EFFECTIVE BOUNDS FOR THE MEASURE OF ROTATIONS

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ABSTRACT. A fundamental question in Dynamical Systems is to identify regions of phase/parameter space satisfying a given property (stability, linearization, etc). Given a family of analytic circle diffeomorphisms depending on a parameter, we obtain effective (almost optimal) lower bounds of the Lebesgue measure of the set of parameters that are conjugated to a rigid rotation. We estimate this measure using an a-posteriori KAM scheme that relies on quantitative conditions that are checkable using computer-assistance. We carefully describe how the hypotheses in our theorems are reduced to a finite number of computations, and apply our methodology to the case of the Arnold family. Hence we show that obtaining non-asymptotic lower bounds for the applicability of KAM theorems is a feasible task provided one has an a-posteriori theorem to characterize the problem. Finally, as a direct corollary, we produce explicit asymptotic estimates in the so called local reduction setting (à la Arnold) which are valid for a global set of rotations.

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CONTENTS

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1. Introduction

One of the most important problems in Hamiltonian Mechanics, and Dynamical Systems in general, is to identify stability (and instability) regions in phase and parameter space. The question of stability goes back to the classical works of outstanding mathematicians during the XVIIIth to early XXth centuries, such as Laplace, Kovalevskaya, Poincaré, Lyapunov, or Birkhoff, who thought about the problem and obtained important results in this direction [5, 21, 52, 59]. During the mid 1950's the birth of KAM theory [2, 39, 49] gave certain hope in the characterization of stable motions, not only by proving the existence of quasi-periodic solutions but also revealing that they are present in regions of positive measure in phase space [42, 50, 53]. Notwithstanding the formidable impact of ideas and results produced in the perturbative context [3, 6, 62], even in recent works [16, 22, 23, 24], the theory was for long time attributed to be seriously limited in the study of concrete and realistic problems. On the one hand, the size of the perturbation admitted in early KAM theorems was dramatically small[1], and on the other hand, as far as the authors know, the only knowledge about measure estimates in phase space is just asymptotic: the union of the surviving invariant tori has relative measure of order at least $1 - \sqrt{\varepsilon}$, where ε is the perturbation parameter (see the above references and also [4, 48], where secondary tori are also considered).

The aim of this paper is to show that the task of obtaining effective bounds for the measure of quasi-periodic solutions in phase space is feasible, and to provide full details in the setting of conjugacy to rotation of (analytic) circle diffeomorphisms. The study of maps of the circle to itself is one of the most fundamental dynamical systems, and many years after Poincaré raised the question of comparing the dynamics of a circle homeomorphism with a rigid rotation, it is perhaps the problem where the global effect of small divisors is best understood. This problem was approached by Arnold himself in [1] who obtained mild conditions for conjugacy to rotation in a perturbative setting (known as *local reduction theorem* in this context), showing also that the existence and smoothness of such conjugacy is closely connected with the existence of an absolutely continuous invariant measure. The first global results (known as *global reduction theorems*) were obtained in [32] and extended later in [61]. Sharp estimates on finite regularity were investigated along different works [35, 37, 58] and finally obtained in [36].

Before going into the details, we think it is convenient to give an overview on the progress to effectively apply KAM theory in particular systems. With the advent of computers and new developed methodologies, the applicability of the theory has been manifested in the pioneering works [13, 20] and in applications to Celestial Mechanics [14, 15, 43, 44]. A recent methodology has been proposed in [26], based on an a-posteriori KAM theorem with quantitative and sharp explicit hypotheses. To check the hypotheses of the theorem, a major difficulty is to control the analytic norm of some complicated functions defined on the torus. This is done using fast Fourier transform (with interval arithmetics) and carrying out an accurate control of the discretization error. The methodology has been applied to low dimensional problems obtaining almost optimal results. But after the previous mentioned works, the most important question remained open, that is:

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[1]Giving a satisfactory account about skeptic criticisms in this direction is far from the scope of this paper, but we refer to [15, 21] for illuminating details and references.

> *Given a particular system with non-perturbative parameters, and given a particular region of interest in phase/parameter space, what is the abundance of quasiperiodic smooth solutions in that region?*

From the perspective of characterizing the Lebesgue measure of such solutions, the above question is a global version of the perturbative (asymptotic) estimates for the measure of KAM tori [2, 42, 50, 53]. In contrast, an analogous global question regarding the topological characterization of instability was formulated by Herman [33] in terms of non-wandering sets. Although non-perturbative and global questions are of the highest interest in the study of a dynamical systems, it is not surprising that they are not often explicitly formulated in the literature (with exception of [5, 59] and some numerical studies, e.g. [40, 54, 57]), since the analytical approaches to the problem were limited by using asymptotic estimates in the perturbation parameter. Then, the work presented in this paper not only is valuable for the fact that it provides a novel tool to use in KAM-like schemes, but also enlarges our vision about how stability can be effectively measured.

In this paper the above question is directly formulated in the context of circle maps. Given any family $\alpha \in A \to f_\alpha$ of analytic circle diffeomorphisms, we answer the following problem:

> *Obtain (almost optimal) lower bounds for the measure of parameters $\alpha \in A$ such that the map f_α is analytically conjugated to a rigid rotation.*

This question was considered by Arnold in [1] (following Poincaré's problem on the study of the rotation number as a function on the space of mappings) for the paradigmatic example

$$(1.1) \qquad \alpha \in [0,1] \longmapsto f_{\alpha,\varepsilon}(x) = x + \alpha + \frac{\varepsilon}{2\pi}\sin(2\pi x),$$

where $|\varepsilon| < 1$ is a fixed parameter. Denoting the rotation number of this family as $\rho_\varepsilon : \alpha \in [0,1] \mapsto \rho(f_{\alpha,\varepsilon})$ and introducing the set $K_\varepsilon = [0,1]\backslash\text{Int}(\rho_\varepsilon^{-1}(\mathbb{Q}))$, he was able to prove that $\text{Leb}(K_\varepsilon) \to 1$ for $|\varepsilon| \to 0$, where $\text{Leb}(\cdot)$ stands for the Lebesgue measure. As global results for $0 < |\varepsilon| < 1$, Herman proved in [32] that K_ε is a Cantor set, $K_\varepsilon \cap \rho_\varepsilon^{-1}(\mathbb{Q})$ is a countable set dense in K_ε, and $\rho_\varepsilon^{-1}(p/q)$ is an interval with non-empty interior for every $p/q \in \mathbb{Q}$. Still, Herman himself proved in [31] that $\text{Leb}(K_\varepsilon) > 0$, but no quantitative estimates for this measure are known. A major difficulty is that, in the light of the previous properties, ρ_ε is not a C^1 function (ρ_ε' blows up in a dense set of points of K_ε). To deal with the task, we resort to an a-posteriori KAM formulation of the problem that combines local and global information and that we informally state as follows:

Theorem. *Let $A, B \subset \mathbb{R}$ be open intervals and $\alpha \in A \mapsto f_\alpha$ be a C^3-family of analytic circle diffeomorphisms. Let $\theta \in B \mapsto h_\theta$ be a Lipschitz family of analytic circle diffeomorphisms and let $\theta \in B \mapsto \alpha(\theta) \in A$ be a Lipschitz function. Under some mild and explicit conditions, if the family of error functions $\theta \in B \mapsto e_\theta$ given by*

$$e_\theta(x) = f_{\alpha(\theta)}(h_\theta(x)) - h_\theta(x + \theta)$$

is small enough, then there exist a Cantor set $\Theta \subset B$ of positive measure, a Lipschitz family of analytic circle diffeomorphisms $\theta \in \Theta \mapsto \bar{h}_\theta$ and a Lipschitz function $\theta \in \Theta \mapsto \bar{\alpha}(\theta) \in A$ such that

$$f_{\bar{\alpha}(\theta)}(\bar{h}_\theta(x)) = \bar{h}_\theta(x + \theta).$$

Moreover, the measure of conjugacies in the space of parameters, $\mathrm{Leb}(\bar{\alpha}(\Theta))$, *is controlled in terms of explicit estimates that depend only on the initial objects and Diophantine properties defining* Θ.

For the convenience of the reader, the above result is presented in two parts. In Section 3 (Theorem 3.1) we present an a-posteriori theorem for the existence of the conjugacy of a fixed rotation number. Then, in Section 4 (Theorem 4.1) we present an a-posteriori theorem to control the existence and measure of conjugacies in a given interval of rotations. Both results could be handled simultaneously, but this splitting is useful when the time comes to produce computer-assisted applications and also allows us to present the ideas in a self-consistent and more accessible way.

An important step to check the hypotheses of the theorem is to put on the ground a solid theory based on Lindstedt series that allows us to perform all necessary computations effectively in a computer-assisted proof. In Section 5 we describe in detail, using analytic arguments, how the hypotheses are thus reduced to a finite amount of computations which can be implemented systematically. In particular, we explain how to control the norms of Fourier-Taylor series using suitable discretizations and taking into account the corresponding remainders analytically. Indeed, the fact that Fourier-Taylor series can be manipulated using fast Fourier methods is *per se* a novel contribution in this paper, so one can outperform the use of symbolic manipulators.

As an illustration of the effectiveness of the methodology we consider the Arnold family (1.1). For example, we prove that

$$0.860748 < \mathrm{Leb}(K_{0.25}) < 0.914161 \,.$$

The lower bound follows from the computer assisted application of our main theorem and the upper bound is obtained by rigorous computation of p/q-periodic orbits up to $q = 20$. Details, and further results, are given in Section 6.

Regarding regularity, we have constrained our result to the analytic case, thus simplifying some intermediate estimates in the analytical part exposed in this paper, but also, this is convenient for the control of the error of Fourier approximations in the computer-assisted application of the method. This simplification only benefits the reader, since the selected problem contains all the technical difficulties associated to small divisors and illustrates very well the method proposed in this paper.

2. Notation and elementary results

We denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the real circle. We introduce a complex strip of $\mathbb{T}$ of width $\rho > 0$ as

$$\mathbb{T}_\rho = \{x \in \mathbb{C}/\mathbb{Z} \ : \ |\mathrm{Im}\, x| < \rho\} \,,$$

denote by $\bar{\mathbb{T}}_\rho$ its closure, and by $\partial \mathbb{T}_\rho = \{|\mathrm{Im}\, x| = \rho\}$ its boundary.

We denote by $\mathrm{Per}(\mathbb{T}_\rho)$ the Banach space of periodic continuous functions $f : \bar{\mathbb{T}}_\rho \to \mathbb{C}$, holomorphic in $\mathbb{T}_\rho$ and such that $f(\mathbb{T}) \subset \mathbb{R}$, endowed with the analytic norm

$$\|f\|_\rho := \sup_{x \in \mathbb{T}_\rho} |f(x)| \,.$$

We denote the Fourier series of a periodic function f as

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} k x}$$

and $\langle f \rangle := \hat{f}_0$ stands for the average.

In this paper we consider circle maps in the affine space

$$\mathcal{A}(\mathbb{T}_\rho) = \{ f \in \mathrm{Hom}(\mathbb{T}) \, , \, f - \mathrm{id} \in \mathrm{Per}(\mathbb{T}_\rho) \} \, .$$

Given an open set $A \subset \mathbb{R}$ and a family of maps $\alpha \in A \to f_\alpha \in \mathcal{A}(\mathbb{T}_\rho)$, we consider the norms

$$\| \partial_{x,\alpha}^{i,j} f \|_{A,\rho} := \sup_{\alpha \in A} \| \partial_{x,\alpha}^{i,j} f_\alpha \|_\rho \, , \qquad i + j > 0 \, ,$$

provided $\partial_{x,\alpha}^{i,j} f_\alpha \in \mathrm{Per}(\mathbb{T}_\rho)$ for every $\alpha \in A$.

We also introduce some useful notation regarding solutions of cohomological equations. Given a zero-average function η, we consider the linear difference equation

$$(2.1) \qquad\qquad \varphi(x + \theta) - \varphi(x) = \eta(x) \, .$$

If equation (2.1) has a unique zero-average solution, it will be denoted by $\mathcal{R}\eta$. To ensure existence and regularity of the solutions of this equation, some arithmetic conditions on the rotation number are required. Given $\gamma > 0$ and $\tau \geq 1$, the set of (γ, τ)-Diophantine numbers is given by

$$\mathcal{D}(\gamma, \tau) := \{ \theta \in \mathbb{R} \, : \, |q\theta - p| \geq \gamma |q|^{-\tau} \, , \, \forall (p, q) \in \mathbb{Z}^2 \, , \, q \neq 0 \}.$$

For the scope of this paper, the following classic lemma is enough.

Lemma 2.1 (Rüssmann estimates [55]). *Let $\theta \in \mathcal{D}(\gamma, \tau)$. Then, for every zero-average function $\eta \in \mathrm{Per}(\mathbb{T}_\rho)$, there exists a unique zero-average solution $\mathcal{R}\eta$ of equation (2.1) such that, for any $0 < \delta \leq \rho$, we have $\mathcal{R}\eta \in \mathrm{Per}(\mathbb{T}_{\rho-\delta})$ and*

$$\| \mathcal{R}\eta \|_{\rho-\delta} \leq \frac{c_R \|\eta\|_\rho}{\gamma \delta^\tau} \, , \qquad \text{with} \qquad c_R = \frac{\sqrt{\zeta(2, 2^\tau) \Gamma(2\tau + 1)}}{2(2\pi)^\tau} \, ,$$

where Γ and ζ are the Gamma and Hurwitz zeta functions, respectively.

Assume that f is a function defined in $\Theta \subset \mathbb{R}$ (not necessarily an interval) and taking values in $\mathbb{C}$. We say that f is Lipschitz in Θ if

$$\mathrm{Lip}_\Theta(f) := \sup_{\substack{\theta_1, \theta_2 \in \Theta \\ \theta_1 \neq \theta_2}} \frac{|f(\theta_2) - f(\theta_1)|}{|\theta_2 - \theta_1|} < \infty \, .$$

We define $\|f\|_\Theta := \sup_{\theta \in \Theta} |f(\theta)|$. Similarly, if we take a family $\theta \in \Theta \mapsto f_\theta \in \mathrm{Per}(\mathbb{T}_\rho)$, we extend the previous notations as

$$\mathrm{Lip}_{\Theta,\rho}(f) := \sup_{\substack{\theta_1, \theta_2 \in \Theta \\ \theta_1 \neq \theta_2}} \frac{\|f_{\theta_2} - f_{\theta_1}\|_\rho}{|\theta_2 - \theta_1|} \, , \qquad \|f\|_{\Theta,\rho} := \sup_{\theta \in \Theta} \|f_\theta\|_\rho.$$

Finally, we say that a function f is Lipschitz from below in Θ if

$$\mathrm{lip}_\Theta(f) := \inf_{\substack{\theta_1, \theta_2 \in \Theta \\ \theta_1 \neq \theta_2}} \frac{|f(\theta_2) - f(\theta_1)|}{|\theta_2 - \theta_1|} > 0 \, .$$

To obtain several estimates required in the paper, we will use the following elementary properties.

Lemma 2.2. *Assume that f, g are Lipschitz functions defined in $\Theta \subset \mathbb{R}$ and taking values in $\mathbb{C}$. Then*

$\mathfrak{P}_1$ $\mathrm{Lip}_\Theta(f + g) \leq \mathrm{Lip}_\Theta(f) + \mathrm{Lip}_\Theta(g),$

$\mathfrak{P}_2$ $\mathrm{Lip}_\Theta(fg) \leq \mathrm{Lip}_\Theta(f)\|g\|_\Theta + \|f\|_\Theta\mathrm{Lip}_\Theta(g),$

$\mathfrak{P}_3$ $\mathrm{Lip}_\Theta(f/g) \leq \mathrm{Lip}_\Theta(f)\|1/g\|_\Theta + \|f\|_\Theta\|1/g\|_\Theta^2\mathrm{Lip}_\Theta(g).$

Assume $0 < \rho \leq \hat{\rho}$, $\delta > 0$, $\rho - \delta > 0$, and that we have families $\theta \in \Theta \mapsto f_\theta \in \mathrm{Per}(\mathbb{T}_{\hat{\rho}})$ and $\theta \in \Theta \mapsto g_\theta \in \mathcal{A}(\mathbb{T}_\rho)$ such that $g_\theta(\bar{\mathbb{T}}_\rho) \subseteq \bar{\mathbb{T}}_{\hat{\rho}}$. Then

$\mathfrak{P}_4$ $\mathrm{Lip}_{\Theta,\rho}(f \circ g) \leq \mathrm{Lip}_{\Theta,\hat{\rho}}(f) + \|\partial_x f\|_{\Theta,\hat{\rho}}\mathrm{Lip}_{\Theta,\rho}(g - \mathrm{id}),$

$\mathfrak{P}_5$ $\mathrm{Lip}_{\Theta,\rho-\delta}(\partial_x f) \leq \frac{1}{\delta}\mathrm{Lip}_{\Theta,\rho}(f).$

Assume that we have a family $\theta \in \Theta \mapsto g_\theta \in \mathrm{Per}(\mathbb{T}_\rho)$, where $\Theta \subset \mathbb{R} \cap \mathcal{D}(\gamma,\tau)$. Then, if we denote $\mathcal{R}_\theta g_\theta = f_\theta$ the zero-average solution of $f_\theta(x+\theta) - f_\theta(x) = g_\theta(x)$ with $\theta \in \Theta$, we have

$\mathfrak{P}_6$ $\mathrm{Lip}_{\Theta,\rho-\delta}(\mathcal{R}g) \leq \frac{c_R}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(g) + \frac{\hat{c}_R}{\gamma^2\delta^{2\tau+1}}\|g\|_{\Theta,\rho}$, *where*

$$\hat{c}_R = \tau^{-2\tau}(2\tau+1)^{2\tau+1}c_R^2.$$

Proof. The properties $\mathfrak{P}_1$ to $\mathfrak{P}_5$ are standard. To obtain the last one, we must control the expression

$$\frac{\mathcal{R}_{\theta_2}g_{\theta_2} - \mathcal{R}_{\theta_1}g_{\theta_1}}{\theta_2 - \theta_1} = \frac{\mathcal{R}_{\theta_2}g_{\theta_2} - \mathcal{R}_{\theta_2}g_{\theta_1}}{\theta_2 - \theta_1} + \frac{\mathcal{R}_{\theta_2}g_{\theta_1} - \mathcal{R}_{\theta_1}g_{\theta_1}}{\theta_2 - \theta_1}.$$

The first term is controlled directly using Rüssmann estimates and the linearity of $\mathcal{R}_{\theta_2}$. To control the second term, we notice that

$$\frac{\mathcal{R}_{\theta_2}g_{\theta_1}(x) - \mathcal{R}_{\theta_1}g_{\theta_1}(x)}{\theta_2 - \theta_1} = \sum_{k \neq 0} \frac{e^{2\pi \mathrm{i}k\theta_1} - e^{2\pi \mathrm{i}k\theta_2}}{2\pi \mathrm{i}k(\theta_2 - \theta_1)} \frac{2\pi \mathrm{i}k\hat{g}_{k,\theta_1}\, e^{2\pi \mathrm{i}kx}}{(e^{2\pi \mathrm{i}k\theta_2} - 1)(e^{2\pi \mathrm{i}k\theta_1} - 1)},$$

which results in the following estimate

(2.2) $$\sup_{\substack{\theta_1,\theta_2 \in \Theta \\ \theta_1 \neq \theta_2}} \frac{\|\mathcal{R}_{\theta_2}g_{\theta_1} - \mathcal{R}_{\theta_1}g_{\theta_1}\|_{\rho-\delta}}{|\theta_2 - \theta_1|} \leq \sup_{\substack{\theta_1,\theta_2 \in \Theta \\ \theta_1 \neq \theta_2}} \|\mathcal{R}_{\theta_2}\mathcal{R}_{\theta_1}g_{\theta_1}'\|_{\rho-\delta},$$

where we used that

$$\left|\frac{e^{2\pi \mathrm{i}k\theta_1} - e^{2\pi \mathrm{i}k\theta_2}}{2\pi \mathrm{i}k(\theta_2 - \theta_1)}\right| = \frac{2|\sin(\pi k(\theta_2 - \theta_1))|}{2\pi|k(\theta_2 - \theta_1)|} \leq 1.$$

Finally, by applying twice Lemma 2.1, reducing the strip by $\tau\delta/(2\tau+1)$, and Cauchy estimates, reducing the strip by $\delta/(2\tau+1)$, we prove the claim. $\qquad\square$

3. An a-posteriori theorem for a single conjugation

Given a map $f \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$ with rotation number $\theta \in \mathcal{D}(\gamma,\tau)$, it is well-known that there exists an analytic circle diffeomorphism that conjugates f to a rigid rotation of angle θ (see [1, 32, 61] and [35, 37, 58, 36] for finite regularity). In this section, we present an a-posteriori result that allows constructing such conjugacy for a given map. For convenience, we express the result for a parametric family of circle maps $\alpha \in A \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$ and, given a target rotation number θ, we obtain a conjugacy $h_\infty \in \mathcal{A}(\mathbb{T}_{\rho_\infty})$ and a parameter $\alpha_\infty \in A$ (such that f_{α_∞} has rotation number θ).

