BILINEAR FORMS ON NON-HOMOGENEOUS SOBOLEV SPACES

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Abstract. In this paper we show that if $b \in L^2(\mathbb{R}^n)$, then the bilinear form defined on the product of the non-homogeneous Sobolev spaces $H^2_s(\mathbb{R}^n) \times H^2_s(\mathbb{R}^n)$, $0 < s < 1$ by

$$(f, g) \in H^2_s(\mathbb{R}^n) \times H^2_s(\mathbb{R}^n) \to \int_{\mathbb{R}^n} (Id - \Delta)^{s/2}(fg)(\mathbf{x})b(\mathbf{x})d\mathbf{x},$$

is continuous if and only if the positive measure $|b(\mathbf{x})|^2 d\mathbf{x}$ is a trace measure for $H^2_s(\mathbb{R}^n)$.

1. Introduction

The space of pointwise multipliers between non-homogeneous Sobolev spaces $H^2_s(\mathbb{R}^n)$ has been object of great atention both for its intrinsic interest and for the study of the operators in which they are involved, in particular in some partial differential equations, [14], [15]. The study and description of these spaces of multipliers from $H^2_s(\mathbb{R}^n)$ and $H^2_t(\mathbb{R}^n)$ when $s \geq t \geq 0$ can be found, for instance, in the book by V.G. Maz'ya and T.O. Shaposhnikova ([12]).

The space $H^2_s(\mathbb{R}^n)$, $s \in \mathbb{R}$ is the completion of the space of compactly supported $\mathcal{C}^\infty$ functions f on $\mathbb{R}^n$, $\mathcal{D}(\mathbb{R}^n)$ (or the Schwartz class $\mathcal{S}(\mathbb{R}^n)$), with respect to the norm

$$\|f\|_{H^2_s(\mathbb{R}^n)} = \|(Id - \Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)}.$$

Here $(Id - \Delta)^{\frac{s}{2}}$ is the Fourier multiplier defined by the function $(1 - |\xi|^2)^{\frac{s}{2}}$.

Observe that via Plancherel's formula the space $H^2_{-t}(\mathbb{R}^n)$, $t > 0$, can be identified as the dual of the space $H^2_t(\mathbb{R}^n)$. Hence, the space of pointwise multipliers (non regular) from $H^2_s(\mathbb{R}^n)$ to $H^2_{-t}(\mathbb{R}^n)$ can be described as the subspace of distributions m such that $\langle mf, g\rangle = \langle m, fg\rangle$, $f, g \in \mathcal{D}(\mathbb{R}^n)$, defines a continuous bilinear form, i.e.,

$$|\langle mf, g\rangle| \leq C\|f\|_{H^2_s(\mathbb{R}^n)}\|g\|_{H^2_t(\mathbb{R}^n)}.$$

In particular, the pointwise multipliers from $H^2_s(\mathbb{R}^n)$ to $H^2_{-s}(\mathbb{R}^n)$ are the weights m (or more generaly the distributions) such that

$$\left|\int_{\mathbb{R}^n} m(\mathbf{x})f(\mathbf{x})g(\mathbf{x})d\mathbf{x}\right| \leq C\|f\|_{H^2_s(\mathbb{R}^n)}\|g\|_{H^2_s(\mathbb{R}^n)},$$

which by polarization, is equivalent to

$$\left|\int_{\mathbb{R}^n} m(\mathbf{x})|f(\mathbf{x})|^2 d\mathbf{x}\right| \leq C\|f\|^2_{H^2_s(\mathbb{R}^n)}. \tag{1.1}$$

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The first known result in this context is due to Verbitsky and Maz'ya ([14]), who gave a complete characterization of the space of multipliers from $H^2_1(\mathbb{R}^n)$ to $H^2_{-1}(\mathbb{R}^n)$). In [15], the same authors considered this problem for $s = \frac{1}{2}$.

In [8] (see also the thesis [7]), Lemarié-Rieusset and Gala gave a characterization for the space of multipliers from homogeneous and non homogeneous Sobolev spaces for the case $0 < r < s < n/2$. They also state a sufficient condition for a distribution φ to be a multiplier for $r = s$ and conjectured that this necessary condition was also necessary.

The purpose of this paper is to give a description for the space of multipliers from the non-homogeneous Sobolev space $H^2_s(\mathbb{R}^n)$ to its dual $H^2_{-s}(\mathbb{R}^n)$. The main result that we obtain is

Theorem 1.1. *Let $0 < s < 1$ and $b \in L^2(\mathbb{R}^n)$. The following assertions are equivalent:*

(i) *For any $f, g \in \mathcal{D}$*

$$\left|\int_{\mathbb{R}^n} (Id-\Delta)^{s/2}(fg)(\mathbf{x})b(\mathbf{x})d\mathbf{x}\right| \lesssim \|f\|_{H^2_s(\mathbb{R}^n)}\|g\|_{H^2_s(\mathbb{R}^n)}.$$

(ii) $d\nu(\mathbf{x}) := |b(\mathbf{x})|^2 d\mathbf{x}$ *is a trace measure for $H^2_s(\mathbb{R}^n)$, that is, $H^2_s(\mathbb{R}^n) \subset L^2(d\nu)$.*

Observe that (i) is equivalent to

(i')

$$\left|\int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x})(Id-\Delta)^{s/2}(b)(\mathbf{x})d\mathbf{x}\right| \lesssim \|f\|_{H^2_s(\mathbb{R}^n)}\|g\|_{H^2_s(\mathbb{R}^n)},$$

that is, $(Id-\Delta)^{s/2}b$ is a pointwise multiplier from $H^2_s(\mathbb{R}^n)$ to its dual $H^2_{-s}(\mathbb{R}^n)$.

We also observe that assertion (ii) can be reformulated by the condition that $b \in Mult(H^2_s(\mathbb{R}^n) \to L^2(\mathbb{R}^n))$.

The first difficulty when dealing with this problem is that, if it is a function, the symbol $(Id-\Delta)^{s/2}(b)$ in condition (i) in Theorem 1.1 may change sign. Observe that the condition (ii) is given in terms of the nonnegative measure $|b(\mathbf{x})|^2 d\mathbf{x}$. The characterization of such positive measures is well known (for example in terms of capacities, see for instance [13]).

The second difficulty is to obtain accurate estimates of the norms of functions in $H^2_s(\mathbb{R}^n)$. This is due to the fact that the operator $(Id-\Delta)^{\frac{s}{2}}$ is non local. To avoid this fact we will use a generalization of the extension operators from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}_+$, introduced by Caffarelli and Silvestre [2]) due to Stinga and Torrea in [17] that reduces the calculus od the non local operator $(Id-\Delta)^{\frac{s}{2}}$ to a limit of local operators. An essential tool here was to obtain an explicit formula for the extension operator.

For the homogeneous Sobolev spaces, this problem was studied by Cascante, Fabrega and Ortega in [3]. Among the main technical differences with the homogeneous case, we first mention that the explicit formula for the kernel of the extension operator Q_s corresponding to $(Id-\Delta)^{\frac{s}{2}}$ is given by $Q_s(\mathbf{x}, y) = C_{n,s}P_s(\mathbf{x}, y)\mathcal{G}_{2n+2s+1}(\mathbf{x}, y)$, where P_s is the extension kernel for the homogeneous case and $\mathcal{G}_{2n+2s+1}$ is the Bessel function. Secondly, we prove a weighted L^p estimate for an area function for some general kernels. Finally, we use a procedure of localization in order to substitute the Bessel kernel for the Riesz kernel, that permits to apply the weighted estimate.

The paper is organized as follows. In Section 2 is devoted to an extension theorem. Each function f in $\mathbb{R}^n$ in the domain of $(Id-\Delta)^s$ can be extended to a function u in

$\mathbb{R}^n \times [0,\infty)$ in such a way that

$$\lim_{y\to 0^+} \frac{1}{2s} y^{1-2s} u_y(\mathbf{x},y) = \frac{\Gamma(-s)}{4^s\Gamma(s)} (Id-\Delta)^s f(\mathbf{x}).$$

This extension gives an isomorphism from $H^2_s(\mathbb{R}^n)$ in a subspace of a weighted Sobolev space $W^2_{1,1-2s}(\mathbb{R}^{n+1}_+)$.

Sections 3 and 4 are instrumental and give the assymptotic behaviour at infinity of the extended functions and its derivatives of the previous section.

In Section 5 we give a theorem on weighted estimates for an area function. This type of results are well known for extensions of functions in terms of a kernel which come from a function in $\mathbb{R}^n$ with integral 0, but we need a version for a more general class of kernels. This theorem may be of interest in other contexts.

In Section 6 it is shown that the kernels that appear naturally as a convolution of the kernel defining the extension given in Section 2 (or its derivatives), with the Bessel kernel are in the conditions of the theorem on weighted estimates for an area function of Section 5. This will be a fundamental step to prove our main result.

Finally, Section 7 is devoted to the proof of Theorem 1.1. By the result in [7], we only have to prove that (i)⇒(ii). It is shown that the measure $|b(\mathbf{x})|^2 d\mathbf{x}$ is a trace measure by checking that it satisfies the capacitary condition. This property will follow by applying the hypothesis (i) to suitable test functions. One of the main steps is the estimate of the norm in $H^2_s(\mathbb{R}^n)$ of these test functions. In doing this, we use a localization process that allows to substitute the Bessel potential by the Riesz potential. We apply the results of Sections 5 and 6 to a power of the Riesz potential which is in the Muckenhoupt class A_2.

Notations: Throughout the paper, the letter C may denote various non-negative numerical constants, possibly different in different places. The notation $f(z) \lesssim g(z)$ means that there exists $C > 0$, which does not depends of z, f and g, such that $f(z) \le Cg(z)$.

2. An extension operator

We recall that the Fourier transform of an integrable function is defined by

$$\mathcal{F}(f)(\mathbf{z}) = \hat{f}(\mathbf{z}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{z}} d\mathbf{x}$$

The fractional Laplacian is defined for $f \in \mathcal{S}$ and $s > 0$

$$(-\Delta)^{\frac{s}{2}} f := \mathcal{F}^{-1}(|\mathbf{z}|^s \mathcal{F}(f)).$$

We also consider the operator defined by

$$(Id-\Delta)^{\frac{s}{2}} f := \mathcal{F}^{-1}\left((1+|\mathbf{z}|^2)^{\frac{s}{2}} \mathcal{F}(f)\right),$$

the homogeneous Sobolev space $\dot{H}^2_s(\mathbb{R}^n)$ is the completion of the space of compactly supported $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$, $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\dot{H}^2_s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}.$$

The non-homogeneous Sobolev space $H^2_s(\mathbb{R}^n)$ is the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^2_s(\mathbb{R}^n)} = \|(Id-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}.$$

The Riesz kernel I_s, $0 < s < n$, is defined by

$$I_s(\mathbf{x}) = a_s \int_0^\infty \delta^{\frac{s-n}{2}} e^{-\frac{\pi|\mathbf{x}|^2}{\delta}} \frac{d\delta}{\delta}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $a_s = \left((4\pi)^{\frac{s}{2}}\Gamma(s/2)\right)^{-1}$.

For $s > 0$, the Bessel function is defined by

$$G_s(\mathbf{x}) = a_s \int_0^{+\infty} \delta^{\frac{s-n}{2}} e^{-\frac{\pi|\mathbf{x}|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \frac{d\delta}{\delta}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

The operator given by the convolution with I_s, $0 < s < n$ is the inverse of $(-\Delta)^{s/2}$ on $\mathcal{S}$, whereas the operator given by the convolution with G_s, is the inverse of $(Id - \Delta)^{s/2}$ (see [16]).

When $0 < 2s < n$, the homogeneous Sobolev space $\dot{H}^2_s(\mathbb{R}^n)$ coincides with the space $I_s(L^2(\mathbb{R}^n))$ of functions $I_s(g)$, $g \in L^2(\mathbb{R}^n)$ where

$$I_s(g)(\mathbf{x}) := \int_{\mathbb{R}^n} I_s(\mathbf{x} - \mathbf{t}) g(\mathbf{t}) d\mathbf{t}.$$

The space $H^2_s(\mathbb{R}^n)$, $0 < s$, coincides with the space $G_s(L^2(\mathbb{R}^n))$ of functions $G_s(g)$, $g \in L^2(\mathbb{R}^n)$, defined by

$$G_s(g)(\mathbf{x}) := \int_{\mathbb{R}^n} G_s(\mathbf{x} - \mathbf{t}) g(\mathbf{t}) d\mathbf{t}.$$

For this reason they are called the space of Riesz and Bessel potentials respectively.

If $0 < s < 1$, we define the Sobolev space with weights

$$W^2_{1,1-2s} := W^2_{1,1-2s}(\mathbb{R}^{n+1}_+) \qquad \text{where} \quad \mathbb{R}^{n+1}_+ := \{(\mathbf{x}, y);\, \mathbf{x} \in \mathbb{R}^n,\, y > 0\},$$

as the completion of $\mathcal{D}\left(\overline{\mathbb{R}^{n+1}_+}\right)$, functions of $\mathcal{C}\left(\overline{\mathbb{R}^{n+1}_+}\right)$, compactly supported, with respect to the norm

$$\|F\|_{W^2_{1,1-2s}} := \int_{\mathbb{R}^{n+1}_+} |\nabla_{\mathbf{x},y} F(\mathbf{x}, y)|^2 y^{1-2s} dx dy + \int_{\mathbb{R}^{n+1}_+} |F(\mathbf{x}, y)|^2 y^{1-2s} dx dy.$$

The following proposition is well known (see, for instance Thm 5 in Chapter 10 in [10])

Proposition 2.1. *Let $0 < s < 1$. We then have: If $f \in \mathcal{D}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |(Id - \Delta)^{\frac{s}{2}} f(\mathbf{x})|^2 d\mathbf{x}$$
$$= \inf_{u \in \mathcal{D}(\overline{\mathbb{R}^{n+1}_+}),\, u_{|\mathbb{R}^n} = f} \int_{\mathbb{R}^{n+1}_+} \left(|\nabla_{\mathbf{x},y} u(\mathbf{x}, y)|^2 + |u(\mathbf{x}, y)|^2\right) y^{1-2s} d\mathbf{x} dy.$$

□

If we consider the Euler-Lagrange equation for the functional

$$J(u) = \int_{\mathbb{R}^{n+1}_+} \left(|\nabla_{\mathbf{x},y} u(\mathbf{x}, y)|^2 + |u(\mathbf{x}, y)|\right) y^{1-2s} d\mathbf{x} dy,$$

that defines $\|u\|_{W^2_{1,1-2s}}$, we obtain that

$$-(Id - \Delta_x) u(\mathbf{x}, y) + \frac{(1-2s)}{y} u_y + u_{yy} = 0, \tag{2.2}$$

or equivalently, $div(y^{1-2s} \nabla u) = y^{1-2s} u$.

On the other hand, in order to study the fractional Laplacian $(-\Delta)^s$, L. Caffarelli and L. Silvestre in [2] consider the PDE on $\mathbb{R}^n \times [0,\infty)$ given by $div(y^{1-2s}\nabla u) = 0$. If $u(\mathbf{x}, y)$ is a solution of this equation such that $u(\mathbf{x}, 0) = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, then $(-\Delta)^s f(\mathbf{x})$ can be obtained as $-\lim_{y\to 0^+} y^{1-2s} u_y(\mathbf{x}, y)$. This fact permits to apply local methods to the study of the fractional Laplacian, which is a non-local operator.

Later, Stinga and Torrea (see [17]) extended this theory in the following form. They considered fractional powers of a linear second order partial differential operator L, non-negative, densely defined, and self-adjoint in $L^2(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. The operator L^σ, $0 < \sigma < 1$, is defined in an spectral way. If E is the unique resolution of the identity, supported on the spectrum of L (which is a subset of $[0,\infty)]$), such that $L = \int_0^\infty \lambda dE(\lambda)$, then $L^\sigma = \int_0^\infty \lambda^\sigma dE(\lambda)$, for $0 < \sigma < 1$. The heat-difussion semigroup generated by L, for $t \geq 0$, is $e^{-tL} = \int_0^\infty e^{-t\lambda} dE(\lambda)$. They described L^σ as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem analogous to the one considered by L. Caffarelli and L. Silvestre in [2]. They also obtained a corresponding Poisson-type formula and introduce the conjugate equation. Precisely they obtained:

Theorem 2.2 ([17]). *Let $f \in Dom(L^\sigma) := \{f \in L^2(\mathbb{R}^n); \int_0^\infty \lambda^{2\sigma} dE_{f,f}(\lambda) < \infty)\}$. A solution of the extension problem*

$$-L_{\mathbf{x}} u + \frac{1-2s}{y} u_y + u_{yy} = 0, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \tag{2.3}$$

is given by the Poisson-type formula

$$u(\mathbf{x}, y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-tL} f(\mathbf{x}) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+s}},$$

where e^{-tL}, $t > 0$ is the heat-difussion semigroup generated by L.

This solution satisfies:

(i) $\frac{1}{2s} \lim_{y\to 0^+} y^{1-2s} u_y(\mathbf{x}, y) = \frac{\Gamma(-s)}{4^s\Gamma(s)} L^\sigma f(\mathbf{x})$. *The limit must be understood in $L^2(\mathbb{R}^n)$.*

(ii) $\|u(\cdot, y)\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$ *and* $\|u(\cdot, y) - f\|_{L^2(\mathbb{R}^n)} \to 0$ *as* $y \to 0^+$.

(iii) *The function* $v(\mathbf{x}, y) = y^{1-2s}\frac{\partial u}{\partial y}(\mathbf{x}, y)$ *is a solution of the following "conjugate" equation:*

$$-L_{\mathbf{x}} v - \frac{1-2s}{y} v_y + v_{yy} = 0.$$

In our situation, $L = Id - \Delta$, which is the infinitessimal generator of the semigroup $T_t = e^{-t} e^{t\Delta}$ and the extension problem (2.3) is now

$$-(Id - \Delta_x) u(\mathbf{x}, y) + \frac{(1-2s)}{y} u_y + u_{yy} = 0, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}). \tag{2.4}$$

We then have that if $f \in Dom((Id - \Delta)^s)$ (in particular, if f is in the Schwarz class $\mathcal{S}$), then (see example 2.14 in [17] for the operator $-\Delta$), $e^{-t(Id-\Delta)} f$ is the convolution of the function f with

$$g^t(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} e^{-t}.$$

Hence, the corresponding Poisson-type kernel that solves (2.3) is defined by

$$Q_s(\mathbf{x}, y) = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-\frac{|\mathbf{x}|^2 + y^2}{4t}} \frac{e^{-t}}{(4\pi t)^{\frac{n}{2}}} \frac{dt}{t^{1+s}}. \tag{2.5}$$

The associated operator is given by

$$Q_s(f)(\mathbf{x},y)=\int_{\mathbb{R}^n} Q_s(\mathbf{x}-\mathbf{u},y)f(\mathbf{u})\,d\mathbf{u}$$

and

$$\frac{1}{2s}\lim_{y\to 0^+} y^{1-2s}\frac{\partial u}{\partial y}(\mathbf{x},y)=\frac{\Gamma(-s)}{4^s\Gamma(s)}(Id-\Delta)^s f(\mathbf{x}),\ \text{in } L^2(\mathbb{R}^n). \tag{2.6}$$

If we make the change of variables $\frac{|\mathbf{x}|^2+y^2}{4t}=r$ in the integral that defines the kernel, we obtain

$$\begin{aligned}&Q_s(\mathbf{x},y)\\&=\frac{1}{\Gamma(s)\pi^{\frac{n}{2}}}\frac{y^{2s}}{(|\mathbf{x}|^2+y^2)^{\frac{n+2s}{2}}}\int_0^\infty e^{-\frac{|\mathbf{x}|^2+y^2}{4r}}e^{-r}r^{\frac{n+2s}{2}}\frac{dr}{r}.\end{aligned}$$

Now, the change $l=4\pi r$ gives

$$\begin{aligned}&Q_s(\mathbf{x},y)\\&=\frac{1}{4^{\frac{n+2s}{2}}\Gamma(s)\pi^{n+s}}\frac{y^{2s}}{(|\mathbf{x}|^2+y^2)^{\frac{n+2s}{2}}}\int_0^\infty e^{-\pi\frac{|\mathbf{x}|^2+y^2}{l}}e^{-\frac{l}{4\pi}}l^{\frac{n+2s}{2}}\frac{dl}{l}.\end{aligned}$$

Here, we recall that the Bessel function G_s is a radial function and, with a little abuse of language, in occasions we will write $G_s(|\mathbf{t}|)=G_s(t_1,\dots,t_n)$. Observe that the function G_s depends on the dimension n.

We observe that this last integral coincides, up to positive constants, with the Bessel function in dimension $n+1$, which corresponds to the parameter $2n+2s+1$, evaluated at the point $(\mathbf{x},y)$. From now on, we will denote the corresponding Bessel function in dimension $n+1$ by $\mathcal{G}$.

Hence

$$Q_s(\mathbf{x},y)=C_{n,s}\frac{y^{2s}}{(|\mathbf{x}|^2+y^2)^{\frac{n+2s}{2}}}\mathcal{G}_{2n+2s+1}(\mathbf{x},y),$$

where

$$C_{n,s}=\frac{4^{\frac{n+1}{2}}\pi^{\frac{1}{2}}}{\Gamma(s)}\Gamma(n+s+\frac{1}{2}).$$

Observe that, up to a constant, $G_s(t_1,\dots,t_n)$ for $s>0$, coincides with $\mathcal{G}_{s+1}(t_1,\dots,t_n,0)$.

Of course, it can be checked directly, without using the results in [17], that if $f\in\mathcal{S}(\mathbb{R}^n)$, then $Q_s(f)(\mathbf{x},y)=\int_{\mathbb{R}^n}Q_s(\mathbf{x}-\mathbf{v},y)f(\mathbf{v})d\mathbf{v}$ satisfies the differential equation $div(y^{1-2s}\nabla u)=y^{1-2s}u$.

Theorem 2.3. (i) $\displaystyle\lim_{y\to 0^+}\int_{\mathbb{R}^n}Q_s(\mathbf{x},y)d\mathbf{x}=1.$

(ii) *If $f\in \mathrm{Dom}((Id-\Delta)^s)$ is bounded and uniformly continuous on a neighbourhood of a compact set $K\subset\mathbb{R}^n$, then*

$$\lim_{y\to 0^+}Q_s(f)(\cdot,y)=f,$$

uniformly on K.

