# IRREDUCIBILITY OF THE MODULI SPACE OF ORTHOGONAL INSTANTON BUNDLES ON $\mathbb{P}^{n}$ 

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#### Abstract

In order to obtain existence criteria for orthogonal instanton bundles on $\mathbb{P}^{n}$, we provide a bijection between equivalence classes of orthogonal instanton bundles with no global sections and symmetric forms. Using such correspondence we are able to provide explicit examples of orthogonal instanton bundles with no global sections on $\mathbb{P}^{n}$ and prove that every orthogonal instanton bundle with no global sections on $\mathbb{P}^{n}$ and charge $c \geq 3$ has rank $r \leq(n-1) c$. We also prove that when the rank $r$ of the bundles reaches the upper bound, $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c, r)$, the coarse moduli space of orthogonal instanton bundles with no global sections on $\mathbb{P}^{n}$, with charge $c \geq 3$ and rank $r$, is affine, reduced and irreducible. Last, we construct Kronecker modules to determine the splitting type of the bundles in $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c, r)$, whenever is non-empty.


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## 1. Introduction

Since the 1970's the "instantons" or pseudo-particle solutions of the classical Yang-Mills equations in the Euclidean 4-space have awaken great interest in the physical and mathematical communities due the link that they provide between algebraic geometry and mathematical physics (see for instance [5] and [6]). In [4] Atiyah, Drinfeld, Hitchin and Manin provided the so called "ADHM contruction of instanton" on $\mathbb{P}^{3}$. In [22] Okonek and Spindler

[^0]generalized the contruction of instanton bundles to $\mathbb{P}^{2 n+1}$, since then the study of this family of bundles and their moduli spaces have been a central topic in algebraic geometry. The moduli space $\mathcal{M}_{\mathbb{P}^{3}}(c)$ of the $c$-instanton bundles on $\mathbb{P}^{3}$, i.e. of stable 2 -bundles $\mathcal{E}$ with Chern classes $\left(c_{1}, c_{2}\right)=(0, c)$ and $\mathrm{H}^{1}(\mathcal{E}(-2))=0$, is expected to be a smooth and irreducible variety with dimension $8 c-3$ for $c \geq 1$. This problem was approached by several authors (see [7], [8], 11], [14, [19], [20]) and while the irreducibility was completely solved by Tikhomirov in [23] and [24, the smoothness was solved on $\mathbb{C P}^{3}$ by Jardim and Verbitsky (see [18]), but remains open for the 3-dimensional projective space over any other algebraically closed field of characteristic 0 .
In order to understand moduli spaces of stable vector bundles over a projective variety, in [16] Jardim extended the definition of instantons to even-dimensional projective spaces and allowed non-locally-free sheaves of arbitrary rank. Jardim defined an instanton sheaf on $\mathbb{P}^{n}(n \geq 2)$ as a torsion-free coherent sheaf $\mathcal{E}$ on $\mathbb{P}^{n}$ with first Chern class $c_{1}(\mathcal{E})=0$ satisfying some cohomological conditions (see Definition 2.1 for details). If $\mathcal{E}$ is locally-free, $\mathcal{E}$ is called an instanton bundle and in addition, if $\mathcal{E}$ is a rank- $2 n$ bundle on $\mathbb{P}^{2 n+1}$ with trivial splitting type, then $\mathcal{E}$ is a mathematical instanton bundle as defined by Okonek and Spindler in [22]. Studying the moduli space of instanton bundles becomes more complicated for higher dimensional projective spaces or higher rank, because of this many authors have considered instanton bundles with some additional structure (special, symplectic and orthogonal). For example, in [9] Costa, Hoffmann, Miró-Roig and Schmitt proved that the moduli space of all symplectic instanton bundles on $\mathbb{P}^{2 n+1}$ with $n \geq 2$ is reducible; in [21] Miró-Roig and Orus-Lacort proved that $\mathcal{M}_{\mathbb{P}^{2 n+1}}(c)$ is singular for $n \geq 2$ and $c \geq 3$; Costa and Ottaviani in [10] proved that $\mathcal{M}_{\mathbb{P}^{2 n+1}}(c)$ is affine and introduced an invariant which allowed Farnik, Frapporti and Marchesi to prove in [12] that there are no orthogonal instanton bundles with rank $2 n$ on $\mathbb{P}^{2 n+1}$. Using the ADHM construction introduced by Henni, Jardim and Martins in [15], Jardim, Marchesi and Wißdorf in [17] consider autodual instantons of arbitrary rank on projective spaces, with focus on symplectic and orthogonal instantons; they described the moduli space of framed autodual instanton bundles and showed that there are no orthogonal instanton bundles with trivial splitting type, arbitrary rank $r$ and charge 2 or odd on $\mathbb{P}^{n}$. While in [1] Abuaf and Boralevi proved that the moduli space of rank $r$ stable orthogonal bundles on $\mathbb{P}^{2}$, with Chern classes $\left(c_{1}, c_{2}\right)=(0, c)$ and trivial splitting type on the general line, is smooth and irreducible for $r=c$ and $c \geq 4$, and $r=c-1$ and $c \geq 8$, the results of Farnik, Frapport and Marchesi in [12] and Jardim, Marchesi and Wißdorf in [17], already mentioned, show us that orthogonal instanton bundles on $\mathbb{P}^{n}, n \geq 3$ are for some reason hard to find and that it is interesting to establish existence criteria for these bundles.

Hence the main goal of this work is to provide existence criteria for orthogonal instanton bundles with higher rank on $\mathbb{P}^{n}$, for $n \geq 3$ and then to study their moduli space and splitting type.

Next, we outline the structure of the paper. In section 2 we introduce some preliminaries necessary through the text. In section 3, in order to establish existence criteria for orthogonal instanton bundles on $\mathbb{P}^{n}$, for $n \geq 3$, we define certain equivalence classes of orthogonal instanton bundles and provide a bijection between these classes and symmetric forms (see Theorem (3.4). Using such correspondence we prove the following result.

Proposition 3.5 Let $c \geq 3$ be an integer. Every orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^{n}$ ( $n \geq 3$ ) with no global sections and charge $c$ has rank $r \leq(n-1) c$. Moreover, there are no orthogonal instanton bundles $\mathcal{E}$ on $\mathbb{P}^{n}$ with no global sections and charge $c$ equal to 1 or 2 .

In section 4, we study the case when the rank reaches the upper bound and prove that there exists an affine coarse moduli space for our problem and addition we prove that this moduli space is irreducible and reduced (see Theorem4.3). Finally, in section 5, given an orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^{n}$ with charge $c$, rank $(n-1) c$ and no global sections, for $c, n \geq 3$, we construct a Kronecker module to determine whether the restriction of $\mathcal{E}$ to a line $L \subset \mathbb{P}^{n}$ is trivial or not (see Theorem 5.4).

## 2. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 . Let us consider $\mathbb{P}^{n}=\mathbb{P}(V)$, where $V$ is a $(n+1)$-dimensional $\mathbb{K}$-vector space, $n \geq 2$. If $\mathcal{E}$ is a vector bundle on $\mathbb{P}^{n}$, then $h^{i}(\mathcal{E}(k))$ denotes the dimension of $\mathrm{H}^{i}(\mathcal{E}(k))$, the $i^{\text {th }}$ cohomology group of $\mathcal{E}$, and $\mathcal{E}^{\vee}$ denotes the dual of $\mathcal{E}$, i. e., $\mathcal{E}^{\vee}=\mathcal{H} \operatorname{om}\left(\mathcal{E}, \mathcal{O}_{\mathbb{P}^{n}}\right)$. We denote by $H_{c}$ a $c$-dimensional $\mathbb{K}$-vector space, with $c \geq 1$ and if $U$ is a $\mathbb{K}$-vector space, we denote by $U^{\vee}$ the dual vector space of $U$.

Definition 2.1. An instanton sheaf on $\mathbb{P}^{n}$ is a torsion-free coherent sheaf $\mathcal{E}$ on $\mathbb{P}^{n}$ with $c_{1}(\mathcal{E})=0$ satisfying the following cohomological conditions:
(1) $\mathrm{H}^{0}(\mathcal{E}(-1))=\mathrm{H}^{n}(\mathcal{E}(-n))=0$;
(2) $\mathrm{H}^{1}(\mathcal{E}(-2))=\mathrm{H}^{n-1}(\mathcal{E}(1-n))=0$, if $n \geq 3$;
(3) $\mathrm{H}^{i}(\mathcal{E}(-k))=0$ for $2 \leq p \leq n-2$ and all $k$, if $n \geq 4$.