Given circle maps f and h, we measure the error of conjugacy of f to the rigid rotation $R(x) = x + \theta$ through h by introducing the (periodic) error function

(3.1) $$e(x) := f(h(x)) - h(x + \theta).$$

The following a-posteriori result gives (explicit and quantitative) sufficient conditions to guarantee the existence of a true conjugacy close to an approximate one.

Theorem 3.1. *Consider an open set $A \subset \mathbb{R}$, a C^2-family $\alpha \in A \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$, with $\hat{\rho} > 0$; and a rotation number $\theta \in \mathbb{R}$ fulfilling:*

$\mathfrak{G}_1$ *For every $1 \le i + j \le 2$, there exist constants $c_{x,\alpha}^{i,j}$ such that $\|\partial_{x,\alpha}^{i,j} f_\alpha\|_{A,\hat{\rho}} \le c_{x,\alpha}^{i,j}$. For convenience we will write $c_x = c_{x,\alpha}^{1,0}$, $c_{x\alpha} = c_{x,\alpha}^{1,1}$, etc.*

$\mathfrak{G}_2$ *We have $\theta \in \mathcal{D}(\gamma, \tau)$.*

Assume that we have a pair $(h, \alpha) \in \mathcal{A}(\mathbb{T}_\rho) \times A$, with $\rho > 0$, fulfilling:

$\mathfrak{H}_1$ *The map h satisfies*

$$\mathrm{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) > 0,$$

and

(3.2) $$\langle h - \mathrm{id} \rangle = 0.$$

Moreover, there exist constants σ_1, and σ_2 such that

$$\|h'\|_\rho < \sigma_1, \qquad \|1/h'\|_\rho < \sigma_2.$$

$\mathfrak{H}_2$ *Given*

(3.3) $$b(x) := \frac{\partial_\alpha f_\alpha(h(x))}{h'(x + \theta)}$$

there exists a constant σ_b such that

$$|1/\langle b \rangle| < \sigma_b.$$

Then, for any $0 < \delta < \rho/2$ and $0 < \rho_\infty < \rho - 2\delta$, there exist explicit constants $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_3$ (depending on the previously defined constants) such that:

$\mathfrak{T}_1$ *Existence: If*

(3.4) $$\frac{\mathfrak{C}_1 \|e\|_\rho}{\gamma^2 \rho^{2\tau}} < 1,$$

with $e(x)$ given by (3.1), then there exists a pair $(h_\infty, \alpha_\infty) \in \mathcal{A}(\mathbb{T}_{\rho_\infty}) \times A$ such that $f_{\alpha_\infty}(h_\infty(x)) = h_\infty(x + \theta)$, that also satisfies $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

$\mathfrak{T}_2$ *Closeness: the pair $(h_\infty, \alpha_\infty)$ is close to the original one:*

(3.5) $$\|h_\infty - h\|_{\rho_\infty} < \frac{\mathfrak{C}_2 \|e\|_\rho}{\gamma \rho^\tau}, \qquad |\alpha_\infty - \alpha| < \mathfrak{C}_3 \|e\|_\rho.$$

The proof of this result is based on a numerical scheme, proposed in [19], to compute Arnold "tongues" with Diophantine rotation number. Indeed, the convergence details are adapted from [12], where the case of torus maps (associated to the inner dynamics of a toroidal normally hyperbolic manifold) is considered. We include here a short (but complete) proof not only for the sake of completeness, but to obtain explicit and sharp formulae for all the constants and conditions involved. Also, the proof of this result is the anteroom of the more involved a-posteriori result, discussed in the next section, which takes into account dependence on parameters.

The construction consists in correcting both the approximate conjugacy $h(x)$ and the parameter α. To this end, we perform an iterative process introducing the corrected objects $\bar{h}(x) = h(x) + \Delta_h(x)$ and $\bar{\alpha} = \alpha + \Delta_\alpha$. These corrections are determined by solving approximately the linearized equation

(3.6) $$\partial_x f_\alpha(h(x))\Delta_h(x) - \Delta_h(x + \theta) + \partial_\alpha f_\alpha(h(x))\Delta_\alpha = -e(x),$$

where the right-hand side is the conjugacy error (3.1). In order to ensure the normalization condition in (3.2), we look for a solution such that

$$\langle \Delta_h \rangle = 0 \,. \tag{3.7}$$

By taking derivatives on both sides of equation (3.1), we obtain

$$\partial_x f_\alpha(h(x))h'(x) - h'(x+\theta) = e'(x) \tag{3.8}$$

and consider the following transformation

$$\Delta_h(x) = h'(x)\varphi(x) \,. \tag{3.9}$$

Then, introducing (3.8) and (3.9) into (3.6), and neglecting the term $e'(x)\varphi(x)$ (we will see that this term is quadratic in the error) we obtain that the solution of equation (3.6) is approximated by the solution of a cohomological equation for φ

$$\varphi(x+\theta) - \varphi(x) = \eta(x) \,, \tag{3.10}$$

where the right-hand side is

$$\eta(x) := a(x) + b(x)\Delta_\alpha \,, \tag{3.11}$$

with

$$a(x) := \frac{e(x)}{h'(x+\theta)} \,, \tag{3.12}$$

and $b(x)$ is given by (3.3).

Equation (3.10) is solved by expressing $\eta(x)$ and $\varphi(x)$ in Fourier series. Then, we obtain a unique zero-average solution, denoted by $\varphi(x) = \mathcal{R}\eta(x)$, by taking

$$\Delta_\alpha = -\frac{\langle a \rangle}{\langle b \rangle} \,, \qquad \hat{\varphi}_k = \frac{\hat{\eta}_k}{e^{2\pi ik\theta} - 1} \,, \qquad k \neq 0 \,, \tag{3.13}$$

so that all solutions of (3.10) are of the form

$$\varphi(x) = \hat{\varphi}_0 + \mathcal{R}\eta(x) \,. \tag{3.14}$$

Finally, the average $\hat{\varphi}_0 = \langle \varphi \rangle$ is selected in order to fulfill condition (3.7). Since this condition is equivalent to $\langle h'\varphi \rangle = 0$ we readily obtain

$$\hat{\varphi}_0 = -\langle h'\mathcal{R}\eta \rangle \,. \tag{3.15}$$

The proof of Theorem 3.1 follows by applying the above correction iteratively (quasi-Newton method), thus obtaining a sequence of corrected objects. The norms of each correction and quantitative estimates for the new objects are controlled by applying the following result.

Lemma 3.2. *Consider an open set $A \subset \mathbb{R}$, a C^2-family $\alpha \in A \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$, with $\hat{\rho} > 0$; and a rotation number $\theta \in \mathbb{R}$ satisfying hypotheses $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in Theorem 3.1. Assume that we have a pair $(h, \alpha) \in \mathcal{A}(\mathbb{T}_\rho) \times A$ fulfilling hypotheses $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in the same theorem. If the following conditions on the error hold:*

$$\frac{\sigma_1 C_1}{\gamma \delta^\tau} \|e\|_\rho < \text{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) \,, \tag{3.16}$$

$$\sigma_b \sigma_2 \|e\|_\rho < \text{dist}\left(\alpha, \partial A\right) \,, \tag{3.17}$$

$$\frac{\sigma_1 C_1}{\gamma \delta^{\tau+1}} \|e\|_\rho < \sigma_1 - \|h'\|_\rho \,, \tag{3.18}$$

$$\frac{(\sigma_2)^2 \sigma_1 C_1}{\gamma \delta^{\tau+1}} \|e\|_\rho < \sigma_2 - \|1/h'\|_\rho \,, \tag{3.19}$$

$$(3.20) \qquad \frac{(\sigma_b)^2 C_2}{\gamma \delta^{\tau+1}} \|e\|_\rho < \sigma_b - |1/\langle b \rangle|\,,$$

where

$$(3.21) \qquad C_1 := (1 + \sigma_1) c_R (1 + c_\alpha \sigma_b \sigma_2) \sigma_2 \,,$$

$$(3.22) \qquad C_2 := \sigma_1 (\sigma_2)^2 c_\alpha C_1 + c_{x\alpha} \sigma_1 \sigma_2 C_1 \delta + c_{\alpha\alpha} \sigma_b (\sigma_2)^2 \gamma \delta^{\tau+1}\,,$$

then there exists a new pair $(\bar{h}, \bar{\alpha}) \in \mathcal{A}(\mathbb{T}_{\rho-\delta}) \times A$, *with* $\bar{h} = h + \Delta_h$ *and* $\bar{\alpha} = \alpha + \Delta_\alpha$, *satisfying also hypotheses* $\mathfrak{H}_1$ *(in the strip* $\mathbb{T}_{\rho-2\delta}$*) and* $\mathfrak{H}_2$*, and the estimates*

$$(3.23) \qquad |\bar{\alpha} - \alpha| = |\Delta_\alpha| < \sigma_b \sigma_2 \|e\|_\rho$$

and

$$(3.24) \qquad \|\bar{h} - h\|_{\rho-\delta} = \|\Delta_h\|_{\rho-\delta} < \frac{\sigma_1 C_1 \|e\|_\rho}{\gamma \delta^\tau}\,.$$

The new conjugacy error, $\bar{e}(x) = f_{\bar{\alpha}}(\bar{h}(x)) - \bar{h}(x + \theta)$*, satisfies*

$$(3.25) \qquad \|\bar{e}\|_{\rho-\delta} < \frac{C_3 \|e\|_\rho^2}{\gamma^2 \delta^{2\tau}}\,,$$

where

$$(3.26) \quad C_3 := C_1 \gamma \delta^{\tau-1} + \frac{1}{2} c_{xx} (\sigma_1 C_1)^2 + c_{x\alpha} \sigma_b \sigma_1 \sigma_2 C_1 \gamma \delta^\tau + \frac{1}{2} c_{\alpha\alpha} (\sigma_b \sigma_2)^2 \gamma^2 \delta^{2\tau}\,.$$

Proof. The result follows by controlling the norms of all the functions involved in the formal scheme described by equations (3.6) to (3.15). Using Cauchy estimates, we have $\|e'\|_{\rho-\delta} \leq \delta^{-1} \|e\|_\rho$. Using hypothesis $\mathfrak{H}_1$ we directly control the functions in (3.3) and (3.12)

$$(3.27) \qquad \|b\|_\rho \leq \|1/h'\|_\rho \|\partial_\alpha f_\alpha\|_{A,\hat{\rho}} < \sigma_2 c_\alpha\,, \qquad \|a\|_\rho \leq \|1/h'\|_\rho \|e\|_\rho < \sigma_2 \|e\|_\rho\,.$$

Then, using also $\mathfrak{H}_2$, the expression for Δ_α in (3.13) is controlled as

$$|\Delta_\alpha| \leq |1/\langle b \rangle| |\langle a \rangle| < \sigma_b \sigma_2 \|e\|_\rho\,,$$

thus obtaining (3.23). Hence we control the function $\eta(x)$ in (3.11) as

$$\|\eta\|_\rho \leq \|a\|_\rho + \|b\|_\rho |\Delta_\alpha| < (1 + c_\alpha \sigma_b \sigma_2) \sigma_2 \|e\|_\rho\,.$$

By decomposing $\varphi(x) = \hat{\varphi}_0 + \mathcal{R}\eta(x)$ and invoking Lemma 2.1, with hypothesis $\mathfrak{G}_2$, we obtain

$$(3.28) \qquad \|\mathcal{R}\eta\|_{\rho-\delta} \leq \frac{c_R}{\gamma \delta^\tau} \|\eta\|_\rho < \frac{c_R (1 + c_\alpha \sigma_b \sigma_2) \sigma_2}{\gamma \delta^\tau} \|e\|_\rho\,,$$

and we control the average $\hat{\varphi}_0$ using the expression (3.15), hypothesis $\mathfrak{H}_1$, and (3.28):

$$(3.29) \qquad |\hat{\varphi}_0| \leq \|h'\|_\rho \|\mathcal{R}\eta\|_{\rho-\delta} < \sigma_1 \frac{c_R (1 + c_\alpha \sigma_b \sigma_2) \sigma_2}{\gamma \delta^\tau} \|e\|_\rho\,.$$

By grouping expressions (3.28) and (3.29) we obtain

$$(3.30) \qquad \|\varphi\|_{\rho-\delta} \leq |\hat{\varphi}_0| + \|\mathcal{R}\eta\|_{\rho-\delta} < \frac{C_1}{\gamma \delta^\tau} \|e\|_\rho\,,$$

where C_1 is given in (3.21). Finally, using (3.9) and $\mathfrak{H}_1$, we obtain (3.24).

Now we control of the distance of the corrected objects to the boundaries of the domains, i.e. we ensure that $\mathrm{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) > 0$ and $\mathrm{dist}(\bar{\alpha}, \partial A) > 0$. Using the assumption in (3.16) we have

$$\mathrm{dist}\left(\bar{h}(\bar{\mathbb{T}}_{\rho-\delta}), \partial \mathbb{T}_{\hat{\rho}}\right) \geq \mathrm{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) - \|\Delta_h\|_{\rho-\delta}$$

$$> \operatorname{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) - \frac{\sigma_1 C_1}{\gamma \delta^\tau} \|e\|_\rho > 0\,,$$

and using the assumption in (3.17) we have

$$\operatorname{dist}\left(\bar{\alpha}, \partial A\right) \geq \operatorname{dist}\left(\alpha, \partial A\right) - |\Delta_\alpha| > \operatorname{dist}\left(\alpha, \partial A\right) - \sigma_b \sigma_2 \|e\|_\rho > 0\,.$$

Next we check that the new approximate conjugacy $\bar{h}(x) = h(x) + \Delta_h(x)$ satisfies $\mathfrak{H}_1$ in the strip $\mathbb{T}_{\rho-2\delta}$:

$$\|\bar{h}'\|_{\rho-2\delta} < \sigma_1\,, \qquad \|1/\bar{h}'\|_{\rho-2\delta} < \sigma_2\,, \qquad |1/\langle \bar{b}\rangle| < \sigma_b\,.$$

The first inequality in $\mathfrak{H}_1$ follows directly using Cauchy estimates in (3.24) and the assumption in (3.18):

$$(3.31) \qquad \|\bar{h}'\|_{\rho-2\delta} \leq \|h'\|_{\rho-2\delta} + \|\Delta_h'\|_{\rho-2\delta} < \|h'\|_\rho + \frac{\sigma_1 C_1}{\gamma \delta^{\tau+1}} \|e\|_\rho < \sigma_1\,.$$

The second inequality follows using Neumann series: in general, if $m \in \mathbb{C}$ satisfies $|1/m| < \sigma$ and $\bar{m} \in \mathbb{C}$ satisfies

$$(3.32) \qquad \frac{\sigma^2 |\bar{m} - m|}{\sigma - |1/m|} < 1\,,$$

then we have

$$(3.33) \qquad |1/\bar{m}| < \sigma\,, \qquad |1/\bar{m} - 1/m| < \sigma^2 |\bar{m} - m|\,.$$

Since hypothesis (3.19) implies

$$(3.34) \qquad \frac{(\sigma_2)^2 \|\bar{h}' - h'\|_{\rho-2\delta}}{\sigma_2 - \|1/h'\|_\rho} \leq \frac{(\sigma_2)^2 \sigma_1 C_1}{\sigma_2 - \|1/h'\|_\rho} \frac{\|e\|_\rho}{\gamma \delta^{\tau+1}} < 1\,,$$

from (3.32) and (3.33) we obtain

$$(3.35) \qquad \|1/\bar{h}' - 1/h'\|_{\rho-2\delta} < (\sigma_2)^2 \|\bar{h}' - h'\|_{\rho-2\delta} < \frac{(\sigma_2)^2 \sigma_1 C_1}{\gamma \delta^{\tau+1}} \|e\|_\rho\,,$$

and also that the control $\|1/\bar{h}'\|_{\rho-2\delta} < \sigma_2$ is preserved.

The control of $|1/\langle \bar{b}\rangle| < \sigma_b$ in $\mathfrak{H}_2$ is similar. To this end, we first write

$$\bar{b}(x) - b(x) = \frac{\partial_\alpha f_{\bar{\alpha}}(\bar{h}(x))}{\bar{h}'(x+\theta)} - \frac{\partial_\alpha f_{\bar{\alpha}}(h(x))}{\bar{h}'(x+\theta)} + \frac{\partial_\alpha f_{\bar{\alpha}}(h(x))}{\bar{h}'(x+\theta)} - \frac{\partial_\alpha f_\alpha(h(x))}{\bar{h}'(x+\theta)}$$
$$+ \frac{\partial_\alpha f_\alpha(h(x))}{\bar{h}'(x+\theta)} - \frac{\partial_\alpha f_\alpha(h(x))}{h'(x+\theta)}\,,$$

so, using (3.23), (3.24) and (3.35), we get the estimate

$$\left|\langle \bar{b}\rangle - \langle b\rangle\right| \leq c_{x\alpha} \sigma_2 \|\Delta_h\|_{\rho-\delta} + c_{\alpha\alpha} \sigma_2 |\Delta_\alpha| + c_\alpha \|1/\bar{h}' - 1/h'\|_{\rho-2\delta} < \frac{C_2}{\gamma \delta^{\tau+1}} \|e\|_\rho\,,$$

where C_2 is given by (3.22). Then, we repeat the computation in (3.34) using the assumption in (3.20). This provides the condition $|1/\langle \bar{b}\rangle| < \sigma_b$ in $\mathfrak{H}_2$.