Proof. We begin with (i). We recall (see [17], example 2.14) that if

$$P_s(\mathbf{x},y)=\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{|x|^2+y^2}{4t}}\frac{1}{(4\pi t)^{\frac{n}{2}}}\frac{dt}{t^{1+s}}=\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}\frac{y^{2s}}{(|x|^2+y^2)^{\frac{n+2s}{2}}},$$

then $\int_{\mathbb{R}^n} P_s(\mathbf{x},y)d\mathbf{x} = 1$. Since by (2.5),

$$Q_s(\mathbf{x},y) = \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-\frac{|x|^2+y^2}{4t}}\frac{e^{-t}}{(4\pi t)^{\frac{n}{2}}}\frac{dt}{t^{1+s}},$$

we then have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} Q_s(\mathbf{x},y)d\mathbf{x} - 1 \\
&= \int_{\mathbb{R}^n}(Q_s(\mathbf{x},y) - P_s(\mathbf{x},y))\,d\mathbf{x} = \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty\left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}d\mathbf{x}\right)\frac{(e^{-t}-1)}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{y^2}{4t}}\frac{dt}{t^{1+s}} \\
&= \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty (e^{-t}-1)e^{-\frac{y^2}{4t}}\frac{dt}{t^{1+s}}
\end{aligned}$$

Since $\int_0^\infty \frac{|e^{-t}-1|dt}{t^{1+s}} < \infty$, the Lebesgue's Dominated Convergence Theorem concludes the proof.

Next we prove (ii). We will then have that for $\mathbf{x} \in K$,

$$\begin{aligned}
&Q_s(f)(\mathbf{x},y) - f(\mathbf{x}) \\
&= \int_{\mathbb{R}^n} Q_s(\mathbf{x}-\mathbf{u},y)(f(\mathbf{u}) - f(\mathbf{x}))d\mathbf{u} + f(\mathbf{x})\left(\int_{\mathbb{R}^n} Q_s(\mathbf{x}-\mathbf{u},y)d\mathbf{u} - 1\right).
\end{aligned}$$

Since f is bounded, (i) gives that the second sumand of the last equality tends to 0 when $y \to 0^+$. For the first summand the argument is straightforward. Indeed, the uniform continuity of f gives that for $\varepsilon > 0$, there exists $\delta > 0$ such that for $|\mathbf{x}-\mathbf{u}| < \delta$, then $|f(\mathbf{x}) - f(\mathbf{u})| < \varepsilon$. Then

$$\begin{aligned}
&|\int_{\mathbb{R}^n} Q_s(\mathbf{x}-\mathbf{u},y)(f(\mathbf{u}) - f(\mathbf{x}))d\mathbf{u}| \\
&\leq |\int_{|\mathbf{x}-\mathbf{u}|<\delta} Q_s(\mathbf{x}-\mathbf{u},y)(f(\mathbf{u}) - f(\mathbf{x}))d\mathbf{u}| + |\int_{|\mathbf{x}-\mathbf{u}|\geq\delta} Q_s(\mathbf{x}-\mathbf{u},y)(f(\mathbf{u}) - f(\mathbf{x}))d\mathbf{u}| \\
&\lesssim \varepsilon + \int_{|\mathbf{x}-\mathbf{u}|\geq\delta} Q_s(\mathbf{x}-\mathbf{u},y)d\mathbf{u} \lesssim \varepsilon + \int_{|\mathbf{t}|\geq\delta}\frac{y^{2s}}{(|\mathbf{t}|^2+y^2)^{\frac{n+2s}{2}}}\mathcal{G}_{2n+2s+1}(|\mathbf{t}|+y)d\mathbf{t} \\
&\lesssim \varepsilon + \int_{|\mathbf{z}|>\frac{\delta}{y}}\frac{d\mathbf{z}}{(|\mathbf{z}|^2+1)^{\frac{n+2s}{2}}}.
\end{aligned}$$

From the integrability of the function $\frac{1}{(|\mathbf{z}|^2+1)^{\frac{n+2s}{2}}}$, we deduce that the above integral tends to 0 as $y \to 0^+$ and that ends the proof. □

Remark 2.4. *Observe that since the function* $y^{1-2s}\frac{\partial u}{\partial y}(\mathbf{x},y)$ *satisfies the conjugate equation in* (iii) *in Theorem 2.2 with boundary values* $C_s(Id-\Delta)^s f(\mathbf{x})$*, the convergence in* (2.6) *is uniform on each compact set where* $(Id-\Delta)^s f$ *is uniformly continuous on a neighborhood of the same compact. In particular, this is the case for each* $f \in \mathcal{S}$.

The following lemma permits to conclude that the extension operator Q_s gives an isomorphism between $H^2_s(\mathbb{R}^n)$ and a subspace in $W^2_{1,1-2s}(\mathbb{R}^{n+1}_+)$.

Proposition 2.5. *Let $0 < s < 1$. There exists $k_s > 0$ such that for every $f \in H^2_s(\mathbb{R}^n)$ and $G \in W^2_{1,1-2s}(\mathbb{R}^{n+1}_+)$,*

$$\int_{\mathbb{R}^{n+1}_+} \nabla G(\mathbf{x},y)\nabla Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy + \int_{\mathbb{R}^{n+1}_+} G(\mathbf{x},y)Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy$$
$$= k_s \int_{\mathbb{R}^n} (Id-\Delta)^{s/2}G(\mathbf{x},0)(Id-\Delta)^{s/2}f(\mathbf{x})d\mathbf{x},$$

In particular, the left hand term only depends on f and the boundary values of $G(\mathbf{x},y)$.

Proof. Since $\mathcal{D}\left(\overline{\mathbb{R}^{n+1}_+}\right)$ and $\mathcal{D}(\mathbb{R}^n)$ are dense in $W^2_{1,1-2s}(\mathbb{R}^{n+1}_+)$ and in H^s, respectively, it is enough to prove the result for $G \in \mathcal{D}\left(\overline{\mathbb{R}^{n+1}_+}\right)$ and $f \in \mathcal{D}(\mathbb{R}^n)$.

Consider the differential form in $\mathbb{R}^2_+$

$$\omega(\mathbf{x},y) = G(\mathbf{x},y)y^{1-2s}\left(\sum_{i=1}^n (-1)^{i+1}\frac{\partial Q_s(f)}{\partial x_i}(\mathbf{x},y)dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\wedge dy\right.$$
$$\left.+(-1)^n\frac{\partial Q_s(f)}{\partial y}(\mathbf{x},y)\,dx_1\wedge\cdots\wedge dx_n\right).$$

Since $div(y^{1-2s}\nabla Q_s(f))(\mathbf{x},y) = y^{1-2s}Q_s(f)(\mathbf{x},y)$, we have

$$d\omega(\mathbf{x},y) = \nabla_{\mathbf{x},y}G(\mathbf{x},y)\nabla_{\mathbf{x},y}Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy + G(\mathbf{x},y)Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy.$$

Thus, Stokes' theorem applied to the region $B_{R,\varepsilon} = \{(\mathbf{x},y)\in B(\mathbf{0},R); y>\varepsilon\}$, with $\varepsilon > 0$ and R are such that the support of G is in $B(\mathbf{0},R)$, gives

$$\int_{\{y>\varepsilon\}} \nabla_{\mathbf{x},y}G(\mathbf{x},y)\nabla_{\mathbf{x},y}Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy + \int_{\{y>\varepsilon\}} G(\mathbf{x},y)Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy$$
$$= -\int_{\mathbb{R}^n\times\{\varepsilon\}} G(\mathbf{x},\varepsilon)\,\varepsilon^{1-2s}\frac{\partial Q_s(f)}{\partial y}(\mathbf{x},\varepsilon)\,d\mathbf{x}.$$

Since, $\nabla_{\mathbf{x},y}G(\mathbf{x},y)\nabla_{\mathbf{x},y}Q_s(f)(\mathbf{x},y)y^{1-2s}, G(\mathbf{x},y)Q_s(f)(\mathbf{x},y)y^{1-2s} \in L^1(\mathbb{R}^{n+1}_+)$, the dominated convergence theorem implies that the first term in the above equality tends when $\varepsilon \to 0$ to

$$\int_{R^{n+1}_+} \nabla_{\mathbf{x},y}G(\mathbf{x},y)\nabla_{\mathbf{x},y}Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy + \int_{R^{n+1}_+} G(\mathbf{x},y)Q_s(f)(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy.$$

By (2.6) and Theorem 2.2, (ii), the second term tends to

$$-\frac{2s\Gamma(-s)}{4^s\Gamma(s)}\int_{\mathbb{R}^n} G(\mathbf{x},0)\,(Id-\Delta)^s(f)(\mathbf{x})d\mathbf{x}$$
$$= -\frac{2s\Gamma(-s)}{4^s\Gamma(s)}\int_{\mathbb{R}^n} (Id-\Delta)^{s/2}G(\mathbf{x},0)(Id-\Delta)^{s/2}(f)(\mathbf{x}),$$

which proves the result. □

Remark 2.6. *The conclusion of the lemma shows that if we take as $G(\mathbf{x},y)$ the function $Q_s(f)(\mathbf{x},y)$, then Q_s gives an isomorphism between $H^2_s(\mathbb{R}^n)$ and a subspace of $W^2_{1,1-2s}(\mathbb{R}^{n+1}_+)$.*

3. Some instrumental lemmas

We begin the section with a recopilation of some of the main properties of the Riesz and Bessel kernels that can be found, for instance, in [16] or in [1].

We recall that if $0 < s < n$, $I_s(f) = \mathcal{F}^{-1}(|\mathbf{z}|^{-s}\mathcal{F}(f))$ and that if $\gamma(s) = \pi^{\frac{n}{2}} 2^s \dfrac{\Gamma(\frac{s}{2})}{\Gamma\left(\frac{n}{2} - \frac{s}{2}\right)}$, then

$$I_s f(\mathbf{x}) = \frac{1}{\gamma(s)} \int_{\mathbb{R}^n} \frac{f(\mathbf{u})}{|\mathbf{x}-\mathbf{u}|^{n-s}} d\mathbf{u}.$$

The following proposition collects the main properties of the Bessel functions that will be used in the forthcoming sections:

Proposition 3.1. (i) *For $s, t > 0$, $G_s * G_t = G_{s+t}$.*

(ii) *For $0 < s < n$, $0 < G_s(\mathbf{x}) < I_s(\mathbf{x})$.*

(iii) *$G_s \in L^1(\mathbb{R}^n)$ and $\|G_s\|_1 = \widehat{G}_s(\mathbf{0}) = 1$.*

(iv) *Let $s \geq 0$. Then:*
   (a) *$G_s(\mathbf{x}) \simeq I_s(\mathbf{x}) = C\frac{1}{|\mathbf{x}|^{n-s}}$ as $|\mathbf{x}| \to 0$ if $0 < s < n$.*
   (b) *$G_s(\mathbf{x}) \simeq 1$ as $|\mathbf{x}| \to 0$ if $s > n$.*
   (c) *$G_s(\mathbf{x}) \simeq \ln\frac{1}{|\mathbf{x}|}$ as $|\mathbf{x}| \to 0$ if $s = n$.*

(v) *For any $0 < c < 1$ and $s > 0$, $G_s(\mathbf{x}) = \mathrm{O}(e^{-c|\mathbf{x}|})$, as $|\mathbf{x}| \to \infty$. In fact this estimate can be improved. Namely:*

$$G_s(\mathbf{x}) \simeq C_s |\mathbf{x}|^{(s-n-1)/2} e^{-|\mathbf{x}|},$$

as $|\mathbf{x}| \to \infty$, for any $s > 0$, where C_s is a positive constant depending on s.

(vi) *$G_s(\mathbf{x}) \lesssim G_s(\mathbf{x}+\mathbf{u})$, $|\mathbf{x}| \geq 2$, $|\mathbf{u}| \leq 1$.*

(vii) *If we write $r = |\mathbf{x}|$, we have that there exists $c > 0$, such that $\lim_{r\to\infty} \dfrac{G_s'(r)}{G_s(r)} = -c$. If $s > 1$, $\lim_{r\to 0} \dfrac{G_s'(r)}{G_{s-1}(r)} = -(n-s)$. In particular, if $s > 1$,*

$$\left|\frac{\partial G_s}{\partial x_i}(\mathbf{x})\right| \lesssim G_s(\mathbf{x}) + G_{s-1}(\mathbf{x}).$$

The next elementary lemma will be used for obtaining estimates for the extension operator Q_s and its derivatives.

Lemma 3.2. *For any $(\mathbf{x}, y) \in \mathbb{R}^{n+1}_+$ and any $\mathbf{t} \in B(\mathbf{0}, 1)$,*

$$e^{(|\mathbf{x}|^2+y^2)^{\frac{1}{2}}} \simeq e^{(|\mathbf{x}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}}.$$

Analogously, $e^{|\mathbf{x}|+y} \simeq e^{|\mathbf{x}-\mathbf{t}|+y}$.

Proof. For the proof of the first equivalence, it is enough to show that

$$(|\mathbf{x}|^2+y^2)^{\frac{1}{2}} - (|\mathbf{x}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}$$

is bounded from above and from below for any $\mathbf{t} \in B(\mathbf{0}, 1)$.

Indeed,

$$\left|(|\mathbf{x}|^2+y^2)^{\frac{1}{2}} - (|\mathbf{x}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}\right| \leq \frac{\|2\mathbf{x}-\mathbf{t}\||\mathbf{t}|}{(|\mathbf{x}|^2+y^2)^{\frac{1}{2}} + (|\mathbf{x}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}} \lesssim 1.$$

□

In order to state the following proposition, we give the following definition.

Definition 3.3. *Let $0 < s < 1$. We define the function $h_s : (0, +\infty) \to \mathbb{R}$ given by*

$$h_s(y) = \begin{cases} 1, & \text{if } y \geq 1, \text{ or if } y \leq 1,\ 1-2s < 0, \\ \frac{1}{y^{1-2s}}, & \text{if } y \leq 1,\ 1-2s > 0 \\ 1 + \ln \frac{1}{y}, & \text{if } y \leq 1,\ 1-2s = 0 \end{cases}$$

Proposition 3.4. *Let $0 < s < 1$. We then have*

(i) $\int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial x_i}(\mathbf{x}, y) d\mathbf{x} = 0,\ i = 1, \dots, n.$

(ii) $\left| \int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial y}(\mathbf{x}, y) d\mathbf{x} \right| \lesssim h_s(y) e^{-\frac{1}{\sqrt{2}} y}.$

Proof. We begin with (i). We write $z_i = x_i/y$, $1 \leq i \leq n$ and $\psi(z_1, \dots, z_n) = \frac{1}{(|\mathbf{z}|^2+1)^{\frac{n+2s}{2}}}$. Let's consider

$$\int_{x_i \in \mathbb{R}} \frac{\partial Q_s}{\partial x_i} dx_i = \frac{C_{n,s}}{y^n} \int_{\mathbb{R}} \left\{ \frac{\partial \psi}{\partial z_i}(\mathbf{z}) \frac{1}{y} \mathcal{G}_{2n+2s+1}((|\mathbf{z}|^2+1)^{\frac{1}{2}} y) \right.$$
$$\left. + \psi(\mathbf{z}) \mathcal{G}'_{2n+2s+1}((|\mathbf{z}|^2+1)^{\frac{1}{2}} y) \frac{z_i}{(|\mathbf{z}|^2+1)^{\frac{1}{2}}} \right\} y dz_i.$$

Now, an integration by parts, using that $\psi(|\mathbf{z}|) \mathcal{G}_{2n+2s+1}((|\mathbf{z}|^2+1)^{\frac{1}{2}} y) \to 0$ if $|z_i| \to \infty$, gives that the first sumand equals to

$$-\int_{\mathbb{R}} \psi(\mathbf{z}) \mathcal{G}'_{2n+2s+1}((|\mathbf{z}|^2+1)^{\frac{1}{2}} y) \frac{z_i}{(|\mathbf{z}|^2+1)^{\frac{1}{2}}} dz_i.$$

Consequently,

$$\int_{x_i \in \mathbb{R}} \frac{\partial Q_s}{\partial x_i} dx_i = 0,$$

and

$$\int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial x_i} d\mathbf{x} = 0.$$

We now prove (ii). Let $(\mathbf{x}, y) \in \mathbb{R}^{n+1}_+$. If we denote $\mathbf{x} = \mathbf{z} y$, we have that $d\mathbf{x} = y^n d\mathbf{z}$. Observe that if we write $|\mathbf{z}| = \rho$ and denote $\varphi(\rho) = \frac{1}{(\rho^2+1)^{\frac{n+2s}{2}}}$, we have that there exists $C_{n,s} > 0$ such that

$$Q_s(\mathbf{x}, y) = C_{n,s} \frac{\varphi(\rho)}{y^n} \mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}} y).$$

Hence, using n-dimensional polar coordinates in $\mathbb{R}^n$,

$$\left| \int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial y}(\mathbf{x}, y) d\mathbf{x} \right|$$
$$\lesssim \left| \int_0^\infty \left\{ \left(-\frac{n\varphi(\rho)}{y} - \frac{\varphi'(\rho)}{y} \rho \right) \mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}} y) \right. \right.$$
$$\left. \left. + \frac{\varphi(\rho)}{(\rho^2+1)^{\frac{1}{2}}} \mathcal{G}'_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}} y) \right\} \rho^{n-1} d\rho. \right|$$

Next, we observe that by (iv) and (v) in Propostition 3.1, for each $y > 0$,

$$\lim_{\rho\to 0^+} \rho^n \varphi(\rho)\mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y) = 0;$$

and

$$\lim_{\rho\to +\infty} \rho^n \varphi(\rho)\mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y) = 0.$$

Hence, an integration by parts gives that

$$\begin{aligned}&-\int_0^\infty \left(\frac{n\rho^{n-1}\varphi(\rho)}{y} + \frac{\varphi'(\rho)}{y}\rho^n\right)\mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y)d\rho \\ &= \int_0^\infty \rho^{n+1}\varphi(\rho)\mathcal{G}'_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y)\frac{1}{(\rho^2+1)^{\frac{1}{2}}}d\rho.\end{aligned}$$

Plugging this formula in the previous sum, we obtain that

$$\left|\int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial y}(\mathbf{x},y)d\mathbf{x}\right| \lesssim \left|\int_0^\infty (\rho^2+1)^{\frac{1}{2}}\rho^{n-1}\varphi(\rho)\mathcal{G}'_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y)d\rho\right|.$$

Next, by Proposition 3.1, (vii), we have that the above is bounded, up to a constant, by

$$\int_0^\infty \frac{\rho^{n-1}}{(\rho^2+1)^{\frac{n-1+2s}{2}}}\left(\mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y) + \mathcal{G}_{2n+2s}((\rho^2+1)^{\frac{1}{2}}y)\right)d\rho \tag{3.7}$$

We split the integral over $[0,\infty)$ in two parts:

(i) The set of $\rho \geq 0$, such that $(\rho^2+1)^{\frac{1}{2}}y \leq 1$, i.e. $\rho^2 \leq \frac{1}{y^2} - 1$.
(ii) The set of $\rho \geq 0$, such that $(\rho^2+1)^{\frac{1}{2}}y > 1$, i.e. $\rho^2 > \frac{1}{y^2} - 1$.

We begin with the first part. Since $n+1 < 2n+2s$, Proposition 3.1, (iv), gives that both $\mathcal{G}_{2n+2s+1}$ and $\mathcal{G}_{2n+2s}$ are bounded. Observe that in addition, $y \leq 1$.

$$\begin{aligned}&\int_0^{((1/y^2)-1)^{\frac{1}{2}}} \frac{\rho^{n-1}}{(\rho^2+1)^{\frac{n-1+2s}{2}}}\left(\mathcal{G}_{2n+2s+1}((\rho^2+1)^{\frac{1}{2}}y) + \mathcal{G}_{2n+2s}((\rho^2+1)^{\frac{1}{2}}y)\right)d\rho \\ &\lesssim \int_0^{((1/y^2)-1)^{\frac{1}{2}}} \frac{\rho^{n-1}}{(\rho^2+1)^{\frac{n-1+2s}{2}}}d\rho \lesssim h_s(y) \lesssim h_s(y)e^{-\frac{y}{\sqrt{2}}},\end{aligned}$$

since $y \leq 1$.

So we are left to estimate the integral over $(\rho^2+1)y > 1$ in (3.7). By Proposition 3.1, (v), the integral in (3.7) is bounded by

$$\begin{aligned}&\int_{((1/y)^2-1)^{\frac{1}{2}}}^\infty \frac{\rho^{n-1}}{(\rho^2+1)^{\frac{n-1+2s}{2}}}(\rho^2+1)^{\frac{n+2s-1}{4}}y^{\frac{n+2s-1}{2}}e^{-((\rho^2+1)^{\frac{1}{2}}y)}d\rho \\ &\leq y^{\frac{n+2s-1}{2}}e^{-\frac{y}{\sqrt{2}}}\int_0^\infty \frac{\rho^{n-1}}{(\rho^2+1)^{\frac{n-1+2s}{4}}}e^{-\frac{\rho y}{\sqrt{2}}}d\rho.\end{aligned} \tag{3.8}$$

The change of variables $\rho y = \lambda$ gives that this is estimate by

$$\frac{1}{y^{1-2s}}e^{-\frac{y}{\sqrt{2}}}\int_0^\infty \lambda^{\frac{n-1}{2}-s}e^{-\frac{\lambda}{\sqrt{2}}}d\lambda \lesssim h_s(y)e^{-\frac{y}{\sqrt{2}}}.$$

□

Proposition 3.5. *Let $f \in L^1(B(\mathbf{0},1))$ and $0 < s < 1$. Let $k > 1$. Then, if $|\mathbf{x}| + y > k$,*

(i)

$$|Q_s(G_s(f))(\mathbf{x},y)| \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)} \int_{B(\mathbf{0},1)} |f(\mathbf{w})| d\mathbf{w}.$$

(ii)

$$|\nabla_{\mathbf{x},y} Q_s(G_s(f))(\mathbf{x},y)| \lesssim h_s(y) e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)} \int_{B(\mathbf{0},1)} |f(\mathbf{w})| d\mathbf{w}.$$

Proof. We begin with the proof of (i). We first observe that it is enough to prove that for $|\mathbf{u}| + y \geq k_1 = k-1$,

$$|Q_s(G_s)(\mathbf{u},y)| \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}. \tag{3.9}$$

Indeed, if (3.9) holds, we have that

$$\begin{aligned} |Q_s(G_s(f))(\mathbf{x},y)| &\leq \int_{B(\mathbf{0},1)} |Q_s(G_s)(\mathbf{x}-\mathbf{t},y)||f(\mathbf{t})| d\mathbf{t} \\ &\lesssim \int_{B(\mathbf{0},1)} e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}-\mathbf{t}|+y)} |f(\mathbf{t})| d\mathbf{t} \end{aligned}$$

which by Lemma 3.2 is in turn bounded by $Ce^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)} \int_{B(\mathbf{0},1)} |f(\mathbf{t})| d\mathbf{t}$, for some $C > 0$.

