BILINEAR FORMS ON POTENTIAL SPACES IN THE UNIT CIRCLE

CARME CASCANTE AND JOAQUÍN M. ORTEGA

Abstract. In this paper we characterize the boundedness on the product of Sobolev spaces $H^s(\mathbb{T})\times H^s(\mathbb{T})$ on the unit circle $\mathbb{T}$, of the bilinear form Λ_b with symbol $b\in H^s(\mathbb{T})$ given by

$$\Lambda_b(\varphi,\psi):=\int_{\mathbb{T}}((-\Delta)^s+I)(\varphi\psi)(\eta)b(\eta)d\sigma(\eta).$$

1. Introduction

In [20], V.G. Maz'ya and I.E. Verbitsky characterize the class of measurable functions V such that the Schrödinger operator $-\Delta+V$ maps the homogeneous Sobolev space $\dot{W}^{1,2}(\mathbb{R}^n)$ to its dual, obtaining necessary and sufficient conditions for the classical inequality

$$\left|\int_{\mathbb{R}^n}(\varphi(x))^2V(x)dx\right|\leq C\int_{\mathbb{R}^n}|\nabla\varphi(x)|^2dx,\quad u\in\mathcal{D}(\mathbb{R}^n),$$

to hold. They also obtained analogous characterizations for the non homogeneous Sobolev space $W^{1,2}(\mathbb{R}^n)$. In this paper we will consider a similar problem on the unit circle $\mathbb{T}$ for the space $W^{s,2}(\mathbb{T})$, $0<s<1/2$.

The space $W^{s,2}(\mathbb{T})$ $s>0$, is the space of functions $\varphi\in L^2(\mathbb{T})$ such that if $(\widehat{\varphi}(k))_{k\in\mathbb{Z}}$ is the sequence of its Fourier coefficients, then

$$\|\varphi\|_{W^{s,2}(\mathbb{T})}:=\left(\sum_{k\in\mathbb{Z}}(1+|k|^s)^2|\widehat{\varphi}(k)|^2\right)^{\frac{1}{2}}<\infty.$$

When $0<s<1$ and for functions in $\mathcal{C}^\infty(\mathbb{T})$, this norm is equivalent to $\|\varphi\|_{L^2(\mathbb{T})}+\|(-\Delta)^s\varphi\|_{L^2(\mathbb{T})}$, where $(-\Delta)^s$ is the fractional laplacian defined, up to a constant,

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by

$$(-\Delta)^s(\varphi)(\zeta) = P.V. \int_{\mathbb{T}} \frac{\varphi(\zeta)-\varphi(\eta)}{|\zeta-\eta|^{1+2s}} d\sigma(\eta)$$

and the space $W^{s,2}(\mathbb{T})$ coincides with the completion of $\mathcal{C}^\infty(\mathbb{T})$ with respect to this norm. In turn, this space coincides with the space of Riesz potentials, that we will denote by $H^s(\mathbb{T}) := I_s(L^2(\mathbb{T}))$, where I_s is the Riesz kernel defined by $I_s(\zeta,\eta) = \frac{\Gamma((1+s)/2)^2}{\Gamma(s)} \frac{1}{|1-\zeta\overline{\eta}|^{1-s}}$ and $\|\varphi\|_{H^s(\mathbb{T})} = \|\psi\|_{L^2(\mathbb{T})}$, if $\varphi = I_s(\psi)$.

We are interested in the case where $0 < s < 1/2$. When $1/2 < s < 1$, $H^s(\mathbb{T})$ is an algebra and the problem that we will consider becomes trivial.

Let Λ_b be the bilinear form with symbol $b \in H^s(\mathbb{T})$ given by

$$\Lambda_b(\varphi,\psi) := \int_{\mathbb{T}} \left((-\Delta)^s + I\right)(\varphi\psi)(\eta) b(\eta) d\sigma(\eta).$$

The main object of this paper is the characterization of the symbols $b \in H^s(\mathbb{T})$ for which the bilinear form Λ_b is bounded on $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, that is,

$$|\Lambda_b(\varphi,\psi)| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})}. \tag{1.1}$$

This problem is equivalent (see Proposition 4.6), to the characterization of the functions $c \in L^2(\mathbb{T})$ that are trace measures (that may change sign) for the space $H^s(\mathbb{T})$, i.e.,

$$\left|\int_{\mathbb{T}} |\varphi|^2 c\, d\sigma\right| \lesssim \|\varphi\|^2_{H^s(\mathbb{T})},$$

In $\mathbb{R}^n$, V.G. Maz'ya and I.E. Verbitsky considered this problem for $s = 1$ (see [20]), showing that the inequality $\left|\int_{\mathbb{R}^n} |\varphi|^2 c\, d\sigma\right| \lesssim \|\varphi\|^2_{H^1(\mathbb{R}^n)}$, is equivalent to the inequality $\left|\int_{\mathbb{R}^n} |\varphi|^2 |(-\Delta)^{-1/2}(c)|^2\, d\sigma\right| \lesssim \|\varphi\|^2_{H^1(\mathbb{R}^n)}$, where $|(-\Delta)^{-1/2}(c)|^2$ is now a non negative measure (see also [21] and [15] for related problems). In [11] it is considered the case $0 < s < 1/2$ in $\mathbb{R}$. We also recall that N. Arcozzi, E. Sawyer and B.D. Wick in [5] have considered a result on the boundedness of a holomorphic version of this problem on the Dirichlet space (see also [8] for a different proof).

Some of the main difficulties when dealing with fractional laplacians arise from the fact that on one hand these operators are non-local and on the other hand, there is a complexity on the computation of fractional laplacians when applied to products of functions. In order to avoid these difficulties, we will follow the ideas in [11] and consider an equivalent bilinear problem on a subspace of a weighted Sobolev space $\mathcal{W}^2_{1,1-2s}(\mathbb{D})$, of extensions of functions on $H^s(\mathbb{T})$ by a generalized Poisson operator P_s whose definition is given in Section 2. For $\mathbb{R}^n$, a similar extension operator was considered by L. Caffarelli and L. Silvestre in [7].

Our main result is the following

Theorem 1.1. *Let $0 < s < 1/2$ and let $b \in H^s(\mathbb{T})$. Then, the following assertions are equivalent:*

(i) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$|\Lambda_b(\varphi,\psi)| \lesssim \|u\|_{H^s(\mathbb{T})}\|v\|_{H^s(\mathbb{T})};$$

(ii) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$\left|\int_{\mathbb{D}} \nabla(P_s(\varphi\psi))\nabla(P_s(b))(1-|z|^2)^{1-2s}dm(z) \right.$$
$$\left. +(1-2s)^2\int_{\mathbb{D}} P_s(\varphi\psi)P_s(b)(1-|z|^2)^{-2s}dm(z)\right| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})};$$

(iii) *For any* $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$,

$$\left|\int_{\mathbb{D}} \nabla(P_s(\varphi)P_s(\psi))\nabla(P_s(b))(1-|z|^2)^{1-2s}dm(z) \right.$$
$$\left. +(1-2s)^2\int_{\mathbb{D}} P_s(\varphi)P_s(\psi)P_s(b)(1-|z|^2)^{-2s}dm(z)\right| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})};$$

(iv) *The measure* $d\nu := \left|(-\Delta)^{\frac{s}{2}}(b)\right|^2 d\sigma$ *is a trace measure for* $H^s(\mathbb{T})$*, that is,* $H^s(\mathbb{T}) \subset L^2(d\nu)$*;*

(v) *The measure* $d\mu := |\nabla(P_s(b))|^2(1-|z|^2)^{1-2s}dm(z)$ *is a Carleson measure for* $P_s(H^s(\mathbb{T}))$*, that is,* $P_s(H^s(\mathbb{T})) \subset L^2(d\mu)$*.*

We observe that as it happens in the real case for $s = 1$ (see [20]) and $n = 1$ and $0 < s < 1/2$ (see [11]), the problem on traces in $H^s(\mathbb{T})$ for measures that may change sign is reduced to a problem of traces of non negative measures on $H^s(\mathbb{T})$, whose characterization is well known.

The fact that we are considering functions defined on $\mathbb{T}$ rather than on $\mathbb{R}$, gives some extra technical difficulties to some of the technical tools used in the proof. Among them it is worth to mention the relationship between the fractional laplacian and a weighted radial derivative of the extensions by P_s. We also check that the operator P_s is an isomorphism from $H^s(\mathbb{T})$ to its image and study a class of Fourier multipliers related to I_s which, are more involved than in the real case, due to the fact that the family of Riesz kernels I_s is not a semigroup with respect to convolution. We also prove a weighted estimate for a weighted area function associated with a Calderon-Zygmund type operator, which may have some interest by itself.

The paper is organized as follows: In Section 2, we introduce the space of weighted Sobolev functions on the unit disc $\mathbb{D}$. In Section 3, we consider the kernels P_s and obtain the main properties. In particular, we prove that P_s is an isomorphism from $H^s(\mathbb{T})$ to its image and we represent $((-\Delta)^s + I)\varphi(\zeta)$, up to a constant, as $\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\zeta)$. In the following Section, we obtain the basic relation $((-\Delta)^s + I)\, I_{2s} = I$ which, in particular, gives a representation of $(-\Delta)^s$ as a Fourier multiplier. We also state a result on discrete Calderon-Zygmund operators, whose proof we postpone to Appendix 1 and we deduce weighted estimates for a class of Fourier multipliers which includes $I_s^{-1} I_{2s}$. In Section 5 we state a weighted estimate for an area function, which will be proved in Appendix 2, which is applied to a vector valued kernel $\mathbf{K}$ associated to the gradient of the convolution of P_s with the Riesz kernel. Section 6 is devoted to introduce the main properties of Riesz capacities and the characterization of trace measures for $H^s(\mathbb{T})$ and Carleson measures for $P(H^s(\mathbb{T}))$. Finally, in Section 7, we give the proof of Theorem 1.1. In particular, for the proof of the necessary contidion (i) $\Rightarrow$ (iv), we construct suitable test functions in $H^s(\mathbb{T})$, based in the ideas in [20].

Throughout the paper, the letter C may denote various non-negative numerical constants, possibly different in different places. The notation $\varphi(x) \lesssim \psi(x)$ means that there exists $C > 0$, which does not depends of x, φ and ψ, such that $\varphi(x) \leq C\psi(x)$. We will write $\varphi(x) \approx \psi(x)$ if $\varphi(x) \lesssim \psi(x)$ and $\psi(x) \lesssim \varphi(x)$. The fact that an estimate holds for $x >> 1$, will mean that it holds for x big enough. All the function spaces considered will be real valued, the points in $\mathbb{T}$ will be denoted either by $\zeta \in \mathbb{C}$ or parametrized by e^{ix}, the points in the unit disc $\mathbb{D}$ will be denoted either by $z \in \mathbb{C}$ or re^{ix}, $0 < r < 1$ and $|1 - z\overline{\zeta}| = |z - \zeta|$ will denote the euclidean distance from $z \in \overline{\mathbb{D}}$ and $\zeta \in \mathbb{T}$.

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2. The weighted Sobolev space $W^2_{1,1-2s}$

If $0 < s < 1$, the space is defined by $\mathcal{W}^2_{1,1-2s} := \mathcal{W}^2_{1,1-2s}(\mathbb{D})$, as the completion of functions Φ in $\mathcal{C}^\infty_{\mathbb{R}}(\overline{\mathbb{D}})$ with respect to the norm

(2.2)
$$\|\Phi\|_{\mathcal{W}^2_{1,1-2s}} := \int_{\mathbb{D}} |\nabla\Phi(z)|^2 (1-|z|^2)^{1-2s} dm(z) + \int_{\mathbb{D}} |\Phi(z)|^2 (1-|z|^2)^{1-2s} dm(z) < \infty.$$

This space coincides with the space of real valued functions Φ defined a.e. on $\mathbb{D}$, such that Φ and its distributional derivatives are in $L^2((1-|z|^2)^{1-2s} dm(z))$ (see, for instance, [17]).

It is well known that we can obtain a trace operator T from $\mathcal{W}^2_{1,1-2s}$ onto $H^s(\mathbb{T})$, by defining the trace for functions $\mathcal{C}^\infty(\overline{\mathbb{D}})$ and extending the definition by continuity (see [14] for the details of the proof), that is,

Lemma 2.1.

$$\|T(\Phi)\|^2_{H^s(\mathbb{T})} \lesssim \int_{\mathbb{D}} |\nabla\Phi(z)|^2 (1-|z|^2)^{1-2s} dm(z) + \int_{\mathbb{D}} |\Phi(z)|^2 (1-|z|^2)^{1-2s} dm(z).$$

□

From our next result we deduce an equivalent norm for $\mathcal{W}^2_{1,1-2s}$.

Lemma 2.2. *Let $0 < s < 1/2$ and let $\Phi \in \mathcal{C}^\infty(\mathbb{D})$. We then have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\begin{aligned}&\int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{-2s} dm(z)\\ &\le C_\varepsilon \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{1-2s} dm(z) + \varepsilon \int_{\mathbb{D}} |\nabla\Phi|^2(z)(1-|z|^2)^{1-2s} dm(z).\end{aligned} \tag{2.3}$$

In particular,

$$\begin{aligned}&\int_{\mathbb{D}} |\nabla\Phi(z)|^2(1-|z|^2)^{1-2s} dm(z) + \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{-2s} dm(z)\\ &\approx \int_{\mathbb{D}} |\nabla\Phi(z)|^2(1-|z|^2)^{1-2s} dm(z) + \int_{\mathbb{D}} \Phi^2(z)(1-|z|^2)^{1-2s} dm(z).\end{aligned} \tag{2.4}$$

Proof. As a consequence of Stokes's Theorem applied to the form

$$\omega = x\Phi^2(x,y)(1-x^2-y^2)^{1-2s}dy - y\Phi^2(x,y)(1-x^2-y^2)^{1-2s}dx$$

and the disc $D_r = \{z \in \mathbb{D};\ |z| \le r\}$, $0 < r < 1$, we obtain

$$\begin{aligned}&\int_{D_r} \left(2(2-2s)\Phi^2(x,y) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\Phi^2(x,y)\right)(1-x^2-y^2)^{1-2s}dxdy\\ &-\int_{D_r} 2(1-2s)\Phi^2(x,y)(1-x^2-y^2)^{-2s}dxdy\\ &= \int_0^{2\pi} \Phi^2(r\cos t, r\sin t)r^2(1-r^2)^{1-2s}dt \ge 0.\end{aligned}$$

Hence, Hölder's inequality gives that

$$\begin{aligned}2(1-2s)&\int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{-2s}dxdy\\ &\lesssim \int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{1-2s}dxdy + \int_{D_r} |\nabla\Phi^2(x,y)|(1-x^2-y^2)^{1-2s}dxdy\\ &\lesssim \int_{D_r} \Phi^2(x,y)(1-x^2-y^2)^{1-2s}dxdy + \varepsilon\int_{D_r} |\nabla\Phi(x,y)|^2(1-x^2-y^2)^{1-2s}dxdy\end{aligned}$$

and (2.3) is then a consequence of the Monotone convergence Theorem. □

3. The generalized Poisson extensions P_s

The Euler-Lagrange equation associated to the functional on the left hand side of (2.4) that corresponds to its stationary values, gives place to the PDE equation

$$(1-(x^2+y^2))\Delta u - 2(1-2s)(x\frac{\partial}{\partial x}u + y\frac{\partial}{\partial y}u) - (1-2s)^2u = 0, \tag{3.5}$$

or equivalently to

$$div\left((\nabla u)(1-x^2-y^2)^{1-2s}\right)-(1-2s)^2(1-x^2-y^2)^{-2s}u=0. \tag{3.6}$$

The next theorem is proved in [2] and establishes that the Dirichlet problem associated to the PDE equation 3.5 has a unique solution. Namely:

Theorem 3.1. *Let $\varphi\in\mathcal{C}(\mathbb{T})$. We then have that the function*

$$u(z)=P_s(\varphi)(z):=\int_{\mathbb{T}}P_s(z,\zeta)\varphi(\zeta)d\sigma(\zeta),$$

where

$$P_s(z,\zeta)=C_s\frac{(1-|z|^2)^{2s}}{|1-z\overline{\zeta}|^{1+2s}},\quad z\in\mathbb{D},\zeta\in\mathbb{T},$$

with $C_s=\frac{\Gamma(1+s/2)^2}{\Gamma(2s)}$ is the unique solution to the PDE given in (3.5), which is a continuous function on $\overline{\mathbb{D}}$ and extends φ to $\overline{\mathbb{D}}$.

In addition, this solution u can be given in terms of its Fourier expansion. If $a,b,c>0$, the hypergeometric function is defined by

$$F(a,b;c;x):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,$$

where $(a)_0=1$, $(a)_n=a(a+1)\dots(a+n-1)$ if $n\ge 1$.