Finally, the new error of invariance is given by

$$\bar{e}(x) = f_{\bar{\alpha}}(\bar{h}(x)) - \bar{h}(x+\theta) = e'(x)\varphi(x) + \Delta^2 f(x)\,,$$

where

$$\Delta^2 f(x) = f_{\bar{\alpha}}(\bar{h}(x)) - f_\alpha(h(x)) - \partial_x f_\alpha(h(x))\Delta_h(x) - \partial_\alpha f_\alpha(h(x))\Delta_\alpha$$

$$(3.36) \qquad = \int_0^1 (1-t)\left(G_{xx}(x)\Delta_h(x)^2 + 2G_{x\alpha}(x)\Delta_h(x)\Delta_\alpha + G_{\alpha\alpha}(x)\Delta_\alpha^2\right) dt\,,$$

and we use the notation $G_{xx}(x) := \partial_{xx} f_{\alpha+t\Delta_\alpha}(h(x)+t\Delta_h(x))$ and similar for $G_{x\alpha}(x)$ and $G_{\alpha\alpha}(x)$. Using $\mathfrak{G}_1$ and the estimates (3.23) and (3.24) for the corrections, we obtain the bound in (3.25). $\qquad\square$

Proof of Theorem 3.1. To initialize the iterative method, we introduce the notation $h_0 = h$, $\alpha_0 = \alpha$, and $e_0 = e$. Notice that Lemma 3.2 provides control of the analytic domains after each iteration. At the s-th iteration, we denote ρ_s the strip of analyticity (with $\rho_0 = \rho$) and δ_s the loss of strip produced at this step (with $\delta_0 = \delta$). Then, we take

$$(3.37) \qquad \rho_s = \rho_{s-1} - 2\delta_{s-1}, \qquad \delta_s = \frac{\delta_{s-1}}{a_1}, \qquad a_1 := \frac{\rho_0 - \rho_\infty}{\rho_0 - 2\delta_0 - \rho_\infty} > 1,$$

where ρ_∞ is the final strip. For convenience, we introduce the auxiliary constants

$$(3.38) \qquad a_2 = \frac{\rho_0}{\rho_\infty} > 1, \qquad a_3 = \frac{\rho_0}{\delta_0} > 2,$$

and observe that the following relation involving a_1, a_2, and a_3 holds

$$a_3 = 2\frac{a_1}{a_1 - 1}\frac{a_2}{a_2 - 1}.$$

In accordance to the previous notation, we denote by h_s, α_s, and e_s the objects at the s-step of the quasi-Newton method.

Existence: We proceed by induction, assuming that we have successfully applied s times Lemma 3.2. At this point, we use (3.25) to control the error of the last conjugacy in terms of the initial one:

$$(3.39) \qquad \begin{aligned} \|e_s\|_{\rho_s} &< \frac{C_3}{\gamma^2 \delta_{s-1}^{2\tau}} \|e_{s-1}\|_{\rho_{s-1}}^2 = \frac{C_3 a_1^{2\tau(s-1)}}{\gamma^2 \delta_0^{2\tau}} \|e_{s-1}\|_{\rho_{s-1}}^2 \\ &< \left(\frac{C_3 a_1^{2\tau}}{\gamma^2 \delta_0^{2\tau}} \|e_0\|_{\rho_0}\right)^{2^s - 1} a_1^{-2\tau s} \|e_0\|_{\rho_0}, \end{aligned}$$

where we used that $1+2+\ldots+2^{s-1} = 2^s - 1$ and $1(s-1)+2(s-2)+\ldots+2^{s-2}1 = 2^s - s - 1$. The above computation motivates the condition

$$(3.40) \qquad \kappa := \frac{C_3 a_1^{2\tau}}{\gamma^2 \delta_0^{2\tau}} \|e_0\|_{\rho_0} < 1,$$

which will be satisfied provided $\mathfrak{C}_1 \geq (a_1 a_3)^{2\tau} C_3$. Under this condition, the sum

$$(3.41) \qquad \Sigma_{\kappa,\lambda} := \sum_{j=0}^{\infty} \kappa^{2^j - 1} a_1^{-\lambda j}$$

is convergent for any $\lambda \in \mathbb{R}$. Along the proof we will need to guarantee a certain number of additional inequalities, so we will proceed by defining a value $\mathfrak{C}_1$ to "include" all of them simultaneously into condition (3.4).

Now, using expression (3.39), we check the inequalities in Lemma 3.2 (that is (3.16), (3.17), (3.18), (3.19), and (3.20)), so that we can perform the step $s + 1$. The required sufficient condition will be also included in (3.4). To simplify the computations, we consider that the constants C_1, C_2, and C_3 are evaluated at the

worst value of δ_s (that is δ_0) so that they can be taken to be equal at all steps. For example, the inequality (3.16) is obtained as follows:

$$\text{dist}\left(h_s(\bar{\mathbb{T}}_{\rho_s}), \partial\mathbb{T}_{\hat{\rho}}\right) - \frac{\sigma_1 C_1 \|e_s\|_{\rho_s}}{\gamma\delta_s^\tau} > \text{dist}\left(h_0(\bar{\mathbb{T}}_{\rho_0}), \partial\mathbb{T}_{\hat{\rho}}\right) - \sum_{j=0}^\infty \frac{\sigma_1 C_1 \|e_j\|_{\rho_j}}{\gamma\delta_j^\tau}$$

$$\tag{3.42} > \text{dist}\left(h_0(\bar{\mathbb{T}}_{\rho_0}), \partial\mathbb{T}_{\hat{\rho}}\right) - \frac{\sigma_1 C_1 \Sigma_{\kappa,\tau} \|e_0\|_{\rho_0}}{\gamma\delta_0^\tau} > 0\,,$$

where we used $h_s = h_{s-1} + \Delta_{h_{s-1}}$, (3.39) and (3.41). The last inequality in (3.42) is included in (3.4). The control of (3.17) is completely analogous:

$$\tag{3.43} \text{dist}\left(\alpha_s, \partial A\right) - \sigma_2\sigma_1\|e_s\|_{\rho_s} > \text{dist}\left(\alpha_0, \partial A\right) - \sigma_2\sigma_1\Sigma_{\kappa,2\tau}\|e_0\|_{\rho_0} > 0\,,$$

and the last inequality in (3.43) is included in (3.4). The condition in (3.18) is obtained using $h_s = h_{s-1} + \Delta_{h_{s-1}}$ and (3.39):

$$\|h_s'\|_{\rho_s} + \frac{\sigma_1 C_1}{\gamma\delta_s^{\tau+1}}\|e_s\|_{\rho_s} < \|h_0'\|_{\rho_0} + \sum_{j=0}^s \frac{\sigma_1 C_1}{\gamma\delta_j^{\tau+1}}\|e_j\|_{\rho_j}$$

$$\tag{3.44} < \|h_0'\|_{\rho_0} + \frac{\sigma_1 C_1 \Sigma_{\kappa,\tau-1}}{\gamma\delta_0^{\tau+1}}\|e_0\|_{\rho_0} < \sigma_1\,,$$

and the last inequality in (3.44) is included in (3.4). Analogous computations allow us to guarantee the conditions in (3.19) and (3.20) for $\|1/h_s'\|_{\rho_s}$ and $|1/\langle b_s\rangle|$, respectively. To this end, we also include in (3.4) the inequalities

$$\tag{3.45} \|1/h_0'\|_{\rho_0} + \frac{(\sigma_2)^2\sigma_1 C_1 \Sigma_{\kappa,\tau-1}}{\gamma\delta_0^{\tau+1}}\|e_0\|_{\rho_0} < \sigma_2\,,$$

$$\tag{3.46} |1/\langle b_0\rangle| + \frac{(\sigma_b)^2 C_2 \Sigma_{\kappa,\tau-1}}{\gamma\delta_0^{\tau+1}}\|e_0\|_{\rho_0} < \sigma_b\,.$$

Putting together the assumptions in (3.40), (3.42), (3.43), (3.44), (3.45), (3.46), and recalling that $\rho/\delta = \rho_0/\delta_0 = a_3$, $h = h_0$, $\alpha = \alpha_0$, we end up with

$$\tag{3.47} \mathfrak{C}_1 := \begin{cases} \max\left\{(a_1 a_3)^{2\tau}C_3, (a_3)^{\tau+1}C_4\gamma\rho^{\tau-1}\right\} & \text{if } \kappa < 1\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

where κ is given by (3.40) and

$$\tag{3.48} C_4 := \max\left\{ \frac{\sigma_1 C_1 \delta \Sigma_{\kappa,\tau}}{\text{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial\mathbb{T}_{\hat{\rho}}\right)}, \frac{\sigma_2\sigma_1 \Sigma_{\kappa,2\tau}\gamma\delta^{\tau+1}}{\text{dist}\left(\alpha, \partial A\right)}, \right.$$

$$\left. \frac{\sigma_1 C_1 \Sigma_{\kappa,\tau-1}}{\sigma_1 - \|h'\|_\rho}, \frac{(\sigma_2)^2\sigma_1 C_1 \Sigma_{\kappa,\tau-1}}{\sigma_2 - \|1/h'\|_\rho}, \frac{(\sigma_b)^2 C_2 \Sigma_{\kappa,\tau-1}}{\sigma_b - |1/\langle b\rangle|} \right\}.$$

As a consequence of the above computations, we can apply Lemma 3.2 again. By induction, we obtain a convergent sequence $\|e_s\|_{\rho_s} \to 0$ when $s \to \infty$. Conclusively, the iterative scheme converges to a true conjugacy $h_\infty \in \mathcal{A}(\mathbb{T}_{\rho_\infty})$ for the map $f_\infty = f_{\alpha_\infty}$.

Closeness: The above computations also prove that the conjugacy h_∞ and the parameter α_∞ are close to the initial objects:

$$\|h_\infty - h\|_{\rho_\infty} < \sum_{j=0}^\infty \|\Delta_{h_j}\|_{\rho_j} < \sum_{j=0}^\infty \frac{\sigma_1 C_1 \|e_j\|_{\rho_j}}{\gamma\delta_j^\tau} < \frac{\sigma_1 C_1 \Sigma_{\kappa,\tau}}{\gamma\delta_0^\tau}\|e_0\|_{\rho_0}\,,$$

$$|\alpha_\infty - \alpha| < \sigma_b \sigma_2 \sum_{j=0}^{\infty} \|e_j\|_{\rho_j} < \sigma_b \sigma_2 \Sigma_{\kappa,2\tau} \|e_0\|_{\rho_0} \,,$$

and we finally obtain

$$(3.49) \qquad \mathfrak{C}_2 = a_3^\tau \sigma_1 C_1 \Sigma_{\kappa,\tau} \,, \qquad \mathfrak{C}_3 = \sigma_b \sigma_2 \Sigma_{\kappa,2\tau} \,. \qquad \square$$

Remark 3.3. Here we summarize how to compute the constants $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ in Theorem 3.1. Given fixed values of the parameters $\rho, \delta, \rho_\infty, \hat{\rho}$ and the distances $\mathrm{dist}\left(h(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right)$ and $\mathrm{dist}\,(\alpha, \partial A)$; the constants c_x, c_α, c_{xx}, $c_{x\alpha}$, $c_{\alpha\alpha}$ in hypothesis $\mathfrak{G}_1$; the constants γ and τ in hypothesis $\mathfrak{G}_2$; the constants σ_1, σ_2 in hypothesis $\mathfrak{H}_1$; and the constant σ_b in hypothesis $\mathfrak{H}_2$, these are computed in the following order:

- a_1, a_2, a_3 using (3.37) and (3.38).
- C_1, C_2, C_3 using (3.21), (3.22) and (3.26).
- κ using (3.40) and check that $\kappa < 1$ (abort the process otherwise).
- $\Sigma_{\kappa,\tau}$, $\Sigma_{\kappa,2\tau}$, $\Sigma_{\kappa,\tau-1}$ using (3.41).
- C_4 using (3.48).
- $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ using (3.47) and (3.49).

4. AN A-POSTERIORI THEOREM FOR THE MEASURE OF CONJUGATIONS

In this section we present an a-posteriori result that extends Theorem 3.1 considering dependence on parameters. The statement is a detailed version of the theorem that was informally exposed in the introduction of the paper. Our aim is to use a global approximation of the conjugacies to rotation, within a range of rotation numbers, in order to obtain a lower bound of the measure of the set of parameters for which a true conjugation exists.

Theorem 4.1. *Consider an open set $A \subset \mathbb{R}$, a C^3-family $\alpha \in A \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$, with $\hat{\rho} > 0$, and a set $\Theta \subset \mathbb{R}$ fulfilling:*

$\mathfrak{G}_1$ *For every $1 \le i + j \le 3$, there exist constants $c_{x,\alpha}^{i,j}$ such that $\|\partial_{x,\alpha}^{i,j} f_\alpha\|_{A,\hat{\rho}} \le c_{x,\alpha}^{i,j}$. For convenience we will write $c_x = c_{x,\alpha}^{1,0}$, $c_{x\alpha} = c_{x,\alpha}^{1,1}$, $c_{\alpha\alpha\alpha} = c_{x,\alpha}^{0,3}$, etc.*

$\mathfrak{G}_2$ *We have $\Theta \subset \mathcal{D}(\gamma, \tau)$.*

Assume that we have a family of pairs

$$\theta \in \Theta \longmapsto (h_\theta, \alpha(\theta)) \in \mathcal{A}(\mathbb{T}_\rho) \times A \,,$$

with $\rho > 0$, such that the following quantitative estimates are satisfied:

$\mathfrak{H}_1$ *For every $\theta \in \Theta$ the map h_θ satisfies*

$$\mathrm{dist}\left(h_\theta(\bar{\mathbb{T}}_\rho), \partial \mathbb{T}_{\hat{\rho}}\right) > 0 \,,$$

and

$$(4.1) \qquad \langle h_\theta - \mathrm{id} \rangle = 0 \,.$$

Moreover, there exist constants σ_1, σ_2 and σ_3 such that

$$\|\partial_x h\|_{\Theta,\rho} < \sigma_1 \,, \qquad \|1/\partial_x h\|_{\Theta,\rho} < \sigma_2 \,, \qquad \|\partial_{xx} h\|_{\Theta,\rho} < \sigma_3 \,.$$

$\mathfrak{H}_2$ *The average of the family of functions*

$$(4.2) \qquad b_\theta(x) := \frac{\partial_\alpha f_{\alpha(\theta)}(h_\theta(x))}{\partial_x h_\theta(x + \theta)}$$

is different from zero for every $\theta \in \Theta$. Moreover, there exist a constant σ_b such that

$$\|1/\langle b \rangle\|_\Theta < \sigma_b \,.$$

$\mathfrak{H}_3$ *There exist constants β_0, β_1, and β_2 such that*

$$\mathrm{Lip}_{\Theta,\rho}(h - \mathrm{id}) < \beta_0 \,, \qquad \mathrm{Lip}_{\Theta,\rho}(\partial_x h) < \beta_1 \,, \qquad \mathrm{Lip}_\Theta(\alpha) < \beta_2 \,.$$

Then, for any $0 < \delta < \rho/2$ and $0 < \rho_\infty < \rho - 2\delta$, there exist constants $\mathfrak{C}_1^{\mathrm{Lip}}$ and $\mathfrak{C}_2^{\mathrm{Lip}}$ such that:

$\mathfrak{T}_1$ *Existence of conjugations: If*

(4.3)
$$\frac{\mathfrak{C}_1^{\mathrm{Lip}} \max \left\{ \|e\|_{\Theta,\rho}, \, \gamma\delta^{\tau+1}\mathrm{Lip}_{\Theta,\rho}(e) \right\}}{\gamma^2 \rho^{2\tau+2}} < 1 \,,$$

where

$$e_\theta(x) := f_{\alpha(\theta)}(h_\theta(x)) - h_\theta(x + \theta)$$

is the associated family of error functions, then there exists a family of pairs $\theta \in \Theta \mapsto (h_{\theta,\infty}, \alpha_\infty(\theta)) \in \mathcal{A}(\mathbb{T}_{\rho_\infty}) \times A$ such that

$$f_{\alpha_\infty(\theta)}(h_{\theta,\infty}(x)) = h_{\theta,\infty}(x + \theta) \,, \qquad \forall \theta \in \Theta$$

and also satisfying $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$.

$\mathfrak{T}_2$ *Closeness: the family $\theta \in \Theta \mapsto (h_{\theta,\infty}, \alpha_\infty(\theta))$ is close to the original one:*

(4.4)
$$\|h_\infty - h\|_{\Theta,\rho_\infty} < \frac{\mathfrak{C}_2\|e\|_{\Theta,\rho}}{\gamma\rho^\tau} \,, \qquad \|\alpha_\infty - \alpha\|_\Theta < \mathfrak{C}_3\|e\|_{\Theta,\rho} \,,$$

where the constants $\mathfrak{C}_2$ and $\mathfrak{C}_3$ are computed in the same way as the analogous constants in Theorem 3.1, i.e., are given by (3.49).

$\mathfrak{T}_3$ *Measure of rotations: the Lebesgue measure of conjucacies to rigid rotation in the space of parameters is bounded from below as follows*

$$\mathrm{Leb}(\alpha_\infty(\Theta)) > \left[\mathrm{lip}_\Theta(\alpha) - \frac{\mathfrak{C}_2^{\mathrm{Lip}} \max \left\{ \|e\|_{\Theta,\rho}, \, \gamma\delta^{\tau+1}\mathrm{Lip}_{\Theta,\rho}(e) \right\}}{\gamma\rho^{\tau+1}} \right] \mathrm{Leb}(\Theta) \,.$$

Remark 4.2. Notice that hypothesis $\mathfrak{H}_2$ plays the role of a *twist condition*, guaranteeing that each rotation number in the domain appears exactly one. If this condition is violated in a particular example, then the interval A should be divided into subintervals and apply the theorem in each of them.

Remark 4.3. Notice that, replacing the norms $\|\cdot\|_\rho$ and $|\cdot|$ by $\|\cdot\|_{\Theta,\rho}$ and $\|\cdot\|_\Theta$ respectively, we have that part of the hypotheses of Theorem 4.1 are in common with those of Theorem 3.1. Taking this into account, we will omit the details associated to the control of the norms $\|\cdot\|_\rho$ and $|\cdot|$, since we can mimic these estimates from those obtained in Section 3.

The proof of this result follows by adapting the construction presented in Section 3, but controlling the Lipschitz constants of the objects involved in the iterative scheme. Notice that our main interest is to show that α_∞ is Lipschitz from below and to obtain sharp lower bounds. To this end, using that $\alpha_s = \alpha_0 + \alpha_s - \alpha_0$, we have that

$$\mathrm{lip}_\Theta(\alpha_s) > \mathrm{lip}_\Theta(\alpha_0) - \mathrm{Lip}_\Theta(\alpha_s - \alpha_0) \,,$$

so the effort is focused in controlling the Lipschitz constants from above.

The following result provides quantitative estimates for the norm of the corrected objects.

