BOUNDEDNESS OF THE BERGMAN PROJECTION ON GENERALIZED FOCK-SOBOLEV SPACES ON $\mathbb{C}^n$

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Abstract. In this paper we solve a problem posed by H. Bommier-Hato, M. Engliš and E.H. Youssfi in [4] on the boundedness of the Bergman-type projections in generalized Fock spaces. It will be a consequence of two facts: a full description of the embeddings between generalized Fock-Sobolev spaces and a complete characterization of the boundedness of the above Bergman type projections between weighted L^p-spaces related to generalized Fock-Sobolev spaces.

1. Introduction

Let $dV = dV_n$ be the Lebesgue measure on $\mathbb{C}^n$ normalized so that the measure of the unit ball $\mathbb{B}^n$ is 1. If $n = 1$ we write $dA = dV_1$. Let $d\sigma$ be the Lebesgue measure on the unit sphere $\mathbb{S}^n$ normalized so that $\sigma(\mathbb{S}^n) = 1$. We denote by $H = H(\mathbb{C}^n)$ the space of entire functions on $\mathbb{C}^n$.

Let $\ell > 0$. For $1 \leq p < \infty$, $\alpha > 0$ and $\rho \in \mathbb{R}$, the space $L^{p,\ell}_{\alpha,\rho} = L^p_{\alpha,\rho}$ consists of all measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|^p_{L^p_{\alpha,\rho}} := \int_{\mathbb{C}^n} \left|f(z)(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}\right|^p dV(z) < \infty,$$

that is, $L^p_{\alpha,\rho} = L^p(\mathbb{C}^n; (1+|z|)^{\rho p} e^{-\frac{\alpha p}{2}|z|^{2\ell}} dV(z))$.

Moreover, $L^{\infty,\ell}_{\alpha,\rho} = L^\infty_{\alpha,\rho}$ consists of all measurable functions f on $\mathbb{C}^n$ such that

$$\|f\|_{L^\infty_{\alpha,\rho}} = \operatorname*{ess\,sup}_{z\in\mathbb{C}^n} |f(z)|(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}} < \infty.$$

We define the generalized Fock-Sobolev spaces as $F^p_{\alpha,\rho} := H \cap L^p_{\alpha,\rho}$.

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When $\rho = 0$, we obtain the generalized Fock spaces $F^p_\alpha = F^p_{\alpha,0}$. According to this notation we write $L^p_\alpha = L^p_{\alpha,0}$.

The space L^2_α is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha := \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-\alpha|z|^{2\ell}}dV(z).$$

and F^2_α is a closed linear subspace of L^2_α. Denote by P_α the orthogonal projection from L^2_α to F^2_α, which is usually called the Bergman projection.

In [9, Theorem 9.1] the authors showed that P_α is bounded from L^p_β to F^p_γ if and only if $\beta < 2\alpha$ and $\beta = \gamma$. In [4] the authors studied the boundedness of P_α between the spaces $\mathcal{L}^p_b := L^p(\mathbb{C}^n; e^{-b|z|^{2\ell}}dV(z))$ and $\mathcal{L}^q_d := L^q(\mathbb{C}^n; e^{-d|z|^{2\ell}}dV(z))$. Observe that $\mathcal{L}^p_a = L^p_{2a/p}$. Since $\mathcal{L}^2_a = L^2_a$ the orthogonal projection $\mathcal{P}_a$ from $\mathcal{L}^2_a$ onto $\mathcal{F}^2_a := H\cap\mathcal{L}^2_a$ coincides with P_a. One advantage of considering the spaces L^p_α is that permits us to include the case $p = \infty$. Their results are given in terms of a parameter c defined by $c := \frac{4d}{a^2q}(a - \frac{b}{p})$. Rewriting the parameters as $a = \alpha$, $b = \beta p/2$ and $d = \gamma q/2$, we have that, in our notations, $c = \gamma\frac{2\alpha-\beta}{\alpha^2}$. Then, the main results in [4] on the boundedness of P_α from L^p_β to F^q_γ are:

(i) If P_α is bounded, then $c \geq 1$.
(ii) If $c > 1$ then P_α is bounded.
(iii) If $c = 1$ and $\ell \leq 1$ then P_α is bounded if and only if $q \geq p$.

For $c = 1$ and $\ell > 1$ the authors only obtain partial results. In particular they prove that if $c = 1$ and $\frac{2n}{2n-1} < \ell < 2$ then P_α is bounded if and only if $q = p$.

The initial motivation of this work was to close the remaining open cases which will be achieved by proving:

(iv) If $c = 1$ and $\ell > 1$ then P_α is bounded if and only if $q = p$.

This result shows that, of the four possible mutually exclusive assertions in [4, Proposition 17], (a) is the valid option.

Note that if $c \geq 1$, then $a - \frac{b}{p} > 0$, which in our notation is equivalent to $\beta < 2\alpha$. The latter condition is necessary in order that the "pointwise evaluation" of the Bergman projection is bounded on L^p_β (see Lemma 2.12 below).

Our main result is the following theorem for generalized Fock-Sobolev spaces.

Theorem 1.1. *Let $\ell \geq 1$, $\alpha, \beta, \gamma > 0$ and $\rho, \eta \in \mathbb{R}$. For $1 \leq p, q \leq \infty$, P_α maps boundedly $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$ if and only if one of the following conditions holds:*

(i) $0 < \alpha^2/(2\alpha - \beta) < \gamma$.

(ii) $\alpha^2/(2\alpha-\beta)=\gamma$, $p\le q$ *and* $\rho-\eta\ge 2n(\ell-1)\left(\frac{1}{p}-\frac{1}{q}\right)$.

(iii) $\alpha^2/(2\alpha-\beta)=\gamma$, $q<p$ *and* $\rho-\eta> 2n\left(\frac{1}{q}-\frac{1}{p}\right)$.

In particular for $\rho=\eta$ we obtain the following generalization of (iv).

Corollary 1.2. *Let $\ell>1$, $\alpha,\beta,\gamma>0$ and $\rho\in\mathbb{R}$. For $1\le p,q\le\infty$, P_α maps boundedly $L^p_{\beta,\rho}$ to $L^q_{\gamma,\rho}$ if and only if either $0<\alpha^2/(2\alpha-\beta)<\gamma$ or $\alpha^2/(2\alpha-\beta)=\gamma$ and $p=q$.*

Our approach to obtain Theorem 1.1 differs from the one in [4]. Instead of proving directly the characterizations, we deduce the results as a consequence of two ingredients: the first is the identity (see Proposition 4.2 below)

$$P_\alpha(L^p_{\beta,\rho})=F^p_{\frac{\alpha^2}{2\alpha-\beta},\rho}\qquad (1\le p\le\infty,\ell\ge1,\beta<2\alpha,\rho>0) \tag{1.1}$$

and the second one is the following embedding result:

Theorem 1.3. *Let $\ell\ge1$, $\beta,\gamma>0$ and $\rho,\eta\in\mathbb{R}$. For $1\le p,q\le\infty$, the embedding $F^p_{\beta,\rho}\hookrightarrow F^q_{\gamma,\eta}$ holds if and only if one of the following three conditions is satisfied:*

(i) $\beta<\gamma$.

(ii) $\beta=\gamma$, $q\ge p$ *and* $2n(\ell-1)\left(\frac{1}{p}-\frac{1}{q}\right)\le\rho-\eta$.

(iii) $\beta=\gamma$, $q<p$ *and* $2n\left(\frac{1}{q}-\frac{1}{p}\right)<\rho-\eta$.

Note that as an immediate consequence of Theorem 1.3 we obtain:

Corollary 1.4.

(i) *If $\ell\ge1$ and the embedding $F^p_{\beta,\rho}\hookrightarrow F^q_{\beta,\eta}$ holds, then $\rho\ge\eta$.*

(ii) *For $\ell=1$, the embedding $F^p_{\beta,\rho}\hookrightarrow F^q_{\beta,\rho}$ holds if and only if $p\le q$.*

(iii) *For $\ell>1$, the embedding $F^p_{\beta,\rho}\hookrightarrow F^q_{\beta,\rho}$ holds if and only if $p=q$.*

The proof of Theorem 1.3 requires of some results which can be of interest by themselves. For instance, assertions (i) and (ii) follow from precise pointwise and $L^p_{\beta,\rho}$-norm estimates of the Bergman kernel. As a consequence, we derive pointwise estimates of the functions in $F^p_{\beta,\rho}$ and some properties on the boundedness of the Bergman projection. The most difficult part is the proof of assertion (iii). In this case, for $1\le q<p<\infty$, we use a technique due to D. Luecking (see [11]), based on Kinchine's inequality, which permits the construction of adequate test functions. Then the case $1\le q<p=\infty$ follows by extrapolation.

The paper is organized as follows: In Section 2 we obtain pointwise and $L^p_{\alpha,\rho}$-norm estimates of the Bergman kernel, from which the boundedness of the Bergman projection P_α on $L^p_{\alpha,\rho}$ is deduced. In Sections 3 and 4 we prove Theorems 1.3 and 1.1 respectively.

Notations: In the next sections we only consider spaces $F^{p,\ell}_{\alpha,\rho} = F^p_{\alpha,\rho}$, with $\ell \geq 1$, $\alpha > 0$ and $\rho \in \mathbb{R}$. So we omit the conditions on ℓ, α and ρ in the statement of the results. We denote by p' the conjugate exponent of $p \in [1,\infty]$.

Let $\mathbb{N}$ be the set of non-negative integer numbers. For a multi-index $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$ and $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, we write, as usual, $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$, $\nu! = \nu_1! \cdots \nu_n!$ and $|\nu| = \nu_1 + \cdots + \nu_n$.

For $z, w \in \mathbb{C}^n$, $z\overline{w} = \sum_{j=1}^n z_j \overline{w}_j$. If $z \in \mathbb{C}^n$ and $r > 0$ then $B(z,r)$ is the open ball in $\mathbb{C}^n$ with center z and radius r. When $n = 1$, $B(z,r)$ is denoted, as usual, by $D(z,r)$.

If $E \subset \mathbb{C}^n$ then $\mathcal{X}_E$ is the characteristic function of E.

If X, Y are normed spaces, the notation $X \hookrightarrow Y$ means that the mapping $f \in X \mapsto f \in Y$ is bounded.

For $\lambda \in \mathbb{C} \setminus \{0\}$, we denote by $\arg\lambda$ the principal branch of the argument of λ, that is, $-\pi < \arg\lambda \leq \pi$. Moreover, $\lambda^\beta = |\lambda|^\beta e^{i\beta\arg\lambda}$, for $\beta \in \mathbb{R}$.

The letter C will denote a positive constant, which may vary from place to place. The notation $\Phi \lesssim \Psi$ means that there exists a constant $C > 0$, which does not depend on the involved variables, such that $\Phi \leq C\,\Psi$. We write $\Phi \simeq \Psi$ when $\Phi \lesssim \Psi$ and $\Psi \lesssim \Phi$.