The integer $c=-\chi(\mathcal{E}(-1))$ is called charge of $\mathcal{E}$.
We will say that a vector bundle $\mathcal{E}$ is autodual if it is isomorphic to its dual, i.e. there exists an isomorphism $\phi: \mathcal{E} \rightarrow \mathcal{E}^{\vee}$. If the isomorphism $\phi$ satisfies $\phi^{\vee}=-\phi$, the vector
bundle is called symplectic. If the isomorphism $\phi$ satisfies $\phi^{\vee}=\phi$, the vector bundle is called orthogonal.

Let $\mathcal{E}$ be an orthogonal instanton bundle over $\mathbb{P}^{n}(n \geq 3)$ with charge $c$, rank $r$ and no global sections $\left(\mathrm{H}^{0}(\mathcal{E})=0\right)$. Considering the following exact sequence

$$
\left.0 \longrightarrow \mathcal{E}(-i-1) \longrightarrow \mathcal{E}(-i) \longrightarrow \mathcal{E}(-i)\right|_{\mathbb{P}^{n}-1} \longrightarrow 0,
$$

by the instanton cohomological conditions in Definition 2.1, the Serre duality and the Hirzebruch-Riemann-Roch theorem, for $0 \leq i \leq n$ and $-n-1 \leq k \leq 0$, one has

$$
h^{i}(\mathcal{E}(k))= \begin{cases}c, & \text { if }(i, k) \in\{(1,-1),(n-1,-n)\} \\ (n-1) c-r, & \text { if }(i, k) \in\{(1,0),(n-1,-n-1)\} \\ 0, & \text { otherwise }\end{cases}
$$

## 3. The equivalence

Consider a triple $(\mathcal{E}, \phi, f)$, where

- $\mathcal{E}$ is an orthogonal instanton bundle on $\mathbb{P}^{n}$ with charge $c$, rank $r$ and no global sections.
- $\phi: \mathcal{E} \xrightarrow{\cong} \mathcal{E}^{\vee}$ is an orthogonal structure of $\mathcal{E}$, i.e. $\phi^{\vee}=\phi$.
- $f: H_{c} \xrightarrow{\cong} \mathrm{H}^{n-1}(\mathcal{E}(-n))$.

Definition 3.1. Two triples $\left(\mathcal{E}_{1}, \phi_{1}, f_{1}\right)$ and $\left(\mathcal{E}_{2}, \phi_{2}, f_{2}\right)$ are called equivalent if there is an isomorphism $g: \mathcal{E}_{1} \xrightarrow{\cong} \mathcal{E}_{2}$ such that the following diagrams commute

where $g_{*}: \mathrm{H}^{n-1}\left(\mathcal{E}_{1}(-n)\right) \xrightarrow{\cong} \mathrm{H}^{n-1}\left(\mathcal{E}_{2}(-n)\right) \quad$ is the induced isomorphism in cohomology and $\lambda \in\{-1,1\}$. We denote by $[\mathcal{E}, \phi, f]$ the equivalence class of a triple $(\mathcal{E}, \phi, f)$.

Fixing the integers $c$ and $r$, we will denote by $\mathbb{E}[c, r]$ the set of all equivalence classes $[\mathcal{E}, \phi, f]$ of orthogonal instanton bundles with charge $c$, rank $r$ and no global sections over $\mathbb{P}^{n}$.

Lemma 3.2. Each triple $(\mathcal{E}, \phi, f) \in \mathbb{E}[c, r]$ defines a morphism $A: \mathrm{H}_{c} \otimes V \longrightarrow \mathrm{H}_{c}^{\vee} \otimes V^{\vee}$, living in $\bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$, that in turn defines a monad.

Proof. We consider the Euler exact sequence and its exterior powers

$$
\begin{equation*}
0 \longrightarrow \Omega_{\mathbb{P}^{n}}^{1} \xrightarrow{i_{1}} V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{e v} \mathcal{O}_{\mathbb{P}^{n}} \longrightarrow 0, \tag{1}
\end{equation*}
$$

(2) $0 \longrightarrow \Omega_{\mathbb{P}^{n}}^{i+1} \longrightarrow \wedge^{i+1} V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-i-1) \longrightarrow \Omega_{\mathbb{P}^{n}}^{i} \longrightarrow 0$, with $1 \leq i \leq n-2$, and

$$
\begin{equation*}
0 \longrightarrow \bigwedge^{n+1} V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-n-1) \longrightarrow \bigwedge^{n} V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-n) \xrightarrow{i_{2}} \Omega_{\mathbb{P}^{n}}^{n-1} \longrightarrow 0 \tag{3}
\end{equation*}
$$

induced by the Koszul complex of $V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{e v} \mathcal{O}_{\mathbb{P}^{n}}$, where ev denotes the canonical evaluation map. Tensoring (1) with $\mathcal{E}$ we obtain $\mathrm{H}^{i}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right)=0$, for $i=0,3, \ldots, n$ and the exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathrm{H}^{1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \xrightarrow{i_{1}} \mathrm{H}^{1}\left(V^{\vee} \otimes \mathcal{E}(-1)\right) \longrightarrow \mathrm{H}^{1}(\mathcal{E}) \longrightarrow H^{2}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \longrightarrow 0 \tag{4}
\end{equation*}
$$

Tensoring (2) with $\mathcal{E}$ we obtain

$$
\begin{equation*}
\mathrm{H}^{j}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{i}\right) \cong \mathrm{H}^{j+1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{i+1}\right) \tag{5}
\end{equation*}
$$

and $\mathrm{H}^{0}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{i}\right)=\mathrm{H}^{n-1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{i+1}\right)=0$, for $1 \leq i \leq n-2$ and $0 \leq j \leq n-1$. Finally tensoring (3) with $\mathcal{E}$ we obtain $\mathrm{H}^{i}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right)=0$, for $i=0, \ldots, n-3, n$ and the exact sequence
$0 \rightarrow \mathrm{H}^{n-2}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right) \rightarrow \mathrm{H}^{n-1}\left(\bigwedge^{n+1} V^{\vee} \otimes \mathcal{E}(-n-1)\right) \rightarrow \mathrm{H}^{n-1}\left(\bigwedge^{n} V^{\vee} \otimes \mathcal{E}(-n)\right) \xrightarrow{i_{2}} \mathrm{H}^{n-1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right) \rightarrow 0$.
Therefore we have

$$
\begin{gathered}
\mathrm{H}^{2}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right)=\mathrm{H}^{3}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{2}\right)=\cdots=\mathrm{H}^{n}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right)=0, \\
\mathrm{H}^{n-2}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right)=\mathrm{H}^{n-3}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-2}\right)=\cdots=\mathrm{H}^{0}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right)=0, \\
h^{n-1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{n-1}\right)=h^{1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right)=h^{1}\left(V^{\vee} \otimes \mathcal{E}(-1)\right)-h^{1}(\mathcal{E})=2 c+r,
\end{gathered}
$$

and the exact sequences

$$
0 \longrightarrow \mathrm{H}^{n-1}(\mathcal{E}(-n-1)) \otimes \bigwedge^{n+1} V^{\vee} \xrightarrow{a} \mathrm{H}^{n-1}(\mathcal{E}(-n)) \otimes \bigwedge^{n} V^{\vee} \xrightarrow{i_{2}} \mathrm{H}^{n-1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P} n}^{n-1}\right) \longrightarrow 0
$$

and

$$
\begin{equation*}
0 \longrightarrow \mathrm{H}^{1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \xrightarrow{i_{1}} \mathrm{H}^{1}(\mathcal{E}(-1)) \otimes V^{\vee} \xrightarrow{b} \mathrm{H}^{1}(\mathcal{E}) \longrightarrow 0 . \tag{7}
\end{equation*}
$$

