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HANKEL BILINEAR FORMS ON GENERALIZED FOCK–SOBOLEV SPACES ON $\mathbf{C}^n$

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Abstract. We characterize the boundedness of Hankel bilinear forms on a product of generalized Fock–Sobolev spaces on $\mathbf{C}^n$ with respect to the weight $(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}$, for $\ell\geq 1$, $\alpha>0$ and $\rho\in\mathbf{R}$. We obtain a weak decomposition of the Bergman kernel with estimates and a Littlewood–Paley formula, which are key ingredients in the proof of our main results. As an application, we characterize the boundedness, compactness and the membership in the Schatten class of small Hankel operators on these spaces.

1. Introduction

The main goal of this work is the characterization of the boundedness of Hankel bilinear forms on generalized Fock–Sobolev spaces.

Given a fixed number $\ell\geq 1$, for $1\leq p<\infty$, $\alpha\geq 0$ and $\rho\in\mathbf{R}$, we consider the space $L^{p,\ell}_{\alpha,\rho}:=L^{p,\ell}_{\alpha,\rho}(\mathbf{C}^n)$ of all measurable functions f on $\mathbf{C}^n$ such that

$$\|f\|^p_{L^{p,\ell}_{\alpha,\rho}}:=\int_{\mathbf{C}^n}\left|f(z)(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}\right|^p dV(z)<\infty,$$

that is, $L^{p,\ell}_{\alpha,\rho}=L^p(\mathbf{C}^n;(1+|z|)^{\rho p}e^{-\frac{\alpha p}{2}|z|^{2\ell}}\,dV(z))$. Here $dV=dV_n$ denotes the Lebesgue measure on $\mathbf{C}^n$ normalized so that the measure of the unit ball $\mathbf{B}^n$ is 1. As usual, if $p=\infty$, $L^{\infty,\ell}_{\alpha,\rho}:=L^{\infty,\ell}_{\alpha,\rho}(\mathbf{C}^n)$ consists of all measurable functions f on $\mathbf{C}^n$ such that $\|f\|_{L^{\infty,\ell}_{\alpha,\rho}}:=\operatorname{ess\,sup}_{z\in\mathbf{C}^n}|f(z)|(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}<\infty$.

For $\alpha>0$, we define the generalized Fock–Sobolev spaces $F^{p,\ell}_{\alpha,\rho}:=H\cap L^{p,\ell}_{\alpha,\rho}$, where $H=H(\mathbf{C}^n)$ is the space of entire functions on $\mathbf{C}^n$. We also consider the little Fock space $\mathfrak{f}^{\infty,\ell}_{\alpha,\rho}$, which is the closure of the space of holomorphic polynomials in $F^{\infty,\ell}_{\alpha,\rho}$. Note that, for any $1\leq p<\infty$, the holomorphic polynomials are also dense in $F^p_{\alpha,\rho}$ (see, for instance, [28, Chapter 2] and Remark 2.13 below).

Since $\ell\geq 1$ is fixed, from now on we will skip it in our notations. If $\rho=0$ we get the generalized Fock spaces $F^p_\alpha=F^p_{\alpha,0}$, and we write $L^p_\alpha=L^p_{\alpha,0}$.

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Note that the space L^2_α is a Hilbert space with the inner product given by the α-pairing

$$\langle f, g\rangle_\alpha := \int_{\mathbf{C}^n} f(z)\overline{g(z)}e^{-\alpha|z|^{2\ell}}dV(z),$$

and F^2_α is a closed linear subspace of L^2_α.

The Fock–Sobolev spaces $F^p_{\alpha,\rho}$ are the natural setting when we are dealing with Fock spaces. For instance, the pointwise estimate of a function in F^p_α as well as the norm estimates of the Bergman kernel are given in terms of weights corresponding to Fock–Sobolev spaces (see Corollary 2.9 and Proposition 2.7). Moreover, each derivative of a Fock function is in a Fock–Sobolev space (see Theorem 1.4). So these spaces have been subject of interest by several authors in recent years, specially for the case $\ell = 1$ (see for instance [7, 5, 6, 18] and the references therein). As it happens for $\ell = 1$ (see, for instance, [13], [4] and the references therein), the model spaces $F^{p,\ell}_{\alpha,\varrho}$, $\ell > 1$, should be useful to solve certain problems in weighted Fock spaces $F^{p,\ell}_\alpha(\omega)$. This will be the object of forthcoming works.

We recall that a Hankel bilinear form on a product of function spaces is a bilinear form Λ satisfying $\Lambda(f, g) = \Lambda(fg, 1)$.

Our first result characterizes the boundedness of the Hankel bilinear forms on $F^p_{\alpha,\rho} \times F^{p'}_{\beta,\eta}$, where $p' = p/(p-1)$, which extends the classical result in [15] for $\ell = 1$ and $\varrho = 0$ (see also the recent paper [24]). In order to state our theorem, we consider the space E of entire functions of order ℓ and finite type, that is, $E = E(\mathbf{C}^n) := \{f \in H(\mathbf{C}^n)\colon |f(z)| = O(e^{\tau|z|^\ell}), \text{ for some } \tau > 0\}$, which is dense in $\mathfrak{f}^\infty_{\alpha,\rho}$ and in $F^p_{\alpha,\rho}$, $1 \le p < \infty$.

Theorem 1.1. *Let $1 \le p \le \infty$, $\alpha, \beta > 0$ and $\rho, \eta \in \mathbf{R}$. A Hankel bilinear form $\Lambda : E \times E \to \mathbf{C}$ satisfies $|\Lambda(f, g)| \lesssim \|f\|_{F^p_{\alpha,\rho}}\|g\|_{F^{p'}_{\beta,\eta}}$ if and only if there exists $b \in F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}$ such that $\Lambda(f, g) = \langle fg, b\rangle_{\frac{\alpha+\beta}{2}}$. In this case, we have $\|\Lambda\| \simeq \|b\|_{F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}}$, and there exists $\varphi \in L^\infty_{0,-\rho-\eta}$ such that the bounded bilinear form $\widetilde{\Lambda}\colon L^p_{\alpha,\rho} \times L^{p'}_{\beta,\eta} \to \mathbf{C}$, defined by $\widetilde{\Lambda}(f, g) = \langle fg, \varphi\rangle_{\frac{\alpha+\beta}{2}}$, coincides with Λ on $E \times E$ and satisfies $\|\widetilde{\Lambda}\| \simeq \|\Lambda\|$.*

As a consequence, we obtain a weak factorization of the space $F^1_{\alpha+\beta,\rho+\eta}$. We recall that the weak product $F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta}$, $1 \le p < \infty$, is the completion of the space of finite sums $h = \sum_j f_j g_j$, $f_j \in F^p_{\alpha,\rho}$ and $g_j \in F^{p'}_{\beta,\eta}$, using the norm

$$\|h\|_{F^p_{\alpha,\rho}\odot F^{p'}_{\beta,\eta}} := \inf\left\{\sum_j \|f_j\|_{F^p_{\alpha,\rho}}\|g_j\|_{F^{p'}_{\beta,\eta}} : h = \sum_j f_j g_j\right\}.$$

We then have:

Corollary 1.2. *For $1 \le p < \infty$, $\alpha, \beta > 0$ and $\rho, \eta \in \mathbf{R}$, $F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta} = F^1_{\alpha+\beta,\rho+\eta}$. Moreover, $F^1_{\alpha,\rho} \odot \mathfrak{f}^\infty_{\beta,\eta} = F^1_{\alpha,\rho} \odot F^\infty_{\beta,\eta} = F^1_{\alpha+\beta,\rho+\eta}$.*

Usually, necessary conditions for the boundedness of a bilinear form Λ are obtained by checking the boundedness on adequate testing functions f and g. This is particularly simple when $\ell = 1$, $\alpha = \beta$, $\rho = \eta = 0$ and $p = 2$ (see [15]). In this classical case, we can take as test functions f and g the square root of the Bergman kernel, that is $f(w) = g(w) = \sqrt{\alpha^n/n!}\, e^{\frac{\alpha}{2}z\overline{w}}$. Here, for $z, w \in \mathbf{C}^n$, $z\overline{w} := \sum_{j=1}^n z_j\overline{w}_j$. Then $|b(z)| = |\langle fg, b\rangle_\alpha| \le \|\Lambda\|\|f\|^2_{F^2_\alpha} = \|\Lambda\|\, e^{\frac{\alpha}{4}|z|^2}$, which proves that $b \in F^\infty_{\alpha/2}$. Observe that the norm estimates of the above test functions f and g are similar to the ones of the Bergman kernel. This is not the situation in the general setting. In fact, although there is a broad literature on pointwise and norm estimates of the Bergman kernel for generalized Fock spaces (see, for instance, [9], [16], [17], [23], [8] and the references therein), it is not at all clear how to derive adequate decompositions of the Bergman kernel from these estimates.

For $\ell > 1$ the choice of the test functions is more delicate because the Bergman kernel $K_\alpha(z,w) = \overline{K_{\alpha,z}(w)}$ is given in terms of derivatives of the so called Mittag-Leffler functions, which have zeros on $\mathbf{C}$ (see, for instance, Lemma 2.5 below and [21, Theorem 2.1.1]). Consequently, it is not clear how to get a strong decomposition as in the previous case. Instead, using the asymptotic behaviour of the Mittag-Leffler functions, we obtain a weak decomposition of the Bergman kernel with accurate pointwise and norm estimates of each factor. This will be a key tool to prove Theorem 1.1.

Theorem 1.3. *Let* $1 \le p \le \infty$ $\alpha, \beta, \gamma > 0$ *and let* $\rho, \eta \in \mathbf{R}$. *Then there exist functions* $G_k = G_{k,\gamma,\alpha,\beta}$, $H_k = H_{k,\gamma,\alpha,\beta} \in E(\mathbf{C})$, $k = 0, \cdots, n$, *such that*

$$K_\gamma(w,z) = \sum_{k=0}^{n} G_k(w\overline{z}) H_k(w\overline{z}), \tag{1.1}$$

$$\|K_{\gamma,z}\|_{F^1_{\alpha+\beta,\rho+\eta}} \simeq \sum_{k=0}^{n} \|G_k(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \|H_k(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \simeq (1+|z|)^{\rho+\eta} e^{\frac{\gamma^2\,|z|^{2\ell}}{2(\alpha+\beta)}}. \tag{1.2}$$

If $\ell = 1$, then $K_\gamma(w,z) = \frac{\gamma^n}{n!} e^{\gamma z \overline{w}}$ and in this case (1.1) reduces to

$$K_\gamma(w,z) = \tfrac{\gamma^n}{n!} e^{\frac{\alpha\gamma}{\alpha+\beta} z\overline{w}} \cdot e^{\frac{\beta\gamma}{\alpha+\beta} z\overline{w}} = \tfrac{n!}{\gamma^n} K_\gamma\left(w, \tfrac{\alpha}{\alpha+\beta} z\right) K_\gamma\left(w, \tfrac{\beta}{\alpha+\beta} z\right). \tag{1.3}$$

For $\ell > 1$, the explicit expression of the functions G_k and H_k is quite involved. A motivated definition of these factors as well as their pointwise and norm estimates are given in Section 4 (see Definition 4.4 and Theorem 4.5 below). In order to prove the norm estimates of the functions G_k and H_k, we use, among other ingredients, the following Littlewood–Paley type formula, which may be of independent interest by itself.

Theorem 1.4. *Let* $1 \le p \le \infty$, $\alpha > 0$ *and* $\rho \in \mathbf{R}$. *For an entire function* f *in* $\mathbf{C}^n$, *let* $|\nabla^m f| = \sum_{|\nu|=m} |\partial^\nu f|$, *where* $|\nu| = \nu_1 + \cdots + \nu_n$. *Then the following assertions are equivalent:*

(i) $f \in F^p_{\alpha,\rho}$.
(ii) *For any* $k \ge 1$, $|\nabla^k f| \in L^p_{\alpha,\rho-k(2\ell-1)}$.
(iii) *For some* $k \ge 1$, $|\nabla^k f| \in L^p_{\alpha,\rho-k(2\ell-1)}$.

Moreover, we have

$$\|f\|_{F^p_{\alpha,\rho}} \simeq \sum_{m=0}^{k-1} |\nabla^m f(0)| + \|\nabla^k f\|_{L^p_{\alpha,\rho-k(2\ell-1)}}.$$

We point out that, in the particular case $\ell = 1$, a fractional derivative version of the Littlewood–Paley formula is given in [5] (see also the references therein).

Finally, as an application of Theorems 1.1 and 1.3, we obtain a characterization of the boundedness, compactness and membership in the Schatten class of the small Hankel operators.