2. The Bergman projection on $L^p_{\alpha,\rho}$

2.1. On the two-parametric Mittag-Leffler functions $E_{a,b}$.

The two-parametric Mittag-Leffler functions are the entire functions

$$E_{a,b}(\lambda) := \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ak+b)} \qquad (\lambda \in \mathbb{C},\ a, b > 0).$$

A good general reference for the Mittag-Leffler functions is the book [8].

In this section we recall the asymptotic expansions of the two-parametric Mittag-Leffler functions and their derivatives. Those expansions will be useful to obtain pointwise and norm estimates of the Bergman kernel.

Theorem 2.1 ([13, Theorem 1.2.1])**.** *Let $a \in (0,1)$ and $b > 0$. Then, for $|\lambda| \to \infty$, we have*

$$E_{a,b}(\lambda) = \begin{cases} \frac{1}{a}\lambda^{(1-b)/a} e^{\lambda^{1/a}} + O(\lambda^{-1}), & \text{if} \quad |\arg \lambda| \le a\pi, \\ O(\lambda^{-1}), & \text{if} \quad |\arg\lambda| \ge a\frac{2\pi}{3}. \end{cases} \tag{2.2}$$

By Cauchy formula (see [12, Theorem 1.4.2]), the asymptotic expansions of the m-th derivatives of $E_{a,b}$ (on "smaller" sectors than the ones involved in (2.2)) can be obtained by differentiating m times the terms in (2.2), that is,

$$E^{(m)}_{a,b}(\lambda) = \begin{cases} \frac{1}{a}\frac{d^m}{d\lambda^m}\left(\lambda^{(1-b)/a}e^{\lambda^{1/a}}\right) + O(\lambda^{-1-m}), & \text{if} \quad |\arg\lambda| \le a\frac{3\pi}{4}, \\ O(\lambda^{-1-m}), & \text{if} \quad |\arg\lambda| \ge a\frac{3\pi}{4}. \end{cases} \tag{2.3}$$

2.2. **The Bergman kernel.**

The next result, which is obtained in [4], gives a description of the Bergman kernel. The main tool to compute the norm of the monomials in F^2_α is the identity

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = 2\ell\gamma^x \int_0^\infty s^{2\ell x-1}e^{-\gamma s^{2\ell}}ds \quad (x>0,\ \gamma>0).$$

Lemma 2.2. *The system $\left\{\frac{w^\nu}{\|w^\nu\|_{F^2_\alpha}}\right\}_{\nu\in\mathbb{N}^n}$ is an orthonormal basis for F^2_α, so the Bergman projection from L^2_α onto F^2_α is*

$$P_\alpha f(z) = \langle f, K_{\alpha,z}\rangle_\alpha = \int_{\mathbb{C}^n} f(w)K_\alpha(\zeta,w)e^{-\alpha|w|^{2\ell}}dV(w),$$

where

$$K_\alpha(z,w) = \overline{K_{\alpha,z}(w)} = \sum_{\nu\in\mathbb{N}^n} \frac{z^\nu\overline{w}^\nu}{\|w^\nu\|^2_{F^2_\alpha}}$$

is the Bergman kernel. Namely, since $\|w^\nu\|^2_{F^2_\alpha} = \frac{\alpha^{-\frac{|\nu|+n}{\ell}}}{\ell}\frac{n!\,\nu!\,\Gamma\left(\frac{|\nu|+n}{\ell}\right)}{(n-1+|\nu|)!}$, $K_\alpha(z,w) = H_\alpha(z\overline{w})$, where

$$H_\alpha(\lambda) := \frac{\ell\alpha^{n/\ell}}{n!}\sum_{k=0}^\infty \frac{(n-1+k)!}{k!}\frac{\alpha^{k/\ell}\lambda^k}{\Gamma\left(\frac{k+n}{\ell}\right)} = \frac{\ell\alpha^{n/\ell}}{n!}E^{(n-1)}_{1/\ell,1/\ell}(\alpha^{1/\ell}\lambda).$$

In particular, for any $\delta > 0$ we have

$$K_\alpha(z,\delta w) = \delta^{-n}K_{\alpha\delta^\ell}(z,w). \tag{2.4}$$

Remark 2.3. *In order to obtain norm estimates of the Bergman kernel it is useful to make the following change of variables. Given $z\in\mathbb{C}^n$, there is a unitary mapping $U : \mathbb{C}^n \to \mathbb{C}^n$ such that $U(z) = (|z|, 0, \dots, 0)$. Then*

$K_\alpha(w,z) = H_\alpha(|z|u_1)$, *where* $U(w) = (u_1, \cdots, u_n)$, *so we may assume* $z = (|z|, 0, \cdots, 0)$.

The remaining part of this section is devoted to derive pointwise and norm estimates of the Bergman kernel, which will be the key tools to obtain our main results.

The following corollaries are consequences of (2.3).

Corollary 2.4. *Let n be a positive integer. For $|\lambda| \to \infty$, we have that*

$$E^{(n-1)}_{1/\ell,1/\ell}(\lambda) = \begin{cases} \ell^n \lambda^{n(\ell-1)} e^{\lambda^\ell}(1 + O(\lambda^{-\ell})) + O(\lambda^{-n}), & \text{if} \quad |\arg\lambda| \le \frac{3\pi}{4\ell}, \\ O(\lambda^{-n}), & \text{if} \quad |\arg\lambda| \ge \frac{3\pi}{4\ell}. \end{cases}$$

Proof. For $\ell = 1$, $E_{1/\ell,1/\ell}(\lambda) = e^\lambda$ so $E^{(n-1)}_{1/\ell,1/\ell}(\lambda) = e^\lambda$, and the above asymptotic identity is obvious in this case.

Next assume $\ell > 1$. By induction on n it is easy to check that

$$\ell \, \frac{d^{n-1}}{d\lambda^{n-1}}\big(\lambda^{\ell-1}e^{\lambda^\ell}\big) = \ell^n \lambda^{n(\ell-1)} e^{\lambda^\ell}(1 + O(\lambda^{-\ell})) \quad (|\lambda| \to \infty,\ |\arg\lambda| < \pi/\ell).$$

By combining this identity with (2.3) we obtain the result. □

Corollary 2.5. *For any $\delta > 0$ and $N > 2$, let $S_N^\delta := D(0,\delta) \cup S_N$, where*

$$S_N := \{\, \lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \le \tfrac{\pi}{N\ell} \,\}.$$

Then there exist $\delta > 0$ and $N > 2$ such that

$$|H_\alpha(\lambda)| \simeq (1 + |\lambda|)^{n(\ell-1)} \left|e^{\alpha\lambda^\ell}\right| \chi_{S_N}(\lambda) + \chi_{D(0,\delta)}(\lambda) \qquad (\lambda \in S_N^\delta), \tag{2.5}$$

$$|H_\alpha(\lambda)| \lesssim (1 + |\lambda|)^{n(\ell-1)}\, e^{\alpha\cos(\frac{\pi}{N})|\lambda|^\ell} \qquad (\lambda \in \mathbb{C} \setminus S_N^\delta). \tag{2.6}$$

In particular,

$$\mathcal{X}_{S_N}(\lambda) \lesssim |H_\alpha(\lambda)| \lesssim (1 + |\lambda|)^{n(\ell-1)}\, e^{\alpha|\lambda|^\ell} \qquad (\lambda \in \mathbb{C}). \tag{2.7}$$

Proof. Corollary 2.4 shows that there is a large $R > 0$ so that

$$|H_\alpha(\lambda)| \simeq (1 + |\lambda|)^{n(\ell-1)} \left|e^{\alpha\lambda^\ell}\right| \qquad (|\lambda| \ge R,\ |\arg\lambda| \le \tfrac{\pi}{3\ell}), \tag{2.8}$$

$$|H_\alpha(\lambda)| \lesssim (1 + |\lambda|)^{n(\ell-1)} e^{\frac{\alpha}{2}|\lambda|^\ell} \qquad (|\lambda| \ge R,\ |\arg\lambda| \ge \tfrac{\pi}{3\ell}). \tag{2.9}$$

Since H_α is a continuous positive function on the interval $[0,\infty)$, we have that there exist a small $\delta > 0$ and a large $N > 2$ such that

$$|H_\alpha(\lambda)| \simeq 1 \simeq (1 + |\lambda|)^{n(\ell-1)} \left|e^{\alpha\lambda^\ell}\right| \chi_{S_N}(\lambda) + \chi_{D(0,\delta)}(\lambda) \qquad (\lambda \in S_N^\delta,\ |\lambda| \le R). \tag{2.10}$$

Therefore (2.5) directly follows from (2.8) and (2.10). Moreover, (2.6) is deduced from (2.8), (2.9) and the fact that H_α is bounded on $D(0,R)$. □

As an immediate consequence of the above results we obtain the following pointwise estimate for the Bergman kernel.

Proposition 2.6. *There exist $\delta > 0$ and $N > 2$ such that*

$$|K_\alpha(w,z)| \simeq (1+|z\overline{w}|)^{n(\ell-1)}\, e^{\alpha\,\mathrm{Re}((z\overline{w})^\ell)} \quad (z\overline{w} \in S_N^\delta),$$
$$|K_\alpha(w,z)| \lesssim (1+|z\overline{w}|)^{n(\ell-1)}\, e^{\alpha\cos(\frac{\pi}{N})|z\overline{w}|^\ell} \quad (z\overline{w} \in \mathbb{C}\setminus S_N^\delta).$$

Now we state norm estimates for the Bergman kernel.

Proposition 2.7. *Let $1 \le p \le \infty$. Then*

$$\|K_\alpha(\cdot,z)\|_{F^p_{\alpha,\rho}} \simeq (1+|z|)^{\rho+2n(\ell-1)/p'} e^{\frac{\alpha}{2}|z|^{2\ell}} \quad (z \in \mathbb{C}^n).$$

This estimate for $1 \le p < \infty$ and $\rho = 0$ is stated without a detailed proof in [4, Section 8.1]. Since this norm estimate of the Bergman kernel is essential in order to obtain our main theorems and it is deduced from several non-trivial technical results, we include its proof. The main tool is the pointwise estimate of H_α given in Corollary 2.5, but we also need the following three technical lemmas.