By the functoriality of the Serre duality, we have $i_{1}=i_{2}^{\vee}$ and the following diagram with exact rows

where $A^{\prime}=i_{2}^{\vee} \circ \partial^{-1} \circ i_{2}$, moreover, the Euler sequence (1) yields the canonical isomorphism $\omega_{\mathbb{P}^{n}} \xrightarrow{\cong} \bigwedge^{n+1} V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-n-1)$. So, fixing an isomorphism $\tau: \mathbb{K} \xrightarrow{\cong} \bigwedge^{n+1} V^{\vee}$, we have the isomorphisms

$$
\begin{equation*}
\tau_{1}: V \xrightarrow{\cong} \bigwedge^{n} V^{\vee} \text { and } \tau_{2}: \omega_{\mathbb{P}^{n}} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^{n}}(-n-1) \tag{9}
\end{equation*}
$$

Thus, each $[\mathcal{E}, f, \phi] \in \mathbb{E}(c, r)$ defines a morphism $A: \mathrm{H}_{c} \otimes V \longrightarrow \mathrm{H}_{c}^{\vee} \otimes V^{\vee}$ through the following composition

$$
\begin{aligned}
A: H_{c} \otimes V \xrightarrow{\mathrm{Id} \otimes \tau_{1}} H_{c} \otimes \Lambda^{n} V^{\vee} \xrightarrow{f \otimes \mathrm{Id}} \mathrm{H}^{n-1}(\mathcal{E}(-n)) \otimes \Lambda^{n} V^{\vee} \xrightarrow{A^{\prime}} \mathrm{H}^{1}(\mathcal{E}(-1)) \otimes V^{\vee} \xrightarrow{\phi \otimes \mathrm{Id}} \mathrm{H}^{1}\left(\mathcal{E}^{\vee}(-1)\right) \otimes V^{\vee} \\
\xrightarrow{\mathrm{SD} \otimes \mathrm{Id}} \mathrm{H}^{n-1}\left(\mathcal{E}(1) \otimes \omega_{\mathbb{P}} n\right)^{\vee} \otimes V^{\vee} \xrightarrow{\tau_{2} \otimes \mathrm{Id}} \mathrm{H}^{n-1}(\mathcal{E}(-n))^{\vee} \otimes V^{\vee} \xrightarrow{f^{\vee} \otimes \mathrm{Id}}
\end{aligned}
$$

where SD denotes the Serre duality isomorphism. Therefore we can write

$$
\begin{equation*}
A=\left(\left(f^{\vee} \circ \tau_{2} \circ \mathrm{SD} \circ \phi\right) \otimes \mathrm{Id}\right) \circ A^{\prime} \circ\left(f \otimes \tau_{1}\right) \tag{10}
\end{equation*}
$$

Note that, since $\tau$ is a multiplication by a scalar, $A$ does not depend on the choice of $\tau$. It is possible to prove that $A$ is symmetric, therefore,

$$
\begin{equation*}
A \in\left(S^{2} H_{c}^{\vee} \otimes S^{2} V^{\vee}\right) \oplus\left(\bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}\right) \tag{11}
\end{equation*}
$$

We will show that actually $A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$. Let us set $W:=\frac{H_{c} \otimes V}{\operatorname{Ker} A}$, and combining (8) and (10) we have the commutative diagram of exact rows

which tells us that $\operatorname{dim} W=2 c+r$ and induces the diagram

where $p$ is the canonical projection and $q_{A}: W \xrightarrow{\cong} W^{\vee}$ is a symmetric isomorphism. So we can define the induced morphism of sheaves

$$
\begin{equation*}
a_{A}^{\vee}: W^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{p^{\vee} \otimes \mathrm{Id}} H_{c}^{\vee} \otimes V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{\mathrm{I} \mathrm{~d} \otimes e v} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \tag{14}
\end{equation*}
$$

which is surjective, therefore $a_{A}$ is injective, and the composition

$$
\psi: H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{q_{A} \otimes \mathrm{Id}} W^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee}} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1)
$$

is zero.
Since $A$ is symmetric, we can write $A=A_{1}+A_{2}$, where $A_{1} \in \Lambda^{2} H_{c}^{\vee} \otimes \Lambda^{2} V^{\vee}$ and $A_{2} \in$ $S^{2} H_{c}^{\vee} \otimes S^{2} V^{\vee}$. By the Euler sequence (1) we have

$$
\begin{equation*}
0 \longrightarrow \bigwedge^{2}(\Omega(1)) \longrightarrow \bigwedge^{2} V^{\vee} \otimes \mathcal{O} \xrightarrow{(\mathrm{Id} \otimes e v) \circ(i \otimes \mathrm{Id})} V^{\vee} \otimes \mathcal{O}(1) \xrightarrow{e v} \mathcal{O}(2) \longrightarrow 0 \tag{15}
\end{equation*}
$$

where $i: \bigwedge^{2} V^{\vee} \hookrightarrow V^{\vee} \otimes V^{\vee}$ is the inclusion. Note that $\psi=(\operatorname{Id} \otimes e v) \circ A \circ\left(\operatorname{Id} \otimes e v^{\vee}\right)$, thus

$$
\psi=(\operatorname{Id} \otimes e v) \circ A_{1} \circ\left(\operatorname{Id} \otimes e v^{\vee}\right)+(\operatorname{Id} \otimes e v) \circ A_{2} \circ\left(\operatorname{Id} \otimes e v^{\vee}\right) .
$$

By the sequence (15), we have $\operatorname{Im} A_{1} \subset$ Ker $e v$ and therefore $\psi=(\operatorname{Id} \otimes e v) \circ A_{2} \circ\left(\operatorname{Id} \otimes e v^{\vee}\right)$; moreover, $\operatorname{Im} A_{2} \subset S^{2} H_{c}^{\vee} \otimes S^{2} V^{\vee} \not \subset$ Ker $e v$, otherwise the evaluation map would be the zero map. Hence, $\psi=0$ implies $A_{2}=0$ and therefore $A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$.
On the other hand, for each $A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$, it follows from the previous paragraph that $(\operatorname{Id} \otimes e v) \circ A \circ\left(\operatorname{Id} \otimes e v^{\vee}\right)=0$, therefore we can associate the monad

$$
\begin{equation*}
\mathcal{M}_{A}: H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee} \circ\left(q_{A} \otimes \mathrm{Id}\right)} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \tag{16}
\end{equation*}
$$

whose cohomology sheaf is defined by

$$
\begin{equation*}
\mathcal{E}_{A}:=\frac{\operatorname{Ker}\left(a_{A}^{\vee} \circ\left(q_{A} \otimes I d\right)\right)}{\operatorname{Im} a_{A}} \tag{17}
\end{equation*}
$$

Recall that asking $\mathcal{E}$ not to have global sections is equivalent to not having trivial summands in the vector bundle $\operatorname{Ker}\left(a_{A}^{\vee} \circ\left(q_{A} \otimes I d\right)\right)$ (see [3] for more details).

Recall also that $A$ is called non-degenerate if $A(h \otimes v) \neq 0$ for any non-zero decomposable tensor $h \otimes v \in H_{c} \otimes V$. Hence, similar to [8], with the notation of the previous proof, the followings are equivalent:
(i) $a_{A}^{\vee} \circ\left(q_{A} \otimes \mathrm{Id}\right)$ is surjective;
(ii) the image of $a_{A}$ is a subbundle;
(iii) $A$ is non-degenerate.

From all the previous observations, the map $A$ defined in (10) has the following properties:
(A1) $\operatorname{rank}\left(A: H_{c} \otimes V \rightarrow H_{c}^{\vee} \otimes V^{\vee}\right)=2 c+r$;
(A2) $A$ is non degenerate;
$(\mathrm{A} 3)$ there exists a symmetric isomorphism $q_{A}: W \xrightarrow{\cong} W^{\vee}$, where $W=\frac{H_{c} \otimes V}{\operatorname{Ker} A}$.
Consider the set

$$
\mathcal{A}[c, r]:=\left\{A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee} ; \text { such that (A1)-(A3) holds }\right\}
$$

our next goal is to prove a bijection between the sets $\mathcal{A}[c, r]$ and $\mathbb{E}[c, r]$. To do so, we will need the next result.