Theorem 1.5. *Let $1 \le p \le \infty$, $\alpha > 0$ and $\rho \in \mathbf{R}$. For $\beta \in (\alpha, 2\alpha)$ and $b \in F^\infty_\beta$, let $\mathfrak{h}_{b,\alpha}$ be the small Hankel operator defined by $\mathfrak{h}_{b,\alpha}(f) := \overline{P_\alpha(\overline{f}\, b)}$, $f \in E$, where P_α is the Bergman projection on F^2_α (see Section 2). Then:*

(i) *$\mathfrak{h}_{b,\alpha}$ extends to a bounded (compact) operator from $F^p_{\alpha,\rho}$ to $\overline{F^p_{\alpha,\rho}}$ if and only if $b \in F^\infty_{\frac{\alpha}{2}}$ (respectively, $b \in \mathfrak{f}^\infty_{\frac{\alpha}{2}}$). Moreover, $\|\mathfrak{h}_{b,\alpha}\|_{F^p_{\alpha,\rho}} \simeq \|b\|_{F^\infty_{\frac{\alpha}{2}}}$.*

(ii) *$\mathfrak{h}_{b,\alpha}$ belongs to the Schatten class $S_p(F^2_{\alpha,\rho}, \overline{F^2_{\alpha,\rho}})$ if and only if $b \in F^p_{\frac{\alpha}{2}, \frac{2n(\ell-1)}{p}}$. Moreover, $\|\mathfrak{h}_{b,\alpha}\|_{S_p(F^2_{\alpha,\rho}, \overline{F^2_{\alpha,\rho}})} \simeq \|b\|_{F^p_{\frac{\alpha}{2}, \frac{2n(\ell-1)}{p}}}$.*

Unlike the case of small Hankel operators, there is a broad bibliography on the characterizations of boundedness, compactness and membership in the Schatten class for Toeplitz operators on large families of weighted Fock spaces (see, for instance, [22, 11, 14, 19, 12] and the references therein). As far as we know, the literature on small Hankel operators is essentially concentrated around the case $\ell = 1$. For instance, in the recent paper [24] the authors characterize the boundedness and compactness of small Hankel operators from $F^{p,1}_\alpha$ to $F^{q,1}_\alpha$, $0 < p, q < \infty$ and $\alpha > 0$. Finally, we remark that for $n = 1$, $\varrho = 0$ and ℓ is a positive integer, the boundedness, compactness and membership in the Hilbert-Schmidt class of the small Hankel operator were studied in an unpublished manuscript written in collaboration with Peláez [3].

The paper is organized as follows: In Section 2 we state the main properties of the Fock–Sobolev spaces $F^p_{\alpha,\rho}$ and the Bergman projection P_α. The Littlewood–Paley formula of Theorem 1.4 and the weak factorization of Theorem 1.3 will be proved in Sections 3 and 4, respectively. Section 5 is devoted to the proof of Theorem 1.1 and Corollary 1.2. Finally, in Section 6 we show Theorem 1.5.

Throughout this paper the notation $\Phi \lesssim \Psi$ means that there exists a constant $C > 0$, which does not depend on the involved variables, such that $\Phi \le C\,\Psi$. We write $\Phi \simeq \Psi$ if $\Phi \lesssim \Psi$ and $\Psi \lesssim \Phi$.

2. The Bergman projection on $L^p_{\alpha,\rho}$

In this section we state some well-known properties of the Bergman projection and the Fock–Sobolev spaces.

2.1. On the two parametric Mittag-Leffler functions $E_{a,b}$. The two parametric Mittag-Leffler functions are the entire functions on $\mathbf{C}$ given by

$$E_{a,b}(\lambda) := \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(ak+b)} \quad (\lambda \in \mathbf{C},\ a, b > 0).$$

Observe that $E_{1,1}(\lambda)$ is just the exponential function e^λ.

A good general reference for the Mittag-Leffler functions is the book [10].

In this section we recall the asymptotic expansions of the two parametric Mittag-Leffler functions and their derivatives. Those expansions will be useful to obtain both pointwise and norm estimates of the Bergman kernel.

Theorem 2.1. [21, Theorem 1.2.1] *Let $a \in (0,1]$ and let $b > 0$. Then, for $|\lambda| \to \infty$, we have*

$$E_{a,b}(\lambda) = \begin{cases} \frac{1}{a}\lambda^{(1-b)/a} e^{\lambda^{1/a}} + O(\lambda^{-1}), & \textit{if } |\arg\lambda| \le \frac{7\pi}{8}\, a, \\ O(\lambda^{-1}), & \textit{if } |\arg\lambda| \ge \frac{5\pi}{8}\, a. \end{cases} \tag{2.4}$$

Here, for $\lambda \in \mathbf{C} \setminus \{0\}$, $\arg\lambda$ denotes the principal branch of the argument of λ, that is, $-\pi < \arg\lambda \leq \pi$. Moreover, for $\beta \in \mathbf{R}$, $\lambda^\beta = |\lambda|^\beta e^{i\beta\arg\lambda}$.

By Cauchy's formula (see [20, Theorem 1.4.2]) we can differentiate the asymptotic expansion (2.4) on smaller sectors to obtain:

Corollary 2.2. *Let $a \in (0,1]$, $b > 0$ and $m \in \mathbf{N}$. Then, for $|\lambda| \to \infty$, we have that*

$$E_{a,b}^{(m)}(\lambda) = \begin{cases} \frac{\lambda^{(m(1-a)+1-b)/a}}{a^{m+1}} e^{\lambda^{1/a}}(1+O(\lambda^{-1/a})) + O(\lambda^{-m-1}), & \text{if } |\arg\lambda| \leq \frac{3\pi}{4}a, \\ O(\lambda^{-m-1}), & \text{if } |\arg\lambda| \geq \frac{3\pi}{4}a. \end{cases}$$

From this result we deduce pointwise estimates of the function $E_{a,b}^{(m)}$. In order to state these estimates we introduce the following function.

Definition 2.3. For $c \geq 0$, let

$$\varphi_c(\lambda) := \begin{cases} |e^{c\lambda^\ell}|, & \text{if } |\arg\lambda| \leq \frac{\pi}{2\ell}, \\ 1, & \text{otherwise.} \end{cases} \tag{2.5}$$

Corollary 2.4. *If $b \in (0,1]$ and $m \in \mathbf{N}$, then*

$$|E_{\frac{1}{\ell},b}^{(m)}(\lambda)| \lesssim (1+|\lambda|)^{m(\ell-1)+(1-b)\ell}\varphi_1(\lambda). \tag{2.6}$$

2.2. The Bergman projection. We denote by P_α the Bergman projection from L_α^2 onto F_α^2 defined by

$$P_\alpha(f)(z) = \langle f, K_{\alpha,z}\rangle_\alpha = \int_{\mathbf{C}^n} f(w)K_\alpha(z,w)e^{-\alpha|w|^{2\ell}}\,dV(w),$$

where K_α is the Bergman kernel and $K_{\alpha,z}(w) := \overline{K_\alpha(z,w)} = K_\alpha(w,z)$.

The first result in this section states that the Bergman kernel can be described in terms of derivatives of the Mittag-Leffler function $E_{1/\ell,1/\ell}$. In order to do that, we recall some standard notations. $\mathbf{N}$ will denote the set of non-negative entire numbers. For a multi-index $\nu = (\nu_1, \cdots, \nu_n) \in \mathbf{N}^n$ and $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$, we use the standard notations $z^\nu = z_1^{\nu_1}\cdots z_n^{\nu_n}$, $\nu! = \nu_1!\cdots\nu_n!$ and $|\nu| = \nu_1 + \cdots + \nu_n$. We then have (see, for instance, [1, §5]):

Lemma 2.5. *The system $\left\{\frac{w^\nu}{\|w^\nu\|_{F_\alpha^2}}\right\}_{\nu\in\mathbf{N}^n}$ is an orthonormal basis for F_α^2, so the Bergman kernel is*

$$K_\alpha(z,w) = \overline{K_{\alpha,z}(w)} = \sum_{\nu\in\mathbf{N}^n} \frac{z^\nu\overline{w}^\nu}{\|w^\nu\|_{F_\alpha^2}^2}.$$

Namely, since $\|w^\nu\|_{F_\alpha^2}^2 = \frac{\alpha^{-\frac{|\nu|+n}{\ell}}}{\ell}\frac{n!\,\nu!\,\Gamma\left(\frac{|\nu|+n}{\ell}\right)}{(n-1+|\nu|)!}$, $K_\alpha(z,w) = H_\alpha(z\overline{w})$, where

$$H_\alpha(\lambda) := \frac{\ell\alpha^{n/\ell}}{n!}\sum_{k=0}^\infty \frac{(n-1+k)!}{k!}\frac{\alpha^{k/\ell}\lambda^k}{\Gamma\left(\frac{k+n}{\ell}\right)} = \frac{\ell\alpha^{n/\ell}}{n!}E_{1/\ell,1/\ell}^{(n-1)}(\alpha^{1/\ell}\lambda).$$

In particular, for any $\delta > 0$ we have

$$K_\alpha(z,\delta w) = K_\alpha(\delta z, w) = \delta^{-n}K_{\alpha\delta^\ell}(z,w). \tag{2.7}$$

As a consequence of (2.6) and the fact that the Taylor coefficients of the function $E_{\frac{1}{\ell},\frac{1}{\ell}}$ are positive, we obtain the following pointwise estimate of the Bergman kernel.

Proposition 2.6. *For $\alpha > 0$ we have*

$$|K_\alpha(z,w)| \lesssim (1+|z|)^{n(\ell-1)}(1+|w|)^{n(\ell-1)}\varphi_\alpha(z\overline{w}).$$

In particular, if $|z| \le M$*, then*

$$|K_\alpha(z,w)| \lesssim (1+|w|)^{n(\ell-1)}\, e^{\alpha M^\ell |w|^\ell} \lesssim e^{\alpha(M+1)^\ell |w|^\ell},$$

so $K_\alpha(\cdot,z) \in E$*, for every* $z \in \mathbf{C}^n$.

The next results will be used to prove our main theorems.

Proposition 2.7. *Let* $1 \le p \le \infty$, $\alpha,\gamma > 0$ *and* $\rho \in \mathbf{R}$*. Then*

$$\|K_\gamma(\cdot,z)\|_{F^p_{\alpha,\rho}} \simeq (1+|z|)^{\rho+2n(\ell-1)/p'} e^{\frac{\gamma^2}{2\alpha}|z|^{2\ell}} \quad (z \in \mathbf{C}^n).$$

The proof of Proposition 2.7 for $\rho = 0$ is in [1], while the general case can be found in [2, Corollary 2.11].

Proposition 2.8. *Let* $1 \le p \le \infty$ *and* $\rho \in \mathbf{R}$*. If* $0 \le \alpha < 2\gamma$ *then the Bergman projection* P_γ *is bounded from* $L^p_{\alpha,\rho}$ *onto* $F^p_{\gamma^2/(2\gamma-\alpha),\rho}$*. Moreover,* P_γ *is the identity operator on* $F^p_{\alpha,\rho}$*. In particular,* $P_\alpha\colon L^p_{\alpha,\rho} \to F^p_{\alpha,\rho}$ *is bounded.*

The condition $\alpha < 2\gamma$ ensures that the projection is well defined, in the sense that if $\varphi \in L^p_{\alpha,\rho}$ then $\varphi\, K_{\gamma,z} \in L^1_{2\gamma}$. The proof of this proposition when $\rho = 0$ can be found in [15] ($\ell = 1$) and in [1] ($\ell > 1$). The general case can be found in [2, Proposition 4.2]. Observe that by Proposition 2.8 $f = P_\alpha(f)$, for any $f \in F^p_{\alpha,\rho}$. Hence Hölder's inequality and Proposition 2.7 give the following elementary pointwise estimate.

Corollary 2.9. *Let* $1 \le p \le \infty$, $\alpha > 0$ *and* $\rho \in \mathbf{R}$*. Then*

$$|f(z)| \lesssim \|f\|_{F^p_{\alpha,\rho}} (1+|z|)^{-\rho+2n(\ell-1)/p} e^{\frac{\alpha}{2}|z|^{2\ell}} \quad (f \in F^p_{\alpha,\rho},\ z \in \mathbf{C}^n),$$

and so $F^p_{\alpha,\rho} \hookrightarrow F^\infty_{\alpha,\rho-2n(\ell-1)/p}$.

Using Corollary 2.9 and simple pointwise estimates of the weights, it is easy to prove the following result. A detailed proof can be found in [2], where we give a complete characterization of the embbedings $F^p_{\alpha,\rho} \hookrightarrow F^q_{\beta,\eta}$.

Corollary 2.10. *Let* $1 \le p,q \le \infty$, $\alpha > 0$ *and* $\rho,\eta \in \mathbf{R}$.

(i) *If* $\beta > \alpha$*, then* $F^p_{\alpha,\rho} \hookrightarrow F^q_{\beta,\eta}$ *and* $F^p_{\alpha,\rho} \hookrightarrow \mathfrak{f}^\infty_{\beta,\eta}$.

(ii) *If* $\rho + 2n(\ell-1)/p' \le \eta$*, then* $F^1_{\alpha,\eta} \hookrightarrow F^p_{\alpha,\rho}$.

The next interpolation result will be used in the forthcoming sections (see, for instance, [2, Lemma 3.10]).

Lemma 2.11. *Let* $1 < p < \infty$*. Then for* $\theta = 1/p'$ *we have*

$$(F^1_{\alpha,\rho}, F^\infty_{\alpha,\rho})_{[\theta]} = F^p_{\alpha,\rho}.$$

Next lemma studies the action of dilations on Fock–Sobolev spaces.

Lemma 2.12. *Let* $1 \le p \le \infty$, $\alpha,\beta > 0$ *and* $\rho \in \mathbf{R}$*. For* $\delta > 0$ *we have:*

(i) *The dilation operator* $f \mapsto f(\delta\cdot)$ *is a topological isomorphism from* $F^p_{\alpha,\rho}$ *onto* $F^p_{\delta^{2\ell}\alpha,\rho}$.

(ii) *If* $f,g \in E$*, then* $\langle f,g\rangle_\alpha = \delta^{2n}\langle f(\cdot), g(\delta^2\cdot)\rangle_{\delta^{2\ell}\alpha}$.

(iii) *If* $f \in E$, $g \in F^p_{\beta,\rho}$ *and* $\delta^{2\ell} < 2\alpha/\beta$*, then* $f(\cdot)\, g(\delta^2\cdot) \in L^1_{2\delta^{2\ell}\alpha}$.