Lemma 2.8. *Let $\alpha > 0$ and let $\beta \in \mathbb{R}$. Then*

$$\sup_{x\ge 0}(1+x)^\beta e^{-\alpha(x-a)^2} \simeq (1+a)^\beta \qquad (a \ge 0).$$

Proof. It is clear that the supremum is greater or equal than $(1+a)^\beta$. The converse estimate for a large enough, say $a > R$, follows by checking that $\alpha(x-a)^2 - \beta\log(1+x)$ attains its minimum value at $x = a + O(1/a)$. Finally, for $a \in [0,R]$ the result is also immediate. □

Lemma 2.9. *Let $a > 0$ and let $b \in \mathbb{R}$. Then*

$$\int_{\mathbb{C}^{n-1}} (1+y+|w|)^b e^{-a(y^2+|w|^2)^\ell} dV_{n-1}(w) \simeq (1+y)^{b-2(n-1)(\ell-1)} e^{-a\,y^{2\ell}} \quad (y \ge 0).$$

Proof. It is clear that the estimate of the statement holds for $0 \le y \le 1$. Thus, by integration in polar coordinates, we only have to prove that

$$I(y) := \int_0^\infty (y+r)^b e^{-a(y^2+r^2)^\ell}\, r^{2n-3} dr \simeq y^{b-2(n-1)(\ell-1)} e^{-a\,y^{2\ell}} \quad (y \ge 1).$$

The change of variables $r = yt$ shows that $I(y) \simeq y^{b+2(n-1)}\, e^{-ay^{2\ell}} J(y)$, where

$$J(y) := \int_0^\infty (1+t)^b e^{-ay^{2\ell}((1+t^2)^\ell - 1)}\, t^{2n-3} dt.$$

We obtain the lower estimate for $I(y)$ by considering the root $t_y > 0$ of the equation $y^{2\ell}((1+t^2)^\ell - 1) = 1$, that is,

$$t_y = \left((1+y^{-2\ell})^{1/\ell} - 1\right)^{1/2} \simeq y^{-\ell},$$

and observing that

$$J(y) \geq \int_0^{t_y} (1+t)^b e^{-ay^{2\ell}((1+t^2)^\ell - 1)}\, t^{2n-3} dt \simeq \int_0^{t_y} t^{2n-3} dt \simeq y^{-2(n-1)\ell}.$$

In order to get the upper estimate, note that if $\ell \geq 1$ then $(1+t^2)^\ell - 1 \geq \ell t^2$, and so

$$J(y) \leq \int_0^\infty (1+t)^b e^{-a\ell y^{2\ell} t^2}\, t^{2n-3} dt \leq 2^{\max(b,0)}(J_1(y) + J_2(y)),$$

where

$$J_1(y) := \int_0^1 e^{-a\ell y^{2\ell} t^2}\, t^{2n-3} dt \quad \text{and} \quad J_2(y) := \int_1^\infty e^{-a\ell y^{2\ell} t^2}\, t^{2n-3+b} dt.$$

By making the change of variables $s = y^\ell t$, we have that

$$J_1(y) = y^{-2(n-1)\ell} \int_0^{y^\ell} e^{-a\ell s^2} s^{2n-3}\, ds \lesssim y^{-2(n-1)\ell} \qquad \text{and}$$

$$J_2(y) = y^{-(2n-2+b)\ell} \int_{y^\ell}^\infty e^{-a\ell s^2} s^{2n-3+b} ds \lesssim y^{-(2n-2+b)\ell} \int_{y^\ell}^\infty e^{-a\ell s} ds \lesssim y^{-2(n-1)\ell},$$

which ends the proof. □

Lemma 2.10. *Let $a > 0$ and let $b \in \mathbb{R}$. Then*

$$I(z) = I_{a,b}(z) := \int_{\mathbb{C}} \frac{e^{-a|v-z|^2}}{(1+|v|)^b}\, dA(v) \simeq \frac{1}{(1+|z|)^b} \qquad (z \in \mathbb{C})$$

and

$$J(z) = J_{a,b}(z) := \int_{\mathbb{C}} \frac{e^{-a(|v|-|z|)^2}}{(1+|v|)^b}\, dA(v) \simeq \frac{1}{(1+|z|)^{b-1}} \qquad (z \in \mathbb{C}).$$

Proof. Since $I_{a,b}(z) \simeq I_{1,b}(za^{1/2})$ and $J_{a,b}(z) \simeq I_{1,b}(za^{1/2})$, we may assume that $a = 1$. Moreover, $I(z) \simeq 1 \simeq J(z)$, when $|z| \leq 1$, so we only have to prove the estimates for $|z| \geq 1$. In this case we split each of the above integrals into the corresponding three integrals on the sets $S_1 = \{v \in \mathbb{C} : |v| < |z|/2\}$, $S_2 = \{v \in \mathbb{C} : |z|/2 \leq |v| \leq 2|z|\}$ and $S_3 = \{v \in \mathbb{C} : |v| > 2|z|\}$, that is, $I(z) = I_1(z) + I_2(z) + I_3(z)$ and $J(z) = J_1(z) + J_2(z) + J_3(z)$, where

$$I_k(z) := \int_{S_k} \frac{e^{-|v-z|^2}}{(1+|v|)^b}\, dA(v) \quad \text{and} \quad J_k(z) := \int_{S_k} \frac{e^{-(|v|-|z|)^2}}{(1+|v|)^b}\, dA(v).$$

If $v \in S_1$ then $|v-z| \geq |z|-|v| > |z|/2$. Thus

$$I_1(z) \leq J_1(z) \lesssim e^{-|z|^2/4} \int_0^{|z|/2} \frac{r\,dr}{(1+r)^b} \lesssim e^{-|z|^2/4}(1+|z|)^{|b|+2} \lesssim \frac{1}{(1+|z|)^b}.$$

If $v \in S_2$ then $(1+|z|)/2 \leq 1+|v| \leq 2(1+|z|)$. Therefore

$$I_2(z) \simeq \frac{1}{(1+|z|)^b} \int_{S_2} e^{-|v-z|^2} dA(v) \text{ and } J_2(z) \simeq \frac{1}{(1+|z|)^b} \int_{S_2} e^{-(|v|-|z|)^2} dA(v).$$

Since $D(z,1/2) \subset S_2$, we have

$$0 < \int_{D(0,1/2)} e^{-|w|^2} dA(w) \leq \int_{S_2} e^{-|v-z|^2} dA(v) \leq \int_{\mathbb{C}} e^{-|w|^2} dA(w) < \infty,$$

and so $I_2(z) \simeq (1+|z|)^{-b}$. On the other hand, $J_2(z) \simeq (1+|z|)^{1-b}$ because

$$\int_{S_2} e^{-(|v|-|z|)^2} dA(v) \simeq \int_{|z|/2}^{2|z|} e^{-(r-|z|)^2} r\,dr \simeq |z| \int_{-|z|/2}^{|z|} e^{-t^2} dt \simeq |z|.$$

If $v \in S_3$ then $|v-z| \geq |v|-|z| > |v|/2$, and hence

$$I_3(z) \leq J_3(z) \lesssim \int_{2|z|}^{\infty} \frac{re^{-r^2/4}}{(1+r)^b}\,dr \leq e^{-|z|^2/2} \int_0^{\infty} \frac{re^{-r^2/8}}{(1+r)^b}\,dr \lesssim \frac{1}{(1+|z|)^b}. \quad \square$$

Proof of Proposition 2.7. Let $p=\infty$. Then the lower estimate follows from (2.5):

$$\begin{aligned} \|K_\alpha(\cdot,z)\|_{F^\infty_{\alpha,\rho}} &\geq K_\alpha(z,z)\,(1+|z|)^\rho\, e^{-\frac{\alpha}{2}|z|^{2\ell}} = H_\alpha(|z|^2)\,(1+|z|)^\rho\, e^{-\frac{\alpha}{2}|z|^{2\ell}} \\ &\gtrsim (1+|z|^2)^{n(\ell-1)}\,(1+|z|)^\rho\, e^{\frac{\alpha}{2}|z|^{2\ell}} \simeq (1+|z|)^{\rho+2n(\ell-1)}\, e^{\frac{\alpha}{2}|z|^{2\ell}}. \end{aligned}$$

In order to obtain the upper estimate, first note that (2.7) and the Cauchy-Schwarz inequality (that is, $|z\overline{w}| \leq |z||w|$, for any $z,w \in \mathbb{C}^n$) show that

$$\begin{aligned} |K_\alpha(w,z)| = |H_\alpha(z\overline{w})| &\lesssim (1+|z\overline{w}|)^{n(\ell-1)}\, e^{\alpha|z\overline{w}|^\ell} \\ &\lesssim (1+|z|)^{n(\ell-1)}(1+|w|)^{n(\ell-1)}\, e^{\alpha|z|^\ell|w|^\ell}. \end{aligned}$$

Therefore $\|K_\alpha(\cdot,z)\|_{F^\infty_{\alpha,\rho}} \lesssim (1+|z|)^{n(\ell-1)}\, e^{\frac{\alpha}{2}|z|^{2\ell}} M(|z|)$, where

$$M(|z|) = \sup_{w\in\mathbb{C}} (1+|w|)^{\rho+n(\ell-1)} e^{-\frac{\alpha}{2}(|w|^\ell-|z|^\ell)^2} \simeq \sup_{x\geq 0} (1+x)^{\frac{\rho+n(\ell-1)}{\ell}} e^{-\frac{\alpha}{2}(x-|z|^\ell)^2}.$$

Since, by Lemma 2.8, $M(|z|) \simeq (1+|z|^\ell)^{\frac{\rho+n(\ell-1)}{\ell}} \simeq (1+|z|)^{\rho+n(\ell-1)}$, we have just proved the upper estimate in this case.

Now assume that $p<\infty$. By making the change of variables $u=Uw$, where $U:\mathbb{C}^n \to \mathbb{C}^n$ is a unitary mapping such that $U(z)=(|z|,0\,\ldots,0)$, we get that

$$\|K_\alpha(\cdot,z)\|^p_{F^p_{\alpha,\rho}} \simeq \int_{\mathbb{C}} |H_\alpha(|z|u_1)|^p\, \Psi(u_1)\, dA(u_1),$$

where

$$\Psi(u_1) := \int_{\mathbb{C}^{n-1}} (1+|u_1|+|u'|)^{\rho p}\, e^{-\frac{\alpha p}{2}(|u_1|^2+|u'|^2)^\ell}\, dV_{n-1}(u').$$

Then Lemma 2.9 implies that

$$\begin{aligned}\|K_\alpha(\cdot,z)\|^p_{F^p_{\alpha,\rho}} \\ \simeq \int_{\mathbb{C}} |H_\alpha(|z|u_1)|^p\,(1+|u_1|)^{\rho p-2(n-1)(\ell-1)}\, e^{-\frac{\alpha p}{2}|u_1|^{2\ell}}\, dA(u_1).\end{aligned} \tag{2.11}$$

Now pick $N>2$ satisfying the statement of Corollary 2.5. Then note that (2.7) implies

$$\mathcal{X}_{S_N}(u_1) \lesssim |H_\alpha(|z|u_1)|^p \lesssim (1+|u_1|)^{np(\ell-1)} e^{\alpha p 2^\ell |u_1|^\ell} \quad (|z|\le 2,\ u_1\in\mathbb{C}).$$

Thus (2.11) shows that

$$\|K_\alpha(\cdot,z)\|^p_{F^p_{\alpha,\rho}} \simeq 1 \simeq (1+|z|)^{\rho+2n(\ell-1)/p'} e^{\frac{\alpha}{2}|z|^{2\ell}} \quad (|z|\le 2),$$

so we only have to prove the norm estimate for $|z|>2$. In order to do that, we split the integral in (2.11) as the sum of the three integrals $\mathcal{I}_1(|z|)$, $\mathcal{I}_2(|z|)$ and $\mathcal{I}_3(|z|)$ on the sets $E_1=\{u_1\in\mathbb{C}: |u_1|>1, |\arg u_1|\le \pi/(N\ell)\}$, $E_2=\{u_1\in\mathbb{C}: |u_1|>1, |\arg u_1|>\pi/(N\ell)\}$ and $E_3=\{u_1\in\mathbb{C}: |u_1|\le 1\}$, respectively.