Lemma 3.3. For any $A \in \mathcal{A}[c, r]$, there are isomorphisms

$$
\begin{array}{ccc}
H_{c} \cong \mathrm{H}^{n-1}\left(\mathcal{E}_{A} \otimes \Omega^{n}(1)\right) & W \cong \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega^{1}\right) & \text { Ker } A^{\vee} \cong \mathrm{H}^{1}\left(\mathcal{E}_{A}\right) \\
H_{c}^{\vee} \cong \mathrm{H}^{1}\left(\mathcal{E}_{A}(-1)\right) & W^{\vee} \cong \mathrm{H}^{n-1}\left(\mathcal{E}_{A} \otimes \Omega^{n-1}\right) &
\end{array}
$$

which are compatible with the Serre duality and the orthogonal structure $\mathcal{E}_{A} \cong \mathcal{E}_{A}^{\vee}$, and give the following commutative diagram


Proof. Given $A \in \mathcal{A}[c, r]$ we have the monad

$$
\begin{equation*}
\mathcal{M}_{A}: H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee} \circ\left(q_{A} \otimes \mathrm{Id}\right)} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1), \tag{18}
\end{equation*}
$$

whose cohomology bundle is $\mathcal{E}_{A}$. On the other hand, applying the Beilinson spectral sequence to $\mathcal{E}_{A}(-1)$, one has the monad

$$
0 \longrightarrow \mathrm{H}^{1}\left(\mathcal{E}_{A}(1) \otimes \Omega_{\mathbb{P}^{n}}^{2}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}}(-2) \xrightarrow{d_{1}^{-2,1}} \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{d_{1}^{-1,1}} \mathrm{H}^{1}\left(\mathcal{E}_{A}(-1)\right) \otimes \mathcal{O}_{\mathbb{P}^{n}} \longrightarrow 0
$$

and tensoring this monad by $\mathcal{O}_{\mathbb{P}^{n}}(1)$, we obtain the monad

$$
\begin{equation*}
0 \longrightarrow \mathrm{H}^{1}\left(\mathcal{E}_{A}(1) \otimes \Omega_{\mathbb{P}^{3}}^{2}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{d_{1}^{-2,1}} \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{d_{1}^{-1,1}} \mathrm{H}^{1}\left(\mathcal{E}_{A}(-1)\right) \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \longrightarrow 0, \tag{19}
\end{equation*}
$$

whose cohomology is isomorphic to $\mathcal{E}_{A}$.

Obviously, $\mathcal{E}_{A} \cong \mathcal{E}_{A}$ thus we have the isomorphism of the monads (18) and (19), which gives us the isomorphisms $H_{c} \cong \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega^{2}(1)\right) \cong \mathrm{H}^{n-1}\left(\mathcal{E}_{A} \otimes \Omega^{n}(1)\right)$, $W \cong \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega^{1}\right)$ and $H_{c}^{\vee} \cong \mathrm{H}^{1}\left(\mathcal{E}_{A}(-1)\right)$. By Serre duality, we have $W^{\vee} \cong \mathrm{H}^{1}\left(\mathcal{E}_{A} \otimes \Omega^{1}\right)^{\vee} \cong \mathrm{H}^{n-1}\left(\mathcal{E}_{A} \otimes \Omega^{n-1}\right)$, and the last isomorphism follows from (13) and (77).
Finally, the commutativity of the diagram follows from the functoriality of Serre-duality.
Thanks to the previous lemma, we have the following result.
Theorem 3.4. There exists a bijection between the equivalence classes $[\mathcal{E}, \phi, f] \in \mathbb{E}[c, r]$ of orthogonal instanton bundles of charge $c$, rank $r$, with no global sections on $\mathbb{P}^{n}(n \geq 3)$ and the elements $A \in \mathcal{A}[c, r]$.

Proof. By Lemma 3.2 given an equivalence class $[\mathcal{E}, \phi, f] \in \mathbb{E}[c, r]$, there exists $A \in \bigwedge^{2} H_{c}^{\vee} \otimes$ $\bigwedge^{2} V^{\vee}$ which satisfies (A1)-(A3). Thus $A \in \mathcal{A}[c, r]$ and there exists a monad

$$
\begin{equation*}
\mathcal{M}_{A}: H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee} \circ\left(q_{A} \otimes \mathrm{Id}\right)} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \tag{20}
\end{equation*}
$$

whose cohomology sheaf is denoted by $\mathcal{E}_{A}$. On the other hand, by ([16] - Theorem 3), $\mathcal{E}$ is cohomology of the monad
(21) $0 \longrightarrow \mathrm{H}^{1}\left(\mathcal{E}(1) \otimes \Omega_{\mathbb{P}^{3}}^{2}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{d_{1}^{-2,1}} \mathrm{H}^{1}\left(\mathcal{E} \otimes \Omega_{\mathbb{P}^{n}}^{1}\right) \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{d_{1}^{-1,1}} \mathrm{H}^{1}(\mathcal{E}(-1)) \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \longrightarrow 0$.

By the Lemma 3.3 the monads (20) and (21) are isomorphic. Thus $A$ defines a monad whose cohomology sheaf $\mathcal{E}_{A}$ is isomorphic to $\mathcal{E}$.

Tensoring $\mathcal{M}_{A}$ by $\mathcal{O}_{\mathbb{P}^{n}}(-n)$ and using (17), we obtain $\mathrm{H}^{n-1}\left(\mathcal{E}_{A}(-n)\right) \cong \mathrm{H}^{n}\left(H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-n-1)\right)$. Note that $\mathrm{h}^{n}\left(H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-n-1)\right)=c$, then there exists $f_{A}: H_{c} \xrightarrow{\cong} \mathrm{H}^{n-1}\left(\mathcal{E}_{A}(-n)\right)$.

Furthermore, the symmetric map $q_{A}$ induces a canonical isomorphism of monads

which induces a symmetric isomorphism of vector bundles $\phi_{A}: \mathcal{E}_{A} \xrightarrow{\cong} \mathcal{E}_{A}^{\vee}$. Thus, the data $\left[\mathcal{E}_{A}, \phi_{A}, f_{A}\right]$ can be recovered from $A$.

By Theorem 3.4 the existence of orthogonal instanton bundles with charge $c$, rank $r$ and no global sections on $\mathbb{P}^{n}$ is related to the existence of symmetric and non-degenerate linear maps. This approach is extremely helpful in the proof of the next result.

Proposition 3.5. Let $c \geq 3$ be an integer. Every orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^{n}$ $(n \geq 3)$ with no global sections and charge $c$ has rank $r \leq(n-1) c$. Moreover, there are no orthogonal instanton bundles $\mathcal{E}$ on $\mathbb{P}^{n}(n \geq 3)$ with no global sections and charge $c$ equal to 1 or 2 .

Proof. First suppose that there exists an orthogonal instanton bundle $\mathcal{E}$ with no global sections, charge $c$ and rank $r$ over $\mathbb{P}^{n}$ and consider its equivalence class $[\mathcal{E}, \phi, f]$. By Theorem 3.4 there exists $A \in \mathcal{A}[c, r]$ associated with $[\mathcal{E}, \phi, f]$.

Given $A \in \mathcal{A}[c, r]$, with some abuse of notation, let us also denote by $A$ the matrix associated with the morphism $A: H_{c} \otimes V \longrightarrow H_{c}^{\vee} \otimes V^{\vee}$. So rank $A=2 c+r \leq(n+1) c$, implying rank $\mathcal{E}=r \leq(n-1) c$.
To conclude the proof, if $c=1$, then $A \in \bigwedge^{2} H_{1}^{\vee} \otimes \bigwedge^{2} V^{\vee} \cong 0$, but the zero map is degenerate.

If $c=2$, then $A \in \bigwedge^{2} H_{2}^{\vee} \otimes \bigwedge^{2} V^{\vee} \cong \mathbb{K} \otimes \bigwedge^{2} V^{\vee}$, so $A$ is skew-symmetric, but $A$ is also symmetric, hence $A$ is the zero map.
So there are no orthogonal instanton bundles, with no global sections and charge 1 or 2 on $\mathbb{P}^{n}$.