Proof. The change of variables $w = \delta z$ easily gives (i). The same change of variables together with the orthogonality of the monomials give (ii), since $\langle f,g\rangle_\alpha = \delta^{2n}\langle f(\delta\cdot), g(\delta\cdot)\rangle_{\delta^{2\ell}\alpha} = \delta^{2n}\langle f, g(\delta^2\cdot)\rangle_{\delta^{2\ell}\alpha}$. Finally, assertion (iii) follows from (i) and (ii). □

Remark 2.13. As it happens in the classical case $\ell = 1$ and $\rho = 0$ (see, for instance, [28, Proposition 2.9]), Lemma 2.12 (i) and Corollary 2.10 allow us to prove the density of the holomorphic polynomials in $F^p_{\alpha,\rho}$, $1 \le p < \infty$. Indeed, if $f \in F^p_{\alpha,\rho}$ and $f_\delta := f(\delta\cdot)$, $0 < \delta < 1$, then $f_\delta \in F^p_{\delta^{2\ell}\alpha,\rho} \subset F^2_{\delta^\ell\alpha,\rho} \subset F^p_{\alpha,\rho}$. Now, standard arguments give $\|f_\delta - f\|_{F^p_{\alpha,\rho}} \to 0$ as $\delta \to 1^-$. Finally, for fixed $0 < \delta < 1$ there is a sequence of polynomials $\{q_{\delta,k}\}_k$ such that $\|f_\delta - q_{\delta,k}\|_{F^2_{\delta^\ell\alpha,\rho}} \to 0$ as $k \to \infty$, so $\|f_\delta - q_{\delta,k}\|_{F^p_{\alpha,\rho}} \to 0$.

We finish this section with a duality result that we will use later. Its proof is standard, but since we have not found an explicit reference, for a sake of completeness we supply a sketch of the proof.

Proposition 2.14. *If* $1 \le p < \infty$ *and* $\alpha/2 \le \gamma < 2\alpha$, *then the dual* $(F^p_{\alpha,\rho})'$ *of* $F^p_{\alpha,\rho}$ *(with respect to the* γ*-pairing) is* $F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}$. *Moreover, the dual of* $\mathfrak{f}^\infty_{\alpha,\rho}$ *is* $F^1_{\frac{\gamma^2}{\alpha},-\rho}$.

Proof. First we prove that if $g \in F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}$, then $f \in E \to \langle f, g\rangle_\gamma$ extends to a bounded linear form on $F^p_{\alpha,\rho}$. Since $0 < \alpha \le 2\gamma$, Proposition 2.8 gives $F^{p'}_{\frac{\gamma^2}{\alpha},-\rho} = P_\gamma(L^{p'}_{2\gamma-\alpha,-\rho})$. Therefore, if $g \in F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}$, then there exists $\varphi \in L^{p'}_{2\gamma-\alpha,-\rho}$ such that $g = P_\gamma(\varphi)$ and $\|\varphi\|_{L^{p'}_{2\gamma-\alpha}} \simeq \|g\|_{F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}}$. As a consequence,

$$|\langle f, g\rangle_\gamma| = |\langle f, \varphi\rangle_\gamma \le \|\varphi\|_{L^{p'}_{2\gamma-\alpha}} \|f\|_{F^p_{\alpha,\rho}} \simeq \|g\|_{F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}} \|f\|_{F^p_{\alpha,\rho}} \quad (f \in E).$$

In order to prove the converse, observe that Lemma 2.12(i) with $\delta^{2\ell} = \gamma/\alpha$ reduces the proof to the case $\gamma = \alpha$. Namely, $b \in F^{p'}_{\alpha,-\rho}$ if and only if $g = b(\delta^2\cdot) \in F^{p'}_{\frac{\gamma^2}{\alpha},-\rho}$, and, since by hypothesis $\gamma^2/\alpha < 2\gamma$, we have that for any $f \in E$ $fg \in L^1_{2\gamma}$ and $\langle f, b\rangle_\alpha = \langle f, g\rangle_\gamma$. From the classical L^p-duality it is easy to check that the dual of $L^p_{\alpha,\rho}$ with respect to the α-pairing is $L^{p'}_{\alpha,-\rho}$. This result together with Proposition 2.8, for $\alpha = \beta$, prove the duality for $F^p_{\alpha,\rho}$.

Next we deal with the duality of $\mathfrak{f}^\infty_{\alpha,\rho}$. Note that if $b \in F^1_{\alpha,-\rho}$ then $\langle\cdot, b\rangle_\alpha \in (\mathfrak{f}^\infty_{\alpha,\rho})^*$ and $\|\langle\cdot, b\rangle_\alpha\|_{(\mathfrak{f}^\infty_{\alpha,\rho})^*} \lesssim \|b\|_{F^1_{\alpha,-\rho}}$.

Conversely, given $u \in (\mathfrak{f}^\infty_{\alpha,\rho})^*$, we are going to prove that there is $b \in F^1_{\alpha,-\rho}$ such that $u = \langle\cdot, b\rangle_\alpha$ and $\|b\|_{F^1_{\alpha,-\rho}} \lesssim \|u\|_{(\mathfrak{f}^\infty_{\alpha,\rho})^*}$. Choose $\alpha/2 < \beta < \alpha$. By Corollary 2.10 we have $F^2_\beta \hookrightarrow \mathfrak{f}^\infty_{\alpha,\rho}$ and so the restriction of u to F^2_β is a bounded linear form on this space. It follows that there is $g \in F^2_\beta$ such that $u(f) = \langle f, g\rangle_\beta$, for every $f \in E$. Now, by Lemma 2.12 with $\delta^{2\ell} = \frac{\alpha}{\beta} < 2$, we have $b = g(\delta^2\cdot) \in F^2_{\delta^{4\ell}\beta} = F^2_{\frac{\alpha^2}{\beta}}$ and $u(f) = \langle f, b\rangle_\alpha$, for any $f \in E$.

Thus it only remains to prove that $\|b\|_{L^1_{\alpha,-\rho}} \lesssim \|u\|_{(\mathfrak{f}^\infty_{\alpha,\rho})^*}$.

For $f \in C_c(\mathbf{C}^n)$, let $Tf(z) := f(z)(1 + |z|)^{-\rho} e^{\frac{\alpha}{2}|z|^{2\ell}} \in L^\infty_{\alpha,\rho}$. Then we have $\|P_\alpha(Tf)\|_{F^\infty_{\alpha,\rho}} \lesssim \|Tf\|_{L^\infty_{\alpha,\rho}} = \|f\|_{L^\infty}$. Since f is compactly supported, Proposition 2.6 gives that $P_\alpha(Tf) \in E$. Then, by duality,

$$\|b\|_{L^1_{\alpha,-\rho}} = \sup_{\substack{f \in C_c(\mathbf{C}^n) \\ \|f\|_{L^\infty}=1}} |\langle Tf, b\rangle_\alpha| = \sup_{\substack{f \in C_c(\mathbf{C}^n) \\ \|f\|_{L^\infty}=1}} |u(P_\alpha(Tf))| \lesssim \|u\|_{(\mathfrak{f}^\infty_\alpha)^*}. \qquad \square$$

3. Proof of Theorem 1.4

We begin the section with the following technical lemma.

Lemma 3.1. *For $c \in \mathbf{R}$, let $\Phi_{c,z}(w) := \varphi_c(w\overline{z})$, where φ_c is defined by* (2.5). *Then, for any $1 \le p \le \infty$, $\alpha > 0$, $\rho \in \mathbf{R}$ and $c \in [0, \alpha]$,*

$$\|\Phi_{c,z}\|_{L^p_{\alpha,\rho}} \simeq (1 + c^{1/\ell}|z|)^{\rho - 2n(\ell-1)/p} e^{\frac{c^2}{2\alpha}|z|^{2\ell}}.$$

Proof. Let $1 \le p < \infty$. Given $z \in \mathbf{C}^n$, pick an unitary mapping U_z on $\mathbf{C}^n$ which maps z to $(|z|, 0) \in \mathbf{C} \times \mathbf{C}^{n-1}$. Then making the change of variables $v = U_z w$ and integrating in polar coordinates (see [2, Lemma 2.9] for a detailed proof of the second equivalence) we get

$$\begin{aligned}
\|\Phi_{c,z}\|^p_{L^p_{\alpha,\rho}} &\simeq \int_{\mathbf{C}} \varphi_c(|z|v_1)^p \int_{\mathbf{C}^{n-1}} (1 + |v_1| + |v'|)^{\rho p}\, e^{-\frac{\alpha p}{2}(|v_1|^2 + |v'|^2)^\ell}\, dV(v')\, dA(v_1) \\
&\simeq \int_{\mathbf{C}} \varphi_c(|z|u)^p (1 + |u|)^{\rho p - 2(n-1)(\ell-1)}\, e^{-\frac{\alpha p}{2}|u|^{2\ell}}\, dA(u) \\
&= \int_{\{|u| \ge 1, |\arg u| \le \frac{\pi}{2\ell}\}} (1 + |u|)^{\rho p - 2(n-1)(\ell-1)} \left| e^{c|z|^\ell u^\ell - \frac{\alpha}{2}|u|^{2\ell}} \right|^p dA(u) \\
&\quad + \int_{\{|u| < 1, |\arg u| \le \frac{\pi}{2\ell}\}} (1 + |u|)^{\rho p - 2(n-1)(\ell-1)} \left| e^{c|z|^\ell u^\ell - \frac{\alpha}{2}|u|^{2\ell}} \right|^p dA(u) \\
&\quad + \int_{\{|\arg u| > \frac{\pi}{2\ell}\}} (1 + |u|)^{\rho p - 2(n-1)(\ell-1)}\, e^{-\frac{\alpha p}{2}|u|^{2\ell}}\, dA(u) =: I_1 + I_2 + I_3.
\end{aligned}$$

For $|\arg u| \le \frac{\pi}{2\ell}$, we have $\operatorname{Re}(cu^\ell |z|^\ell) - \frac{\alpha}{2}|u|^{2\ell} = \frac{c^2}{2\alpha}|z|^{2\ell} - \frac{\alpha}{2}\left|\frac{c}{\alpha}|z|^\ell - u^\ell\right|^2$. Hence, the change $\lambda = u^\ell$ gives

$$\begin{aligned}
I_1 &= e^{\frac{c^2 p}{2\alpha}|z|^{2\ell}} \int_{\{|u| \ge 1, |\arg u| \le \frac{\pi}{2\ell}\}} (1 + |u|)^{\rho p - 2(n-1)(\ell-1)}\, e^{-\frac{\alpha p}{2}\left|\frac{c}{\alpha}|z|^\ell - u^\ell\right|^2} dA(u) \\
&\lesssim e^{\frac{c^2 p}{2\alpha}|z|^{2\ell}} \int_{\mathbf{C}} (1 + |\lambda|)^{\frac{\rho p - 2n(\ell-1)}{\ell}}\, e^{-\frac{\alpha p}{2}\left|\frac{c}{\alpha}|z|^\ell - \lambda\right|^2} dA(\lambda) \\
&\lesssim (1 + |z|)^{\rho p - 2n(\ell-1)} e^{\frac{c^2 p}{2\alpha}|z|^{2\ell}}.
\end{aligned}$$

The proof of the last inequality for $|z| \le 1$ is clear. For $|z| > 1$, splitting the integral over $\mathbf{C}$ as a sum of the integral on the set

$$A = \left\{\lambda \in \mathbf{C} \colon \tfrac{c}{2\alpha}|z|^\ell \le |\lambda| \le \tfrac{2c}{\alpha}|z|^\ell\right\}$$

and the integral on $\mathbf{C} \setminus A$, it is easy to check that $I_1 \lesssim (1 + |z|)^{\rho p - 2n(\ell-1)} + e^{-\varepsilon|z|^{2\ell}}$ for some $\varepsilon > 0$, which proves the result (see [2, Lemma 2.10] for more details).

The estimates of I_2 and I_3 are much easier. Clearly $I_3 \lesssim 1$ and, since $|e^{c|z|^\ell u^\ell}| \le e^{c|z|^\ell}$, for $|u| < 1$, we also have $I_2 \lesssim e^{cp|z|^\ell}$, which completes the case $p < \infty$.