Lemma 4.4. *Consider an open set $A \subset \mathbb{R}$, a C^3-family $\alpha \in A \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_{\hat{\rho}})$, with $\hat{\rho} > 0$, and a set $\Theta \subset \mathbb{R}$ satisfying the hypotheses $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of Theorem 4.1. Assume that we have a family of pairs*

$$\theta \in \Theta \longmapsto (h_\theta, \alpha(\theta)) \in \mathcal{A}(\mathbb{T}_\rho) \times A$$

fulfilling hypotheses $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ of the same theorem. Assume that, adapting the expressions for the norms $\|\cdot\|_{\Theta,\rho}$ and $\|\cdot\|_\Theta$, the family of errors $\theta \in \Theta \mapsto e_\theta$ satisfies conditions like (3.16), (3.17), (3.18), (3.19), (3.20), and the additional conditions:

$$(4.5) \qquad \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+1}} \|e\|_{\Theta,\rho} + \frac{\sigma_1 C_1}{\gamma \delta^\tau} \mathrm{Lip}_{\Theta,\rho}(e) < \beta_0 - \mathrm{Lip}_{\Theta,\rho}(h - \mathrm{id}),$$

$$(4.6) \qquad \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+2}} \|e\|_{\Theta,\rho} + \frac{\sigma_1 C_1}{\gamma \delta^{\tau+1}} \mathrm{Lip}_{\Theta,\rho}(e) < \beta_1 - \mathrm{Lip}_{\Theta,\rho}(\partial_x h),$$

$$(4.7) \qquad C_0^{\mathrm{Lip}} \|e\|_{\Theta,\rho} + \sigma_b \sigma_2 \mathrm{Lip}_{\Theta,\rho}(e) < \beta_2 - \mathrm{Lip}_\Theta(\alpha),$$

$$(4.8) \qquad \frac{2\sigma_1 C_1}{\gamma \delta^{\tau+2}} \|e\|_{\Theta,\rho} < \sigma_3 - \|\partial_{xx} h\|_{\Theta,\rho},$$

where constant C_1 is computed as in (3.21) and we introduce the new constants

$$(4.9) \qquad C_0^{\mathrm{Lip}} := \sigma_b(\sigma_2)^2(\beta_1 + \sigma_3)(1 + c_\alpha \sigma_b \sigma_2) + (\sigma_2)^2(\sigma_b)^2[c_{\alpha\alpha}\beta_2 + c_{x\alpha}\beta_0]$$

$$(4.10) \qquad C_1^{\mathrm{Lip}} := \sigma_2[(c_{\alpha\alpha}\beta_2 + c_{x\alpha}\beta_0)\sigma_b\sigma_2 + c_\alpha C_0^{\mathrm{Lip}} + (1 + c_\alpha\sigma_b\sigma_2)\sigma_2(\beta_1 + \sigma_3)],$$

$$(4.11) \qquad C_2^{\mathrm{Lip}} := c_R C_1^{\mathrm{Lip}}\gamma\delta^{\tau+1} + \hat{c}_R(1 + c_\alpha\sigma_b\sigma_2)\sigma_2,$$

$$(4.12) \qquad C_3^{\mathrm{Lip}} := C_2^{\mathrm{Lip}}(\sigma_1 + 1) + \beta_1 c_R(1 + c_\alpha\sigma_b\sigma_2)\sigma_2\gamma\delta^{\tau+1},$$

$$(4.13) \qquad C_4^{\mathrm{Lip}} := \beta_1 C_1\gamma\delta^{\tau+1} + \sigma_1 C_3^{\mathrm{Lip}}.$$

Then, there is a new family $\theta \in \Theta \mapsto \bar{h}_\theta \in \mathcal{A}(\mathbb{T}_{\rho-\delta})$, with $\bar{h}_\theta = h_\theta + \Delta_{h_\theta}$, and a new function $\theta \in \Theta \mapsto \bar{\alpha} \in A$, with $\bar{\alpha}(\theta) = \alpha(\theta) + \Delta_\alpha(\theta)$, satisfying also hypotheses $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ of Theorem 4.1 in the strip $\mathbb{T}_{\rho-2\delta}$; and the estimates

$$(4.14) \qquad \|\bar{\alpha} - \alpha\|_\Theta = \|\Delta_\alpha\|_\Theta < \sigma_b\sigma_2\|e\|_{\Theta,\rho},$$

$$(4.15) \qquad \|\bar{h} - h\|_{\Theta,\rho-\delta} = \|\Delta_h\|_{\Theta,\rho-\delta} < \frac{\sigma_1 C_1 \|e\|_{\Theta,\rho}}{\gamma\delta^\tau},$$

$$(4.16) \qquad \mathrm{Lip}_\Theta(\Delta_\alpha) < C_0^{\mathrm{Lip}}\|e\|_{\Theta,\rho} + \sigma_b\sigma_2\mathrm{Lip}_{\Theta,\rho}(e),$$

and

$$(4.17) \qquad \mathrm{Lip}_{\Theta,\rho-\delta}(\Delta_h) < \frac{C_4^{\mathrm{Lip}}}{\gamma^2\delta^{2\tau+1}}\|e\|_{\Theta,\rho} + \frac{\sigma_1 C_1}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(e).$$

The new error $\bar{e}_\theta(x) = f_{\bar{\alpha}(\theta)}(\bar{h}_\theta(x)) - \bar{h}_\theta(x + \theta)$, satisfies

$$(4.18) \qquad \|\bar{e}\|_{\Theta,\rho-\delta} < \frac{C_3\|e\|^2_{\Theta,\rho}}{\gamma^2\delta^{2\tau}},$$

and

$$(4.19) \qquad \mathrm{Lip}_{\Theta,\rho-\delta}(\bar{e}) < \frac{C_5^{\mathrm{Lip}}\|e\|^2_{\Theta,\rho}}{\gamma^3\delta^{3\tau+1}} + \frac{2C_3\mathrm{Lip}_{\Theta,\rho}(e)\|e\|_{\Theta,\rho}}{\gamma^2\delta^{2\tau}},$$

where C_3 is computed as in (3.26) and

$$(4.20) \qquad C_5^{\mathrm{Lip}} := C_3^{\mathrm{Lip}}\gamma\delta^{\tau-1} + \tfrac{1}{2}(c_{xx\alpha}\beta_2 + c_{xxx}\beta_0)(\sigma_1 C_1)^2\gamma\delta^{\tau+1}$$

$$+ (c_{x\alpha\alpha}\beta_2 + c_{xx\alpha}\beta_0)\sigma_1 C_1 \sigma_b \sigma_2 \gamma^2 \delta^{2\tau+1}$$
$$+ \tfrac{1}{2}(c_{\alpha\alpha\alpha}\beta_2 + c_{x\alpha\alpha}\beta_0)(\sigma_b\sigma_2)^2 \gamma^3 \delta^{3\tau+1}$$
$$+ C_4^{\mathrm{Lip}}(c_{xx}\sigma_1 C_1 + c_{x\alpha}\sigma_b\sigma_2\gamma\delta^\tau)$$
$$+ C_0^{\mathrm{Lip}}\gamma^2\delta^{2\tau+1}(c_{x\alpha}\sigma_1 C_1 + c_{\alpha\alpha}\sigma_b\sigma_2\gamma\delta^\tau),$$

Proof. Following the proof of Lemma 3.2 we consider the objects

$$(4.21) \qquad \eta_\theta(x) := a_\theta(x) + b_\theta(x)\Delta_\alpha(\theta), \qquad \varphi_\theta(x) = \hat\varphi_0(\theta) + \mathcal{R}_\theta \eta_\theta(x),$$

where

$$(4.22) \qquad a_\theta(x) = \frac{e_\theta(x)}{\partial_x h_\theta(x+\theta)}, \qquad \Delta_\alpha(\theta) = -\frac{\langle a_\theta \rangle}{\langle b_\theta \rangle},$$

and $b_\theta(x)$ is given by (4.2). Since we are assuming the hypotheses of Lemma 3.2, we can use the intermediate estimates obtained in the proof of that result (see Remark 4.3). In particular, the estimates in (4.14), (4.15), (4.18) and the control on the distances

$$\mathrm{dist}\left(\bar h_\theta(\bar{\mathbb{T}}_{\rho-\delta}), \partial\mathbb{T}_{\hat\rho}\right) > 0, \qquad \mathrm{dist}\left(\bar\alpha(\theta), \partial A\right) > 0,$$

follow straightly.

Using property $\mathfrak{P}_5$ (of Lemma 2.2) we readily obtain that

$$(4.23) \qquad \mathrm{Lip}_{\Theta,\rho-\delta}(\partial_x e) \leq \frac{\mathrm{Lip}_{\Theta,\rho}(e)}{\delta}.$$

Then, we control the function $\Delta_\alpha(\theta)$ in (4.22) using property $\mathfrak{P}_3$ and hypotheses $\mathfrak{H}_1, \mathfrak{H}_2$

$$\mathrm{Lip}_\Theta(\Delta_\alpha) \leq \mathrm{Lip}_{\Theta,\rho}(a)\,\|1/\langle b\rangle\|_\Theta + \|a\|_{\Theta,\rho}\,\|1/\langle b\rangle\|_\Theta^2\,\mathrm{Lip}_\Theta(b)$$
$$\leq \sigma_b\mathrm{Lip}_{\Theta,\rho}(a) + \|e\|_{\Theta,\rho}\sigma_2(\sigma_b)^2\mathrm{Lip}_{\Theta,\rho}(b).$$

Now we control the Lipschitz bound of the map $a_\theta(x)$ in (4.22), writing the rigid rotation as $R_\theta(x) = x + \theta$ and using $\mathfrak{P}_3, \mathfrak{P}_4$, and $\mathfrak{H}_3$:

$$\mathrm{Lip}_{\Theta,\rho}(a) \leq \mathrm{Lip}_{\Theta,\rho}(e)\|1/\partial_x h\|_{\Theta,\rho} + \|e\|_{\Theta,\rho}\mathrm{Lip}_{\Theta,\rho}(1/\partial_x h \circ R)$$
$$\leq \mathrm{Lip}_{\Theta,\rho}(e)\sigma_2 + \|e\|_{\Theta,\rho}\left(\mathrm{Lip}_{\Theta,\rho}(1/\partial_x h) + \|\partial_x(1/\partial_x h)\|_{\Theta,\rho}\mathrm{Lip}_{\Theta,\rho}(R-\mathrm{id})\right)$$
$$(4.24) \qquad \leq \mathrm{Lip}_{\Theta,\rho}(e)\sigma_2 + \|e\|_{\Theta,\rho}(\sigma_2)^2(\beta_1 + \sigma_3),$$

where we used that $\mathrm{Lip}_{\Theta,\rho}(R) = 1$,

$$(4.25) \qquad \mathrm{Lip}_{\Theta,\rho}(1/\partial_x h) \leq \|1/\partial_x h\|_{\Theta,\rho}^2\mathrm{Lip}_{\Theta,\rho}(\partial_x h) < (\sigma_2)^2\beta_1,$$

and

$$\|\partial_x(1/\partial_x h)\|_{\Theta,\rho} \leq \|1/\partial_x h\|_{\Theta,\rho}^2\|\partial_{xx}h\|_{\Theta,\rho} < (\sigma_2)^2\sigma_3.$$

The Lipschitz control of the function $b_\theta(x)$ is analogous:

$$\mathrm{Lip}_{\Theta,\rho}(b) \leq \mathrm{Lip}_{\Theta,\rho}(\partial_\alpha f_\alpha \circ h)\|1/\partial_x h\|_{\Theta,\rho} + \|\partial_\alpha f_\alpha\|_{\Theta,\hat\rho}\mathrm{Lip}_{\Theta,\rho}(1/\partial_x h \circ R)$$
$$(4.26) \qquad \leq (c_{\alpha\alpha}\beta_2 + c_{x\alpha}\beta_0)\sigma_2 + c_\alpha(\sigma_2)^2(\beta_1 + \sigma_3).$$

Putting together the above computations we obtain (4.16) with the constant C_0^{Lip} in (4.9).

The control of η_θ, given in (4.21), yields to

$$\mathrm{Lip}_{\Theta,\rho}(\eta) \leq \mathrm{Lip}_{\Theta,\rho}(a) + \mathrm{Lip}_{\Theta,\rho}(b)\|\Delta_\alpha\|_\Theta + \|b\|_{\Theta,\rho}\mathrm{Lip}_\Theta(\Delta_\alpha),$$

where we used properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Using the previous estimates in (3.27), (4.14), (4.16), (4.24), (4.26), $\mathfrak{G}_1$, $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and the expression of C_1^{Lip} in (4.10), we obtain

$$\mathrm{Lip}_{\Theta,\rho}(\eta) \le C_1^{\mathrm{Lip}}\|e\|_{\Theta,\rho} + (1 + c_\alpha \sigma_b \sigma_2)\sigma_2 \mathrm{Lip}_{\Theta,\rho}(e) \, ,$$

where C_1^{Lip} is given in (4.10). Then, we estimate the Lipschitz constant of the zero-average solution $\mathcal{R}_\theta \eta_\theta(x)$ using $\mathfrak{P}_6$ and $\mathfrak{G}_2$

$$(4.27) \qquad \mathrm{Lip}_{\Theta,\rho-\delta}(\mathcal{R}\eta) \le \frac{C_2^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+1}}\|e\|_{\Theta,\rho} + \frac{c_R(1 + c_\alpha \sigma_b \sigma_2)\sigma_2}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(e) \, .$$

Next, we control the average $\hat{\varphi}_0(\theta) = -\langle \partial_x h_\theta \mathcal{R}\eta_\theta \rangle$ (we use $\mathfrak{H}_1$, $\mathfrak{H}_3$ and the estimates (3.28) and (4.27))

$$\mathrm{Lip}_\Theta(\hat{\varphi}_0) \le \mathrm{Lip}_{\Theta,\rho}(\partial_x h)\|\mathcal{R}\eta\|_{\Theta,\rho-\delta} + \|\partial_x h\|_{\Theta,\rho}\mathrm{Lip}_{\Theta,\rho-\delta}(\mathcal{R}\eta)$$

$$\le \frac{\beta_1 c_R(1 + c_\alpha \sigma_b \sigma_2)\sigma_2 \gamma \delta^{\tau+1} + \sigma_1 C_2^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+1}}\|e\|_{\Theta,\rho}$$

$$+ \frac{\sigma_1 c_R(1 + c_\alpha \sigma_b \sigma_2)\sigma_2}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(e) \, ,$$

and we put together the previous two expressions as follows (using property $\mathfrak{P}_1$):

$$(4.28) \quad \mathrm{Lip}_{\Theta,\rho-\delta}(\varphi) \le \mathrm{Lip}_\Theta(\hat{\varphi}_0) + \mathrm{Lip}_{\Theta,\rho-\delta}(\mathcal{R}\eta) =: \frac{C_3^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+1}}\|e\|_{\Theta,\rho} + \frac{C_1}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(e) \, ,$$

where C_3^{Lip} is given by (4.12) and C_1 is given by (3.21).

Then, the estimate in (4.17) is straightforward using that $\Delta_{h_\theta}(x) = \partial_x h_\theta(x)\varphi_\theta(x)$, the estimates (3.30) and (4.28), together with $\mathfrak{P}_2$, $\mathfrak{H}_1$ and $\mathfrak{H}_3$. In addition, we control the new maps $\bar{h}_\theta(x) = h_\theta(x) + \Delta_{h_\theta}(x)$ as

$$\mathrm{Lip}_{\Theta,\rho-\delta}(\bar{h} - \mathrm{id}) \le \mathrm{Lip}_{\Theta,\rho}(h - \mathrm{id}) + \mathrm{Lip}_{\Theta,\rho-\delta}(\Delta_h)$$

$$\le \mathrm{Lip}_{\Theta,\rho}(h - \mathrm{id}) + \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+1}}\|e\|_{\Theta,\rho} + \frac{\sigma_1 C_1}{\gamma\delta^\tau}\mathrm{Lip}_{\Theta,\rho}(e) < \beta_0 \, ,$$

where we used hypothesis (4.5). The control of $\partial_x \bar{h}_\theta(x)$ is analogous:

$$\mathrm{Lip}_{\Theta,\rho-2\delta}(\partial_x \bar{h}) \le \mathrm{Lip}_{\Theta,\rho}(\partial_x h) + \mathrm{Lip}_{\Theta,\rho-2\delta}(\partial_x \Delta_h)$$

$$\le \mathrm{Lip}_{\Theta,\rho}(\partial_x h) + \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+2}}\|e\|_{\Theta,\rho} + \frac{\sigma_1 C_1}{\gamma\delta^{\tau+1}}\mathrm{Lip}_{\Theta,\rho}(e) < \beta_1 \, ,$$

where we used hypothesis (4.6). An analogous computation shows that the smallness hypothesis in (4.7) guarantees that $\mathrm{Lip}_\Theta(\bar{\alpha}) < \beta_2$. Then, the smallness hypothesis in (4.8) guarantees that $\|\partial_{xx}\bar{h}\|_{\theta,\rho-2\delta} < \sigma_3$. The computations are similar to (3.31):

$$\|\partial_{xx}\bar{h}\|_{\rho-2\delta} \le \|\partial_{xx}h\|_{\rho-2\delta} + \|\partial_{xx}\Delta_h\|_{\rho-2\delta} < \|\partial_{xx}h\|_\rho + \frac{2\sigma_1 C_1}{\gamma\delta^{\tau+2}}\|e\|_\rho < \sigma_3 \, .$$

Finally, we control the new error of invariance

$$\bar{e}_\theta(x) = f_{\bar{\alpha}(\theta)}(\bar{h}_\theta(x)) - \bar{h}_\theta(x + \theta) = \partial_x e_\theta(x)\varphi_\theta(x) + \Delta^2 f_\theta(x)$$

using (3.30), (4.23), and (4.28):

$$\mathrm{Lip}_{\Theta,\rho-\delta}(\bar{e}) \le \mathrm{Lip}_{\Theta,\rho-\delta}(\partial_x e)\|\varphi\|_{\Theta,\rho-\delta} + \|\partial_x e\|_{\Theta,\rho-\delta}\mathrm{Lip}_{\Theta,\rho-\delta}(\varphi) + \mathrm{Lip}_{\Theta,\rho-\delta}(\Delta^2 f)$$

$$(4.29) \qquad \le \frac{C_3^{\mathrm{Lip}}}{\gamma^2 \delta^{2\tau+2}}\|e\|_{\Theta,\rho}^2 + \frac{2C_1}{\gamma\delta^{\tau+1}}\|e\|_{\Theta,\rho}\mathrm{Lip}_{\Theta,\rho}(e) + \mathrm{Lip}_{\Theta,\rho-\delta}(\Delta^2 f) \, ,$$

and it remains to control the map $\Delta^2 f_\theta(x)$, given by (3.36). This last term is controlled as follows

$$
\begin{aligned}
\mathrm{Lip}_{\Theta,\rho-\delta}(\Delta^2 f) \leq {} & \frac{1}{2}\Big(\|\partial_{xx\alpha}f\|_{\Theta,\rho}\beta_2 + \|\partial_{xxx}f\|_{\Theta,\rho}\beta_0\Big)\|\Delta_h\|_{\Theta,\rho-\delta}^2 \\
& + \|\partial_{xx}f\|_{\Theta,\rho}\mathrm{Lip}_{\Theta,\rho-\delta}(\Delta_h)\|\Delta_h\|_{\Theta,\rho-\delta} \\
& + \Big(\|\partial_{x\alpha\alpha}f\|_{\Theta,\rho}\beta_2 + \|\partial_{xx\alpha}f\|_{\Theta,\rho}\beta_0\Big)\|\Delta_h\|_{\Theta,\rho-\delta}\|\Delta_\alpha\|_\Theta \\
& + \|\partial_{x\alpha}f\|_{\Theta,\rho}\Big(\mathrm{Lip}_{\Theta,\rho-\delta}(\Delta_h)\|\Delta_\alpha\|_\Theta + \|\Delta_h\|_{\Theta,\rho-\delta}\mathrm{Lip}_\Theta(\Delta_\alpha)\Big) \\
& + \frac{1}{2}\Big(\|\partial_{\alpha\alpha\alpha}f\|_{\Theta,\rho}\beta_2 + \|\partial_{x\alpha\alpha}f\|_{\Theta,\rho}\beta_0\Big)\|\Delta_\alpha\|_\Theta^2 \\
& + \|\partial_{\alpha\alpha}f\|_{\Theta,\rho}\mathrm{Lip}_\Theta(\Delta_\alpha)\|\Delta_\alpha\|_{\Theta,\rho}\,,
\end{aligned}
$$

where we used that

$$
\sup_{t\in[0,1]} \mathrm{Lip}_{\Theta,\rho-\delta}(h + t\Delta_h) \leq \mathrm{Lip}_{\Theta,\rho}(h) + \mathrm{Lip}_{\Theta,\rho-\delta}(\Delta_h) < \beta_0\,,
$$

$$
\sup_{t\in[0,1]} \mathrm{Lip}_\Theta(\alpha + t\Delta_\alpha) \leq \mathrm{Lip}_\Theta(\alpha) + \mathrm{Lip}_\Theta(\Delta_\alpha) < \beta_2\,.
$$

Then, we use hypotheses $\mathfrak{G}_1$, $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, and the previous estimates for the objects $\Delta_{h_\theta}(x)$ and $\Delta_\alpha(\theta)$. Finally, we introduce this estimate into equation (4.29) and after some tedious computations we obtain that (4.19) holds using the constants C_3 and C_5^{Lip} given respectively by (3.26) and (4.20). $\qquad\square$

Proof of Theorem 4.1. We reproduce the analysis of the convergence performed in Theorem 3.1. To this end, we consider the same sequences ρ_s, δ_s and denote the corresponding objects at the s-th step by h_s, α_s and e_s.