Now, with the help of Macaulay2, see [13], we will construct explicit examples of orthogonal instanton bundles on $\mathbb{P}^{n}$ when $r$ reaches the upper bound. Let us start by explaining the consequences of the results obtained.
Proposition 3.5 and diagram (13) imply that $A \cong q_{A}$. Moreover, we have

$$
a_{A}^{\vee}: W^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{3}} \xrightarrow{\mathrm{Id}} H_{n}^{\vee} \otimes V^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{3}} \xrightarrow{e v} H_{n}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{3}}(1) .
$$

So if $\left\{x_{0}, x_{1}, \ldots, x_{n}\right\}$ is a basis of $V^{\vee}$, we have the monad

$$
\begin{equation*}
\mathcal{M}_{A}: H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee} \circ(A \otimes \mathrm{Id})} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1), \tag{22}
\end{equation*}
$$

where $a_{A}^{\vee}$ is given by

$$
a_{A}^{\vee}=\left(\begin{array}{ccccccccccccccccccc}
x_{0} & x_{1} & \cdots & x_{n} & 0 & 0 & \ldots & 0 & \cdots & & & & & & \\
0 & 0 & \cdots & 0 & x_{0} & x_{1} & \cdots & x_{n} & \cdots & & & & & & \\
& & & & & & & & \ddots & & & & & & & \\
& & & & & & & & \cdots & 0 & 0 & \cdots & 0 & x_{0} & x_{1} & \ldots & x_{n}
\end{array}\right)_{(n+1) \times(n+1) c .}
$$

Theorem 3.4 simplifies the search for orthogonal instanton bundles and translates our existence problem in a linear algebra problem: we have to look for invertible matrices in $\bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$. Recall that every skew-symmetric matrix $M$ can be written as a block diagonal matrix

$$
\left(\begin{array}{ccccccccc}
0 & & \lambda_{1} & & & & & &  \tag{23}\\
& & & 0 & & & & \\
-\lambda_{1} & & 0 & & & & \cdots & & 0 \\
& & & 0 & & \lambda_{2} & & & \\
& 0 & & & & & & 0 & \\
& & & -\lambda_{2} & & 0 & & & \\
& \vdots & & & & & \ddots & & \vdots \\
& & & & & & 0 & & \lambda_{l} \\
& 0 & & 0 & & \cdots & & & \\
& & & & & & -\lambda_{l} & & 0
\end{array}\right)_{2 l \times 2 l}
$$

where $\pm i \lambda_{i}$ are the non-zero eigenvalues of $M$. In order to build examples of orthogonal instanton bundles with even charge $c$ on $\mathbb{P}^{n}$, with $n$ odd, we can take two matrices $B$ and $C$ as in (23), where:

- $B$ is a $c \times c$ skew-symmetric matrix;
- $C$ is a $(n+1) \times(n+1)$ skew-symmetric matrix.

So if we consider $A=B \otimes C$, then $A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$.
Example 3.6. Let us construct an example of orthogonal instanton bundle with no global sections and charge 6 on $\mathbb{P}^{3}$. Let $\left\{x_{0}, x_{1}, x_{2}, x_{3}\right\}$ be a basis for $V^{\vee}$. Consider

$$
B=\left(\begin{array}{cccccc}
0 & 2 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{array}\right) \text { and } C=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 \\
0 & 0 & 3 & 0
\end{array}\right)
$$

Let $A=B \otimes C \in \bigwedge^{2} H_{6}^{\vee} \otimes \bigwedge^{2} V^{\vee}$. We have that rank $A=24$, so $A$ is invertible and therefore non degenerate. By Theorem 3.4 and Proposition 3.5 we have the linear monad

$$
\mathcal{O}_{\mathbb{P}^{3}}^{6}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^{3}}^{24} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{3}}^{6}(1)
$$

where
and whose cohomology bundle is an orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^{3}$ with no global sections, charge 6 and rank 12 .

As we can see in the next example, when $c$ is odd or $n$ is even, we need to be a little more careful, because skew-symmetric matrices of odd order do not have complete rank.

Example 3.7. For $c=5$ and $n=3$, we consider

$$
\begin{gathered}
B_{1}=\left(\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right) B_{2}=\left(\begin{array}{ccccc}
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 & 0
\end{array}\right) B_{3}=\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{array}\right) \\
C_{1}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{array}\right) C_{2}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right) C_{3}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0
\end{array}\right) .
\end{gathered}
$$

Let $A=B_{1} \otimes C_{1}+B_{2} \otimes C_{2}+B_{3} \otimes C_{3}$. We have that rank $A=20$, so $A$ is invertible and non degenerate. By Theorem 3.4 and Proposition 3.5 we have the linear monad

$$
\mathcal{O}_{\mathbb{P}^{3}}^{5}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^{3}}^{20} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{3}}^{5}(1)
$$

where
as constructed before. The vector bundle which is the cohomology of the monad (3.7) is an orthogonal instanton bundle $\mathcal{E}$ on $\mathbb{P}^{3}$ with no global sections, charge 5 and rank 10 .

## 4. Moduli space

In this section we will keep focusing on orthogonal instanton bundles with maximal possible rank. Our goal is to use geometric invariant theory (GIT) to construct $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c, r)$, the moduli
space of orthogonal instanton bundles with charge $c$, rank $r$ and no global sections on $\mathbb{P}^{n}$, for $n, c \geq 3$. First notice when $r=(n-1) c$, the conditions (A1) and (A3) are superfluous, and we have

$$
\mathcal{A}[c,(n-1) c]=\left\{A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee} ; A \text { is non degenerate }\right\}
$$

Denote $\mathbb{E}_{c}=\mathbb{E}[c,(n-1) c], \mathcal{A}_{c}=\mathcal{A}[c,(n-1) c], G=G L\left(H_{c}\right)$, and let $\widetilde{\mathbb{E}_{c}}$ be the set of isomorphism classes $[\mathcal{E}, \phi]$ such that $[\mathcal{E}, \phi, f] \in \mathbb{E}_{c}$. Consider the action

$$
\begin{aligned}
\alpha: G \times \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee} & \rightarrow \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee} \\
(h, A) & \mapsto(h \otimes \mathrm{Id}) A\left(h^{\vee} \otimes \mathrm{Id}\right) .
\end{aligned}
$$

Lemma 4.1. The set $\mathcal{A}_{c}$ is $G$-invariant subset of $\bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$.
Proof. Let $h \in G, A \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$ and $B=\alpha(h, A)$ the image of $h$ and $A$ by the previous action, that means

$$
B=(h \otimes \mathrm{Id}) A\left(h^{\vee} \otimes \mathrm{Id}\right)
$$

We can write $A=\sum_{i}\left(C_{i} \otimes D_{i}\right)$, where $C_{i} \in \Lambda^{2} H_{c}^{\vee}$ and $D_{i} \in \Lambda^{2} V^{\vee}$ for all integers $i$. Thus

$$
\begin{aligned}
B & =(h \otimes \mathrm{Id})\left(\sum_{i}\left(C_{i} \otimes D_{i}\right)\right)\left(h^{\vee} \otimes \mathrm{Id}\right) \\
& =\sum_{i}\left(\left(h C_{i} h^{\vee}\right) \otimes D_{i}\right)
\end{aligned}
$$

Since $h C_{i} h^{\vee} \in \bigwedge^{2} H_{c}^{\vee}$ for all integers $i$, it follows that $B \in \bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}$.
The bijection given in the next theorem is the key ingredient to construct $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c)$.
Theorem 4.2. There is a bijection between the set of isomorphism classes $\widetilde{\mathbb{E}_{c}}$ and the orbit space $\mathcal{A}_{c} / G$. The isotropy group in each point is $\left\{ \pm \operatorname{Id}_{H_{c}}\right\}$.