To estimate $\mathcal{I}_1(|z|)$ recall that (2.5) gives

$$|H_\alpha(|z|u_1)|^p \simeq (|z||u_1|)^{np(\ell-1)} e^{\alpha p|z|^\ell \operatorname{Re} u_1^\ell} \quad (u_1\in E_1,\ |z|>2),$$

so

$$\mathcal{I}_1(|z|) \simeq |z|^{np(\ell-1)} e^{\frac{\alpha p}{2}|z|^{2\ell}} \int_{E_1} |u_1|^{np(\ell-1)+\rho p-2(n-1)(\ell-1)} e^{-\frac{\alpha p}{2}|u_1^\ell-|z|^\ell|^2} dA(u_1).$$

By making the change of variables $v=u_1^\ell$ we have that

$$\mathcal{I}_1(|z|) \simeq |z|^{np(\ell-1)} e^{\frac{\alpha p}{2}|z|^{2\ell}} \int_{\{|v|\ge 1, |\arg v|\le \pi/N\}} |v|^\beta e^{-\frac{\alpha p}{2}|v-|z|^\ell|^2} dA(v),$$

where $\beta := (n(\ell-1)(p-2)+\rho p)/\ell$. Since for $|z|>2$ we have the inclusions

$$\begin{aligned} D(|z|^\ell, \sin(\pi/N)) &\subset \{v\in\mathbb{C}: |v|>1\} \cap D(|z|^\ell, |z|^\ell \sin(\pi/N)) \\ &\subset \{v\in\mathbb{C}: |v|>1, |\arg v|\le \pi/N\},\end{aligned}$$

the preceding integral $\mathcal{I}_1'(|z|)$ satisfies

$$\mathcal{I}_1'(|z|) \ge \int_{D(|z|^\ell,\sin(\pi/N))} |v|^\beta e^{-\frac{\alpha p}{2}|v-|z|^\ell|^2} dA(v) \simeq |z|^{\beta\ell}.$$

Moreover, Lemma 2.10 shows that $\mathcal{I}_1'(|z|) \lesssim I_{\alpha p/2,-\beta}(|z|^\ell) \simeq |z|^{\beta\ell}$. It follows that $\mathcal{I}_1'(|z|) \simeq |z|^{\beta\ell} = |z|^{n(\ell-1)(p-2)+\rho p}$, and hence

$$\mathcal{I}_1(|z|) \simeq |z|^{np(\ell-1)} e^{\frac{\alpha p}{2}|z|^{2\ell}} \mathcal{I}_1'(|z|) \simeq (1+|z|)^{2n(\ell-1)(p-1)+\rho p}\, e^{\frac{\alpha p}{2}|z|^{2\ell}}. \tag{2.12}$$

Now we estimate $\mathcal{I}_2(|z|)$. By (2.6),

$$|H_\alpha(|z|u_1)|^p \lesssim (|z||u_1|)^{np(\ell-1)} e^{\alpha p \cos(\frac{\pi}{N})\,|z|^\ell |u_1|^\ell} \quad (u_1 \in E_2,\ |z|>2),$$

so $\mathcal{I}_2(|z|) \lesssim |z|^{np(\ell-1)} e^{\frac{\alpha p}{2}\cos^2(\frac{\pi}{N})\,|z|^{2\ell}} \mathcal{I}_2'(|z|)$, where

$$\begin{aligned}\mathcal{I}_2'(|z|) &:= \int_{E_2} |u_1|^{np(\ell-1)+\rho p-2(n-1)(\ell-1)} e^{-\frac{\alpha p}{2}\{|u_1|^\ell - |z|^\ell \cos(\frac{\pi}{N})\}^2} dA(u_1) \\ &\simeq \int_1^\infty r^{1+np(\ell-1)+\rho p-2(n-1)(\ell-1)} e^{-\frac{\alpha p}{2}\{r^\ell - |z|^\ell \cos(\frac{\pi}{N})\}^2} dr.\end{aligned}$$

Then we make the change of variables $t = r^\ell$ to get that

$$\mathcal{I}_2'(|z|) \simeq \int_1^\infty t^{\beta+1} e^{-\frac{\alpha p}{2}\{t-|z|^\ell \cos(\frac{\pi}{N})\}^2} dt,$$

so Lemma 2.10 shows that $\mathcal{I}_2'(|z|) \lesssim J_{\frac{\alpha p}{2},-\beta}(|z|^\ell \cos(\frac{\pi}{N})) \simeq |z|^{\beta\ell+\ell}$. Hence

$$\mathcal{I}_2(|z|) \lesssim |z|^{np(\ell-1)+\beta\ell+\ell} e^{\frac{\alpha p}{2}\cos^2(\frac{\pi}{N})\,|z|^{2\ell}} \lesssim (1+|z|)^{2n(\ell-1)(p-1)+\rho p}\, e^{\frac{\alpha p}{2}|z|^{2\ell}}. \tag{2.13}$$

Finally, since by (2.7) we have that

$$|H_\alpha(|z|u_1)|^p \lesssim (1+|z|)^{np(\ell-1)} e^{\alpha p|z|^\ell} \quad (u_1 \in E_3, |z|>2),$$

we obtain that

$$\mathcal{I}_3(|z|) \lesssim (1+|z|)^{n(\ell-1)} e^{\alpha|z|^\ell} \lesssim (1+|z|)^{2n(\ell-1)(p-1)+\rho p} e^{\frac{\alpha p}{2}|z|^{2\ell}}. \tag{2.14}$$

Taking into account (2.12), (2.13) and (2.14), we conclude that

$$\|K_\alpha(\cdot,z)\|^p_{F^p_{\alpha,\rho}} \simeq (1+|z|)^{2n(\ell-1)(p-1)+\rho p} e^{\frac{\alpha p}{2}|z|^{2\ell}} \quad (|z|>2),$$

which ends the proof. □

Corollary 2.11. *Let $1 \le p \le \infty$. Then*

$$\|K_\alpha(\cdot,z)\|_{F^p_{\beta,\rho}} \simeq (1+|z|)^{\rho+2n(\ell-1)/p'} e^{\frac{\alpha^2}{2\beta}|z|^{2\ell}} \quad (z \in \mathbb{C}^n).$$

Proof. Since $K_\alpha(\delta z, w) = \delta^{-n} K_{\delta^\ell \alpha}(z,w)$, for $\delta = (\beta/\alpha)^{1/\ell}$, we have

$$\begin{aligned}\|K_\alpha(\cdot,z)\|_{F^p_{\beta,\rho}} &\simeq \|K_\beta(\cdot, z/\delta)\|_{F^p_{\beta,\rho}} \\ &\simeq (1+|z|/\delta)^{\rho+2n(\ell-1)/p'} e^{\frac{\beta}{2}|z/\delta|^{2\ell}} \\ &\simeq (1+|z|)^{\rho+2n(\ell-1)/p'} e^{\frac{\alpha^2}{2\beta}|z|^{2\ell}}.\end{aligned}$$

This ends the proof. □

2.3. **The Bergman projection.**

The next lemma shows that the Bergman projection P_α is pointwise well-defined on $L^p_{\beta,\rho}$ if and only if $\beta < 2\alpha$.

Lemma 2.12. *Let $\zeta \in \mathbb{C}^n$ and assume $1 \leq p \leq \infty$.*

(i) *If for $\zeta \neq 0$ the linear functional $U_\zeta : L^2_\alpha \to \mathbb{C}$, defined by $U_\zeta(f) = P_\alpha(f)(\zeta)$, is bounded on the normed space $(L^2_\alpha \cap L^p_{\beta,\rho}, \|\cdot\|_{L^p_{\beta,\rho}})$ then $\beta < 2\alpha$.*

(ii) *Conversely, if $\beta < 2\alpha$ then $U_\zeta : L^p_{\beta,\rho} \to \mathbb{C}$, defined by*

$$U_\zeta(f) = \int_{\mathbb{C}^n} f(w) K_\alpha(\zeta, w) e^{-\alpha|w|^{2\ell}} dV(w),$$

is bounded and

$$\|U_\zeta\| \lesssim (1+|\zeta|)^{-\rho+2n(\ell-1)/p} e^{\frac{1}{2}\frac{\alpha^2}{2\alpha-\beta}|\zeta|^{2\ell}}.$$

Proof. Assume that U_ζ is bounded on $(L^2_\alpha \cap L^p_{\beta,\rho}, \|\cdot\|_{L^p_{\beta,\rho}})$. Then, by Hahn-Banach theorem's, U_ζ extends to a bounded operator on $L^p_{\beta,\rho}$, which we also denote by U_ζ.

Let ν be a multi-index. It is clear that the function

$$f(z) := \frac{z^\nu}{(1+|z|)^{|\nu|+\rho+2n+1}} e^{\frac{\beta}{2}|z|^{2\ell}} \tag{2.15}$$

belongs to $L^p_{\beta,\rho}$. Let $\mathcal{X}_R$ be the characteristic function of the open ball B_R centered at 0 with radius R. Then the function $f_R = \mathcal{X}_R \cdot f$ is in $L^2_\alpha \cap L^p_{\beta,\rho}$ and $\|f_R - f\|_{F^p_{\beta,\rho}} \to 0$ as $R \to \infty$. Since

$$K_{\alpha,z}(w) = \sum_{\mu\in\mathbb{N}^n} \frac{w^\mu \overline{z}^\mu}{\|w^\mu\|^2_{F^2_\alpha}},$$

where the series converges in L^2_α,

$$P_\alpha(f_R)(z) = \langle f_R, K_{\alpha,z}\rangle_\alpha = \sum_{\mu\in\mathbb{N}^n} \frac{z^\mu}{\|w^\mu\|^2_{F^2_\alpha}} \langle f_R, w^\mu\rangle_\alpha.$$

By integration in polar coordinates we have $\langle f_R, w^\mu\rangle^\ell_\alpha = \delta_{\mu,\nu} c_\nu(R)$, where

$$c_\nu(R) := \int_{B_R} \frac{|w^\nu|^2}{(1+|w|)^{|\nu|+\rho+2n+1}} e^{(\frac{\beta}{2}-\alpha)|w|^{2\ell}} dV(w).$$

Thus $U_\zeta(f_R) = P_\alpha(f_R)(\zeta) = c_\nu(R)\,\zeta^\nu/\|w^\nu\|^2_{F^2_\alpha}$. So, by the hypothesis and the monotone convergence theorem,

$$U_\zeta(f) = \lim_{R\to\infty} U_\zeta(f_R) = \frac{\zeta^\nu}{\|w^\nu\|^2_{F^2_\alpha}} \int_{\mathbb{C}^n} \frac{|w^\nu|^2}{(1+|w|)^{|\nu|+\rho+2n+1}} e^{(\frac{\beta}{2}-\alpha)|w|^{2\ell}} dV(w).$$

It follows that for any ν such that $\zeta^\nu \neq 0$ we have that the above integral is finite. Choosing ν such that $|\nu| \geq 1+\rho$ we obtain that $\beta < 2\alpha$.

