ESSENTIAL NORMS AND SCHATTEN(-HERZ) CLASSES OF INTEGRATION OPERATORS FROM BERGMAN SPACES TO HARDY SPACES

JIALE CHEN*, JORDI PAU AND MAOFA WANG

Abstract. In this paper, we completely characterize the compactness of the Volterra type integration operators J_b acting from weighted Bergman spaces A^p_α to Hardy spaces H^q for all $0 < p, q < \infty$. It is quite surprising that the boundedness and the compactness of $J_b : A^p_\alpha \to H^q$ are not equivalent when $0 < q < p \leq 2$, while, due to the well-known results, the compactness and boundedness of $J_b : A^p_\alpha \to A^q_\beta$ (resp. $J_b : H^p \to H^q$) are always equivalent when $p > q$. Furthermore, we give some estimates for the essential norms of $J_b : A^p_\alpha \to H^q$ in the case $0 < p \leq q < \infty$. We finally describe the membership in the Schatten(-Herz) class of the Volterra type integration operators.

1. Introduction

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$ and $\mathcal{H}(\mathbb{B}_n)$ denote the algebra of holomorphic functions on $\mathbb{B}_n$. A function $b \in \mathcal{H}(\mathbb{B}_n)$ induces a Volterra type integration operator J_b given by the formula:

$$J_b f(z) = \int_0^1 f(tz) Rb(tz) \frac{dt}{t}, \quad z \in \mathbb{B}_n, \tag{1.1}$$

where $f \in \mathcal{H}(\mathbb{B}_n)$ and Rb is the radical derivative of b:

$$Rb(z) = \sum_{k=1}^{n} z_k \frac{\partial b}{\partial z_k}(z), \quad z = (z_1, z_2, \dots, z_n) \in \mathbb{B}_n.$$

A fundamental property of the operator J_b is the following formula involving the radical derivative R:

$$R(J_b f)(z) = f(z) Rb(z), \quad z \in \mathbb{B}_n.$$

Date: December 24, 2020.

* Corresponding author .

2010 *Mathematics Subject Classification.* 47G10, 32A35, 32A36.

Key words and phrases. Integration operator, Hardy space, Bergman space, Compactness, Essential norm, Schatten-Herz class.

J. Pau was partially supported by the grants MTM2017-83499-P (Ministerio de Educación y Ciencia) and 2017SGR358 (Generalitat de Catalunya). M. Wang was partially supported by NSFC (No. 11771340) of China.

For $0 < p < \infty$, the Hardy space H^p consists of those holomorphic functions f in $\mathbb{B}_n$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} M_p^p(f,r) = \sup_{0<r<1} \int_{\mathbb{S}_n} |f(r\xi)|^p d\sigma(\xi) < \infty,$$

where $d\sigma$ is the surface measure on the unit sphere $\mathbb{S}_n = \partial\mathbb{B}_n$ normalized so that $\sigma(\mathbb{S}_n) = 1$. Given $\alpha > -1$ and $0 < p < \infty$, a function $f \in \mathcal{H}(\mathbb{B}_n)$ belongs to the weighted Bergman space A^p_α, if

$$\|f\|_{A^p_\alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty.$$

Here $dv = dv_0$ is the Lebesgue measure on $\mathbb{B}_n$, normalized so that $v(\mathbb{B}_n) = 1$. The measure dv_α is given by $dv_\alpha(z) = c_{n,\alpha}(1-|z|^2)^\alpha dv(z)$ with normalized constant $c_{n,\alpha}$ so that $v_\alpha(\mathbb{B}_n) = 1$.

The operator J_b has been studied by many authors, see [9, 13, 15, 18] and the references therein. In particular, Wu [18] partially solved the boundedness of $J_b : A^p_\alpha \to H^q$ in the setting of the unit disk. Recently, Miihkinen, Pau, Perälä and Wang [13] completely characterized the boundedness of $J_b : A^p_\alpha \to H^q$ for all dimensions n. In this paper, we follow the line of research to completely characterize the compactness of the Volterra type integration operators J_b acting from weighted Bergman spaces A^p_α to Hardy spaces H^q for all $0 < p, q < \infty$. Furthermore, we give some estimates for the essential norms of $J_b : A^p_\alpha \to H^q$ in the case $0 < p \le q < \infty$. We finally describe the membership in Schatten(-Herz) classes of the Volterra type integration operators and give some descriptions of asymptotic property of singular values.

Our first result is the following little version of the main result in [13].

Theorem 1.1. *Let $\alpha > -1$, $0 < p, q < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) If $0 < p \le \min\{2, q\}$ or $2 < p < q < \infty$, then $J_b : A^p_\alpha \to H^q$ is compact if and only if

$$\lim_{|z|\to 1^-} Rb(z)(1-|z|^2)^{\frac{n}{q}+1-\frac{n+1+\alpha}{p}} = 0.$$

(2) If $2 < p = q < \infty$, then $J_b : A^p_\alpha \to H^q$ is compact if and only if

$$|Rb(z)|^{\frac{2p}{p-2}}(1-|z|^2)^{\frac{p-2\alpha}{p-2}} dv(z)$$

is a vanishing Carleson measure.

(3) If $p > \max\{2, q\}$, then $J_b : A^p_\alpha \to H^q$ is compact if and only if

$$\xi \mapsto \left(\int_{\Gamma(\xi)} |Rb(z)|^{\frac{2p}{p-2}}(1-|z|^2)^{\frac{2-2\alpha}{p-2}+1-n} dv(z) \right)^{\frac{p-2}{2p}}$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{S}_n)$.

(4) If $0 < q < p \le 2$, then $J_b : A^p_\alpha \to H^q$ is compact if and only if

$$\xi \mapsto \sup_{z\in\Gamma(\xi)\cap\{|z|\ge r\}} |Rb(z)|(1-|z|^2)^{\frac{p-1-\alpha}{p}}$$

converges to zero in $L^{\frac{pq}{p-q}}(\mathbb{S}_n)$ *as* $r \to 1^-$.