Next assume $p = \infty$. In this case, arguing as above,

$$\|\Phi_{c,z}\|_{L^\infty_{\alpha,\rho}} \simeq \sup_{v_1 \in \mathbf{C}} \varphi_c(|z|v_1) \sup_{v' \in \mathbf{C}^{n-1}} (1 + |v_1| + |v'|)^\rho\, e^{-\frac{\alpha}{2}(|v_1|^2 + |v'|^2)^\ell}$$

It is easy to check that

$$\sup_{v' \in \mathbf{C}^{n-1}} (1 + |v_1| + |v'|)^\rho\, e^{-\frac{\alpha}{2}(|v_1|^2 + |v'|^2)^\ell} \simeq (1 + |v_1|)^\rho\, e^{-\frac{\alpha}{2}|v_1|^{2\ell}},$$

so $\|\Phi_{c,z}\|_{L^\infty_{\alpha,\rho}} \simeq M_c(z) + L_c(z)$, where

$$M_c(z) = \sup_{|\arg u|\le\frac{\pi}{2\ell}} \varphi_c(|z|u)(1+|u|)^\rho\, e^{-\frac{\alpha}{2}|u|^{2\ell}},$$
$$L_c(z) = \sup_{|\arg u|>\frac{\pi}{2\ell}} \varphi_c(|z|u)(1+|u|)^\rho\, e^{-\frac{\alpha}{2}|u|^{2\ell}}.$$

Now

$$M_c(z) \simeq e^{\frac{c^2}{2\alpha}|z|^{2\ell}} \sup_{|\arg\lambda|\le\frac{\pi}{2}} (1+|\lambda|)^{\rho/\ell}\, e^{-\frac{\alpha}{2}\left|\frac{c}{\alpha}|z|^\ell-\lambda\right|^2}$$
$$= e^{\frac{c^2}{2\alpha}|z|^{2\ell}} \sup_{r>0} (1+r)^{\rho/\ell}\, e^{-\frac{\alpha}{2}\left|\frac{c}{\alpha}|z|^\ell-r\right|^2}.$$

It is easy to check that the last supremum is equivalent to $(1 + c^{1/\ell}|z|)^\rho$ (see for instance [2, Lemma 2.8]). Moreover, $L_c(z) \simeq 1$. Hence

$$\|\Phi_{c,z}\|_{L^\infty_{\alpha,\rho}} \simeq (1+c^{1/\ell}|z|)^\rho e^{\frac{c^2}{2\alpha}|z|^{2\ell}} + 1 \simeq (1+c^{1/\ell}|z|)^\rho e^{\frac{c^2}{2\alpha}|z|^{2\ell}},$$

which ends the proof. □

Proof of Theorem 1.4. The proof of the theorem is a consequence of the following assertions:

1) The linear operators $f \mapsto \partial_{z_j} f$ are bounded from $F^p_{\alpha,\rho}$ to $F^p_{\alpha,\rho+1-2\ell}$.
2) The linear operators

$$S_j(g)(z) := z_j \int_0^1 g(tz)\, dt, \quad j = 1, \cdots, n,$$

are bounded from $F^p_{\alpha,\rho+1-2\ell}$ to $F^p_{\alpha,\rho}$.

Taking for granted these results it is easy to prove the case $k = 1$. Indeed, assertion 1) shows that if $f \in F^p_{\alpha,\rho}$, then $|\nabla f| \in L^p_{\alpha,\rho+1-2\ell}$. Moreover, the identity

$$f(z) = f(0) + \sum_{j=1}^n \int_0^1 z_j \partial_{z_j} f(tz)\, dt, \quad f \in H(\mathbf{C}^n),$$

together with assertion 2) give the converse. Combining these results we have

$$\|f\|_{F^p_{\alpha,\rho}} \simeq |f(0)| + \|\nabla f\|_{L^p_{\alpha,\rho+1-2\ell}} \simeq |f(0)| + \sum_{j=1}^n \|\partial_{z_j} f\|_{F^p_{\alpha,\rho+1-2\ell}}.$$

Iterating this argument we prove the general case.

Next, we prove the two assertions. We begin showing that the linear operator $f \mapsto \partial_{z_j} f$ is bounded from $F^p_{\alpha,\rho}$ to $F^p_{\alpha,\rho+1-2\ell}$. By interpolation (see Lemma 2.11) it is sufficient to prove this result for $p = 1$ and $p = \infty$.

By Proposition 2.8, $f = P_\alpha(f)$, so

$$\partial_{z_j} f(z) = \int_{\mathbf{C}^n} f(w)\, \partial_{z_j} K_\alpha(z,w)\, e^{-\alpha|w|^{2\ell}}\, dV(w).$$

Therefore Lemma 2.5 and Corollary 2.4 imply

$$|\partial_{z_j} K_\alpha(z,w)| \simeq \left|\overline{w}_j\, E^{(n)}_{1/\ell,1/\ell}(\alpha^{1/\ell} z\overline{w})\right| \lesssim |w|(1+|z\overline{w}|)^{(n+1)(\ell-1)} \varphi_\alpha(z\overline{w}),$$

where φ_α is defined by (2.5). Hence

$$|\partial_{z_j} f(z)| \lesssim \int_{\mathbf{C}^n} |f(w)| T_\alpha(z,w) e^{-\alpha|w|^{2\ell}}\, dV(w),$$

where $T_\alpha(z,w) := (1+|z|)^{(n+1)(\ell-1)}(1+|w|)^{(n+1)(\ell-1)+1}\varphi_\alpha(z\overline{w})$. Thus

$$\|\partial_{z_j} f\|_{F^1_{\alpha,\rho-2\ell+1}} \lesssim \int_{\mathbf{C}^n} |f(w)| \|T_\alpha(\cdot,w)\|_{F^1_{\alpha,\rho-2\ell+1}} e^{-\alpha|w|^{2\ell}}\, dV(w),$$

$$\|\partial_{z_j} f\|_{F^\infty_{\alpha,\rho-2\ell+1}} \lesssim \|f\|_{F^\infty_{\alpha,\rho}} \sup_{z\in\mathbf{C}^n} (1+|z|)^{\rho-2\ell+1} e^{-\frac{\alpha}{2}|z|^{2\ell}} \|T_\alpha(z,\cdot)\|_{L^1_{\alpha,-\rho}}.$$

Now Lemma 3.1 shows

$$\|T_\alpha(\cdot,w)\|_{L^1_{\alpha,\rho-2\ell+1}} = (1+|w|)^{n(\ell-1)+\ell}\|\Phi_{\alpha,w}\|_{L^1_{\alpha,\rho+n(\ell-1)-\ell}} \simeq (1+|w|)^\rho e^{\frac{\alpha}{2}|w|^{2\ell}}$$

and

$$\|T_\alpha(z,\cdot)\|_{L^1_{\alpha,-\rho}} = (1+|z|)^{(n+1)(\ell-1)}\|\Phi_{\alpha,z}\|_{L^1_{\alpha,-\rho+n(\ell-1)+\ell}} \simeq (1+|z|)^{-\rho+2\ell-1} e^{\frac{\alpha}{2}|z|^{2\ell}}.$$

Hence $\|\partial_{z_j} f\|_{F^p_{\alpha,\rho-2\ell+1}} \lesssim \|f\|_{F^p_{\alpha,\rho}}$ for $p=1$ and $p=\infty$. Consequently, for any p, we have

$$|f(0)| + \|\nabla f\|_{L^p_{\alpha,\rho+1-2\ell}} \lesssim \|f\|_{F^p_{\alpha,\rho}}.$$

To complete the proof we show that the operators S_j map $F^p_{\alpha,\rho+1-2\ell}$ to $F^p_{\alpha,\rho}$, $p=1$ and $p=\infty$. Indeed, Proposition 2.8 gives

$$S_j(g)(z) = z_j \int_0^1 \int_{\mathbf{C}^n} g(w) K_\alpha(tz,w) e^{-\alpha|w|^{2\ell}}\, dV(w)\, dt.$$

Therefore Proposition 2.7 gives

$$\begin{aligned}\|S_j(g)\|_{F^1_{\alpha,\rho}} &\le \int_0^1 \int_{\mathbf{C}^n} |g(w)| \|K_\alpha(\cdot,tw)\|_{F^1_{\alpha,\rho+1}} e^{-\alpha|w|^{2\ell}}\, dV(w)\, dt \\ &\lesssim \int_{\mathbf{C}^n} |g(w)| e^{-\alpha|w|^{2\ell}} \int_0^1 (1+t|w|)^{\rho+1}\, e^{\frac{\alpha}{2}(t|w|)^{2\ell}}\, dt\, dV(w),\end{aligned}$$

and

$$\begin{aligned}\|S_j(g)\|_{F^\infty_{\alpha,\rho}} &\lesssim \|g\|_{L^\infty_{\alpha,\rho+1-2\ell}} \sup_z (1+|z|)^{\rho+1} e^{-\frac{\alpha}{2}|z|^{2\ell}} \int_0^1 \|K_\alpha(\cdot,tz)\|_{F^1_{\alpha,2\ell-\rho-1}}\, dt \\ &\lesssim \|g\|_{L^\infty_{\alpha,\rho+1-2\ell}} \sup_z (1+|z|)^{\rho+1} e^{-\frac{\alpha}{2}|z|^{2\ell}} \int_0^1 (1+t|z|)^{2\ell-\rho-1}\, e^{\frac{\alpha}{2}(t|z|)^{2\ell}}\, dt.\end{aligned}$$

Therefore, the norm estimates $\|S_j(g)\|_{F^1_{\alpha,\rho}} \lesssim \|g\|_{F^1_{\alpha,\rho+1-2\ell}}$ and $\|S_j(g)\|_{F^\infty_{\alpha,\rho}} \lesssim \|g\|_{F^\infty_{\alpha,\rho+1-2\ell}}$ follow from

$$\int_0^1 (1+ta)^\tau\, e^{(ta)^{2\ell}}\, dt \le c_\tau (1+a)^{\tau-2\ell}\, e^{a^{2\ell}} \quad (a>0),$$

which can be easily derived by splitting the integral as a sum of the integrals from 0 to 1/2 and from 1/2 and 1.

Altogether gives that

$$\|f\|_{F^p_{\alpha,\rho}} \lesssim |f(0)| + \|\nabla f\|_{L^p_{\alpha,\rho+1-2\ell}},$$

which ends the proof of Theorem 1.4. □

As a consequence, we deduce the following result that will be used in the next section.

Corollary 3.2. *Let $f \in E(\mathbf{C})$ and let $k = 0, 1, \cdots$. Then:*

(i) *There exists* $\tau=\tau(k)>0$ *such that* $|f^{(k)}(w\overline{z})|=O\big(e^{\tau|z|^\ell|w|^\ell}\big)$. *In particular,* $f^{(k)}(\cdot\,\overline{z})\in E(\mathbf{C}^n)$, *for every* $z\in\mathbf{C}^n$.

(ii) *For any* $1\le p\le\infty$ *we have*

$$\|f^{(k)}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho-k(2\ell-1)}}\lesssim(1+|z|)^{-k}\|f(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}}\quad(|z|\ge 1).$$

Proof. Let us begin by observing that if $f\in E(\mathbf{C})$, then $f^{(k)}\in E(\mathbf{C})$. This proves (i).

For $z\neq 0$, pick a unitary mapping U_z which maps z to $(|z|,0)\in\mathbf{C}\times\mathbf{C}^{n-1}$. Then making the change of variables $v=U_zw$ and defining $g_z(v_1,v')=f(|z|v_1)$, Theorem 1.4 gives

$$\begin{aligned}\|f^{(k)}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho-k(2\ell-1)}}&=|z|^{-k}\left\|\frac{\partial^k g_z}{\partial v_1^k}\right\|_{F^p_{\alpha,\rho-k(2\ell-1)}}\\&\lesssim|z|^{-k}\,\|g_z\|_{F^p_{\alpha,\rho}}=|z|^{-k}\,\|f(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}}.\end{aligned}$$

□

4. Proof of Theorem 1.3

From the asymptotic expansion (2.4) it is easy to check the following result.

Lemma 4.1. *For* $0<\theta<1$ *there exists* $R_{\ell,\theta}\in H(\mathbf{C})$ *such that*

$$E_{1/\ell,1/\ell}(\lambda)=c_{\ell,\theta}E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(\theta^{1/\ell}\lambda)\,E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}((1-\theta)^{1/\ell}\lambda)\,+R_{\ell,\theta}(\lambda),\tag{4.8}$$

where $c_{\ell,\theta}=\frac{(\theta(1-\theta))^{\frac{1-\ell}{2\ell}}}{\ell}$. *Moreover, by* (2.6),

$$|E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(\theta^{1/\ell}\lambda)|\lesssim(1+|\lambda|)^{\frac{\ell-1}{2}}\varphi_\theta(\lambda),\tag{4.9}$$

$$|E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}((1-\theta)^{1/\ell}\lambda)|\lesssim(1+|\lambda|)^{\frac{\ell-1}{2}}\varphi_{1-\theta}(\lambda),\tag{4.10}$$

$$|R_{\ell,\theta}(\lambda)|\lesssim(1+|\lambda|)^{\frac{\ell-3}{2}}(\varphi_\theta(\lambda)+\varphi_{1-\theta}(\lambda)),\tag{4.11}$$

where φ_c *is the function defined by* (2.5).

For $\ell=1$ the identity (4.8) reduces to $e^\lambda=e^{\theta\lambda}e^{(1-\theta)\lambda}$ and $R_{1,\theta}=0$.

Corollary 4.2. *Let* $\theta\in(0,1)$ *and let* $\tilde\theta=1-\theta$. *Then*

$$\begin{aligned}K_\gamma(z,w)=C_{\ell,\gamma,\theta}&\sum_{k=0}^{n-1}\tbinom{n-1}{k}\theta^{\frac{k}{\ell}}E^{(k)}_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}((\theta\gamma)^{1/\ell}\,z\overline{w})\tilde\theta^{\frac{n-1-k}{\ell}}E^{(n-1-k)}_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(\tilde\theta\gamma)^{\frac{1}{\ell}}z\overline{w})\\&+\frac{\ell\gamma^{n/\ell}}{n!}R^{(n-1)}_{\ell,\theta}(\gamma^{\frac{1}{\ell}}z\overline{w}),\end{aligned}$$

where $C_{\ell,\gamma,\theta}=\frac{\ell\gamma^{n/\ell}c_{\ell,\theta}}{n!}$.

Observe that if $\ell=1$, and $\theta=\frac{\alpha}{\alpha+\beta}$ this decomposition is just (1.3).