Proof. Given $A \in \mathcal{A}_{c}$ by Theorem 3.4 there exists $\left[\mathcal{E}_{A}, \phi_{A}, f_{A}\right] \in \mathbb{E}_{c}$ and we can define

$$
\begin{aligned}
\Psi: \mathcal{A}_{c} & \rightarrow \widetilde{\mathbb{E}_{c}} \\
A & \mapsto\left[\mathcal{E}_{A}, \phi_{A}\right] .
\end{aligned}
$$

We will prove that $\Psi / G: A_{c} / G \rightarrow \widetilde{\mathbb{E}_{c}}$ is a bijection. First note that $\Psi$ factors through $\mathcal{A}_{c} / G$; indeed, consider $A, B \in \mathcal{A}_{c}$ such that there exists $h \in G$ with $\alpha(h, A)=B$. We have the following commutative diagram


Since $A, B \in \mathcal{A}_{c}$, we have $A$ and $B$ invertible and by diagram (13), we have the following commutative diagram,

so, we have the isomorphism of monads


Considering the cohomology of the monads $\mathcal{M}_{A}$ and $\mathcal{M}_{B}$, we get $\Psi(A)=\left[\mathcal{E}_{A}, \phi_{A}\right]=$ $\left[\mathcal{E}_{B}, \phi_{B}\right]=\Psi(B)$ and we have the following commutative diagram


The projection $\pi$ is surjective by definition and we have by Theorem 3.4 and Proposition 3.5 that $\Psi$ is surjective as well. This implies that $\Psi / G$ is surjective.
We need now to prove that $\Psi / G$ is injective. Indeed, let $A, B \in \mathcal{A}_{c}$ such that $\Psi(A)=$ $\left[\mathcal{E}_{A}, \phi_{A}\right]=\left[\mathcal{E}_{B}, \phi_{B}\right]=\Psi(B)$, we will show that there exists $h \in G$ such that $A=\alpha(h, B)$. If $\left[\mathcal{E}_{A}, \phi_{A}\right]=\left[\mathcal{E}_{B}, \phi_{B}\right]$, then by definition there exists an isomorphism $g: \mathcal{E}_{A} \xrightarrow{\cong} \mathcal{E}_{B}$ such that the following diagram is commutative


Hence by Lemma 3.3 we have the commutative diagram, which works in a more general case,

where $g^{*}$ denotes the morphisms induced by $g$ on the cohomology groups, but recall that in our case $H_{c} \otimes V \cong W_{A}$ and Ker $A \cong 0$. Thus, the middle blocks are commutative and the commutativity of the top and bottom blocks follows from Lemma 3.3. Therefore, there exists $h \in G$ such that $B=\alpha(h, A)$.
Finally, we will prove that the isotropy group is $\left\{ \pm \operatorname{Id}_{H_{c}}\right\}$. Let $h \in G$ and $A \in \mathcal{A}_{c}$, such that $A=\alpha(h, A)$. By Theorem 3.4 we have $\left[\mathcal{E}_{A}, \phi_{A}, f_{A}\right]=\left[\mathcal{E}_{\alpha(h, A)}, \phi_{\alpha(h, A)}, f_{\alpha(h, A)}\right]$, since they come from the same symmetric map; hence there exists an isomorphism $g: \mathcal{E}_{A} \xrightarrow{\cong} \mathcal{E}_{\alpha(h, A)}$ such that the following diagrams commute

with $\lambda \in\{-1,1\}$. Thus

$$
\begin{equation*}
g^{*} \circ f_{A}= \pm f_{\alpha(h, A)} . \tag{24}
\end{equation*}
$$

On the other hand, since $A=\alpha(h, A)$ by Lemma 3.3 we have the commutative diagram

and therefore, looking at the left column we have

$$
\begin{align*}
h & =\left(f_{\alpha(h, A)}\right)^{-1} \circ g^{*} \circ f_{A} \\
& =\left(f_{\alpha(h, A)}\right)^{-1} \circ\left( \pm f_{\alpha(h, A)}\right)  \tag{24}\\
& = \pm \operatorname{Id}_{H_{c}},
\end{align*}
$$

and the isotropy group is $\left\{ \pm \operatorname{Id}_{H_{c}}\right\}$.

Since $G=G L\left(H_{c}\right)$ is a reductive group and its isotropy group $\left\{ \pm \mathrm{Id}_{H_{c}}\right\}$ is a discrete subgroup, the quotient $G_{0}=G /\left\{ \pm \operatorname{Id}_{H_{c}}\right\}$ is also reductive. Moreover, the action of $G_{0}$ on $\mathcal{A}_{c}$ is free and we have:

Theorem 4.3. The geometric quotient $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-1) c):=\mathcal{A}_{c} / / G_{0}$ is reduced and irreducible affine coarse moduli space of dimension $\binom{c}{2}\binom{n+1}{2}-c^{2}$ for orthogonal instanton bundles with charge $c$, rank $(n-1) c$ and no global sections on $\mathbb{P}^{n}$, for $n, c \geq 3$.

Proof. First note that $\mathcal{A}_{c}$ is an open dense subset of $\bigwedge^{2} H_{c}^{\vee} \otimes \Lambda^{2} V^{\vee}$ which is affine, reduced and irreducible, thus $\mathcal{A}_{c}$ is also affine, reduced and irreducible. Moreover being $G_{0}$ a reductive group, then $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-1) c):=\mathcal{A}_{c} / / G_{0}$ is an affine good quotient and therefore $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-$ $1) c$ ) is an affine, reduced and irreducible categorical quotient. Since the action is free all orbits of the action are closed and it follows that $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-1) c)$ is an affine, reduced and irreducible coarse moduli space.
The dimension of $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-1) c)$ can be computed as

$$
\begin{aligned}
\operatorname{dim} \mathcal{A}_{c}-\operatorname{dim} G_{0} & =\operatorname{dim}\left(\bigwedge^{2} H_{c}^{\vee} \otimes \bigwedge^{2} V^{\vee}\right)-\operatorname{dim} G \\
& =\binom{c}{2}\binom{n+1}{2}-c^{2} .
\end{aligned}
$$

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Remark 4.4. Since $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c,(n-1) c)$ is reduced and irreducible it follows that $\mathcal{M}_{\mathbb{P} n}^{\mathcal{O}}(c,(n-$ $1) c$ ) is generically smooth. But a question arises naturally in this context is:

Question 4.5. Is $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c)$ smooth?

## 5. Splitting type

The goal of this section is to determine the type of splitting of orthogonal instanton bundles in $\mathcal{M}_{\mathbb{P}^{n}}^{\mathcal{O}}(c, r)$ for $n, c \geq 3$, whenever non-empty.

Jardim, Marchesi and Wißdorf proved in ([17] - Lemma 4.3 and Theorem 4.4) that there are no orthogonal instanton bundles of trivial splitting type, arbitrary rank $r$, and charge 2 or odd on $\mathbb{P}^{n}$. In order to determine the splitting type of the orthogonal instanton bundles, with no global section, charge $c$ and rank $r$ on $\mathbb{P}^{n}$ we will associate these bundles to Kronecker modules.

Definition 5.1. A Kronecker module of rank $r$ is a linear map

$$
\gamma: \bigwedge^{2} V \rightarrow \operatorname{Hom}\left(H_{c}, H_{c}^{\vee}\right)
$$

such that for the associated linear map,

$$
\hat{\gamma}: V \otimes H_{c} \rightarrow V^{\vee} \otimes H_{c}^{\vee}
$$

defined by $\hat{\gamma}\left(v_{1} \otimes h_{1}\right)\left(v_{2} \otimes h_{2}\right)=\left[\gamma\left(v_{1} \wedge v_{2}\right)\left(h_{1}\right)\right]\left(h_{2}\right)$, the following statements hold
(K1) $\hat{\gamma}(v \otimes-): H_{c} \rightarrow V^{\vee} \otimes H_{c}^{\vee}$ is injective for all $v \neq 0$.
(K2) If $v^{\vee \vee}: V^{\vee} \otimes H_{c}^{\vee} \rightarrow H_{c}^{\vee}$ is the evaluation map associated to $v \in V$, then $v^{\vee \vee} \circ \hat{\gamma}$ : $V \otimes H_{c} \rightarrow H_{c}^{\vee}$ is surjective for all $v \neq 0$.
(K3) $\operatorname{rank} \hat{\gamma}=2 n+r$.
In section 3 , we saw that given $[\mathcal{E}, \phi, f] \in \mathbb{E}[c, r]$, with $n, c \geq 3$, by Theorem 3.4 there exists $A \in \mathcal{A}[c, r]$ and the monad below

$$
\begin{equation*}
\mathcal{M}_{A}: 0 \longrightarrow H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{a_{A}^{\vee} \circ(A \otimes \mathrm{Id})} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \longrightarrow 0 \tag{25}
\end{equation*}
$$

whose cohomological bundle $\mathcal{E}_{A}$ is isomorphic to $\mathcal{E}$.
Now let us use the maps $a_{A}$ and $b_{A}=a_{A}^{\vee} \circ(A \otimes \mathrm{Id})$ in (25) to construct a Kronecker module associated to $\mathcal{E}$. We can associate to $a_{A}$ and $b_{A}$ the linear maps $\alpha \in \operatorname{Hom}\left(V, \operatorname{Hom}\left(H_{c}, W\right)\right)$ and $\beta \in \operatorname{Hom}\left(V, \operatorname{Hom}\left(W, H_{c}^{\vee}\right)\right)$ as follows