So let us prove (3.9). Let $|\mathbf{u}| + y \geq k_1 = k-1$ and fix $0 < K < \frac{k_1}{8}$. We decompose $Q_s(G_s)(\mathbf{u},y)$ as

$$\begin{aligned} Q_s(G_s)(\mathbf{u},y) &= C \int_{|\mathbf{u}-\mathbf{t}|+y \leq K} \frac{y^{2s}}{(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{n+2s}{2}}} \mathcal{G}_{2n+2s+1}((|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}) G_s(\mathbf{t}) d\mathbf{t} \\ &+ C \int_{|\mathbf{u}-\mathbf{t}|+y \geq K} \frac{y^{2s}}{(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{n+2s}{2}}} \mathcal{G}_{2n+2s+1}((|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}) G_s(\mathbf{t}) d\mathbf{t} \\ &= A(\mathbf{u},y) + B(\mathbf{u},y). \end{aligned}$$

Let us estimate $A(\mathbf{u},y)$. We first check that since for $|\mathbf{u}-\mathbf{t}| + y \leq K < \frac{k_1}{8}$, we have that $|\mathbf{u}-\mathbf{t}| \leq \frac{k_1}{8}$ and $y \leq \frac{k_1}{8}$, then $|\mathbf{u}| \geq k_1 - y \geq \frac{7}{8}k_1$. Consequently,

$$|\mathbf{t}| \geq |\mathbf{u}| - |\mathbf{u}-\mathbf{t}| \geq |\mathbf{u}| - \frac{k_1}{8} \geq |\mathbf{u}| - \frac{|\mathbf{u}|}{7} = \frac{6}{7}|\mathbf{u}| \geq \frac{6}{8}k_1.$$

By Proposition 3.1, (v),

$$G_s(\mathbf{t}) \lesssim \frac{e^{-|\mathbf{t}|}}{|\mathbf{t}|^{\frac{n+1-s}{2}}} \lesssim e^{-\frac{6}{7}|\mathbf{u}|} \lesssim e^{-\frac{1}{\sqrt{2}}|\mathbf{u}|},$$

Hence, since y is bounded and $\mathcal{G}_{2n+2s+1}((|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}) \lesssim 1$, we have,

$$A(\mathbf{u},y) \lesssim \int_{|\mathbf{u}-\mathbf{t}|+y \leq K} \frac{y^{2s}}{(|\mathbf{u}-\mathbf{t}|+y)^{n+2s}} e^{-\frac{|\mathbf{u}|}{\sqrt{2}}} e^{\frac{-y}{\sqrt{2}}} e^{\frac{y}{\sqrt{2}}} d\mathbf{t} \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}.$$

Next we estimate the integral $B(\mathbf{u},y)$. Using (iv) and (v) in Proposition 3.1, for $\frac{1}{\sqrt{2}} < c < 1$ and $c_1 = c - \frac{1}{\sqrt{2}}$ we then have that

$$\begin{aligned} B(\mathbf{u},y) &\lesssim \int_{|\mathbf{u}-\mathbf{t}|+y \geq K} y^{2s} e^{-\frac{(|\mathbf{u}-\mathbf{t}|+y)}{\sqrt{2}}} \frac{e^{-c|\mathbf{t}|}}{|\mathbf{t}|^{n-s}} d\mathbf{t} \\ &\lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)} \int_{\mathbb{R}^n} \frac{e^{-c_1|\mathbf{t}|}}{|\mathbf{t}|^{n-s}} d\mathbf{t} \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}. \end{aligned}$$

Now we deal with the estimate of (ii). As in the previous case, it is now enough to prove that for $|\mathbf{u}| + y \geq k_1 = k - 1$,

$$|\nabla_{x,y}(Q_s(G_s))(\mathbf{u}, y)| \lesssim h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}.$$

We begin with the estimates for

$$\frac{\partial}{\partial x_i} Q_s(G_s(f))(\mathbf{ux}, y).$$

If we deal directly with $\int_{\mathbb{R}^n} |\frac{\partial}{\partial x_i} Q_s(\mathbf{u} - \mathbf{t}, y)||G_s(\mathbf{t})|d\mathbf{t}$, we would not obtain the desired estimate. We must try to reduce the "singularity" of $\frac{\partial}{\partial x_i} Q_s(\mathbf{u} - \mathbf{t}, y)$ in $\mathbf{t} = \mathbf{u}$, independently of y. We will use that by Proposition 3.4,

$$\int_{\mathbb{R}^n} \frac{\partial Q_s}{\partial x_i}(\mathbf{u} - \mathbf{t}, y)d\mathbf{t} = 0.$$

As before, let $K \leq \frac{k_1}{8}$. We have that

$$\begin{aligned}
&\frac{\partial}{\partial x_i} Q_s(G_s)(\mathbf{u}, y) \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} Q_s(\mathbf{u} - \mathbf{t}, y) \left(G_s(\mathbf{t}) - G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}\right) d\mathbf{t} \\
&= \int_{|\mathbf{u}-\mathbf{t}|+y \leq K} \frac{\partial}{\partial x_i} Q_s(\mathbf{u} - \mathbf{t}, y) \left(G_s(\mathbf{t}) - G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}\right) d\mathbf{t} \\
&+ \int_{|\mathbf{u}-\mathbf{t}|+y > K} \frac{\partial}{\partial x_i} Q_s(\mathbf{u} - \mathbf{t}, y) \left(G_s(\mathbf{t}) - G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}\right) d\mathbf{t} = C(\mathbf{u}, y) + D(\mathbf{u}, y).
\end{aligned} \tag{3.10}$$

We begin with the estimate of $C(\mathbf{u}, y)$. Now $|\mathbf{u}-\mathbf{t}|+y \leq K$. We have that $|\mathbf{u}-\mathbf{t}| \leq \frac{k_1}{8}$ and $y < \frac{k_1}{8}$. As before, $|\mathbf{u}| \geq \frac{7k_1}{8}$ and $|\mathbf{t}| \geq \frac{6}{8}k_1$. Hence, observing that $G_s(\mathbf{t}) \simeq \mathcal{G}_{s+1}(\mathbf{t}, 0)$, Proposition 3.1 (vii), gives that

$$\begin{aligned}
\left|G_s(\mathbf{t}) - G_s(\mathbf{u})\chi_{|\mathbf{u}|>\frac{k_1}{8}}\right| &= |G_s(\mathbf{t}) - G_s(\mathbf{u})| \leq |\mathbf{u} - \mathbf{t}| \sup_{\mathbf{v}=\mathbf{u}+\rho(\mathbf{t}-\mathbf{u});\, 0\leq\rho\leq 1} |\nabla \mathcal{G}_{s+1}(\mathbf{v}, 0)| \\
&\lesssim |\mathbf{u} - \mathbf{t}| \sup_{\mathbf{v}=\mathbf{u}+\rho(\mathbf{t}-\mathbf{u});\, 0\leq\rho\leq 1} \left(\frac{e^{-|\mathbf{v}|}}{|\mathbf{v}|^{\frac{n+1-s}{2}}} + \frac{e^{-|\mathbf{v}|}}{|\mathbf{v}|^{\frac{n+1-s-1}{2}}} \right).
\end{aligned}$$

Since $|\mathbf{v}| \geq |\mathbf{u}| - \rho|\mathbf{t} - \mathbf{u}| \geq \frac{7k_1}{8} - \frac{k_1}{8} = \frac{6k_1}{8}$ we have that

$$\left|G_s(\mathbf{t}) - G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}\right| \lesssim |\mathbf{u} - \mathbf{t}|e^{-|\mathbf{u}|}.$$

Since y is bounded, we have that $e^{\frac{y}{\sqrt{2}}} \simeq 1$. Hence

$$C(\mathbf{u}, y) \lesssim \int_{\mathbb{R}^n} \frac{y^{2s}}{(|\mathbf{u} - \mathbf{t}| + y)^{n+2s}} e^{-|\mathbf{u}|}d\mathbf{t} \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}.$$

In order to estimate $D(\mathbf{u},y)$, we use Proposition 3.1, (iv) and (v) for $c>0$ such that $1/\sqrt{2}<c<1$. We have

$$\begin{aligned}
&\left|\int_{|\mathbf{u}-\mathbf{t}|+y\geq K} \frac{\partial}{\partial x_i} Q_s(\mathbf{u}-\mathbf{t},y)\left(G_s(\mathbf{t})-G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}\right)d\mathbf{t}\right| \\
&\lesssim \int_{|\mathbf{u}-\mathbf{t}|+y\geq K} e^{-(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac12}}\left(G_s(\mathbf{t})+G_s(\mathbf{u})\chi_{|\mathbf{u}|\geq k_1/8}\right)d\mathbf{t} \\
&\lesssim \int_{|\mathbf{u}-\mathbf{t}|+y\geq K} e^{-\frac{|\mathbf{u}-\mathbf{t}|+y}{\sqrt{2}}}\left(\frac{e^{-c|\mathbf{t}|}}{|\mathbf{t}|^{n-s}}+e^{-c|\mathbf{u}|}\right)d\mathbf{t} \\
&\lesssim e^{-\frac{(|\mathbf{u}|+y)}{\sqrt{2}}}\left(\int_{\mathbb{R}^n}\frac{e^{-(c-\frac{1}{\sqrt{2}})|\mathbf{t}|}}{|\mathbf{t}|^{n-s}}d\mathbf{t}+\int_{\mathbb{R}^n} e^{-\frac{|\mathbf{u}-\mathbf{t}|}{\sqrt{2}}}d\mathbf{t}\right)\lesssim e^{-\frac{(|\mathbf{u}|+y)}{\sqrt{2}}}.
\end{aligned}$$

We next deal with the derivative with respect to y. In the corresponding step in (3.10) when we now reduce the "singularity" of $\frac{\partial}{\partial y}Q_s(\mathbf{u}-\mathbf{t},y)$, it appears a suplementary term, that by Propositions 3.1 (v) and 3.4, satisfies

$$\left|\int_{\mathbb{R}^n}\frac{\partial}{\partial y}Q_s(\mathbf{u}-\mathbf{t},y)G_s(\mathbf{u})\chi_{|\mathbf{u}|>k_1/8}d\mathbf{t}\right|\lesssim h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}.$$

Arguing as in (3.10), we observe that the terms when we derivate with respect to y the denominator of P_s or $\mathcal{G}_{2n+2s+1}$, are completely analogous to the derivatives with respecto to x_i. So, in order to finish the proof of the proposition, we need to estimate the derivative with respect to y of the numerator of P_s. We then have to estimate

$$\frac{1}{y^{1-2s}}\int_{\mathbb{R}^n}\frac{1}{(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{n+2s}{2}}}\mathcal{G}_{2n+2s+1}((|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac12})|G_s(\mathbf{t})-G_s(\mathbf{u})\chi_{|\mathbf{u}|>k/8}|d\mathbf{t}.$$

Spliting again the integral in two terms, when $|\mathbf{u}-\mathbf{t}|+y\leq K$ and when $|\mathbf{u}-\mathbf{t}|+y>K$, and denoting each integral by $C_1(\mathbf{u},y)$ and $D_1(\mathbf{u},y)$ respectively, we have that

$$C_1(\mathbf{u},y)\lesssim\frac{1}{y^{1-2s}}\int_{|\mathbf{u}-\mathbf{t}|+y\leq K}\frac{1}{(|\mathbf{u}-\mathbf{t}|+y)^{n+2s}}|G_s(\mathbf{t})-G_s(\mathbf{u})\chi_{|\mathbf{u}|>k/8}|d\mathbf{t},$$

that arguing as before is bounded by $h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}$.

Next,

$$D_1(\mathbf{u},y)\lesssim\frac{1}{y^{1-2s}}\int_{|\mathbf{u}-\mathbf{t}|+y\geq K}e^{-(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac12}}\left(\frac{e^{-c|\mathbf{t}|}}{|\mathbf{t}|^{n-s}}+e^{-c|\mathbf{u}|}\right)d\mathbf{t},$$

that with analogous arguments, is bounded by $h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{u}|+y)}$.

□

Proposition 3.6. *Let $0<s<1$ and let μ be a nonnegative Borel measure. For each $\lambda>1$, there exists $k>0$ such that if $|\mathbf{x}|+y\geq k$, then*

$$|Q_s(G_{2s}(\mu))(\mathbf{x},y)|\gtrsim e^{-\lambda(|\mathbf{x}|+y)}\mu(B(\mathbf{0},1)).$$

Proof. We consider first the case $n-2s>0$. Observe that since we are assuming that $s<1$, $n-2s\leq 0$ is only possible when $n=1$. Let $|\mathbf{u}|+y\geq k>2$.

If $y \geq k/4$, then using (v) in Proposition 3.1 and Lemma 3.2, we have

$$|Q_s(G_{2s})(\mathbf{u},y)| \gtrsim \int_{|\mathbf{t}|\leq 1} \frac{y^{2s}(|\mathbf{u}-\mathbf{t}|+y)^{\frac{2n+2s-n-1}{2}}}{(|\mathbf{u}-\mathbf{t}|+y)^{n+2s}} e^{-(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}} \frac{d\mathbf{t}}{|\mathbf{t}|^{n-2s}}$$
$$= \int_{|\mathbf{t}|\leq 1} \frac{y^{2s}}{(|\mathbf{u}-\mathbf{t}|+y)^{\frac{n+2s+1}{2}}} e^{-(|\mathbf{u}-\mathbf{t}|^2+y^2)^{\frac{1}{2}}} \frac{d\mathbf{t}}{|\mathbf{t}|^{n-2s}}$$
$$\simeq \frac{y^{2s}}{(|\mathbf{u}|+y)^{\frac{n+2s+1}{2}}} e^{-(|\mathbf{u}|^2+y^2)^{\frac{1}{2}}} \int_{|\mathbf{t}|\leq 1} \frac{d\mathbf{t}}{|\mathbf{t}|^{n-2s}} \gtrsim e^{-\lambda(|\mathbf{u}|^2+y^2)^{\frac{1}{2}}} \gtrsim e^{-\lambda(|\mathbf{u}|+y)},$$

for any $\lambda > 1$.

Assume now that $y < k/4$. Since $|\mathbf{u}| + y \geq k$, we have that $|\mathbf{u}| > 3k/4$. We also have that if $|\mathbf{u}-\mathbf{t}| < y$, then $|\mathbf{u}-\mathbf{t}|+y \lesssim y < k/4$ and consequently, $\mathcal{G}_{2n+2s+1}((|\mathbf{u}-\mathbf{t}|+y)^{\frac{1}{2}}) \simeq 1$. Hence,

$$|Q_s(G_{2s})(\mathbf{u},y)| \gtrsim \int_{|\mathbf{u}-\mathbf{t}|\leq y} \frac{y^{2s}}{y^{n+2s}} G_{2s}(\mathbf{t})d\mathbf{t}.$$

But, we observe that in this region $|\mathbf{t}| \simeq |\mathbf{u}|$. Indeed,

$$|\mathbf{t}| \geq |\mathbf{u}| - |\mathbf{u}-\mathbf{t}| \geq |\mathbf{u}| - y \geq |\mathbf{u}| - k/4 \geq |\mathbf{u}| - \frac{|\mathbf{u}|}{3} = \frac{2|\mathbf{u}|}{3}.$$

Conversely,

$$|\mathbf{t}| \leq |\mathbf{u}| + |\mathbf{u}-\mathbf{t}| \leq |\mathbf{u}| + y \leq |\mathbf{u}| + k/4 \leq (1+1/3)|\mathbf{u}|.$$

Note that, in particular, $|\mathbf{t}| \leq |\mathbf{u}| + y$ and $|\mathbf{t}| \geq \frac{2}{3}\frac{3k}{4}$, then by (v) in Proposition 3.1

$$|Q_s(G_{2s})(\mathbf{u},y)| \gtrsim \int_{|\mathbf{u}-\mathbf{t}|\leq y} \frac{y^{2s}}{y^{n+2s}} |\mathbf{u}|^{\frac{-n-1+2s}{2}} e^{-(|\mathbf{u}|+y)} d\mathbf{t}$$
$$\gtrsim \frac{e^{-(|\mathbf{u}|+y)}}{|\mathbf{u}|^{\frac{n+1-2s}{2}}} \gtrsim e^{-\lambda(|\mathbf{u}|+y)},$$

for any $\lambda > 1$.

Hence, in any case,

$$|Q_s(G_{2s})(\mathbf{u},y)| \gtrsim e^{-\lambda(|\mathbf{u}|+y)},$$

and, consequently,

$$|Q_s(G_{2s}(\mu))(\mathbf{x},y)| \gtrsim \left| \int_{B(\mathbf{0},1)} e^{-\lambda(|\mathbf{x}-\mathbf{t}|+y)} d\mu(t) \right| \gtrsim e^{-\lambda(|\mathbf{x}|+y)} \mu(B(\mathbf{0},1)),$$

where we have used Lemma 3.2 since $|\mathbf{t}| \leq 1$.

The remaining case $1 - 2s \leq 0$ is proved with minor changes using the corresponding estimates in Proposition 3.1, (iv). □

4. Capacities, trace measures and potentials of equilibrium measures

Definition 4.1. *Let $E \subset \mathbb{R}^n$. The Bessel capacity of E is defined by*

$$Cap_s(E) := \inf\{\|f\|^2_{L^2(\mathbb{R}^n)} : G_s(|f|) \geq 1 \text{ on } E\}.$$

We list some properties of the equilibrium measure for a compact set in $\mathbb{R}^n$, which will be used below and that can be found, for instance, in [1], Thm. 2.2.7.

Theorem 4.2. *Given a compact set $E \subset \mathbb{R}^n$, there exists a positive measure ν_E on $\mathbb{R}^n$ (that is called capacitary or equilibrium measure), such that:*

(i) *ν_E is supported on E and $\nu_E(E) = Cap_s(E)$.*
(ii) *$p_E := G_{2s}(\nu_E) \geq 1$ a.e. on E.*
(iii) *$p_E \in H^2_s(\mathbb{R}^n)$ and $\|p_E\|^2_{H^2_s(\mathbb{R}^n)} \lesssim Cap_s(E)$.*
(iv) *There is a constant $C > 0$ independent of E, such that $p_E(x) \leq C$ for any $x \in \mathbb{R}^n$.*

The following result is well known (see Theorem 3.1.4 and Remark 3.1.1 in [12]) and gives a characterization of the positive trace measures on the space of Bessel potentials.

Proposition 4.3. *Let μ be a positive Borel measure on $\mathbb{R}^n$. Then, μ is a trace measure for $H^2_s(\mathbb{R}^n)$, that is, $\int_{\mathbb{R}^n} |f|^2 d\mu \lesssim \|f\|^2_{H^2_s(\mathbb{R}^n)}$ for every $f \in H^2_s(\mathbb{R}^n)$, if and only if there exists $C_\mu > 0$ such that for any compact set E of diameter less or equal to 1, $\mu(E) \leq C_\mu Cap_s(E)$.*

Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a $\mathcal{C}^\infty$ radial function, nonincreasing in $|\mathbf{x}|$, with support on $B(\mathbf{0}, 1)$ and such that $\int \varphi = 1$. For $\delta > 0$, let $\varphi_\delta(\mathbf{x}) = \frac{1}{\delta}\varphi(\mathbf{x}/\delta)$. We write $\nu_{E,\delta} = \nu_E * \varphi_\delta$, the regularizations of the measure ν_E. We then have that $\nu_{E,\delta}$ are functions in $\mathcal{D}$ satisfying that $d\nu_{E,\delta} := \nu_{E,\delta}\, d\mathbf{x} \to \nu_E$ in the sense of distributions and such that $\|\nu_{E,\delta}\|_1 = Cap_s(E)$.

We denote by

$$p_{E,\delta} := G_{2s} * \nu_{E,\delta}, \quad \delta > 0.$$

As a corollary of Proposition 3.5 we have that

Proposition 4.4. *Let ν_E be the equilibrium measure of E and $0 < s < 1$. If $|\mathbf{x}| + y > k > 1$, then*

(i) $|Q_s(p_E)(\mathbf{x}, y)| \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)}\nu_E(E)$.
(ii) $|\nabla Q_s(p_E)(\mathbf{x}, y)| \lesssim h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)}\nu_E(E)$.

We also have that if $\delta > 0$, then

(i) $|Q_s(p_{E,\delta})(\mathbf{x}, y)| \lesssim e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)}\nu_E(E)$.
(ii) $|\nabla Q_s(p_{E,\delta})(\mathbf{x}, y)| \lesssim h_s(y)e^{-\frac{1}{\sqrt{2}}(|\mathbf{x}|+y)}\nu_E(E)$,

with constants independent of $\delta > 0$.

We will need the following result:

Proposition 4.5 ([11], Chapter 2, Lemma 3)**.** *If $2s < n$ and $\beta \in (1, \frac{n}{n-2s})$, ν a nonnegative Borel measure on $\mathbb{R}$. We then have that if $I_{2s}(\nu)$ is the Riesz potential of the measure ν, then $I_{2s}(\nu)^\beta$ is in the Muckenhoupt class A_1, with A_1-constant independent of ν. For $n = 1$ and $s = \frac{1}{2}$, we must replace I_{2s} by the logarithmic potential.*

Remark 4.6. *We observe that if we instead consider the Bessel potential of a nonnegative measure, proposition 4.5 is no longer true. In fact G^β_{2s} is not even a doubling weight for any $\beta > 0$, since the exponential function is not a doubling weight. For instance, if $n = 1$, $G_{2s}(\mathbf{x}) \simeq \frac{e^{-|x|}}{|x|^{\frac{2-2s}{2}}}$, when $|x| \to \infty$. Then, for any $R > 0$*

$$\left(\int_R^{5R} \frac{e^{-\beta x}}{x^{\beta(1-s)}}dx\right)\left(\int_{2R}^{4R} \frac{e^{-\beta x}}{x^{\beta(1-s)}}dx\right)^{-1} \simeq \frac{e^{-\beta R} - e^{-\beta 5R}}{e^{-\beta 2R} - e^{-\beta 4R}} \to \infty,$$

when $R \to \infty$.