In order to prove (1.1) we introduce the following definitions. For $k=0,\cdots,n-1$, let

$$G_{k,\gamma,\theta}(\lambda):=\tbinom{n-1}{k}\theta^{\frac{k}{\ell}}\frac{\ell\gamma^{n/\ell}c_{\ell,\theta}}{n!}E^{(k)}_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}((\theta\gamma)^{1/\ell}\,\lambda),\tag{4.12}$$

$$H_{k,\gamma,\theta}(\lambda):=(1-\theta)^{\frac{n-1-k}{\ell}}E^{(n-1-k)}_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(((1-\theta)\gamma)^{1/\ell}\,\lambda),\tag{4.13}$$

$$R_{n,\gamma,\theta}(\lambda):=\frac{\ell\gamma^{n/\ell}}{n!}R^{(n-1)}_{\ell,\theta}(\gamma^{\frac{1}{\ell}}\lambda).\tag{4.14}$$

We claim that:

Proposition 4.3. *Let $\gamma > 0$ and $\theta \in (0,1)$. Then, for any $z \in \mathbf{C}^n$, the functions $G_{k,\gamma,\theta}(\cdot\,\overline{z})$, $H_{k,\gamma,\theta}(\cdot\,\overline{z})$, $k = 0,\cdots,n-1$, and $R_{n,\gamma,\theta}(\cdot\,\overline{z})$ belong to E. Moreover, for $1 \le p \le \infty$, $\alpha > 0$ and $\rho \in \mathbf{R}$ we have*

$$\|G_{k,\gamma,\theta}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \lesssim (1+|z|)^{\rho+(\ell-1)(2k+1-2n/p)} e^{\frac{\theta^2\gamma^2}{2\alpha}|z|^{2\ell}}, \quad k = 0,\cdots,n-1, \tag{4.15}$$

$$\|R_{n,\gamma,\theta}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \lesssim (1+|z|)^{\rho+(\ell-1)(2n/p'-1)} \big(e^{\frac{\theta^2\gamma^2}{2\alpha}|z|^{2\ell}} + e^{\frac{(1-\theta)^2\gamma^2}{2\alpha}|z|^{2\ell}}\big). \tag{4.16}$$

Observe that replacing $G_{k,\gamma,\theta}$ by $H_{k,\gamma,\theta}$ and p,α,ρ,θ,k by $p',\beta,\eta,1-\theta,n-1-k$, respectively, we obtain

$$\|H_{k,\gamma,\theta}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\eta+(\ell-1)(2(n-1-k)+1-2n/p')} e^{\frac{(1-\theta)^2\gamma^2}{2\beta}|z|^{2\ell}}, \tag{4.17}$$

$$\|R_{n,\gamma,\theta}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\eta+(\ell-1)(2n/p-1)} \big(e^{\frac{\theta^2\gamma^2}{2\beta}|z|^{2\ell}} + e^{\frac{(1-\theta)^2\gamma^2}{2\beta}|z|^{2\ell}}\big). \tag{4.18}$$

Taking for granted these estimates, we conclude the proof of Theorem 1.3.

We first state the following definition.

Definition 4.4. For $\alpha,\beta,\gamma > 0$ and $k = 0,\cdots,n-1$, we define the following entire functions on $\mathbf{C}$ given by:

$$\begin{aligned} &G_k(\lambda) := G_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\lambda), \quad H_k(\lambda) := H_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\lambda), \\ &G_n(\lambda) := R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}(\lambda), \quad H_n(\lambda) := 1, \quad \text{if } \alpha \ge \beta, \\ &G_n(\lambda) := 1, \quad H_n(\lambda) := R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}(\lambda), \quad \text{if } \alpha < \beta. \end{aligned}$$

Proof of Theorem 1.3. By Corollary 4.2, it is clear that the functions G_k and H_k in Definition 4.4 satisfy equation (1.1).

Next we prove (1.2). By Proposition 2.7 and Hölder's inequality we have

$$(1+|z|)^{\rho+\eta} e^{\frac{\gamma^2}{2(\alpha+\beta)}|z|^{2\ell}} \simeq \|K_\gamma(\cdot,z)\|_{F^1_{\alpha+\beta,\rho+\eta}} \lesssim \sum_{k=0}^{n} \|G_{k,\gamma,\theta}(\cdot\overline{z})\|_{F^p_{\alpha,\rho}} \|H_{k,\gamma,\theta}(\cdot\overline{z})\|_{F^{p'}_{\beta,\eta}}.$$

By (4.15) and (4.17),

$$\sum_{k=0}^{n-1} \|G_{k,\gamma,\theta}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \|H_{k,\gamma,\theta}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\rho+\eta} e^{\frac{\gamma^2}{2}\psi(\theta)|z|^{2\ell}},$$

where $\psi(\theta) = \frac{\theta^2}{\alpha} + \frac{(1-\theta)^2}{\beta}$. Since $\psi(\theta) \ge \psi\big(\frac{\alpha}{\alpha+\beta}\big) = \frac{1}{\alpha+\beta}$,

$$\sum_{k=0}^{n-1} \|G_k(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \|H_k(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\rho+\eta} e^{\frac{\gamma^2}{2(\alpha+\beta)}|z|^{2\ell}}.$$

Assume $\alpha \ge \beta$. Now (4.16), with $\theta = \frac{\alpha}{\alpha+\beta}$, shows that

$$\begin{aligned} \|G_n(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \|H_n(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} &\lesssim (1+|z|)^{\rho+(\ell-1)(2n/p'-1)} \big(e^{\frac{\alpha\gamma^2}{2(\alpha+\beta)^2}|z|^{2\ell}} + e^{\frac{\beta^2\gamma^2}{2\alpha(\alpha+\beta)^2}|z|^{2\ell}}\big) \\ &\lesssim (1+|z|)^{\rho+\eta} e^{\frac{\gamma^2}{2(\alpha+\beta)}|z|^{2\ell}}. \end{aligned}$$

By using (4.18) we obtain the same estimate for $\alpha < \beta$. $\square$

For further references, we consider convenient to state the following more precise version of Theorem 1.3, which provides an explicit decomposition of the Bergman kernel K_γ with norm-estimates of the factors.

Theorem 4.5. *Let* $1 \le p \le \infty$ α, $\beta, \gamma > 0$ *and let* $\rho, \eta \in \mathbf{R}$. *Then,*

$$K_\gamma(w,z) = \sum_{k=0}^{n-1} G_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(w\overline{z}) H_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(w\overline{z}) + R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}(w\overline{z}), \tag{4.19}$$

and, for $k = 0, \cdots, n-1$,

$$\|G_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \lesssim (1+|z|)^{\rho+(\ell-1)(2k+1-2n/p)} e^{\frac{\alpha\gamma^2}{2(\alpha+\beta)^2}|z|^{2\ell}}, \tag{4.20}$$

$$\|H_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\eta+(\ell-1)(2n/p-2k-1)} e^{\frac{\beta\gamma^2}{2(\alpha+\beta)^2}|z|^{2\ell}}, \tag{4.21}$$

$$\|R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \lesssim (1+|z|)^{\rho+(\ell-1)(2n/p'-1)} e^{\frac{\alpha\gamma^2}{2(\alpha+\beta)^2}|z|^{2\ell}}, \quad \text{if } \alpha \ge \beta, \tag{4.22}$$

$$\|R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \lesssim (1+|z|)^{\eta+(\ell-1)(2n/p-1)} e^{\frac{\beta\gamma^2}{2(\alpha+\beta)^2}|z|^{2\ell}}, \quad \text{if } \alpha < \beta. \tag{4.23}$$

Therefore, defining

$$\begin{aligned} G_{n,\gamma,\frac{\alpha}{\alpha+\beta}} &= R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}, \quad \text{and} \quad H_{n,\gamma,\frac{\alpha}{\alpha+\beta}} = 1, \quad \text{if } \alpha \ge \beta, \\ G_{n,\gamma,\frac{\alpha}{\alpha+\beta}} &= 1, \quad \text{and} \quad H_{n,\gamma,\frac{\alpha}{\alpha+\beta}} = R_{n,\gamma,\frac{\alpha}{\alpha+\beta}}, \quad \text{if } \alpha < \beta, \end{aligned}$$

we obtain

$$\|K_{\gamma,z}\|_{F^1_{\alpha+\beta,\rho+\eta}} \simeq \sum_{k=0}^{n} \|G_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^p_{\alpha,\rho}} \|H_{k,\gamma,\frac{\alpha}{\alpha+\beta}}(\cdot\,\overline{z})\|_{F^{p'}_{\beta,\eta}} \simeq (1+|z|)^{\rho+\eta} e^{\frac{\gamma^2}{2(\alpha+\beta)}|z|^{2\ell}}.$$

Next we prove Proposition 4.3.

Proof of Proposition 4.3. In order to simplify the notations, for $k = 0, \cdots, n-1$ we write G_k and H_k instead of $G_{k,\gamma,\theta}$ and $H_{k,\gamma,\theta}$, respectively, and R_n instead of $R_{n,\gamma,\theta}$.

By Corollary 2.4 all the Mittag-Leffler functions in the identity (4.8) are in $E(\mathbf{C})$, so $R_{\ell,\theta} \in E(\mathbf{C})$. Therefore, Corollary 3.2(i) shows that $G_k(\cdot\,\overline{z})$, $H_k(\cdot\,\overline{z})$ and $R_n(\cdot\,\overline{z})$ are in $E(\mathbf{C}^n)$.

Next, we prove (4.15) and (4.16). Since G_k and R_n are in E, there exists $\tau > 0$ such that, for every $|z| \le 1$, $|G_k(\cdot\,\overline{z})|$, $|R_n(\cdot\,\overline{z})| \lesssim e^{\tau|w|^\ell}$ and consequently $\|G_k(\cdot\overline{z})\|_{F^p_{\alpha,\rho}}$, $\|R_n(\cdot\overline{z})\|_{F^p_{\alpha,\rho}} \lesssim 1$.

Next consider $|z| > 1$. The estimate (4.15) follows from Corollary 3.2(ii), (4.9) and Lemma 3.1. Indeed,

$$\begin{aligned} \|E^{(k)}_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(\cdot\,(\theta\gamma)^{1/\ell}\overline{z})\|_{F^p_{\alpha,\rho}} &\lesssim (1+|z|)^{-k} \|E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}(\cdot\,(\theta\gamma)^{1/\ell}\overline{z})\|_{F^p_{\alpha,\rho+k(2\ell-1)}} \\ &\lesssim (1+|z|)^{\frac{\ell-1}{2}-k} \|\Phi_{\theta\gamma,z}\|_{L^p_{\alpha,\rho+\frac{\ell-1}{2}+k(2\ell-1)}} \\ &\lesssim (1+|z|)^{\rho+(\ell-1)(2k+1-2n/p)} e^{\frac{\theta^2\gamma^2}{2\alpha}|z|^{2\ell}}. \end{aligned}$$

In order to prove (4.16) for $|z| > 1$, we follow the same arguments used to prove (4.15). Note that (4.11) shows that $R_{\ell,\theta}$satisfies an estimate similar to the one satisfied by $E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}$:

$$|R_{\ell,\theta}(\lambda)| \lesssim (1+|\lambda|)^{\frac{\ell-1}{2}} (\varphi_\theta(\lambda) + \varphi_{1-\theta}(\lambda)). \tag{4.24}$$

Then Corollary 3.2(ii), (4.24) and Lemma 3.1 give

$$\begin{aligned}\|R^{(n-1)}_{\ell,\theta}(\cdot\,\gamma^{\frac{1}{\ell}}\overline{z})\|_{F^p_{\alpha,\rho}} &\lesssim (1+|z|)^{1-n}\|R_{\ell,\theta}(\cdot\,\gamma^{\frac{1}{\ell}}\overline{z})\|_{F^p_{\alpha,\rho+(n-1)(2\ell-1)}}\\ &\lesssim (1+|z|)^{\frac{\ell-1}{2}+1-n}\|\Phi_{\theta\gamma,z}\|_{L^p_{\alpha,\rho+\frac{\ell-1}{2}+(n-1)(2\ell-1)}}\\ &\quad + (1+|z|)^{\frac{\ell-1}{2}+1-n}\|\Phi_{(1-\theta)\gamma,z}\|_{L^p_{\alpha,\rho+\frac{\ell-1}{2}+(n-1)(2\ell-1)}}\\ &\lesssim (1+|z|)^{\rho+(\ell-1)(2n/p'-1)}\big(e^{\frac{\theta^2\gamma^2}{2\alpha}|z|^{2\ell}}+e^{\frac{(1-\theta)^2\gamma^2}{2\alpha}|z|^{2\ell}}\big). \qquad \square\end{aligned}$$

Remark 4.6. If ℓ is a positive integer, then e^{λ^ℓ} is a zero-free entire function. Therefore, we have the strong decomposition

$$K_\gamma(w,z)=\left[e^{\frac{\alpha\gamma}{\alpha+\beta}(z\overline{w})^\ell}\right]\cdot\left[e^{-\frac{\alpha\gamma}{\alpha+\beta}(z\overline{w})^\ell}\,K_\gamma(w,z)\right],$$

whose terms can be estimated with the same methods used above (for $n=1$, $\alpha=\beta$ and $\varrho=\eta=0$, see [3].)