If $\varphi(e^{ix})=\sum_{k\in\mathbb{Z}}\widehat{\varphi}(k)e^{ikx}$ is the Fourier expansion of φ, then the function $u=P_s(\varphi)$ can be expressed as

$$u(z)=\sum_{k=0}^{\infty}f_k(r^2)r^k(\hat{\varphi}(k)e^{ikx}+\hat{\varphi}(-k)e^{-ikx}),\qquad z=re^{ix}, \tag{3.7}$$

where $f_k(x)=\frac{F_k(x)}{F_k(1)}$ and $F_k(x)=F(k-s+\frac{1}{2},-s+\frac{1}{2};k+1;x)$, with uniform and absolute convergence on compact sets in $\mathbb{D}$. In particular, if $\varphi\equiv 1$, we have that

$$\int_{\mathbb{T}}P_s(z,\zeta)d\sigma(\zeta)=f_0(|z|^2). \tag{3.8}$$

We will see (see Corollary 3.6) that P_s gives an isomorphism between the space $H^s(\mathbb{T})$ and its image in $\mathcal{W}^2_{1,1-2s}$.

Observe that (3.8) gives that, unlike what happens in the classical case for the Poisson kernel ($s=1/2$), $P_s(1)$ is not constant.

Definition 3.2. *Let $0<s<1$. If $\varphi\in\mathcal{C}^1(\mathbb{T})$, the fractional derivative of order s is defined by*

$$(-\Delta)^s(\varphi)(\zeta)=\frac{2s\Gamma(1/2-s)^2}{\Gamma(1-2s)}P.V.\int_{\mathbb{T}}\frac{\varphi(\zeta)-\varphi(\eta)}{|\zeta-\eta|^{1+2s}}d\sigma(\eta).$$

As it is usual, if the function φ is regular enough, the principal value of the integral reduces to an ordinary integral of a function. We will see that this operator can be also defined by a Fourier multiplier.

In $\mathbb{R}^n$, in [7] it is proved that an operator analogous to P_s satisfies that

$$\lim_{y\to 0} y^{1-2s}\frac{\partial}{\partial y}P_s(\varphi)(x,y) = -(-\Delta)^s\varphi(x)$$

and that this operator P_s is an isometry. Our next theorems establish a version of these results for the unit circle, which, in particular, permits to study the fractional laplacian on $\mathbb{T}$ through ordinary derivatives on $\mathcal{W}^2_{1,1-2s}$.

Theorem 3.3. *Let $0 < s < \frac{1}{2}$ and let $\varphi \in \mathcal{C}^1(\mathbb{T})$. We then have that*

$$\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\zeta) = 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left((-\Delta)^s(\varphi)(\zeta)+\varphi(\zeta)\right).$$

Proof. Let, as before, $f_k(x)$ be the function defined by $f_k(x) = \frac{F_k(x)}{F_k(1)}$. We will first prove that

(3.9)
$$\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{d}{dr}f_k(r^2)r^k = 2\frac{\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)}\frac{\Gamma(k+s+1/2)}{\Gamma(k-s+1/2)} \qquad k\geq 0.$$

Indeed, we have that

$$\frac{d}{dr}f_k(r^2)r^k = 2f_k'(r^2)r^{k+1} + kr^{k-1}f_k(r^2) = \frac{2F_k'(r^2)r^{k+1}+kr^{k-1}F_k(r^2)}{F_k(1)}.$$

Hence, using that by [6], p.58,

$$F_k'(t) = \frac{(k-s+1/2)(1/2-s)}{k+1}F(k-s+3/2, 3/2-s; k+2; t),$$

we have that

$$\begin{aligned}&\frac{d}{dr}f_k(r^2)r^k\\ &= \frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)}F(k-s+3/2,3/2-s;k+2;r^2)2r^{k+1} + \frac{kr^{k-1}}{F_k(1)}F_k(r^2).\end{aligned}$$

Next, it is well known (see, for instance, [6], p.64) that if $|z|<1$, then

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z).$$

Thus the above equals to

$$\begin{aligned}&\frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)}(1-r^2)^{2s-1}F(s+1/2,k+s+1/2;k+2;r^2)2r^{k+1}\\ &+\frac{kr^{k-1}}{F_k(1)}F_k(r^2)\end{aligned}$$

and

$$
\begin{aligned}
&(1-r^2)^{1-2s}\frac{d}{dr}f_k(r^2)r^k\\
(3.10)\qquad &=\frac{(k-s+1/2)(1/2-s)}{(k+1)F_k(1)}F(s+1/2,k+s+1/2;k+2;r^2)2r^{k+1}\\
&+(1-r^2)^{1-2s}\frac{kr^{k-1}}{F_k(1)}F_k(r^2):=A_k(r)+B_k(r).
\end{aligned}
$$

Since $k+1-(k-s+1/2)-(1/2-s)=2s>0$, the series that defines the function $F_k(r^2)$ converges absolutely in $r=1$ ([6], p.57), so we have that $\lim_{r\to1^-}F_k(r^2)=F_k(1)$. Hence, $\lim\limits_{r\to1^-}B_k(r)=0$. Next, by hypothesis, we have that $k+2-(s+1/2)-(k+s+1/2)=1-2s>0$, so using the preceding argument, we obtain that

$$
\lim_{r\to1^-}A_k(r)=\frac{2(k-s+1/2)(1/2-s)}{(k+1)}\frac{F(s+1/2,k+s+1/2;k+2;1)}{F(k-s+\frac12,-s+\frac12;k+1;1)}.
$$

But (see [6], p.61), if $\operatorname{Re}c>\operatorname{Re}b>0$ and $\operatorname{Re}(c-a-b)>0$, we have that

$$
F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
$$

Thus, using that $\Gamma(z+1)=z\Gamma(z)$, we obtain that

(3.11)

$$
\begin{aligned}
&\lim_{r\to1^-}A_k(r)=\\
&\frac{2(k-s+1/2)(1/2-s)}{(k+1)}\frac{\Gamma(k+2)\Gamma(1-2s)}{\Gamma(k-s+3/2)\Gamma(3/2-s)}\frac{\Gamma(s+1/2)\Gamma(k+s+1/2)}{\Gamma(k+1)\Gamma(2s)}\\
&=\frac{2(k-s+1/2)\Gamma(s+1/2)\Gamma(k+s+1/2)\Gamma(1-2s)}{(k-s+1/2)\Gamma(k-s+1/2)\Gamma(2s)\Gamma(1/2-s)}\\
&=2\frac{\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)}\frac{\Gamma(k+s+1/2)}{\Gamma(k-s+1/2)}.
\end{aligned}
$$

That gives (3.9).

Next, we finishes the proof of the theorem. Using (3.9) for $k=0$, we have that

$$
\begin{aligned}
&\lim_{r\to1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r\zeta,\eta)d\sigma(\eta)\\
&=\lim_{r\to1^-}(1-r^2)^{1-2s}\frac{d}{dr}f_0(r^2)=2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}.
\end{aligned}
$$

Hence,

(3.12)

$$
\begin{aligned}
&\lim_{r\to1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(r\zeta)=\lim_{r\to1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r\zeta,\eta)\varphi(\eta)d\sigma(\eta)\\
&=\lim_{r\to1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r\zeta,\eta)\left(\varphi(\eta)-\varphi(\zeta)\right)d\sigma(\eta)+2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\varphi(\zeta).
\end{aligned}
$$

We have that if $\zeta\overline{\eta} = e^{ix}$, $|1-\zeta\overline{\eta}| \approx |x|$, and

$$(1-r^2)^{1-2s}\frac{\partial}{\partial r}\frac{(1-r^2)^{2s}}{|1-r\zeta\overline{\eta}|^{1+2s}} = (1-r^2)^{1-2s}\frac{\partial}{\partial r}\frac{(1-r^2)^{2s}}{(1+r^2-2r\cos x)^{\frac{1+2s}{2}}}$$
$$= \frac{-4sr}{(1+r^2-2r\cos x)^{\frac{1+2s}{2}}} - (1+2s)\frac{(1-r^2)(r-\cos x)}{(1+r^2-2r\cos x)^{\frac{1+2s}{2}+1}}$$

Consequently, since $\varphi \in \mathcal{C}^1(\mathbb{T})$, $|\varphi(\eta)-\varphi(\zeta)| = \mathrm{O}(|x|)$ if $x \to 0$ and $|1-r\zeta\overline{\eta}| \approx (1-r)+|x|$, we have that

$$(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(r,\eta)\,|\varphi(\eta)-\varphi(\zeta)| \lesssim \frac{1}{|x|^{2s}} \in L^1(\mathbb{T}).$$

Hence, the Dominated Convergence Theorem gives that

$$\begin{aligned}(3.13)\qquad &\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}\int_{\mathbb{T}}P_s(r,\eta)\,(\varphi(\eta)-\varphi(\zeta))\,d\sigma(\eta)\\ &= -4sC_s\int_{\mathbb{T}}\frac{\varphi(\eta)-\varphi(\zeta)}{|\zeta-\eta|^{1+2s}}d\sigma(\eta) = 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}(-\Delta)^s(\varphi(\zeta)).\end{aligned}$$

This calculation and (3.12) finishes the proof of the theorem. □

Theorem 3.4. *Let φ be a $\mathcal{C}^\infty$ function on $\mathbb{T}$. We then have that*

$$2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_{\mathbb{T}}\varphi^2 + \int_{\mathbb{T}}\varphi(-\Delta)^s\varphi\right)$$
$$= \int_{\mathbb{D}}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}dxdy + (1-2s)^2\int_{\mathbb{D}}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}dxdy.$$

Proof. Let ω be the form defined on $\mathbb{D}$ by

$$\omega(z) = P_s(\varphi)(z)\left(\frac{\partial P_s(\varphi)}{\partial x}dy - \frac{\partial P_s(\varphi)}{\partial y}dx\right)(1-x^2-y^2)^{1-2s}, \qquad z = x+iy.$$

Let $r<1$ be fixed, and let D_r be the disc centered at the origin and of radius $r>0$. Stokes's Theorem and (3.6) give that

(3.14)
$$\int_{\partial D_r}\omega$$
$$= \int_{D_r}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}dxdy + (1-2s)^2\int_{D_r}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}dxdy.$$

The Lebesgue's monotone convergence Theorem gives that

(3.15)
$$\lim_{r\to 1^-}\int_{D_r}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}dxdy$$
$$+(1-2s)^2\int_{D_r}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}dxdy$$
$$=\int_{\mathbb{D}}|\nabla P_s(\varphi)|^2(1-x^2-y^2)^{1-2s}dxdy+(1-2s)^2\int_{\mathbb{D}}|P_s(\varphi)|^2(1-x^2-y^2)^{-2s}dxdy.$$

On the other hand, we have

$$\int_{\partial D_r}\omega$$
$$=\int_0^{2\pi}P_s(\varphi)_{|\partial D_r}(1-r^2)^{1-2s}\left(\frac{\partial P_s(\varphi)}{\partial x}_{|\partial D_r} r\cos x+\frac{\partial P_s(\varphi)}{\partial y}_{|\partial D_r} r\sin x\right)dx$$
$$=\int_0^{2\pi}(1-r^2)^{1-2s}P_s(\varphi)_{|\partial D_r} r\frac{\partial}{\partial r}P_s(\varphi)_{|\partial D_r}dx.$$

In order to pass to the limit when $r\to 1^-$, we next check that we can apply the Lebesgue's dominated convergence Theorem. Since $P_s(\varphi)\in\mathcal{C}(\overline{\mathbb{D}})$, we just have to check that the function $r(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)_{|\partial D_r}$ is uniformly bounded for $r\in[0,1]$ by an integrable function. Using (3.7) and (3.10), we have that

$$r(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(re^{ix})$$
$$=\sum_{k\geq 0} r\left(A_k(r)+B_k(r)\right)\left(\widehat{\varphi}(k)e^{ikx}+\widehat{\varphi}(-k)e^{-ikx}\right).$$

Since the hypergeometric function is an increasing function on r, we have that

$$A_k(r)\lesssim\frac{F(s+1/2,k+s+1/2;k+2;1)}{F_k(1)}$$
$$=\frac{\Gamma(k+2)\Gamma(1-2s)}{\Gamma(k-s+3/2)\Gamma(3/2-s)}\frac{\Gamma(1/2+s)\Gamma(k+s+1/2)}{\Gamma(k+1)\Gamma(2s)}\approx k^{2s},\quad k>>1.$$

And

$$B_k(r)\lesssim k.$$

Since $\varphi\in\mathcal{C}^\infty(\mathbb{T})$, we deduce that $|\widehat{\varphi}(k)|\lesssim\frac{1}{|k|^l}$, for each $l\geq 1$. Hence,

$$\sup_{r\leq 1} r(1-r^2)^{1-2s}|\frac{\partial}{\partial r}P_s(\varphi)(re^{ix})|<\infty.$$

and consequently applying Theorem 3.3,

$$\lim_{r\to 1^-}\int_0^{2\pi} P_s(\varphi)(re^{ix}) r(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(re^{ix})dx$$
$$= 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_0^{2\pi}\varphi^2+\int_0^{2\pi}\varphi(-\Delta)^s(\varphi)\right).$$

Thus, using (3.14) and (3.15), we have proved the theorem. □

Corollary 3.5. *Let $\varphi\in\mathcal{C}^\infty(\mathbb{T})$. Let $\psi=((-\Delta)^s+I)\varphi$. For each $k\in\mathbb{Z}$,*

$$\widehat{\psi}(k)=\frac{\Gamma(1/2-s)\Gamma(|k|+s+1/2)}{\Gamma(1/2+s)\Gamma(|k|-s+1/2)}\widehat{\varphi}(k). \tag{3.16}$$

In particular, $(-\Delta)^s$ can be defined as a Fourier multiplier.

Proof. By (3.10) and (3.11), $\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(re^{ix})$ has as a sequence of Fourier multipliers

$$\left(\frac{2\Gamma(s+1/2)\Gamma(1-2s)}{\Gamma(2s)\Gamma(1/2-s)}\frac{\Gamma(|k|+s+1/2)}{\Gamma(|k|-s+1/2)}\right)_{k\in\mathbb{Z}}.$$

Since by (3.10) and (3.13),

$$\lim_{r\to 1^-}(1-r^2)^{1-2s}\frac{\partial}{\partial r}P_s(\varphi)(re^{ix})$$
$$=2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left((-\Delta)^s\varphi+\varphi\right)(e^{ix}),$$

we obtain (3.16). □

Corollary 3.6. *For any $\varphi\in H^s(\mathbb{T})$,*

$$\|\varphi\|_{H^s(\mathbb{T})}\approx\|P_s(\varphi)\|_{W^2_{1,1-2s}}.$$

Proof. It is a consequence of last theorem, the density of the functions $\mathcal{C}^\infty(\mathbb{T})$ in $H^s(\mathbb{T})$ and Lemma 2.2, since by Theorem 3.3 and Stirling's formula we that $\|((-\Delta)^s)^{1/2}(\varphi)\|_{L^2(\mathbb{T})}\approx\|(-\Delta)^{s/2}(\varphi)\|_{L^2(\mathbb{T})}$. □

Proposition 3.7. *Let $\varphi\in H^s(\mathbb{T})$ and let $\Psi\in\mathcal{W}^2_{1,1-2s}$ and ψ its restriction to $\mathbb{T}$. We then have that*

$$2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_{\mathbb{T}}\psi\varphi+\int_{\mathbb{T}}\psi(-\Delta)^s\varphi\right)$$
$$=\int_{\mathbb{D}}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}dxdy+(1-2s)^2\int_{\mathbb{D}}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}dxdy.$$

Proof. Since $\mathcal{C}^\infty(\overline{D})$ and $\mathcal{C}^\infty(\mathbb{T})$ are dense in $\mathcal{W}^2_{1,1-2s}$ and in $H^s(\mathbb{T})$, respectively, we may assume, without loss of generality, that $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ and $\Psi \in \mathcal{C}^\infty(\overline{\mathbb{D}})$. Let ω be the form defined on $\mathbb{D}$ by

$$\omega(z) = \Psi(z)\left(\frac{\partial P_s(\varphi)}{\partial x}dy - \frac{\partial P_s(\varphi)}{\partial y}dx\right)(1-x^2-y^2)^{1-2s}, \qquad z = x+iy.$$

Let $r<1$ be fixed. Stokes's Theorem and (3.6) give that

$$\begin{aligned}&\int_{\partial D_r}\omega\\ &= \int_{D_r}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}dxdy + (1-2s)^2\int_{D_r}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}dxdy.\end{aligned}$$

We now observe that since both $\nabla\Psi$ and Ψ are bounded on $\overline{\mathbb{D}}$ and by Theorem 3.4, $\nabla P_s(\varphi)$ is in the vector valued space $\mathbf{L}^2((1-|z|^2)^{1-2s}dm(z))$ and $P_s(\varphi) \in L^2((1-|z|^2)^{-2s}dm(z))$, the Lebesgue's dominated convergence Theorem gives that

$$\begin{aligned}&\lim_{r\to 1^-}\left(\int_{D_r}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}dxdy\right.\\ &\left.+(1-2s)^2\int_{D_r}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}dxdy\right)\\ &= \int_{\mathbb{D}}\nabla\Psi\nabla P_s(\varphi)(1-x^2-y^2)^{1-2s}dxdy + (1-2s)^2\int_{\mathbb{D}}\Psi P_s(\varphi)(1-x^2-y^2)^{-2s}dxdy.\end{aligned}$$

A similar argument to the one used in the proof of Theorem 3.4, gives that

$$\begin{aligned}&\lim_{r\to 1^-}\int_0^{2\pi} r(1-r^2)^{1-2s}\Psi_{|\partial D_r}\frac{\partial}{\partial r}P_s(\varphi)_{|\partial D_r}dx\\ &= 2C_s\frac{\Gamma(1-2s)}{\Gamma(1/2-s)^2}\left(\int_0^{2\pi}\psi\varphi + \int_0^{2\pi}\psi(-\Delta)^s(\varphi)\right).\end{aligned}$$

□

Finally, we recall the following estimate that was implicit in [2], p.130, and whose proof we include for a sake of completeness.