Theorem 4.7. *Let $E\subset\mathbb{R}^n$ be a compact set, ν_E its capacitary measure and let $p_E=G_{2s}(\nu_E)$ be its capacitary or equilibrium potential and $p_{E,\delta}$ its regularization Then, if $\alpha>1/2$, $p_{E,\delta}^\alpha\in H^2_s(\mathbb{R}^n)$ and $\|p_{E,\delta}^\alpha\|^2_{H^2_s(\mathbb{R}^n)}\lesssim Cap_s(E)$.*

Proof. It is enough to show (see Proposition 2.1) that

$$\int_{\mathbb{R}^{n+1}_+}|\nabla(Q_s(p_{E,\delta}))^\alpha|^2y^{1-2s}d\mathbf{x}dy+\int_{\mathbb{R}^{n+1}_+}|(Q_s(p_{E,\delta}))^\alpha|^2y^{1-2s}d\mathbf{x}dy\lesssim Cap_s(E).$$

In order to estimate the first integral over $\mathbb{R}^{n+1}_+$, we apply Stoke's Theorem to the following domain and form. Let $\eta>0$ and $R>0$ and let $\Omega_{\eta,R}$ be the region in $\mathbb{R}^{n+1}_+$ defined by

$$\Omega_{\eta,R}=\{(\mathbf{x},y)\in\overline{B(\mathbf{0},R)}\,;\ y\ge\eta\}.$$

Let ω_δ be the form defined by

$$\omega_\delta=(Q_s(p_{E,\delta}))^{2\alpha-1}y^{1-2s}\left(\sum_{i=1}^n(-1)^{i+1}\frac{\partial Q_s(p_{E,\delta})}{\partial x_i}(\mathbf{x},y)dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\wedge dy\right.$$
$$\left.+(-1)^n\frac{\partial Q_s(p_{E,\delta})}{\partial y}(\mathbf{x},y)\,dx_1\wedge\cdots\wedge dx_n\right).$$

Since $div(y^{1-2s}\nabla Q_s(p_{E,\delta}))(\mathbf{x},y)=y^{1-2s}Q_s(p_{E,\delta})(\mathbf{x},y)$, we have

$$\begin{aligned}&d\omega(\mathbf{x},y)\\&=(2\alpha-1)(Q_s(p_{E,\delta}))^{2\alpha-2}(\mathbf{x},y)|\nabla Q_s(p_{E,\delta})(\mathbf{x},y)|^2y^{1-2s}d\mathbf{x}dy\\&+(Q_s(p_{E,\delta}))^{2\alpha}(\mathbf{x},y)y^{1-2s}d\mathbf{x}dy.\end{aligned}$$

To conclude the proof we need the following lemmas.

Lemma 4.8.

$$\lim_{R\to\infty}\int_{\partial\Omega_{\eta,R}}\omega_\delta=\int_{\mathbb{R}^n\times\{\eta\}}\omega_\delta.$$

Proof. By Proposition 4.4, we have that, provided $|\mathbf{x}|+y$ is big enough,

$$|(Q_s(p_{E,\delta}))^{2\alpha-1}y^{1-2s}|\nabla Q_s(p_{E,\delta})(\mathbf{x},y)|\lesssim y^{1-2s}h_s(y)e^{-\frac{((2\alpha-1)+1)(|\mathbf{x}|+y)}{\sqrt2}}\nu(E).$$

Hence, for $R\to\infty$,

$$\left|\int_{\partial\Omega_{\eta,R}\cap\{y>\eta\}}\omega_\delta\right|\lesssim R^{n+1}e^{-\frac{((2\alpha-1)+1)R}{\sqrt2}}\to0.$$

Analogously, $\int_{\partial\Omega_{\eta,R}\cap\{y=\eta\}}\omega_\delta\to\int_{\mathbb{R}^n\times\{\eta\}}\omega_\delta$ as $R\to\infty$. □

Lemma 4.9. *There exists a positive constant $C>0$ such that*

$$\lim_{\eta\to0}\int_{\mathbb{R}^n\times\{\eta\}}\omega_\delta=(-1)^nC\int_{\mathbb{R}^n}(p_{E,\delta})^{2\alpha-1}d\nu_{E,\delta}.$$

Proof. Assume that ν is supported in $B(\mathbf{0},k)$ and let $K=k+2$. We write $\mathbb{R}\times\{\eta\}=I_1\cup I_2$, where $I_1=\{|\mathbf{x}|\le K\}\times\{\eta\}$ and $I_2=\{|\mathbf{x}|>K\}\times\{\eta\}$.

We have that by Theorems 2.3 and 2.2, $\lim\limits_{\eta\to0}P_s(p_{E,\delta})(\mathbf{x},\eta)=p_{E,\delta}(\mathbf{x})$, and

$$\lim_{\eta\to0}\eta^{1-2s}\frac{\partial}{\partial y}Q_s(p_{E,\delta})(\mathbf{x})=-C(Id-\Delta)^sp_{E,\delta}(\mathbf{x})=-C\nu_{E,\delta}(\mathbf{x}),$$

where $C > 0$.

On I_1, Theorem 2.3 gives that $\lim_{\eta\to 0} Q_s(p_{E,\delta})(\mathbf{x},\eta) = p_{E,\delta}(\mathbf{x})$, uniformly on $B(\mathbf{0}, K)$ and by Remark 2.4,

$$(-1)^n (Q_s(p_{E,\delta}))^{2\alpha-1}\eta^{1-2s}\frac{\partial}{\partial y}Q_s(p_{E,\delta})(\mathbf{x},\eta) \to -(-1)^n C p_{E,\delta}^{2\alpha-1}\nu_{E,\delta}(\mathbf{x}),$$

when $\eta \to 0$, uniformly on $B(\mathbf{0}, K)$. We then have that

$$\lim_{\eta\to 0}\int_{I_1}\omega_\delta = (-1)^{n+1}C\int_{|\mathbf{x}|\le K}(p_{E,\delta})^{2\alpha-1}d\nu_{E,\delta} = (-1)^{n+1}C\int_{\mathbb{R}}(p_{E,\delta})^{2\alpha-1}d\nu_{E,\delta}.$$

If $x \in I_2$ and $0 < \eta < 1$,

$$|(Q_s(p_{E,\delta}))^{2\alpha-1}\eta^{1-2s}\frac{\partial}{\partial y}Q_s(p_{E,\delta})(\mathbf{x},\eta)| \lesssim e^{-\frac{((2\alpha-1)+1)(|\mathbf{x}|+\eta)}{\sqrt{2}}}.$$

And this last function is integrable on $|\mathbf{x}| \ge K$. In addition, since $\operatorname{supp}\nu_{E,\delta} \subset I_1$, we have that for $\mathbf{x} \in I_2$, $\lim_{\eta\to 0}\eta^{1-2s}\frac{\partial}{\partial y}Q_s(p_{E,\delta})(\mathbf{x},\eta) = 0$. Hence, Lebesgue's Dominated Convergence Theorem gives that

$$\lim_{\eta\to 0}\int_{I_2}\omega_\delta = 0.$$

This concludes the proof of Lemma 4.9. □

Now, we continue with the proof of Theorem 4.7, proving first that

$$\int_{\mathbb{R}^{n+1}_+}|\nabla_{\mathbf{x},y}(Q_s(p_{E,\delta})^\alpha)|^2 y^{1-2s}d\mathbf{x}dy + \int_{\mathbb{R}^{n+1}_+}|Q_s(p_{E,\delta})^\alpha|^2 y^{1-2s}d\mathbf{x}dy \lesssim \int_{\mathbb{R}^n}(p_{E,\delta})^{2\alpha-1}d\nu_\delta.$$

Using Lebesgue's Monotone Convergence Theorem, Stoke's Theorem, and Lemmas 4.8 and 4.9, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+}|\nabla_{\mathbf{x},y}(Q_s(p_{E,\delta})^\alpha)|^2 y^{1-2s}d\mathbf{x}dy + \int_{\mathbb{R}^{n+1}_+}|Q_s(p_{E,\delta})^\alpha|^2 y^{1-2s}d\mathbf{x}dy\\
&= \lim_{\eta\to 0, R\to\infty}\int_{\Omega_{\eta,R}}\left(|\nabla_{\mathbf{x},y}(Q_s(p_{E,\delta})^\alpha)|^2 + |Q_s(p_{E,\delta})^\alpha|^2\right)y^{1-2s}d\mathbf{x}dy\\
&\lesssim \lim_{\eta\to 0}\lim_{R\to\infty}\int_{\partial\Omega_{\eta,R}}\omega_\delta = C\int_{\mathbb{R}^n}(p_{E,\delta})^{2\alpha-1}d\nu_{E,\delta}.
\end{aligned}$$

Using that $p_{E,\delta} \lesssim 1$, and that $2\alpha > 1$, we deduce that this integral is bounded, up to a constant by

$$\int_{\mathbb{R}^n}\nu_{E,\delta} = Cap_s(E).$$

□

5. Weighted L^p-estimates for an area function

Let $\mathbf{K} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$ be a vector-valued kernel (in fact we will consider only two type of kernels, one a vector-valued kernel in $\mathbb{R}^{n+1}$ and the other a scalar one). This kernel defines a vector valued operator from functions on $\mathbb{R}^n$ to vector-valued functions on $\mathbb{R}^n \times [0,\infty)$ given by

$$\mathbf{K}(f)(\mathbf{x},y) = \int_{\mathbb{R}^n}\mathbf{K}(\mathbf{x}-\mathbf{z},y)f(\mathbf{z})d\mathbf{z}.$$

The area function associated to $\mathbf{K}$ is

$$A_{\mathbf{K}}(f)(\mathbf{t}) := \left(\int_{\Gamma(\mathbf{t})} |\mathbf{K}(f)(\mathbf{x},y)|^2 \frac{d\mathbf{x}d\mathbf{y}}{y^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma(\mathbf{t}) = \{(\mathbf{x},y) \in \mathbb{R}^{n+1}_+ \,;\, |\mathbf{x}-\mathbf{t}| < y\}$ is the cone with vertex $\mathbf{t}$.

We will need a result on a weighted L^2- estimate for the area function associated to some convenient kernels.

We will work with a class of vector-valued kernels that with a little abuse of notation we will denote by $A_{\mathbf{K}}$-Calderón-Zygmund type kernels, defined by:

Definition 5.1. *A vector-valued kernel $\mathbf{K} : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}^m$ is an $A_{\mathbf{K}}$-Calderón-Zygmund type kernel if it satisfies that there exist constants $\varepsilon, \eta > 0$ such that:*

(i) $\|A_{\mathbf{K}}(f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$.
(ii) $|\mathbf{K}(\mathbf{x},y)| \lesssim \frac{y^\eta}{(|\mathbf{x}|^2+y^2)^{\frac{n+\eta}{2}}}$.
(iii) *For* $|\mathbf{x}-\widetilde{\mathbf{x}}| \leq \varepsilon(|\mathbf{x}|^2+y^2)^{\frac{1}{2}}$,

$$|\mathbf{K}(\mathbf{x},y)) - \mathbf{K}(\widetilde{\mathbf{x}},y)| \lesssim \frac{y^\eta |\mathbf{x}-\widetilde{\mathbf{x}}|^\eta}{(|\mathbf{x}|^2+y^2)^{\frac{n+2\eta}{2}}}.$$

For a function g defined on $\mathbb{R}^n \times [0,\infty)$, we define

$$\|g\|^2_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} := \int_{\Gamma(\mathbf{0})} |g(\mathbf{u},\mathbf{y})|^2 \frac{d\mathbf{u}dy}{y^{n+1}}$$

Theorem 5.2. *Let $\mathbf{K} : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}^{n+1}$ be an $A_{\mathbf{K}}$-Calderón-Zygmund type kernel. Then, for any $\omega \in A_p(\mathbb{R}^n)$, $p > 1$, we have that,*

$$\|A_{\mathbf{K}}(f)\|_{L^p(\omega d\mathbf{x})} \lesssim \|f\|_{L^p(\omega d\mathbf{x})}.$$

The proof is based in the ideas in [5], where the authors obtain a weighted L^p-estimate for a Littlewood-Paley square function associated to a function $\varphi \in \mathcal{S}$ with zero integral. We also use some results in [9]. We recall that the Fefferman-Stein sharp function $f^\sharp$ is given by

$$f^\sharp(\mathbf{x}) = \sup_{\mathbf{x}\in Q} \inf_c \frac{1}{|Q|} \int_Q |f(\mathbf{y}) - c| d\mathbf{y}.$$

If $0 < \delta < 1$, we write $f^\sharp_\delta(\mathbf{x}) = \sup_{\mathbf{x}\in Q} \inf_c \left(\frac{1}{|Q|}\int_Q |f(\mathbf{y}) - c|^\delta d\mathbf{y}\right)^{\frac{1}{\delta}}$. Applying [9], Theorem 3.1 together with (4.1) in the same paper, we have that:

Theorem 5.3 ([9])**.** *Let $1 < p < \infty$. For any $\omega \in A_p$, $\delta < 1$ and for any compactly supported function f, integrable on $\mathbb{R}^n$, we have*

$$\|Mf\|_{L^p(\omega d\mathbf{x})} \lesssim \||f|^\sharp_\delta\|_{L^p(\omega d\mathbf{x})},$$

where Mf is the usual Hardy-Littlewood maximal function and the constants depend only on p, n and the weight ω. In particular,

$$\|f\|_{L^p(\omega d\mathbf{x})} \lesssim \||f|^\sharp_\delta\|_{L^p(\omega d\mathbf{x})} \tag{5.11}$$

5.1. **Proof of Theorem 5.2.** If we denote $\mathbf{K}(f)_{\mathbf{t}}((\mathbf{x},y)) := \mathbf{K}(f)((\mathbf{x}+\mathbf{t},y))$, we have

$$\begin{aligned} A_{\mathbf{K}}(f)(\mathbf{t}) &= \left(\int_{\Gamma(\mathbf{t})} |\mathbf{K}(f)((\mathbf{x},y))|^2 \frac{d\mathbf{x}dy}{y^{n+1}}\right)^{\frac12} \\ &= \left(\int_{\Gamma(\mathbf{0})} |\mathbf{K}(f)((\mathbf{x}+\mathbf{t},y))|^2 \frac{d\mathbf{x}dy}{y^{n+1}}\right)^{\frac12} = \left(\int_{\Gamma(\mathbf{0})} |\mathbf{K}(f)_{\mathbf{t}}((\mathbf{x},y))|^2 \frac{d\mathbf{x}dy}{y^{n+1}}\right)^{\frac12}. \end{aligned}$$

We claim that it is enough to show that for $0<\delta<1$,

$$A_{\mathbf{K}}(f)^{\sharp}_{\delta} \lesssim M(f). \tag{5.12}$$

Postponing the proof of the claim, we first finish the proof of the theorem. By (5.11) and (5.12), we have that if $\omega \in A_p$,

$$\|A_{\mathbf{K}}(f)\|_{L^p(\omega)} \lesssim \|A_{\mathbf{K}}(f)^{\sharp}_{\delta}\|_{L^p(\omega)} \lesssim \|Mf\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)},$$

which gives the desired estimate.

So, we are left to prove (5.12), or equivalently, we will show that for any cube Q, with sides paralels to the coordinate axes, and centered at $\mathbf{t_0}$, there exists a constant c_Q such that

$$\left(\frac{1}{|Q|}\int_Q |A_{\mathbf{K}}(f)(\mathbf{t}) - c_Q|^{\delta} d\mathbf{t}\right)^{\frac{1}{\delta}} \lesssim Mf(\mathbf{t_0}).$$

We need a lemma that shows that $A_{\mathbf{K}}$ is of $(1,1)$-weak type and which is based in a well known technique of splitting functions of A.P. Calderon and A. Zygmund.

Lemma 5.4. *There exists $C>0$ such that for any $\lambda>0$, $f\in L^1(\mathbb{R}^n)$,*

$$|\{\mathbf{x}\in\mathbb{R}^n;\ A_{\mathbf{K}}(f)(\mathbf{x})>\lambda\}| \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}.$$

Proof. If $\lambda>0$ and $f\in L^1(\mathbb{R}^n)$, we denote $\Omega_\lambda = \{x\in\mathbb{R}^n;\ M(f)(\mathbf{x})>\lambda\}$.

Since the Hardy-Littlewood maximal operator is of weak type $(1,1)$, we have that

$$|\Omega_\lambda| \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} |f(\mathbf{x})|d\mathbf{x}.$$

We must then estimate

$$|\{\mathbf{x}\notin\Omega_\lambda;\ A_{\mathbf{K}}(f)(\mathbf{x})>\lambda\}|.$$

Let $(Q_k)_k$ be a Whitney decomposition of the set Ω_λ. We write $f=g+b$, where

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x}\notin\Omega_\lambda \\ \frac{1}{|Q_k|}\int_{Q_k} f(\mathbf{x})d\mathbf{x} & \mathbf{x}\in Q_k. \end{cases}$$

Since $(Q_k)_k$ is a Whitney decomposition of Ω_λ, there exists $R>0$ such that the dilated cube RQ satisfies $RQ\cap\Omega_\lambda^c\neq\emptyset$. We then have that $\|g\|_\infty \lesssim \lambda$. We denote $b_k = b\chi_{Q_k} = (f-f_{Q_k})\chi_{Q_k}$, where $f_{Q_k} = \dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} f(\mathbf{x})\mathbf{dx}$. We then have that b_k is supported in Q_k, $\int_{Q_k} b_k d\mathbf{x} = 0$ and $\|b_k\|_{L^1(\mathbb{R}^n)} \lesssim \int_{Q_k}|f|d\mathbf{x}$ and $b=\sum_k b_k$.

If we decompose

$$\begin{aligned} &|\{\mathbf{x}\notin\Omega_\lambda;\ A_{\mathbf{K}}(f)(\mathbf{x})>\lambda\}| \\ &\le |\{\mathbf{x}\notin\Omega_\lambda;\ A_{\mathbf{K}}(g)(\mathbf{x})>\lambda/2\}| + |\{\mathbf{x}\notin\Omega_\lambda;\ A_{\mathbf{K}}(b)(\mathbf{x})>\lambda/2\}| = \mathcal{A}+\mathcal{B}, \end{aligned}$$

Chebyshev's inequality, (i) in Definition 5.1 and the fact that $\|g\|_\infty \lesssim \lambda$ gives easily that

$$\mathcal{A} \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} |f(\mathbf{x})|d\mathbf{x},$$

We now estimate $\mathcal{B}$. Let $\Omega^* = \cup_k Q_k^*$, where Q_k^* is the cube with the same center that Q_k and such that $|Q_k^*| = R^n|Q_k|$. We then have that $|\Omega^*| \lesssim |\Omega|$ and

$$|\{\mathbf{x} \notin \Omega_\lambda;\, A_{\mathbf{K}}(b)(\mathbf{x}) > \lambda/2\}| \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\Omega^*} A_{\mathbf{K}}(b)(\mathbf{x})d\mathbf{x}$$

$$\leq \frac{1}{\lambda}\sum_k \int_{\mathbb{R}^n\setminus Q_k^*} A_{\mathbf{K}}(b_k)(\mathbf{x})d\mathbf{x}.$$

So, it is enough to prove that

$$\int_{\mathbb{R}^n\setminus Q_k^*} A_{\mathbf{K}}(b_k)(\mathbf{x})d\mathbf{x} \lesssim \|b_k\|_{L^1(\mathbb{R}^n)}.$$

Let $\mathbf{x} \in \mathbb{R}^n \setminus Q_k^*$. Denote by $\mathbf{x}^{\mathbf{k}}$ the center of Q_k, $k \geq 1$. The fact that for each $k \geq 1$, $\int_{\mathbb{R}^n} b_k = 0$, gives that

$$\mathbf{K}(b_k)(\mathbf{x}) = \int_{Q_k} \left(K(\mathbf{x}-\mathbf{u}, y) - K(\mathbf{x}-\mathbf{x}^{\mathbf{k}}, y)\right) b_k(\mathbf{u})d\mathbf{u}.$$

Next, observe that if we choose R big enough so that for any $\mathbf{u} \in Q_k$ and $\mathbf{x} \in \mathbb{R}^n \setminus Q_k^*$, we have that by (iii) in 5.1,

$$\left|K(\mathbf{x}-\mathbf{u}, y) - K(\mathbf{x}-\mathbf{x}^{\mathbf{k}}, y)\right| \lesssim \frac{|\mathbf{u}-\mathbf{x}^{\mathbf{k}}|^\eta y^\eta}{(|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + y)^{n+2\eta}}.$$

Thus, the above integral is bounded by

$$\int_{Q_k} \frac{|\mathbf{u}-\mathbf{x}^{\mathbf{k}}|^\eta y^\eta}{(|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + y)^{n+2\eta}} |b_k(\mathbf{u})|d\mathbf{u} \lesssim \frac{r_{Q_k}^\eta y^\eta}{(|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + y)^{n+2\eta}} \int_{Q_k} |b_k(\mathbf{u})|d\mathbf{u}.$$

Hence

$$A_{\mathbf{K}}(b_k)^2(\mathbf{t}) \lesssim r_{Q_k}^{2\eta}\|b_k\|^2_{L^1(\mathbb{R}^n)} \int_{\Gamma(\mathbf{t})} \frac{y^{2\eta}}{(|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + y)^{2n+4\eta}} \frac{d\mathbf{x}dy}{y^{n+1}}.$$

But if $|\mathbf{x}-\mathbf{t}| < y$, we have that $|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + y \geq |\mathbf{x}^{\mathbf{k}} - \mathbf{t}|$.