Next assume $\beta < 2\alpha$. Let $F_\zeta(w) := G(w)H_\zeta(w)$, where

$$G(w) := |f(w)|(1+|w|)^\rho e^{-\frac{\beta}{2}|w|^{2\ell}} \quad \text{and}$$
$$H_\zeta(w) := |K_\alpha(\zeta,w)|(1+|w|)^{-\rho} e^{-(\alpha-\frac{\beta}{2})|w|^{2\ell}}.$$

Since $\|G\|_{L^p} = \|f\|_{L^p_{\beta,\rho}}$, we obtain

$$|U_\zeta(f)| \leq \|F_\zeta\|_{L^1} \leq \|K_\alpha(\cdot,\zeta)\|_{L^{p'}_{2\alpha-\beta,-\rho}} \|f\|_{L^p_{\beta,\rho}}.$$

Hence Corollary 2.11 ends the proof. □

Remark 2.13. *From the pointwise estimate of* $|K_\alpha(z,w)|$ *with* $z\overline{w} \in S_N^\delta$*, given in Proposition 2.6, it is easy to check that if* $\beta \geq 2\alpha$ *and* f *is the function defined in* (2.15) *with* $\nu = 0$*, then* $F_\zeta \notin L^1$*. So* $U_\zeta(f)$ *is not well defined.*

Corollary 2.14. *Let* $1 \leq p < \infty$*. Then*

$$F^p_{\alpha,\rho} \hookrightarrow F^\infty_{\alpha,\rho-2n(\ell-1)/p},$$

that is,

$$|f(z)| \lesssim \|f\|_{F^p_{\alpha,\rho}} (1+|z|)^{-\rho+(2n(\ell-1))/p} e^{\alpha|z|^{2\ell}/2} \quad (f \in F^p_{\alpha,\rho},\ z \in \mathbb{C}^n).$$

Lemma 2.15. *Let* $1 \leq p \leq \infty$ *and let* $\beta < 2\alpha$*. Then* $P_\alpha L^p_{\beta,\rho} \subset H(\mathbb{C}^n)$*, and* $P_\alpha f = f$*, for every* $f \in F^p_{\beta,\rho}$*.*

Proof. By Proposition 2.6, if $R > 0$ there is $C > 0$ such that $|K_\alpha(z,w)| \leq C^{|w|^\ell}$, for every $|z| < R$ and $w \in \mathbb{C}^n$. Then it follows that $P_\alpha L^p_{\beta,\rho} \subset H(\mathbb{C}^n)$.

Let $f \in F^p_{\beta,\rho}$. In order to prove that $P_\alpha f = f$ it is enough to check that $P_\alpha f$ and f share the same Taylor coefficients, that is,

$$\int_{\mathbb{S}^n} P_\alpha f(\zeta)\overline{\zeta}^\gamma d\sigma(\zeta) = \int_{\mathbb{S}^n} f(\zeta)\overline{\zeta}^\gamma d\sigma(\zeta) \qquad (\gamma \in \mathbb{N}^n).$$

Indeed, if $f(z)=\sum_\gamma f_\gamma z^\gamma$ is the Taylor expansion of f, then, by Fubini's theorem and Lemma 2.2,

$$\begin{aligned}\int_{\mathbb{S}^n} P_\alpha f(\zeta)\overline{\zeta}^\gamma d\sigma(\zeta) &= \int_{\mathbb{S}^n}\int_{\mathbb{C}^n} f(w)K_\alpha(\zeta,w)e^{-\alpha|w|^{2\ell}}dV(w)\overline{\zeta}^\gamma d\sigma(\zeta)\\ &= \int_{\mathbb{C}^n} \frac{f(w)\overline{w}^\gamma\|\zeta^\gamma\|^2_{L^2(\mathbb{S}^n)}}{\|w^\gamma\|^2_{F^2_\alpha}} e^{-\alpha|w|^{2\ell}}dV(w)\\ &= f_\gamma \frac{\|\zeta^\gamma\|^2_{L^2(\mathbb{S}^n)}}{\|w^\gamma\|^2_{F^2_\alpha}}\int_{\mathbb{C}^n}|w^\gamma|^2 e^{-\alpha|w|^{2\ell}}dV(w)\\ &= \int_{\mathbb{S}^n} f(\zeta)\overline{\zeta}^\gamma d\sigma(\zeta). \qquad \square\end{aligned}$$

Proposition 2.16. *For $1\le p\le\infty$ the Bergman operator P_α is a bounded projection from $L^p_{\alpha,\rho}$ onto $F^p_{\alpha,\rho}$.*

Proof. By Lemma 2.15 we only have to prove that P_α is bounded on $L^p_{\alpha,\rho}$.

First we consider the case $1<p<\infty$. By Proposition 2.7, the function

$$\Omega_\alpha(z,w) := e^{-\frac{\alpha}{2}|z|^{2\ell}}|K_\alpha(z,w)|e^{-\frac{\alpha}{2}|w|^{2\ell}}$$

satisfies

$$\int_{\mathbb{C}^n}\Omega_\alpha(z,w)(1+|w|)^c dV(w) \simeq e^{-\frac{\alpha}{2}|z|^{2\ell}}\|K_\alpha(\cdot,z)\|_{L^1_{\alpha,c}} \simeq (1+|z|)^c. \tag{2.16}$$

If $\varphi\in L^p_{\alpha,\rho}$, then Hölder's inequality and (2.16) with $c=0$ give

$$\begin{aligned} e^{-\frac{p\alpha}{2}|z|^{2\ell}}|P_\alpha(\varphi)(z)|^p &\le \left(\int_{\mathbb{C}^n}|\varphi(w)|e^{-\frac{\alpha}{2}|w|^{2\ell}}\Omega_\alpha(z,w)dV(w)\right)^p\\ &\lesssim \int_{\mathbb{C}^n}|\varphi(w)|^p e^{-\frac{p\alpha}{2}|w|^{2\ell}}\Omega_\alpha(z,w)dV(w).\end{aligned} \tag{2.17}$$

So Fubini's theorem and (2.16) with $c=\rho p$ imply $\|P_\alpha(\varphi)\|_{L^p_{\alpha,\rho}}\lesssim\|\varphi\|_{L^p_{\alpha,\rho}}$.

If $p=1$ then (2.17) is obvious and, as in the above case, we obtain the result.

If $p=\infty$ then

$$(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}|P_\alpha(\varphi)(z)| \lesssim \|\varphi\|_{L^\infty_{\alpha,\rho}}(1+|z|)^\rho\int_{\mathbb{C}^n}\frac{\Omega_\alpha(z,w)}{(1+|w|)^\rho}dV(w).$$

So (2.16) shows that $\|P_\alpha(\varphi)\|_{L^\infty_{\alpha,\rho}}\lesssim\|\varphi\|_{L^\infty_{\alpha,\rho}}$. $\square$

Corollary 2.17. *Let $1\le p<\infty$. Then the dual of $F^p_{\alpha,\rho}$, with respect to the pairing $\langle\cdot,\cdot\rangle_\alpha$, is $F^{p'}_{\alpha,-\rho}$.*

Proof. From the classical L^p-duality it is easy to check that the dual of $L^p_{\alpha,\rho}$, with respect to the pairing $\langle\cdot,\cdot\rangle^\ell_\alpha$, is $L^{p'}_{\alpha,-\rho}$. This result together with Proposition 2.16 prove the corollary. □

3. Proof of Theorem 1.3

The case $\ell = 1$ and $\rho = \eta = 0$ is well known (see [9]). For $n = 1$, the theorem can be deduced from the characterization of Carleson measures obtained in [6, Theorem 1].

3.1. Necessary conditions for all p and q.

Lemma 3.1. *If $F^p_{\beta,\rho} \hookrightarrow F^q_{\gamma,\eta}$, then either $\beta < \gamma$ or $\beta = \gamma$ and*

$$2n(\ell-1)\left(\tfrac{1}{p}-\tfrac{1}{q}\right) \le \rho - \eta.$$

Proof. By Corollary 2.11 the ratio

$$\frac{\|K_\alpha(\cdot,z)\|_{F^q_{\gamma,\eta}}}{\|K_\alpha(\cdot,z)\|_{F^p_{\beta,\rho}}} \simeq \frac{(1+|z|)^{\eta+2n(\ell-1)/q'}e^{\frac{\alpha^2}{2\gamma}|z|^{2\ell}}}{(1+|z|)^{\rho+2n(\ell-1)/p'}e^{\frac{\alpha^2}{2\beta}|z|^{2\ell}}}$$

is bounded if and only if β, γ, ρ and η satisfy the above conditions. □

3.2. Proof of Theorem 1.3 for $1 \le p \le q \le \infty$.

The next lemma shows that the necessary conditions obtained in the above section are also sufficient, which proves Theorem 1.3 for $1 \le p \le q \le \infty$.

Lemma 3.2. *If either $\beta < \gamma$ or $\beta = \gamma$ and*

$$2n(\ell-1)\left(\tfrac{1}{p}-\tfrac{1}{q}\right) \le \rho - \eta,$$

then $F^p_{\beta,\rho} \hookrightarrow F^q_{\gamma,\eta}$, provided that $1 \le p \le q \le \infty$.

Proof. If $p = q$ then $\eta \le \rho$. Hence $(1+|z|)^\eta e^{-\frac{\gamma}{2}|z|^{2\ell}} \lesssim (1+|z|)^\rho e^{-\frac{\beta}{2}|z|^{2\ell}}$ which proves the embedding $F^p_{\beta,\rho} \hookrightarrow F^p_{\gamma,\eta}$.

The case $p < q = \infty$ is a consequence of Corollary 2.14 and the case $p = q$. Indeed, $F^p_{\beta,\rho} \hookrightarrow F^\infty_{\beta,\rho-2n(\ell-1)/p} \hookrightarrow F^\infty_{\gamma,\eta}$.