Let

$$\nabla_{\mathbb{D}}\varphi(z) = \left((1-|z|^2)\mathcal{R}\varphi, (1-|z|^2)\overline{\mathcal{R}}\varphi\right). \tag{3.17}$$

where $\mathcal{R}\varphi(z) = z\frac{\partial\varphi}{\partial z}(z)$, $\overline{\mathcal{R}}\varphi(z) = \overline{z}\frac{\partial\varphi}{\partial\overline{z}}(z)$ are the radial derivatives.

Lemma 3.8. *Let $0<s<1/2$. Then*

$$\left|\int_{\mathbb{D}}\nabla_{\mathbb{D}}P_s(z,\zeta)d\sigma(\zeta)\right| \lesssim (1-|z|^2)^{2s}, \quad z\in\mathbb{D}.$$

Proof. Since (see [2], p.130), $|\int_{\mathbb{D}} \nabla_{\mathbb{D}} P_s(z,\zeta) d\sigma(\zeta)| \lesssim (1-|z|^2)|F'(1/2-s, 1/2-s; 1; |z|^2)|$ and , by [6], p.58, we have that $F'(1/2-s, 1/2-s; 1; |z|^2) = (1/2-s)^2 F(3/2-s, 3/2-s; 2; |z|^2) = (1-|z|^2)^{2s-1} F(1/2+s, 1/2+s; 1; |z|^2)$. The assertion is a consequence of the continuity on $\overline{\mathbb{D}}$ of the function $F(1/2+s, 1/2+s; 1; |z|^2)$ (see [6], p.57).

□

4. The space $H^s(\mathbb{T})$ and weighted estimates for a Fourier multiplier

4.1. The space $H^s(\mathbb{T})$.

Definition 4.1. *Let $0 < s < 1$. The Riesz kernel I_s on the unit circle is defined by*

$$I_s(\zeta,\eta) = \frac{\Gamma((1+s)/2)^2}{\Gamma(s)} \frac{1}{|1-\zeta\overline{\eta}|^{1-s}}, \qquad \zeta,\eta \in \mathbb{T}. \tag{4.18}$$

If f is an integrable function on $\mathbb{T}$, the Riesz operator is defined by

$$I_s(f)(\zeta) = \int_{\mathbb{T}} I_s(\zeta,\eta) f(\eta) d\sigma(\eta).$$

The space $I_s(L^2(\mathbb{T}))$ is the space of functions $\psi = I_s(\varphi)$, $\varphi \in L^2(\mathbb{T})$, normed by $\|\psi\|_{I_s(L^2(\mathbb{T}))} = \|\varphi\|_{L^2(\mathbb{T})}$.

The Fourier coefficients of the Riesz kernel in $\mathbb{T}$ are the following (see for instance, [3]):

Lemma 4.2. *Let $0 < s < 1$. Then for any $k \in \mathbb{Z}$,*

$$\widehat{I_s}(k) = \frac{\Gamma(|k| + \frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1-s}{2})\Gamma(|k| + \frac{1+s}{2})}. \quad \square$$

Theorem 4.3. *Let $0 < s < 1/2$.*

$$((-\Delta)^s + I)\, I_{2s} = I. \tag{4.19}$$

Proof. Indeed, by Corollary 3.5, we deduce that if $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ has as the sequence of Fourier coefficients $(\widehat{\varphi}(k))_{k\in\mathbb{Z}}$, then the function$((-\Delta)^s + I)\,\varphi$ has as sequence of Fourier coefficients $\left(\frac{\Gamma(|k|+s+1/2)\Gamma(1/2-s)}{\Gamma(|k|-s+1/2)\Gamma(1/2+s)}\widehat{\varphi}(k)\right)_{k\in\mathbb{Z}}$. The proof of the proposition follows then from the density of $\mathcal{C}^\infty(\mathbb{T})$ in $L^2(\mathbb{T})$ and Lemma 4.2. □

Corollary 4.4. *Let $0 < s < 1/2$. We then have that $\varphi \in H^s(\mathbb{T})$ (that is $(-\Delta)^{s/2}\varphi, \varphi \in L^2(\mathbb{T})$) if and only if $\varphi = I_s(\psi)$, $\psi \in L^2(\mathbb{T})$ and $\|\varphi\|_{L^2(\mathbb{T})} \approx \|\psi\|_{L^2(\mathbb{T})}$.*

Corollary 4.5. *From Lemma 4.2, the above Corollary and Stirling's formula, we deduce that if $\varphi(e^{ix}) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)e^{ikx}$, then $\|\varphi\|^2_{H^s(\mathbb{T})} \approx \sum_{k\in\mathbb{Z}} |(|k|^s + 1)\widehat{\varphi}(k)|^2$.* □

From Theorem 4.3 we deduce a reformulation of the bilinear problem (1.1). Namely,

Proposition 4.6. *Let $c \in L^2(\mathbb{T})$. We then have that the boundedness of the bilinear form*

$$\left|\int_{\mathbb{T}} \varphi\psi c d\sigma\right| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})}, \tag{4.20}$$

is equivalent to the boundedness of the bilinear form

$$|\Lambda_{I_{2s}(c)}(\varphi,\psi)| = \left|\int_{\mathbb{T}} \left((-\Delta)^s + I\right)(\varphi\psi) I_{2s}(c) d\sigma\right| \lesssim \|\varphi\|_{H^s(\mathbb{T})}\|\psi\|_{H^s(\mathbb{T})}.$$

□

4.2. **Weighted estimates for the operator $I_s^{-1}I_{2s}^{1/2}$.** On the real line, the analogous to the operator I_s on $\mathbb{T}$ is the Riesz operator with kernel $\dfrac{1}{|x-y|^{1-s}}$. This family of operators on $\mathbb{R}$ is a semigroup and, in particular, $I_sI_s = I_{2s}$. This fact gives that, as multipliers on $L^2(\mathbb{R})$, we have that $I_s = I_{2s}^{1/2}$ or, equivalently, $I_s^{-1}I_{2s}^{1/2} = Id$. Here, in the unit disc, we can also define $I_{2s}^{1/2}$ as a Fourier multiplier, but it does not coincide with I_s. Nevertheless, the asymptotic behavior is the same and, in particular, $I_s^{-1}I_{2s}^{1/2}$ defines a bounded operator in $L^2(\mathbb{T})$. We will need to check that it also defines a bounded operator on $L^2(\omega)$, where ω is a weight in the Muckenhoupt class A_2. This will be a consequence of the fact that $T = I_s^{-1}I_{2s}^{1/2}$ realizes as a Calderon-Zygmund operator of type zero.

Then, Coiffman and Feferman's theorem gives that T is bounded from $L^p(\omega)$ to $L^p(\omega)$, for any ω in A_p (see [p. 205,[22]] for a proof of this result).

We prove a discrete version for the unit circle of a well known result in $\mathbb{R}^n$ of a realization as a singular integral operator of an adequate pseudodifferential operator, see [Ch. VI,[22]]. Namely, if m is the operator defined as a Fourier multiplier by

$$m(k) := \left(\frac{\Gamma(|k|+1/2-s/2)}{\Gamma(|k|+1/2+s/2)}\right)^{-1}\left(\frac{\Gamma(|k|+1/2-s)}{\Gamma(|k|+1/2+s)}\right)^{1/2}, \tag{4.21}$$

which corresponds up to constants to $I_s^{-1}I_{2s}^{\frac{1}{2}}$, is a Calderon-Zygmund of type zero.

4.2.1. *Discrete Calderon-Zygmund Operators.*

Definition 4.7. *An operator T defined on $L^2(\mathbb{T})$ as a convolution operator by a function T_K on $\mathbb{T}\setminus\{1\}$ as*

$$T(\varphi)(\zeta) := \int_{\mathbb{T}} T_K(\zeta\overline{\eta})\varphi(\eta)d\sigma(\eta),$$

is a Calderon-Zygmund operator if:

(i) $\|T(\varphi)\|_{L^2(\mathbb{T})} \lesssim \|\varphi\|_{L^2(\mathbb{T})}$*;*
(ii) $|T_K(\zeta)| \lesssim \frac{1}{|1-\zeta|}$, $\zeta \neq 1$*;*
(iii) *There exists $\delta > 0$ such that $|T_K(\zeta\overline{\alpha}) - T_K(\zeta\overline{\alpha_1})| \lesssim \frac{|\alpha-\alpha_1|}{|\zeta-\alpha|^{1+\delta}}$, if $|\zeta-\alpha| >> |\alpha-\alpha_1|$, $\zeta,\alpha,\alpha_1 \in \mathbb{T}$ and $\zeta \neq \alpha$, $\zeta \neq \alpha_1$.*

Definition 4.8. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function. The finite difference of length* 1 *is* $\boldsymbol{\Delta}^1(\varphi)(x) = \varphi(x+1) - \varphi(x)$ *and for* $j \geq 1$, $\boldsymbol{\Delta}^j(\varphi)(x) = \boldsymbol{\Delta}^1(\boldsymbol{\Delta}^{j-1})(\varphi)(x)$.

The proof of the following theorem is postponed to Appendix 1.

Theorem 4.9. *Assume that m is a bounded function on $\mathbb{Z}$ satisfying that for any $j \geq 1$, there exists $C > 0$ such that*

$$\left|\boldsymbol{\Delta}^j(m)(k)\right| \leq \frac{C}{|k|^j} \quad |k| >> 1. \tag{4.22}$$

Let T be the bounded operator on $L^2(\mathbb{T})$ defined as

$$\widehat{T(f)}(k) = m(k)\widehat{f}(k).$$

Then the operator T agrees with a convolution operator by a function $T_K \in \mathcal{C}^\infty(\mathbb{T} \setminus \{1\})$ that identifying $\mathbb{T}$ in the usual way by e^{ix}, $x \in (-\pi, \pi]$ satisfies that for each $j \geq 0$,

$$\left|T_K^{(j)}(x)\right| \lesssim \frac{1}{|x|^{j+1}}, \quad x \neq 0. \tag{4.23}$$

Consequently, T is a Calderon-Zygmund operator. In particular, if ω is in the Muckenhoupt class A_p, $p > 1$,

$$\|T(\varphi)\|_{L^p(\omega)} \lesssim \|\varphi\|_{L^p(\omega)}. \tag{4.24}$$

Our next goal is to prove that the operator $I_s^{-1} I_{2s}^{\frac{1}{2}}$ defined on $L^2(\mathbb{T})$ satisfies the hypothesis of Theorem 4.9.

4.3. **Weighted estimates for the operator $I_s^{-1} I_{2s}^{\frac{1}{2}}$.** We recall that the operator $I_s^{-1} I_{2s}^{\frac{1}{2}}$ is defined as a Fourier multiplier operator, up to a constant, by

$$m(k) = \Psi_s(|k|), \quad k \in \mathbb{Z},$$

where

$$\Psi_s(x) = \left(\frac{\Gamma(x + 1/2 - s)}{\Gamma(x + 1/2 + s)}\right)^{1/2} \left(\frac{\Gamma(x + 1/2 - s/2)}{\Gamma(x + 1/2 + s/2)}\right)^{-1}.$$

By Stirling's formula, it follows easily that the sequence $(\Psi_s(|j|))_{j \in \mathbb{Z}}$ is bounded.

We begin with a technical lemma that deduces estimates on the iterated finite differences of a function from estimates on its iterated derivatives.

Lemma 4.10. *Let $C > 0$ be a constant such that for every $\varphi \in \mathcal{C}^\infty$ function on $[0, +\infty)$ and every $r \geq 0$,*

$$|\varphi^{(j)}(x)| \leq \frac{C}{x^{j+r}}, \quad x >> 1, \quad j \geq 1.$$

Then we have that for every $\varphi \in \mathcal{C}^\infty$ function on $[0, +\infty)$ and every $r \geq 0$,

$$|\boldsymbol{\Delta}^j(\varphi)(x)| \leq \frac{C}{x^{j+r}}, \quad x >> 1, \quad j \geq 1.$$

In particular, we have that for every $\varphi \in \mathcal{C}^\infty$ function on $[0,+\infty)$ and every $r \geq 0$,

$$|\mathbf{\Delta}^j(\varphi)(k)| \leq \frac{C}{|k|^{j+r}}, \quad k >> 1, \quad j \geq 1.$$

Proof. The proof follow easily from induction on j and the Mean-Value Theorem. □

Proposition 4.11. *Let $0 < s < 1/2$. We then have that the function*

$$\Psi_s(x) := \left(\frac{\Gamma(x+1/2-s)}{\Gamma(x+1/2+s)}\right)^{\frac{1}{2}} \left(\frac{\Gamma(x+1/2-s/2)}{\Gamma(x+1/2+s/2}\right)^{-1}, \quad x \geq 0$$

satisfies that for any $j \in \mathbb{N}$, there exists $C = C(j,s)$ such that

$$|\Psi_s^{(j)}(x)| \lesssim \frac{C}{x^j}, \quad x >> 1.$$

Proof. We will first obtain estimates of the derivatives of the quotient of Gamma functions involved in the definition of the function Ψ. Let Φ_{2s} be the function defined by $\Phi_{2s}(x) = \frac{\Gamma(x+1/2-s)}{\Gamma(x+1/2+s)}$, $x \geq 0$. We then have:

(i) For any $j \in \mathbb{N} \cup \{0\}$, there exists $C = C(j,s)$ such that

$$\Phi_{2s}^{(j)}(x) \leq C\frac{1}{x^{2s+j}}, \qquad x >> 1.$$

(ii) For any $j \in \mathbb{N} \cup \{0\}$, there exists $C = C(j,s)$ such that

$$(\Phi_s^{-1})^{(j)}(x) \leq C\frac{1}{x^{j-s}}, \qquad x >> 1.$$

We begin with the proof of (i). The proof will follow by induction on $j \geq 0$. Stirling's formula gives that, provided x is big enough, $\frac{\Gamma(x-s+1/2)}{\Gamma(x+s+1/2)} \approx \frac{1}{x^{2s}}$. Then (i) holds for $j = 0$. Next, if we denote by $\mathcal{P} = \dfrac{\Gamma'}{\Gamma}$, we have that

$$\left(\frac{\Gamma(x-s+1/2)}{\Gamma(x+s+1/2)}\right)' = \frac{\Gamma(x-s+1/2)}{\Gamma(x+s+1/2)}\left(\mathcal{P}(x-s+1/2) - \mathcal{P}(x+s+1/2)\right). \tag{4.25}$$

Let us now estimate the differences $\mathcal{P}(x-s+1/2) - \mathcal{P}(x+s+1/2)$. Observe that

$$\mathcal{P}^{(j)}(z) = (-j)_{j-1}\frac{1}{z^{1+j}} + \sum_{n=1}^{\infty}(-j)_{j-1}\frac{1}{(n+z)^{j+1}}. \tag{4.26}$$

Hence,

$$|\mathcal{P}(x-s+1/2) - \mathcal{P}(x+s+1/2)| \lesssim \sup_{x-s+1/2<y<x+s+1/2} |\mathcal{P}'(y)|.$$

For x big enough we have that

$$\sum_n \frac{1}{(n+x+t+1/2)^2} \lesssim \frac{1}{x}, \qquad -s < t < s.$$

Hence

$$|\mathcal{P}(x-s+1/2)-\mathcal{P}(x+s+1/2)|\lesssim\frac{1}{x}$$

and in general,

$$\left|\mathcal{P}^{(j)}(x-s+1/2)-\mathcal{P}^{(j)}(x+s+1/2)\right|\lesssim\frac{1}{x^{j+1}}.$$

Next, assume that the estimate (i) is true for $l\leq j$, and we will check that it also holds for $j+1$. By Leibniz's formula, and the induction hypothesis applied to (4.25),

$$\Phi_{2s}^{(j+1)}(x)=\left(\frac{\Gamma(x-s+1/2)}{\Gamma(x+s+1/2)}\right)^{(j+1)}\lesssim\sum_{i=0}^{j}\frac{1}{x^{2s+i}}\frac{1}{x^{j+1-i}}\approx\frac{1}{x^{2s+j+1}}.$$

A similar argument on induction proves (ii).