Consequently,

$$A_{\mathbf{K}}(b_k)^2(\mathbf{t}) \lesssim r_{Q_k}^{2\eta}\|b_k\|^2_{L^1(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \frac{d\mathbf{x}}{(|\mathbf{x}-\mathbf{x}^{\mathbf{k}}| + |\mathbf{x}^{\mathbf{k}}-\mathbf{t}|)^{2n+4\eta}|\mathbf{x}-\mathbf{t}|^{n-2\eta}}\right)$$

$$\lesssim \frac{r_{Q_k}^{2\eta}\|b_k\|^2_{L^1(\mathbb{R}^n)}}{|\mathbf{x}^{\mathbf{k}}-\mathbf{t}|^{2n+2\eta}}$$

and

$$\int_{\mathbb{R}^n\setminus Q_k^*} A_{\mathbf{K}}(b_k)(\mathbf{x})d\mathbf{x} \lesssim r_{Q_k}^\eta \|b_k\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus Q_k^*} \frac{d\mathbf{t}}{|\mathbf{x}^{\mathbf{k}}-\mathbf{t}|^{n+\eta}} \lesssim \|b_k\|_{L^1(\mathbb{R}^n)}.$$

□

We need a second lemma:

Lemma 5.5. *Let* $\mathbf{K} : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}^m$ *be an* $A_{\mathbf{K}}$*-Calderón-Zygmund type kernel. Then there exists* $0 < \eta$ *such that if* $|\mathbf{r}-\mathbf{t}| \le \varepsilon|\mathbf{r}|$,

$$\|\mathbf{K}(\mathbf{u}+\mathbf{t},y)-\mathbf{K}(\mathbf{u}+\mathbf{r},y)\|^2_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} \lesssim \frac{|\mathbf{r}-\mathbf{t}|^{2\eta}}{|\mathbf{r}|^{2n+2\eta}}.$$

Proof. Let $|\mathbf{r}-\mathbf{t}| \le \varepsilon|\mathbf{r}|$. We then have that for any $|\mathbf{u}|<y$, $|\mathbf{u}+\mathbf{r}-(\mathbf{u}+\mathbf{t})| = |\mathbf{r}-\mathbf{t}| \le \varepsilon|\mathbf{r}| \le \varepsilon(|\mathbf{u}+\mathbf{r}|+y)$. Since $\mathbf{K}$ is an $A_{\mathbf{K}}$-Calderón-Zygmund type kernel, we have that by (iii) in 5.1,

$$\|\mathbf{K}(\mathbf{u}+\mathbf{t},y)-\mathbf{K}(\mathbf{u}+\mathbf{r},y)\|^2_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} \lesssim \int_{|\mathbf{u}|<y} \frac{|\mathbf{t}-\mathbf{r}|^{2\eta}y^{2\eta}}{(|\mathbf{u}+\mathbf{r}|+y)^{2n+4\eta}}\frac{d\mathbf{u}dy}{y^{n+1}}.$$

It is easy to check that

$$\int_{|\mathbf{u}|<y} \frac{1}{(|\mathbf{u}+\mathbf{r}|+y)^{2n+4\eta}}\frac{d\mathbf{u}dy}{y^{n+1-2\eta}} \lesssim \frac{1}{|\mathbf{r}|^{2n+2\eta}}.$$

And that ends the lemma. □

Now we prove (5.12). Let $f = f_1 + f_2$, where $f_1 = f\chi_{Q^*}$ and where, as before, Q^* is the cube with the same center that Q and such that $|Q^*| = R^n|Q|$. If we denote as usual $g_Q = \frac{1}{|Q|}\int_Q g(\mathbf{t})d\mathbf{t}$, let

$$c_Q := \|\frac{1}{|Q|}\int_{\mathbf{r}\in Q}\left(\int_{\mathbf{z}\in\mathbb{R}^n\setminus Q^*}\mathbf{K}(\mathbf{u}+\mathbf{r}-\mathbf{z},y)f_2(\mathbf{z})d\mathbf{z}\right)d\mathbf{r}\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)},.$$

which we wil write briefly as $\|(\mathbf{K}(f_2))_{u+Q}\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)}$. We then have,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |A_{\mathbf{K}}(f)(\mathbf{t})-c_Q|^\delta d\mathbf{t}\right)^{\frac{1}{\delta}}\\
&=\left(\frac{1}{|Q|}\int_Q \left|\|\mathbf{K}(f)_{\mathbf{t}}\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} - \|(\mathbf{K}(f_2))_{u+Q}\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)}\right|^\delta d\mathbf{t}\right)^{\frac{1}{\delta}}\\
&\lesssim\left(\frac{1}{|Q|}\int_Q \|\mathbf{K}(f_1)_{\mathbf{t}}\|^\delta_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)}d\mathbf{t}\right)^{\frac{1}{\delta}}\\
&+\left(\frac{1}{|Q|}\int_Q \|\mathbf{K}(f_2)_{\mathbf{t}}-(\mathbf{K}(f_2))_{u+Q}\|^\delta_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)}d\mathbf{t}\right)^{\frac{1}{\delta}} = I + II.
\end{aligned}$$

The estimate of I follows from Kolmogorov's inequality, since by Lemma 5.4, $A_{\mathbf{K}}(f)$ is of weak-type $(1,1)$. We then have,

$$\frac{1}{|Q|}\int_Q \|(\mathbf{K}(f_1))_t\|^\delta_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} \lesssim \frac{1}{|Q|}\int_{\mathbb{R}^n}|A_{\mathbf{K}}(f_1)(\mathbf{t})|^\delta d\mathbf{t} \lesssim \left(\frac{1}{|Q^*|}\int_{Q^*}|f_1(\mathbf{t})|d\mathbf{t}\right)^\delta$$

and consequently,

$$I \lesssim \frac{1}{|Q^*|}\int_{Q^*}|f_1(\mathbf{t})|d\mathbf{t} \lesssim Mf(\mathbf{t}_0).$$

Let us now estimate II. We have that since $\delta < 1$

(5.13)

$$
\begin{aligned}
II &\le \frac{1}{|Q|}\int_{Q} \| \int_{\mathbf{z}\in\mathbb{R}^n\setminus Q^*} \left(\mathbf{K}(\mathbf{u}+\mathbf{t}-\mathbf{z},y) - \left(\frac{1}{|Q|}\int_{\mathbf{r}\in Q}\mathbf{K}(\mathbf{u}+\mathbf{r}-\mathbf{z},y)d\mathbf{r}\right)\right) f(\mathbf{z})d\mathbf{z}\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} d\mathbf{t} \\
&\le \frac{1}{|Q|}\int_{\mathbf{t}\in Q}\left\|\frac{1}{|Q|}\int_{\mathbf{z}\in\mathbb{R}^n\setminus Q^*}\int_{\mathbf{r}\in Q}|\mathbf{K}(\mathbf{u}+\mathbf{t}-\mathbf{z},y)-\mathbf{K}(\mathbf{u}+\mathbf{r}-\mathbf{z},y)|\,|f(\mathbf{z})|d\mathbf{z}d\mathbf{r}\right\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} d\mathbf{t} \\
&\le \frac{1}{|Q|^2}\int_{\mathbf{t}\in Q}\int_{\mathbf{z}\in\mathbb{R}^n\setminus Q^*}\int_{\mathbf{r}\in Q}\|\mathbf{K}(\mathbf{u}+\mathbf{t}-\mathbf{z},y)-\mathbf{K}(\mathbf{u}+\mathbf{r}-\mathbf{z},y)\|_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)}|f(\mathbf{z})|d\mathbf{r}d\mathbf{z}d\mathbf{t},
\end{aligned}
$$

where in the last estimate we have used Minkowski's inequality. Next, Lemma 5.5 gives that there exists $0 < \eta < 1$ such that

$$
\|\mathbf{K}(\mathbf{u}+\mathbf{t}-\mathbf{z},y)-\mathbf{K}(\mathbf{u}+\mathbf{r}-\mathbf{z},y)\|^2_{L^2\left(\Gamma(\mathbf{0}),\frac{d\mathbf{u}dy}{y^{n+1}}\right)} \lesssim \frac{|\mathbf{r}-\mathbf{t}|^{2\eta}}{|\mathbf{r}-\mathbf{z}|^{2n+2\eta}}
$$

Hence, (5.13) is bounded, up to a constant, by

$$
\frac{1}{|Q|^2}\int_{\mathbf{t}\in Q}\int_{\mathbf{z}\in\mathbb{R}^n\setminus Q^*}\int_{\mathbf{r}\in Q}\frac{|\mathbf{r}-\mathbf{t}|^{2\eta}}{|\mathbf{r}-\mathbf{z}|^{2n+2\eta}}|f(\mathbf{z})|d\mathbf{r}d\mathbf{z}d\mathbf{t} \le Mf(\mathbf{t_0}),
$$

where the last inequality follows easily by a standard discretization method.

6. The kernels $\mathbf{K}_s$ and $\mathbf{J}_s$

Let $\mathbf{K}_s(\mathbf{x},y)$ and $\mathbf{J}_s(\mathbf{x},y)$ be the kernels defined respectively by

$$
\mathbf{K}_s(\mathbf{x},y) = y^{1-s}\int_{\mathbb{R}^n}\nabla_{\mathbf{x},y}Q_s(\mathbf{x}-\mathbf{u},y)G_s(|\mathbf{u}|)d\mathbf{u},
$$

$$
\mathbf{J}_s(\mathbf{x},y) = y^{1-s}\int_{\mathbb{R}^n}Q_s(\mathbf{x}-\mathbf{u},y)G_s(|\mathbf{u}|)d\mathbf{u},
$$

and the associated integral operators $\mathbf{K}_s(f)(\mathbf{x},y) = \int_{\mathbb{R}^n}\mathbf{K}_s(\mathbf{x}-\mathbf{z},y)f(\mathbf{z})d\mathbf{z}$ and $\mathbf{J}_s(f)(\mathbf{x},y) = \int_{\mathbb{R}^n}\mathbf{J}_s(\mathbf{x}-\mathbf{z},y)f(\mathbf{z})d\mathbf{z}$ respectively.

Theorem 6.1. *Let $0 < s < 1$. Let $\mathbf{T}_s$ be either $\mathbf{K}_s$ or $\mathbf{J}_s$. We then have that for some $\eta > 0$,*

(i) $\|A_{\mathbf{T}_s}(f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$.

(ii) $|\mathbf{T}_s(\mathbf{x},y)| \lesssim \frac{y^\eta}{(|\mathbf{x}|^2+y^2)^{\frac{n+\eta}{2}}}$.

(iii) *If $|\mathbf{x}-\widetilde{\mathbf{x}}| \le \varepsilon(|\mathbf{x}|^2+y^2)^{\frac{1}{2}}$, for some $\varepsilon > 0$,*

$$
|\mathbf{T}_s(\mathbf{x},y)-\mathbf{T}_s(\widetilde{\mathbf{x}},y)| \lesssim \frac{y^\eta|\mathbf{x}-\widetilde{\mathbf{x}}|^\eta}{(|\mathbf{x}|^2+y^2)^{\frac{n+2\eta}{2}}}.
$$

Proof of (i).

We start with the kernel $\mathbf{K}_s$. Observe that Fubini's Theorem and Remark 2.6 give that

$$
\begin{aligned}
\|A_{\mathbf{K}_s}(f)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\Gamma(\mathbf{t})}|y^{1-s}\nabla Q_s(G_sf)(\mathbf{x},y)|^2\frac{d\mathbf{x}dy}{y^{n+1}}d\mathbf{t} \\
&= \int_{\mathbb{R}^{n+1}_+} y^{1-2s}|\nabla Q_s(G_sf)(\mathbf{x},y)|^2d\mathbf{x}dy \lesssim \|G_sf\|^2_{H^2_s(\mathbb{R}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

The estimate (i) for $\mathbf{T}_s = \mathbf{J}_s$ is analogously a consequence of Fubini's Theorem and Remark 2.6.

Proof of (ii).

First, we obtain the estimate for the kernel $\mathbf{K}_s$. We consider two cases: $|\mathbf{x}| \le y$ and $|\mathbf{x}| \ge y$.

We prove (ii) for $\mathbf{K}_s$ when $|\mathbf{x}| \le y$. We will check that

$$|\mathbf{K}_s(\mathbf{x}, y)| \lesssim \frac{1}{y^n} = \frac{y^s}{y^{n+s}} \simeq \frac{y^s}{(|\mathbf{x}|^2 + y^2)^{\frac{n+s}{2}}}.$$

We have that

$$\begin{aligned} &\mathbf{K}_s(\mathbf{x}, y) \\ &= y^{1-s} \int_{\mathbb{R}^n} \nabla_{\mathbf{x},y} \left(\frac{y^{2s}}{(|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{n+2s}{2}}} \right) \mathcal{G}_{2n+2s+1}((|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{1}{2}}) G_s(|\mathbf{u}|) d\mathbf{u} \\ &+ y^{1-s} \int_{\mathbb{R}^n} \frac{y^{2s}}{(|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{n+2s}{2}}} \nabla_{\mathbf{x},y} \left(\mathcal{G}_{2n+2s+1}((|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{1}{2}}) \right) G_s(|\mathbf{u}|) d\mathbf{u} \end{aligned}$$

We observe that by Proposition 3.1, $\mathcal{G}_{2n+2s+1}$ and $\nabla \mathcal{G}_{2n+2s+1}$ are bounded functions. Hence

$$|\mathbf{K}_s(\mathbf{x}, y)| \lesssim \int_{\mathbb{R}^n} \frac{y^s}{(|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{n+2s}{2}}} G_s(|\mathbf{u}|) d\mathbf{u}. \tag{6.14}$$

Next,

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{y^s}{(|\mathbf{x}-\mathbf{u}|^2 + y^2)^{\frac{n+2s}{2}}} G_s(|\mathbf{u}|) d\mathbf{u} \\ &\lesssim \int_{|\mathbf{x}-\mathbf{u}| \le 2y} \frac{y^s}{y^{n+2s}} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u} + \int_{|\mathbf{x}-\mathbf{u}| > 2y} \frac{y^s}{|\mathbf{x}-\mathbf{u}|^{n+2s}} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u} \lesssim \frac{1}{y^n}. \end{aligned} \tag{6.15}$$

Now, we prove (ii) for $\mathbf{K}_s$ when $y \le |\mathbf{x}|$. We will check that in this case,

$$|\mathbf{K}_s(\mathbf{x}, y)| \lesssim \frac{y^\eta}{|\mathbf{x}|^{n+\eta}},$$

for some $\eta > 0$. In order to avoid the "singularity" of the integral near $\mathbf{x}$, indepently of y, of the term $\frac{1}{(|\mathbf{x}-\mathbf{u}|+y)^{n+1+2s}}$, we write

$$\begin{aligned} \mathbf{K}_s(\mathbf{x}, y) &= y^{1-s} \int_{\mathbb{R}^n} \nabla_{\mathbf{x},y} Q_s(\mathbf{x}-\mathbf{u}, y) G_s(|\mathbf{x}|) d\mathbf{u} \\ &+ y^{1-s} \int_{\mathbb{R}^n} \nabla_{\mathbf{x},y} Q_s(\mathbf{x}-\mathbf{u}, y) \left(G_s(|\mathbf{u}|) - G_s(|\mathbf{x}|) \right) d\mathbf{u} = \mathbf{I}_1(\mathbf{x}, y) + \mathbf{I}_2(\mathbf{x}, y). \end{aligned}$$

We begin estimating $\mathbf{I}_1(\mathbf{x}, y)$. By Proposition 3.4, we have that

$$\left| \int_{\mathbb{R}^n} \nabla_{\mathbf{x},y} Q_s(\mathbf{x}, y) d\mathbf{x} \right| \lesssim h_s(y) e^{-\frac{1}{\sqrt{2}} y}.$$

Hence, for any $0 < c < 1$, $|\mathbf{I}_1(\mathbf{x}, y)| \lesssim y^{1-s} h_s(y) e^{-\frac{y}{\sqrt{2}}} \frac{e^{-c|\mathbf{x}|}}{|\mathbf{x}|^{n-s}} \lesssim \frac{y^s}{|\mathbf{x}|^{n+s}}$.

Now, we deal with $\mathbf{I}_2(\mathbf{x}, y)$. We recall that $y \le |\mathbf{x}|$.

We consider first the integral in $|\mathbf{x}-\mathbf{u}| \ge \frac{|\mathbf{x}|}{2}$. Here there is no singularity near $\mathbf{x}$, and it is enough to estimate separately each of the two integrals, corresponding to $G_s(|\mathbf{u}|)$ and

$G_s(|\mathbf{x}|)$ respectively. Indeed,

$$\left| y^{1-s} \int_{|\mathbf{x}-\mathbf{u}|\geq \frac{|\mathbf{x}|}{2}} \nabla_{\mathbf{x},y} Q_s(\mathbf{x}-\mathbf{u},y) G_s(|\mathbf{u}|) d\mathbf{u} \right|$$
$$\lesssim \int_{|\mathbf{x}-\mathbf{u}|\geq \frac{|\mathbf{x}|}{2}} \frac{y^s}{|\mathbf{x}-\mathbf{u}|^{n+2s}} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u} =$$
$$\int_{|\mathbf{x}-\mathbf{u}|\geq \frac{|\mathbf{x}|}{2},\, |\mathbf{u}|\leq 2|\mathbf{x}|} \frac{y^s}{|\mathbf{x}-\mathbf{u}|^{n+2s}} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u} + \int_{|\mathbf{x}-\mathbf{u}|\geq \frac{|\mathbf{x}|}{2},\, |\mathbf{u}|> 2|\mathbf{x}|} \frac{y^s}{|\mathbf{x}-\mathbf{u}|^{n+2s}} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u}$$
$$\lesssim \frac{y^s}{|\mathbf{x}|^{n+2s}} \int_{|\mathbf{u}|\leq 2|\mathbf{x}|} \frac{1}{|\mathbf{u}|^{n-s}} d\mathbf{u} + y^s \int_{|\mathbf{u}|> 2|\mathbf{x}|} \frac{1}{|\mathbf{u}|^{2n+s}} d\mathbf{u} \lesssim \frac{y^s}{|\mathbf{x}|^{n+s}}.$$

Next, consider the integral corresponding to $G_s(|\mathbf{x}|)$, if $0 < c < 1$

$$\left| y^{1-s} \int_{|\mathbf{x}-\mathbf{u}|\geq \frac{|\mathbf{x}|}{2}} \nabla_{\mathbf{x},y} Q_s(\mathbf{x}-\mathbf{u},y) G_s(|\mathbf{x}|) du \right|$$
$$\lesssim \frac{e^{-c|\mathbf{x}|}}{|\mathbf{x}|^{n-s}} \int_{|\mathbf{x}-\mathbf{u}|\geq |\mathbf{x}|/2} \frac{y^s}{(|\mathbf{x}-\mathbf{u}|^2+y^2)^{\frac{n+2s}{2}}} d\mathbf{u} \lesssim \frac{y^s}{|\mathbf{x}|^{n+s}}.$$

So, in order to finish the estimate of $\mathbf{I}_2(\mathbf{x},y)$, we are left to obtain the estimate when we integrate in $|\mathbf{x}-\mathbf{u}| \leq \frac{|\mathbf{x}|}{2}$. By the mean value theorem,

$$|G_s(|\mathbf{u}|) - G_s(|\mathbf{x}|)| \leq |\mathbf{x}-\mathbf{u}| \sup_{\mathbf{z}=\mathbf{x}+\lambda(\mathbf{u}-\mathbf{x});\, 0\leq\lambda\leq 1} |\nabla G_s(\mathbf{z})|\,.$$

Applying (vii) in Proposition 3.1, and using that since $|\mathbf{x}-\mathbf{u}| \leq \frac{|\mathbf{x}|}{2}$, we have that for any $\mathbf{z} = \mathbf{x}+\lambda(\mathbf{u}-\mathbf{x})$, $0 \leq \lambda \leq 1$, $|\mathbf{z}| \simeq |\mathbf{x}|$, we deduce that for some $c > 0$,

$$|\nabla G_s(\mathbf{z})| \lesssim \frac{1}{|\mathbf{x}|^{n+1-s}} e^{-c|\mathbf{x}|}.$$

Let $0 < \eta < \min(s, 1-s)$. We then have

$$\int_{|\mathbf{x}-\mathbf{u}|\leq \frac{|\mathbf{x}|}{2}} \frac{y^s}{(|\mathbf{x}-\mathbf{u}|^2+y^2)^{\frac{n+2s}{2}}} |G_s(|\mathbf{u}|) - G_s(|\mathbf{x}|)| \, d\mathbf{u}$$
$$\lesssim \frac{y^\eta}{|\mathbf{x}|^{n+1-s}} \int_{|\mathbf{x}-\mathbf{u}|\leq \frac{|\mathbf{x}|}{2}} \frac{d\mathbf{u}}{|\mathbf{x}-\mathbf{u}|^{n+2s-(s-\eta)-1}} \lesssim \frac{y^\eta}{|\mathbf{x}|^{n+\eta}},$$

Estimate (ii) for the kernel $\mathbf{J}_s$.

If $|\mathbf{x}| \leq y$, we have that since

$$|Q_s(\mathbf{x},y)| \lesssim \frac{y^{2s}}{(|\mathbf{x}|^2+y^2)^{\frac{n+2s}{2}}} e^{-cy},$$

$$|\mathbf{J}_s(\mathbf{x},y)| \lesssim \int_{\mathbb{R}^n} \frac{y^s}{(|\mathbf{x}-\mathbf{u}|^2+y^2)^{\frac{n+2s}{2}}} G_s(|\mathbf{u}|) d\mathbf{u}$$

and we can continue arguing as in (6.14).

If $|\mathbf{x}| \geq y$, an analogous argument to the one used for the derivatives of Q_s gives the desired estimate, using that now

$$\int_{\mathbb{R}^n} Q_s(\mathbf{x},y) dx \lesssim e^{-cy}$$

and taking into account that in comparison with the estimate of $\nabla_{\mathbf{x},y}Q_s$, in the estimate of Q_s appears in the numerator an extra y, that together with e^{-cy} gives that $ye^{-cy} \lesssim 1$. As before, this observation allows to reduce the arguments to estimate $\mathbf{J}_s$ to the ones used in the estimates of $\mathbf{K}_s$.

Proof of (iii).

Let's begin with the kernel $\mathbf{K}_s$. We must show that for $|\mathbf{x}-\widetilde{\mathbf{x}}| \leq \varepsilon(|\mathbf{x}|^2+y^2)^{\frac{1}{2}}$ and some $\eta>0$,

$$|\mathbf{K}_s(\mathbf{x},y)-\mathbf{K}_s(\widetilde{\mathbf{x}},y)| \lesssim \frac{y^\eta|\mathbf{x}-\widetilde{\mathbf{x}}|^\eta}{(|\mathbf{x}|^2+y^2)^{\frac{n+2\eta}{2}}}.$$

We consider separately, as before, the cases $|\mathbf{x}|\leq y$ and $|\mathbf{x}|>y$.