Two remarks are in order. (1) Comparing the above theorem with [13, Theorem 1], we find that the boundedness and compactness of $J_b : A^p_\alpha \to H^q$ are not equivalent when $0 < q < p \le 2$. This is quite different from the cases $J_b : A^p_\alpha \to A^q_\beta$ and $J_b : H^p \to H^q$, where J_b is compact if and only if it is bounded whenever $p > q$, see [9, 15].

(2) By Theorem 1.1 (4), we know that if $0 < q < p \le 2$, $J_b : A^p_\alpha \to H^q$ may be not compact even when b is a polynomial. It is quite surprising because the corresponding operators $J_b : A^p_\alpha \to A^q_\beta$ and $J_b : H^p \to H^q$ are always compact for any polynomial b if $p > q$, see [9, 15]. See Section 3 for more details.

Let X, Y be (quasi-)Banach spaces and $T : X \to Y$ a bounded operator. The essential norm of T, denoted by $\|T\|_e$, is its distance from the space of compact operators. It is clear that T is compact if and only if $\|T\|_e = 0$. Our next result gives some estimates for the essential norm of $J_b : A^p_\alpha \to H^q$ in the case $0 < p \le q < \infty$.

Theorem 1.2. *Let* $\alpha > -1$, $0 < p \le q < \infty$ *and* $b \in \mathcal{H}(\mathbb{B}_n)$ *such that* $J_b : A^p_\alpha \to H^q$ *is bounded.*

(1) If $0 < p \le \min\{2, q\}$ *or* $2 < p < q < \infty$, *then*

$$\|J_b\|_e \asymp \limsup_{|a|\to 1^-} |Rb(a)|(1-|a|^2)^{\frac{n}{q}+1-\frac{n+1+\alpha}{p}}.$$

(2) If $2 < p = q < \infty$, *then*

$$\|J_b\|_e \lesssim \limsup_{|a|\to 1^-} \left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z, a\rangle|^{2n}} |Rb(z)|^{\frac{2p}{p-2}} (1-|z|^2)^{\frac{p-2\alpha}{p-2}} \, dv(z) \right)^{\frac{p-2}{2p}}.$$

Recall that if T is a compact operator acting on a separable Hilbert space H, then there exist a nonincreasing sequence $\{s_k(T)\}$ of nonnegative numbers tending to 0 and orthonormal sets $\{e_k\}$, $\{\sigma_k\}$ in H such that

$$Tx = \sum_k s_k(T)\langle x, e_k\rangle \sigma_k$$

for all $x \in H$. This is the so-called canonical decomposition of the compact operator T. The number $s_k(T)$ is the kth singular value of T, which is exactly the square root of the kth eigenvalue of the positive operator T^*T if we rearrange the eigenvalues in nonincreasing order, where T^* is the Hilbert adjoint of T. For $0 < p < \infty$, the compact operator T belongs to the Schatten class $S_p(H)$ if $\{s_k(T)\}$ is in the sequence space l^p. If H_1 and H_2 are two separable Hilbert spaces and $T : H_1 \to H_2$ is a compact operator, we say that T is in the Schatten class $S_p(H_1, H_2)$ if T^*T is in $S_{p/2}(H_1)$. We refer to [21, Chapter 1] for a brief account on Schatten classes.

Our third result is a complete characterization of the membership in the Schatten class $S_p(A^2_\alpha, H^2)$ of the Volterra type integration operator J_b.

Theorem 1.3. *Let $\alpha > -1$, $0 < p < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) If $p(1-\alpha)/2 > n$, then J_b belongs to $S_p(A^2_\alpha, H^2)$ if and only if

$$\int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{\frac{p}{2}(1-\alpha)} d\lambda_n(z) < \infty,$$

where

$$d\lambda_n(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the invariant measure on $\mathbb{B}_n$.

(2) If $p(1-\alpha)/2 \leq n$, then J_b is in $S_p(A^2_\alpha, H^2)$ if and only if b is constant.

Loaiza, López-García and Pérez-Esteva introduced the Schatten-Herz class of Toeplitz operators in [10], which is a generalization of the Schatten class. Recall that, given a positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator T_μ on A^2_α is defined by

$$T_\mu f(z) = \int_{\mathbb{B}_n} f(w) K^\alpha(z,w) d\mu(w), \quad z \in \mathbb{B}_n,$$

where $K^\alpha(z,w)$ is the reproducing kernel of A^2_α. See [22] for more information about Toeplitz operators. For $0 < p, q < \infty$, the Toeplitz operator T_μ is said to be in the Schatten-Herz class $S_{p,q}(A^2_\alpha)$ if each $T_{\mu\chi_k}$ is in $S_p(A^2_\alpha)$ and the sequence $\{\|T_{\mu\chi_k}\|_{S_p}\}_{k\geq 0}$ is in l^q, where χ_k is the characteristic function of the annulus $A_k = \{z \in \mathbb{B}_n : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}\}$ for $k \geq 0$. Following [7], we say that $J_b : A^2_\alpha \to H^2$ is in the Schatten-Herz class $S_{p,q}(A^2_\alpha, H^2)$ if $J_b^* J_b \in S_{\frac{p}{2},\frac{q}{2}}(A^2_\alpha)$. Our next result characterizes the Schatten-Herz class of integration operators.

Theorem 1.4. *Let $\alpha > -1$, $0 < p, q < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) If $p(1-\alpha)/2 > n$, then J_b belongs to $S_{p,q}(A^2_\alpha, H^2)$ if and only if

$$\int_0^1 M_p^q(Rb, r)(1-r)^{\frac{q}{2}(1-\alpha) - n\frac{q}{p} - 1} dr < \infty.$$

(2) If $p(1-\alpha)/2 \leq n$, then J_b is in $S_{p,q}(A^2_\alpha, H^2)$ if and only if b is constant.