Assume $1 \le p < q < \infty$ and let $f \in F^p_{\beta,\rho}$. Consider F the function defined by

$$F(z) := |f(z)|(1+|z|)^\eta e^{-\frac{\gamma}{2}|z|^{2\ell}} = G(z)^{p/q}H(z)^{(q-p)/q},$$

where

$$G(z) := |f(z)|(1+|z|)^\rho e^{-\frac{\beta}{2}|z|^{2\ell}}$$

and

$$H(z) := |f(z)|(1+|z|)^{\frac{\eta q-\rho p}{q-p}} e^{-\frac{\gamma q-\beta p}{2(q-p)}|z|^{2\ell}}.$$

By Corollary 2.14 and the hypotheses on ρ and η, we have

$$\begin{aligned}|H(z)| &\lesssim \|f\|_{F^p_{\beta,\rho}}(1+|z|)^{\frac{\eta q-\rho p}{q-p}-\rho+\frac{2n(\ell-1)}{p}}e^{\left(-\frac{\gamma q-\beta p}{2(q-p)}+\frac{\beta}{2}\right)|z|^{2\ell}}\\ &= \|f\|_{F^p_{\beta,\rho}}(1+|z|)^{(\eta-\rho)\frac{q}{q-p}+\frac{2n(\ell-1)}{p}}e^{-\frac{(\gamma-\beta)q}{2(q-p)}|z|^{2\ell}}\\ &\lesssim \|f\|_{F^p_{\beta,\rho}}.\end{aligned}$$

Hence

$$\|f\|^q_{F^q_{\gamma,\eta}} = \|F\|^q_{L^q} \lesssim \|f\|^{q-p}_{F^p_{\beta,\rho}}\|G\|^p_{L^p} = \|f\|^q_{F^p_{\beta,\rho}}. \qquad \square$$

Observe that, for $1\le p\le q\le\infty$, by Lemmas 3.1 and 3.2, the fact that the embedding $F^p_{\beta,\rho}\hookrightarrow F^q_{\gamma,\eta}$ holds is only a question of growth, that is, $F^p_{\beta,\rho}\hookrightarrow F^q_{\gamma,\eta}$ if and only if $F^\infty_{\beta,\rho-2n(\ell-1)/p}\hookrightarrow F^\infty_{\gamma,\eta-2n(\ell-1)/q}$.

3.3. **Sufficient conditions for $1\le q<p\le\infty$.**

Lemma 3.3. *If either $\beta<\gamma$ or $\beta=\gamma$ and $2n\left(\frac{1}{q}-\frac{1}{p}\right)<\rho-\eta$, then we have $F^p_{\beta,\rho}\hookrightarrow F^q_{\gamma,\eta}$, provided that $1\le q<p\le\infty$.*

Proof. Let $f\in F^p_{\beta,\rho}$. Assume first $p=\infty$. In this case $q(\rho-\eta)>2n$, so the hypotheses on the parameters give

$$\begin{aligned}\|f\|^q_{F^q_{\gamma,\eta}} &= \int_{\mathbb{C}^n}|f(z)|^q(1+|z|)^{\eta q}e^{-\frac{\gamma q}{2}|z|^{2\ell}}dV(z)\\ &\lesssim \|f\|^q_{F^\infty_{\beta,\rho}}\int_{\mathbb{C}^n}(1+|z|)^{-(\rho-\eta)q}e^{-\frac{(\gamma-\beta)q}{2}|z|^{2\ell}}dV(z)\lesssim\|f\|^q_{F^\infty_{\beta,\rho}}.\end{aligned}$$

Next assume p finite. In this case $(\rho-\eta)\frac{pq}{p-q}>2n$. Consider the function

$$F(z):=|f(z)|(1+|z|)^\eta e^{-\frac{\gamma}{2}|z|^{2\ell}}=G(z)H(z),$$

where

$$G(z):=|f(z)|(1+|z|)^\rho e^{-\frac{\beta}{2}|z|^{2\ell}} \quad\text{and}\quad H(z):=(1+|z|)^{(\eta-\rho)}e^{-\frac{\gamma-\beta}{2}|z|^{2\ell}}.$$

By Hölder's inequality with exponent $p/q>1$ we have

$$\begin{aligned}\|f\|_{F^q_{\gamma,\eta}} &= \|F\|_{L^q}\le\|G\|_{L^p}\|H\|_{L^{pq/(p-q)}}\\ &= \|f\|_{F^p_{\beta,\rho}}\left(\int_{\mathbb{C}^n}(1+|z|)^{-(\rho-\eta)\frac{pq}{p-q}}e^{-\frac{\gamma-\beta}{2}\frac{pq}{p-q}|z|^{2\ell}}dV(z)\right)^{\frac{p-q}{pq}}.\end{aligned}$$

Therefore $\|f\|_{F^q_{\gamma,\eta}}\lesssim\|f\|_{F^p_{\beta,\rho}}$. $\square$

3.4. **Necessary conditions for $1 \le q < p < \infty$ and $\beta = \gamma$.**

Proposition 3.4. *If $1 \le q < p < \infty$ and $F^p_{\beta,\rho} \hookrightarrow F^q_{\beta,\eta}$ then $2n\left(\frac{1}{q} - \frac{1}{p}\right) < \rho - \eta$.*

The proof of Proposition 3.4 follows from the ideas in [11]. We need some technical results.

For $r > 0$, let $\tau_r : \mathbb{C} \to (0,\infty)$ be the function defined by

$$\tau_r(z) := r(1+|z|)^{1-\ell} \tag{3.18}$$

and let $B_r(z) := B(z, \tau_r(z))$.

Note that τ_r is a radius function in the sense of [7, p.1617-1618], that is,

$$1+|z| \simeq 1+|w| \quad (z \in \mathbb{C}^n,\ w \in B_r(z)). \tag{3.19}$$

Then we have:

Lemma 3.5 ([7, Proposition 7])**.** *For any $r > 0$ there exists a sequence $\{z_k\}$ in $\mathbb{C}^n$ such that the Euclidean balls $B_k := B_r(z_k)$ satisfy:*

(i) $\cup_k B_k = \mathbb{C}^n$.

(ii) *The overlapping of the balls B_k is finite, that is, there exists $N_r \in \mathbb{N}$ such that $\sum_k \mathcal{X}_{B_k}(z) \le N_r$ for any $z \in \mathbb{C}^n$.*

The following lemma states a subharmonic type estimate.

Lemma 3.6.

(i) *There exists $r > 0$ such that*

$$|K_\alpha(z,w)| e^{-\frac{\alpha}{2}|w|^{2\ell}} e^{-\frac{\alpha}{2}|z|^{2\ell}} \simeq (1+|z|)^{2n(\ell-1)} \quad (w \in B_r(z)).$$

(ii) *Let $1 \le p < \infty$, $\rho \in \mathbb{R}$ and $r > 0$. There exists $C = C_{\alpha,p,\rho,r} > 0$ such that*

$$|f(z)|^p (1+|z|)^{\rho p - 2n(\ell-1)} e^{-\frac{\alpha p}{2}|z|^{2\ell}} \le C \int_{B_r(z)} |f(w)|^p (1+|w|)^{\rho p} e^{-\frac{\alpha p}{2}|w|^{2\ell}}\, dV(w),$$

for any $f \in H(\mathbb{C}^n)$ and any $z \in \mathbb{C}^n$.

Proof. We begin proving (i). By Remark 2.3, we may assume that $z = (|z|, 0, \cdots, 0)$. Then we have to prove that

$$|H_\alpha(|z|w_1)| e^{-\frac{\alpha}{2}|w|^{2\ell}} e^{-\frac{\alpha}{2}|z|^{2\ell}} \simeq (1+|z|)^{2n(\ell-1)} \quad (w \in B_r(z)). \tag{3.20}$$

By Corollary 2.5, there exist $\delta > 0$ and $N > 2$ satisfying (2.5). For $r > 0$ small enough we have $|z|w_1 \in S^\delta_N$, for any $z \in \mathbb{C}^n$ and $w \in B_r(z)$. By (2.5),

$$|H_\alpha(|z|w_1)| \simeq (1+|z||w_1|)^{n(\ell-1)} e^{\alpha|z|^\ell \operatorname{Re} w_1^\ell} \quad (w \in B_r(z)). \tag{3.21}$$

In particular for $|z| \leq 2r$ the terms in (3.20) are comparable to a positive constant and there is nothing to prove.

Now assume $|z| > 2r$. In this case, $|w_1| \simeq |z|$ for $w \in B_r(z)$. Hence, by (3.21), the equivalence (3.20) will be a consequence of

$$e^{\alpha|z|^\ell \operatorname{Re} w_1^\ell} e^{-\frac{\alpha}{2}|w|^{2\ell}} e^{-\frac{\alpha}{2}|z|^{2\ell}} \simeq 1 \quad (w \in B_r(z)). \tag{3.22}$$

First note that

$$\begin{aligned} e^{\alpha|z|^\ell \operatorname{Re} w_1^\ell} e^{-\frac{\alpha}{2}|w|^{2\ell}} e^{-\frac{\alpha}{2}|z|^{2\ell}} &= e^{\alpha|z|^\ell \operatorname{Re} w_1^\ell} e^{-\frac{\alpha}{2}(|w_1|^2+|w'|^2)^\ell} e^{-\frac{\alpha}{2}|z|^{2\ell}} \\ &= e^{-\frac{\alpha}{2}||z|^\ell - w_1^\ell|^2} e^{-\frac{\alpha}{2}[(|w_1|^2+|w'|^2)^\ell - |w_1|^{2\ell}]}. \end{aligned}$$

By mean value theorem, for $w \in B_r(z)$ we have

$$0 \leq ||z|^\ell - w_1^\ell| \lesssim (|z| + r(1+|z|)^{1-\ell})^{\ell-1}(1+|z|)^{1-\ell} \simeq 1$$

and

$$(|w_1|^2 + |w'|^2)^\ell - |w_1|^{2\ell} \lesssim (|w_1|^2 + |w'|^2)^{\ell-1}|w'|^2 \lesssim |z|^{2(\ell-1)}(1+|z|)^{2(1-\ell)} \simeq 1,$$

we obtain (3.22).

In order to prove part (ii), note that, by (3.19), the case $\rho \neq 0$ follows from the result for $\rho = 0$. This last case can be deduced using the arguments in the proofs of Proposition 12 and of Lemma 13 in [7].