If $|\mathbf{x}|\leq y$, it is enough to see that for any $|\mathbf{x}-\widetilde{\mathbf{x}}|\leq \varepsilon y$, we have that

$$|\mathbf{K}_s(\mathbf{x},y)-\mathbf{K}_s(\widetilde{\mathbf{x}},y)| \lesssim \frac{|\mathbf{x}-\widetilde{\mathbf{x}}|^s}{y^{n+s}}.$$

Indeed,

$$\begin{aligned}
&\mathbf{K}_s(\mathbf{x},y)-\mathbf{K}_s(\widetilde{\mathbf{x}},y)\\
&= y^{1-s}\int_{\mathbb{R}^n}\{\nabla_{\mathbf{x},y}Q_s(\mathbf{x}-\mathbf{u},y)-\nabla_{\mathbf{x},y}Q_s(\widetilde{\mathbf{x}}-\mathbf{u},y)\}\,G_s(|\mathbf{u}|)du\\
&= y^{1-s}\int_{|\mathbf{x}-\mathbf{u}|\leq Cy}\{\nabla_{\mathbf{x},y}Q_s(\mathbf{x}-\mathbf{u},y)-\nabla_{\mathbf{x},y}Q_s(\widetilde{\mathbf{x}}-\mathbf{u},y)\}\,G_s(|\mathbf{u}|)d\mathbf{u}\\
&+ y^{1-s}\int_{|\mathbf{x}-\mathbf{u}|> Cy}\{\nabla_{\mathbf{x},y}Q_s(\mathbf{x}-\mathbf{u},y)-\nabla_{\mathbf{x},y}Q_s(\widetilde{\mathbf{x}}-\mathbf{u},y)\}\,G_s(|\mathbf{u}|)d\mathbf{u}\\
&= A_1(\mathbf{x},\widetilde{\mathbf{x}},y)+A_2(\mathbf{x},\widetilde{\mathbf{x}},y).
\end{aligned}$$

We begin with the estimate of $A_1(\mathbf{x},\widetilde{\mathbf{x}},y)$. Since $|\mathbf{x}-\widetilde{\mathbf{x}}|\leq\varepsilon y$ and $|\mathbf{x}-\mathbf{u}|\leq Cy$, we have that for any $\mathbf{z}=\lambda\mathbf{x}+(1-\lambda)\widetilde{\mathbf{x}}$, $|\mathbf{z}-\mathbf{u}|\lesssim y$ and, in particular, $|\mathbf{z}-\mathbf{u}|^2+y^2\simeq y^2$. On the other hand, if $|\mathbf{x}-\mathbf{u}|\leq Cy$, we have that $|\mathbf{u}|\leq|\mathbf{x}-\mathbf{u}|+|\mathbf{x}|\leq C_1y$ Hence, Mean Value Theorem applied to the functions $\mathbf{x}\to\nabla_{\mathbf{x},y}Q_s(\mathbf{x}-\mathbf{u},y)$ gives that

$$|A_1(\mathbf{x},\widetilde{\mathbf{x}},y)| \lesssim \frac{y^s}{y^{n+2s+1}}|\mathbf{x}-\widetilde{\mathbf{x}}|\int_{|\mathbf{u}|\leq C_1y}\frac{d\mathbf{u}}{|\mathbf{u}|^{n-s}} \lesssim \frac{y^s}{y^{n+2s+1}}|\mathbf{x}-\widetilde{\mathbf{x}}|y^s \lesssim \frac{|\mathbf{x}-\widetilde{\mathbf{x}}|^s}{y^{n+s}}.$$

Next, we estimate $A_2(\mathbf{x},\widetilde{\mathbf{x}},y)$. Since $|\mathbf{x}|\leq y$ and $|\mathbf{x}-\mathbf{u}|\geq Cy$, if we choose $C>0$ big enough, we deduce that $|\mathbf{u}|\geq C_1y$, with $C_1>1$. We also have since $|\mathbf{x}-\widetilde{\mathbf{x}}|\leq\varepsilon y$ that for any $\mathbf{z}=\lambda\mathbf{x}+(1-\lambda)\widetilde{\mathbf{x}}$, $0<\lambda<1$, $|\mathbf{z}-\mathbf{x}|\leq\varepsilon y$ and

$$|\mathbf{z}-\mathbf{u}|\geq|\mathbf{x}-\mathbf{u}|-|\mathbf{z}-\mathbf{x}|\geq|\mathbf{x}-\mathbf{u}|-\varepsilon y\geq(1-\frac{\varepsilon}{C})|\mathbf{x}-\mathbf{u}|.$$

Proceeding as in the previous case, we have that

$$\begin{aligned}
&|A_2(\mathbf{x},\widetilde{\mathbf{x}},y)|\\
&\lesssim \int_{|\mathbf{x}-\mathbf{u}|>Cy}\frac{y^s|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}-\mathbf{u}|^{n+2s+1}}\frac{1}{|\mathbf{u}|^{n-s}}d\mathbf{u} \lesssim \frac{y^s|\mathbf{x}-\widetilde{\mathbf{x}}|}{y^{n-s}}\int_{|\mathbf{x}-\mathbf{u}|>C_2y}\frac{1}{|\mathbf{x}-\mathbf{u}|^{n+2s+1}}d\mathbf{u}\\
&\simeq \frac{y^s|\mathbf{x}-\widetilde{\mathbf{x}}|}{y^{n-s+2s+1}} \lesssim \frac{|\mathbf{x}-\widetilde{\mathbf{x}}|^s}{y^{n+s}}.
\end{aligned}$$

Assume now that $|\mathbf{x}| \geq y$. We must show that for $|\mathbf{x}-\widetilde{\mathbf{x}}| \leq \varepsilon|\mathbf{x}|$ and some $0<\eta$,

$$|\mathbf{K}_s(\mathbf{x},y)-\mathbf{K}_s(\widetilde{\mathbf{x}},y)| \lesssim \frac{y^{\eta}|\mathbf{x}-\widetilde{\mathbf{x}}|^{\eta}}{|\mathbf{x}|^{n+2\eta}}.$$

The change of variables $\mathbf{u}=\mathbf{x}-\mathbf{v}$ and $\mathbf{u}=\widetilde{\mathbf{x}}-\mathbf{v}$ respectively, give that

$$|\mathbf{K}_s(\mathbf{x},y)-\mathbf{K}_s(\widetilde{\mathbf{x}},y)| = \left| y^{1-s}\int_{\mathbb{R}^n} \nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)\left\{G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)\right\}d\mathbf{v}\right|. \tag{6.16}$$

Since $|y^{1-s}\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)| \lesssim \frac{y^s}{(|\mathbf{v}|+y)^{n+2s}}$, and in order to avoid the eventual "singularity" when integrating with respect to $\mathbf{v}$ indepoendent of y, we bound the expression (6.16) by

$$\begin{aligned}
&y^{1-s}\left|\int_{\mathbb{R}^n}\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)\left(G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)\right)d\mathbf{v}\right|\\
&+y^{1-s}\left|\int_{\mathbb{R}^n}\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)\left\{(G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|))-(G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|))\right\}d\mathbf{v}\right|\\
&=B_1(\mathbf{x},\widetilde{\mathbf{x}},y)+B_2(\mathbf{x},\widetilde{\mathbf{x}},y).
\end{aligned}$$

Let us begin with the estimate of $B_1(\mathbf{x},\widetilde{\mathbf{x}},y)$. By the mean value theorem,

$$|G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)| \leq |\mathbf{x}-\widetilde{\mathbf{x}}| \sup_{\mathbf{z}=\lambda\mathbf{x}+(1-\lambda)\widetilde{\mathbf{x}};\,0\leq\lambda\leq 1} |\nabla G_s(\mathbf{z})\|\,.$$

But since $|\mathbf{x}-\widetilde{\mathbf{x}}|\leq\varepsilon|\mathbf{x}|$, we have that for any $\mathbf{z}=\lambda\mathbf{x}+(1-\lambda)\widetilde{\mathbf{x}}$, $0\leq\lambda\leq1$, $|\mathbf{z}|\simeq|\mathbf{x}|$. Hence, for some $c>0$,

$$|G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)| \lesssim \frac{|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+1-s}}e^{-c|\mathbf{x}|},$$

Applying Proposition 3.4 we have that

$$\begin{aligned}
&B_1(\mathbf{x},\widetilde{\mathbf{x}},y)\\
&\lesssim |G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)|y^{1-s}\left|\int_{\mathbb{R}^n}\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)d\mathbf{v}\right|\\
&\lesssim y^{1-s}h_s(y)\frac{e^{-\frac{1}{\sqrt{2}}y}|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+1-s}}e^{-c|\mathbf{x}|} \lesssim \frac{y^{\eta}|\mathbf{x}-\widetilde{\mathbf{x}}|^{\eta}}{|\mathbf{x}|^{n+2\eta}},
\end{aligned}$$

for some $\eta>0$.

Now we estimate $B_2(\mathbf{x},\widetilde{\mathbf{x}},y)$. In the following arguments we will see why we choose the next decomposition of the integral that defines $B_2(\mathbf{x},\widetilde{\mathbf{x}},y)$.

We recall that we are assuming that $|\mathbf{x}-\widetilde{\mathbf{x}}|\le\varepsilon|\mathbf{x}|$ and in addition $|\mathbf{x}|\ge y$. We write

$$\begin{aligned}
&|B_2(\mathbf{x},\widetilde{\mathbf{x}},y)|\\
&\le y^{1-s}\int_{|\mathbf{v}|\ge\frac12|\mathbf{x}|}|\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)|\,|G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)|\,d\mathbf{v}\\
&+y^{1-s}\int_{|\mathbf{v}|\ge 2|\mathbf{x}|}|\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)|\,|(G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|))|\,d\mathbf{v}\\
&+y^{1-s}\int_{\frac12|\mathbf{x}|\le|\mathbf{v}|<2|\mathbf{x}|}|\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)|\,|(G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|))|\,d\mathbf{v}\\
&+y^{1-s}\int_{|\mathbf{v}|<\frac12|\mathbf{x}|}|\nabla_{\mathbf{v},y}Q_s(\mathbf{v},y)|\,|(G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|))-(G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|))|\,d\mathbf{v}\\
&=B_{2,1}+B_{2,2}+B_{2,3}+B_{2,4}.
\end{aligned}$$

The estimates of $B_{2,1}$ and $B_{2,2}$ are similar. Applying the Mean Value Theorem in $B_{2,1}$,

$$|G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)|\lesssim\frac{|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n-s+1}}$$

and in $B_{2,2}$, for some $\mathbf{z}$ in the segment with extrems $\mathbf{x}$ and $\widetilde{\mathbf{x}}$,

$$|G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)|\lesssim\frac{|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{v}-\mathbf{z}|^{n-s+1}}\lesssim\frac{|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n-s+1}}.$$

We then have

$$B_{2,1}+B_{2,2}\lesssim y^s|\mathbf{x}-\widetilde{\mathbf{x}}|\frac{1}{|\mathbf{x}|^{n-s+1}}\int_{|\mathbf{v}|\ge\frac12|\mathbf{x}|}\frac{d\mathbf{v}}{|\mathbf{v}|^{n+2s}}\lesssim\frac{y^s|\mathbf{x}-\widetilde{\mathbf{x}}|^s}{|\mathbf{x}|^{n+2s}}.$$

Next we deal with the term B_{23}. If we just argue as in the case B_{21}, the intermediate point that appear when we apply the Mean Value Theorem to

$$G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)$$

is not easy to estimate when $\frac12|\mathbf{x}|\le|\mathbf{v}|<2|\mathbf{x}|$. For this reason, we will use a different argument, approximating the Bessel kernels by Riesz kernels, approximation that allows a more precise estimate.

We express $G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)$ as a sum of the difference between the corresponding Riesz kernels and the difference of the remaining parts. Due to the simmetry of the position of the points $\mathbf{x}$ and $\widetilde{\mathbf{x}}$, we will just consider the part of the integral where $|\mathbf{v}-\widetilde{\mathbf{x}}|\ge|\mathbf{v}-\mathbf{x}|$. We will assume from now on that we have this extra assumption. We have that there exist $a_s,b_s>0$ such that

$$\begin{aligned}
&G_s(\mathbf{v}-\mathbf{x})-G_s(\mathbf{v}-\widetilde{\mathbf{x}})=a_s\int_0^\infty\left(e^{-\pi\frac{|\mathbf{x}-\mathbf{v}|^2}{\delta}}-e^{-\pi\frac{|\widetilde{\mathbf{x}}-\mathbf{v}|^2}{\delta}}\right)\left(e^{-\frac{\delta}{4\pi}}-1\right)\delta^{\frac{-n+s}{2}}\frac{d\delta}{\delta}\\
&+b_s\left(\frac{1}{|\mathbf{x}-\mathbf{v}|^{n-s}}-\frac{1}{|\widetilde{\mathbf{x}}-\mathbf{v}|^{n-s}}\right)=D_1+D_2.
\end{aligned}$$

We begin with the estimate of D_1. We have that

$$\frac{\left|e^{-\frac{\delta}{4\pi}}-1\right|}{\delta}\lesssim\begin{cases}1, & \text{if }\delta\le1,\\ \frac{1}{\delta}, & \text{if }\delta\ge1.\end{cases}$$

We write

$$\begin{aligned}&\left|e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}-e^{-\pi\frac{|\widetilde{\mathbf{x}}-v|^2}{\delta}}\right|\\&=e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}\left(1-e^{-\frac{\pi}{\delta}(|\mathbf{v}-\widetilde{\mathbf{x}}|+|\mathbf{v}-\mathbf{x}|)(|\mathbf{v}-\widetilde{\mathbf{x}}|-|\mathbf{v}-\mathbf{x}|)}\right)\leq e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}\left(1-e^{-\frac{2\pi}{\delta}|\mathbf{v}-\widetilde{\mathbf{x}}||\mathbf{x}-\widetilde{\mathbf{x}}|}\right)\end{aligned}$$

Next, since $1-e^{-\lambda}\leq\lambda$ if $\lambda\geq 0$,

$$\left|e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}-e^{-\pi\frac{|\widetilde{\mathbf{x}}-\mathbf{v}|^2}{\delta}}\right|\lesssim e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}\frac{2\pi}{\delta}|\mathbf{v}-\widetilde{\mathbf{x}}||\mathbf{x}-\widetilde{\mathbf{x}}|.$$

Then

$$\begin{aligned}D_1&\lesssim|\mathbf{x}-\widetilde{\mathbf{x}}|\int_0^1 e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}|\mathbf{v}-\widetilde{\mathbf{x}}|\delta^{\frac{-n+s}{2}-1}d\delta\\&+|\mathbf{x}-\widetilde{\mathbf{x}}|\int_1^\infty e^{-\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}}|\mathbf{v}-\widetilde{\mathbf{x}}|\delta^{\frac{-n+s}{2}-2}d\delta.\end{aligned}$$

The change of variables $\lambda=\pi\frac{|\mathbf{v}-\mathbf{x}|^2}{\delta}$, $d\delta=-\frac{\pi}{\lambda^2}|\mathbf{v}-\mathbf{x}|^2d\lambda$ gives that the above coincides, up to a constant, with

$$\begin{aligned}D_1&\lesssim|\mathbf{x}-\widetilde{\mathbf{x}}||\mathbf{v}-\widetilde{\mathbf{x}}||\mathbf{v}-\mathbf{x}|^{-n+s}\int_{|\mathbf{v}-\mathbf{x}|^2\pi}^\infty e^{-\lambda}\lambda^{\frac{n-s}{2}}\frac{d\lambda}{\lambda}\\&+|\mathbf{x}-\widetilde{\mathbf{x}}||\mathbf{v}-\widetilde{\mathbf{x}}||\mathbf{v}-\mathbf{x}|^{-n+s-2}\int_0^{|\mathbf{v}-\mathbf{x}|^2\pi}e^{-\lambda}\lambda^{\frac{n-s}{2}+1}\frac{d\lambda}{\lambda}\\&\lesssim|\mathbf{x}-\widetilde{\mathbf{x}}||\mathbf{v}-\widetilde{\mathbf{x}}|\left(|\mathbf{v}-\mathbf{x}|^{-n+s}+1\right).\end{aligned}\tag{6.17}$$

For the estimate of D_2, we will use the following lemmas to estimate the integrals involving $\left(\frac{1}{|\mathbf{v}-\mathbf{x}|^{n-s}}-\frac{1}{|\widetilde{\mathbf{x}}-v|^{n-s}}\right)$.

Lemma 6.2. *Let $n\geq 1$ and $0<s<1$. We then have that if $|\widetilde{\mathbf{h}}|\geq|\mathbf{h}|$,*

$$\left|\frac{1}{|\mathbf{h}|^{n-s}}-\frac{1}{|\widetilde{\mathbf{h}}|^{n-s}}\right|\lesssim|\mathbf{h}-\widetilde{\mathbf{h}}|\frac{1}{|\widetilde{\mathbf{h}}||\mathbf{h}|^{n-s}}.$$

Proof. The case $n=1$ was proved in [4]. If $n\geq 2$, Mean Value Theorem gives that

$$\left|\frac{1}{|\mathbf{h}|^{n-s}}-\frac{1}{|\widetilde{\mathbf{h}}|^{n-s}}\right|=\frac{\left||\widetilde{\mathbf{h}}|^{n-s}-|\mathbf{h}|^{n-s}\right|}{|\mathbf{h}|^{n-s}|\widetilde{\mathbf{h}}|^{n-s}}\lesssim\sup_{\mathbf{z}=\lambda\mathbf{h}+(1-\lambda)\widetilde{\mathbf{h}};\,0\leq\lambda\leq1}\frac{|\mathbf{h}-\widetilde{\mathbf{h}}||\mathbf{z}|^{n-s-1}}{|\mathbf{h}|^{n-s}|\widetilde{\mathbf{h}}|^{n-s}}.$$

But any $\mathbf{z}=\lambda\mathbf{h}+(1-\lambda)\widetilde{\mathbf{h}}$, $0\leq\lambda\leq1$ satisfies that $|\mathbf{z}|\leq\sup\left(|\mathbf{h}|,|\widetilde{\mathbf{h}}|\right)=|\widetilde{\mathbf{h}}|$. Since $n-s-1>0$, we deduce that the above is bounded by

$$|\mathbf{h}-\widetilde{\mathbf{h}}|\frac{1}{|\widetilde{\mathbf{h}}||\mathbf{h}|^{n-s}}.$$

□

The estimate (6.17) together with Lemma 6.2 and that we are assuming that $|\mathbf{v}-\widetilde{\mathbf{x}}|\geq|\mathbf{v}-\mathbf{x}|$ give that

$$\begin{aligned}
&\int_{\frac{1}{2}|\mathbf{x}|\le|\mathbf{v}|\le 2|\mathbf{x}|;\,|\mathbf{v}-\widetilde{\mathbf{x}}|\ge|\mathbf{v}-\mathbf{x}|}(D_1+D_2)d\mathbf{v}\\
&\lesssim |\mathbf{x}-\widetilde{\mathbf{x}}|\int_{\frac{1}{2}|\mathbf{x}|\le|\mathbf{v}|\le 2|\mathbf{x}|;\,|\mathbf{v}-\widetilde{\mathbf{x}}|\ge|\mathbf{v}-\mathbf{x}|}\left(\frac{|\mathbf{v}-\widetilde{\mathbf{x}}|}{|\mathbf{v}-\mathbf{x}|^{n-s}}+|\mathbf{v}-\widetilde{\mathbf{x}}|+\frac{1}{|\mathbf{v}-\widetilde{\mathbf{x}}||\mathbf{v}-\mathbf{x}|^{n-s}}\right)d\mathbf{v}\\
&\lesssim |\mathbf{x}-\widetilde{\mathbf{x}}|\left(|\mathbf{x}|^{1+s}+|\mathbf{x}|^{n+1}+\frac{1}{|\mathbf{x}-\widetilde{\mathbf{x}}|^{1-s}}\right).
\end{aligned}$$

Then, for some $0<c<1$,

$$B_{23}\lesssim \frac{y^s e^{-c|\mathbf{x}|}|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+2s}}\left(|\mathbf{x}|^{1+s}+|\mathbf{x}|^{n+1}+\frac{1}{|\mathbf{x}-\widetilde{\mathbf{x}}|^{1-s}}\right)\lesssim \frac{y^s|\mathbf{x}-\widetilde{\mathbf{x}}|^s}{|\mathbf{x}|^{n+2s}}.$$

Now we estimate B_{24}. Here we are assuming that $|\mathbf{v}|\le\frac{1}{2}|\mathbf{x}|$, $|\mathbf{x}-\widetilde{\mathbf{x}}|\le\varepsilon|\mathbf{x}|$ and $|\mathbf{x}|\ge y$. Assume first that $|\mathbf{x}-\widetilde{\mathbf{x}}|\le|\mathbf{v}|$.

We apply the Mean Value Theorem to both $G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)$ and $G_s(|\mathbf{x}|)-G_s(|\widetilde{\mathbf{x}}|)$. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be intermediate points corresponding to the intervals of extreme points $\mathbf{v}-\mathbf{x}$ and $\mathbf{v}-\widetilde{\mathbf{x}}$ and $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ respectively. We then have that any point $\mathbf{w}$ in the interval joinning $\mathbf{z}_1$ and $\mathbf{z}_2$ will satisfy that $|\mathbf{w}|\simeq|\mathbf{x}|$ and $|\mathbf{z}_1-\mathbf{z}_2|\lesssim|\mathbf{x}-\widetilde{\mathbf{x}}|+|\mathbf{v}|\lesssim|\mathbf{v}|$. Applying again the Mean Value Theorem, we will get

$$B_{24}\lesssim\int_{|\mathbf{v}|\le\frac{1}{2}|\mathbf{x}|}\frac{y^s}{(|\mathbf{v}|+y)^{n+2s}}\frac{|\mathbf{v}||\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+2-s}}d\mathbf{v}.$$

We arrive to the same estimate if $|\mathbf{v}|\le|\mathbf{x}-\widetilde{\mathbf{x}}|$ applying first the Mean Value Theorem to both $G_s(|\mathbf{v}-\mathbf{x}|)-G_s(|\mathbf{x}|)$ and $G_s(|\mathbf{v}-\widetilde{\mathbf{x}}|)-G_s(|\widetilde{\mathbf{x}}|)$.

Next, let $0<\eta<s$ and $\eta<1-s$. We then have

$$B_{24}\lesssim\int_{|\mathbf{v}|\le\frac{1}{2}|\mathbf{x}|}\frac{y^\eta}{(|\mathbf{v}|+y)^{n+s+\eta-1}}\frac{|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+2-s}}d\mathbf{v}\lesssim\frac{y^\eta|\mathbf{x}|^{1-s-\eta}|\mathbf{x}-\widetilde{\mathbf{x}}|}{|\mathbf{x}|^{n+2-s}}\lesssim\frac{y^\eta|\mathbf{x}-\widetilde{\mathbf{x}}|^\eta}{|\mathbf{x}|^{n+2\eta}}.$$

We finish with the proof of estimate (iii) for $\mathbf{J}_s$. We also consider the cases $|\mathbf{x}|\le y$ and $|\mathbf{x}|\ge y$.