It is also interesting to consider the speed of $s_k(T)$ converging to zero if T is a compact operator on a separable Hilbert space. See [8] and references there for more details. Based on the work concerning Toeplitz operators in [5], our next result (see Theorem 5.4 in Section 5) gives a description of the decay of singular values of $J_b : A^2_\alpha \to H^2$.

The paper is organized as follows. Some background and preliminary results are given in Section 2. In Section 3 we consider the compactness of integration operators $J_b : A^p_\alpha \to H^q$. In Section 4 we estimate the essential norms. Section 5 is devoted to the proof of Theorem 1.3, Theorem 1.4 and some descriptions of asymptotic property of singular values of $J_b : A^2_\alpha \to H^2$. We also consider the essential norms and membership in Schatten(-Herz) classes of integration operators from Hardy spaces to Bergman spaces in Section 6.

Notation. For $1 < p < \infty$, we let p' denote the conjugate exponent of p. The notation $A \lesssim B$ means that $A \leq CB$ for some inessential constant $C > 0$. The converse relation $A \gtrsim B$ is defined in an analogous manner, and if $A \lesssim B$ and $A \gtrsim B$ both hold, we write $A \asymp B$. In addition, we always let r be a positive number less than 1.

2. Preliminaries

In this section we introduce some well-known results that will be used throughout the paper.

2.1. **Carleson measures.** For $\xi \in \mathbb{S}_n$ and $\delta > 0$, the non-isotropic metric ball $B_\delta(\xi)$ is defined by

$$B_\delta(\xi) = \{z \in \mathbb{B}_n : |1 - \langle z, \xi\rangle| < \delta\}.$$

A positive Borel measure μ on $\mathbb{B}_n$ is said to be a Carleson measure if there is a constant $C > 0$ such that

$$\mu(B_\delta(\xi)) \leq C\delta^n$$

for all $\xi \in \mathbb{S}_n$ and $\delta > 0$. Obviously every Carleson measure is finite. Hörmander [6] extended to several complex variables the famous Carleson measure embedding theorem [3, 4] asserting that, for $0 < p < \infty$, the embedding $I_d : H^p \to L^p(\mathbb{B}_n, d\mu)$ is bounded if and only if μ is a Carleson measure. We denote by $\|\mu\|_{CM}$ the infimum of all possible C above. It is well-known (see [19, Theorem 45]) that μ is a Carleson measure if and only if for each (some) $t > 0$ one has

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1 - |a|^2)^t}{|1 - \langle z, a\rangle|^{n+t}} d\mu(z) < \infty. \tag{2.1}$$

Moreover, with constant depending on t, the supremum of the above integral is comparable to $\|\mu\|_{CM}$.

A positive Borel measure μ on $\mathbb{B}_n$ is called a vanishing Carleson measure if

$$\lim_{\delta\to 0} \frac{\mu(B_\delta(\xi))}{\delta^n} = 0$$

uniformly for $\xi \in \mathbb{S}_n$. Equivalently, one may require that for each (some) $t > 0$ one has

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_n} \frac{(1 - |a|^2)^t}{|1 - \langle z, a\rangle|^{n+t}} d\mu(z) = 0, \tag{2.2}$$

or for $0 < p < \infty$, the embedding $I_d : H^p \to L^p(\mathbb{B}_n, d\mu)$ is compact.

2.2. **Separated sequences and lattices.** A sequence of points $\{z_j\} \subset \mathbb{B}_n$ is said to be separated if there exists $\delta_0 > 0$ such that $\beta(z_i, z_j) \geq \delta_0$ for all i and j with $i \neq j$, where $\beta(z, w)$ denotes the Bergman metric on $\mathbb{B}_n$. This implies that there is $\delta > 0$ such that the Bergman metric balls $D(z_j, \delta) = \{z \in \mathbb{B}_n : \beta(z, z_j) < \delta\}$ are pairwise disjoint.

We need a well-known result on decomposition of the unit ball $\mathbb{B}_n$. By Theorem 2.23 in [20], there exists a positive integer N such that for any $0 < \delta < 1$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ with the following properties:

(i) $\mathbb{B}_n = \bigcup_k D(a_k, \delta)$;
(ii) The sets $D(a_k, \delta/4)$ are mutually disjoint;
(iii) Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 4\delta)$.
Any sequence $\{a_k\}$ satisfying the above conditions is called a δ-lattice (in the Bergman metric). Obviously any δ-lattice is a separated sequence.

2.3. **Area methods and equivalent norms.** For $\xi \in \mathbb{S}_n$ and $\gamma > 1$, recall that the admissible approach region $\Gamma_\gamma(\xi)$ is defined as

$$\Gamma_\gamma(\xi) = \left\{ z \in \mathbb{B}_n : |1 - \langle z, \xi \rangle| < \frac{\gamma}{2}(1 - |z|^2) \right\}.$$

In this paper we agree that $\Gamma(\xi) := \Gamma_2(\xi)$. It is known that for every $\delta > 0$ and $\gamma > 1$, there exists $\gamma' > 1$ so that

$$\bigcup_{z \in \Gamma_\gamma(\xi)} D(z, \delta) \subset \Gamma_{\gamma'}(\xi). \tag{2.3}$$

We will write $\widetilde{\Gamma}(\xi)$ to indicate this change of aperture. If $I(z) = \{\xi \in \mathbb{S}_n : z \in \Gamma(\xi)\}$, then $\sigma(I(z)) \asymp (1 - |z|^2)^n$, and it follows from Fubini's theorem that, for a positive measurable function φ, and a finite positive measure ν, one has

$$\int_{\mathbb{B}_n} \varphi(z) d\nu(z) \asymp \int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)} \varphi(z) \frac{d\nu(z)}{(1 - |z|^2)^n} \right) d\sigma(\xi). \tag{2.4}$$

This fact will be used repeatedly throughout the paper.