Next, Faà di Bruno formula, (see for instance [12]) gives that

$$\begin{aligned}&\left(\Phi_{2s}^{1/2}\right)^{(j)}(x)\\&=\sum\frac{j!}{m_1!\cdots m_j!(1!)^{m_1}\cdots(j!)^{m_j}}(1/2-1)\cdots(1/2-(m_1+\cdots+m_j))\\&\times\Phi_{2s}(x)^{1/2-(m_1+\cdots+m_j)}\prod_{l=1}^{j}(\Phi_{2s}^{(l)}(x))^{m_l},\end{aligned}$$

where the sum is over all the l-tuples of non negative integers $(m_1,\dots,m_l)$ satisfying that $1\cdot m_1+2\cdot m_2+\cdots+j\cdot m_j=j$.

Applying that by (i), $|(\Phi_{2s})^{(l)}(x)|\lesssim\frac{1}{x^{2s+l}}$, for $x>>1$, we then have

$$|\left(\Phi_{2s}^{1/2}\right)^{(j)}(x)|\lesssim\frac{1}{x^{s+j}},\qquad x>>1.$$

Finally, Leibnitz's rule together with this estimate and (ii) give that

$$|\Psi_s^{(j)}(x)|\lesssim\sum_{i=0}^{j}\frac{1}{x^{i-s}}\frac{1}{x^{s+j-i}}=\frac{1}{x^{j}},\qquad x>>1.$$

□

As a consequence of the above result, Lemma 4.10 and Theorem 4.9, we have

Theorem 4.12. *For any weight ω in the Muckenhoupt class A_p,*

$$\|I_s^{-1}*I_{2s}^{1/2}(\varphi)\|_{L^p(\omega)}\lesssim\|\varphi\|_{L^p(\omega)},\quad\varphi\in L^2(\omega).$$

5. Weighted estimates for a weighted area function

Let $\mathbf{K}:\mathbb{D}\times\mathbb{T}\to\mathbb{C}\times\mathbb{C}$ be a vector-valued kernel. The area function associated to $\mathbf{K}$ is

$$G_{\mathbf{K}}(\varphi)(\zeta):=\left(\int_{\Gamma_\zeta}|\mathbf{K}(\varphi)(z)|^2\frac{dm(z)}{(1-|z|^2)^2}\right)^{\frac12}, \tag{5.27}$$

where Γ_ζ, where $\Gamma(\zeta)=\{z\in\mathbb{D};\,|z-\zeta|<\alpha(1-|z|^2)\}$, $\alpha>1$, is the cone with vertex ζ.

We will need a result on a weighted L^2- estimate for the area function associated to a convenient kernel. In $\mathbb{R}^n$ and for the area function associated to the classical Poisson kernel a proof of the weighted estimate can be found in [13]. In [23] it is obtained a deep generalization of the classical result that heavily relies in the fact that the kernels K considered in this extension derive from some fixed functions ϕ with integral zero, considering $K(x,t)=\frac1t\phi(x/t)$. Since this is not our case, we have chosen a proof of the weighted estimated, based in the arguments in [16], which were applied in a different context in [11].

We will use the following theorem which will be proved in Appendix 2:

Theorem 5.1. *Let $\mathbf{K}:\mathbb{D}\times\mathbb{T}\to\mathbb{C}\times\mathbb{C}$ be a vector-valued kernel satisfying that there exist constants $C,c>0$ such that:*

(i) $\|G_{\mathbf{K}}(\varphi)\|_{L^2(\mathbb{T})}\lesssim\|\varphi\|_{L^2(\mathbb{T})}$, *for any $\varphi\in L^2(\mathbb{T})$.*
(ii) $|\mathbf{K}(z,\zeta)|\le C\frac{(1-|z|^2)^\varepsilon}{|1-z\bar\zeta|^{1+\varepsilon}}$, *for some $\varepsilon>0$.*
(iii) *For $\alpha_1,\alpha_2,\zeta\in\mathbb{T}$, $0<r<1$ such that $|\alpha_1-\alpha_2|\le c|1-r\zeta\overline{\alpha_1}|$,*

$$|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\le C\frac{(1-r^2)^\varepsilon|\alpha_1-\alpha_2|^\varepsilon}{|1-r\zeta\overline{\alpha_1}|^{1+2\varepsilon}},$$

Then, for any $\omega\in A_p$, we have,

$$\|G_{\mathbf{K}}(\varphi)\|_{L^p(\omega)}\lesssim\|\varphi\|_{L^p(\omega)}.$$

Our next goal is to check that the (vector-valued) kernel

$$\mathbf{K}(z,\zeta)=(1-|z|^2)^{-s}\int_{\mathbb{T}}\nabla_{\mathbb{D}}P_s(z,\eta)\frac{d\sigma(\eta)}{|1-\zeta\overline{\eta}|^{1-s}}, \tag{5.28}$$

is in the hypothesis of Theorem 5.1 with $\varepsilon=s$.

Proposition 5.2. *Let $\mathbf{K}(z,\zeta)$ be the vector-valued kernel defined in* (5.28) *and $G_{\mathbf{K}}$ as in* (5.27)*. We have that there exist constants $C,c>0$ such that:*

(i') $\|G_{\mathbf{K}}(\varphi)\|_{L^2(\mathbb{T})}\lesssim\|\varphi\|_{L^2(\mathbb{T})}$, *for any $\varphi\in L^2(\mathbb{T})$.*
(ii') $|\mathbf{K}(z,\zeta)|\le C\frac{(1-|z|^2)^s}{|1-z\bar\zeta|^{1+s}}$.

(iii') *For* $\alpha_1, \alpha_2, \zeta \in \mathbb{T}$, $0 < r < 1$ *such that* $|\alpha_1 - \alpha_2| \le c|1 - r\zeta\overline{\alpha_1}|$,

$$|\mathbf{K}(r\alpha_1, \zeta) - \mathbf{K}(r\alpha_2, \zeta)| \le C\frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}},$$

5.1. **Proof of (i').** It is an immediate consequence of Fubini's Theorem, Corollary 3.6 and Corollary 4.4.

5.2. **Estimate (ii').** We write

$$\mathbf{K}(z,\zeta) = (1-|z|^2)^{-s}\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\eta)\left(\frac{1}{|1-\zeta\overline{\eta}|^{1-s}} - \frac{1}{|1-z\overline{\zeta}|^{1-s}}\right) d\sigma(\eta)$$
$$+ (1-|z|^2)^{-s}\int_{\mathbb{T}} \nabla_{\mathbb{D}} P_s(z,\eta)\frac{d\sigma(\eta)}{|1-z\overline{\zeta}|^{1-s}} := \mathbf{A}(z,\zeta) + \mathbf{B}(z,\zeta).$$

Let us begin obtaining the desired estimate for $|\mathbf{A}|$. Computing $\nabla_{\mathbb{D}}$ and using that we have that

$$\left|\frac{1}{a^\alpha} - \frac{1}{b^\alpha}\right| \lesssim \frac{|a-b|(a^\alpha + b^\alpha)}{a^\alpha b^\alpha (a+b)}, \qquad 0<\alpha<1 \ a,b>0. \tag{5.29}$$

we obtain, considering separately the points $\eta \in \mathbb{T}$ such that $|1-\zeta\overline{\eta}| \le |1-z\overline{\zeta}|$ and $|1-\zeta\overline{\eta}| > |1-z\overline{\zeta}|$ respectively (see (3.17))

$$|\mathbf{A}(z,\zeta)| \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{||1-z\overline{\zeta}| - |1-\zeta\overline{\eta}||}{|1-z\overline{\eta}|^{1+2s}}\frac{1}{|1-\zeta\overline{\eta}|^{1-s}} d\sigma(\eta)$$
$$+ \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s}}\int_{\mathbb{T}} \frac{||1-z\overline{\zeta}| - |1-\zeta\overline{\eta}||}{|1-z\overline{\eta}|^{1+2s}|1-\zeta\overline{\eta}|} d\sigma(\eta) := A_1(z,\zeta) + A_2(z,\zeta).$$

$$A_1(z,\zeta) \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{|1-z\overline{\eta}|}{|1-z\overline{\eta}|^{1+2s}|1-\zeta\overline{\eta}|^{1-s}} d\sigma(\eta)$$
$$\le \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|}\int_{\mathbb{T}} \frac{d\sigma(\eta)}{|1-z\overline{\eta}|^{2s}|1-\zeta\overline{\eta}|^{1-s}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

Next, we bound A_2. Let $0 < \delta < 2s$ be fixed. We then have:

$$A_2(z,\zeta) \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s+\delta}}\int_{\mathbb{T}} \frac{1}{|1-z\overline{\eta}|^{2s}|1-\zeta\overline{\eta}|^{1-\delta}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

For the estimate of $|\mathbf{B}(z,\zeta)|$, we use that by Lemma 3.8, $|\int_{\mathbb{D}} \nabla_{\mathbb{D}} P_s(z,\eta) d\sigma(\eta)| \lesssim (1-|z|^2)^{2s}$. Hence

$$|\mathbf{B}(z,\zeta)| \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1-s}} \lesssim \frac{(1-|z|^2)^s}{|1-z\overline{\zeta}|^{1+s}}.$$

Altogether gives that $\mathbf{K}$ satisfies (ii').

5.3. **Estimate (iii') when $|\alpha_1-\alpha_2|\ge\delta$, for some $\delta>0$.** This case is immediate since if $|\alpha_1-\alpha_2|\le c|1-r\zeta\overline{\alpha_1}|$, with $2c<1$, we have that $|1-r\zeta\overline{\alpha_1}|\approx|1-r\zeta\overline{\alpha_2}|$ and $|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|\le|\mathbf{K}(r\alpha_1,\zeta)|+|\mathbf{K}(r\alpha_2,\zeta)|\lesssim|\mathbf{K}(r\alpha_1,\zeta)|$. Hence, using estimate (ii') and the condition $|\alpha_1-\alpha_2|\ge\delta$, we deduce that

$$|\mathbf{K}(r\alpha_1,\zeta)|\lesssim\frac{(1-r^2)^s}{|1-r\alpha_1\overline{\zeta}|^{1+s}}\lesssim\frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\alpha_1\overline{\zeta}|^{1+2s}}. \tag{5.30}$$

5.4. **Estimate (iii') when $|\alpha_1-\alpha_2|<\delta$, for some $\delta>0$.** We recall that as before, if $2c<1$ and $|\alpha_1-\alpha_2|\le c|1-r\zeta\overline{\alpha_1}|$, then $|1-r\zeta\overline{\alpha_1}|\approx|1-r\zeta\overline{\alpha_2}|$.

$$\begin{aligned}
&|\mathbf{K}(r\alpha_1,\zeta)-\mathbf{K}(r\alpha_2,\zeta)|=|\mathbf{K}(r\zeta,\alpha_1)-\mathbf{K}(r\zeta,\alpha_2)|\\
&=(1-r^2)^{-s}\left|\int_{\mathbb{T}}\nabla_{\mathbb{D}}P_s(r\zeta,\eta)\left(\frac{1}{|1-\eta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-\eta\overline{\alpha}_2|^{1-s}}\right)d\sigma(\eta)\right|\\
&\le(1-r^2)^{-s}\int_{\mathbb{T}}|\nabla_{\mathbb{D}}P_s(r\zeta,\eta)|\left|\frac{1}{|1-\eta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-\eta\overline{\alpha}_2|^{1-s}}-\right.\\
&\left.-\left(\frac{1}{|1-r\zeta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-r\zeta\overline{\alpha}_2|^{1-s}}\right)\right|d\sigma(\eta)\\
&+(1-r^2)^{-s}\left|\int_{\mathbb{T}}\nabla_{\mathbb{D}}P_s(r\zeta,\eta)\left(\frac{1}{|1-r\zeta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-r\zeta\overline{\alpha}_2|^{1-s}}\right)d\sigma(\eta)\right|\\
&=D+E.
\end{aligned}$$

This decomposition permits to avoid integrability problems when we introduce the modulus inside the integral.

We first observe that by Lemma 3.8, and using again (5.29), we have that since $|1-r\zeta\overline{\alpha_1}|\approx|1-r\zeta\overline{\alpha_2}|$,

$$\begin{aligned}
&E\lesssim(1-r^2)^s\left|\frac{1}{|1-r\zeta\overline{\alpha_1}|^{1-s}}-\frac{1}{|1-r\zeta\overline{\alpha_2}|^{1-s}}\right|\lesssim\frac{(1-r^2)^s|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{2-s}}\\
&\lesssim\frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{2-s+s-1}}\lesssim\frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}.
\end{aligned} \tag{5.31}$$

In order to obtain the desired estimate for D, we consider separately the integration regions $\Omega_1:=\{\eta\in\mathbb{T},\ |1-r\zeta\overline{\eta}|\ge\varepsilon|1-r\zeta\overline{\alpha_1}|\}$ and $\Omega_2:=\{\eta\in\mathbb{T},\ |1-r\zeta\overline{\eta}|\le\varepsilon|1-r\zeta\overline{\alpha_1}|\}$, where $\varepsilon<1$ will be fixed later on. We denote the corresponding integrals by D_1 and D_2.

5.4.1. *Estimate of D_1 (integration on $|1-r\zeta\overline{\eta}|\geq\varepsilon|1-r\zeta\overline{\alpha_1}|$).* Here there are not integrability problems and we decompose the integral in two parts. We have that

$$D_1\leq(1-r^2)^{-s}\int_{\Omega_1}|\nabla_{\mathbb{D}}P_s(r\zeta,\eta)|\left|\frac{1}{|1-\eta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-\eta\overline{\alpha}_2|^{1-s}}\right|d\sigma(\eta)$$

$$+(1-r^2)^{-s}\int_{\Omega_1}|\nabla_{\mathbb{D}}P_s(r\zeta,\eta)|\left|\frac{1}{|1-r\zeta\overline{\alpha}_1|^{1-s}}-\frac{1}{|1-r\zeta\overline{\alpha}_2|^{1-s}}\right|d\sigma(\eta)=D_{11}+D_{12}.$$

On D_{11} we assume without loss of generality that $|1-\eta\overline{\alpha_1}|\leq|1-\eta\overline{\alpha_2}|$. Then, using (5.29) and choosing $0<\varepsilon<s$ we have that

$$D_{11}\lesssim(1-r^2)^s\frac{|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}\int_{\Omega_1}\frac{d\sigma(\eta)}{|1-\eta\overline{\alpha_1}|^{1-s+\varepsilon}|1-\eta\overline{\alpha_2}|^{1-\varepsilon}},$$

where ε satisfies that $0<\varepsilon<s$. Hence,

$$D_{11}\lesssim(1-r^2)^s\frac{|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}\frac{1}{|\alpha_1-\alpha_2|^{1-s}}=(1-r^2)^s\frac{|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}.\tag{5.32}$$

Next, we estimate D_{12}. By (5.29) and the fact that since $2c<1$ for $|\alpha_1-\alpha_2|\leq c|1-r\zeta\overline{\alpha_1}|$, we have that $|1-r\zeta\overline{\alpha_1}|\approx|1-r\zeta\overline{\alpha_2}|$, then

$$\begin{aligned}D_{12}&\lesssim\frac{(1-r^2)^s}{|1-r\zeta\overline{\alpha_1}|^{2-s}}\int_{|1-r\zeta\overline{\eta}|\geq|1-r\zeta\overline{\alpha_1}|/2}\frac{|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\eta}|^{1+2s}}d\sigma(\eta)\\&\lesssim\frac{(1-r^2)^s|\alpha_1-\alpha_2|}{|1-r\zeta\overline{\alpha_1}|^{2-s}}\frac{1}{|1-r\zeta\overline{\alpha_1}|^{2s}}\leq\frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-r\zeta\overline{\alpha_1}|^{1+2s}}.\end{aligned}\tag{5.33}$$

5.4.2. *Estimate of D_2 (integration on $|1-r\zeta\overline{\eta}|\leq\varepsilon|1-r\zeta\overline{\alpha_1}|$).* In this case, it is convenient to use the parametrization of the unit circle $\mathbb{T}$ by e^{it}, $t\in(-\pi,\pi]$. We denote $\zeta=e^{ix}$, $\eta=e^{iy}$, $\alpha_1=e^{ia_1}$, and $\alpha_2=e^{ia_2}$, where $x,y,a_1,a_2\in(-\pi,\pi]$. With this notation we have

$$D_2\leq\int_{|1-re^{i(x-y)}|\leq\varepsilon|1-re^{i(x-a_1)}|}\frac{(1-r^2)^s}{|1-re^{i(x-y)}|^{1+2s}}\left|\frac{1}{|1-e^{i(y-a_1)}|^{1-s}}-\frac{1}{|1-e^{i(y-a_2)}|^{1-s}}-\right.$$
$$\left.\frac{1}{|1-re^{i(x-a_1)}|^{1-s}}+\frac{1}{|1-re^{i(x-a_2)}|^{1-s}}\right|dy$$