$$
\begin{array}{rlll}
\alpha: V & \rightarrow \operatorname{Hom}\left(H_{c}, W\right) & & \\
v & \mapsto \alpha(v): H_{c} & \rightarrow W \\
h & \mapsto a_{A}(x)(h \otimes v)
\end{array}
$$

and

$$
\begin{array}{rlrl}
\beta: V & \rightarrow & \operatorname{Hom}\left(W, H_{c}^{\vee}\right) & \\
v & \mapsto & \beta(v): W & \rightarrow \\
H_{c}^{\vee} \\
w & \mapsto & b_{A}(x)(w)(h \otimes v)
\end{array}
$$

where $x=\mathbb{P}(\mathbb{K} v)$. First note that $b_{A}(x)(w)(h \otimes v)=A(w)(\alpha(v)(h))$, this is why $\beta$ it is also known as the transpose map of $\alpha$ (with respect to $A$ ).
This pair of maps $(\alpha, \beta)$ has the following properties:
(P1) $\alpha(v): H_{c} \rightarrow W$ is injective for all $v \neq 0$;
(P2) $\beta(v) \circ \alpha(v): H_{c} \rightarrow H_{c}^{\vee}$ is the zero mapping for all $v \in V$;
(P3) the map $\hat{\alpha}: V \otimes H_{c} \rightarrow W$ is surjective, with $\hat{\alpha}(v \otimes h)=\alpha(v)(h)$.
The property ( P 1 ) holds if and only if $a_{A}$ is injective in each fiber.
The property ( P 2 ) is equivalent for the composition $a_{A}^{\vee} \circ(A \otimes \mathrm{Id}) \circ a_{A}$ to be the zero mapping in each fiber. Indeed, for each $x \in \mathbb{P}(\mathbb{K} v) \in \mathbb{P}^{n}$, we have

$$
\begin{aligned}
\left(b_{A} \circ a_{A}\right)(x)(h \otimes v) & =\left[a_{A}^{\vee} \circ(A \otimes \mathrm{Id}) \circ a_{A}\right](x)(h \otimes v) \\
& =A(\alpha(v)(h))(\alpha(v)(h)) \\
& =\beta(v) \circ \alpha(v) .
\end{aligned}
$$

Now, let us prove that (P3) holds if and only if the cohomology bundle of (25) has no global sections. Indeed, by the display of the monad (25), we have the following exact sequences

$$
\begin{aligned}
& 0 \longrightarrow H_{c} \otimes \mathcal{O}_{\mathbb{P}^{n}}(-1) \xrightarrow{a_{A}} \operatorname{Ker}\left(b_{A}\right) \longrightarrow \mathcal{E}_{A} \longrightarrow \operatorname{Ker}\left(b_{A}\right) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^{n}} \xrightarrow{b_{A}} H_{c}^{\vee} \otimes \mathcal{O}_{\mathbb{P}^{n}}(1) \longrightarrow 0 \\
& 0 \longrightarrow
\end{aligned}
$$

thus,

$$
H^{0}\left(\mathcal{E}_{A}\right) \cong H^{0}\left(\operatorname{Ker}\left(b_{A}\right)\right) \cong \operatorname{Ker}\left(W \rightarrow H_{c}^{\vee} \otimes V^{\vee}\right)
$$

that means $H^{0}\left(\mathcal{E}_{A}\right)=0$ if and only if $H^{0}\left(b_{A}\right): W \rightarrow H_{c}^{\vee} \otimes V^{\vee}$ is injective, if and only if $\hat{\alpha}: V \otimes H_{c} \rightarrow W$ is surjective.
Let us consider

$$
\begin{align*}
\gamma^{\prime}: V \times V & \rightarrow \operatorname{Hom}\left(H_{c}, H_{c}^{\vee}\right)  \tag{26}\\
\left(v_{1}, v_{2}\right) & \mapsto \beta\left(v_{2}\right) \circ \alpha\left(v_{1}\right),
\end{align*}
$$

which defines an element $\gamma \in \operatorname{Hom}\left(\bigwedge^{2} V, \operatorname{Hom}\left(H_{c}, H_{c}^{\vee}\right)\right)$. We now will prove that the map $\gamma$ is a Kronecker module of rank $r$.

Lemma 5.2. The element $\gamma \in \operatorname{Hom}\left(\bigwedge^{2} V, \operatorname{Hom}\left(H_{c}, H_{c}^{\vee}\right)\right)$ constructed as above is a Kronecker module of rank $r$.

Proof. Let $\hat{\alpha}: V \otimes H_{c} \rightarrow W, \hat{\beta}: W \rightarrow V^{\vee} \otimes H_{c}^{\vee}$ and $\hat{\gamma}: H_{c} \otimes V \rightarrow H_{c}^{\vee} \otimes V^{\vee}$ be the linear maps associated to $\alpha, \beta$ and $\gamma$, respectively. By the definition of $\gamma^{\prime}$ in (26) we have $\hat{\gamma}=\hat{\beta} \circ \hat{\alpha}$. Now let us prove that $\gamma$ satisfies the properties (K1)-(K3) of Definition 5.1.
For each $v \neq 0$ we have

$$
\hat{\gamma}(v \otimes h)\left[\gamma\left(v \wedge v_{1}\right)(h)\right]\left(h_{1}\right),
$$

but

$$
\begin{aligned}
{\left[\gamma^{\prime}\left(v \wedge v_{1}\right)(h)\right]\left(h_{1}\right) } & =\left[\beta\left(v_{1}\right) \circ \alpha(v)\right](h)\left(h_{1}\right) \\
& =A(\alpha(v)(h))\left(\alpha\left(v_{1}\right)\left(h_{1}\right)\right) .
\end{aligned}
$$

Thus (K1) follows by property (P1). Also observe that $v^{\vee \vee} \circ \hat{\gamma}=\hat{\gamma}(v \otimes-)^{\vee}$, therefore (K1) implies (K2).
By (P3) we have that $\hat{\alpha}$ is surjective and $\hat{\beta}$ is injective, thus it follows that rank $\hat{\gamma}=(n+1) c$, moreover, since $\hat{\gamma}=\hat{\beta} \circ \hat{\alpha}$, we have rank $\gamma=r$ and the property (P3) and therefore $\gamma$ is a Kronecker module of rank $r$.

Remark 5.3. Note that the linear map $\hat{\gamma}$ associated to the Kronecker module $\gamma$ is in fact the map $A$. Fixing the basis $\left\{v_{1}, \cdots, v_{n+1}\right\}$ and $\left\{h_{1}, \cdots, h_{c}\right\}$ for $V$ and $H_{c}$, repectively, for any $v_{j} \otimes h_{i} \in V \otimes H_{c}$, we have $\hat{\gamma}\left(v_{j} \otimes h_{i}\right)=A\left(v_{j} \otimes h_{i}\right)$. Indeed, for any $v_{k} \otimes h_{l} \in V \otimes H_{c}$ it follows

$$
\begin{aligned}
\hat{\gamma}\left(v_{j} \otimes h_{i}\right)\left(v_{k} \otimes h_{l}\right) & =(\hat{\beta} \circ \hat{\alpha})\left(v_{j} \otimes h_{i}\right)\left(v_{k} \otimes h_{l}\right) \\
& =\hat{\beta}\left(\hat{\alpha}\left(v_{j} \otimes h_{i}\right)\right)\left(v_{k} \otimes h_{l}\right) \\
& =\hat{\beta}\left(\alpha\left(v_{j}\right)\left(h_{i}\right)\right)\left(v_{k} \otimes h_{l}\right) \\
& =\hat{\beta}\left(a_{A}\left(x_{j}\right)\left(h_{i} \otimes v_{j}\right)\right)\left(v_{k} \otimes h_{l}\right) \\
& =\hat{\beta}\left(h_{i} \otimes v_{j}\right)\left(v_{k} \otimes h_{l}\right) \\
& \left.=\beta\left(v_{k}\right)\left(h_{i} \otimes v_{j}\right)\right)\left(h_{l}\right) \\
& =b_{A}\left(x_{k}\right)\left(h_{i} \otimes v_{j}\right)\left(h_{l} \otimes v_{k}\right) \\
& =A\left(h_{i} \otimes v_{j}\right)\left(\alpha\left(v_{k}\right)\left(h_{l}\right)\right) \\
& =A\left(h_{i} \otimes v_{j}\right)\left(h_{l} \otimes v_{k}\right),
\end{aligned}
$$

where $x_{j}=\mathbb{P}\left(\mathbb{K} v_{j}\right)$ and $x_{k}=\mathbb{P}\left(\mathbb{K} v_{k}\right)$.