Let us recall the following Littlewood-Paley identity, which can be found in [20].

Theorem A. *Suppose* $0 < p < \infty$. *Then*

$$\|f - f(0)\|_{H^p}^p = \frac{p^2}{2n} \int_{\mathbb{B}_n} |Rf(z)|^2 |f(z) - f(0)|^{p-2} |z|^{-2n} \log \frac{1}{|z|} dv(z)$$

for all $f \in H^p$. *In particular, if* $f(0) = 0$,

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{B}_n} |Rf(z)|^2 |f(z)|^{p-2} (1 - |z|^2) dv(z).$$

The next estimate is the Calderón's area theorem [2, 12]. The variant we will use can be found in [1] or in [15].

Theorem B. *Let* $0 < p < \infty$. *If* $f \in \mathcal{H}(\mathbb{B}_n)$ *and* $f(0) = 0$, *then*

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)} |Rf(z)|^2 (1 - |z|^2)^{1-n} dv(z) \right)^{p/2} d\sigma(\xi).$$

We will also need the following result essentially due to Luecking [11] (see also [15]) describing those positive Borel measures for which the embedding from H^p into $L^s(\mu)$ is bounded when $s < p$.

Theorem C. *Let $0 < s < p < \infty$ and let μ be a positive Borel measure on $\mathbb{B}_n$. Then the identity $I_d : H^p \to L^s(\mu)$ is bounded if and only if the function defined on $\mathbb{S}_n$ by*

$$\widetilde{\mu}(\xi) = \int_{\Gamma(\xi)} (1 - |z|^2)^{-n} d\mu(z)$$

belongs to $L^{p/(p-s)}(\mathbb{S}_n)$. Moreover, one has $\|I_d\|_{H^p \to L^s(\mu)} \asymp \|\widetilde{\mu}\|^{1/s}_{L^{p/(p-s)}(\mathbb{S}_n)}$.

3. Compactness

In this section, we will prove Theorem 1.1. We need the following two lemmas first.

Lemma 3.1. *If μ is a vanishing Carleson measure, then*

$$\lim_{r \to 1^-} \|\chi_{(r\mathbb{B}_n)^c} \mu\|_{CM} = 0,$$

where $(r\mathbb{B}_n)^c = \mathbb{B}_n \setminus r\mathbb{B}_n$, and $\chi_{(r\mathbb{B}_n)^c}$ is the characteristic function of the set $(r\mathbb{B}_n)^c$.

Proof. Since $\|\chi_{(r\mathbb{B}_n)^c} \mu\|_{CM} \asymp \sup_{a \in \mathbb{B}_n} \int_{(r\mathbb{B}_n)^c} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu(z)$, it is sufficient to show that

$$\lim_{r \to 1^-} \sup_{a \in \mathbb{B}_n} \int_{(r\mathbb{B}_n)^c} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu(z) = 0.$$

We complete the proof by contradiction. Suppose that

$$\lim_{r \to 1^-} \sup_{a \in \mathbb{B}_n} \int_{(r\mathbb{B}_n)^c} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu(z) \neq 0,$$

and then there exist $\epsilon_0 > 0$, an increasing sequence of positive numbers $\{r_k\}$, $r_k \to 1^-$, and $\{a_k\} \subset \mathbb{B}_n$, such that

$$\int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^n \chi_{(r_k\mathbb{B}_n)^c}(z)}{|1-\langle a_k,z\rangle|^{2n}} d\mu(z) \geq \epsilon_0, \quad \forall k \geq 1. \tag{3.1}$$

There are two possibilities about the sequence $\{a_k\}$: $|a_k| \to 1^-$ and $|a_k| \nrightarrow 1^-$. If $|a_k| \to 1^-$, then by (2.2) we have

$$\int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^n \chi_{(r_k\mathbb{B}_n)^c}(z)}{|1-\langle a_k,z\rangle|^{2n}} d\mu(z) \leq \int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^n}{|1-\langle a_k,z\rangle|^{2n}} d\mu(z) \to 0$$

as $k \to \infty$. This is contradictory with (3.1). If $|a_k| \nrightarrow 1^-$, there are a point $a_0 \in \mathbb{B}_n$ and a subsequence of $\{a_k\}$ converging to a_0. Without loss of generality, we assume $a_k \to a_0$, then there exists $\delta_0 > 0$ such that $\overline{B(a_0,\delta_0)} = \{z : |z - a_0| \leq \delta_0\} \subset \mathbb{B}_n$ and $a_k \in \overline{B(a_0,\delta_0)}$ if k is large enough. Therefore, we have

$$\frac{(1-|a_k|^2)^n \chi_{(r_k\mathbb{B}_n)^c}(z)}{|1-\langle a_k,z\rangle|^{2n}} \leq \frac{1}{[1-(|a_0|+\delta_0)]^{2n}}$$

for all $z \in \mathbb{B}_n$ whenever k is large enough, and it is clear that $\frac{(1-|a_k|^2)^n \chi_{(r_k\mathbb{B}_n)^c}(z)}{|1-\langle a_k,z\rangle|^{2n}} \to 0$ for all $z \in \mathbb{B}_n$. Then we get

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} \frac{(1-|a_k|^2)^n \chi_{(r_k\mathbb{B}_n)^c}(z)}{|1-\langle a_k,z\rangle|^{2n}} d\mu(z) = 0$$

by dominated convergence theorem since μ is a vanishing Carleson measure (in particular, μ is finite). This, again, is contradictory with (3.1). Thus the proof is finished. $\square$