Next, for $x,y\in(-\pi,\pi]$, such that $|1-re^{i(x-y)}|\leq\varepsilon|1-re^{i(x-a_1)}|$, the function

$$\begin{aligned}\Phi(t)&:=\frac{1}{|1-e^{i(y-t)}|^{1-s}}-\frac{1}{|1-re^{i(x-t)}|^{1-s}}\\&=\frac{1}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{\frac{1-s}{2}}}-\frac{1}{\left((1-r)^2+4r\sin^2((x-t)/2)\right)^{\frac{1-s}{2}}}\end{aligned}$$

is differentiable for $t \in [a_1, a_2]$. By the Mean-Value Theorem, we deduce that

$$D_2 \lesssim \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a)}|} \frac{(1-r^2)^s}{|1-re^{i(x-y)}|^{1+2s}} \left|\int_{a_1}^{a_2} \frac{d}{dt}\Phi(t)dt\right| dy$$
$$= \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a_1)}|} \frac{(1-r^2)^s(1-s)}{|1-re^{i(x-y)}|^{1+2s}} \frac{1}{2} \left|\int_{a_1}^{a_2} \left(\frac{\sin((y-t)/2)\cos((y-t)/2)}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{\frac{1-s}{2}+1}} \right.\right.$$
$$\left.\left. - \frac{r\sin((x-t)/2)\cos((x-t)/2)}{\left((1-r)^2+4r\sin^2((x-t)/2\right)^{\frac{1-s}{2}+1}} \right) dt \right| dy.$$

We first observe that if we choose ε small enough, then the condition $|1-re^{i(x-y)}| \le \varepsilon|1-re^{i(x-a_1)}|$ gives that $1-r \lesssim |1-e^{i(x-a_1)}|$, and consequently, we have that $|1-e^{i(x-a_1)}| \approx |1-re^{i(x-a_1)}|$. Adding and substracting the intermediate term $\dfrac{\sin((x-t)/2)\cos((x-t)/2)}{4^{(1-s)/2}\left(\sin^2((x-t)/2)\right)^{\frac{1-s}{2}+1}}$, we have that

$$D_2 \lesssim \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a)}|} \frac{(1-r^2)^s(1-s)}{|1-re^{i(x-y)}|^{1+2s}} \int_{a_1}^{a_2} \left| \frac{\sin((y-t)/2)\cos((y-t)/2)}{4^{(1-s)/2}\left(\sin^2((y-t)/2)\right)^{\frac{1-s}{2}+1}} \right.$$
$$\left. - \frac{\sin((x-t)/2)\cos((x-t)/2)}{4^{(1-s)/2}\left(\sin^2((x-t)/2)\right)^{\frac{1-s}{2}+1}} \right| dtdy$$
$$+ \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a)}|} \frac{(1-r^2)^s(1-s)}{|1-re^{i(x-y)}|^{1+2s}} \int_{a_1}^{a_2} \left| \frac{\sin((x-t)/2)\cos((x-t)/2)}{4^{(1-s)/2}\left(\sin^2((x-t)/2)\right)^{\frac{1-s}{2}+1}} \right.$$
$$\left. - \frac{r\sin((x-t)/2)\cos((x-t)/2)}{\left((1-r)^2+4r\sin^2((x-t)/2\right)^{\frac{1-s}{2}+1}} \right| dtdy := D_{21} + D_{22}.$$

We begin with D_{21}. We apply the Mean Value Theorem, and for each $t \in [a_1, a_2]$, there exists l_t between x and y such that

$$D_{21} \lesssim \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a_1)}|} \frac{(1-r^2)^s|x-y|}{|1-re^{i(x-y)}|^{1+2s}} \int_{a_1}^{a_2} \left\{ \left| \frac{\cos^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{3-s}} \right| \right.$$
$$\left. + \left| \frac{\sin^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{3-s}} \right| + \left| \frac{\sin^2((l_t-t)/2)\cos^2((l_t-t)/2)}{|\sin((l_t-t)/2)|^{5-s}} \right| \right\} dtdy$$
$$\lesssim |1-re^{i(x-a_1)}|^{1-2s} \frac{(1-r^2)^s|a_1-a_2|}{|1-re^{i(x-a_1)}|^{3-s}} \le \frac{(1-r^2)^s|a_1-a_2|^s}{|1-re^{i(x-a_1)}|^{1+2s}},$$

where we have used that for any $t \in [a_1, a_2]$, $|\sin((l_t-t)/2)| \approx |1-re^{i(x-a_1)}|$ and $|a_1-a_2| < c|1-r\zeta\overline{a_1}|$.

Finally, for the estimate of D_{22}, we use again the Mean Value Theorem and we obtain that for each $0<r<1$ and each $t\in[a_1,a_2]$, there exists $l\in[r,1)$ such that

$$D_{22}$$

$$\lesssim \int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a_1)}|} \frac{(1-r^2)^s(1-r)}{|1-re^{i(x-y)}|^{1+2s}} \int_{a_1}^{a_2} \left(\frac{|\sin((x-t)/2)\cos((x-t)/2)|}{\big((1-l)^2+4l\sin^2((x-t)/2)\big)^{\frac{1-s}{2}+1}} \right.$$

$$\left. + \frac{l|\sin((x-t)/2)\cos((x-t)/2)|(2(1-l)+4\sin^2((x-t)/2)}{\big((1-l)^2+4l\sin^2((x-t)/2)\big)^{\frac{1-s}{2}+2}} \right) dtdy.$$

Since $|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a_1)}|$, we have that $|1-le^{i(x-a_1)}|\approx|1-re^{i(x-a_1)}|$ for any $l\in[r,1)$. Hence the above is bounded by

$$\int_{|1-re^{i(x-y)}|\le\varepsilon|1-re^{i(x-a_1)}|} \int_{a_1}^{a_2} \frac{(1-r^2)^s(1-r)}{|1-re^{i(x-y)}|^{1+2s}} \left(\frac{1}{|1-le^{i(x-a_1)}|^{2-s}} + \frac{1}{|1-le^{i(x-a_1)}|^{3-s}} \right) dtdy$$

$$\lesssim \frac{(1-r^2)^s(1-r^2)^{1-2s}|\alpha_1-\alpha_2|}{|1-z\overline{\alpha_1}|^{3-s}} \lesssim \frac{(1-r^2)^s|\alpha_1-\alpha_2|^s}{|1-z\overline{\alpha_1}|^{1+2s}},$$

where in the last estimate we have used that $|\alpha_1-\alpha_2|\lesssim|1-z\overline{\alpha_1}|$ and $(1-r)\lesssim|1-z\overline{\alpha_1}|$. □

6. Capacities, trace measures for $H^s(\mathbb{T})$ and Carleson measures for $P_s(H^s(\mathbb{T}))$

Definition 6.1. *Let $E\subset\mathbb{T}$. The Riesz capacity of E is defined by*

$$Cap_s(E) := \inf\{\|f\|_2^2 :\ I_s(|f|)\ge 1 \text{ on } E\}.$$

We list some properties of the equilibrium measure for a compact set in $\mathbb{T}$, which will be used below and that are essentially due to O. Frostman (see [1], Thm. 2.2.7).

Theorem 6.2. *Given a closed set $E\subset\mathbb{T}$, there exists a positive capacitary measure ν_E on $\mathbb{T}$, such that:*

(i) *ν_E is supported on E and $\nu_E(E)=Cap_s(E)$.*
(ii) *$q_E := I_s * I_s(\nu_E)\ge 1$ a.e. on E.*
(iii) *$q_E\in H^s(\mathbb{T})$ and $\|q_E\|^2_{H^s(\mathbb{T})}\lesssim Cap_s(E)$.*
(iv) *There is a constant $C>0$ independent of E, such that for any $\zeta\in\mathbb{T}$, $q_E(\zeta)\le C$.* □

Remark 6.3. *Since $2s<1$, we have that $I_s*I_s\approx I_{2s}$. This fact and Corollary 4.5 give that the function $p_E := I_{2s}(\nu_E)$ satisfies properties* (iii) *and* (iv)*, with property* (ii) *replaced by $p_E\gtrsim 1$ a.e. on E. For our purposes this is the function we will use when constructing appropriate test functions.*

Let $\varphi : (-\pi,\pi) \to [0,\infty)$ be a $\mathcal{C}^\infty$ function on $(-\pi,\pi]$, nonincreasing in $|x|$, with compact support on $(-\pi,\pi)$ and such that $\int_{-\pi}^{\pi}\varphi = 1$. For $\delta > 0$, let $\varphi_\delta(x) = \frac{1}{\delta}\varphi(x/\delta)$. We write $\nu_{E,\delta} := \nu_E * \varphi_\delta$, the regularizations of the measure ν_E. We then have that $\nu_{E,\delta}$ are functions in $\mathcal{C}^\infty$ on $\mathbb{T}$ satisfying that $d\nu_{E,\delta} := \nu_{E,\delta}\,dx \to d\nu$ in the sense of distributions and such that $\|\nu_{E,\delta}\|_1 = Cap_s(E)$.

We denote by $p_{E,\delta} := I_{2s} * \nu_{E,\delta}, \quad \delta > 0$.

Lemma 6.4 ([19], Chapter 2, Lemma 3.6)**.** *If $0 < s < 1/2$ and $\beta \in (1, 1/(1-2s)]$, then $p_{E,\delta}^\beta$ is in the Muckenhoupt class A_1, with A_1-constant independent of E and δ.*

Theorem 6.5. *Let $E \subset \mathbb{T}$ be a closed set and let p_E be the function given in Remark 6.3 and $p_{E,\delta}$ the regularization considered before. Let $\alpha > 1/2$. Then,*

(i) $\|p_{E,\delta}^\alpha\|_{H^s(\mathbb{T})}^2 \lesssim Cap_s(E)$.

Proof. We define the form ω_δ by

$$\omega_\delta = (P_s(p_{E,\delta}))^{2\alpha-1}(1-r^2)^{1-2s}\left(\frac{\partial}{\partial x}P_s(p_{E,\delta})dy - \frac{\partial}{\partial y}P_s(p_{E,\delta})dx\right).$$

Arguing as in Theorem 3.4, using that $p_{E,\delta}$ is bounded we can pass to the limit under the integral sign. Then using Theorem 3.3 and Proposition 4.3, we have that

$$\begin{aligned}
&\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta \\
&= \lim_{r\to 1^-}\int_0^{2\pi}(1-r^2)^{1-2s}(P_s(p_{E,\delta})^{2\alpha-1}_{|\partial D_r} r\frac{\partial}{\partial r}P_s(p_{E,\delta})_{|\partial D_r}dx = \int_{\mathbb{T}} p_{E,\delta}^{2\alpha-1}(I+(-\Delta)^s)p_{E,\delta} \\
&= \int_{\mathbb{T}} p_{E,\delta}^{2\alpha-1}d\nu_{E,\delta} \lesssim \int_{\mathbb{T}} d\nu_{E,\delta} = Cap_s(E),
\end{aligned}$$

Next, Stokes's Theorem and the Monotone's Lebesgue's convergence Theorem, give that

$$\begin{aligned}
\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta &= \frac{2\alpha-1}{\alpha^2}\int_{\mathbb{D}}(1-r^2)^{1-2s}|\nabla(P_s(p_{E,\delta}))^\alpha|^2 dm(z) \\
&+ (1-2s)^2\int_{\mathbb{D}}(1-r^2)^{-2s}|(P_s(p_{E,\delta}))^\alpha|^2 dm(z).
\end{aligned}$$

On the other hand, the function $(P_s(p_{E,\delta}))^\alpha$ has bondary values $p_{E,\delta}^\alpha$. Consequently, by Lemma 2.1, we have that

$$\begin{aligned}
&\|p_{E,\delta}^\alpha\|_{H^s(\mathbb{T})}^2 \lesssim \int_{\mathbb{D}}(1-r^2)^{1-2s}|\nabla(P_s(p_{E,\delta}))^\alpha|^2 dm(z) + \int_{\mathbb{D}}(1-r^2)^{-2s}|(P_s(p_{E,\delta}))^\alpha|^2 dm(z) \\
&\approx \lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta \lesssim Cap_s(E).
\end{aligned}$$

□

6.1. **Trace measures for $H^s(\mathbb{T})$ and Carleson measures for $P_s(H^s(\mathbb{T}))$.**

The characterization of the positive trace measures for $H^s(\mathbb{T})$ is well known (see, for instance the book [18] or Theorem 7.2.1 in [1] for a proof). Namely

Proposition 6.6. *Let $0 < s < 1/2$ and let μ be a positive Borel measure on $\mathbb{T}$. Then, μ is a trace measure for $H^s(\mathbb{T})$, that is, $\int_{\mathbb{T}} |f|^2 d\mu \lesssim \|f\|^2_{H^s(\mathbb{T})}$ for every $f \in H^s(\mathbb{T})$, if and only if there exists $C_\mu > 0$ such that for any compact set $E \subset \mathbb{T}$ $\mu(E) \leq C_\mu Cap_s(E)$.*

Definition 6.7. *Let $E \subset \mathbb{T}$. Then the tent over E, $T(E)$ is defined by $T(E) = \mathbb{D} \setminus \bigcup_{\xi \notin E} \Gamma(\xi)$.*

The arguments for the proof of the next elementary lemma can be found, for instance, in Lemma 3.25 in [11].

Lemma 6.8. *Let $0 < s < 1/2$, and let $E \subset \mathbb{T}$. Let f be a nonnegative measurable function on $\mathbb{T}$ such that $f \geq 1$ a.e. on E. Then $P_s(f) \gtrsim 1$ on $T(E)$.* □

Our next result gives a characterization of the Carleson measures for the space $P_s(H^s(\mathbb{T}))$. The proof heavily relies on Hanson's strong capacitary estimate (see, for instance Theorem 7.1.1 in [1] for a proof) and in Lemma 6.8 (see Theorem 3.26 in [11] for the details of the arguments of the proof).

Theorem 6.9. *Let $0 < s < 1/2$ and let μ be a positive Borel measure on $\mathbb{D}$. Then, μ is a Carleson measure for $P_s(H^s(\mathbb{T}))$, that is, $\int_{\mathbb{D}} P_s(\varphi)^2 d\mu \lesssim \|\varphi\|^2_{H^s(\mathbb{T})}$ for every $\varphi \in H^s(\mathbb{T})$ if and only if, there exists $C_\mu > 0$ such that for any compact set $E \subset \mathbb{R}$, $\mu(T(E)) \leq C_\mu Cap_s(E)$.*

We finish the section with two results that will give equivalent reformulations to (iv) and (v) in Theorem 1.1 and that will be used when needed in the proof of this Theorem.

Lemma 6.10. *Assume that the measure $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$, then the measure $|P_s(b)|^2(1-|z|^2)^{-2s}dm(z)$ is also a Carleson measure for $P_s(H^s(\mathbb{T}))$.*

In particular, $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$ if and only if $(|\nabla P_s(b)|^2(1-|z|^2)^{1-2s} + |P_s(b)|^2(1-|z|^2)^{-2s})\, dm(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T}))$.

Proof. Let $\varphi \in H^s(\mathbb{T})$. Applying Lemma 2.2, we deduce that

$$\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s} dm(z) \lesssim \int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{1-2s} dm(z)$$
$$+ \int_{\mathbb{D}} |P_s(\varphi)||\nabla P_s(\varphi)|P_s(b)|^2 (1-|z|^2)^{1-2s} dm(z)$$
$$+ \int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)||\nabla P_s(b)|(1-|z|^2)^{1-2s} dm(z) = I + II + III.$$

Now, we use the pointwise estimate for extensions of L^2 functions in $\mathbb{T}$ given in Lemma 2.8 in [2], which gives in particular that $|P_s(b)(z)| \lesssim \frac{1}{(1-|z|^2)^{\frac{1}{2}}}$, together with Corollary 3.6 to obtain

$$I \lesssim \int_{\mathbb{D}} |P_s(\varphi)|^2 (1-|z|^2)^{-2s} dm(z) \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}.$$

Next, Hölder's inequality and the same pointwise estimate $|P_s(b)(z)| \lesssim \frac{1}{(1-|z|^2)^{\frac{1}{2}}}$, give that

$$II$$
$$\le \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s} dm(z)\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |\nabla P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{2-2s} dm(z)\right)^{\frac{1}{2}}$$
$$\le \frac{\varepsilon}{2} \int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s} dm(z) + \frac{1}{2\varepsilon} \|P_s(\varphi)\|^2_{W^2_{1,1-2s}},$$

where $\varepsilon < 1$. Hence, $II \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}$.