5. Proof of Theorem 1.1 and Corollary 1.2

5.1. Proof of Theorem 1.1. Assume that $|\Lambda(f,g)|\lesssim\|f\|_{F^p_{\alpha,\rho}}\|f\|_{F^{p'}_{\beta,\eta}}$, for $f,g\in E$. The first observation is that if there exists $\gamma>0$, $\tau\in\mathbf{R}$ and $b\in F^\infty_{\gamma,\tau}$ such that $\Lambda(f,g)=\langle fg,b\rangle_\gamma$, for $f,g\in E$, then Proposition 2.8 and Theorem 1.3 give

$$|b(z)|=|\langle K_\gamma(\cdot,z),b\rangle_\gamma|\le\sum_{k=0}^n|\Lambda(G_k(\cdot\,\overline{z}),H_k(\cdot\,\overline{z}))|\lesssim\|\Lambda\|(1+|z|)^{\rho+\eta}e^{\frac{\gamma^2}{2(\alpha+\beta)}|z|^{2\ell}}.$$

Thus $b\in F^\infty_{\frac{\gamma^2}{\alpha+\beta},-\rho-\eta}$ and $\|b\|_{F^\infty_{\frac{\gamma^2}{\alpha+\beta},-\rho-\eta}}\lesssim\|\Lambda\|$.

Therefore it is enough to prove that there exists $b\in F^\infty_{\frac{\alpha+\beta}{2},\tau}$ such that $\Lambda(f,g)=\langle fg,b\rangle_{\frac{\alpha+\beta}{2}}$, for every $f,g\in E$.

Let $E^p_{\alpha,\rho}=(E,\|\cdot\|_{F^p_{\alpha,\rho}})$ and assume $\alpha\ge\beta$. The boundedness of the bilinear form Λ on $E^p_{\alpha,\rho}\times E^{p'}_{\beta,\eta}$ implies that $f\mapsto\Lambda(f,1)$ is a bounded linear form on $E^p_{\alpha,\rho}$. Since $\frac{\alpha+\beta}{2}\le\alpha$, Corollary 2.10 shows that $E^1_{\frac{\alpha+\beta}{2},-\tau}\hookrightarrow E^p_{\alpha,\rho}$, for any $\tau\le-\rho-2n(\ell-1)/p'$. In particular, $f\mapsto\Lambda(f,1)$ is a bounded linear form on $E^1_{\frac{\alpha+\beta}{2},-\tau}$. Therefore, using that Λ is a Hankel form and the fact that de dual of $E^1_{\frac{\alpha+\beta}{2},-\tau}$ with respect to the $\frac{\alpha+\beta}{2}$-pairing is $F^\infty_{\frac{\alpha+\beta}{2},\tau}$ (see Proposition 2.14), we obtain the result. The case $\alpha<\beta$ can be proved in a similar way.

Next we prove the converse. By Proposition 2.8, if $b\in F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}$ then there exists $\varphi\in L^\infty_{0,-\rho-\eta}$ such that $P_{\frac{\alpha+\beta}{2}}(\varphi)=b$ and $\|\varphi\|_{L^\infty_{0,-\rho-\eta}}\simeq\|b\|_{F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}}$. Therefore $\Lambda(f,g)=\langle fg,b\rangle_{\frac{\alpha+\beta}{2}}=\langle fg,\varphi\rangle_{\frac{\alpha+\beta}{2}}$, for $f,g\in E$. Hence Fubini's theorem and Hölder's inequality give

$$|\Lambda(f,g)|\le\|\varphi\|_{L^\infty_{0,-\rho-\eta}}\|f\|_{F^p_{\alpha,\rho}}\|g\|_{F^{p'}_{\beta,\eta}}.$$

So if we consider the form $\widetilde{\Lambda}:L^p_{\alpha,\rho}\times L^{p'}_{\beta,\eta}\to\mathbf{C}$ defined by $\widetilde{\Lambda}(f,g)=\langle fg,\varphi\rangle_\alpha$ we have $\widetilde{\Lambda}=\Lambda$ on $E\times E$ and

$$\|\varphi\|_{L^\infty_{0,-\rho-\eta}}\simeq\|b\|_{F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}}\simeq\|\Lambda\|\le\|\widetilde{\Lambda}\|\le\|\varphi\|_{L^\infty_{0,-\rho-\eta}}.$$

5.2. Proof of Corollary 1.2. First we consider the case $1 < p < \infty$.

By Hölder's inequality it is clear that $F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta} \hookrightarrow F^1_{\alpha+\beta,\rho+\eta}$. Since E is dense in both spaces, in order to prove that they coincide it is enough to prove that $\|h\|_{F^p_{\alpha,\rho}\odot F^{p'}_{\beta,\eta}} \simeq \|h\|_{F^1_{\alpha+\beta,\rho+\eta}}$, for $h \in E$.

It is easy to check that the dual of $F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta}$ is isometrically isomorphic to the space of bounded Hankel bilinear forms on $F^p_{\alpha,\rho} \times F^{p'}_{\beta,\eta}$, which we denote by $\mathcal{H}$. Namely, any $\Psi \in (F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta})'$ defines a bounded bilinear form on $F^p_{\alpha,\rho} \times F^{p'}_{\beta,\eta}$ by $\Lambda(f,g) = \Psi(fg)$, $f, g \in E$, which satisfies $\|\Lambda\| = \|\Psi\|$. Conversely, each $\Lambda \in \mathcal{H}$ defines a form Ψ on $F^p_{\alpha,\rho} \odot F^{p'}_{\beta,\eta}$ by $\Psi(\sum_j f_j\, g_j) = \sum_j \Lambda(f_j, g_j)$ and $\|\Psi\| = \|\Lambda\|$. By Theorem 1.1, the map $b \mapsto \Lambda_b = \langle \cdot, b\rangle_{\frac{\alpha+\beta}{2}}$ is a topological isomorphism from $F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}$ onto $\mathcal{H}$. Therefore the duality $(F^1_{\alpha+\beta,\rho+\eta})' = F^\infty_{\frac{\alpha+\beta}{4},-\rho-\eta}$ with respect to the $\frac{\alpha+\beta}{2}$-pairing (see Proposition 2.14) gives

$$\|h\|_{F^p_{\alpha,\rho}\odot F^{p'}_{\beta,\eta}} = \sup_{\|\Psi\|=1} |\Psi(h)| \simeq \sup_{\|b\|_{F^\infty_{\frac{\alpha+\beta}{4},-\varrho-\eta}}=1} |\langle h, b\rangle_{\frac{\alpha+\beta}{2}}| \simeq \|h\|_{F^1_{\alpha+\beta,\rho+\eta}}.$$

The proof of the case $p = 1$ is similar. It is clear that $F^1_{\alpha,\rho} \odot F^\infty_{\beta,\eta} \hookrightarrow F^1_{\alpha+\beta,\rho+\eta}$. By Proposition 2.14 we have $(\mathfrak{f}^\infty_{\alpha+\beta,\rho+\eta})' = F^1_{\frac{\alpha+\beta}{4},-\rho-\eta}$ with respect to the $\frac{\alpha+\beta}{2}$-pairing. Hence, arguing as above, we have $\|h\|_{F^1_{\alpha,\rho}\odot \mathfrak{f}^\infty_{\beta,\eta}} \simeq \|h\|_{F^1_{\alpha+\beta,\rho+\eta}}$ and

$$F^1_{\alpha+\beta,\rho+\eta} = F^1_{\alpha,\rho} \odot \mathfrak{f}^\infty_{\beta,\eta} \hookrightarrow F^1_{\alpha,\rho} \odot F^\infty_{\beta,\eta} \hookrightarrow F^1_{\alpha+\beta,\rho+\eta},$$

which ends the proof.

6. Proof of Theorem 1.5

We begin observing that by dilation we can reduce the proof of Theorem 1.5 to the case $\alpha = 1$. As usual, we denote $S_p(F^2_{\alpha,\rho}, \overline{F^2_{\alpha,\rho}})$ by $S_p(F^2_{\alpha,\rho})$.

By Lemma 2.12(i), the dilation operator $\Psi_\alpha(f)(z) := f(\alpha^{\frac{-1}{2\ell}} z)$ is a topological isomorphism from $L^p_{\tau\alpha,\rho}$ onto $L^p_{\tau,\rho}$, $1 \le p \le \infty$, $\tau > 0$, such that $\Psi_\alpha(F^p_{\tau\alpha,\rho}) = F^p_{\tau,\rho}$ and $\Psi_\alpha(\overline{F^p_{\tau\alpha,\rho}}) = \overline{F^p_{\tau,\rho}}$. Moreover, for $f \in E$,

$$\Psi_\alpha(\mathfrak{h}_{b,\alpha}(f))(z) = \langle K_\alpha(\cdot, \alpha^{\frac{-1}{2\ell}} z)\, f, b\rangle_\alpha = \langle K_\alpha(\alpha^{\frac{-1}{2\ell}} \cdot, z)\, f, b\rangle_\alpha.$$

Therefore Lemma 2.12(ii) and (2.7) give

$$\begin{aligned}\Psi_\alpha(\mathfrak{h}_{b,\alpha}(f))(z) &= \alpha^{\frac{-n}{\ell}} \langle K_\alpha(\alpha^{\frac{-1}{\ell}} \cdot, z)\, \Psi_\alpha(f), \Psi_\alpha(b)\rangle_1 \\ &= \langle K_1(\cdot, z)\, \Psi_\alpha(f), \Psi_\alpha(b)\rangle_1 = \mathfrak{h}_{\Psi_\alpha(b),1}(\Psi_\alpha(f))(z).\end{aligned}$$

So the boundedness (compactness) of the operator $\mathfrak{h}_{b,\alpha}$ on $F^p_{\alpha,\rho}$ is equivalent to the boundedness (respectively, compactness) of $\mathfrak{h}_{\Psi_\alpha(b),1}$ on $F^p_{1,\rho}$ and

$$\|\mathfrak{h}_{b,\alpha}\|_{F^p_{\alpha,\rho}} \simeq \|\mathfrak{h}_{\Psi_\alpha(b),1}\|_{F^p_{1,\rho}}.$$

Similarly, $\mathfrak{h}_{b,\alpha} \in S_p(F^2_{\alpha,\rho})$ if and only if $\mathfrak{h}_{\Psi_\alpha(b),1} \in S_p(F^2_{1,\rho})$, with equivalent norms (see, for instance, [25, Theorem 7.8]). Moreover,

$$\|\mathfrak{h}_{\Psi_\alpha(b),1}\|_{F^p_{1,\rho}} \simeq \|\Psi_\alpha(b)\|_{F^\infty_{\frac{1}{2}}} \Longleftrightarrow \|\mathfrak{h}_{b,\alpha}\|_{F^\infty_{\frac{\alpha}{2}}} \simeq \|b\|_{F^\infty_{\frac{\alpha}{2}}},$$

and

$$\|\mathfrak{h}_{\Psi_\alpha(b),1}\|_{S_p(F^2_{1,\rho})} \simeq \|\Psi_\alpha(b)\|_{F^p_{\frac{1}{2},\rho+2n(\ell-1)/p}} \Longleftrightarrow \|\mathfrak{h}_{b,\alpha}\|_{S_p(F^2_{\alpha,\rho})} \simeq \|b\|_{F^p_{\frac{\alpha}{2},\rho+2n(\ell-1)/p}}.$$

Hence from now on we only consider the case $\alpha = 1$ and we will simplify the notations by writing $\langle\cdot,\cdot\rangle$, $\mathfrak{h}_{\mathfrak{b}}$, K, P, ..., instead of $\langle\cdot,\cdot\rangle_1$, $\mathfrak{h}_{\mathfrak{b},1}$, K_1, P_1, ...

In order to prove Theorem 1.5, we will use (4.19) with $\gamma = 1$ and $\alpha = \beta = 1$, that is,

$$K(w,z) = \sum_{k=0}^{n-1} G_{k,1,\frac{1}{2}}(w\,\overline{z})H_{k,1,\frac{1}{2}}(w\,\overline{z}) + R_{n,1,\frac{1}{2}}(w\,\overline{z}). \tag{6.25}$$

According to the choice in the statement of Theorem 4.5, we write $G_k = G_{k,1,1/2}$, $H_k = H_{k,1,1/2}$, $k = 0, \cdots, n-1$, $G_n = R_{n,1,1/2}$ and $H_n = 1$.

Let $b \in F^\infty_\beta$, $0 < \beta < 2$. Since $\overline{b(z)} = \langle K(\cdot,z), b\rangle$, (6.25) shows that

$$\overline{b(z)} = \sum_{k=0}^{n} \langle H_k(\cdot\overline{z}), \overline{\mathfrak{h}_b(G_k(\cdot\overline{z}))}\rangle = \sum_{k=0}^{n} \langle G_k(\cdot\overline{z}), \overline{\mathfrak{h}_b(H_k(\cdot\overline{z}))}\rangle. \tag{6.26}$$

This representation formula is the main tool to prove Theorem 1.5.

6.1. Proof of Theorem 1.5(i). In this section we prove that $\mathfrak{h}_b$ extends to a bounded (compact) operator from $F^p_{1,\rho}$ to $\overline{F^p_{1,\rho}}$ if and only if $b \in F^\infty_{\frac{1}{2}}$ (respectively, $b \in \mathfrak{f}^\infty_{\frac{1}{2}}$), and in this case $\|\mathfrak{h}_b\|_{F^p_{1,\rho}} \simeq \|b\|_{F^\infty_{\frac{1}{2}}}$.