If $|\mathbf{x}|\le y$, we have that

$$\begin{aligned}
&y^{1-s}\left|Q_s(\mathbf{x}-\mathbf{u},y)-Q_s(\widetilde{\mathbf{x}}-\mathbf{u},y)\right|\\
&\lesssim|\mathbf{x}-\widetilde{\mathbf{x}}|\sup_{\mathbf{v}=\mathbf{x}+\lambda(\mathbf{x}-\widetilde{\mathbf{x}});\,0\le\lambda\le1}y^{1-s}\left\|\nabla_{\mathbf{v},y}Q_s(\mathbf{v}-\mathbf{u},y)\right\|,
\end{aligned}$$

and use this estimate to proceed analogously to the case of the kernel $\mathbf{K}_s$.

Finally, if $|\mathbf{x}|\ge y$, a simmilar type of arguments, absorving the bigger powers of y with the exponential e^{-cy} give the desired estimate for the kernel $\mathbf{J}_s$. □

7. Proof of Theorem 1.1

7.1. **Proof of** (ii) $\Rightarrow$ (i)**.**

In [7], Theorem 3.1.5, it is proved that if $f\in\mathcal{D}'(\mathbb{R}^n)$ and $0<s<1$ and we have that $f=(Id-\Delta)^{\frac{s}{2}}F$ with $F\in\mathcal{M}(H^2_s(\mathbb{R}^n)\to L^2(\mathbb{R}^n))$, then $f\in\mathcal{M}(H^2_s(\mathbb{R}^n)\to H^2_{-s}(\mathbb{R}^n))$.

Hence, if (ii) in Theorem 1.1 holds, i.e., $b \in \mathcal{M}(H^2_s(\mathbb{R}^n) \to L^2(\mathbb{R}^n))$, this theorem gives that $(Id-\Delta)^{s/2}b \in \mathcal{M}(H^2_s(\mathbb{R}^n) \to H^2_{-s}(\mathbb{R}^n))$. Equivalently,

$$\left|\int_{\mathbb{R}^n} fg(Id-\Delta)^{s/2} b d\mathbf{x}\right| \lesssim \|f\|_{H^2_s(\mathbb{R}^n)}\|g\|_{H^2_s(\mathbb{R}^n)},$$

which gives (i).

7.2. **Proof of** (i) $\Rightarrow$ (ii)**.**

We begin the proof assuming first that $n-2s \geq 0$.

We will show that the measure $d\mu = |b(\mathbf{x})|^2 d\mathbf{x}$ satisfies the capacitary condition given in Proposition 4.3, i.e. there exists $C>0$ such that for any compact set $E \subset \mathbb{R}^n$ with $\operatorname{diam} E \leq 1$, then $\mu(E) \leq CCap_s(E)$. This will be a consequence of the hypothesis (i) applied to adequate test functions. We consider suitable regularizations associated to $\frac{G_s(\chi_E b)}{p_E^\alpha}$ and to p_E^α. We notice that in Theorem 4.7, we have already obtained an estimate of the norm of some regularizations of p_E^α. The following step is to give an estimate of the norm of some regularizations of the function $\frac{G_s(\chi_E b)}{p_E^\alpha}$.

Theorem 7.1. *Let $0<s<1$ and $n-2s \geq 0$ and fix $\alpha \in (\frac{1}{2}, \frac{1}{2}\frac{n}{n-2s})$ and $\alpha < \frac{1}{\sqrt{2}}$. Let E be a compact set of diameter less than or equal to one. Let $b \in L^2(\mathbb{R}^n)$, and consider the function $g_E = \chi_E b$. For $\varepsilon > 0$ and $\delta > 0$, let $g_{E,\varepsilon} = g_E * \varphi_\varepsilon$ and $p_{E,\delta} = G_{2s} * \nu_{E,\delta}$, where $\nu_{E,\delta}$ is the regularization of the extremal capacitary measure ν of E. Let $f_{E,\varepsilon,\delta} = \frac{G_s(g_{E,\varepsilon})}{p^\alpha_{E,\delta}}$.*

Then,

$$\|f_{E,\varepsilon,\delta}\|^2_{H^2_s(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{p^{2\alpha}_{E,\delta}}.$$

Proof. We have that by Proposition 2.1

$$\begin{aligned}
\|f_{E,\varepsilon,\delta}\|^2_{H^2_s(\mathbb{R}^n)} &\lesssim \left\|\frac{Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)}{Q_s(p_{E,\delta})^\alpha}\right\|^2_{W^2_{1,1-2s}} \\
&\lesssim \left\|\frac{Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)}{Q_s(p_{E,\delta})^\alpha}\right\|^2_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s}d\mathbf{x}dy)} + \left\|\frac{\nabla Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)}{Q_s(p_{E,\delta})^\alpha}\right\|^2_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s}d\mathbf{x}dy)} \\
&+ \left\|\frac{Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)}{Q_s(p_{E,\delta})^{\alpha+1}}\nabla Q_s(p_{E,\delta})\right\|^2_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s}d\mathbf{x}dy)} =: \mathcal{A}+\mathcal{B}+\mathcal{C}.
\end{aligned}$$

We will estimate separately each term $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

7.2.1. *Estimate of the term $\mathcal{A}$.*

Here we use a localization argument in order to substitute the Bessel kernel by the Riesz kernel, togeher with the results obtained in Section 5 and refsection 6. Since $diam(E) \leq 1$, $E \subset B(\mathbf{x}_0, 1)$ and without restriction we may assume that $\mathbf{x}_0$ is $\mathbf{0}$.

$$\int_{\mathbb{R}^{n+1}_+} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-2s} d\mathbf{x}dy = \int_{\mathbb{R}^n} \int_{\Gamma(\mathbf{u})} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u}$$
$$= \int_{\mathbb{R}^n\setminus B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u}$$
$$+ \int_{B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u}$$
$$= \mathcal{A}_1 + \mathcal{A}_2,$$

where k is as in Propositions 3.5 and 3.6. Let us begin estimating $\mathcal{A}_1$. We first observe that if $\mathbf{u} \notin B(\mathbf{0}, 2k)$ and $(\mathbf{x}, y) \in \Gamma(\mathbf{u})$, then $|\mathbf{x}| + y \geq k$.

Hence, applying Propositions 3.5 and 3.6 and using that $\nu_{E,\delta}$ is supported in $B(\mathbf{0}, 1)$ if δ is small enough, we have that for any $\lambda > 1$

$$\mathcal{A}_1 \lesssim \frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}} \int_{\mathbb{R}^n\setminus B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} y^{1-n-2s} \frac{e^{-\frac{2}{\sqrt{2}}(|\mathbf{x}|+y)}}{e^{-2\lambda\alpha(|\mathbf{x}|+y)}} d\mathbf{x}dyd\mathbf{u}.$$

Choosing $\lambda > 1$, so that $\rho = \frac{1}{\sqrt{2}} - \lambda\alpha > 0$ (which is possible since $\alpha < \frac{1}{\sqrt{2}}$), we deduce that the above is bounded by

$$\frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}} \int_{\mathbb{R}^n} \int_{\Gamma(\mathbf{u})} y^{1-n-2s} e^{-2\rho(|\mathbf{x}|+y)} d\mathbf{x}dyd\mathbf{u}$$
$$\lesssim \frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}} \int_{\mathbb{R}^{n+1}_+} y^{1-2s} e^{-2\rho(|\mathbf{x}|+y)} d\mathbf{x}dy \lesssim \frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}}.$$

But Hölder's inequality, together with Minkowski's inequality ($1 < 2\alpha < \frac{n}{n-2s}$) give that

$$\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2 \leq \int_{B(\mathbf{0},1)} \frac{|g_{E,\varepsilon}|^2}{p_{E,\delta}^{2\alpha}} \int_{B(\mathbf{0},1)} p_{E,\delta}^{2\alpha} \lesssim \nu_{E,\delta}(E)^{2\alpha} \int_{B(\mathbf{0},1)} \frac{|g_{E,\varepsilon}|^2}{p_{E,\delta}^{2\alpha}}.$$

Hence, $\mathcal{A}_1 \lesssim \int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{p_{E,\delta}^{2\alpha}}$.

We now estimate

$$\mathcal{A}_2 = \int_{B(\mathbf{0},2k)} \left(\int_{\Gamma(\mathbf{u})} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dy\right) d\mathbf{u}$$
$$= \int_{B(\mathbf{0},2k)} \left(\int_{(\mathbf{x},y)\in\Gamma(\mathbf{u});\, y>k} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dy\right) d\mathbf{u}$$
$$+ \int_{B(\mathbf{0},2k)} \left(\int_{(\mathbf{x},y)\in\Gamma(\mathbf{u});\, y\leq k} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dy\right) d\mathbf{u} = \mathcal{A}_{21} + \mathcal{A}_{22}.$$

To estimate $\mathcal{A}_{21}$, simmilar arguments to the ones used for $\mathcal{A}_1$ give, since $|\mathbf{x}| + y \geq k$, that $\mathcal{A}_{21} \lesssim \int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{p_{E,\delta}^{2\alpha}}$.

Now we estimate $\mathcal{A}_{22}$. Denote $\Gamma^k(\mathbf{u}) = \Gamma(\mathbf{u}) \cap \{y \leq k\}$. Let $(\mathbf{x}, y) \in \Gamma^k(\mathbf{u})$. We then have,

$$Q_s(p_{E,\delta})(\mathbf{x}, y) \gtrsim \int_{\mathbf{v}\in B(\mathbf{x},y)} \frac{y^{2s}}{(|\mathbf{v}-\mathbf{x}|^2 + y^2)^{\frac{n+2s}{2}}} \mathcal{G}_{2n+2s+1}(\mathbf{v}-\mathbf{x}, y) p_{E,\delta}(\mathbf{v}) d\mathbf{v}.$$

But for any $\mathbf{v} \in B(\mathbf{x}, y)$, $|\mathbf{v}-\mathbf{x}| \leq y$. Hence, since in the integral defining $\mathcal{A}_{22}$, $\mathbf{u} \in B(\mathbf{0}, 2k)$,

$$|\mathbf{v}| \leq |\mathbf{v}-\mathbf{x}| + |\mathbf{x}-\mathbf{u}| + |\mathbf{u}| \lesssim 1.$$

Consequently, we have that for any $\mathbf{v} \in B(\mathbf{x}, y)$, $p_{E,\delta}(\mathbf{v}) \simeq I_{2s}(\nu_{E,\delta})(\mathbf{v})$ (when $n-2s=0$, we have that $n=1$ and $s=\frac{1}{2}$ and $I_{2s}(\nu_{E,\delta})$ must be replaced by the logarithmic potential of the measure $\nu_{E,\delta}$). We also have that for $|\mathbf{v}-\mathbf{x}| \leq y < k$, $\mathcal{G}_{2n+2s+1}(\mathbf{v}-\mathbf{x}, y) \simeq 1$. Thus,

$$Q_s(p_{E,\delta})(\mathbf{x}, y) \gtrsim \int_{\mathbf{v}\in B(\mathbf{x},y)} \frac{y^{2s}}{(|\mathbf{v}-\mathbf{x}|^2+y^2)^{\frac{n+2s}{2}}} I_{2s}(\nu_{E,\delta})(\mathbf{v})d\mathbf{v} \gtrsim \int_{B(\mathbf{x},y)} \frac{1}{y^n} I_{2s}(\nu_{E,\delta})(\mathbf{v})d\mathbf{v}.$$

Hence,

$$\begin{aligned}
\mathcal{A}_{22} &= \int_{B(\mathbf{0},2k)} \left(\int_{\Gamma^k(\mathbf{u})} \frac{|Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dy \right) d\mathbf{u} \\
&\lesssim \int_{B(\mathbf{0},2k)} \int_{\Gamma^k(\mathbf{u})} |Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)|^2 y^{1-n-2s} \left(\frac{1}{|B(\mathbf{x},y)|} \int_{B(\mathbf{x},y)} I_{2s}(\nu_{E,\delta})(\mathbf{v})d\mathbf{v} \right)^{-2\alpha} d\mathbf{x}dyd\mathbf{u} \\
&\lesssim \int_{B(\mathbf{0},2k)} \int_{\Gamma^k(\mathbf{u})} |Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)|^2 y^{1-n-2s} \left(\frac{1}{|B(\mathbf{x},y)|} \int_{B(\mathbf{x},y)} I_{2s}(\nu_{E,\delta})^{-1}(\mathbf{v})d\mathbf{v} \right)^{2\alpha} d\mathbf{x}dyd\mathbf{u} \\
&\lesssim \int_{\mathbf{u}\in B(\mathbf{0},2k)} \int_{\Gamma^k(\mathbf{u})} |Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)|^2 y^{1-n-2s} \frac{1}{|B(\mathbf{x},y)|} \int_{\mathbf{v}\in B(\mathbf{x},y)} I_{2s}(\nu_{E,\delta})^{-2\alpha}(\mathbf{v})d\mathbf{v}\, d\mathbf{x}dyd\mathbf{u} \\
&\lesssim \int_{\mathbb{R}^{n+1}_+} |Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)|^2 y^{1-2s} \frac{1}{y^n} \int_{\mathbf{v}\in B(\mathbf{x},y)} \frac{1}{I_{2s}(\nu_{E,\delta})^{2\alpha}(\mathbf{v})} d\mathbf{v}d\mathbf{x}dy \\
&\lesssim \int_{\mathbb{R}^n} \int_{\Gamma(\mathbf{v})} |Q_s\left(G_s\left(g_{E,\varepsilon}\right)\right)|^2 y^{1-n-2s} \frac{1}{I_{2s}(\nu_{E,\delta})^{2\alpha}(\mathbf{v})} d\mathbf{x}dyd\mathbf{v}.
\end{aligned}$$

Next, Lemma 4.5 gives that $I_{2s}(\nu_{E,\delta})^{2\alpha}$ is in A_2, with constants independent of $\delta > 0$ and consequently, $I_{2s}(\nu_{E,\delta})^{-2\alpha}$ is also in A_2. Hence, Theorem 6.1 applied to the operator $\mathbf{J}_s$ gives that the above is bounded by

$$\int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{I_{2s}(\nu_{E,\delta})^{2\alpha}} \simeq \int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{p_{E,\delta}^{2\alpha}},$$

since $g_{E,\varepsilon}$ is supported in a set with diameter less than or equal to 2, then, $I_{2s}(\nu_{E,\delta}) \simeq p_{E,\delta}$.

7.2.2. *Estimate of the term $\mathcal{B}$.*

As in the estimate of $\mathcal{A}$, we decompose $\mathcal{B}$ in two integrals. Namely

$$\begin{aligned}
\mathcal{B} &= \int_{\mathbb{R}^{n+1}_+} \frac{|\nabla Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-2s} d\mathbf{x}dy = \int_{\mathbf{u}\in\mathbb{R}^n} \int_{\Gamma(\mathbf{u})} \frac{|\nabla Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u} \\
&= \int_{\mathbb{R}^n\setminus B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} \frac{|\nabla Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u} \\
&+ \int_{B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} \frac{|\nabla Q_s(G_s(g_{E,\varepsilon}))|^2}{|Q_s(p_{E,\delta})|^{2\alpha}} y^{1-n-2s} d\mathbf{x}dyd\mathbf{u} = \mathcal{B}_1 + \mathcal{B}_2.
\end{aligned}$$

We begin estimating $\mathcal{B}_1$. Applying now (ii) in Proposition 3.5 and Proposition 3.6, give that

$$\mathcal{B}_1 \lesssim \frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}} \int_{\mathbb{R}^n\setminus B(\mathbf{0},2k)} \int_{\Gamma(\mathbf{u})} h_s(y)^2 y^{1-n-2s} \frac{e^{-\frac{2}{\sqrt{2}}(|\mathbf{x}|+y)}}{e^{-2\lambda\alpha(|\mathbf{x}|+y)}} d\mathbf{x}dyd\mathbf{u}$$

and following the arguments used in the estimate of $\mathcal{A}_1$, we get $\mathcal{B}_1 \lesssim \frac{\left(\int_{B(\mathbf{0},1)} |g_{E,\varepsilon}|\right)^2}{\nu_{E,\delta}(E)^{2\alpha}}$.

The estimate of $\mathcal{B}_2$ is done as in the estimate of $\mathcal{A}_2$ decomposing the integral in two parts, one corresponding to $y \geq k$, which we will denote by $\mathcal{B}_{21}$, and the other corresponding to $y < k$, which we will denote by $\mathcal{B}_{22}$. The estimate of $\mathcal{B}_{21}$ for $y > k$ it is done in a simmilar way to the estimate of $\mathcal{A}_{21}$. The estimate of $\mathcal{B}_{22}$, follows exactly as in the estimate of $\mathcal{A}_{22}$ replacing in the application of Theorem 6.1 the operator $\mathbf{J}_s$ by $\mathbf{K}_s$.

7.2.3. *Estimate of the term $\mathcal{C}$.*

We will reduce the estimate the term $\mathcal{C}$ to the preceeding two terms $\mathcal{A}$ and $\mathcal{B}$, using Stokes' Theorem. Given $\eta, R > 0$, let $\Omega_{\eta,R}$ the region in $\mathbb{R}^{n+1}_+$ defined by

$$\Omega_{\eta,R} = \{(\mathbf{x},y) \in \mathbb{R}^{n+1}_+ ;\, (\mathbf{x},y) \in \overline{B(\mathbf{0},R)},\, y \geq \eta\}.$$

Let ω be the form defined by

$$\begin{aligned}
&\omega_{\delta,\varepsilon}(\mathbf{x},y)\\
&= \frac{(Q_s(G_s(g_{E,\varepsilon})))^2}{(Q_s(p_{E,\delta}))^{2\alpha+1}} y^{1-2s} \left(\sum_{i=1}^n (-1)^{i-1} \frac{\partial Q_s(p_{E,\delta})}{\partial x_i} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dy \right.\\
&\left. + (-1)^n \frac{\partial Q_s(p_{E,\delta})}{\partial y} dx_1 \wedge \cdots \wedge dx_n \right).
\end{aligned} \tag{7.18}$$

Using that $div(y^{1-2s}\nabla u) = y^{1-2s}u$, Stokes' Theorem (orienting $\partial\Omega_{\eta,R}$ adequately) gives that

$$\begin{aligned}
&\int_{\partial\Omega_{\eta,R}\cap\{y=\eta\}} \omega_{\delta,\varepsilon}(\mathbf{x},y) + \int_{\partial\Omega_{\eta,R}\cap\{y>\eta\}} \omega_{\delta,\varepsilon}(\mathbf{x},y)\\
&= -(2\alpha+1) \int_{\Omega_{\eta,R}} \frac{(Q_s(G_s(g_{E,\varepsilon})))^2}{(Q_s(p_{E,\delta}))^{2\alpha+2}} |\nabla Q_s(p_{E,\delta})|^2 y^{1-2s} d\mathbf{x}dy\\
&+ 2\int_{\Omega_{\eta,R}} Q_s(G_s(g_{E,\varepsilon})) \frac{\nabla Q_s(G_s) \cdot \nabla Q_s(p_{E,\delta})}{(Q_s(p_{E,\delta}))^{2\alpha+1}} y^{1-2s} d\mathbf{x}dy\\
&+ \int_{\Omega_{\eta,R}} \frac{(Q_s(G_s(g_{E,\varepsilon})))^2}{(Q_s(p_{E,\delta}))^{2\alpha+1}} Q_s(p_{E,\delta}) y^{1-2s} d\mathbf{x}dy
\end{aligned} \tag{7.19}$$

In order to pass to the limit when $\eta \to 0$ and $R \to \infty$, we will need the following lemmas:

Lemma 7.2. *Let $0 < s < 1$, $n - 2s > 0$, $1/2 < \alpha < \min(\frac{n}{2(n-2s)}, \frac{1}{2}(\frac{3}{\sqrt{2}} - 1))$ and let $\omega_{\delta,\varepsilon}$ be as in* (7.18). *Then we have that*

$$\lim_{R\to\infty} \int_{\partial\Omega_{\eta,R}\cap\{y\geq\eta\}} \omega_{\delta,\varepsilon}(\mathbf{x},y) = 0.$$

Proof. By Propositions (3.5) and (3.6), we have that, if R is big enough and $\lambda > 1$ such that $\lambda(2\alpha+1) - \frac{3}{\sqrt{2}} < 0$, the coefficients of the form $\omega_{\delta,\varepsilon}$ are bounded by $h_s(y)y^{1-2s}e^{(\lambda(2\alpha+1)-\frac{3}{\sqrt{2}})(|\mathbf{x}|+y)}$ and consequently,

$$\lim_{R\to\infty}\int_{\{(\mathbf{x},y)\,;\,|\mathbf{x}|^2+y^2=R^2\,,\,y\geq\eta\}}\omega_{\delta,\varepsilon}=0.$$

□

Lemma 7.3. *Let* $0<s<1$ *and* $n-2s>0$, $\frac{1}{2}<\alpha<\min(\frac{n}{2(n-2s)},\frac{1}{2}(\frac{3}{\sqrt{2}}-1))$ *and let* $\omega_{\delta,\varepsilon}$ *be as in* (7.18). *Then we have that*

$$\lim_{R\to\infty}\int_{\partial\Omega_{\eta,R}\cap\{y=\eta\}}\omega_{\delta,\varepsilon}=\int_{\partial(\{(\mathbf{x},y),\,\mathbf{x}\in\mathbb{R}^n,\,y\geq\eta\})}\omega_{\delta,\varepsilon}.$$

Proof. As in Lemma 7.2, let $\lambda>1$ such that $\lambda(2\alpha+1)-\frac{2}{\sqrt{2}}<0$. Then the function $\mathbf{x}\to h_s(\eta)\eta^{1-2s}e^{\left(\lambda(2\alpha+1)-\frac{3}{\sqrt{2}}\right)(|\mathbf{x}|+\eta)}\in L^1(\mathbb{R}^n)$ and consequently, the Lebesgue's Dominated Convergence Theorem finishes the proof.