Existence of conjugations: The conditions for the convergence of the sequence $\|e_s\|_{\Theta,\rho_s}$ and the control of the norms of the objects are the same that we already obtained in Theorem 3.1, so we must include them into (4.3). In the following, we focus our attention in the control of the Lipschitz constants. It is convenient to introduce the following weighted error:

$$
\mathcal{E}_s := \max\left\{\|e_s\|_{\Theta,\rho_s}\,,\ \gamma\delta_s^{\tau+1}\mathrm{Lip}_{\Theta,\rho_s}(e_s)\right\}\,,
$$

so that

$$
(4.30) \qquad\qquad \|e_s\|_{\Theta,\rho_s} \leq \mathcal{E}_s\,, \qquad \mathrm{Lip}_{\Theta,\rho_s}(e_s) \leq \frac{\mathcal{E}_s}{\gamma\delta_s^{\tau+1}}\,.
$$

Using (4.18), (4.19) and the properties in (4.30), we have

$$
\begin{aligned}
\mathcal{E}_s < {} & \max\left\{\frac{C_3\|e_{s-1}\|_{\Theta,\rho_{s-1}}^2}{\gamma^2\delta_{s-1}^{2\tau}}\,,\ \frac{\gamma\delta_s^{\tau+1}C_5^{\mathrm{Lip}}\|e_{s-1}\|_{\Theta,\rho_{s-1}}^2}{\gamma^3\delta_{s-1}^{3\tau+1}}\right. \\
& \left. + \frac{2C_3\gamma\delta_s^{\tau+1}}{\gamma^2\delta_{s-1}^{2\tau}}\mathrm{Lip}_{\Theta,\rho_{s-1}}(e_{s-1})\|e_{s-1}\|_{\Theta,\rho_{s-1}}\right\}
\end{aligned}
$$

$$
(4.31) \qquad < \frac{\max\{C_3\,,\ (C_5^{\mathrm{Lip}} + 2C_3)(a_1)^{-\tau-1}\}}{\gamma^2\delta_{s-1}^{2\tau}}\mathcal{E}_{s-1}^2 =: \frac{C_6^{\mathrm{Lip}}}{\gamma^2\delta_{s-1}^{2\tau}}\mathcal{E}_{s-1}^2\,.
$$

Then, we reproduce the computations in (3.39)-(3.40) asking for the condition

$$(4.32) \qquad \mu := \frac{C_6^{\mathrm{Lip}} a_1^{2\tau}}{\gamma^2 \delta_0^{2\tau}} \mathcal{E}_0 < 1$$

where the constant C_6^{Lip} is evaluated at the worst case δ_0. We obtain that

$$\mathcal{E}_s < \mu^{2^s - 1} a_1^{-2\tau s} \mathcal{E}_0 \,.$$

Then, we must check that the inequalities in (4.5), (4.6) and (4.7) are preserved along the iterative procedure. Again, we assume that the constants C_1^{Lip} and C_3^{Lip}, given by (4.10) and (4.12) respectively, are evaluated at δ_0. For example, we compute the following

$$\mathrm{Lip}_{\Theta, \rho_s}(h_s - \mathrm{id}) + \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta_s^{2\tau+1}} \|e_s\|_{\Theta, \rho_s} + \frac{\sigma_1 C_1}{\gamma \delta_s^{\tau}} \mathrm{Lip}_{\Theta, \rho_s}(e_s)$$

$$< \mathrm{Lip}_{\Theta, \rho_s}(h_s - \mathrm{id}) + \frac{C_4^{\mathrm{Lip}} + \sigma_1 C_1}{\gamma^2 \delta_s^{2\tau+1}} \mathcal{E}_s$$

$$< \mathrm{Lip}_{\Theta, \rho_0}(h_0 - \mathrm{id}) + \sum_{j=0}^{\infty} \frac{C_4^{\mathrm{Lip}} + \sigma_1 C_1}{\gamma^2 \delta_j^{2\tau+1}} \mathcal{E}_j$$

$$(4.33) \qquad < \mathrm{Lip}_{\Theta, \rho_0}(h_0 - \mathrm{id}) + \frac{C_7^{\mathrm{Lip}} \Sigma_{\mu, -1}}{\gamma^2 \delta_0^{2\tau+1}} \mathcal{E}_0 < \beta_0$$

where we used (3.41), (4.30) and introduced the constant

$$(4.34) \qquad C_7^{\mathrm{Lip}} := C_4^{\mathrm{Lip}} + \sigma_1 C_1 \,.$$

Proceeding as in the proof of Theorem 3.1, the last inequality in (4.33) must be included in condition (4.3). An analogous computation leads us to

$$\mathrm{Lip}_{\Theta, \rho_s}(\partial_x h_s) + \frac{C_4^{\mathrm{Lip}}}{\gamma^2 \delta_s^{2\tau+2}} \|e_s\|_{\Theta, \rho_s} + \frac{\sigma_1 C_1}{\gamma \delta_s^{\tau+1}} \mathrm{Lip}_{\Theta, \rho_s}(e_s)$$

$$(4.35) \qquad < \mathrm{Lip}_{\Theta, \rho_0}(\partial_x h_0) + \frac{C_7^{\mathrm{Lip}} \Sigma_{\mu, -2}}{\gamma^2 \delta_0^{2\tau+2}} \mathcal{E}_0 < \beta_1$$

which is also included in (4.3). Finally, we have

$$(4.36) \qquad \mathrm{Lip}_{\Theta}(\alpha_s) < \mathrm{Lip}_{\Theta}(\alpha_0) + \frac{C_8^{\mathrm{Lip}} \Sigma_{\mu, \tau-1}}{\gamma \delta_0^{\tau+1}} \mathcal{E}_0 < \beta_2 \,,$$

which is also included in (4.3), where we introduced the constant

$$(4.37) \qquad C_8^{\mathrm{Lip}} := C_0^{\mathrm{Lip}} \gamma \delta_0^{\tau+1} + \sigma_b \sigma_2 \,.$$

Putting together (4.32), (4.33), (4.35), (4.36), and (3.42), (3.43), (3.44), (3.45), (3.46), and using that $\rho/\delta = \rho_0/\delta_0 = a_3$, we end up with

$$(4.38) \qquad \mathfrak{c}_1^{\mathrm{Lip}} := \begin{cases} \max\left\{ (a_1 a_3)^{2\tau} \rho^2 C_6^{\mathrm{Lip}}, (a_3)^{\tau+1} C_4 \gamma \rho^{\tau+1}, C_9^{\mathrm{Lip}} \right\} & \text{if } \mu < 1 \,, \\ \infty & \text{otherwise} \,, \end{cases}$$

where μ is given by (4.32), C_4 is given by (3.48) and we have introduced the new constant

$$(4.39) \qquad C_9^{\mathrm{Lip}} := \max\left\{ \frac{C_7^{\mathrm{Lip}} \Sigma_{\mu, -1}(a_3)^{2\tau+1} \rho}{\beta_0 - \mathrm{Lip}_{\Theta, \rho}(h - \mathrm{id})}, \frac{C_7^{\mathrm{Lip}} \Sigma_{\mu, -2}(a_3)^{2\tau}}{\beta_1 - \mathrm{Lip}_{\Theta, \rho}(\partial_x h)}, \right.$$

$$\left.\frac{C_8^{\mathrm{Lip}}\Sigma_{\mu,\tau-1}\gamma(a_3\rho)^{\tau+1}}{\beta_2 - \mathrm{Lip}_\Theta(\alpha)}\right\}.$$

Notice that, since $C_6^{\mathrm{Lip}} \geq C_3$, it turns out that the condition (4.3) in Theorem 4.1 includes the condition (3.4) in Theorem 3.1. As a consequence, we can apply Lemmata 3.2 and 4.4 again. Therefore, the sequence $\mathcal{E}_s$ tends to zero when $s \to \infty$. The iterative scheme converges to a family $\theta \in \Theta \mapsto h_{\theta,\infty} \in \mathcal{A}(\mathbb{T}_{\rho_\infty})$ and a function $\theta \in \Theta \mapsto \alpha_\infty(\theta) \in A$, such that

$$f_{\alpha_\infty(\theta)}(h_{\theta,\infty}(x)) = h_{\theta,\infty}(x + \theta), \qquad \forall \theta \in \Theta$$

and $h_{\theta,\infty}$ also satisfies (3.2).

Measure of rotations: As it was mentioned just after the statement of Theorem 4.1, we can write

$$\alpha_s(\theta) = \alpha_0(\theta) + \alpha_s(\theta) - \alpha_0(\theta) = \alpha_0(\theta) + \sum_{j=0}^{s-1} \Delta_{\alpha_j}(\theta).$$

Then, reproducing the computation in (4.36), we obtain

$$(4.40) \qquad \mathrm{lip}_\Theta(\alpha_\infty) > \mathrm{lip}_\Theta(\alpha_0) - \sum_{j=0}^{\infty} \mathrm{Lip}_\Theta(\Delta_{\alpha_j}) > \mathrm{lip}_\Theta(\alpha_0) - \frac{C_8^{\mathrm{Lip}}\Sigma_{\mu,\tau-1}}{\gamma\delta_0^{\tau+1}}\mathcal{E}_0.$$

Finally, thesis $\mathfrak{T}_3$ holds using this estimation (of the Lipschitz constant from below) to transport the measure of the set Θ through the function $\theta \in \Theta \mapsto \alpha_\infty(\theta) \in A$. More concretely, we use (4.40) in

$$\mathrm{Leb}(\alpha_\infty(\Theta)) \geq \mathrm{lip}_\Theta(\alpha_\infty)\,\mathrm{Leb}(\Theta),$$

taking

$$(4.41) \qquad \mathfrak{C}_2^{\mathrm{Lip}} := (a_3)^{\tau+1}C_8^{\mathrm{Lip}}\Sigma_{\mu,\tau-1},$$

and recalling that $a_3 = \rho_0/\delta_0$. $\qquad\qquad\square$

Remark 4.5. Here we summarize how to compute constants $\mathfrak{C}_1^{\mathrm{Lip}}$, $\mathfrak{C}_2^{\mathrm{Lip}}$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ of Theorem 4.1. Given fixed values of the parameters $\rho, \delta, \rho_\infty, \hat{\rho}$ and the distances $\mathrm{dist}\left(h_\theta(\bar{\mathbb{T}}_\rho), \partial\mathbb{T}_{\hat{\rho}}\right)$ and $\mathrm{dist}\left(\alpha(\Theta), \partial A\right)$; the constants c_x, c_α, c_{xx}, $c_{x\alpha}$, $c_{\alpha\alpha}$, c_{xxx}, $c_{xx\alpha}$, $c_{x\alpha\alpha}$, $c_{\alpha\alpha\alpha}$ in hypothesis $\mathfrak{G}_1$; the constants γ and τ in hypothesis $\mathfrak{G}_2$; the constants σ_1, σ_2, σ_3 in hypothesis $\mathfrak{H}_1$; the constant σ_b in hypothesis $\mathfrak{H}_2$; and the constants β_0, β_1, β_2 in hypothesis $\mathfrak{H}_3$, are computed in the following order:

- a_1, a_2, a_3 using (3.37) and (3.38).
- C_1, C_2, C_3 using (3.21), (3.22) and (3.26).
- $C_0^{\mathrm{Lip}}, C_1^{\mathrm{Lip}}, C_2^{\mathrm{Lip}}, C_3^{\mathrm{Lip}}, C_4^{\mathrm{Lip}}, C_5^{\mathrm{Lip}}$ using (4.9), (4.10), (4.11), (4.12), (4.13) and (4.20).
- κ, μ using (3.40) and (4.32), and check that $\mu < 1$ (abort the process otherwise).
- $\Sigma_{\kappa,\tau}, \Sigma_{\kappa,2\tau}, \Sigma_{\kappa,\tau-1}, \Sigma_{\mu,-1}, \Sigma_{\mu,-2}, \Sigma_{\mu,\tau-1}$ using (3.41), replacing $\|\cdot\|_\rho$ by $\|\cdot\|_{\Theta,\rho}$.
- C_4 using (3.48).
- $C_6^{\mathrm{Lip}}, C_7^{\mathrm{Lip}}, C_8^{\mathrm{Lip}}, C_9^{\mathrm{Lip}}$ using (4.31), (4.34), (4.37) and (4.39).
- $\mathfrak{C}_2$, $\mathfrak{C}_3$ using (3.49).
- $\mathfrak{C}_1^{\mathrm{Lip}}, \mathfrak{C}_2^{\mathrm{Lip}}$ using (4.38) and (4.41).

5. On the verification of the hypotheses

In this section we show a systematic approach, tailored to be implemented in a computer-assisted proof, to verify the assumptions of our a-posteriori theorems. To do so, we perform an analytic study of the hypotheses with the goal of providing formulae that satisfy the following requirements: they are computable with a finite number operations, they give sharp bounds of the involved estimates, and the computational time is fast: all computations are performed with a complexity of order $N \log N$, N being the number of Fourier modes used to represent the conjugacies. We focus in the hypotheses of Theorem 4.1, since the discourse can be in fact simplified to deal with Theorem 3.1.

In Section 5.1 we discuss the global hypotheses $\mathfrak{G}_1$ and $\mathfrak{G}_2$, mainly how to obtain a suitable subset Θ of Diophantine numbers contained in a given interval B of rotation numbers, giving a sharp estimate on the measure of Θ.

In Section 5.2 we discuss the hypotheses $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$ which depend on the initial objects, which are taken as Fourier-Taylor polynomials. Denoting θ_0 the center of the interval B, we assume that $h_\theta(x) = h(x, \theta)$ is of the form

$$(5.1) \qquad h(x, \theta) = x + \sum_{s=0}^{m} h^{[s]}(x)(\theta - \theta_0)^s \,, \qquad h^{[s]}(x) = \sum_{k=-N/2}^{N/2-1} \hat{h}_k^{[s]} \mathrm{e}^{2\pi \mathrm{i} k x} \,,$$

where $\hat{h}_0^{[s]} = 0$, $\hat{h}_{-N/2}^{[s]} = 0$ and $\hat{h}_k^{[s]} = (\hat{h}_{-k}^{[s]})^*$ otherwise, and that $\alpha(\theta)$ is of the form

$$(5.2) \qquad \alpha(\theta) = \sum_{s=0}^{m} \alpha^{[s]}(\theta - \theta_0)^s \,, \qquad \alpha^{[s]} \in \mathbb{R} \,;$$

for certain degree m and $N = 2^q$. Notice that the symmetries in the Fourier coefficients of h have been selected so the corresponding function is real-analytic and satisfies the normalization condition

$$\langle h_\theta - \mathrm{id} \rangle = 0 \,.$$

We will see that the fact that these objects are chosen to be polynomials plays an important role to obtain sharp values of the constants σ_1, σ_2, σ_3, σ_b, β_0, β_1 and β_2.

In Section 5.3 we present the main result of this section, which allows us to obtain a fine control of the norm of the error of conjugacy of the initial family. This requires new tools since the corresponding error function, denoted $e(x, \theta)$, is no longer a Fourier-Taylor polynomial, so we combine the jet-propagation in the variable θ with the control of the discrete Fourier approximation in the variable x.

For convenience, we briefly recall here some standard notation used along this section, and an approximation theorem to control the error of the discrete Fourier transform when approximating periodic functions. Given a function $g : \mathbb{T} \to \mathbb{C}$, we consider its Fourier series

$$g(x) = \sum_{k \in \mathbb{Z}} \hat{g}_k \mathrm{e}^{2\pi \mathrm{i} k x} \,.$$

Let us consider $N = 2^q$ with $q \in \mathbb{N}$, and the discretization $\{x_j\}$, $x_j = j/N$, $0 \le j < N$, that defines a sampling $\{g_j\}$, with $g_j = g(x_j)$. Then, the discrete Fourier transform (DFT) is

$$\tilde{g}_k = \frac{1}{N} \sum_{j=0}^{N-1} g_j \mathrm{e}^{-2\pi \mathrm{i} k x_j} =: \mathrm{DFT}_k(\{g_j\})$$

and the function $g(x)$ is approximated by the trigonometric polynomial

$$\tilde{g}(x) := \sum_{k=-N/2}^{N/2-1} \tilde{g}_k e^{2\pi i k x} .$$

We use the notation

$$(5.3) \qquad\qquad \{\tilde{g}_k\} = \{\mathrm{DFT}_k(\{g_j\})\} ,$$

$$(5.4) \qquad\qquad \{g_j\} = \{\mathrm{DFT}_j^{-1}(\{\tilde{g}_k\})\} .$$

Notice that formulae (5.3) and (5.4) are exact if $g(x)$ is a trigonometric polynomial of degree at most N. In this case, we write $\tilde{g}_k = \hat{g}_k$.

Theorem 5.1 (See [26]). *Let $g : \mathbb{T}_{\tilde{\rho}} \to \mathbb{C}$ be an analytic and bounded function in the complex strip $\mathbb{T}_{\tilde{\rho}}$, with $\tilde{\rho} > 0$. Let $\tilde{g}$ be the DFT approximation of g using N nodes. Then*

$$(5.5) \qquad\qquad |\tilde{g}_k - \hat{g}_k| \leq s_N(k, \tilde{\rho}) \|g\|_{\tilde{\rho}} ,$$

$$(5.6) \qquad\qquad \|\tilde{g} - g\|_\rho \leq C_N(\rho, \tilde{\rho}) \|g\|_{\tilde{\rho}} ,$$

for any $0 \leq \rho \leq \tilde{\rho}$, where

$$(5.7) \qquad\qquad s_N(k, \tilde{\rho}) := \frac{e^{-2\pi\tilde{\rho}N}}{1 - e^{-2\pi\tilde{\rho}N}} \left(e^{2\pi\tilde{\rho}k} + e^{-2\pi\tilde{\rho}k} \right)$$

and

$$(5.8) \qquad\qquad C_N(\rho, \tilde{\rho}) = S_N^1(\rho, \tilde{\rho}) + S_N^2(\rho, \tilde{\rho}) + S_N^3(\rho, \tilde{\rho})$$

with

$$S_N^1(\rho, \tilde{\rho}) = \frac{e^{-2\pi\tilde{\rho}N}}{1 - e^{-2\pi\tilde{\rho}N}} \frac{e^{-2\pi(\tilde{\rho}+\rho)} + 1}{e^{-2\pi(\tilde{\rho}+\rho)} - 1} \left(1 - e^{\pi(\tilde{\rho}+\rho)N} \right) ,$$

$$S_N^2(\rho, \tilde{\rho}) = \frac{e^{-2\pi\tilde{\rho}N}}{1 - e^{-2\pi\tilde{\rho}N}} \frac{e^{2\pi(\tilde{\rho}-\rho)} + 1}{e^{2\pi(\tilde{\rho}-\rho)} - 1} \left(1 - e^{-\pi(\tilde{\rho}-\rho)N} \right) ,$$

$$S_N^3(\rho, \tilde{\rho}) = \frac{e^{2\pi(\tilde{\rho}-\rho)} + 1}{e^{2\pi(\tilde{\rho}-\rho)} - 1} \, e^{-\pi(\tilde{\rho}-\rho)N} .$$

The following Fourier norm will be useful

$$\|g\|_\rho^{\mathcal{F}} := \sum_{k\in\mathbb{Z}} |\hat{g}_k| e^{2\pi|k|\rho} ,$$

since it can be evaluated with a finite amount of computations if $g(x)$ is a trigonometric polynomial. Notice that $\|g\|_\rho \leq \|g\|_\rho^{\mathcal{F}}$.