Finally, Hölder's inequality, the hypothesis and the pointwise estimate $|P_s(b)(z)| \lesssim \frac{1}{(1-|z|^2)^{\frac{1}{2}}}$ give

$$III$$
$$\le \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{1-2s} dm(z)\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 |\nabla P_s(b)|^2 (1-|z|^2)^{1-2s} dm(z)\right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_{\mathbb{D}} |P_s(\varphi)|^2 (1-|z|^2)^{-2s} dm(z)\right)^{\frac{1}{2}} \|P_s(\varphi)\|_{W^2_{1,1-2s}} \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}.$$

Altogether gives finally that

$$\int_{\mathbb{D}} |P_s(\varphi)|^2 |P_s(b)|^2 (1-|z|^2)^{-2s} dm(z) \lesssim \|P_s(\varphi)\|^2_{W^2_{1,1-2s}} \approx \|\varphi\|^2_{H^s(\mathbb{T})}.$$

□

Lemma 6.11. *The following assertions are equivalent:*

(i) *The measure* $d\nu := \left|(-\Delta)^{s/2}(b)\right|^2 d\sigma$ *is a trace measure for* $H^s(\mathbb{T})$.

(ii) *The measure* $d\widetilde{\nu} := \left|((-\Delta)^s + I)^{\frac{1}{2}}(b)\right|^2 d\sigma$ *is a trace measure for* $H^s(\mathbb{T})$.

Proof. The proof is based in the following result by V. Maz'ya and I.E. Verbitsky(see [Ma-Ve]):

Proposition 6.12. *Let g be an integrable function on $\mathbb{T}$ such that $|g|^p d\sigma$ is a trace measure for $I_s[L^p]$. Let h be a measurable function on $\mathbb{T}$ satisfying that there exists $C>0$ such that for any weight w in A_1,*

$$\int_{\mathbb{T}} |h|^p w \leq C \int_{\mathbb{T}} |g|^p w. \tag{6.34}$$

We then have that $|h|^p d\sigma$ is a trace measure for $I_s[L^p]$.

Assume that (i) holds, that is $\left|(-\Delta)^{s/2}(b)\right|^2 d\sigma$ is a trace measure for $H^s(\mathbb{T})$. Let $h=(-\Delta)^{s/2}(b)$ and $g=((-\Delta)^s+I)^{\frac{1}{2}}(b)$.

Then $g=((-\Delta)^s+I)^{\frac{1}{2}}(-\Delta)^{-s/2}(-\Delta)^{s/2}b = Th$, where $T=((-\Delta)^s+I)^{\frac{1}{2}}(-\Delta)^{-s/2}$. Applying Corollary 3.5 and Lemma 4.2 and using an argument similar to the one used in Proposition 4.11, we deduce that T is an operator of Calderon Zygmund.

Hence, applying Theroem 4.12, we have that for any $\omega \in A_1 \subset A_2$,

$$\int_{\mathbb{T}} |g|^2 d\omega = \int_{T} |Th|^2 d\omega \lesssim \int_{\mathbb{T}} |h|^2 d\omega.$$

Now, Proposition 6.12 gives that $d\widetilde{\nu}$ is a trace measure for $H^s(\mathbb{T})$, which is (ii).

The implication in the other sense is proved in an analogous way. □

7. Proof of the main result (Theorem 1.1)

7.1. **Proof of** (i)⇔ (ii)⇔(iii)**.** If $\varphi,\psi \in \mathcal{C}^\infty(\mathbb{T})$, then $P_s(\varphi)$, $P_s(\varphi)$ and $P_s(\varphi\psi)$ are in $\mathcal{W}^2_{1,1-2s}\cap L^\infty$. Hence $P_s(\varphi)P_s(\varphi)\in \mathcal{W}^2_{1,1-2s}$ with the same boundary values, $\varphi\psi$, than the function $P_s(\varphi\psi)$. Consequently, the equivalences between (i), (ii) and (iii) follow from Proposition 3.7.

7.2. **Proof of** (v)⇒ (iii)**.** We first observe that if

$$|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)$$

is a Carleson measure for $P(H^s(\mathbb{T}))$, Lemma 6.10 gives that the measure

$$\left(|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}+|P_s(b)|^2(1-|z|^2)^{-2s}\right)dm(z)$$

is also a Carleson measure for $P(H^s(\mathbb{T}))$.

Next, in order to prove (iii), it is enough to consider the case $\varphi = \psi$. Then Hölder's inequality and the above observation gives that

$$\begin{aligned}
&\left|\int_{\mathbb{D}} \nabla(P_s(\varphi)^2)(z)\nabla P_s(b)(z)(1-|z|^2)^{1-2s}dm(z)\right| \\
&\lesssim |\int_{\mathbb{D}} |P_s(\varphi)(z)||\nabla P_s(\varphi)(z)||\nabla P_s(b)(z)|(1-|z|^2)^{1-2s}dm(z) \\
&\le \left(\int_{\mathbb{D}} |P_s(\varphi)(z)|^2|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{D}} |\nabla P_s(\varphi)(z)|^2(1-|z|^2)^{1-2s}dm(z)\right)^{\frac{1}{2}} \lesssim \|\varphi\|^2_{H^s(\mathbb{T})}.
\end{aligned}$$

Similarly,

$$\left|\int_{\mathbb{D}} (P_s(\varphi))^2(z)P_s(b)(z)(1-|z|^2)^{-2s}dm(z)\right| \lesssim \|\varphi\|^2_{H^s(\mathbb{T})},$$

7.3. **Proof of** (iv)$\Rightarrow$ (v)**.** By Lemmas 6.10 and (6.11) it is enough to show that if $(|(-\Delta)^s+I)^{\frac{1}{2}}(b)|^2d\sigma$ is a trace measure for H^s, then $|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)$ is a Carleson measure for $P_s(H^s(\mathbb{T})(\mathbb{T}))$. Using Theorem 6.9, we must show that for any closed set $E\subset\mathbb{T}$, $\int_{T(E)}|\nabla P_s(b)|^2(1-|z|^2)^{1-2s}dm(z)\lesssim Cap_s(E)$.

Let $E\subset\mathbb{T}$ be closed and let p_E be the potential of the extremal measure for the set E. For $z\in\mathbb{D}$, let $I_z=\{\zeta\in\mathbb{T}\,;\,z\in\Gamma(\zeta)\}$. We have that if $z\in T(E)$, then $I_z\subset E$ and $|I_z|\approx(1-|z|^2)$. Let $\alpha\in(1/2,1/2(1-2s)]$. Then, Lemma 6.4, gives that $p_E^{2\alpha}\in A_1$, and, in particular, $p_E^{2\alpha}\in A_2$.

Since $p_E\gtrsim 1$ a.e. on E, Fubini's theorem gives,

$$\begin{aligned}
&\int_{T(E)} |\nabla P_s(b)(z)|^2(1-|z|^2)^{1-2s}dm(z) \\
&\lesssim \int_{\mathbb{D}} |\nabla P_s(b)(z)|^2(1-|z|^2)^{1-2s}\frac{1}{(1-|z|^2)}\int_{I_z} p_E^{2\alpha}(\zeta)d\sigma(\zeta)dm(z) \\
&\lesssim \int_{\mathbb{T}}\int_{\Gamma(\zeta)} |\nabla P_s(b)(z)|^2(1-|z|^2)^{-2s}p_E^{2\alpha}(\zeta)dm(z)d\sigma(\zeta) \\
&= \int_{\mathbb{T}}\int_{\Gamma(\zeta)} |(1-|z|^2)^{-s}\nabla P_s(I_s(I_s^{-1}(b)))(z)|^2p_E^{2\alpha}(\zeta)dm(z)d\sigma(\zeta) \\
&= \|G_{\mathbf{K}}(I_s^{-1}(b))\|^2_{L^2(p_E^{2\alpha})}.
\end{aligned}$$

Since $p_E^{2\alpha}\in A_2$, Proposition 5.2 and Theorem 5.1 give that the above is bounded by

$$\left(\int_{\mathbb{T}} |I_s^{-1}(b)(\zeta)|^2p_E^{2\alpha}(\zeta)d\sigma(\zeta)\right)^2 = \|I_s^{-1}(b)\|^2_{L^2(p_E^{2\alpha})}.$$

Since $I_s^{-1} = I_{2s}^{1/2} I_s^{-1} I_{2s}^{-1/2}$, Theorem 4.12 gives

$$\begin{aligned}\|I_s^{-1}(b)\|^2_{L^2(p_E^{2\alpha})} &= \|I_{2s}^{1/2} I_s^{-1} I_{2s}^{-1/2}(b)\|^2_{L^2(p_E^{2\alpha})} \lesssim \|I_{2s}^{-1/2}(b)\|^2_{L^2(p_E^{2\alpha})} \\ &\leq \liminf_{\delta\to 0^+} \int_{\mathbb{T}} p_{E,\delta}^{2\alpha}(\zeta)|I_{2s}^{-1/2}(b)(\zeta)|^2 d\sigma(\zeta)\end{aligned}$$

But $I_{2s}^{-1/2} = ((-\Delta)^s + I)^{\frac{1}{2}}$ and $|I_{2s}^{-1/2}(b)(\zeta)|^2 d\sigma$ is by hypothesis a trace measure for $H^s(\mathbb{T})$. Then we have that the above is bounded by $\liminf_{\delta\to 0} \|p_{E,\delta}^{\alpha}\|^2_{H^s(\mathbb{T})}$, which by Theorem 6.5, is in turn bounded by $Cap_s(E)$.

7.4. **Proof of** (i)⇒ (iv)**.** By Lemma (6.11), we have that proving condition (iv) is equivalent to proving that $d\mu(\zeta) = |((-\Delta)^s + I)^{\frac{1}{2}}(b)(\zeta)|^2 d\sigma(\zeta) = |I_{2s}^{-1/2}(b)(\zeta)|^2 d\sigma(\zeta)$ is a trace measure for H^s. This will be checked by proving that it satisfies the capacitary characterization given in Proposition 6.6, that is, we will show that for each compact set $E \subset \mathbb{T}$,

$$\int_E |I_{2s}^{-1/2}(b)(\zeta)|^2 d\sigma(\zeta) \lesssim Cap_s(E). \tag{7.35}$$

Let $E \subset \mathbb{T}$ be a closed subset of $\mathbb{T}$ and let $p_{E,\delta} = I_{2s} * \nu_{E,\delta}$, $\delta > 0$, where $\nu_{E,\delta}$ is a regularization of the extremal capacitary measure of E. Let $\alpha \in (1/2, 1/(2(1-2s))$ be fixed. We consider the test functions

$$\varphi_\delta := \frac{I_{2s}^{1/2}(\chi_E I_{2s}^{-1/2}(b))}{p_{E,\delta}^{\alpha}}, \qquad \psi_\delta := p_{E,\delta}^{\alpha}.$$

We write $g_E = \chi_E \left(I_{2s}^{-1/2}(b)\right)$. Applying the hypothesis (i), we have that

$$\int_E |I_{2s}^{-1/2}(b)|^2 d\sigma = \int_{\mathbb{T}} I_{2s}^{-1/2}(\varphi_\delta \psi_\delta) I_{2s}^{-1/2}(b) d\sigma \lesssim \|\varphi_\delta\|_{H^s(\mathbb{T})} \|\psi_\delta\|_{H^s(\mathbb{T})}. \tag{7.36}$$

We next estimate each of these last norms. First, we have that by Theorem 6.5, $\|\psi_\delta\|^2_{H^s(\mathbb{T})} = \|p_{E,\delta}^{\alpha}\|^2_{H^s(\mathbb{T})} \lesssim Cap_s(E)$. Our next objective is to prove that

$$\lim_{\delta\to 0} \|\varphi_\delta\|^2_{H^s(\mathbb{T})} \lesssim \int_{\mathbb{T}} |g_E|^2 d\sigma. \tag{7.37}$$

If this estimate holds, we will have by (7.36) that $\int_E |I_{2s}^{-1/2}(b)|^2 d\sigma \lesssim Cap_s(E)$, which is the estimate we wanted to prove.

Using Lemma 2.1,

$$\|\varphi_\delta\|^2_{H^s(\mathbb{T})} \lesssim \int_{\mathbb{D}} \left| \nabla \left(\frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right) \right|^2 (1-|z|^2)^{1-2s} dm(z)$$

$$+ \int_{\mathbb{D}} \left| \frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} dm(z)$$

$$\lesssim \int_{\mathbb{D}} \left| \frac{\nabla\left(P_s\left(I_{2s}^{1/2}(g_E)\right)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} dm(z)$$

$$+ \int_{\mathbb{D}} \frac{\left| P_s\left(I_{2s}^{1/2}(g_E)\right) \nabla\left(P_s(p_{E,\delta})\right)\right|^2}{(P_s(p_{E,\delta}))^{2\alpha+2}} (1-|z|^2)^{1-2s} dm(z)$$

$$+ \int_{\mathbb{D}} \left| \frac{P_s\left(I_{2s}^{1/2}(g_E)\right)}{(P_s(p_{E,\delta}))^\alpha} \right|^2 (1-|z|^2)^{1-2s} dm(z)$$

$$= I + II + III.$$

We begin with the estimate of I. Let $z \in \mathbb{D}$. We have that

$$P_s(p_{E,\delta})(z) \gtrsim \frac{1}{|I_z|} \int_{I_z} p_{E,\delta}.$$

Using this estimate and Hölder's inequality twice, we obtain that

$$I \lesssim \int_{\mathbb{D}} |\nabla\left(P_s(I_{2s}^{1/2}(g_E))\right)|^2 (1-|z|^2)^{1-2s} \left(\frac{1}{|I_z|} \int_{I_z} p_{E,\delta}(\eta) d\sigma(\eta) \right)^{-2\alpha} dm(z)$$

$$\lesssim \int_{\mathbb{D}} |\nabla\left(P_s(I_{2s}^{1/2}(g_E))\right)|^2 (1-|z|^2)^{1-2s} \left(\frac{1}{|I_z|} \int_{I_z} p^{-1}_{E,\delta}(\eta) d\sigma(\eta) \right)^{2\alpha} dm(z)$$

$$\lesssim \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |\nabla\left(P_s(I_s I_s^{-1} I_{2s}^{1/2}(g_E))\right)|^2 (1-|z|^2)^{-2s} dm(z) \frac{1}{p^{2\alpha}_{E,\delta}}(\eta) d\sigma(\eta).$$

Since by Lemma 6.4, $p^{2\alpha}_{E,\delta} \in A_2$ with constants independent of E and δ (and hence also $p^{-2\alpha}_{E,\delta} \in A_2$), Proposition 5.2 and Theorem 5.1 give that the above is bounded, up to a constant, by

$$\int_{\mathbb{T}} |I_s^{-1} I_{2s}^{1/2}(g_E)|^2 \frac{1}{p^{2\alpha}_{E,\delta}}(\eta) d\sigma(\eta) \lesssim \int_{\mathbb{T}} g_E^2(\eta) \frac{1}{p^{2\alpha}_{E,\delta}}(\eta) d\sigma(\eta),$$

where in the last estimate we have used Theorem 4.12, since $p^{-2\alpha}_{E,\delta} \in A_2$. Altogether we deduce that

$$I \lesssim \int_{\mathbb{T}} g_E^2(\eta) \frac{1}{p^{2\alpha}_{E,\delta}}(\eta) d\sigma(\eta). \tag{7.38}$$

Now we proceed to estimate II. We consider the form given by

$$\omega_\delta(z) = \frac{(P_s(I_{2s}^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}(1-|z|^2)^{1-2s}\left(\frac{\partial P_s(p_{E,\delta})}{\partial x}dy - \frac{\partial P_s(p_{E,\delta})}{\partial y}dx\right). \tag{7.39}$$

Integrating on the circle of radius $r<1$, taking polar coordinates and letting $r\to 1^-$, we have (see Theorem 3.3) that

$$\begin{aligned}
&\lim_{r\to 1^-}\int_{\partial D_r}\omega_\delta = \lim_{r\to 1^-}\int_{\partial D_r}(1-r^2)^{1-2s} r \frac{(P_s(I_{2s}^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}\frac{\partial}{\partial r}P_s(p_{E,\delta})\\
&= \int_{\mathbb{T}}\frac{(I_{2s}^{1/2} g_{E_\delta})^2}{p_{E,\delta}^{2\alpha+1}}((-\Delta)^s + I)p_{E,\delta} = \int_{\mathbb{T}}\frac{(I_{2s}^{1/2} g_{E_\delta})^2}{p_{E,\delta}^{2\alpha+1}}d\nu_\delta \geq 0.
\end{aligned}$$