Let φ be a real $\mathcal{C}^2$-function on the closed unit ball $\overline{B(0,1)}$ of $\mathbb{C}^n$. It is well known (see for instance [1]) that there exists a real $\mathcal{C}^2$-function ψ on $B(0,1)$ such that

$$\partial\overline{\partial}\psi = \partial\overline{\partial}\varphi \quad \text{and} \quad \|\psi\|_{L^\infty(B(0,1))} \leq C\|\partial\overline{\partial}\varphi\|_{L^\infty(B(0,1))}.$$

By rescaling, we get that if φ is a real $\mathcal{C}^2$-function on the closed ball $\overline{B(z,R)}$, then there is a real $\mathcal{C}^2$-function ψ on $B(z,R)$ such that

$$\partial\overline{\partial}\psi = \partial\overline{\partial}\varphi \quad \text{and} \quad \|\psi\|_{L^\infty(B(z,R))} \leq CR^2\|\partial\overline{\partial}\varphi\|_{L^\infty(B(z,R))}.$$

Applying this result to the function $\varphi(w) = |w|^{2\ell}$ and to the ball $B_r(z)$ there exists a real $\mathcal{C}^2$-function ψ_z on $B_r(z)$ such that $\partial\overline{\partial}\psi_z = \partial\overline{\partial}\varphi$ and, by (3.19),

$$\|\psi_z\|_{L^\infty(B_r(z))} \leq Cr^2(1+|z|)^{2(1-\ell)} \sup_{w\in B_r(z)} |w|^{2(\ell-1)} \leq C' r^2.$$

Since $\psi_z - \varphi$ is a pluriharmonic function on $B_r(z)$, it is the real part of a holomorphic function h_z on $B_r(z)$. Thus we have

$$\begin{aligned}|f(z)|^p e^{-\frac{\alpha p}{2}|z|^{2\ell}} \simeq |f(z)e^{\frac{\alpha}{2}h_z(z)}|^p &\le \frac{1}{|B_r(z)|}\int_{B_r(z)} |f(w)e^{\frac{\alpha}{2}h_z(w)}|^p\,dV(w)\\ &\simeq (1+|z|)^{2n(\ell-1)}\int_{B_r(z)} |f(w)|^p e^{-\frac{\alpha p}{2}|z|^{2\ell}} dV(w).\end{aligned}$$

□

Lemma 3.7. *Let $\{z_k\}$ be a sequence satisfying the properties in Lemma 3.5. Then, for $1 \le p < \infty$ the map*

$$\{c_k\} \longmapsto \Phi(\{c_k\})(z) := \sum_k c_k \frac{K_\beta(z,z_k)}{\|K_\beta(z,z_k)\|_{F^p_{\beta,\rho}}}$$

is bounded from the sequence space ℓ^p to $F^p_{\beta,\rho}$.

Proof. For $p = 1$ the result is clear. Assume $p > 1$. By Corollary 2.17, the dual of the space $F^{p'}_{\beta,-\rho}$ with respect to the pairing $\langle\cdot,\cdot\rangle_\beta$ is $F^p_{\beta,\rho}$. Since the overlapping of the balls B_k is finite, Proposition 2.7 and Lemma 3.6(ii) show that the map

$$g \longmapsto T_{p'}(g) := \big\{g(z_k)/\|K_\beta(z,z_k)\|_{F^p_{\beta,\rho}}\big\}$$

is bounded from $F^{p'}_{\beta,-\rho}$ to $\ell^{p'}$. Indeed,

$$\begin{aligned}\|T_{p'}(g)\|^{p'}_{\ell^{p'}} &\simeq \sum_k |g(z_k)|^{p'}(1+|z_k|)^{-\rho p'-2n(\ell-1)}e^{-\frac{\beta}{2}|z_k|^{2\ell}}\\ &\lesssim \sum_k \int_{B_r(z_k)} |g(z)|^{p'}(1+|z|)^{-\rho p'}e^{-\frac{\beta}{2}|z|^{2\ell}}dV(z) \simeq \|g\|^{p'}_{F^{p'}_{\beta,-\rho}}.\end{aligned}$$

So the adjoint map $T^*_{p'}$ of $T_{p'}$, with respect to the pairing $\langle\cdot,\cdot\rangle_\beta$, is bounded from ℓ^p to $F^p_{\beta,\rho}$. We are going to show that $T^*_{p'} = \Phi$. For $\{c_k\} \in c_{oo}$ (the space of sequences with a finite number of non-zero terms) and $g \in F^{p'}_{\beta,-\rho}$ we have

$$\begin{aligned}\langle T^*_{p'}\{c_k\}, g\rangle_\beta &= \langle \{c_k\}, g(z_k)/\|K_\beta(z,z_k)\|_{F^p_{\beta,\rho}}\rangle_{\ell^2}\\ &= \Big\langle \sum_k c_k K_\beta(z,z_k)/\|K_\beta(z,z_k)\|_{F^p_{\beta,\rho}}, g\Big\rangle_\beta,\end{aligned}$$

since $g(z_k) = \int_{\mathbb{C}^n} g(z)K_\beta(z_k,z)e^{-\frac{\beta}{2}|z|^{2\ell}}dV(z)$. Therefore

$$T^*_{p'}\{c_k\} = \sum_k c_k \frac{K_\beta(z,z_k)}{\|K_\beta(z,z_k)\|_{F^p_{\beta,\rho}}} \quad (\{c_k\} \in c_{oo}).$$

Since c_{oo} is dense in ℓ^p we conclude that $T^*_{p'} = \Phi$. □

Proof of Proposition 3.4. Pick $r > 0$ satisfying Lemma 3.6 (i), and let $\{z_k\}$ be a sequence as in Lemma 3.5. Let $\{c_k\} \in \ell^p$ and consider the function

$$\Phi_t(\{c_k\})(z) := \sum_k c_k r_k(t) \frac{K_\beta(z, z_k)}{\|K_\beta(z, z_k)\|_{F^p_{\beta,\rho}}}, \quad 0 \le t \le 1,$$

where $\{r_k(t)\}$ is a sequence of Rademacher functions (see [11, p.336]). By the hypothesis and Lemma 3.7,

$$\|\Phi_t(\{c_k\})\|_{F^q_{\beta,\eta}} \lesssim \|\Phi_t(\{c_k\})\|_{F^p_{\beta,\rho}} \lesssim \|\{c_k r_k(t)\}\|_{\ell^p} = \|\{c_k\}\|_{\ell^p}.$$

So, by Fubini's theorem and Khinchine's inequality (see [11, p.336])

$$\begin{aligned} &\int_{\mathbb{C}^n} \left(\sum_k |c_k|^2 \frac{|K_\beta(z, z_k)|^2}{\|K_\beta(z, z_k)\|^2_{F^p_{\beta,\rho}}} (1 + |z|)^{2\eta} e^{-\beta|z|^{2\ell}} \right)^{q/2} dV(z) \\ &\quad \simeq \int_0^1 \|\Phi_t(\{c_k\})\|^q_{F^q_{\beta,\eta}} \, dt \lesssim \|\{c_k\}\|^q_{\ell^p}. \end{aligned}$$

By Proposition 2.7 this is equivalent to the fact that $I(\{c_k\}) \lesssim \|\{c_k\}\|^q_{\ell^p}$, where

$$I(\{c_k\}) := \int_{\mathbb{C}^n} \left(\sum_k |c_k|^2 \frac{|K_\beta(z, z_k)|^2 e^{-\beta|z_k|^{2\ell}} e^{-\beta|z|^{2\ell}}}{(1 + |z_k|)^{2(\rho-\eta)+4n(\ell-1)/p'}} \right)^{q/2} dV(z).$$

Now

$$I(\{c_k\}) \gtrsim \int_{\mathbb{C}^n} \left(\sum_k |c_k|^2 \frac{|K_\beta(z, z_k)|^2 e^{-\beta|z_k|^{2\ell}} e^{-\beta|z|^{2\ell}}}{(1 + |z_k|)^{2(\rho-\eta)+4n(\ell-1)/p'}} \mathcal{X}_{B_k}(z) \right)^{q/2} dV(z).$$

Since, by Lemma 3.5, any point $z \in \mathbb{C}^n$ is at most in N balls B_k, the equivalence of the ℓ^2-norm and $\ell^{q/2}$-norm on $\mathbb{C}^N$ give

$$I(\{c_k\}) \gtrsim \sum_k |c_k|^q \int_{B_k} \frac{|K_\beta(z, z_k)|^q e^{-\frac{\beta q}{2}|z_k|^{2\ell}} e^{-\frac{\beta q}{2}|z|^{2\ell}}}{(1 + |z_k|)^{(\rho-\eta)q+2n(\ell-1)q/p'}} dV(z).$$

By Lemma 3.6(i)

$$|K_\beta(z, z_k)|^q e^{-\frac{\beta q}{2}|z_k|^{2\ell}} e^{-\frac{\beta q}{2}|z|^{2\ell}} \simeq (1 + |z_k|)^{2n(\ell-1)q} \quad (z \in B_k).$$

Hence

$$\begin{aligned} \|\{c_k\}\|^q_{\ell^p} &\gtrsim \sum_k |c_k|^q (1 + |z_k|)^{-(\rho-\eta)q-2n(\ell-1)(q/p'-q+1)} \\ &= \sum_k |c_k|^q (1 + |z_k|)^{-(\rho-\eta)q-2n(\ell-1)(p-q)/p}, \end{aligned}$$

and consequently for any $\{d_k\} \in \ell^{p/q}$,

$$\sum_k |d_k|(1+|z_k|)^{-(\rho-\eta)q-2n(\ell-1)(p-q)/p} \lesssim \|d_k\|_{\ell^{p/q}}.$$

By the duality of the sequence spaces $(\ell^{p/q})^* = \ell^{p/(p-q)}$, we obtain

$$\sum_k (1+|z_k|)^{-(\rho-\eta)\frac{pq}{p-q}-2n(\ell-1)} < \infty$$

Since

$$\begin{aligned}\infty > \sum_k (1+|z_k|)^{-(\rho-\eta)\frac{pq}{p-q}-2n(\ell-1)} &\simeq \sum_k \int_{B_k} (1+|z|)^{-(\rho-\eta)\frac{pq}{p-q}} dV(z) \\ &\simeq \int_{\mathbb{C}^n} (1+|z|)^{-(\rho-\eta)\frac{pq}{p-q}} dV(z),\end{aligned}$$

we conclude that $-(\rho-\eta)\frac{pq}{p-q} < -2n$. This ends the proof. □

3.5. **Necessary condition for $1 \le q < p = \infty$ and $\beta = \gamma$.**

In this section we extend Proposition 3.4 to the case $p = \infty$.

Proposition 3.8. *If $1 \le q < \infty$ and $F^\infty_{\beta,\rho} \hookrightarrow F^q_{\beta,\eta}$ then $\frac{2n}{q} < \rho - \eta$.*

The necessary condition will be obtained from the case $1 \le q < p < \infty$ by complex interpolation. In particular we will use the Riesz-Thorin theorem and the following well-known result (see for instance [10, Lemma 7.11]).