For convenience, we use the suitable language of interval analysis, with the only aim of controlling truncation and discretization errors. In particular, for any closed interval $Z \subset \mathbb{R}$ we use the standard notation

$$(5.9) \qquad\qquad Z = [\underline{Z}, \overline{Z}] , \qquad \mathrm{rad}(Z) = \frac{\overline{Z} - \underline{Z}}{2}$$

for the boundaries and the radius of an interval. The error produced when evaluating the proposed expressions using a computer (with finite precision arithmetics) is easily controlled performing computations with interval arithmetics.

5.1. **Controlling the global hypotheses $\mathfrak{G}_1$ and $\mathfrak{G}_2$.** Obtaining the bounds on the derivatives of the map, $\mathfrak{G}_1$, is problem dependent and does not suppose a big challenge. If the map f_α is given in an explicit form then the bounds can be obtained by hand, as we illustrate in Section 6 with an example.

The global hypothesis $\mathfrak{G}_2$ has to do with finding (positive measure) sets of Diophantine numbers in a closed interval.

Lemma 5.2. *Given an interval $B = [\underline{B}, \overline{B}] \subset \mathbb{R}$, constants $\gamma < \frac{1}{2}$ and $\tau > 1$, and $Q \in \mathbb{N}$ such that $\frac{2}{Q} \leq \overline{B} - \underline{B}$, then the relative measure of $\Theta = B \cap \mathcal{D}(\gamma, \tau)$ satisfies*

$$(5.10) \qquad \frac{\mathrm{Leb}(\Theta)}{\mathrm{Leb}(B)} \geq 1 - \frac{4\gamma}{(\tau - 1)Q^{\tau - 1}} - \sum_{q=1}^{Q} \sum_{\substack{p=\lfloor \underline{B}q \rfloor \\ \gcd(p,q)=1}}^{\lceil \overline{B}q \rceil} \Delta(p, q),$$

where

$$(5.11) \qquad \Delta(p, q) = \begin{cases} \max\left(\frac{p}{q} + \frac{\gamma}{q^{\tau+1}} - \underline{B}, 0\right) & \text{if } p = \lfloor \underline{B}q \rfloor, \\ \max\left(\overline{B} - \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}, 0\right) & \text{if } p = \lceil \overline{B}q \rceil, \\ \min\left(\overline{B}, \frac{p}{q} + \frac{\gamma}{q^{\tau+1}}\right) - \max\left(\underline{B}, \frac{p}{q} - \frac{\gamma}{q^{\tau+1}}\right) & \text{otherwise}. \end{cases}$$

Here we use the notation $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ for the ceil and floor functions, respectively.

Proof. Since $\gamma < \frac{1}{2}$, the (p, q)-resonant sets

$$\mathrm{Res}_{p,q}(B, \gamma, \tau) = \left\{ \theta \in B \ : \ |q\theta - p| < \frac{\gamma}{q^\tau} \right\},$$

for each fixed $q > 0$, are pairwise disjoint. Notice also that, for any $k \in \mathbb{N}$, $\mathrm{Res}_{kp,kq}(B, \gamma, \tau) \subset \mathrm{Res}_{p,q}(B, \gamma, \tau)$, so that the full resonant set of type (γ, τ) is

$$\mathrm{Res}(B, \gamma, \tau) = \bigcup_{\gcd(p,q)=1} \mathrm{Res}_{p,q}(B, \gamma, \tau),$$

and so, finding a lower bound of $\mathrm{Leb}(\Theta)$ is equivalent to finding an upper bound of $\mathrm{Leb}(\mathrm{Res}(B, \gamma, \tau))$.

Given a fixed number Q, we consider the disjoint union

$$\mathrm{Res}(B, \gamma, \tau) = \mathrm{Res}_{\leq Q}(B, \gamma, \tau) \cup \mathrm{Res}_{>Q}(B, \gamma, \tau)$$

where $\mathrm{Res}_{\leq Q}(B, \gamma, \tau)$ and $\mathrm{Res}_{>Q}(B, \gamma, \tau)$ are, respectively, the sets of resonances with denominator q satisfying $q \leq Q$ and $q > Q$. The measure of the first set is controlled as

$$\mathrm{Leb}(\mathrm{Res}_{\leq Q}(B, \gamma, \tau)) \leq \sum_{q=1}^{Q} \sum_{\substack{p=\lfloor \underline{B}q \rfloor \\ \gcd(p,q)=1}}^{\lceil \overline{B}q \rceil} \mathrm{Leb}(\mathrm{Res}_{p,q}(B, \gamma, \tau))$$

$$\leq (\overline{B} - \underline{B}) \sum_{q=1}^{Q} \sum_{\substack{p=\lfloor \underline{B}q \rfloor \\ \gcd(p,q)=1}}^{\lceil \overline{B}q \rceil} \Delta(p, q),$$

where $\Delta(p, q)$, given in (5.11), is obtained by estimating the measure of each resonance strip, taking into account if the strip is strictly or partially included in B.

The measure of the second set is controlled as

$$\text{Leb}(\text{Res}_{>Q}(B,\gamma,\tau)) \leq \sum_{q=Q+1}^{\infty} \frac{2\gamma}{q^{\tau+1}}(\lceil \overline{B}q \rceil - \lfloor \underline{B}q \rfloor)$$

$$\leq \sum_{q=Q+1}^{\infty} \frac{2\gamma}{q^{\tau+1}}(\overline{B}q - \underline{B}q + 2)$$

$$\leq 2\gamma \left(\frac{\overline{B} - \underline{B}}{(\tau-1)Q^{\tau-1}} + \frac{2}{\tau Q^{\tau}} \right).$$

Finally, since $\frac{2}{Q} \leq \overline{B} - \underline{B}$, we get the upper bound

$$\text{Leb}(\text{Res}_{>Q}(B,\gamma,\tau)) \leq \frac{4\gamma(\overline{B} - \underline{B})}{(\tau-1)Q^{\tau-1}}.$$

Then, the bound (5.10) holds by combining both estimates. $\qquad\square$

Remark 5.3. When $B = [0,1]$ we have $\Delta(p,q) = \frac{2\gamma}{q^{\tau+1}}$ for every p. Then, taking $Q \to \infty$ we obtain a Dirichlet series as a lower bound

$$\text{Leb}(\Theta) = \frac{\text{Leb}(\Theta)}{\text{Leb}(B)} \geq 1 - 2\gamma \sum_{q=1}^{\infty} \frac{\phi(q)}{q^{\tau+1}} = 1 - 2\gamma \frac{\zeta(\tau)}{\zeta(\tau+1)},$$

where ϕ is the Euler function and ζ is the Riemann zeta function.

5.2. **Controlling the hypotheses $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$.** Given an interval B, centered at θ_0, we consider $h_\theta(x) = h(x,\theta)$ and $\alpha(\theta)$ given by (5.1) and (5.2), respectively. In this section we present a procedure to control the hypothesis $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$ corresponding to these objects. To this end, it is convenient to introduce some notation to enclose the dependence of the variable θ.

Definition 5.4. *Given a function of the form*

$$F(x,\theta) = \sum_{s \geq 0} F^{[s]}(x)(\theta - \theta_0)^s, \qquad F^{[s]}(x) = \sum_{k \in \mathbb{Z}} \hat{F}_k^{[s]} e^{2\pi i k x},$$

we introduce the enclosing interval function and its formal derivative as follows:

$$(5.12) \qquad F_B(x) := \sum_{k \in \mathbb{Z}} \hat{F}_{B,k} e^{2\pi i k x}, \qquad F_B'(x) := \sum_{k \in \mathbb{Z}} (2\pi i k) \hat{F}_{B,k} e^{2\pi i k x},$$

where $\hat{F}_{B,k}$ are given by

$$\hat{F}_{B,k} := \sum_{s \geq 0} \hat{F}_k^{[s]} (B - \theta_0)^s = \left\{ \sum_{s \geq 0} \hat{F}_k^{[s]} (\theta - \theta_0)^s \ : \ \theta \in B \right\}.$$

Abusing notation we write

$$\|F_B\|_\rho := \max_{x \in \mathbb{T}_\rho} \overline{\left| \sum_{k \in \mathbb{Z}} \hat{F}_{B,k} e^{2\pi i k x} \right|}, \qquad \|F_B\|_\rho^{\mathcal{F}} := \overline{\sum_{k \in \mathbb{Z}} |\hat{F}_{B,k}| e^{2\pi |k| \rho}}.$$

Using the enclosing interval function associated to a Fourier-Taylor polynomial of the form

$$F(x,\theta) = \sum_{s=0}^{m} F^{[s]}(x)(\theta - \theta_0)^s, \qquad F^{[s]}(x) = \sum_{k=-N/2}^{N/2-1} \hat{F}_k^{[s]} e^{2\pi i k x},$$

we introduce the following notation:

$$\mathfrak{M}_{B,\rho}(F) := \max_{j=0,\ldots,N-1}\left\{\left|\overline{\mathrm{DFT}_j^{-1}(\{\hat{F}_{B,k}\mathrm{e}^{2\pi\mathrm{i}k\rho}\})}\right|\right\} + \frac{1}{2N}\|F_B'\|_\rho^{\mathcal{F}}\,.$$

and

$$\mathfrak{m}_{B,\rho}(F) := \min_{j=0,\ldots,N-1}\left\{\left|\underline{\mathrm{DFT}_j^{-1}(\{\hat{F}_{B,k}\mathrm{e}^{2\pi\mathrm{i}k\rho}\})}\right|\right\} - \frac{1}{2N}\|F_B'\|_\rho^{\mathcal{F}}\,.$$

Proposition 5.5. *Take $h(x,\theta)$ and $\alpha(\theta)$ of the form (5.1) and (5.2), respectively, and consider the objects f_α, $\hat{\rho}$ and ρ in Theorem 4.1. Assume that*

(5.13) $$\hat{\rho} > \rho + \mathfrak{M}_{B,\rho}(h - \mathrm{id})\,, \quad \mathfrak{M}_{B,\rho}(\partial_x h - 1) < 1\,, \quad \mathfrak{m}_{B,\rho}(\partial_x h) > 0\,.$$

Then:

(1) Hypothesis $\mathfrak{H}_1$ holds by taking

$$\sigma_1 > \mathfrak{M}_{B,\rho}(\partial_x h),$$
$$\sigma_2 > 1/\mathfrak{m}_{B,\rho}(\partial_x h),$$
$$\sigma_3 > \mathfrak{M}_{B,\rho}(\partial_{xx} h)\,.$$

(2) Consider the function

$$b(x,\theta) = \frac{\partial_\alpha f_{\alpha(\theta)}(h(x,\theta))}{\partial_x h(x+\theta,\theta)}\,,$$

and let $\{b_B(x_j)\}$ be the corresponding enclosing function evaluated in the grid $x_j = j/N$. Assume that

(5.14) $$c_b := \frac{s_N(0,\rho)c_\alpha\overline{|1/\tilde{b}_{B,0}|}}{\mathfrak{m}_{B,\rho}(\partial_x h)} < 1\,,$$

where $s_N(0,\rho)$ is given in (5.7) and

$$\tilde{b}_{B,0} = \frac{1}{N}\sum_{j=0}^{N-1} b_B(x_j)\,.$$

Then, hypothesis $\mathfrak{H}_2$ holds by taking

$$\sigma_b > \frac{\overline{|1/\tilde{b}_{B,0}|}}{1-c_b}\,.$$

(3) Assume that the interval $\alpha'(B)$ does not contain 0. Then, hypothesis $\mathfrak{H}_3$ holds by taking

$$\beta_0 > \mathfrak{M}_{B,\rho}(\partial_\theta h)\,,$$
$$\beta_1 > \mathfrak{M}_{B,\rho}(\partial_{x\theta} h)\,,$$
$$\beta_2 > \overline{\alpha'(B)}\,.$$

Furthermore, we observe that $\underline{\alpha'(B)} < \mathrm{lip}_\Theta(\alpha)$.

Remark 5.6. Since $h(x,\theta)$ is a Fourier-Taylor polynomial, we could directly use the Fourier norm to produce

$$\sigma_1 > 1 + \sum_{s=0}^m \|\partial_x h^{[s]}\|_\rho^{\mathcal{F}}\,\mathrm{rad}(B)^s\,, \qquad \sigma_3 > \sum_{s=0}^m \|\partial_{xx} h^{[s]}\|_\rho^{\mathcal{F}}\,\mathrm{rad}(B)^s\,.$$

This approach not only produces substantial overestimation (which is propagated in the KAM constants $\mathfrak{C}_1$, $\mathfrak{C}_2$, etc), but does not give information to control σ_2. Notice that the overestimation (produced by the Fourier norm) of the derivative F_B' in $\mathfrak{M}_{B,\rho}(F)$ and $\mathfrak{m}_{B,\rho}(F)$ is mitigated both by the factor $1/2N$ and by enclosing the dependence of θ (since cancellations are taken into account).

Proof. We first observe that, since $h(x, \theta)$ is a Fourier-Taylor polynomial, it is indeed analytic in $B \supset \Theta$, so we consider the bounds

$$\|\partial_x h\|_{\Theta,\rho} \le \|\partial_x h\|_{B,\rho}, \quad \|1/\partial_x h\|_{\Theta,\rho} \le \|1/\partial_x h\|_{B,\rho}, \quad \|\partial_{xx} h\|_{\Theta,\rho} \le \|\partial_{xx} h\|_{B,\rho}.$$

To control $\|\partial_x h\|_{B,\rho}$ and $\|\partial_{xx} h\|_{B,\rho}$, we use the maximum modulus principle for analytic functions. By hypothesis, the functions are real-analytic, so it suffices to consider one component of the boundary, say $\{\mathrm{Im}(x) = \rho\}$.

Using the enclosing operation in Definition 5.4 we reduce the discussion to manipulate formal Fourier series $\partial_x h_B(x)$ and $\partial_{xx} h_B(x)$, with interval coefficients, that include the dependence of the variable θ. Hence, to estimate the maximum of a function $F(x, \theta)$ in $\{\mathrm{Im}(x) = \rho\} \times B$ we construct the bound $\mathfrak{M}_{B,\rho}(F)$ as follows:

- The restriction of the enclosing function $F_B(x)$ to the boundary is obtained by multiplying each kth Fourier coefficient by $\mathrm{e}^{2\pi \mathrm{i} k \rho}$.
- The evaluation of $F_B(x + \mathrm{i}\rho)$ in the uniform grid of N intervals is performed using DFT.
- The maximum of $F_B(x + \mathrm{i}\rho)$ in each interval of length $1/N$, centered at the grid points, is bounded above by the value of the function at the grid plus a global bound of the derivative.
- The bound of the mentioned derivative is obtained using the immediate inequality

$$\|F_B'\|_\rho \le \|F_B'\|_\rho^{\mathcal{F}}.$$

The above discussion allows us to control σ_1 and σ_3.

To control σ_2 we perform an analogous argument for the minimum modulus principle. Notice that the condition $\mathfrak{M}_{B,\rho}(\partial_x h - \mathrm{id}) < 1$ ensures that the function $\partial_x h(x, \theta)$ is non-zero at all points in $\mathbb{T}_\rho \times B$. The fact that $\mathfrak{m}_{B,\rho}(\partial_x h) > 0$ ensures that we can take $\sigma_2 < \infty$.

As the last condition in hypothesis $\mathfrak{H}_1$, we must see that for every $\theta - \theta_0 \in B$ the map $h(x, \theta)$ satisfies

(5.15) $$\mathrm{dist}\left(h(\bar{\mathbb{T}}_\rho, \theta), \partial \mathbb{T}_{\hat{\rho}}\right) > 0.$$

To this end, we compute

$$\max_{x \in \mathbb{T}_\rho} \max_{\theta \in B} |\mathrm{Im}(h(x, \theta))| \le \rho + \|h - \mathrm{id}\|_{B,\rho} \le \rho + \mathfrak{M}_{B,\rho}(h - \mathrm{id})$$

and the inequality (5.15) holds from the first assumption in (5.13). This completes item (1).

Regarding item (2), let us recall that we are interested in controlling $\|1/\langle b \rangle\|_\Theta$ where $\langle b \rangle(\theta)$ is the actual average with respect to x. Notice that

$$\langle b \rangle(\theta) = \sum_{s \ge 0} \langle b^{[s]} \rangle (\theta - \theta_0)^s \in \int_0^1 b_B(x) \mathrm{d}x =: \langle b_B \rangle$$

and so

$$\|1/\langle b \rangle\|_\Theta \le \overline{|1/\langle b_B \rangle|}.$$

Using the notation in the statement, and Theorem 5.1, we have

$$\left|\overline{\widetilde{b}_{B,0} - \langle b_B \rangle}\right| \leq s_N(0,\rho)\|b\|_{B,\rho} \leq s_N(0,\rho)\|f_\alpha\|_{A,\hat\rho}\|1/\partial_x h\|_{B,\rho} \leq \frac{s_N(0,\rho)\ c_\alpha}{\mathfrak{m}_{B,\rho}(\partial_x h)}\ .$$

Then, we compute

$$\overline{|1/\langle b_B \rangle|} \leq \frac{\overline{\left|1/\widetilde{b}_{B,0}\right|}}{1 - \overline{\left|1/\widetilde{b}_{B,0}\right|\left|\overline{\widetilde{b}_{B,0} - \langle b_B \rangle}\right|}} \leq \frac{\overline{\left|1/\widetilde{b}_{B,0}\right|}}{1 - c_b}\ ,$$

where we used (5.14).

Finally, item (3) follows reproducing the argument for item (1), but controlling the Lipschitz norms in terms of the norm of the corresponding derivative with respect to θ. $\qquad\square$

5.3. Controlling the error of conjugacy.

Given an interval B, centered at θ_0, we consider again $h_\theta(x) = h(x,\theta)$ and $\alpha(\theta)$ given by (5.1) and (5.2), respectively. In this section we propose suitable (sharp and computable) estimates to control the norm $\|\cdot\|_{\Theta,\rho}$ of the Fourier-Taylor series

$$(5.16) \qquad e(x,\theta) = \sum_{s \geq 0} e^{[s]}(x)(\theta - \theta_0)^s := f_{\alpha(\theta)}(h(x,\theta)) - h(x+\theta,\theta)\ .$$

Notice that we have to compose $f_\alpha(x)$ with the objects $h(x,\theta)$ and $\alpha(\theta)$ given by (5.1) and (5.2). Assuming that the family $f_\alpha(x) = f(x,\alpha)$ is C^∞ in α, we can express the composition as follows

$$(5.17) \qquad F(x,\theta) = f(h(x,\theta),\alpha(\theta)) = x + \sum_{s \geq 0} F^{[s]}(x)(\theta - \theta_0)^s\ ,$$

where

$$F^{[0]}(x) = \mathfrak{F}_0(x + h^{[0]}(x), \alpha^{[0]}) = f_{\alpha^{[0]}}(x + h^{[0]}(x)) - x\ ,$$

and the remaining coefficients, for $s \geq 1$, are given by recurrence formulae

$$(5.18) \qquad \begin{aligned} F^{[s]}(x) = \mathfrak{F}_s\Big(&x + h^{[0]}(x), \alpha^{[0]}; h^{[1]}(x), \ldots, h^{[s]}(x), \alpha^{[1]}, \ldots, \alpha^{[s]},\\ &F^{[1]}(x), \ldots, F^{[s-1]}(x)\Big)\ . \end{aligned}$$

Notice that the recurrences $\mathfrak{F}_s$ are explicit in terms of Faà di Bruno formulae or, if the function $f_\alpha(x)$ is elementary, using Automatic Differentiation rules (see [38]). In particular, formula (5.18) depends polynomially with respect to $h^{[1]}(x), \ldots, h^{[s]}(x)$, $\alpha^{[1]}, \ldots, \alpha^{[s]}$, $F^{[1]}(x), \ldots, F^{[s-1]}(x)$.