□

Lemma 7.4. *Let* $0<s<1$, $n-2s>0$, $1/2<\alpha<\min(\frac{n}{2(n-2s)},\frac{1}{2}(\frac{3}{\sqrt{2}}-1))$ *and let* $\omega_{\delta,\varepsilon}$ *be as in* (7.18). *Then there exists* $C>0$ *such that*

$$\lim_{\eta\to0}\int_{\mathbb{R}^n\times\{\eta\}}\omega_{\delta,\varepsilon}=-(-1)^nC\int_{\mathbb{R}^n}\frac{(G_s(g_{E,\varepsilon}))^2}{(p_{E,\delta})^{2\alpha+1}}d\nu_{E,\delta}.$$

Proof. By Theorem 2.2, we have that there exists a positive constant $C>0$ such that for any $\mathbf{x}\in\mathbb{R}^n$,

$$\lim_{\eta\to0}\omega_{\delta,\varepsilon}(\mathbf{x},\eta)=-(-1)^nC\frac{(G_s(g_{E,\varepsilon})(\mathbf{x}))^2}{(p_{E,\delta}(\mathbf{x}))^{2\alpha+1}}(Id-\Delta)^s(p_{E,\delta})(\mathbf{x}).$$

In order to finish the proof of the lemma, we denote $I_\eta=I^1_\eta\cup I^2_\eta$, where $I^1_\eta=\{(\mathbf{x},\eta);\,|\mathbf{x}|\leq k_0\}$ and $I^2_\eta=\{(\mathbf{x},\eta);\,|\mathbf{x}|\geq k_0\}$, where k_0 is big enough such that Proposition 3.6 holds. We also may assume that for any $0<\delta<1$, the support of $\nu_{E,\delta}$ is in $|\mathbf{x}|\leq k_0$.

Observe that if $|\mathbf{t}|\leq k_0+1$, then

$$p_{E,\delta}(\mathbf{t})=\int_{\mathbb{R}}G_{2s}(\mathbf{t}-\mathbf{x})d\nu_{E,\delta}(\mathbf{x})\gtrsim1.$$

By continuity, we will have that $Q_s(p_{E,\delta})\gtrsim1$ on $\{(\mathbf{x},y);\,|\mathbf{x}|\leq k_0,\,y\in[0,\rho]\}$, for some $\rho>0$.

On the other hand, the function $G_s(g_{E,\varepsilon})$ is uniformly continuous on $\{|\mathbf{x}|\leq k_0\}$ (see Theorem 2.3). Hence,

$$\lim_{\eta\to0}Q_s(G_s(g_{E,\varepsilon}))(\mathbf{x},\eta)=G_s(g_{E,\varepsilon})(\mathbf{x}),$$

uniformly on $\{|\mathbf{x}|\leq k_0\}$. Analogously, $\lim_{\eta\to0}Q_s(p_{E,\delta})^{2\alpha+1}(\mathbf{x},\eta)=p^{2\alpha+1}_{E,\delta}(\mathbf{x})$ uniformly on $\{|\mathbf{x}|\leq k_0\}$.

In addition, $(Id-\Delta)^sp_{E,\delta}=\nu_{E,\delta}\in\mathcal{D}$ and by Theorem 2.3,

$$\lim_{\eta\to0}\eta^{1-2s}\frac{\partial Q_s(p_{E,\delta})}{\partial y}(\mathbf{x},\eta)=-C(Id-\Delta)^sp_{E,\delta}(\mathbf{x})$$

uniformly on $\{|\mathbf{x}| \le k_0\}$, where $C > 0$. Then, we have that

$$\begin{aligned}\lim_{\eta\to 0}\int_{I^1_\eta}\omega_{\delta,\varepsilon} &= -(-1)^n C\int_{|\mathbf{x}|\le k_0}\frac{(G_s(g_{E,\varepsilon}))^2}{(p_{E,\delta})^{2\alpha+1}}(Id-\Delta)^s p_{E,\delta}\,d\mathbf{x}\\ &= -(-1)^n C\int_{|\mathbf{x}|\le k_0}\frac{(G_s(g_{E,\varepsilon}))^2}{(p_{E,\delta})^{2\alpha+1}}d\nu_{E,\delta}.\end{aligned}$$

Next, if $(\mathbf{x},\eta)\in I^2_\eta$, and denoting by $|\omega_{\delta,\varepsilon}(\mathbf{x},\eta)|$ the modulus of the coefficients of the form $\omega_{\delta,\varepsilon}(\mathbf{x},\eta)$, we have that

$$|\omega_{\delta,\varepsilon}(\mathbf{x},\eta)| \lesssim e^{\left(\lambda(2\alpha+1)-\frac{3}{\sqrt{2}}\right)|\mathbf{x}|}\chi_{\{\|\mathbf{x}\|\ge k_0\}} \in L^1(\mathbb{R}^n).$$

Thus, passing to the limit, Lebesgue's Dominated Convergence Theorem finishes the lemma.

□

Now we can finish the proof of the boundedness of term $\mathcal{C}$. We have that by (7.19) and Hölder's inequality,

$$\begin{aligned}&(2\alpha+1)\int_{\Omega_{\eta,R}}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{Q_s(p_{E,\delta})^{2\alpha+2}}|\nabla Q_s(p_{E,\delta})|^2 y^{1-2s}d\mathbf{x}dy\\ &\le -\int_{\partial\Omega_{\eta,R}\cap\{y=\eta\}}\omega_{\delta,\varepsilon}(\mathbf{x},y) - \int_{\partial\Omega_{\eta,R}\cap\{y>\eta\}}\omega_{\delta,\varepsilon}(\mathbf{x},y)\\ &+\int_{\Omega_{\eta,R}}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{\left(Q_s(p_{E,\delta})\right)^{2\alpha+1}}Q_s(p_{E,\delta})y^{1-2s}d\mathbf{x}dy\\ &+2\left(\int_{\Omega_{\eta,R}}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{Q_s(p_{E,\delta})^{2\alpha+2}}|\nabla Q_s(p_{E,\delta})|^2 y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}}\\ &\times\left(\int_{\Omega_{\eta,R}}\frac{|\nabla Q_s\left((G_s(g_{E,\varepsilon})\right)|^2}{Q_s(p_{E,\delta})^{2\alpha}}y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}},\end{aligned}$$

Next, we will pass to the limit when $R\to\infty$ and $\eta\to 0$, using Lemmas 7.3, 7.4 and 7.2, the Lebesgue's Monotone Convergence Theorem. Observe that the orientation on $\partial\Omega_{R,\eta}$ given by Stokes's Theorem when passing to the limit corresponds to the usual orientation on $\mathbb{R}^n$ with a factor $(-1)^{n+1}$. We will then have that:

$$(2\alpha+1)\int_{\mathbb{R}^{n+1}_+}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{Q_s(p_{E,\delta})^{2\alpha+2}}|\nabla Q_s(p_{E,\delta})|^2y^{1-2s}d\mathbf{x}dy$$
$$= -C\int_{\mathbb{R}^n}\frac{\left(G_sg_{E,\varepsilon}\right)^2}{(p_{E,\delta})^{2\alpha+1}}d\nu_{E,\delta}+\int_{\mathbb{R}^{n+1}_+}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{\left(Q_s(p_{E,\delta})\right)^{2\alpha+1}}Q_s(p_{E,\delta})y^{1-2s}d\mathbf{x}dy$$
$$+2\left(\int_{\mathbb{R}^{n+1}_+}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{Q_s(p_{E,\delta})^{2\alpha+2}}|\nabla Q_s(p_{E,\delta})|^2y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{R}^{n+1}_+}\frac{|\nabla Q_s\left(G_s(g_{E,\varepsilon})\right)|^2}{Q_s(p_{E,\delta})^{2\alpha}}|y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}}\leq\int_{\mathbb{R}^{n+1}_+}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{\left(Q_s(p_{E,\delta})\right)^{2\alpha+1}}Q_s(p_{E,\delta})y^{1-2s}d\mathbf{x}dy$$
$$+2\left(\int_{\mathbb{R}^{n+1}_+}\frac{(Q_s\left(G_s(g_{E,\varepsilon})\right))^2}{Q_s(p_{E,\delta})^{2\alpha+2}}|\nabla Q_s(p_{E,\delta})|^2y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{R}^{n+1}_+}\frac{|\nabla Q_s\left(G_s(g_{E,\varepsilon})\right)|^2}{Q_s(p_{E,\delta})^{2\alpha}}y^{1-2s}d\mathbf{x}dy\right)^{\frac{1}{2}},$$

since $\displaystyle\int_{\mathbb{R}^n}\frac{\left(G_s(g_{E,\varepsilon})\right)^2}{(p_{E,\delta})^{2\alpha+1}}d\nu_{E,\delta}\geq 0$.

We have proved then that $\mathcal{C}\lesssim\mathcal{A}+\mathcal{C}^{\frac{1}{2}}\mathcal{B}^{\frac{1}{2}}$ and consequently, that $\mathcal{C}^2\lesssim\mathcal{A}^2+\mathcal{C}\,\mathcal{B}$. Thus, either $\frac{1}{2}\mathcal{C}^2\leq k\mathcal{C}\,\mathcal{B}$ or $\frac{1}{2}\mathcal{C}^2\leq k\mathcal{A}^2$. Equivalently, either $\frac{1}{2}\mathcal{C}\leq k\mathcal{B}$ or $\frac{1}{\sqrt{2}}\mathcal{C}\leq\sqrt{k}\mathcal{A}$. Hence, using the estimates obtained for $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{C}\lesssim\mathcal{A}+\mathcal{B}\lesssim\int_E\frac{|g_{E,\varepsilon}|^2}{p^{2\alpha}_{E,\delta}}.$$

□

7.2.4. *End of proof of* (i) ⇒ (ii) *for* $n-2s\geq 0$.

We begin with a technical lemma.

Lemma 7.5. *Let $R>0$ and let $\phi_R(x,y)=\phi(|\mathbf{x}|^2+y^2)$ be a $\mathcal{C}^\infty(\overline{\mathbb{R}^{n+1}_+})$ function such that $\phi_R\equiv 1$ on $\overline{D(0,R)}\cap\mathbb{R}^{n+1}_+$, $\phi_R\equiv 0$ on $\mathbb{R}^2_+\setminus D(0,2R)$, and $0\leq\phi_R\leq 1$ on $\overline{\mathbb{R}^{n+1}_+}$, such that for $R<|\mathbf{x}|^2+y^2<2R$, $|\nabla\phi_R(\mathbf{x},y)|\lesssim\frac{1}{R}$, with constants independent of R. We then have,*

(i) $\lim_{R\to\infty}\phi_Rf_{E,\varepsilon,\delta}=f_{E,\varepsilon,\delta}$ *in* $H^2_s(\mathbb{R}^n)$.
(ii) $\lim_{R\to\infty}\phi_Rp_{E,\delta}=p_{E,\delta}$ *in* $H^2_s(\mathbb{R}^n)$.

Proof. We begin with (i). We must show that

$$\|(\phi_R-1)f_{E,\varepsilon,\delta}\|_{H^2_s(\mathbb{R}^n)}\to 0\qquad R\to\infty.$$

But,

$$\begin{aligned}
&\|(\phi_R-1)f_{E,\varepsilon,\delta}\|^2_{H^2_s(\mathbb{R}^n)} \le \|(\phi_R-1)\frac{Q_s(G_s(g_{E,\varepsilon}))}{(Q_s(p_{E,\delta}))^\alpha}\|^2_{L^2(\mathbb{R}^{n+1}_+,y^{1-2s}d\mathbf{x}dy)}\\
&\le \|(\phi_R-1)\frac{Q_s(G_s(g_{E,\varepsilon}))}{(Q_s(p_{E,\delta}))^\alpha}\|^2_{L^2(\mathbb{R}^{n+1}_+),y^{1-2s}d\mathbf{x}dy)}\\
&+\|(\phi_R-1)\nabla\left(\frac{Q_s(G_s(g_{E,\varepsilon}))}{(Q_s(p_{E,\delta}))^\alpha}\right)\|^2_{L^2(\mathbb{R}^{n+1}_+,y^{1-2s}d\mathbf{x}dy)}\\
&+\|\nabla\phi_R\frac{Q_s(G_s(g_{E,\varepsilon}))}{(Q_s(p_{E,\delta}))^\alpha}\|^2_{L^2(\mathbb{R}^{n+1}_+,y^{1-2s}d\mathbf{x}dy)} = A(R)+B(R)+C(R).
\end{aligned}$$

Since $|\phi_R-1|\le 1$, the Lebesgue's Dominated Convergence Theorem together with Theorem 7.1 gives that $\lim_{R\to\infty} A(R)=0$ and $\lim_{R\to\infty} B(R)=0$. Finally, since for $R<|\mathbf{x}|^2+y^2<2R$, $|\nabla\phi_R(x,y)|\lesssim\frac{1}{R}$, and ϕ_R is supported in $D(0,2R)$, we deduce, using Propositions 3.5 and 3.6 that $\lim_{R\to\infty} C(R)=0$.

A similar argument, using now Proposition 4.4 and Theorem 4.7, proves (ii). □

Assume that (i) holds. If we apply this hypothesis to $\phi_R\frac{G_s(g_{E,\varepsilon})}{p^\alpha_{E,\delta}}$ (see notations in Theorem 7.1) and to $\phi_R p^\alpha_{E,\delta}$, we obtain

$$\left|\int_{\mathbb{R}^n}(Id-\Delta)^{s/2}\left(\phi_R^2 G_s(g_{E,\varepsilon})\right)(\mathbf{x})b(\mathbf{x})\,d\mathbf{x}\right|\lesssim\|\phi_R f_{E,\varepsilon,\delta}\|_{H^2_s(\mathbb{R}^n)}\|\phi_R p^\alpha_{E,\delta}\|_{H^2_s(\mathbb{R}^n)}.$$

We claim that

$$\left|\int_{\mathbb{R}^n}(Id-\Delta)^{s/2}\left(\left(\phi_R^2-1\right)G_s(g_{E,\varepsilon})\right)(\mathbf{x})b(\mathbf{x})\,d\mathbf{x}\right|\to 0,\qquad R\to\infty. \tag{7.20}$$

Indeed, using Hölder's inequality, it is enough to show that

$$\int_{\mathbb{R}^n}\left|(Id-\Delta)^{s/2}\left(\left(\phi_R^2-1\right)G_s(g_{E,\varepsilon})\right)(\mathbf{x})\right|^2 d\mathbf{x}\to 0,\qquad R\to\infty.$$

But, the above is bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+}\left|(\phi_R^2-1)Q_s(G_s(g_{E,\varepsilon}))(\mathbf{x})\right|^2 y^{1-2s}d\mathbf{x}dy\\
&+\int_{\mathbb{R}^{n+1}_+}\left|\nabla\left((\phi_R^2-1)Q_s(G_s(g_{E,\varepsilon}))\right)(\mathbf{x})\right|^2 y^{1-2s}d\mathbf{x}dy.
\end{aligned}$$

A similar argument to the one used in Lemma 7.5, finishes the proof of (7.20).

Using (7.20) and Lemma 7.5, we deduce that

$$\left|\int_{\mathbb{R}^n}(Id-\Delta)^{s/2}\left(G_s(g_{E,\varepsilon})\right)(\mathbf{x})b(\mathbf{x})\,d\mathbf{x}\right|\lesssim\|f_{E,\varepsilon,\delta}\|_{H^2_s(\mathbb{R}^n)}\|p^\alpha_{E,\delta}\|_{H^2_s(\mathbb{R}^n)}.$$

By Theorems 7.1 and 4.7, we have that $\|f_{E,\varepsilon,\delta}\|^2_{H^2_s(\mathbb{R}^n)}\lesssim\int_n\frac{|g_{E,\varepsilon}|^2}{p^{2\alpha}_{E,\delta}}$ and $\|p^\alpha_{E,\delta}\|^2_{H^2_s(\mathbb{R}^n)}\lesssim Cap_s(E)$. Hence, using that since $g_{E,\varepsilon}$ is compactly supported in a neighborhood of E of diameter les that or equal to 1,we have that in its support, $p_{E,\delta} \simeq I_{2s}(\nu_{E,\delta})$,

$$\begin{aligned}(7.21)\quad & \left|\int_{\mathbb{R}^n} g_{E,\varepsilon}(\mathbf{x})b(\mathbf{x})\,d\mathbf{x}\right| \\ & \lesssim \left(\int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}(\mathbf{x})|^2}{p^{2\alpha}_{E,\delta}(\mathbf{x})}d\mathbf{x}\right)^{1/2} Cap_s(E)^{1/2} \simeq \left(\int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}(\mathbf{x})|^2}{I_{2s}(\nu_{E,\delta})^{2\alpha}(\mathbf{x})}d\mathbf{x}\right)^{1/2} Cap_s(E)^{1/2}.\end{aligned}$$

But $g_{E,\varepsilon} \lesssim M_{HL}(g_E) = M_{HL}(\chi_E\, b)$. Hence, since $I_{2s}(\nu_{E,\delta})^{-2\alpha} \in A_2$ with constant independent of δ, the integral $\int_{\mathbb{R}^n} \frac{|g_{E,\varepsilon}|^2}{I_{2s}(\nu_{E,\delta})^{2\alpha}}$ can be bounded by

$$(7.22)\qquad \int_{\mathbb{R}^n} \frac{(M_{HL}\chi_E\, b)^2(\mathbf{x})}{I_{2s}(\nu_{E,\delta})^{2\alpha}(\mathbf{x})}d\mathbf{x} \lesssim \int_E \frac{|b(\mathbf{x})|^2}{I_{2s}(\nu_{E,\delta})^{2\alpha}(\mathbf{x})}d\mathbf{x} \lesssim \int_E \frac{|b(\mathbf{x})|^2}{p^{2\alpha}_{E,\delta}(\mathbf{x})}d\mathbf{x}.$$

In addition $\lim_{\varepsilon\to 0} g_{E,\varepsilon} = \chi_E\, b$ in L^2. Consequently, from (7.21) and (7.22) we deduce,

$$(7.23)\qquad \left|\int_E |b(\mathbf{x})|^2 d\mathbf{x}\right| \lesssim \left(\int_E \frac{|b(\mathbf{x})|^2}{p^{2\alpha}_{E,\delta}(\mathbf{x})}d\mathbf{x}\right)^{1/2} Cap_s(E)^{1/2}.$$

Next, recall that for $\mathbf{t}\in E$, $p_{E,\delta}(\mathbf{t}) \simeq I_{2s}(\nu_{E,\delta})(\mathbf{t})$ and $\int I_{2s}(\mathbf{t}-\mathbf{v})\nu_{E,\delta}(\mathbf{v})d\mathbf{v} \gtrsim \nu(E)$. Consequently, the Lebesgue's Dominated Convergence Theorem gives that,

$$\lim_{\delta\to 0} \int_E \frac{|b(\mathbf{x})|^2}{p^{2\alpha}_{E,\delta}(\mathbf{t})}\,d\mathbf{x} = \int_E \frac{|b(\mathbf{x})|^2}{p^{2\alpha}_{E}(\mathbf{t})}d\mathbf{x} \le \int_E |b(\mathbf{x})|^2\,d\mathbf{x},$$

where in the last estimate we have used that by Theorem 4.2, (ii), a.e. on E, $p_E(\mathbf{x})^{2\alpha} \ge 1$.

Hence, from (7.23) we have that

$$\int_E |b(\mathbf{x})|^2 d\mathbf{x} \lesssim Cap_s(E).$$

And that proves (ii) for $n-2s\ge 0$. □

7.2.5. *The case $n-2s<0$.*

Observe that since $s<1$, we have that $n-2s<0$ implies that $n=1$. In this case the proof is much easier since all the functions in $H^s_2(\mathbb{R})$ are bounded and continuous. In the choice of the test functions we simply take $\alpha=1$. The same arguments of the previous case give the estimates of all the terms except the ones corresponding to $\mathcal{A}_{22}$, that in this particular case is, in fact, easier. Indeed, if $E\subset B(\mathbf{0},1)$ and $\mathbf{v}$ is bounded, then $p_{E,\delta}(\mathbf{v}) \simeq Cap(E) \simeq 1$ and consequently, $Q_s(p_{E,\delta})(\mathbf{v}) \gtrsim 1$. Then, the estimate of the term $\mathcal{A}_{22}$ follows directly from the unweighted L^2-estimate of the involved area functions. And the conclusion of the proof is then analogous to the case $n-2s\ge 0$.

References

[1] Adams, D.R. and Hedberg, L.I.: *Function spaces and potential theory.* Grundlehren der Mathematischen Wissenschaften, Vol. 314 (1999), Springer-Verlag, Berlin.

[2] Caffarelli, L., Silvestre, L. An extension problem related to the fractional laplacian. *Communications in Partial Differential Equations* **32**, (2007), 1245-1260.

[3] Cascante, C., Fàbrega, J.; Ortega, J.M.: Sharp norm estimates for the Bergman operator from weighted mixed-norm spaces to weighted Hardy spaces. Canad. J. Math. **68** (2016), 1257–1284.

[4] Cascante, C., Fàbrega, J.; Ortega, J.M.: Bilinear forms on homogeneous Sobolev spaces, *J. Math. Anal. Applications*, **457**, (2018), 722-750.

[5] Cruz-Uribe, D.; Pérez, C.: On the two-weight problem for singular integral operators, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* Vol. I (2002), 821-849.

[6] Duoandikoetxea, J.: *Fourier Analysis*, Graduate Studies in Mathematics, vol. 29 (2001), Rhode Island.

[7] Gala, S.: Operateurs de multiplication ponctuelle entre espaces de Sobolev, Thesis, https://tel.archives-ouvertes.fr/tel-00009577/document.

[8] Lemarié-Rieusset, P.G., Gala, S. Multipliers between Sobolev spaces and fractional differentiation. *J. Math. Anal. Appl.* **322**, (2006), 1030-1054.

[9] Lerner, A.K.: On some sharp weighted norm inequalities, *Journal of Functional Aalysis*, **232**, (2006), 477-494.

[10] Maz'ya, V.: *Sobolev spaces with applications to elliptic partial differential equations.* Grundlehren der Mathematischen Wissenschaften, **342**, (2006), Springer-Verlag, Berlin.

[11] Maz'ya, V., Shaposhnikova, T.O.: *Theory of multipliers in spaces of differentiable functions.* Monographs and Studies in Mathematics, **23**, (1985), Pitman, Boston.

[12] Maz'ya, V., Shaposhnikova, T.O.: *Theory of Sobolev multipliers with applications to differential and integral operators.* Grundlehren der Mathematischen Wissenschaften, **33**, (2009), Springer-Verlag, Berlin.

[13] Maz'ya, V., Verbitsky, I.E.: *Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers.* Ark. Mat. **33**, (1995), 81-115.

[14] Maz'ya, V., Verbitsky, I.E.: *The Schrödinger operator on the energy space: boundedness and compactness criteria. Acta Math.* **188**, (2002), 263-302.

[15] Maz'ya, V., Verbitsky, I.E. The form boundedness criterion for the relativistic Schrödinger operator. *Ann. de l'Inst. Fourier*, **54**, (2004), 317-339.

[16] Stein, E.M.: *Singular integrals and differentiability properties of functions.* Princeton University Press, (1970), Princeton, New Jersey.

[17] Stinga, P. R.; Torrea, J. L.: Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations.* **35**, (2010), 2092-2122.

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