Lemma 3.2. *Let $0 < p < \infty$, $\beta \geq 0$ and $Z = \{a_k\}$ be a δ-lattice. Suppose $f \in \mathcal{H}(\mathbb{B}_n)$ and $\int_{\mathbb{S}_n} \sup_{z\in\Gamma(\xi)} |f(z)|^p(1-|z|^2)^\beta d\sigma(\xi) < \infty$. If*

$$\lim_{r\to 1^-} \int_{\mathbb{S}_n} \sup_{a_k\in\Gamma(\xi)\cap\{|w|\geq r\}} |f(a_k)|^p(1-|a_k|^2)^\beta d\sigma(\xi) = 0 \tag{3.2}$$

whenever $\delta > 0$ is small enough, then

$$\lim_{r\to 1^-} \int_{\mathbb{S}_n} \sup_{z\in\Gamma(\xi)\cap\{|w|\geq r\}} |f(z)|^p(1-|z|^2)^\beta d\sigma(\xi) = 0.$$

Proof. Given $\epsilon > 0$, choose $\delta > 0$ small enough so that $\delta^p < \epsilon$. Fix this δ, and let

$$\tilde{r} = \inf\{|a_k| : a_k \in Z, a_k \in D(z,\delta) \text{ for some } |z| \geq r\}.$$

By the same method as in the proof of [13, Lemma 3], we have

$$\begin{aligned}
&\sup_{z\in\Gamma(\xi)\cap\{|w|\geq r\}} |f(z)|^p(1-|z|^2)^\beta \\
&\qquad \lesssim \delta^p \sup_{z\in\tilde{\tilde{\Gamma}}(\xi)} |f(z)|^p(1-|z|^2)^\beta + \sup_{a_k\in\tilde{\Gamma}(\xi)\cap\{|w|\geq\tilde{r}\}} |f(a_k)|^p(1-|a_k|^2)^\beta.
\end{aligned}$$

Integrating over $\mathbb{S}_n$ yields

$$\begin{aligned}
&\int_{\mathbb{S}_n} \sup_{z\in\Gamma(\xi)\cap\{|w|\geq r\}} |f(z)|^p(1-|z|^2)^\beta d\sigma(\xi) \\
&\lesssim \delta^p \int_{\mathbb{S}_n} \sup_{z\in\tilde{\tilde{\Gamma}}(\xi)} |f(z)|^p(1-|z|^2)^\beta d\sigma(\xi) + \int_{\mathbb{S}_n} \sup_{a_k\in\tilde{\Gamma}(\xi)\cap\{|w|\geq\tilde{r}\}} |f(a_k)|^p(1-|a_k|^2)^\beta d\sigma(\xi) \\
&\lesssim \epsilon + \int_{\mathbb{S}_n} \sup_{a_k\in\Gamma(\xi)\cap\{|w|\geq\tilde{r}\}} |f(a_k)|^p(1-|a_k|^2)^\beta d\sigma(\xi).
\end{aligned}$$

It is easy to see $\tilde{r} \to 1^-$ as $r \to 1^-$. Combining the above estimate with (3.2), we get

$$\limsup_{r\to 1^-} \int_{\mathbb{S}_n} \sup_{z\in\Gamma(\xi)\cap\{|w|\geq r\}} |f(z)|^p(1-|z|^2)^\beta d\sigma(\xi) \lesssim \epsilon.$$

The proof is finished since $\epsilon > 0$ is arbitrary. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We only prove (2) and the necessary part of (4). The rest parts can be proved by standard modifications of the corresponding parts in [13] and so are omitted. Fix $0 < \epsilon < 1$.

(2) For any $f \in A^p_\alpha$, consider the measure $d\mu_{f,b}(z) = |f(z)|^2|Rb(z)|^2 dv_1(z)$. Then

$$\|J_b f\|^p_{H^p} \asymp \int_{\mathbb{S}_n} \widetilde{\mu_{f,b}}(\xi)^{p/2} d\sigma(\xi) \tag{3.3}$$

by Theorem B. For the sake of simplicity, we denote

$$d\mu_b(z) = |Rb(z)|^{\frac{2p}{p-2}} (1-|z|^2)^{\frac{p-2\alpha}{p-2}} dv(z).$$

We first consider the sufficiency part. Suppose μ_b is a vanishing Carleson measure and $\{f_k\}$ is a bounded sequence in A^p_α converging to 0 uniformly on compact subsets of $\mathbb{B}_n$. We need to show $\|J_b f_k\|_{H^p} \to 0$. For any $h \in H^{p/(p-2)}$, by Hölder's inequality and the fact that μ_b is a (vanishing) Carleson measure we have that

$$\begin{aligned}
&\int_{\mathbb{B}_n} |h(z)| d\mu_{f_k,b}(z) \\
&\quad = \left(\int_{r\mathbb{B}_n} + \int_{(r\mathbb{B}_n)^c} \right) |h(z)||f_k(z)|^2 |Rb(z)|^2 dv_1(z) \\
&\quad \lesssim \|h\|_{H^{\frac{p}{p-2}}} \cdot \left(\int_{r\mathbb{B}_n} |f_k(z)|^p dv_\alpha(z) \right)^{\frac{2}{p}} + \|h\|_{H^{\frac{p}{p-2}}} \cdot \|\chi_{(r\mathbb{B}_n)^c}\mu_b\|^{\frac{p-2}{p}}_{CM}.
\end{aligned}$$

Since μ_b is a vanishing Carleson measure, there exists r_0 $(0 < r_0 < 1)$ such that

$$\|\chi_{(r_0\mathbb{B}_n)^c}\mu_b\|^{\frac{p-2}{p}}_{CM} < \epsilon$$

by Lemma 3.1. Fix this r_0, we have $\left(\int_{r_0\mathbb{B}_n} |f_k(z)|^p dv_\alpha(z) \right)^{2/p} < \epsilon$ when k is large enough since $f_k \to 0$ uniformly on $r_0\mathbb{B}_n$. Thus

$$\int_{\mathbb{B}_n} |h(z)| d\mu_{f_k,b}(z) \lesssim \epsilon \|h\|_{H^{\frac{p}{p-2}}} \tag{3.4}$$

when k is large enough. Denote by $I_{d_k} : H^{p/(p-2)} \to L^1(d\mu_{f_k,b})$ the embedding operator, and then we get $\lim_{k\to\infty} \|I_{d_k}\|_{H^{p/(p-2)} \to L^1(d\mu_{f_k,b})} = 0$ by (3.4). Therefore, by Theorem C, we have

$$\int_{\mathbb{S}_n} \widetilde{\mu_{f_k,b}}(\xi)^{p/2} d\sigma(\xi) \asymp \|I_{d_k}\|^{p/2}_{H^{p/(p-2)} \to L^1(d\mu_{f_k,b})} \to 0.$$