Applying Stokes's Theorem on D_r, and letting $r\to 1^-$ as in Theorem 3.4, we have that

$$\begin{aligned}
&\int_{\mathbb{T}}\omega_\delta = \int_{\mathbb{T}}\frac{I_{2s}^{1/2} g_{E_\delta}^2}{p_{E,\delta}^{2\alpha+1}}d\nu_\delta\\
&= -(2\alpha+1)\int_{\mathbb{D}}\frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+2}}|\nabla P_s(p_{E,\delta})|^2(1-|z|^2)^{1-2s}dm(z)\\
&+ 2\int_{\mathbb{D}}\frac{P_s(I_{2s}^{1/2} g_E)\nabla P_s(I_{2s}^{1/2} g_E)\nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}}(1-|z|^2)^{1-2s}dm(z)\\
&+ (1-2s)^2\int_{\mathbb{D}}\frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha+1}}\int_{\mathbb{T}}\frac{p_{E,\delta}(\zeta)}{|z-\zeta|^{1+2s}}d\sigma(\zeta)dm(z).
\end{aligned}$$

Since we have shown that $\int_{\mathbb{T}}\omega_\delta \geq 0$, we deduce that

$$\begin{aligned}
&II \lesssim \int_{\mathbb{D}}\frac{P_s(I_{2s}^{1/2} g_E)\nabla P_s(I_{2s}^{1/2} g_E)\nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}}(1-|z|^2)^{1-2s}dm(z)\\
&+ \int_{\mathbb{D}}\frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha}}(1-|z|^2)^{-2s}dm(z).
\end{aligned}$$

Next, we proceed to estimate the first term on the right. Hölder's inequality gives that

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{P_s(I_{2s}^{1/2} g_E)\nabla P_s(I_{2s}^{1/2} g_E)\nabla P_s(p_{E,\delta})}{P_s(p_{E,\delta})^{2\alpha+1}}(1-|z|^2)^{1-2s}dm(z)\\
&\lesssim \left(\int_{\mathbb{D}} \frac{|P_s\left(I_{2s}^{1/2}(g_E)\right)|^2|\nabla\left(P_s(p_{E,\delta})\right)|^2}{|P_s(p_{E,\delta})|^{2\alpha+2}}(1-|z|^2)^{1-2s}dm(z)\right)^{\frac12}\\
&\times\left(\int_{\mathbb{D}} \frac{|\nabla P_s\left(I_{2s}^{1/2}(g_E)\right)|^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}dm(z)\right)^{\frac12}\\
&= II^{\frac12} I^{\frac12} \lesssim (1/\varepsilon)I + \varepsilon II.
\end{aligned}$$

In addition, (2.3) gives that

$$\int_{\mathbb{D}} \frac{(P_s(I_s^{1/2} g_{E_\delta}))^2}{(P_s(p_{E,\delta}))^{2\alpha}}(1-|z|^2)^{-2s}dm(z) \lesssim III + \varepsilon(I+II).$$

Consequently, we have shown that

$$II \lesssim I + III. \tag{7.40}$$

Next, if we now choose $0<\varepsilon'<1$, we have

$$III \lesssim$$

$$= \int_{1-|z|^2<\varepsilon'} \frac{P_s(I_{2s}^{\frac12} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}dm(z) + \int_{1-|z|^2\ge\varepsilon'} \frac{P_s(I_{2s}^{\frac12} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}dm(z).$$

Since in the first integral $1-|z|^2<\varepsilon'$, using (2.3) and (7.40), it is bounded by $\varepsilon'(I+II+III)$. We pass that to the left hand side and obtain that

$$I + II + III \lesssim I + \int_{1-|z|^2\ge\varepsilon'} \frac{P_s(I_{2s}^{\frac12} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}dm(z). \tag{7.41}$$

But, when $1-|z|^2\ge\varepsilon'$, we have that $P_s(I_{2s}^{\frac12}(g_E)) \approx \int_{\mathbb{T}} I_{2s}^{\frac12}(g_E)$ and $P_s(p_{E,\delta}) \approx \int_{\mathbb{T}} p_{E,\delta}$.

Hence, using Hölder's inequality and that by Lemma 6.4, $p_{E,\delta}^{2\alpha}\in A_2$, with constants independent of E and δ we have that

$$\begin{aligned}
&\int_{1-|z|^2\ge\varepsilon'} \frac{P_s(I_{2s}^{\frac12} g_E)^2}{P_s(p_{E,\delta})^{2\alpha}}(1-|z|^2)^{1-2s}dm(z)\\
&\lesssim \frac{\left(\int_{\mathbb{T}} I_{2s}^{\frac12}(g_E)\right)^2}{\left(\int_{\mathbb{T}} p_{E,\delta}\right)^{2\alpha}} \lesssim \frac{\left(\int_{\mathbb{T}} I_{2s}^{\frac12}(g_E)\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}}.
\end{aligned} \tag{7.42}$$

Fubini's Theorem and the fact that the operator $I_{2s}^{\frac{1}{2}}$ can be represented as a convolution by a kernel $T_{\mathbf{K}}(\zeta,\eta)$ satisfying that $T_{\mathbf{K}}(\zeta,\eta)\lesssim\frac{1}{|\zeta-\eta|^{1-s}}$ (see Remark 8.3), give that

$$\left|\int_{\mathbb{T}} I_{2s}^{\frac{1}{2}}(g_E)\right|\lesssim\left|\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|\zeta-\eta|^{1-s}}d\sigma(\eta)g_E(\zeta)d\sigma(\zeta)\right|\lesssim\int_{\mathbb{T}}|g_E(\zeta)|d\sigma(\zeta).$$

Plugging this estimate in (7.42), and using that g_E is supported on E, we have that Hölder's inequality gives that

$$\frac{\left(\int_{\mathbb{T}} I_{2s}^{\frac{1}{2}}(g_E)d\sigma\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}d\sigma}\lesssim\frac{\left(\int_{\mathbb{T}}|g_E(\zeta)|d\sigma(\zeta)\right)^2}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}d\sigma}\lesssim\frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)d\sigma(\zeta)}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}d\sigma}.$$

So, we have just proved that, using (7.36) and 7.41, that

$$\|\varphi_\delta\|_{H^s(\mathbb{T})}^2\lesssim I+II+III\lesssim\int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p_{E,\delta}^{2\alpha}}(\eta)d\sigma(\eta)+\frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)d\sigma(\zeta)}{\int_{\mathbb{T}} p_{E,\delta}^{2\alpha}d\sigma}.$$

We next have that $p_{E,\delta}^{2\alpha}$ is bounded by above, and also bounded by below (with constant depending on E) since,

$$p_{E,\delta}(\zeta)=\int_{\mathbb{T}}\frac{d\nu_\delta(\eta)}{|1-\zeta\overline{\eta}|^{1-2s}}\gtrsim\int_{\mathbb{T}} d\nu_\delta(\eta)=\nu(\mathbb{T}).$$

Hence, we can apply the Dominated Lebesgue's Theorem and deduce that

$$\lim_{\delta\to0}\|\varphi_\delta\|_{H^s(\mathbb{T})}^2\lesssim\int_{\mathbb{T}} g_E^2(\eta)\frac{1}{p_E^{2\alpha}}(\eta)d\sigma(\eta)+\frac{m(E)\int_{\mathbb{T}} g_E^2(\zeta)d\sigma(\zeta)}{\int_{\mathbb{T}} p_E^{2\alpha}d\sigma}.$$

Next, since g_E is supported on E and $p_E\gtrsim1$ on E, we deduce that

$$\lim_{\delta\to0}\|\varphi_\delta\|_{H^s(\mathbb{T})}^2\lesssim\int_{\mathbb{T}} g_E^2(\eta)d\sigma(\eta),$$

which prove (7.37) and, as it was pointed out, finishes the proof of the theorem. □

8. Appendix 1: Proof of Theorem 4.9

We begin the section with two technical results on finite differences that will be used in the proof of Theorem 4.9. The first one is a summation by parts formula that can be proved using induction

Proposition 8.1. *Let $\varphi,\psi:\mathbb{Z}\to\mathbb{C}$. We define $\widetilde{\mathbf{\Delta}}(\varphi)(k):=\varphi(k-1)-\varphi(k)$ and $\widetilde{\mathbf{\Delta}}^n(\varphi)(k):=\widetilde{\mathbf{\Delta}}(\widetilde{\mathbf{\Delta}}^{n-1}(\varphi))(k)$, $n\in\mathbb{N}$.*

Let N be a positive integer. We then have that for any $l\in\mathbb{N}$,

$$\begin{aligned}\sum_{2^l\le|k|<2^{l+1}}\mathbf{\Delta}^N(\varphi)(k)\psi(k)&=\sum_{2^l\le|k|<2^{l+1}}\varphi(k)\widetilde{\mathbf{\Delta}}^N(\psi)(k)\\&+\sum_{|k|=2^{t+l},\,t\in\{0,1\},\,m=0,\cdots,N-1}(-1)^{t+1}\widetilde{\mathbf{\Delta}}^m(\varphi)(k)(\widetilde{\mathbf{\Delta}})^{N-1-m}(\psi)(k-1).\end{aligned}\tag{8.43}$$

□

Lemma 8.2. *Let $C>0$ such that for each $\varphi:\mathbb{Z}\to\mathbb{Z}$ that*

$$|\boldsymbol{\Delta}^j(\varphi)(k)|\le \frac{C}{|k|^{j+r}},\quad k\in\mathbb{Z}\setminus\{0\}\quad \textit{and for some}\quad r>-1.$$

Then, for each non negative integer $M\ge 1$ and $n\in\mathbb{N}$, there exists C_1, depending on C, M and n, such that,

$$\left|\boldsymbol{\Delta}^M(k^n\boldsymbol{\Delta}^j(\varphi))(k)\right|\le C_1|k|^{n-j-r-M},\quad k\in\mathbb{Z}\setminus\{0\}. \tag{8.44}$$

Proof. The proof is deduced by induction on M. □

Proof of Theorem 4.9. The proof is a version for the group $\mathbb{T}$ of a well known result on $\mathbb{R}^n$ (see for instance [22]). Formally, the operator T will be given by a convolution with $T_K=\sum_{k\in\mathbb{Z}}m(k)e^{ikx}$. Let us check that $T_K(x)$ is a function that satisfies the estimates (4.23) of the theorem.

Since m is by hypothesis bounded, we may assume that

$$|\boldsymbol{\Delta}^j(m)(k)|\le\frac{C}{|k|^j},\quad k\in\mathbb{Z}\setminus\{0\}.$$

Let l be a non negative integer. For $x\in[-\pi,\pi]$, let

$$T_{K,l}(x)=\sum_{2^l\le|k|<2^{l+1}}m(k)e^{ikx}.$$

Let $n\ge 0$ be fixed. We claim that

$$\begin{aligned}&\left|T_{K,l}^{(n)}(x)\right|\lesssim\frac{2^{ln}}{|x|};\\ &\left|T_{K,l}^{(n)}(x)\right|\lesssim\frac{2^{l(n+1-M)}}{|x|^M},\quad M>1+n,\end{aligned} \tag{8.45}$$

with constants independent of l. Postponing the proof of the claim, we finish the proof of the estimate (4.23) in Theorem 4.9. Given $x\in[-\pi,\pi]\setminus\{0\}$, we write,

$$\left|T_K^{(n)}(x)\right|\le\sum_{2^l<\frac{1}{|x|}}|T_{K,l}^{(n)}(x)|+\sum_{2^l\ge\frac{1}{|x|}}|T_{K,l}^{(n)}(x)|=A+B.$$

For the estimate of A, we use the first assertion of the claim and we obtain

$$A=\sum_{2^l<\frac{1}{|x|}}|T_{K,l}^{(n)}(x)|\lesssim\sum_{2^l<\frac{1}{|x|}}\frac{2^{ln}}{|x|}\lesssim\frac{1}{|x|^{n+1}}.$$

For the estimate of B, we use the second assertion of the claim for $M>1+n$ fixed and we obtain

$$B=\sum_{2^l\ge\frac{1}{|x|}}|T_{K,l}^{(n)}(x)|\lesssim\sum_{2^l\ge\frac{1}{|x|}}\frac{2^{l(n+1-M)}}{|x|^M}\le\frac{|x|^{M-1-n}}{|x|^M}=\frac{1}{|x|^{n+1}}.$$

So we are left to prove the claim (8.45).

We begin with the first assertion.

$$T_{K,l}^{(n)}(x) = \sum_{2^l \le |k| < 2^{l+1}} i^n k^n m(k) e^{ikx},$$

and since $\mathbf{\Delta_1}(e^{ikx}) = (e^{ix} - 1)e^{ikx}$, we have by Lemma 8.1, that

$$\left|(e^{ix} - 1)T_{K,l}^{(n)}(x)\right| \le |\sum_{2^l \le |k| < 2^{j+1}} \mathbf{\Delta_1}(e^{ikx}) i^n k^n m(k)|$$

$$\le \sum_{2^l \le |k| < 2^{j+1}} |\tilde{\mathbf{\Delta}}_\mathbf{1}(k^m m(k))|$$

$$+ \sum_{|k| = 2^{t+l},\, t \in \{0,1\}} |(k-1)^n m(k-1)|.$$

Next, applying Lemma 8.2 to $\widetilde{\Delta}$, we deduce that $|\tilde{\mathbf{\Delta}}_\mathbf{1}(k^n m(k))| \lesssim |k|^{n-1}$. So we have that,

$$\left|(e^{ix} - 1)T_{K,l}^{(n)}(x)\right| \lesssim \sum_{2^l \le |k| < 2^{l+1}} |k|^{n-1} + 2^{ln} \lesssim 2^{ln},$$

and consequently, that $|T_{K,l} l^{(n)}(x)| \lesssim \frac{2^{ln}}{|x|}$. For the second part of the claim, let $M > 1 + n$ be fixed. Since $\mathbf{\Delta_1}^M(e^{ikx}) = (e^{ix} - 1)^M e^{ikx}$, Lemma 8.1 gives that

$$\left|(e^{ix} - 1)^M T_{K,l}^{(n)}(x)\right| \le |\sum_{2^l \le |k| < 2^{j+1}} \mathbf{\Delta_1}^M(e^{ikx}) k^n m(k)|$$

$$\le \sum_{2^l \le |k| < 2^{j+1}} |\tilde{\mathbf{\Delta}}_\mathbf{1}^M(k^n m(k))|$$

$$+ \sum_{|k| = 2^{q+l},\, q \in \{0,1\},\, p = 0, \cdots, M-1} |\mathbf{\Delta_1}^p(e^{ikx}) \widetilde{\mathbf{\Delta}}_\mathbf{1})^{M-1-p}(k^n m)(k-1)|.$$

By hypothesis, $|\tilde{\mathbf{\Delta}}_\mathbf{1}^M(k^n m(k))| \lesssim |k|^{(n-M)}$, and consequently,

$$\left|(e^{ix} - 1)^M T_{K,l}^{(n)}(x)\right| \lesssim \sum_{2^l \le |k| < 2^{l+1}} 2^{k(n-M)} + 2^{l(n+1-M)}.$$

Hence,

$$|T_{K,l}^{(n)}(x)| \lesssim \frac{2^{l(n+1-M)}}{|x|^M}.$$

And that finishes the proof of the claim.

Observe that if

$$\left|T_K^{(j)}(x)\right| \lesssim \frac{1}{|x|^{j+1}}, \quad x \neq 0,$$

then , identifying $\mathbb{T}$ in the usual way by e^{ix}, $x \in (-\pi, \pi]$, T_K satisfies (ii) and (iii) of Definition 4.7, and consequently, T is a Calderon-Zygmund operator.

Next, since T is a Calderon-Zygmund operator, we have that for any $\omega \in A_p$, $1 < p < \infty$, T_K is bounded on $L^p(\omega)$ (see for instance Theorem 7.11 in [13]) . And that finishes the proof of (4.24) of the theorem. $\square$

Remark 8.3. *We observe that the same arguments of* (4.9) *show that if the Fourier multiplier m satisfies that $\Delta^{(j)}m(k) = \mathrm{O}\left(\frac{1}{|k|^{j-\alpha}}\right)$, $0 \le \alpha < 1$ then the kernel T_K satisfies $|T_K^{(n)}(x)| \lesssim \frac{1}{|x|^{n+1-\alpha}}$.*

9. Appendix 2: Proof of Theorem 5.1

The proof of Theorem 5.1 follows the scheme given in [16]. In consequence, we will just sketch the specific parts of the proof for our situation and remit to this paper to find the proofs of the remaining parts used here. We recall some definitions.

If f is a measurable function on $\mathbb{T}$ and Q is an interval on $\mathbb{T}$, the local mean oscillation of f on Q is given by

$$\omega_\lambda(f;Q) = \inf_{c\in\mathbb{R}}((f-c)\chi_Q)^*(\lambda|Q|), \qquad 0<\lambda<1,$$

where $((f-c)\chi_Q)^*$ is the non-increasing rearrangement of $(f-c)\chi_Q$.