Furthermore, a natural way to enclose the power series (5.17) is the truncated Taylor model (recall that m is the fixed order in $\theta - \theta_0$ of the initial objects $h(x,\theta)$ and $\alpha(\theta)$)

$$F(x,\theta) \in x + \sum_{s=0}^{m} F^{[s]}(x)(\theta - \theta_0)^s + F_B^{[m+1]}(x)[-1,1]\,\mathrm{rad}(B)^{m+1}\ ,$$

where $F_B^{[m+1]}(x)$ is obtained evaluating the same recurrences

$$(5.19) \qquad \begin{aligned} F_B^{[s]}(x) = \mathfrak{F}_s\Big(&x + h_B^{[0]}(x), \alpha_B^{[0]}; h_B^{[1]}(x), \ldots, h_B^{[s]}(x), \alpha_B^{[1]}, \ldots, \alpha_B^{[s]},\\ &F_B^{[1]}(x), \ldots, F_B^{[s-1]}(x)\Big)\ , \end{aligned}$$

for $1 \leq s \leq m + 1$, with the fattened objects

$$(5.20) \qquad h_B^{[s]}(x) = \frac{1}{s!} \left\{ \frac{\partial^s h(x, \theta)}{\partial \theta^s} : \theta \in B \right\}, \qquad \alpha_B^{[s]} = \frac{1}{s!} \left\{ \frac{\mathrm{d}^s \alpha(\theta)}{\mathrm{d}\theta^s} : \theta \in B \right\}.$$

See [60] for further details.

In the following theorem we propose an explicit estimate for the norm of the error (5.16) using the above idea. A major obstacle is that the space of trigonometric polynomials of degree at most N is not an algebra. This is overcome by combining recurrences (5.18) and (5.19) with control on the discretization error in Fourier space. Thus, we obtain an additional source of error that, remarkably, is estimated using recursive formulae that depend only on the family $f_\alpha(x)$.

Theorem 5.7. *Take $h(x, \theta)$ and $\alpha(\theta)$ of the form (5.1) and (5.2), respectively, and consider the objects f_α, $\hat\rho$ and ρ in Theorem 4.1. Assume that, given $r > \hat\rho$, we have $f_\alpha(x) \in \mathcal{A}(\mathbb{T}_r)$ and the maps are C^{m+1} with respect to α. Then, for any $\tilde\rho > \rho$ such that*

$$(5.21) \qquad\qquad r > \tilde\rho + \mathfrak{M}_{\theta_0, \tilde\rho}(h^{[0]}),$$

the error (5.16) satisfies

$$\|e\|_{\Theta, \rho} \leq \sum_{s=0}^{m} \|\widetilde{e}^{[s]}\|_\rho^{\mathcal{F}} \, \mathrm{rad}(B)^s + C_T \, \mathrm{rad}(B)^{m+1} + C_F \, C_N(\rho, \tilde\rho),$$

$$\mathrm{Lip}_{\Theta, \rho}(e) \leq \sum_{s=1}^{m} s \|\widetilde{e}^{[s]}\|_\rho^{\mathcal{F}} \, \mathrm{rad}(B)^{s-1} + (m+1) C_T \, \mathrm{rad}(B)^m + C_F' \, C_N(\rho, \tilde\rho),$$

where $\widetilde{e}^{[s]}(x)$ is the discrete Fourier approximation given by

$$(5.22) \qquad\qquad \widetilde{e}_k^{[s]} = \mathrm{DFT}_k(\{e^{[s]}(x_j)\}),$$

with $x_j = j/N$, and the computable constants C_T, C_F, C_F' that depend on $m, B, \rho, \tilde\rho$ and the initial objects.

Remark 5.8. Note that the functions $\widetilde{e}^{[s]}(x)$, for $0 \leq s \leq m$, are expected to be small if the candidates $h(x, \theta)$ and $\alpha(\theta)$ are good enough approximations of the Lindstedt series at θ_0. Moreover, the constant $C_N(\rho, \tilde\rho) = O(e^{-\pi N(\tilde\rho - \rho)})$ can be taken very small for a suitable choice of $\tilde\rho$. Hence, the limiting factor of the estimate is the term $C_T \, \mathrm{rad}(B)^{m+1}$ (see (5.24)).

Proof. The error (5.16) can be enclosed as

$$e(x, \theta) \in \sum_{s=0}^{m} e^{[s]}(x)(\theta - \theta_0)^s + e_B^{[m+1]}(x)[-1, 1] \, \mathrm{rad}(B)^{m+1},$$

which yields the control

$$\|e\|_{\Theta, \rho} \leq \sum_{s=0}^{m} \|e^{[s]}\|_\rho \, \mathrm{rad}(B)^s + \left\|e_B^{[m+1]}\right\|_\rho \, \mathrm{rad}(B)^{m+1}.$$

Since the functions $e^{[s]}(x)$ and $e_B^{[m+1]}(x)$ have infinitely many harmonics, we approximate them using suitable trigonometric polynomials $\tilde{e}^{[s]}(x)$ (*Step 1*) and $\tilde{e}_B^{[m+1]}(x)$ (*Step 2*), thus obtaining the bound

$$\|e\|_{\Theta, \rho} \leq \sum_{s=0}^{m} \|\tilde{e}^{[s]}\|_\rho \, \mathrm{rad}(B)^s + \left\|\tilde{e}_B^{[m+1]}\right\|_\rho \, \mathrm{rad}(B)^{m+1}$$

$$+ \sum_{s=0}^{m} \|e^{[s]} - \tilde{e}^{[s]}\|_\rho \, \mathrm{rad}(B)^s + \left\| e_B^{[m+1]} - \tilde{e}_B^{[m+1]} \right\|_\rho \mathrm{rad}(B)^{m+1} .$$

Then, we deal with the error committed by approximating $e^{[s]}(x)$ and $e_B^{[m+1]}(x)$ with $\tilde{e}^{[s]}(x)$ and $\tilde{e}_B^{[m+1]}(x)$ (*Step 3*). After controlling the norm $\|e\|_{\Theta,\rho}$, we use that the function is indeed smooth in $B \supset \Theta$ and control $\mathrm{Lip}_{\Theta,\rho}(e)$ (*Step 4*).

Step 1: The Taylor coefficients of the first term in (5.16), $F(x,\theta) = f_{\alpha(\theta)}(h(x,\theta))$, satisfy the recurrences (5.18). In particular, we evaluate them pointwise in the grid $x_j = j/N$, $0 \le j < N$, thus obtaining $\{F^{[s]}(x_j)\}$. Notice that the evaluations

$$\{h^{[s]}(x_j)\} = \{\mathrm{DFT}_j^{-1}(\{\tilde{h}_k^{[s]}\})\} ,$$

are exact, since $h^{[s]}(x)$ are trigonometric polynomials.

For the second term in (5.16), $H(x,\theta) = h(x+\theta,\theta)$, we have

$$(5.23) \quad H(x,\theta) = x + \sum_{s \ge 0} H^{[s]}(x)(\theta - \theta_0)^s$$

$$:= x + \theta + \sum_{s=0}^{m} h^{[s]}(x + \theta_0 + \theta - \theta_0)(\theta - \theta_0)^s$$

$$= x + \theta + \sum_{s=0}^{m} \left(\sum_{j \ge 0} \frac{1}{j!} \frac{\mathrm{d}^j h^{[s]}(x+\theta_0)}{\mathrm{d}x^j}(\theta - \theta_0)^j \right)(\theta - \theta_0)^s$$

$$= x + \theta_0 + (\theta - \theta_0) + \sum_{s \ge 0} \left(\sum_{\substack{j=0 \\ j \ge s-m}}^{s} \frac{1}{j!} \frac{\mathrm{d}^j h^{[s-j]}(x+\theta_0)}{\mathrm{d}x^j} \right)(\theta - \theta_0)^s .$$

Notice that the Fourier coefficients of $H^{[s]}(x)$, $\widehat{H}_k^{[s]}$, are just finite linear combinations (obtained from derivatives and shifts of angle θ_0) of the Fourier coefficients of $h^{[l]}(x)$, $0 \le l \le s$.

Putting together the two terms, we obtain (notice that the affine part in the $[0]^{\mathrm{th}}$ coefficient cancels out)

$$e^{[s]}(x) = F^{[s]}(x) - H^{[s]}(x), \qquad 0 \le s \le m.$$

Therefore, we obtain the Fourier coefficients of the approximations $\tilde{e}^{[s]}$ as

$$\tilde{e}_k^{[s]} = \mathrm{DFT}_k(\{F^{[s]}(x_j)\}) - \widehat{H}_k^{[s]}, \qquad 0 \le s \le m,$$

which corresponds to (5.22) by linearity.

Step 2: On the one hand, we approximate $F_B^{[m+1]}(x)$ by evaluating the recurrences (5.19) and (5.20) in the grid x_j. On the other hand, we take

$$H_B^{[m+1]}(x) := \sum_{j=1}^{m+1} \frac{1}{j!} \frac{\mathrm{d}^j h^{[m+1-j]}(x+B)}{\mathrm{d}x^j} ,$$

thus obtaining

$$\tilde{e}_{B,k}^{[m+1]} = \mathrm{DFT}_k(\{F_B^{[m+1]}(x_j)\}) - \widehat{H}_{B,k}^{[m+1]} .$$

We thus define

$$C_T := \left\| \tilde{e}_B^{[m+1]} \right\|_\rho^{\mathcal{F}} .$$
(5.24)

Step 3: We take into account the error produced when approximating $e^{[s]}(x)$ using discrete Fourier approximation. We control first the term $e^{[0]}(x) - \tilde{e}^{[0]}(x)$ using Theorem 5.1, obtaining

$$\|e^{[0]} - \tilde{e}^{[0]}\|_\rho = \|F^{[0]} - \tilde{F}^{[0]}\|_\rho \le C_N(\rho, \tilde{\rho}) \|F^{[0]}\|_{\tilde{\rho}} = C_N(\rho, \tilde{\rho}) \mathcal{F}_0 ,$$

where, for convenience, we have introduced the notation

$$\mathcal{F}_0 := \|f_{\alpha_0}(\mathrm{id} + h^{[0]}) - \mathrm{id}\|_{\tilde{\rho}} .$$

Notice that $\mathcal{F}_0 < \infty$ due to the assumption in (5.21). Since $H^{[s]}(x)$ are trigonometric polynomials, we have

$$\|e^{[s]} - \tilde{e}^{[s]}\|_\rho = \|F^{[s]} - \tilde{F}^{[s]}\|_\rho \le C_N(\rho, \tilde{\rho}) \|F^{[s]}\|_{\tilde{\rho}} .$$

Recalling that $F^{[s]}(x)$ satisfy the recurrences (5.18) we obtain

$$\|F^{[s]}\|_{\tilde{\rho}} = \left\| \mathfrak{F}_s(\mathrm{id} + h^{[0]}, \alpha^{[0]}; h^{[1]}, \dots, h^{[s]}, \alpha^{[0]}, \dots, \alpha^{[s]}, F^{[1]}, \dots, F^{[s-1]}) \right\|_{\tilde{\rho}}$$

(5.25)
$$\le \sup_{x \in \mathbb{T}_{\tilde{\rho}}} \mathfrak{G}_s \Big(x + h^{[0]}(x), \alpha^{[0]}; \|h^{[1]}\|_{\tilde{\rho}}^{\mathcal{F}}, \dots, \|h^{[s]}\|_{\tilde{\rho}}^{\mathcal{F}}, |\alpha^{[1]}|, \dots, |\alpha^{[s]}|, $$

$$\mathcal{F}_1, \dots, \mathcal{F}_{s-1} \Big) =: \mathcal{F}_s ,$$

where the majorant recurrences $\mathfrak{G}_s$ are obtained by applying triangular inequalities, Banach algebra properties, and $\|\cdot\|_{\tilde{\rho}} \le \|\cdot\|_{\tilde{\rho}}^{\mathcal{F}}$ in the expression of the recurrence $\mathfrak{F}_s$. Notice that the control of the supremum can be performed using the ideas in Proposition 5.5 and optimal bounds are easily obtained for each particular problem at hand.

Similarly, we control the term $e_B^{[m+1]}(x) - \tilde{e}_B^{[m+1]}(x)$ as follows

$$\|e_B^{[m+1]} - \tilde{e}_B^{[m+1]}\|_\rho = \|F_B^{[m+1]} - \tilde{F}_B^{[m+1]}\|_\rho \le C_N(\rho, \tilde{\rho}) \|F_B^{[m+1]}\|_{\tilde{\rho}} \le C_N(\rho, \tilde{\rho}) \mathcal{F}_{B,m+1}$$

where $\mathcal{F}_{B,m+1}$ is obtained using analogous recurrences

$$\mathcal{F}_{B,s} := \overline{\mathcal{G}_{B,s}} ,$$

(5.26)
$$\mathcal{G}_{B,s} := \sup_{x \in \mathbb{T}_{\tilde{\rho}}} \mathfrak{G}_s \Big(x + h_B^{[0]}(x), \alpha_B^{[0]}; \|h_B^{[1]}\|_{\tilde{\rho}}^{\mathcal{F}}, \dots, \|h_B^{[s]}\|_{\tilde{\rho}}^{\mathcal{F}}, |\alpha_B^{[1]}|, \dots, |\alpha_B^{[s]}|, $$

$$\mathcal{F}_{B,1}, \dots, \mathcal{F}_{B,s-1} \Big) \subset \mathbb{R} ,$$

where we used the notation in (5.9) for the right boundary of an interval. Notice that these recurrences are initialized as

$$\mathcal{F}_{B,0} := \|f_{\alpha_0}(\mathrm{id} + h_B^{[0]}) - \mathrm{id}\|_{\tilde{\rho}} ,$$

and that the last term is evaluated taking $h^{[m+1]}(x) = 0$ and $\alpha^{[m+1]} = 0$.

We finally define

$$C_F := \sum_{s=0}^{m} \mathcal{F}_s \, \mathrm{rad}(B)^s + \mathcal{F}_{B,m+1} \, \mathrm{rad}(B)^{m+1} ,$$

and complete the estimate for $\|e\|_{\Theta, \rho}$.

Step 4: Since the objects $h(x, \theta)$ and $\alpha(\theta)$ are polynomials with respect to θ, we have that $e_\theta(x)$ is smooth in the domain $B \supset \Theta$. Hence, we can control $\mathrm{Lip}_{\Theta,\rho}(e) \leq \|\partial_\theta e\|_{B,\rho}$ taking derivatives in our Taylor-model. The estimate in the statement follows directly with the constant C_F' given by

$$C_F' := \sum_{s=1}^{m} s\mathcal{F}_s \, \mathrm{rad}(B)^{s-1} + (m+1)\mathcal{F}_{B,m+1} \, \mathrm{rad}(B)^m \, . \qquad \square$$

Remark 5.9. A quite technical observation is that along the proof we propose the use of the Fourier norm to control several trigonometric polynomials (see (5.24), (5.25) and (5.26)), rather than using the estimate $\mathfrak{M}_{B,\rho}(\cdot)$. The reason is that these objects have quite large analytic norms so both approaches produce equivalent estimates, but the advantage of the Fourier norm is that it is faster to evaluate. However, in condition (5.21) we use $\mathfrak{M}_{\theta_0,\rho}(\cdot)$ instead since it produces a sharper estimate.

6. APPLICATION IN AN EXAMPLE

To complement the exposition and the effectiveness of the estimates, we illustrate the performance of our rigorous estimates with an example. For a given value of $\varepsilon \in [0, 1)$, we consider the Arnold family

$$(6.1) \qquad \alpha \in [0, 1) \longmapsto f_\alpha(x) = x + \alpha + \frac{\varepsilon}{2\pi} \sin(2\pi x)$$

and apply our a-posteriori theorem to obtain effective bounds for the measure of parameters α that correspond to conjugacy to rigid rotation.

Obtaining candidates for $h(x, \theta)$ and $\alpha(\theta)$: Given a fixed rotation number θ_0, the functions $h^{[s]}(x)$ and the numbers $\alpha^{[s]}$, $0 \leq s \leq m$, are determined by performing Lindstedt-series at the point $\theta = \theta_0$:

- We compute a trigonometric polynomial $h^{[0]}(x)$ and a constant $\alpha^{[0]}$, that approximate the objects $h_0(x) \simeq x + h^{[0]}(x)$ and $\alpha_0 \simeq \alpha^{[0]}$ corresponding to the conjugacy of the map $f_{\alpha_0}(x)$, at a selected value of ε, to a rigid rotation of angle θ_0. The pair is obtained by numerical continuation with respect to $\varepsilon \in [0, \varepsilon_0]$ of the initial objects $h_0(x) = x$ and $\alpha_0 = \theta_0$ that conjugate the case $\varepsilon = 0$ from the case $\varepsilon = 0$ (with initial objects $h_0(x) = x$ and $\alpha_0 = \theta_0$). The interested reader is referred to [19] for implementation details of this numerical method and to [8, 9, 10, 11, 28, 34] for other contexts where numerical algorithms have been designed from a-posteriori KAM-like theorems.
- We then compute approximations for the higher order terms of the Lindstedt series, i.e. trigonometric polynomials $h^{[s]}(x)$ and numbers $\alpha^{[s]}$, for $1 \leq s \leq m$. Specifically, we assume inductively that we have computed the exact Lindstedt series up to order ℓ

$$(6.2) \qquad h_\ell(x, \theta) = x + \sum_{s=0}^{\ell} h^{[s]}(x)(\theta - \theta_0)^s \, , \qquad \alpha_\ell(\theta) = \sum_{s=0}^{\ell} \alpha^{[s]}(\theta - \theta_0)^s \, .$$

Then, we compute the partial error of conjugacy

$$e_\ell(x, \theta) = \sum_{s \geq \ell+1} e_\ell^{[s]}(x)(\theta - \theta_0)^s := f_{\alpha_\ell(\theta)}(h_\ell(x, \theta)) - h_\ell(x + \theta, \theta)$$

using the rules of Automatic Differentiation for the composition with the sinus function. Then, the terms of order $\ell + 1$ satisfy the equation

$$\partial_x f_{\alpha_0}(h_0(x))h^{[\ell+1]}(x) - h^{[\ell+1]}(x+\theta_0) + \partial_\alpha f_{\alpha_0}(h_0(x))\alpha^{[\ell+1]} = -e_\ell^{[\ell+1]}(x)$$

which has the same structure as the linearized equation (3.6), so its solutions are approximated using trigonometric polynomials by evaluating formulae (3.13) in Fourier space.

Unless otherwise stated, all computations discussed from now onwards are performed using at most $N = 2048$ Fourier coefficients, requesting that the error of invariance at θ_0 satisfies $\|e\|_0 < 10^{-35}$, and the Lindstedt series is computed up to order $m = 9$.

Following Section 5, since the objects (6.2) are polynomials in θ, we denote by B the interval (centered at θ_0) were they are evaluated. Regarding the length of the interval B, it is clear that the application of the KAM theory fails if $\mathrm{rad}(B)$ is too large, so we may need to split the interval B into subintervals with non-overlapping interior. We carry out a *branch and bound* procedure, applying the KAM theorem in a subinterval $B_0 \subset B$ and repeating the procedure by splitting the set $B \backslash B_0$ into two smaller intervals. We stop when the intervals are small enough.

Given the numerical approximation described above, the estimates described in Section 5 are evaluated. In the terminology of validated computations, if we can apply Theorem 3.1 successfully, obtaining explicit control of the pair $h_0(x)$ and α_0, we say that the numerical computation has been rigorously validated. Moreover, if we can apply Theorem 4.1 we say that we have validated a family of conjugacies $h(x, \theta)$ obtaining a rigorous lower bound of the measure of parameters α which leads to conjugation. It is worth mentioning that we are not limited to use Lindstedt series to obtain a candidate for $h(x, \theta)$ and $\alpha(\theta)$. One should notice (see Sections 4 and 5) that the arguments do not depend on how the candidates are obtained.