Thus

$$\|J_b f_k\|^p_{H^p} \asymp \int_{\mathbb{S}_n} \widetilde{\mu_{f_k,b}}(\xi)^{p/2} d\sigma(\xi) \to 0$$

by (3.3) and the proof of the sufficiency is complete.

Next we consider the necessity part. Assume that $J_b : A^p_\alpha \to H^p$ is compact. Then for any $f \in A^p_\alpha$, by (3.3), we have

$$\int_{\mathbb{S}_n} \widetilde{\mu_{f,b}}(\xi)^{p/2} d\sigma(\xi) \asymp \|J_b f\|^p_{H^p} < \infty.$$

We see that the measure $\mu_{f,b}$ satisfies the conditions of Theorem C for parameters 1 and $\frac{p}{p-2}$, and this implies $I_{d_f} : H^{p/(p-2)} \to L^1(d\mu_{f,b})$ is compact. Moreover, for any $h \in H^{p/(p-2)}$,

$$\int_{\mathbb{B}_n} |h(z)| d\mu_{f,b}(z) \lesssim \|J_b f\|^2_{H^p} \cdot \|h\|_{H^{\frac{p}{p-2}}}.$$

Let $B_{A^p_\alpha} = \{f \in A^p_\alpha : \|f\|_{A^p_\alpha} \le 1\}$ be the closed unit ball of A^p_α, then $J_b(B_{A^p_\alpha})$ is relatively compact in H^p. We can find $f_1, \cdots, f_m \in B_{A^p_\alpha}$, such that for any $f \in B_{A^p_\alpha}$, there are some $j \in \{1, 2, \cdots, m\}$ with

$$\|J_b f - J_b f_j\|_{H^p} < \epsilon.$$

Suppose $\{h_k\} \subset H^{p/(p-2)}$ is a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{B}_n$. Since $I_{d_f} : H^{p/(p-2)} \to L^1(d\mu_{f,b})$ is compact for any $f \in A^p_\alpha$, then for every $j \in \{1, \cdots, m\}$, there is $K_j > 0$ such that $k > K_j$ implies

$$\int_{\mathbb{B}_n} |h_k(z)| d\mu_{f_j,b}(z) < \epsilon^2.$$

Define $d\nu_{h_k,b}(z) = |h_k(z)||Rb(z)|^2 dv_1(z)$. We now suppose $f \in B_{A^p_\alpha}$ and $k > K := \max\{K_1, \cdots, K_m\}$, then there is $j \in \{1, \cdots, m\}$ such that

$$\begin{aligned}
&\left(\int_{\mathbb{B}_n} |f(z)|^2 d\nu_{h_k,b}(z)\right)^{1/2} \\
&\quad\le \left(\int_{\mathbb{B}_n} |f(z) - f_j(z)|^2 d\nu_{h_k,b}(z)\right)^{1/2} + \left(\int_{\mathbb{B}_n} |f_j(z)|^2 d\nu_{h_k,b}(z)\right)^{1/2} \\
&\quad= \left(\int_{\mathbb{B}_n} |h_k(z)| d\mu_{f-f_j,b}(z)\right)^{1/2} + \left(\int_{\mathbb{B}_n} |h_k(z)| d\mu_{f_j,b}(z)\right)^{1/2} \\
&\quad\lesssim \|J_b(f - f_j)\|_{H^p} \cdot \|h_k\|^{1/2}_{H^{p/(p-2)}} + \left(\int_{\mathbb{B}_n} |h_k(z)| d\mu_{f_j,b}(z)\right)^{1/2} \\
&\quad\lesssim \epsilon.
\end{aligned}$$

This implies that the embeddings $I_{d_k} : A^p_\alpha \to L^2(d\nu_{h_k,b})$ are bounded and

$$\lim_{k\to\infty} \|I_{d_k}\|_{A^p_\alpha \to L^2(d\nu_{h_k,b})} = 0.$$

Therefore, since $p > 2$, by [19, Theorem 54], we have for any $\delta > 0$ and $k \ge 1$,

$$\hat{\nu}_{h_k,b}(z) := \frac{\nu_{h_k,b}(D(z,\delta))}{(1 - |z|^2)^{n+1+\alpha}} \in L^{p/(p-2)}(\mathbb{B}_n, dv_\alpha)$$

and

$$\|\hat{\nu}_{h_k,b}\|_{L^{p/(p-2)}(dv_\alpha)} \lesssim \|I_{d_k}\|^2_{A^p_\alpha \to L^2(d\nu_{h_k,b})}. \tag{3.5}$$

By subharmonic property, we have

$$\nu_{h_k,b}(D(z,\delta)) \gtrsim |h_k(z)||Rb(z)|^2 (1 - |z|^2)^{n+2}. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\int_{\mathbb{B}_n} |h_k(z)|^{\frac{p}{p-2}} |Rb(z)|^{\frac{2p}{p-2}} (1-|z|^2)^{\frac{p-2\alpha}{p-2}} dv(z) \lesssim \|I_{d_k}\|^{\frac{2p}{p-2}}_{A^p_\alpha \to L^2(d\nu_{h_k,b})}.$$

Therefore, we have

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |h_k(z)|^{\frac{p}{p-2}} d\mu_b(z) = 0.$$

This implies that $I_d : H^{p/(p-2)} \to L^{p/(p-2)}(d\mu_b)$ is compact. Consequently, μ_b is a vanishing Carleson measure.