Let $m(f,Q)$ be the median value of f over Q, as a (possibly non unique) real number such that

$$\max\left(|\{\zeta\in Q\,;\, f(\zeta) > m(f,Q)\}|, |\{\zeta\in Q\,;\, f(\zeta) < m_f(Q)\}|\right) \le |Q|/2.$$

Next, given an interval Q_0, let us denote $\mathcal{D}(Q_0)$ the dyadic intervals with respect to Q_0. The dyadic local sharp maximal function $m^{\#,d}_{\lambda;Q_0}f$ is defined by

$$m^{\#}_{\lambda;Q_0}f(\zeta) = \sup_{\zeta\in Q'\in\mathcal{D}(Q_0)} \omega_\lambda(f;Q'). \tag{9.46}$$

One key ingredient in the proof of the theorem is the decomposition of A.K. Lerner in terms of the local mean oscillation. In [4], it is proved the following version of Lerner's estimate for homogeneous spaces:

Theorem 9.1. *Let f a measurable function on $\mathbb{T}$, $\mathcal{D}$ a dyadic decomposition of intervals of $\mathbb{T}$. Let $Q_0 \in \mathcal{D}$. Then there exists $\varepsilon > 0$ and a (possibly empty) sparse family $\mathcal{S}(Q_0)$ of intervals in $\mathcal{D}$ included in Q_0 such that for a.e. $\zeta \in Q_0$,*

$$|f(\zeta) - m(f,Q_0)| \le m^{\#}_{\varepsilon,Q_0}(f)(\zeta) + \sum_{Q\in\mathcal{S}(Q_0)} \omega_\varepsilon(f,Q)\chi_Q(\zeta). \tag{9.47}$$

We would like to apply this theorem to the function $f = G_{\mathbf{K}}(\varphi)^2$, and we will need to obtain estimates for $m^{\#}_{\varepsilon,Q_0}(G_{\mathbf{K}}(\varphi)^2)$ and $\omega_\varepsilon(G_{\mathbf{K}}(\varphi)^2, Q)$.

The following lemma follows from well-known techniques of splitting functions in "good" and "bad" parts, which come from a method stated by A.P. Calderon and A. Zygmund (see [16]).

Lemma 9.2. *There exists $C>0$ such that for any $\lambda>0$, $f\in L^1(\mathbb{T})$,*

$$|\{\eta;\, G_{\mathbf{K}}(\varphi)(\eta)>\lambda\}|\lesssim \frac{\|\varphi\|_{L^1(d\sigma)}}{\lambda}.$$

We now can prove the following version of Lerner's estimate:

Lemma 9.3. *Let $0<\lambda<1$. Then, for any cube $Q\in\mathcal{D}_j$,*

$$\omega_\lambda(G_{\mathbf{K}}(\varphi)^2;Q)\lesssim \sum_{k\geq 0}\frac{1}{2^{ks}}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi|\right)^2.$$

Proof. Let $Q\in\mathcal{D}_j$. We decompose $G_{\mathbf{K}}(\varphi)^2(\eta)$ in two terms given by

$$\begin{aligned}&G_{\mathbf{K}}(\varphi)^2(\eta)\\&=\int_{T(2Q)}\chi_{\Gamma_\eta}(z)\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta)\right|^2\frac{dm(z)}{(1-|z|^2)^2}\\&+\int_{\mathbb{D}\setminus T(2Q)}\chi_{\Gamma_\eta}(z)\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta)\right|^2\frac{dm(z)}{(1-|z|^2)^2}=I_1(\varphi)(\eta)+I_2(\varphi)(\eta).\end{aligned}$$

We will then have that if ζ_1 is an arbitrary point in Q,

$$\begin{aligned}&\omega_\lambda(G_{\mathbf{K}}(\varphi)^2;Q)\\&\leq\left((G_{\mathbf{K}}(\varphi)^2-I_2(\varphi)(\zeta_1))\chi_Q\right)^*(\lambda|Q|)\\&\lesssim (I_1(\varphi)\chi_Q)^*((\lambda|Q|/2)+((I_2(\varphi)-I_2(\varphi)(\zeta_1))\chi_Q)^*((\lambda|Q|/2))\\&\lesssim (I_1(\varphi)\chi_Q)^*((\lambda|Q|)/2)+\|I_2(\varphi)-I_2(\varphi)(\zeta_1)\|_{L^\infty(Q)}.\end{aligned}\tag{9.48}$$

We will first show that

$$(I_1(\varphi)\chi_Q)^*(\lambda|Q|/2)\lesssim\sum_{2^kl_Q\leq 1}\frac{1}{2^k}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi|d\sigma\right)^2.\tag{9.49}$$

Since $(x+y)^2\leq 2(x^2+y^2)$, we have that for any $\eta\in Q$,

$$I_1(\varphi)(\eta)\leq 2\left(I_1(\varphi\chi_{4Q})(\eta)+I_1(\varphi\chi_{\mathbb{T}\setminus 4Q})(\eta)\right),$$

and consequently,

$$(I_1(\varphi)\chi_Q)^*(\lambda|Q|/2)\lesssim (I_1(\varphi\chi_{4Q}))^*(\lambda|Q|/4)+\left(I_1(\varphi\chi_{\mathbb{T}\setminus 4Q})\right)^*(\lambda|Q|/4).\tag{9.50}$$

By Lemma 9.2 we have that

$$(I_1(\varphi\chi_{4Q}))^*(\lambda|Q|/4)\leq\left((G_{\mathbf{K}}(\varphi\chi_{4Q}))^2\right)^*(\lambda|Q|/2)\lesssim\left(\frac{1}{|4Q|}\int_{4Q}|\varphi|d\sigma\right)^2.$$

Consider now the term $\left(I_1(\varphi\chi_{\mathbb{T}\setminus 4Q})\right)^*(\lambda|Q|/2)$. It will be enough to obtain, for $z\in T(2Q)$, the following pointwise estimate:

$$|\mathbf{K}(\varphi\chi_{\mathbb{T}\setminus 4Q})(z)|\lesssim\left(\frac{1-|z|^2}{l_Q}\right)^s\sum_{k\geq 1;\,2^kl_Q\lesssim 1}\frac{1}{2^{ks}}\frac{1}{2^kl_Q}\int_{2^kQ}|\varphi|d\sigma. \tag{9.51}$$

Indeed, if (9.51) holds, then we will have that by Chebyshev's inequality

$$\begin{aligned}
&\left(I_1(\varphi\chi_{\mathbb{D}\setminus 4Q})\chi_Q\right)^*(\lambda|Q|/4)\lesssim\frac{\|I_1(\varphi\chi_{\mathbb{T}\setminus 4B(Q)})\chi_Q\|_{L^1}}{(\lambda|Q|)/4}\\
&\lesssim\frac{4}{\lambda|Q|}\int_{\mathbb{T}}\int_{\mathbb{D}}\chi_{\Gamma_\eta}(z)\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)\chi_{\mathbb{D}\setminus 4Q}(\zeta)d\sigma(\zeta)\right|^2(1-|z|^2)^{-2}dV(z)d\sigma(\eta)\\
&\lesssim\left(\sum_{k\geq 2,\,2^kl_Q\leq 1}\frac{1}{2^{ks}}\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi|d\sigma\right)^2\int_0^{2l_Q}\frac{l^{2s}}{l_Q^{2s}}\frac{dl}{l}\\
&\lesssim\left(\sum_{k\geq 2,\,2^kl_Q\leq 1}\frac{1}{2^{ks}}\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi|d\sigma\right)^2.
\end{aligned}$$

Consequently, applying Schwartz's inequality,

$$\left(I_1(f\chi_{\mathbb{T}\setminus 4Q})\chi_Q\right)^*(\lambda|Q|/4)\lesssim\sum_{k\geq 2,\,2^kl_Q\leq 1}\frac{1}{2^{ks}}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\varphi(\eta)|d\sigma(\eta)\right)^2. \tag{9.52}$$

Let us prove (9.51). If $z\in T(2Q)$, we then have that

$$\begin{aligned}
|\mathbf{K}(\varphi\chi_{\mathbb{T}\setminus 4Q})(z)|&=|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)\chi_{\mathbb{T}\setminus 4Q}(\zeta)d\sigma(\zeta)|\\
&\lesssim\left(\frac{(1-|z|^2)}{l_Q}\right)^s\sum_{k\geq 2,\,2^kl_Q\leq 1}\frac{1}{2^{ks}}\frac{1}{2^kl_Q}\int_{2^kQ}|\varphi|
\end{aligned}$$

So, in order to finish the proof of (9.48), we are left to estimate $\|I_2(\varphi)-I_2(\varphi)(\zeta_1)\|_{L^\infty(Q)}$. Let $\omega_1,\omega_2\in Q$. Then,

$$\begin{aligned}
&|I_2(\varphi)(\omega_1)-I_2(\varphi)(\omega_2)|\\
&=\left|\int_{\mathbb{D}\setminus T(2Q)}\chi_{\Gamma_{\omega_1}}(z)\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta)\right|^2\frac{dV(z)}{(1-|z|^2)^2}\right.\\
&\left.-\int_{\mathbb{D}\setminus T(2Q)}\chi_{\Gamma_{\omega_2}}(z)\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta)\right|^2\frac{dV(z)}{(1-|z|^2)^2}\right|\\
&\leq\sum_{k\geq 1,\,2^kl_Q\leq 1}\int_{T(2^{k+1}Q)\setminus T(2^kQ)}\left|\chi_{\Gamma_{\omega_1}}(z)-\chi_{\Gamma_{\omega_2}}(z)\right|\left|\int_{\mathbb{T}}\mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta)\right|^2\frac{dV(z)}{(1-|z|^2)^2}.
\end{aligned} \tag{9.53}$$

We split the points $z \in T(2^{k+1}Q) \setminus T(2^kQ)$ such that $\chi_{\Gamma_{\omega_1}}(z) - \chi_{\Gamma_{\omega_2}}(z) \neq 0$ in two connected sets, Ω_1^k, Ω_2^k. We will obtain estimates for the integrals over one of them, say Ω_1^k, being the estimates over Ω_2^k analogous. Then,

$$|I_2(\varphi)(\omega_1) - I_2(\varphi)(\omega_2)| \lesssim \sum_{k\geq 1,\, 2^k l_Q \leq 1} \int_{\Omega_1^k} \left| \int_{\mathbb{T}} \mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta) \right|^2 \frac{dV(z)}{(1-|z|^2)^2}$$

$$\lesssim \sum_{k\geq 1,\, 2^k l_Q \leq 1} \int_{\Omega_1^k} \left| \int_{\mathbb{T}} \mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta) \right|^2 \frac{dV(z)}{(1-|z|^2)^2}$$

$$= \sum_{k\geq 1,\, 2^k l_Q \leq 1} \int_{\Omega_1^k} \left| \int_{2^kQ} \mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta) \right.$$

$$\left. + \sum_{j>k,\, 2^j l_Q \leq 1} \int_{2^jQ\setminus 2^{j-1}Q} \mathbf{K}(z,\zeta)\varphi(\zeta)d\sigma(\zeta) \right|^2 \frac{dV(z)}{(1-|z|^2)^2}.$$

Next, observe that if $\zeta \in 2^kQ$ and $z \in \Omega_1^k$, we have that $(1-|z|^2) \approx 2^k l_Q$ and $|1-z\bar{\zeta}| \approx 2^k l_Q$. On the other hand, if $\zeta \in 2^jQ \setminus 2^{j-1}Q$, $j > k$, we have that $|1-z\bar{\zeta}| \gtrsim 2^j l_Q$. Altogether gives, integrating in polar coordinates on Ω_1^k and using the fact that the angle width is of order l_Q, whereas the line integral on r is of order $2^k l_Q$, that the above is bounded, up to constant, by

$$\frac{(2^k l_Q)^{2s}}{(2^k l_Q)^{2(1+s)}} \frac{l_Q}{(2^k l_Q)} \left(\int_{2^k l_Q} |\varphi| d\sigma \right)^2 + \frac{l_Q}{(2^k l_Q)} \left(\sum_{j>k,\, 2^j l_Q \leq 1} \frac{(2^k l_Q)^s \int_{2^jQ} |\varphi| d\sigma}{(2^j l_Q)^{1+s}} \right)^2.$$

Hence, adding up in k, we will have that (9.53) is bounded, up to a constant, by

$$\sum_{k\geq 1,\, 2^k l_Q \leq 1} \frac{1}{2^k} \left(\frac{1}{2^k l_Q} \int_{2^k l_Q} |\varphi| d\sigma \right)^2 + \sum_{k\geq 1,\, 2^k l_Q \leq 1} \frac{1}{2^{k(1-2s)}} \left(\sum_{j>k,\, 2^j l_Q \leq 1} \frac{1}{2^{js}} \frac{1}{2^j l_Q} \int_{2^jQ} |\varphi| \right)^2$$

By Hölder's inequality, the above is bounded by

$$\sum_{k\geq 1,\, 2^k l_Q \leq 1} \frac{1}{2^k} \left(\frac{1}{2^k l_Q} \int_{2^k l_Q} |\varphi)| \right)^2$$

$$+ \sum_{k\geq 1,\, 2^k l_Q \leq 1} \frac{1}{2^{k(1-2s)}} \left(\sum_{j>k,\, 2^j l_Q \leq 1} \frac{1}{2^{js}} \right) \left(\sum_{j>k,\, 2^j l_Q \leq 1} \frac{1}{2^{js}} \left(\frac{1}{2^j l_Q} \int_{2^jQ} |\varphi| \right)^2 \right)$$

$$\lesssim \sum_{k\geq 1,\, 2^k l_Q \leq 1} \frac{1}{2^{ks}} \left(\frac{1}{2^k l_Q} \int_{2^k l_Q} |\varphi)| \right)^2.$$

We now sketch how to finish the proof of Theorem 5.1. First, Lemma 9.3 gives that a.e. $\zeta \in Q$, $m^{\#}_{\lambda,Q}G(\psi)^2(\zeta) \lesssim M(\psi)(\zeta)^2$, where $M(\psi)$ denotes the Hardy-Littlewood maximal function. Next, we have that for any $Q \in \mathcal{D}^i$, there exists a sparse family $\mathcal{S}(Q) = \left(Q_j^k\right)$, $Q_j^k \in \mathcal{D}^i$ so that if we denote by

$$\mathcal{T}_l^{\mathcal{S}}(\psi)(\zeta) = \left(\sum_{Q_j^k \in \mathcal{S}(Q)} (\psi_{2^l B(Q_j^k)})^2 \chi_{Q_j^k}(\zeta) \right)^{1/2},$$

then by Theorem 9.1, we have that if

$$\mathcal{T}^{\mathcal{S}}(\psi)(\zeta) = \sum_{l \geq 0} \frac{1}{2^{l/4}} \mathcal{T}_l^{\mathcal{S}}(\psi)(\zeta),$$

then for a.e $\zeta \in Q$,

$$|G(\psi)(\zeta)^2 - m_Q(G(\psi)^2)| \lesssim \left(M(\psi)(\zeta)^2 + \sum_{l \geq 0} \frac{1}{2^{l/2}} \left(\mathcal{T}_l^{\mathcal{S}}(\psi)\right)^2 \right). \tag{9.54}$$

Hence

$$|G(\psi)(\zeta)^2 - m_Q(G(\psi)^2)|^{1/2} \lesssim M(\psi)(\zeta) + \mathcal{T}^{\mathcal{S}}(\psi)(\zeta), \tag{9.55}$$

where M is the Hardy-Littlewood maximal function.

It is proved in [16] that for any $\omega \in A_3$,

$$\|\mathcal{T}^{\mathcal{S}}(\psi)\|_{L^3(\omega)} \lesssim \|\psi\|_{L^3(\omega)}.$$

Observe that here we are not interested in obtaining sharpest estimates and in consequence, we could have chosen other index $p_0 > 2$ instead of $p_0 = 3$.

On the other hand, the Hardy-Littlewood maximal function maps $L^3(\omega)$ to $L^3(\omega)$, so, $\|M(\psi)\|_{L^3(\omega)} \lesssim \|\psi\|_{L^3(\omega)}$. Altogether gives that

$$\| \left(G(\psi)^2 - m_Q(G(\psi)^2)\right)^{1/2} \|_{L^3(\omega)} \lesssim \|\psi\|_{L^3(\omega)}.$$

Hence

$$\begin{aligned}
\|G(\psi)\|_{L^3(\omega)} &= \|G(\psi)^2\|_{L^{3/2}(\omega)}^{1/2} \\
&\lesssim \|G(\psi)^2 - m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2} + \|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2} \\
&\lesssim \|\psi\|_{L^3(\omega)} + \|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2}.
\end{aligned}$$

Finally, it is proved in [11] that

$$\|m_Q(G(\psi)^2)\|_{L^{3/2}(\omega)}^{1/2} \lesssim \|\psi\|_{L^3(\omega)}.$$

Rubio de Francia's extrapolation theorem gives then that $\|G(\psi)\|_{L^p(\omega)} \lesssim \|\psi\|_{L^p(\omega)}$.

□

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C. Cascante: Dept. Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

E-mail address: cascante@ub.edu

Joaquín M. Ortega: Dept. Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

E-mail address: ortega@ub.edu