6.1.1. Proof of the sufficient condition. Assume $b \in F^\infty_{\frac{1}{2}}$. By Proposition 2.8, there exists $\varphi \in L^\infty$ such that $P(\varphi) = b$ and $\|\varphi\|_{L^\infty} \simeq \|b\|_{F^\infty_{\frac{1}{2}}}$. Therefore $\mathfrak{h}_b(f)(z) = \langle f\,K(\cdot,z),\, b\rangle = \langle f\,K(\cdot,z),\, \varphi\rangle$, and consequently

$$|\mathfrak{h}_b(f)(z)| \le \|\varphi\|_{L^\infty}\langle |f|,\, |K(\cdot,z)|\rangle. \tag{6.27}$$

By Proposition 2.7,

$$|\mathfrak{h}_b(f)(z)| \le \|\varphi\|_{L^\infty}\|f\|_{F^\infty_{1,\rho}}\,\|K(\cdot,z)\|_{F^1_{1,-\rho}} \lesssim \|\varphi\|_{L^\infty}\|f\|_{F^\infty_{1,\rho}}(1+|z|)^{-\rho}e^{\frac{1}{2}|z|^{2\ell}},$$

so $\|\mathfrak{h}_{b,\alpha}\|_{F^\infty_{1,\rho}} \lesssim \|b\|_{F^\infty_{\frac{1}{2}}}$. Next, (6.27), Fubini's theorem and Proposition 2.7 give

$$\begin{aligned}\|\mathfrak{h}_{b,\alpha}(f)\|_{F^1_{1,\rho}} &\lesssim \|\varphi\|_{L^\infty}\langle |f(w)|,\, \|K(\cdot,w)\|_{F^1_{1,\rho}}\rangle\\ &\lesssim \|\varphi\|_{L^\infty}\langle |f(w)|,\, (1+|w|)^\rho e^{\frac{1}{2}|w|^{2\ell}}\rangle = \|\varphi\|_{L^\infty}\|f\|_{F^1_{1,\rho}},\end{aligned}$$

which proves that $\|\mathfrak{h}_{b,\alpha}\|_{F^1_{1,\rho}} \lesssim \|b\|_{F^\infty_{\frac{1}{2}}}$.

By Lemma 2.11 we obtain $\|\mathfrak{h}_{b,\alpha}\|_{F^p_{1,\rho}} \lesssim \|b\|_{F^\infty_{\frac{1}{2}}}$, for $1 \le p \le \infty$.

Now assume that $b \in \mathfrak{f}^\infty_{\frac{1}{2}}$. Since $\mathfrak{f}^\infty_{\frac{1}{2}}$ is the closure of the polynomials in $F^\infty_{\frac{1}{2}}$, there is a sequence of polynomials $\{q_k\}_{k\in\mathbf{N}}$ such that $\|q_k - b\|_{F^\infty_{\frac{1}{2}}} \to 0$. Therefore $\|\mathfrak{h}_{q_k} - \mathfrak{h}_b\|_{F^p_{1,\rho}} \to 0$, because

$$\|\mathfrak{h}_{q_k} - \mathfrak{h}_b\|_{F^p_{1,\rho}} = \|\mathfrak{h}_{q_k-b}\|_{F^p_{1,\rho}} \lesssim \|q_k - b\|_{F^\infty_{\frac{1}{2}}}.$$

Since $\{\mathfrak{h}_{q_k}\}_{k\in\mathbf{N}}$ is a sequence of finite rank operators, it follows that $\mathfrak{h}_b : F^p_{1,\rho} \to \overline{F^p_{1,\rho}}$ is compact.

Remark 6.1. Using the above arguments we have that if $b \in F^\infty_{\frac{1}{2}}$ then $\mathfrak{h}_b$ is bounded on $\mathfrak{f}^\infty_{1,\rho}$. Indeed, by (6.27) and Proposition 2.6,

$$\begin{aligned}|\mathfrak{h}_b(f)(z)| &\lesssim \langle \chi_R|f|,\, |K(\cdot,z)|\rangle + \langle (1-\chi_R)|f|,\, |K(\cdot,z)|\rangle\\ &\lesssim e^{(R+1)^\ell|z|^\ell}\|f\|_{L^1_{\frac{1}{2}}} + \|(1-\chi_R)f\|_{L^\infty_{1,\rho}},\, \|K(\cdot,z)\|_{F^1_{1,-\rho}}\end{aligned}$$

where $f\in\mathfrak{f}^\infty_{1,\rho}$ and χ_R denotes the characteristic function of the ball centered at 0 and radius R. By Proposition 2.7

$$(1+|z|)^\rho\, e^{-\frac12|z|^{2\ell}}|\mathfrak{h}_b(f)(z)|\lesssim\|f\|_{F^\infty_{1,\rho}}\, e^{-\frac12|z|^{2\ell}+(R+1)^\ell|z|^\ell}+\|(1-\chi_R)f\|_{L^\infty_{1,\rho}}.$$

Since $f\in\mathfrak{f}^\infty_{1,\rho}$, $\|(1-\chi_R)f\|_{L^\infty_{1,\rho}}\to0$ as $R\to\infty$. Moreover, for any $R>0$, $(1+|z|)^\rho\, e^{-\frac12|z|^{2\ell}+(R+1)^\ell|z|^\ell}\to0$ as $|z|\to\infty$. That proves that $\mathfrak{h}_b(f)\in\mathfrak{f}^\infty_{1,\rho}$.

6.1.2. Proof of the necessary condition. First we prove that if $\mathfrak{h}_b$ is bounded on $F^p_{1,\rho}$, then $b\in F^\infty_{\frac12}$.

For $k=0,\cdots,n$, let us consider the "normalized" functions

$$\begin{aligned}\widetilde{G}_{k,z}(w)&:=(1+|z|)^{-\rho-(\ell-1)(2k+1-2n/p)}e^{-\frac{|z|^{2\ell}}{8}}G_k(w\overline{z}),\\ \widetilde{H}_{k,z}(w)&:=(1+|z|)^{\rho-(\ell-1)(2n/p-2k-1)}e^{-\frac{|z|^{2\ell}}{8}}H_k(w\overline{z}).\end{aligned}$$

By (4.20)–(4.22) we have $\|\widetilde{G}_{k,z}\|_{F^p_{1,\rho}}\lesssim1$ and $\|\widetilde{H}_{k,z}\|_{F^{p'}_{1,-\rho}}\lesssim1$. Using the representation formula (6.26) we have

$$e^{-\frac{|z|^{2\ell}}{4}}\,\overline{b(z)}=\sum_{k=0}^n\langle\widetilde{H}_{k,z},\overline{\mathfrak{h}_b(\widetilde{G}_{k,z})}\rangle,\tag{6.28}$$

so, by Schwarz's inequality, $\|b\|_{F^\infty_{\frac12}}\lesssim\|\mathfrak{h}_b\|_{F^p_{1,\rho}}$.

Now assume that $\mathfrak{h}_b$ is compact on $F^p_{1,\rho}$, $1<p<\infty$. By (6.28) we have

$$e^{-\frac{|z|^{2\ell}}{4}}|b(z)|\lesssim\sum_{k=0}^n\big\|\mathfrak{h}_b(\widetilde{G}_{k,z})\big\|_{F^p_{1,\rho}}.$$

Consequently, in order to show that $b\in\mathfrak{f}^\infty_{\frac12}$, it is enough to prove that $\|\mathfrak{h}_b(\tilde{G}_{k,z})\|_{F^p_{1,\rho}}\to0$ as $|z|\to\infty$. Since, for $|w|\le R$, $|G_k(w\overline{z})|\lesssim e^{(R^\ell+1)|z|^\ell}$, $\widetilde{G}_{k,z}$ converges uniformly to 0 on compact sets as $|z|\to\infty$. This fact together with $\|\widetilde{G}_{k,z}\|_{F^p_{1,\rho}}\lesssim1$ easily shows that $\tilde{G}_{k,z}\to0$ weakly in $F^p_{1,\rho}$ as $|z|\to\infty$ (see, for instance, [3, Lemma 5.1]). Therefore, the compactness of $\mathfrak{h}_b$ implies that $\|\mathfrak{h}_b(\tilde{G}_{k,z})\|_{F^p_{1,\rho}}\to0$ as $|z|\to\infty$.

The same argument proves that if $\mathfrak{h}_b$ is compact on $\mathfrak{f}^\infty_{1,\rho}$, then $b\in\mathfrak{f}^\infty_{\frac12}$.

Next we use this result to prove that if $\mathfrak{h}_b$ is compact on $F^1_{1,\rho}$ then $b\in\mathfrak{f}^\infty_{\frac12}$.

If $\mathfrak{h}_b$ is compact on $F^1_{1,\rho}$, then it is bounded, so $b\in F^\infty_{\frac12}$. By Remark 6.1 we have that $\mathfrak{h}_b$ is bounded on $\mathfrak{f}^\infty_{1,-\rho}$. The duality $(\mathfrak{f}^\infty_{1,-\rho})'=F^1_{1,\rho}$ together with the fact that $\langle\overline{\mathfrak{h}_b(f)},g\rangle=\langle f,\overline{\mathfrak{h}_b(g)}\rangle=\langle fg,b\rangle$, give that $\mathfrak{h}_b$ is compact in $F^1_{1,\rho}$ if and only if it is compact in $\mathfrak{f}^\infty_{1,-\rho}$, which implies $b\in\mathfrak{f}^\infty_{\frac12}$.

Finally, if $\mathfrak{h}_b$ is compact in $F^\infty_{1,\rho}$ then $b\in F^\infty_{\frac12}$ and $\mathfrak{h}_b$ is bounded on $F^1_{1,-\rho}$. So it is compact on $F^\infty_{1,\rho}$ if and only if it is compact on $F^1_{1,-\rho}$, which implies that $b\in\mathfrak{f}^\infty_{\frac12}$.

6.2. Proof of Theorem 1.5(ii). In this section we prove that $\mathfrak{h}_b$ is in $S_p(F^2_{1,\rho})=S_p(F^2_{1,\rho},\overline{F^2_{1,\rho}})$ if and only if $b\in F^p_{\frac12,2n(\ell-1)/p}$, and, in this case, $\|\mathfrak{h}_b\|_{S_p(F^2_{1,\rho})}\simeq\|b\|_{F^p_{\frac12,2n(\ell-1)/p}}$.

We start this section recalling some well-known results concerning to the Schatten class $S_p(H_0,H_1)$, where H_0 and H_1 are separable complex Hilbert spaces. See, for instance, [25, Chapter 7].

Let T be a compact linear operator from H_0 to H_1. Then $|T| := (T^*T)^{1/2}$ is a compact positive operator on H_0, so we may consider its sequence of eigenvalues $\{s_k(T)\}_{k\in\mathbf{N}}$, which are usually called the singular values of T.

For $0 < p < \infty$, the Schatten class $S_p(H_0, H_1)$ consists of all compact linear operators T from H_0 to H_1 such that

$$\|T\|^p_{S_p(H_0,H_1)} := \sum_{k=1}^{\infty} s_k(T)^p < \infty.$$

Moreover, $S_\infty(H_0, H_1)$ is the space of all the bounded linear operators from H_0 to H_1.

Note that $(S_p(H_0, H_1), \|\cdot\|_{S_p(H_0,H_1)})$ is a Banach space for $p \ge 1$ and a quasi-Banach space for $p < 1$. Moreover, since $\|T\|_{S_q(H_0,H_1)} \le \|T\|_{S_p(H_0,H_1)}$ for $p < q$ and $T \in S_p(H_0, H_1)$, we have the embedding

$$S_p(H_0, H_1) \hookrightarrow S_q(H_0, H_1), \quad (0 < p < q \le \infty).$$

By using the polar decomposition of T, it turns out that there exist two orthonormal systems $\{u_k\}_{k\in\mathbf{N}}$ and $\{v_k\}_{k\in\mathbf{N}}$ of H_0 and H_1, respectively, such that

$$T(f) = \sum_{k=1}^{\infty} s_k(T)\langle f, u_k\rangle_{H_0} v_k.$$

Note that if $T_k(f) := s_k(T)\langle f, u_k\rangle_{H_0} v_k$, then $\|T_k\|_{S_p(H_0,H_1)} = s_k(T)$. So if $T \in S_1(H_0, H_1)$, then the rank one operators T_k satisfy

$$\sum_{k=1}^{n} T_k \to T \ \text{ in } S_1(H_0, H_1) \quad \text{and} \quad \Big\|\sum_{k=1}^{n} T_k\Big\|_{S_1(H_0,H_1)} = \sum_{k=1}^{n} \|T_k\|_{S_1(H_0,H_1)}. \tag{6.29}$$

We end this section by recalling the interpolation identity

$$(S_1(H_0, H_1), S_\infty(H_0, H_1))_{[\theta]} = S_{1/(1-\theta)}(H_0, H_1) \quad (0 < \theta < 1). \tag{6.30}$$

See, for instance, [27, Theorem 2.6].

The following lemma is easy to check.

Lemma 6.2. *$T\colon F^2_{1,\rho} \to \overline{F^2_{1,\rho}}$ is a bounded linear operator of rank one if and only if there are non zero functions $g \in F^2_{1,-\rho}$ and $h \in F^2_{1,\rho}$ such that $T(f) = \langle f, g\rangle \overline{h}$, for any $f \in F^2_{1,\rho}$. Moreover, in this case, $\|T\|_{S_p(F^2_{1,\rho})} \simeq \|g\|_{F^2_{1,-\rho}}\|h\|_{F^2_{1,\rho}}$, for $1 \le p \le \infty$.*

6.2.1. Proof of the sufficient condition. The sufficient condition is a direct consequence of the following result.

Proposition 6.3. *For $1 \le p \le \infty$, the operator $b \mapsto \mathfrak{h}_b$ is bounded from $F^p_{\frac{1}{2},\frac{2n(\ell-1)}{p}}$ to $S_p(F^2_{1,\rho})$.*

In order to prove Proposition 6.3, we will need the following interpolation Lemma.