The following result describes how we can obtain the splitting type of an orthogonal instanton bundle.

Theorem 5.4. Let $\mathcal{E}$ be an orthogonal instanton bundle on $\mathbb{P}^{n}$ with charge $c$, rank $r$ and no global sections, for $n, c \geq 3$, and let $\gamma$ be its associated Kronecker module. If $L \subset \mathbb{P}^{n}$ is the line defined by $v_{1}, v_{2} \in V, v_{1} \wedge v_{2} \neq 0$, the restriction $\left.\mathcal{E}\right|_{L}$ is trivial if and only if $\gamma\left(v_{1} \wedge v_{2}\right)$ is an isomorphism.

Proof. Let $\mathcal{E}$ be an orthogonal instanton bundle on $\mathbb{P}^{n}$ with charge $c$, rank $r$ and no global sections, for $n, c \geq 3$. Consider the maps $(\alpha, \beta)$ and the Kronecker module $\gamma$ as before. Let $v_{1}, v_{2} \in V$ such that $v_{1} \wedge v_{2} \neq 0$, and consider the $\mathbb{K}$-subspace $K=\mathbb{K} v_{1}+\mathbb{K} v_{2}$. The restriction of the monad (25) to $L=\mathbb{P}(K)$ is the monad

$$
\begin{equation*}
\mathcal{M}_{A}: 0 \longrightarrow H_{c} \otimes \mathcal{O}_{L}(-1) \xrightarrow{\left.a_{A}\right|_{L}} W \otimes \mathcal{O}_{L} \xrightarrow{\left.b_{A}\right|_{L}} H_{c}^{\vee} \otimes \mathcal{O}_{L}(1) \longrightarrow 0 . \tag{27}
\end{equation*}
$$

The display of the monad (27) gives the exact sequences

$$
\begin{gathered}
\left.0 \longrightarrow H_{c} \otimes \mathcal{O}_{L}(-1) \stackrel{\left.a_{A}\right|_{L}}{\longrightarrow} \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right) \longrightarrow \mathcal{E}\right|_{L} \longrightarrow 0, \\
0 \longrightarrow \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right) \longrightarrow W \otimes \mathcal{O}_{L} \xrightarrow{\left.b_{A}\right|_{L}} H_{c}^{\vee} \otimes \mathcal{O}_{L}(1) \longrightarrow 0
\end{gathered}
$$

thus

$$
H^{0}\left(L,\left.\mathcal{E}\right|_{L}\right) \cong H^{0}\left(L, \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right)\right) \cong \operatorname{Ker}\left(W \rightarrow H_{c}^{\vee} \otimes K^{\vee}\right)
$$

Observe that $\left.\mathcal{E}\right|_{L}$ has trivial splitting type if and only if no section $s \in H^{0}\left(L,\left.\mathcal{E}\right|_{L}\right) \backslash\{0\}$ has zeros, so our goal is to prove that this holds if and only if $\gamma\left(v_{1} \wedge v_{2}\right)$ is invertible. Consider the inclusions

$$
H_{c} \otimes \mathcal{O}_{L}(-1) \stackrel{i}{\hookrightarrow} \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right) \stackrel{j}{\hookrightarrow} W \otimes \mathcal{O}_{L} .
$$

Let $\left.s \in H^{0}\left(L, \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right)\right)\right) \cong H^{0}\left(L,\left.\mathcal{E}\right|_{L}\right)$ be a section; being $H^{0}\left(W \otimes \mathcal{O}_{L}\right) \cong W$, there exists $w \in W$ with $j \circ s(x)=w$ for all $x \in L$. So the section $s^{\prime} \in H^{0}\left(L,\left.\mathcal{E}\right|_{L}\right)$ defined by $s$ has zeros at $x=\mathbb{P}(\mathbb{K} v) \in L$ if and only if $s(x)$ lies in the image of the inclusion $i(x): H_{c} \otimes \mathcal{O}_{L}(-1) \hookrightarrow \operatorname{Ker}\left(\left.b_{A}\right|_{L}\right)(x)$, i.e. if and only if there exists $h \in H_{c}$ with $\alpha(v)(h)=w$. Because $s$ is a section in $\operatorname{Ker}\left(\left.b_{A}\right|_{L}\right)$, for every $v^{\prime} \in K$ we must have $\beta\left(v^{\prime}\right)(w)=0$, thus $\left.\mathcal{E}\right|_{L}$ has no trivial section with zeros if and only if

$$
\operatorname{Im} \alpha(v) \subset \bigcap_{v^{\prime} \in K} \operatorname{Ker} \beta\left(v^{\prime}\right)
$$

for at least one vector $v \in K \backslash\{0\}$, which means that for any basis $v, v^{\prime} \in K$ of $K$ the map

$$
\gamma\left(v \wedge v^{\prime}\right)=\beta\left(v^{\prime}\right) \circ \alpha(v)
$$

is not an isomorphism.
Now let us use the Theorem 5.4 to determine the type of splitting of the bundle constructed in Example 3.6

Example 5.5. Let $\mathcal{E}$ be the orthogonal instanton bundle on $\mathbb{P}^{3}$ of Example 3.6. Let $L \in \mathbb{P}^{3}$ be the line joining two general points $P=[a: b: c: d]$ and $Q=[e: f: g: h]$. By Theorem 5.4, $\left.\mathcal{E}\right|_{L}$ is trivial if and only if $\beta(Q) \alpha(P)$ is invertible. In our case,

$$
\beta(Q) \alpha(P)=\left(\begin{array}{cccccc}
0 & \lambda_{1} & 0 & 0 & 0 & 0 \\
-\lambda_{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda_{2} & 0 & 0 \\
0 & 0 & \lambda_{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_{3} \\
0 & 0 & 0 & 0 & -\lambda_{3} & 0
\end{array}\right)
$$

where $\lambda_{1}=2 b e-2 a f-6 d g+6 c h, \lambda_{2}=-b e+a f+3 d g-3 c h$ and $\lambda_{3}=b e-a f-3 d g+3 c h$. Thus $\beta(Q) \alpha(P)$ is invertible and therefore $\left.\mathcal{E}\right|_{L}$ is trivial. Since the points are general this is also true for every general line, therefore $\mathcal{E}$ has trivial splitting type.
On the other hand, let $L_{0}$ be the line joining the points $P=[1: 0: 0: 0]$ and $Q=[0: 0:$ $0: 1$ ]. By the previous construction $\beta(Q) \alpha(P)=0$, therefore by Theorem 5.4 $\left.\mathcal{E}\right|_{L_{0}}$ is not trivial, hence $L_{0}$ is a jumping line for $\mathcal{E}$.

Finally, let us prove that the bundle presented in Example 3.7 has no trivial splitting type, as expected from ([17] - Lemma 4.3).

Example 5.6. Let $\mathcal{E}$ be the orthogonal instanton bundle on $\mathbb{P}^{3}$ Example 3.7, Let $L \in \mathbb{P}^{3}$ be the line joint the points $P=[a: b: c: d]$ and $Q=[e: f: g: h]$. We have

$$
\beta(Q) \alpha(P)=\left(\begin{array}{ccccc}
0 & \lambda_{1} & \lambda_{2} & 0 & \lambda_{2} \\
-\lambda_{1} & 0 & \lambda_{1} & \lambda_{2} & 0 \\
-\lambda_{2} & -\lambda_{1} & 0 & 0 & \lambda_{2} \\
0 & -\lambda_{2} & 0 & 0 & \lambda_{2} \\
-\lambda_{2} & 0 & -\lambda_{2} & -\lambda_{2} & 0
\end{array}\right)
$$

where $\lambda_{1}=b e-a f+d g-c h$ and $\lambda_{2}=d e+c f-b g-a h$. Since $\beta(Q) \alpha(P)$ is a skew-symmetric matrix of odd order, $\beta(Q) \alpha(P)$ is not invertible for all $P$ and $Q$. Thus by Theorem [5.4, $\left.\mathcal{E}\right|_{L}$ is not trivial for every line $L \in \mathbb{P}^{3}$, i.e. $\mathcal{E}$ has no trivial splitting type.

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