\left(\frac{1}{1-r^2}\right) + \dots \end{aligned}$$

By Lemma 7 we get then

$$\text{Var } E_{f_L}(r) = \frac{2^{n-2} \left(\frac{n}{2}\right)^2}{\sqrt{\pi} (n-1)! (n-2)!} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} (1-r^2)^{n-2} \log\left(\frac{1}{1-r^2}\right) + \dots$$

The duplication formula (17) with $z = \frac{n-1}{2}$ yields $\frac{2^{n-2} \Gamma(\frac{n-1}{2})}{\sqrt{\pi} (n-2)!} = \frac{1}{\Gamma(\frac{n}{2})}$ and thus the stated result.

Case $L > n/2$. Here $K(L, r)$ tends, as $r \rightarrow 1^-$, to the constant

$$K(L, 1) = \int_0^1 \frac{s^{2L-n}}{1-s^{2L}} \int_0^{\arccos s} (2 \cos \theta - 2s)^{n-2} \left(s + \frac{1}{s} - 2 \cos \theta\right) d\theta ds.$$

As before, we expand the power and apply Fubini's theorem:

$$\begin{aligned} K(L, 1) &= 2^{n-2} \int_0^1 \int_0^{\arccos s} \sum_{k=0}^{\infty} s^{2L+2Lk-n} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j s^j (\cos \theta)^{n-2-j} \left(s + \frac{1}{s} - 2 \cos \theta\right) d\theta ds \\ &= 2^{n-2} \sum_{k=0}^{\infty} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\pi/2} (\cos \theta)^{n-2-j} \int_0^{\cos \theta} s^{2L(1+k)-n+j} \left(s + \frac{1}{s} - 2 \cos \theta\right) ds d\theta. \end{aligned}$$

Re-indexing the sum in k and performing the integral in s we get

$$\begin{aligned} K(L, 1) &= 2^{n-2} \sum_{k=1}^{\infty} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \left[\int_0^{\pi/2} \frac{(\cos \theta)^{2Lk}}{2Lk - n + j + 2} d\theta + \int_0^{\pi/2} \frac{(\cos \theta)^{2Lk-2}}{2Lk - n + j} d\theta \right. \\ &\quad \left. - 2 \int_0^{\pi/2} \frac{(\cos \theta)^{2Lk}}{2Lk - n + j + 1} d\theta \right] \end{aligned}$$

Using (16) to compute the integrals in θ , and (12) to compute the sum in j , we obtain

$$\begin{aligned} K(L, 1) &= \sqrt{\pi} (n-2)! 2^{n-3} \sum_{k=1}^{\infty} \left[\frac{\Gamma(Lk + 1/2) \Gamma(2Lk - n + 2)}{\Gamma(Lk + 1) \Gamma(2Lk + 1)} + \frac{\Gamma(Lk - 1/2) \Gamma(2Lk - n)}{\Gamma(Lk) \Gamma(2Lk - 1)} \right. \\ &\quad \left. - 2 \frac{\Gamma(Lk + 1/2) \Gamma(2Lk - n)}{\Gamma(Lk + 1) \Gamma(2Lk)} \right]. \end{aligned}$$

Lemma 8. For $M > 0$

$$\begin{aligned} \frac{\Gamma(M+1/2)\Gamma(2M-n+2)}{\Gamma(M+1)\Gamma(2M+1)} + \frac{\Gamma(M-1/2)\Gamma(2M-n)}{\Gamma(M)\Gamma(2M-1)} - 2\frac{\Gamma(M+1/2)\Gamma(2M-n+1)}{\Gamma(M+1)\Gamma(2M)} \\ = 2^{-n}\frac{\Gamma(M-\frac{n}{2})\Gamma(M-\frac{n-1}{2})}{(\Gamma(M+1))^2}\left(M+\frac{n(n-1)}{2}\right) \end{aligned}$$

This with $M = Lk$ yields

$$K(L, 1) = \frac{\sqrt{\pi}(n-2)!}{8} \sum_{k=1}^{\infty} \frac{\Gamma(Lk-n/2)\Gamma(Lk-(n-1)/2)}{(\Gamma(Lk+1))^2} \left(LK + \frac{n(n-1)}{2}\right),$$

which together with the identity of Lemma 7 gives the stated result.