Global constants for hypothesis $\mathfrak{G}_1$*:* As it was mentioned in Section 5.1, this part is problem dependent. Given $\hat{\rho} > 0$, the derivatives of the Arnold map (6.1) are controlled as

$$\begin{aligned}
c_x &= 1 + \varepsilon \cosh(2\pi\hat{\rho})\,, \\
c_\alpha &= 1\,, \\
c_{xx} &= 2\pi\varepsilon \cosh(2\pi\hat{\rho})\,, \\
c_{xxx} &= 4\pi^2\varepsilon \cosh(2\pi\hat{\rho})\,, \\
c_{x\alpha} &= c_{\alpha\alpha} = c_{x\alpha\alpha} = c_{\alpha\alpha\alpha} = 0\,.
\end{aligned}$$

Global constants for hypothesis $\mathfrak{G}_2$*:* Given an interval of rotation numbers $B \subset (0, 1)$ centered at θ_0, the parameters γ and τ are selected to guarantee that the set of Diophantine numbers in the set $\Theta = B \cap \mathcal{D}(\gamma, \tau)$ reaches a prefixed relative measure. To this end, we use Lemma 5.2 asking for a relative measure of 99%. This fulfills the hypothesis $\mathfrak{G}_2$. For example, for an interval with $\mathrm{rad}(B) = 1/2^{14}$ centered on $\theta_0 = (\sqrt{5} - 1)/2$, such relative measure is achieved taking $\gamma = 0.0009765625$ and $\tau = 1.2$. Note that, when dividing the interval B into subintervals, suitable constants γ and τ are recomputed. Thus, we take into account the resonances that affect only each subinterval.

Hypotheses $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$: Using the numerical candidates, we proceed to invoke Proposition 5.5. The finite amount of computations are rigorously performed using computer interval arithmetics. For example, for $\varepsilon = 0.25$, $\theta_0 = (\sqrt{5} - 1)/2$ and $\mathrm{rad}(B) = 1/2^{14}$, we obtain

$$\mathfrak{M}_{B,\rho}(\partial_x h) < 1.16741651\,,$$
$$1/\mathfrak{m}_{B,\rho}(\partial_x h) < 1.16623515\,,$$
$$\mathfrak{M}_{B,\rho}(\partial_{xx} h) < 1.14231325\,,$$
$$\overline{|1/\tilde{b}_{B,0}|}/(1 - c_b) < 0.990541647\,,$$
$$\mathfrak{M}_{B,\rho}(\partial_\theta h) < 0.114433851\,,$$
$$\mathfrak{M}_{B,\rho}(\partial_{x\theta} h) < 0.842060023\,,$$
$$\overline{\alpha'(B)} < 0.990774718\,,$$
$$\underline{\alpha'(B)} > 0.990772454\,.$$

Rigorous control of the error of invariance: Let us first describe the recurrences $\mathfrak{F}_s$ and $\mathfrak{G}_s$, associated to the family (6.1), which are used to compute the estimate produced in Theorem 5.7. Given series

$$h(x,\theta) = x + \sum_{s=0}^{m} h^{[s]}(x)(\theta - \theta_0)^s\,, \qquad \alpha(\theta) = \sum_{s=0}^{m} \alpha^{[s]}(\theta - \theta_0)^s$$

we have that the coefficients of

$$F(x,\theta) = f(h(x,\theta), \alpha(\theta)) = x + \sum_{s \geq 0} F^{[s]}(x)(\theta - \theta_0)^s\,,$$

are obtained using the formula $\mathfrak{F}_s$, which in this case corresponds to evaluate the recurrences

$$F^{[s]}(x) = h^{[s]}(x) + \alpha^{[s]} + \frac{\varepsilon}{2\pi} S^{[s]}(x)\,,$$

where

$$S^{[0]}(x) = \sin(2\pi(x + h^{[0]}(x)))\,,$$
$$C^{[0]}(x) = \cos(2\pi(x + h^{[0]}(x)))\,,$$

and, for $s \geq 1$,

$$S^{[s]}(x) = \frac{2\pi}{s} \sum_{j=0}^{s-1} (s - j) h^{[s-j]}(x) C^{[j]}(x)\,,$$

$$C^{[s]}(x) = -\frac{2\pi}{s} \sum_{j=0}^{s-1} (s - j) h^{[s-j]}(x) S^{[j]}(x)\,.$$

When the time comes to control the error produced with Fourier discretization, these formulae lead to (in this case the functions $\sin(\cdot)$ and $\cos(\cdot)$ have the same bounds)

$$\|F^{[s]}\|_{\tilde{\rho}} \leq \|h^{[s]}\|_{\tilde{\rho}} + |\alpha^{[s]}| + \frac{\varepsilon}{2\pi} \mathcal{S}_s = \mathcal{F}_s\,,$$

for $s \geq 0$, where the constants $\mathcal{S}_s$ are initialized as

$$\mathcal{S}_0 = \cosh(2\pi(\tilde{\rho} + \|h^{[0]}\|_{\tilde{\rho}}^{\mathcal{F}}))$$

and then obtained recursively, for $s \geq 1$, as

$$\mathcal{S}_s = \frac{2\pi}{s} \sum_{j=0}^{s-1} (s-j)\|h^{[s-j]}\|_{\tilde{\rho}}^{\mathcal{F}} \, \mathcal{S}_j \,.$$

Analogous formulae are used to compute $\mathcal{F}_{B,m+1}$.

Selection of the remaining KAM parameters: Finally, there is a set of parameters that must be selected in order the apply the a-posteriori theorems: ρ, δ, ρ_∞, $\hat{\rho}$, $\tilde{\rho}$ and parameters to control the initial objects. To choose the parameters σ_1, σ_2, σ_3, σ_b, β_0, β_1 and β_2, we use a single parameter $\sigma > 1$ together with the sharp estimates produced in Proposition 5.5. For example, we take $\sigma_1 = \mathfrak{M}_{B,\rho}(\partial_x h)\sigma$. Suitable KAM parameters are selected following a heuristic procedure, adapted mutatis mutandis from [26, Appendix A], that allows us to optimize the values of the constants $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_1^{\mathrm{Lip}}$, and $\mathfrak{C}_2^{\mathrm{Lip}}$.

Some specific results: Now that the implementation details have been specified, our goal is to use Theorem 4.1 to estimate the measure of rotations in the full interval $B = (0,1)$ for a specific value of ε. For the sake of concreteness we set $\varepsilon = 0.25$. This parameter value is large enough, so that asymptotic measure estimates are of little use, while keeping the computational cost at a reasonable level for the purpose of illustration. Moreover, a large range of values of ε in $(0,1)$ is considered later for a restricted range of rotations (see Table 1).

For these computations we restrict the estimates to a set of Diophantine numbers $\Theta \subset B$ such that $\mathrm{Leb}(\Theta) > 0.99$. After applying Theorem 4.1 we obtain a lower bound $\mathrm{Leb}(\alpha_\infty(\Theta)) > 0.860748$ for the absolute measure of parameters which correspond to rotation. To see how sharp is this estimate, we compute also a lower bound of the measure of the phase-locking intervals, thus obtaining

$$0.085839 < \mathrm{Leb}([0,1]\backslash\alpha_\infty(\Theta)) \,.$$

This lower bound follows by computing (using standard rigorous Newton method) two p/q-periodic orbits close to the boundaries of each interval of rotation $p/q \in \mathbb{Q}$, for $q \leq 20$. To summarize, in terms of the notation in the introduction, we have

$$0.860748 < \mathrm{Leb}(K_{0.25}) < 0.914161 \,.$$

Notice that, assuming that the measure was $\mathrm{Leb}(K_{0.25}) \simeq 0.914161$, our rigorous lower bounds predicts the 94.15% of the measure.

Of course, most part of underestimation corresponds to the resonances $0/1$, $1/2$, $1/3$, $2/3$, $1/4$ and $3/4$. If we remove the set of subintervals (of total measure 0.082046) were we fail to apply the theorem (mostly around the mentioned resonances), then it turns out that the produced lower bound corresponds to a relative measure of 93.76% in the considered set of rotations B (which now has measure 0.917954). Indeed, for the interval $B = (391/1024, 392/1024)$ (which contains $(3 - \sqrt{5})/2$) we obtain a relative lower bound of 98.26% for the existence of conjugacies. These lower bounds can be improved by increasing the number of Fourier coefficients and the tolerances, but these numbers serve as an illustration with moderate computational effort. For example, using a single desktop computer, the validation of the interval $B = (391/1024, 392/1024)$ takes around 10 minutes (49 subdivisions of B are required) and the validation of the interval $B = (0,1)$ takes around 30 days (161891 subdivisions of B are required). As expected, the

bottleneck are resonant rotation numbers which do not contribute in practice. The memory requirements are negligible ($\simeq 20$ MB per interval) since they are proportional to the number of used Fourier coefficients N, and we are using at most $N = 2048$.

Table 1 shows how the lower bound depends on the selected value of ε_0. In order to illustrate the computational cost of the validation, we show the largest interval B (of length of the form $1/2^n$), centered at the golden number, such that the theorem can be applied without subdividing the interval. This, together with the number of Fourier coefficients required, depicts the technological difficulty to apply the KAM theory when one approaches the critical limit.

ε_0	N	$\mathrm{Leb}(\alpha_\infty(\Theta))/\mathrm{Leb}(B) >$
$1/2^7 = 0.0078125$	64	0.999533 $(\mathrm{Leb}(B) = 1/2^{12})$
$10/2^7 = 0.078125$	128	0.998661 $(\mathrm{Leb}(B) = 1/2^{12})$
$20/2^7 = 0.15625$	256	0.995996 $(\mathrm{Leb}(B) = 1/2^{14})$
$30/2^7 = 0.234375$	256	0.991461 $(\mathrm{Leb}(B) = 1/2^{14})$
$40/2^7 = 0.3125$	256	0.984921 $(\mathrm{Leb}(B) = 1/2^{15})$
$50/2^7 = 0.390625$	512	0.976080 $(\mathrm{Leb}(B) = 1/2^{17})$
$60/2^7 = 0.46875$	512	0.964547 $(\mathrm{Leb}(B) = 1/2^{17})$
$70/2^7 = 0.546875$	512	0.949686 $(\mathrm{Leb}(B) = 1/2^{18})$
$80/2^7 = 0.625$	1024	0.930482 $(\mathrm{Leb}(B) = 1/2^{19})$
$90/2^7 = 0.703125$	1024	0.905233 $(\mathrm{Leb}(B) = 1/2^{19})$
$100/2^7 = 0.78125$	1024	0.870752 $(\mathrm{Leb}(B) = 1/2^{20})$
$110/2^7 = 0.859375$	2048	0.819862 $(\mathrm{Leb}(B) = 1/2^{22})$
$120/2^7 = 0.9375$	4096	0.728697 $(\mathrm{Leb}(B) = 1/2^{25})$
$123/2^7 = 0.9609375$	8192	0.678925 $(\mathrm{Leb}(B) = 1/2^{26})$
$125/2^7 = 0.9765625$	16384	0.627992 $(\mathrm{Leb}(B) = 1/2^{28})$

TABLE 1. Rigorous lower bounds for the measure of the conjugacy of rotations for the Arnold map (for different values of ε_0) corresponding to a small interval B of rotation numbers centered at $\theta_0 = (\sqrt{5} - 1)/2$. For each ε_0 we choose the largest interval that allows us to apply Theorem 4.1 without subdividing it. We include also the number of Fourier coefficients used.

For the sake of completeness, we finally include a list with some parameters and intermediate estimates associated to the computer-assisted proof corresponding to $\varepsilon_0 = 0.25$ and an interval with $\mathrm{rad}(B) = 1/2^{14}$ centered on $\theta_0 = (\sqrt{5} - 1)/2$:

$$\rho = 1.060779991992726 \cdot 10^{-2},$$
$$\rho_\infty = 1.060779991992726 \cdot 10^{-5},$$
$$\delta = 2.651949979981816 \cdot 10^{-3},$$
$$\hat{\rho} = 3.352069177291399 \cdot 10^{-2},$$
$$\tilde{\rho} = 1.272935990391272 \cdot 10^{-1},$$
$$\sigma = 1.000107420067662,$$
$$C_T = 5.898764259376722 \cdot 10^{17},$$
$$C_F = 7.510700397297086 \cdot 10^{-1},$$
$$\|e\|_{\Theta,\rho} < 5.555151225281469 \cdot 10^{-24},$$

$$\mathrm{Lip}_{\Theta,\rho}(e) < 9.101559767500465 \cdot 10^{-19} \,,$$
$$\kappa < 3.000359067414863 \cdot 10^{-11} \,,$$
$$\mu < 5.097018817302322 \cdot 10^{-11} \,,$$
$$\mathfrak{C}_1^{\mathrm{Lip}} < 5.364258457375897 \cdot 10^5 \,,$$
$$\mathfrak{C}_2^{\mathrm{Lip}} < 2.440549276176043 \cdot 10^1 \,,$$
$$\frac{\mathfrak{C}_1^{\mathrm{Lip}}}{\gamma^2 \rho^{2\tau+2}} \mathcal{E} < 6.778257281684347 \cdot 10^{-1} \,,$$
$$\frac{\mathfrak{C}_2^{\mathrm{Lip}}}{\gamma \rho^{\tau+1}} \mathcal{E} < 6.466059566862642 \cdot 10^{-14} \,,$$

where we recall that

$$\mathcal{E} := \max\left\{ \|e\|_{\Theta,\rho} \,,\; \gamma \delta^{\tau+1} \mathrm{Lip}_{\Theta,\rho}(e) \right\} \,.$$

Then, we apply Theorem 4.1 and obtain a lower bound $\mathrm{Leb}(\alpha_\infty(\Theta)) > 0.000120024$ of the absolute measure, which corresponds to a relative measure of 98.32% in the selected interval.

7. Final remarks

To finish we include some comments regarding direct applications and generalizations of the results presented in the paper. Our aim is to present a global picture of our approach and to establish connections with different contexts.

Asymptotic estimates in the local reduction case. Although Theorem 4.1 has been developed with the aim of performing computer-assisted applications in non-perturbative regimes, it is clear it allows recovering the perturbative setting. Indeed, as a direct corollary, we obtain asymptotic measures (à la Arnold) of conjugacies close to rigid rotation in large regions of parameters.

Corollary 7.1. *Consider a family of the form*

$$f_\alpha(x) = x + \alpha + \varepsilon g(x) \,,$$

with $g \in \mathrm{Per}(\mathbb{T}_{\hat\rho})$, $\hat\rho > 0$. Then there is constants $\mathfrak{C}_1^{\mathrm{Lip}}$, $\mathfrak{C}_2^{\mathrm{Lip}}$ (which are directly computed using Theorem 4.1) such that if ε satisfies

$$\varepsilon \frac{\mathfrak{C}_1^{\mathrm{Lip}} \|g\|_\rho}{\gamma^2 \rho^{2\tau+2}} < 1 \,,$$

with $\rho < \hat\rho$, $\gamma < 1/2$ and $\tau > 1$, then we have

$$\mathrm{Leb}(\alpha_\infty(\Theta)) \geq \left(1 - \varepsilon \frac{\mathfrak{C}_2^{\mathrm{Lip}} \|g\|_\rho}{\gamma \rho^{\tau+1}}\right) \left(1 - 2\gamma \frac{\zeta(\tau)}{\zeta(\tau+1)}\right)$$

$$> 1 - \gamma \left(\frac{\mathfrak{C}_2^{\mathrm{Lip}} \rho^{\tau+1}}{\mathfrak{C}_1^{\mathrm{Lip}}} - 2 \frac{\zeta(\tau)}{\zeta(\tau+1)}\right) + \mathcal{O}(\gamma^2) \,,$$

where $\Theta = [0,1] \cap \mathcal{D}(\gamma,\tau)$ and ζ is the Riemann zeta function.

Proof. We consider the candidates

$$h(x,\theta) = x \,, \qquad \alpha(\theta) = \theta \,,$$

and apply Theorem 4.1. First, we notice that the error of conjugacy is of the form $e(x) = \varepsilon g(x)$, independent of θ, so we have the estimates

$$\|e\|_{\Theta,\rho} \le \varepsilon \|g\|_\rho \,, \qquad \mathrm{Lip}_{\Theta,\rho}(e) = 0 \,.$$

To satisfy the hypothesis $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ we introduce an uniform parameter $\sigma > 1$ and we take

$$\sigma_1 = \sigma_2 = \sigma_b = \beta_2 = \sigma > 1 \,,$$
$$\sigma_3 = \beta_0 = \beta_1 = \sigma - 1 > 0 \,.$$

Then, we compute the constant $\mathfrak{C}_1^{\mathrm{Lip}}$ and assume that ε is small enough so condition (4.3) holds. Indeed, the largest value of ε that saturates this condition can be obtained by selecting a suitable value of σ. Then, the measure of parameters is given by the formula

$$\mathrm{Leb}(\alpha_\infty(\Theta)) \ge \left[1 - \frac{\mathfrak{C}_2^{\mathrm{Lip}} \varepsilon \|g\|_\rho}{\gamma \rho^{\tau+1}} \right] \mathrm{Leb}(\Theta) \,.$$

The statement follows recalling the estimate for $\mathrm{Leb}(\Theta)$ in Remark 5.3.   □

Conjugation of maps on $\mathbb{T}^d$. Theorem 3.1 and Theorem 4.1 can be readily extended to consider a family $\alpha \in A \subset \mathbb{R}^d \mapsto f_\alpha \in \mathcal{A}(\mathbb{T}_\rho^d)$. In this case, a map $g : \mathbb{T}^d \to \mathbb{T}^d$ is viewed as a vector, and the norm $\|g\|_\rho$, for $g \in \mathrm{Per}(\mathbb{T}_\rho)$, is taken as the induced norm. Then, all the arguments and computations are extended to matrix object with no special difficulty. For example, the torsion matrix becomes

$$b(x) = \mathrm{D}_x h(x + \theta)^{-1} \mathrm{D}_\alpha f(h(x))$$

where $\theta \in \mathbb{R}^d$ is the selected Diophantine vector. The interested reader is referred to [12] for details regarding the corresponding Theorem 3.1 and to [26] for details regarding the approximation of functions in $\mathbb{T}^d$ using discrete Fourier transform.

Other KAM contexts. We have paid special attention to present Theorem 3.1 and Theorem 4.1 separately. The important message is that our methodology can be readily extended to any problem, as long as there exists an a-posteriori KAM theorem (which replaces Theorem 3.1) for the existence of quasi-periodic dynamics: Lagrangian tori in Hamiltonian systems or symplectic maps [18, 28, 30], dissipative systems [8], skew-product systems [25, 28, 29] or lower dimensional tori [27, 47], just to mention a few. For each of such theorems, a judicious revision of the corresponding KAM scheme must be performed in order to obtain sharp estimates of the Lipschitz dependence on parameters (which replaces Theorem 4.1). It is worth mentioning that we have devoted a significant effort to explain, in common analytic terms, the technical issues that permits the computer-assisted validation of the hypothesis of our theorems. This is an important step that prevents the computer-assisted proof to be a "black box" full of tricks that can be only appreciated by a few experts.

Rigorous validation of rotation numbers. Last but not least, another significant corollary of the methodology developed in this paper is that it allows to rigorously enclose the rotation number of a circle map. Due to the relevance of this topological invariant, during the last years, many numerical methods have been developed for this purpose. We refer for example to the works [7, 41, 45, 46, 51, 56] and to [17] for

a remarkable method with infinite order convergence, based on Birkhoff averages. As illustrated in Section 6, the KAM theorems presented in this paper allows us to obtain a rigorous enclosure (as tight as required) for the rotation number of a map, which is interesting in order to rigorously validate the numerical approximations performed with any of the mentioned numerical methods.

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