Lemma 6.4. *Let $1 < p < \infty$. Then*

$$(L^1_{1/2,2n(\ell-1)}, L^\infty_{1/2})_{[1/p']} = L^p_{1/2,2n(\ell-1)/p}, \quad \textit{and} \tag{6.31}$$

$$(F^1_{1/2,2n(\ell-1)}, F^\infty_{1/2})_{[1/p']} = F^p_{1/2,2n(\ell-1)/p}, \tag{6.32}$$

Proof. We begin with the proof of (6.31). Since $f \mapsto f(z)e^{-\frac{|z|^{2\ell}}{2}}$ is an isometric isomorphism from $L^p_{1/2,2n(\ell-1)/p}$ onto $L^p((1+|z|)^{2n(\ell-1)}\,dV(z))$, Riesz-Thorin theorem gives (6.31).

By Proposition 2.8, $P_{\frac{1}{2}}$ is bounded from $L^p_{1/2,2n(\ell-1)/p}$ to $F^p_{1/2,2n(\ell-1)/p}$ and it is the identity on $F^p_{1/2,2n(\ell-1)/p} \hookrightarrow L^p_{1/2,2n(\ell-1)/p}$. Thus $F^p_{1/2,2n(\ell-1)/p}$ is a retract of $L^p_{1/2,2n(\ell-1)/p}$, $1 \le p \le \infty$ and, consequently, (6.32) follows from (6.31). $\square$

Proof of Proposition 6.3. By the interpolation identities (6.32) and (6.30) it is enough to prove the result for $p = 1$ and $p = \infty$. Since the last case has been done in the previous section, we only have to deal with the case $p = 1$.

Assume $b \in F^1_{\frac{1}{2},2n(\ell-1)}$. By Corollary 2.9, $b \in F^\infty_{\frac{1}{2}}$ and $b = P_{\frac{1}{2}} b$. Therefore, for $f \in E$ we have

$$\begin{aligned}(\mathfrak{h}_b(f))\,(z) &= \int_{\mathbf{C}^n} f(u)\,\overline{b(u)}\,K(u,z)\,e^{-|u|^{2\ell}}\,dV(u)\\ &= \int_{\mathbf{C}^n} f(u) \int_{\mathbf{C}^n} \overline{b(w)}\,K_{\frac{1}{2}}(w,u)\,e^{-\frac{|w|^{2\ell}}{2}}\,dV(w)\,K(u,z)\,e^{-|u|^{2\ell}}\,dV(u),\end{aligned}$$

and Fubini's theorem gives

$$(\mathfrak{h}_b(f))\,(z) = \int_{\mathbf{C}^n} \overline{b(w)}\,(\mathfrak{h}_{K_{\frac{1}{2}}(\cdot,w)} f)(z)\,e^{-\frac{|w|^{2\ell}}{2}}\,dV(w). \tag{6.33}$$

This allows us to consider the following Bochner integral

$$\int_{\mathbf{C}} \overline{b(w)}\,\mathfrak{h}_{K_{1/2}(\cdot,w)}\,e^{-\frac{|w|^{2\ell}}{2}}\,dV(w). \tag{6.34}$$

By Bochner's integrability theorem (see for instance [26, p. 133]), the $S_1(F^2_{1,\rho})$-convergence of the Bochner's integral (6.34) means that the integrand

$$S(w) := \overline{b(w)}\,\mathfrak{h}_{K_{1/2}(\cdot,w)}$$

is an $S_1(F^2_{1,\rho})$-valued strongly measurable function on $\mathbf{C}$ which satisfies

$$\int_{\mathbf{C}} \|S(w)\|_{S_1(F^2_{1,\rho})}\,e^{-\frac{|w|^{2\ell}}{2}}\,dV(w) < \infty. \tag{6.35}$$

We are going to show that $S(w)$ is an operator of rank at most one, for every $w \in \mathbf{C}$, and next we estimate its $S_1(F^{2,\ell}_{1,\rho})$-norm.

For any $w \in \mathbf{C}$ and $f \in E$, we have

$$\mathfrak{h}_{K_{1/2}(\cdot,w)}(f)(z) = 2^{-n/\ell}\langle f, K(\cdot,2^{-1/\ell}w)\rangle\,K(2^{-1/\ell}w,z). \tag{6.36}$$

Indeed, by (2.7), $K_{1/2}(\cdot,w) = 2^{-n/\ell}K(\cdot,2^{-1/\ell}w)$. Therefore

$$\begin{aligned}\mathfrak{h}_{K_{1/2}(\cdot,w)}(f)(z) &= 2^{-n/\ell}\langle fK(\cdot,z),\,K(\cdot,2^{-1/\ell}w)\rangle = 2^{-n/\ell}f(2^{-1/\ell}w)K(2^{-1/\ell}w,z)\\ &= 2^{-n/\ell}\langle f, K(\cdot,2^{-1/\ell}w)\rangle\,K(2^{-1/\ell}w,z).\end{aligned}$$

So $\mathfrak{h}_{K_{1/2}(\cdot,w)}$ is an operator of rank one and, by Lemma 6.2 and Proposition 2.7, we obtain

$$\begin{aligned}\|\mathfrak{h}_{K_{1/2}(\cdot,w)}\|_{S_1(F^2_{1,\rho})} &\simeq \|K(\cdot,2^{-1/\ell}w)\|_{F^2_{1,-\rho}}\|K(\cdot,2^{-1/\ell}w)\|_{F^2_{1,\rho}}\\ &\simeq (1+|w|)^{2n(\ell-1)}\,e^{\frac{|w|^{2\ell}}{4}}.\end{aligned} \tag{6.37}$$

Observe that (6.36) shows that S is an $S_1(F^2_{1,\rho})$-valued function on $\mathbf{C}$. Moreover, it is $S_1(F^2_{1,\rho})$-strongly measurable because

$$w \in \mathbf{C} \longmapsto \mathfrak{h}_{K_{1/2}(\cdot,w)} \in S_1(F^2_{1,\rho})$$

is continuous. That follows because $\mathfrak{h}_{K_{1/2}(\cdot,w)} - \mathfrak{h}_{K_{1/2}(\cdot,v)}$ has rank at most 2 and so

$$\begin{aligned}\|\mathfrak{h}_{K_{1/2}(\cdot,w)} - \mathfrak{h}_{K_{1/2}(\cdot,v)}\|_{S_1(F^2_{1,\rho})} &\le 2\,\|\mathfrak{h}_{\{K_{1/2}(\cdot,w)-K_{1/2}(\cdot,v)\}}\|_{S_\infty(F^2_{1,\rho})}\\ &\overset{(1)}{\lesssim} \|K_{1/2}(\cdot,w)-K_{1/2}(\cdot,v)\|_{F^\infty_{1/2}}\\ &\overset{(2)}{\lesssim} \|K_{1/2}(\cdot,w)-K_{1/2}(\cdot,v)\|_{F^1_{\frac{1}{2},2n(\ell-1)}} \xrightarrow{(3)} 0,\end{aligned}$$

as $w \to v$, where (1), (2) and (3) are consequences of Theorem 1.5(i), Corollary 2.9 and the dominated convergence theorem, respectively.

Now (6.37) gives (6.35):

$$\int_{\mathbf{C}} \|S(w)\|_{S_1(F^{2,\ell}_1)}\, e^{-\frac{|w|^{2\ell}}{2}}\, dV(w) \lesssim \int_{\mathbf{C}} |b(w)|\,(1+|w|)^{2n(\ell-1)} e^{-\frac{|w|^{2\ell}}{4}}\, dV(w).$$

Therefore, by (6.33), $\mathfrak{h}_b \in S_1(F^2_{1,\rho})$ and $\|\mathfrak{h}_b\|_{S_1(F^2_{1,\rho})} \lesssim \|b\|_{F^1_{1/2,2n(\ell-1)}}$. □

6.2.2. Proof of necessary condition. The following definition is motived by (6.26).

Definition 6.5. For $T \in S_\infty(F^2_{1,\rho})$, let

$$\Phi_T(z) := \sum_{k=0}^{n} \langle H_k(\cdot\,\overline{z}), \overline{T\big(G_k(\cdot\,\overline{z})\big)}\,\rangle \quad (z \in \mathbf{C}).$$

Observe that $\Phi_{\mathfrak{h}_b} = \overline{b}$, by (6.26). Therefore the necessary part in Theorem 1.5(ii) is a direct consequence of the following proposition.

Proposition 6.6. *For $1 \le p \le \infty$, the linear operator $T \mapsto \Phi_T$ is bounded from $S_p(F^2_{1,\rho})$ to $L^p_{1/2,2n(\ell-1)/p}$.*

Proof. It is easy to check that Φ_T is a continuous function on $\mathbf{C}$. Indeed, if $z_j \to z$ in $\mathbf{C}$, estimates (4.15) and (4.16) and the dominated convergence theorem imply that

$$H_k(\cdot\,\overline{z}_j) \to H_k(\cdot\,\overline{z}) \quad \text{in } F^2_{1,-\rho} \quad \text{and} \quad G_k(\cdot\,\overline{z}_j) \to G_k(\cdot\,\overline{z}) \ \text{ in } F^2_{1,\rho}.$$

So, taking into account the interpolation identities (6.30) and (6.31), it is enough to prove the proposition for $p = 1$ and $p = \infty$.

The case $p = \infty$ follows from Schwarz inequality, the boundedness of T and (1.2):

$$|\Phi_T(z)| \lesssim \|T\|_{S_\infty(F^2_{1,\rho})} \sum_{k=0}^{n} \|G_k(\cdot\overline{z})\|_{F^2_{1,\rho}} \|H_k(\cdot\overline{z})\|_{F^2_{1,-\rho}} \simeq \|T\|_{S_\infty(F^2_{1,\rho})} e^{\frac{|z|^{2\ell}}{4}}.$$

Now we prove the case $p = 1$, that is,

$$\|\Phi_T\|_{L^1_{1/2,2n(\ell-1)}} \lesssim \|T\|_{S_1(F^2_{1,\rho})} \quad (T \in S_1(F^2_{1,\rho})). \tag{6.38}$$

By (6.29) we only have to prove (6.38) for operators of rank one. So, taking into account Lemma 6.2, we may assume that T satisfies

$$T(f) = \langle f, g\rangle\, \overline{h} \quad (f \in F^2_{1,\rho}),$$

for some functions $g \in F^2_{1,-\rho}$ and $h \in F^2_{1,\rho}$.

In this case,

$$\Phi_T(z) = \sum_{k=0}^{n} \langle G_k(\cdot\,\overline{z}), g\rangle\, \langle H_k(\cdot\,\overline{z}), h\rangle,$$

and Schwarz inequality gives

$$\|\Phi_T\|_{L^1_{\frac{1}{2},2n(\ell-1)}} \lesssim \sum_{k=0}^{n} I_k\, J_k,$$

where

$$\begin{aligned} I_k^2 &:= \int_{\mathbf{C}^n} |\langle G_k(\cdot\overline{z}), g\rangle|^2\, (1+|z|)^{-2\rho+2(\ell-1)(2n-2k-1)} e^{-\frac{|z|^{2\ell}}{4}}\, dV(z) \\ J_k^2 &:= \int_{\mathbf{C}^n} |\langle H_k(\cdot\overline{z}), h\rangle|^2\, (1+|z|)^{2\rho+2(\ell-1)(2k+1)} e^{-\frac{|z|^{2\ell}}{4}}\, dV(z). \end{aligned}$$

Next we prove that $I_k \lesssim \|g\|_{F^2_{1,-\rho}}$ and $J_k \lesssim \|h\|_{F^2_{1,\rho}}$, which, by Lemma 6.2, give

$$\|\Phi_T\|_{L^1_{\frac{1}{2},2n(\ell-1)}} \lesssim \|g\|_{F^2_{1,-\rho}}\, \|h\|_{F^2_{1,\rho}} \simeq \|T\|_{S_1(F^2_{1,\rho})}.$$

In order to prove the estimate $I_k \lesssim \|g\|_{F^2_{1,-\rho}}$, first note that Schwarz's inequality gives

$$|\langle G_k(\cdot\overline{z}), g\rangle|^2 \lesssim \int_{\mathbf{C}^n} |g(w)|^2 |G_k(w\overline{z})| e^{-\frac{3|w|^{2\ell}}{2}}\, dV(w) \int_{\mathbf{C}^n} |G_k(w\overline{z})| e^{-\frac{|w|^{2\ell}}{2}}\, dV(w).$$

Then, by (4.15) and (4.16), we obtain

$$|\langle G_k(\cdot\,\overline{z}), g\rangle|^2 \lesssim (1+|z|)^{(\ell-1)(2k+1-2n)}\, e^{\frac{|z|^{2\ell}}{8}} \int_{\mathbf{C}^n} |g(w)|^2 |G_k(\overline{z}w)| e^{-\frac{3|w|^{2\ell}}{2}}\, dV(w).$$

Therefore Proposition 4.3 with $\gamma = 1$, $\alpha = \frac{1}{4}$ and $\theta = \frac{1}{2}$ gives

$$\begin{aligned} I_k^2 &\lesssim \int_{\mathbf{C}^n} |g(w)|^2 \|G_k(\cdot\overline{w})\|_{L^1_{\frac{1}{4},-2\rho+(\ell-1)(2n-2k-1)}}\, e^{-\frac{3|w|^{2\ell}}{2}}\, dV(w) \\ &\lesssim \int_{\mathbf{C}^n} |g(w)|^2 (1+|w|)^{-2\rho} e^{-|w|^{2\ell}}\, dV(w) = \|g\|_{F^2_{1,-\rho}}. \end{aligned}$$

Similarly, replacing ρ and k by $-\rho$ and $n-1-k$, respectively, we obtain $J_k \lesssim \|h\|_{F^2_{1,\rho}}$. □

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