Proof of Lemma 8. Denote by S the left hand side of identity in the lemma. Then

$$\begin{aligned} S &= \frac{\Gamma(M+1/2)\Gamma(2M-n+2)}{\Gamma(M+1)\Gamma(2M+1)} - \frac{\Gamma(M+1/2)\Gamma(2M-n+1)}{\Gamma(M+1)\Gamma(2M)} + \\ &\quad + \frac{\Gamma(M-1/2)\Gamma(2M-n)}{\Gamma(M)\Gamma(2M-1)} - \frac{\Gamma(M+1/2)\Gamma(2M-n+1)}{\Gamma(M+1)\Gamma(2M)} \\ &= \frac{\Gamma(M+1/2)\Gamma(2M-n+2)}{\Gamma(M+1)\Gamma(2M)} \left(\frac{1}{2M} - \frac{1}{2M-n+1}\right) + \\ &\quad + \frac{\Gamma(M+1/2)\Gamma(2M-n+1)}{\Gamma(M+1)\Gamma(2M)} \left(\frac{2M}{2M-n} - 1\right) \\ &= \frac{\Gamma(M+1/2)\Gamma(2M-n)}{\Gamma(M+1)\Gamma(2M+1)} (2M+n(n-1)). \end{aligned}$$

Using the duplication formula (17) for $z = M - n$ and $z = M + 1$ we get finally

$$\begin{aligned} S &= \frac{\Gamma(M+1/2)}{\Gamma(M+1)} \frac{2^{2M-n-1}\Gamma(M-\frac{n}{2})\Gamma(M-\frac{n-1}{2})}{2^{2M}\Gamma(M+1/2)\Gamma(M+1)} (2M+n(n-1)) \\ &= \frac{1}{2^n} \frac{\Gamma(M-\frac{n}{2})\Gamma(M-\frac{n-1}{2})}{(\Gamma(M+1))^2} \left(M + \frac{n(n-1)}{2}\right). \end{aligned}$$

■

REFERENCES

- [Buc13] Jeremiah Buckley, *Fluctuations in the zero set of the hyperbolic Gaussian analytic function*, Int. Math. Res. Not. IMRN to appear (2013), 18. ↑3, 5, 6
- [BMP14] Jeremiah Buckley, Xavier Massaneda, and Bharti Pridhnani, *Gaussian analytic functions in the ball*, Preprint (2014). ↑2
- [HKPV09] John Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series, vol. 51, American Mathematical Society, Providence, RI, 2009. MR2552864 (2011f:60090) ↑2, 6

- [NS11] Fedor Nazarov and Mikhail Sodin, *Fluctuations in random complex zeroes: asymptotic normality revisited*, Int. Math. Res. Not. IMRN **24** (2011), 5720–5759. MR2863379 (2012k:60103) ↑
- [Rud08] Walter Rudin, *Function theory in the unit ball of \mathbb{C}^n* , Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1980 edition. MR2446682 (2009g:32001) ↑2
- [SZ06] Bernard Shiffman and Steve Zelditch, *Number variance of random zeros*, ArXiv: arxiv.org/pdf/math/0608743v3 (2006). ↑3, 6
- [SZ08] ———, *Number variance of random zeros on complex manifolds*, Geom. Funct. Anal. **18** (2008), no. 4, 1422–1475, DOI 10.1007/s00039-008-0686-3. MR2465693 (2009k:32019) ↑4, 6
- [Sod00] Mikhail Sodin, *Zeros of Gaussian analytic functions*, Math. Res. Lett. **7** (2000), no. 4, 371–381. MR1783614 (2002d:32030) ↑2
- [ST04] Mikhail Sodin and Boris Tsirelson, *Random complex zeroes. I. Asymptotic normality*, Israel J. Math. **144** (2004), 125–149, DOI 10.1007/BF02984409. MR2121537 (2005k:60079) ↑2
- [Sto94] Manfred Stoll, *Invariant potential theory in the unit ball of \mathbb{C}^n* , London Mathematical Society Lecture Note Series, vol. 199, Cambridge University Press, Cambridge, 1994. MR1297545 (96f:31011) ↑2

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