(4) Let

$$V_{b,r}(\xi) = \sup_{z\in\Gamma(\xi)\cap\{|z|\geq r\}} |Rb(z)|(1-|z|^2)^{\frac{p-1-\alpha}{p}}.$$

Suppose that $J_b : A^p_\alpha \to H^q$ is compact, and then it follows from [13, Theorem 1] that $V_{b,0} \in L^{pq/(p-q)}(\mathbb{S}_n)$. We want to show that

$$\lim_{r\to 1^-} \|V_{b,r}\|_{L^{pq/(p-q)}(\mathbb{S}_n)} = 0. \tag{3.7}$$

If $p-1-\alpha < 0$, then $V_{b,0} \in L^{pq/(p-q)}(\mathbb{S}_n)$ implies b is constant. So (3.7) is obvious. We now assume that $p-1-\alpha \geq 0$. By Lemma 3.2, we only need to prove that

$$\xi \mapsto \sup_{a_k\in\Gamma(\xi)\cap\{|z|\geq r\}} |Rb(a_k)|^q (1-|a_k|^2)^{\frac{q}{p}(p-1-\alpha)}$$

converges to zero in $L^{p/(p-q)}(\mathbb{S}_n)$ as $r \to 1^-$, where $Z = \{a_k\}$ is a δ-lattice with δ small enough.

Let $B_{T^p_p(Z)} = \{c \in T^p_p(Z) : \|c\|_{T^p_p(Z)} \leq 1\}$ be the closed unit ball of $T^p_p(Z)$. For any $c = \{c_k\} \in B_{T^p_p(Z)}$, define

$$S(c)(z) = \sum_k (1-|a_k|^2)^{n/p} c_k g_k(z), \quad z \in \mathbb{B}_n,$$

where

$$g_k(z) = \frac{(1-|a_k|^2)^{b-(n+1+\alpha)/p}}{(1-\langle z, a_k\rangle)^b}$$

with $b > n\max\{1, 1/p\} + (\alpha+1)/p$. Then we have $S(c) \in A^p_\alpha$ with $\|S(c)\|_{A^p_\alpha} \lesssim 1$ for any $c \in B_{T^p_p(Z)}$ by [20, Theorem 2.30].

Since $J_b : A^p_\alpha \to H^q$ is compact, the set $J_b \circ S(B_{T^p_p(Z)})$ is relatively compact in H^q. We can choose $0 < r_0 < 1$ such that

$$\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\cap\{|z|\geq r_0\}} |Rb(z)|^2 |S(c)(z)|^2 (1-|z|^2)^{1-n} dv(z) \right)^{q/2} d\sigma(\xi) \lesssim \epsilon^q$$

for any $c \in B_{T^p_p(Z)}$. That is,

$$\int_{\mathbb{S}_n} \left(\int_{\Gamma(\xi)\cap\{|z|\geq r_0\}} \left| Rb(z) \sum_k (1-|a_k|^2)^{n/p} c_k g_k(z) \right|^2 dv_{1-n}(z) \right)^{q/2} d\sigma(\xi) \lesssim \epsilon^q.$$

By a same process as in the proof of [13, Theorem 7], we can establish

$$\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)\cap\{|z|\geq r_0'\}}|c_k|^2|Rb(a_k)|^2(1-|a_k|^2)^{2-2(1+\alpha)/p}\right)^{q/2}d\sigma(\xi)\lesssim\epsilon^q,$$

where $r_0'=\inf\{|a_k|:D(a_k,\delta)\subset\{|z|\geq r_0\}\}$. Thus, for any $c\in T_p^p(Z)$, we have

$$\begin{aligned}\int_{\mathbb{S}_n}\left(\sum_{a_k\in\Gamma(\xi)\cap\{|z|\geq r_0'\}}|c_k|^2|Rb(a_k)|^2(1-|a_k|^2)^{2-2(1+\alpha)/p}\right)^{q/2}d\sigma(\xi)\\ \lesssim\epsilon^q\|c\|^q_{T_p^p(Z)}.\end{aligned}\tag{3.8}$$

Using the dual and factorization of sequence tent spaces (see the proof of Theorem 7 and Theorem 8 in [13]), by (3.8), we get

$$\xi\mapsto\sup_{a_k\in\Gamma(\xi)\cap\{|z|\geq r\}}|Rb(a_k)|^q(1-|a_k|^2)^{\frac{q}{p}(p-1-\alpha)}$$

converges to zero in $L^{p/(p-q)}(\mathbb{S}_n)$ as $r\to1^-$ and then the necessity part holds. The proof of Theorem 1.1 is now finished. □

As mentioned in Introduction, here we give an example indicating that the integration operator $J_b:A^p_\alpha\to H^q$ may be not compact for some polynomials b and some parameters p,q and α. For simplicity, we consider the case $n=1$. Let $b(z)=z$ and

$$e_k(z)=\sqrt{\frac{(k+1)(k+2)}{2}}z^k.$$

Then $\|e_k\|_{A^2_1}=1$ and $e_k\to0$ weakly in A^2_1. By an elementary calculation, we get

$$J_be_k(z)=\int_0^z e_k(\zeta)b'(\zeta)d\zeta=\sqrt{\frac{k+2}{2(k+1)}}z^{k+1}.$$

Thus

$$\|J_be_k\|_{H^q}=\sup_{0<r<1}\left(\int_{\mathbb{S}_1}|J_be_k(r\xi)|^qd\sigma(\xi)\right)^{1/q}=\sqrt{\frac{k+2}{2(k+1)}}\to\frac{1}{\sqrt{2}}$$

as $k\to\infty$. Consequently, $J_b:A^2_1\to H^q$ is not compact for any $0<q<\infty$. However, if $0<q<2$, it is easy to see that $b(z)=z$ satisfies the integrable condition in [13, Theorem 1(4)]. That is, $J_b:A^2_1\to H^q$ is bounded if $0<q<2$. Therefore, the boundedness and compactness of J_b are not equivalent for $0<q<p\leq2$. This is in sharp contrast with the classical case where the boundedness and the compactness of $J_b:H^p\to H^q$ (or $J_b:A^p_\alpha\to A^q_\beta$) are always equivalent for any $0<q<p<\infty$.