Lemma 3.9. *Let (Y_0, Y_1) and (X_0, X_1) be admissible pairs of Banach spaces. Assume that (Y_0, Y_1) is a retract of (X_0, X_1), that is, there exist bounded linear operators $E : Y_j \to X_j$ and $R : X_j \to Y_j$ such that $R \circ E$ is the identity operator on Y_j, $j = 0, 1$. Then $(Y_0, Y_1)_{[\theta]} = R((X_0, X_1)_{[\theta]})$.*

Lemma 3.10. *Let $1 \le q < \infty$ and let $\theta \in (0,1)$. If $\frac{1}{s} = \frac{1-\theta}{q}$ then*

$$(F^q_{\beta,\rho}, F^\infty_{\beta,\rho})_{[\theta]} = F^s_{\beta,\rho} \quad \text{and} \quad (F^q_{\beta,\rho}, F^q_{\beta,\eta})_{[\theta]} = F^q_{\beta,(1-\theta)\rho+\theta\eta}.$$

Proof. Observe that the map $\Phi(f)(z) := f(z)e^{\frac{\beta}{2}|z|^{2\ell}}(1+|z|)^{-\rho}$ is a linear isometry from L^r onto $L^r_{\beta,\rho}$, $1 \le r \le \infty$. So by Lemma 3.10 and the Riesz-Thorin theorem, we obtain

$$(L^q_{\beta,\rho}, L^\infty_{\beta,\rho})_{[\theta]} = \Phi((L^q, L^\infty)_{[\theta]}) = \Phi(L^s) = L^s_{\beta,\rho}$$

By Proposition 2.16, for $1 \le r \le \infty$, $(F^q_{\beta,\rho}, F^\infty_{\beta,\rho})$ is a retract of $(L^q_{\beta,\rho}, L^\infty_{\beta,\rho})$ and so

$$(F^q_{\beta,\rho}, F^\infty_{\beta,\rho})_{[\theta]} = P_\beta((L^q_{\beta,\rho}, L^\infty_{\beta,\rho})_{[\theta]}) = P_\beta(L^s_{\beta,\rho}) = F^s_{\beta,\rho},$$

which proves the first interpolation identity.

In order to prove the second identity, by Theorem [3, Theorem 5.5.3] we have

$$\begin{aligned}(L^q_{\beta,\rho}, L^q_{\beta,\eta})_{[\theta]} &= (L^q(e^{-\frac{q\beta}{2}|z|^{2\ell}}(1+|z|)^{q\rho}), L^q(e^{-\frac{q\beta}{2}|z|^{2\ell}}(1+|z|)^{q\eta}))_{[\theta]}\\ &= L^q(e^{-\frac{q\beta}{2}|z|^{2\ell}}(1+|z|)^{q((1-\theta)\rho+\theta\eta)}) = L^q_{\beta,(1-\theta)\rho+\theta\eta}.\end{aligned}$$

Therefore, as above,

$$(F^q_{\beta,\rho}, F^q_{\beta,\eta})_{[\theta]} = P_\beta(L^q_{\beta,(1-\theta)\rho+\theta\eta}) = F^q(e^{-\frac{\beta}{2}|z|^{2\ell}}(1+|z|)^{1-\theta)\rho+\theta\eta}).$$

This ends the proof. □

Proof of Proposition 3.8. Assume $F^\infty_{\beta,\rho} \hookrightarrow F^q_{\beta,\eta}$. By Lemma 3.10,

$$F^s_{\beta,\rho} = (F^q_{\beta,\rho}, F^\infty_{\beta,\rho})_{[\theta]} \hookrightarrow (F^q_{\beta,\rho}, F^q_{\beta,\eta})_{[\theta]} = F^q_{\beta,(1-\theta)\rho+\theta\eta},$$

with $\frac{1}{s} = \frac{1-\theta}{q}$. Since $q = (1-\theta)s < s < \infty$, Proposition 3.4 gives

$$2n(\tfrac{1}{q} - \tfrac{1}{s}) < \rho - ((1-\theta)\rho + \theta\eta) = q(\tfrac{1}{q} - \tfrac{1}{s})(\rho - \eta),$$

and so $\frac{2n}{q} < \rho - \eta$. □

3.6. **Proof of Theorem 1.3 for $1 \le q < p \le \infty$.**

The sufficient conditions follow from Lemma 3.3.

If $\beta \neq \gamma$ the necessary condition $\beta < \gamma$ follows from Lemma 3.1. If $\beta = \gamma$ the necessary condition follows from Propositions 3.4 and 3.8.

4. Proof of Theorem 1.1

First we prove the necessary condition $\beta < 2\alpha$. For the case $\rho = 0$ next lemma corresponds to [4, Lemma 3].

Lemma 4.1. *Let $1 \le p, q \le \infty$. If P_α is bounded from $(L^2_\alpha \cap L^p_{\beta,\rho}, \|\cdot\|_{L^p_{\beta,\rho}})$ to $L^q_{\gamma,\eta}$ then $\beta < 2\alpha$.*

Proof. For any $\zeta \in \mathbb{C}^n$, the linear form $g \mapsto g(\zeta)$ is bounded on $F^q_{\gamma,\eta}$ (see Corollary 2.14). Then the boundedness of $P_\alpha : (L^2_\alpha \cap L^p_{\beta,\rho}, \|\cdot\|_{L^p_{\beta,\rho}}) \mapsto L^q_{\gamma,\eta}$ implies the boundedness of the form $U_\zeta(f) = P_\alpha(f)(\zeta)$ on $(L^2_\alpha \cap L^p_{\beta,\rho}, \|\cdot\|_{L^p_{\beta,\rho}})$. Hence Lemma 2.12 gives $\beta < 2\alpha$. □

Now the proof of Theorem 1.1 follows from the next proposition and its corollary.

Proposition 4.2. *Let $1 \le p \le \infty$. If $0 < \beta < 2\alpha$ then the Bergman projection P_α is bounded from $L^p_{\beta,\rho}$ onto $F^p_{\alpha^2/(2\alpha-\beta),\rho}$.*

Corollary 4.3. *Let $1 \le p, q \le \infty$ and let $0 < \beta < 2\alpha$. Then the Bergman projection P_α is bounded from $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$ if and only if $F^p_{\alpha^2/(2\alpha-\beta),\rho} \hookrightarrow F^q_{\gamma,\eta}$.*

Taking for granted these results, we finish the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.1 it is clear that $\beta < 2\alpha$ is a necessary condition for the boundedness of P_α from $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$.

If $\beta < 2\alpha$, Corollary 4.3 shows that P_α is bounded from $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$ if and only if $F^p_{\alpha^2/(2\alpha-\beta),\rho} \hookrightarrow F^q_{\gamma,\eta}$. Thus Theorem 1.1 is a consequence of Theorem 1.3. □

We conclude this section with the proofs of Proposition 4.2 and Corollary 4.3. To do so, we introduce the following notations which will used in the next results. For $\beta < 2\alpha$, let

$$\delta := \big(\tfrac{\alpha}{2\alpha-\beta}\big)^{1/\ell} \quad \text{and} \quad \kappa := \alpha\delta^\ell = \tfrac{\alpha^2}{2\alpha-\beta}.$$

The next lemma follows from (2.4).

Lemma 4.4. *If $f \in L^p_{\beta,\rho}$, then $P_\alpha(f) = P_\kappa(T_\delta(f))$, where*

$$T_\delta(f)(z) = \delta^n f(\delta z) e^{(\alpha-\beta)|\delta z|^{2\ell}}.$$

Proof. Using the change of variables $w = \delta u$ and (2.4), we obtain

$$\begin{aligned} P_\alpha(f)(z) &= \delta^{2n} \int_{\mathbb{C}^n} f(\delta u) K_\alpha(z, \delta u) e^{-\alpha|\delta u|^{2\ell}} dV(u) \\ &= \delta^n \int_{\mathbb{C}^n} [f(\delta u) e^{(-\alpha+\kappa\delta^{-2\ell})|\delta u|^{2\ell}}] K_\kappa(z, u) e^{-\kappa|u|^{2\ell}} dV(u). \end{aligned}$$

Since $-\alpha+\kappa\delta^{-2\ell} = -\alpha+\alpha\delta^{-\ell} = -\alpha+2\alpha-\beta = \alpha-\beta$ we obtain the result. □

Lemma 4.5. *The operator T_δ is a topological isomorphism from $L^p_{\beta,\rho}$ onto $L^p_{\kappa,\rho}$.*

Proof. Since $\alpha - \beta = -\frac{\beta}{2} + \frac{2\alpha-\beta}{2} = -\frac{\beta}{2} + \frac{\kappa}{2}\delta^{-2\ell}$, we have

$$T_\delta(f)(z) = \delta^n f(\delta z) e^{-\frac{\beta}{2}|\delta z|^{2\ell}} e^{\frac{\kappa}{2}|z|^{2\ell}}.$$

Therefore

$$\begin{aligned} \|T_\delta(f)\|_{L^p_{\kappa,\rho}} &\simeq \|f(\delta z) e^{-\frac{\beta}{2}|\delta z|^{2\ell}} (1+|z|)^\rho\|_{L^p} \\ &\simeq \|f(\delta z) e^{-\frac{\beta}{2}|\delta z|^{2\ell}} (1+|\delta z|)^\rho\|_{L^p} \simeq \|f\|_{L^p_{\beta,\rho}}. \end{aligned}$$

So to conclude the proof we only need to show that the operator T_δ is surjective. This follows from the fact that the unique solution of the equation $T_\delta(f) = g$ is $f(z) = \delta^{-n} g(z/\delta) e^{(\beta-\alpha)|z|^{2\ell}}$ and

$$\|f\|_{L^p_{\beta,\rho}} \simeq \|T_\delta(f)\|_{L^p_{\kappa,\rho}} = \|g\|_{L^p_{\kappa,\rho}}. \qquad \square$$

Proof of Proposition 4.2. By Proposition 2.16, P_κ is a bounded operator from $L^p_{\kappa,\rho}$ onto $F^p_{\kappa,\rho}$. So Lemmas 4.4 and 4.5 give

$$P_\alpha(L^p_{\beta,\rho}) = P_\kappa(T_\delta(L^p_{\beta,\rho})) = P_\kappa(L^p_{\kappa,\rho}) = F^p_{\kappa,\rho}. \qquad \square$$

Proof of Corollary 4.3. By Proposition 4.2, it is clear that if $F^p_{\alpha^2/(2\alpha-\beta),\rho} \hookrightarrow F^q_{\gamma,\eta}$, then P_α is bounded from $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$.

Conversely, if P_α is bounded from $L^p_{\beta,\rho}$ to $L^q_{\gamma,\eta}$ then, by Proposition 4.2,

$$F^p_{\alpha^2/(2\alpha-\beta),\rho} = P_\alpha(L^p_{\beta,\rho}) \hookrightarrow F^q_{\gamma,\eta}. \qquad \square$$

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