4. Essential norms

In order to estimate the essential norm of $J_b : A^p_\alpha \to H^q$, we need some auxiliary results.

For $\gamma \geq 0$, let $\mathcal{B}_\gamma$ be the γ-Bloch space, that is, the space of holomorphic functions $b \in \mathcal{H}(\mathbb{B}_n)$ such that

$$\|b\|_{\mathcal{B}_\gamma} := \sup_{z\in\mathbb{B}_n} |Rb(z)|(1-|z|^2)^\gamma < \infty.$$

It becomes a Banach space provided that we identify functions that differ by a constant. Let $\mathcal{B}_{\gamma,0}$ be the closed subspace of $\mathcal{B}_\gamma$ consisting of functions $b \in \mathcal{H}(\mathbb{B}_n)$ such that

$$\lim_{|z|\to 1^-} |Rb(z)|(1-|z|^2)^\gamma = 0.$$

We have the following distance formula for the space $\mathcal{B}_\gamma$.

Lemma 4.1. *Let $\gamma \geq 0$ and $b \in \mathcal{B}_\gamma$. Then*

$$\operatorname{dist}(b, \mathcal{B}_{\gamma,0}) \asymp \limsup_{|z|\to 1^-} |Rb(z)|(1-|z|^2)^\gamma.$$

Proof. If $\gamma = 0$, by maximum modulus principle, we have

$$\operatorname{dist}(b, \mathcal{B}_{0,0}) = \|b\|_{\mathcal{B}_0} = \sup_{z\in\mathbb{B}_n} |Rb(z)| = \limsup_{|z|\to 1^-} |Rb(z)|.$$

We now assume $\gamma > 0$. The lower estimate can be easily deduced by triangle inequality. We consider the upper estimate. It is clear that $b_r \in \mathcal{B}_{\gamma,0}$ for any $0 < r < 1$. Here, $b_r(z) = b(rz)$. Hence, for any $0 < \delta < 1$,

$$\begin{aligned}\operatorname{dist}(b, \mathcal{B}_{\gamma,0}) &\leq \limsup_{r\to 1^-} \|b - b_r\|_{\mathcal{B}_\gamma} \\ &\leq \limsup_{r\to 1^-} \sup_{|z|\leq\delta} |Rb(z) - Rb(rz)|(1-|z|^2)^\gamma + \\ &\qquad \limsup_{r\to 1^-} \sup_{|z|>\delta} |Rb(z) - Rb(rz)|(1-|z|^2)^\gamma \\ &\leq \sup_{|z|>\delta} |Rb(z)|(1-|z|^2)^\gamma + \limsup_{r\to 1^-} \sup_{|z|>\delta} |Rb(rz)|(1-|z|^2)^\gamma.\end{aligned}$$

Let $\delta \to 1^-$, and we get

$$\operatorname{dist}(b, \mathcal{B}_{\gamma,0}) \lesssim \limsup_{|z|\to 1^-} |Rb(z)|(1-|z|^2)^\gamma,$$

which completes the lemma. □

Given $1 \leq p < \infty$ and $\gamma > -1$, let $\mathcal{CM}^p_\gamma$ be the space of holomorphic functions $b \in \mathcal{H}(\mathbb{B}_n)$ such that

$$d\mu_{b,p,\gamma}(z) = |Rb(z)|^p(1-|z|^2)^\gamma dv(z)$$

is a Carleson measure, and define

$$\|b\|_{\mathcal{CM}^p_\gamma} = \|\mu_{b,p,\gamma}\|^{\frac{1}{p}}_{CM}.$$

We also define the space $\mathcal{CM}^p_{\gamma,0}$ to be the subspace of $\mathcal{CM}^p_\gamma$ consisting of $b \in \mathcal{H}(\mathbb{B}_n)$ such that $\mu_{b,p,\gamma}$ is a vanishing Carleson measure. We have the following distance formulas for the space $\mathcal{CM}^p_\gamma$. Here we denote $Q(0) = \mathbb{B}_n$ and for $a \in \mathbb{B}_n \backslash \{0\}$,

$$Q(a) = \{z \in \mathbb{B}_n : |1 - \langle z, \frac{a}{|a|}\rangle| < 1 - |a|^2\}.$$

Lemma 4.2. *Let $1 \le p < \infty$, $\gamma > -1$ and $b \in \mathcal{CM}^p_\gamma$. Then*

$$\begin{aligned}\operatorname{dist}(b, \mathcal{CM}^p_{\gamma,0}) &\asymp \limsup_{|a|\to 1^-}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} |Rb(z)|^p (1-|z|^2)^\gamma dv(z)\right)^{\frac{1}{p}} \\ &\asymp \limsup_{|a|\to 1^-}\left(\frac{1}{(1-|a|^2)^n}\int_{Q(a)} |Rb(z)|^p (1-|z|^2)^\gamma dv(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Proof. The lower estimate follows from triangle inequality. We deduce the upper estimate. For any $0 < r < 1$, it is easy to see $b_r \in \mathcal{CM}^p_{\gamma,0}$. Moreover, for any $0 < \delta < 1$,

$$\begin{aligned}&\sup_{|a|\le\delta}\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_{b-b_r,p,\gamma}(z) \\ &\quad\le \frac{1}{(1-\delta)^{2n}}\int_{\mathbb{B}_n} |Rb(z) - Rb(rz)|^p (1-|z|^2)^\gamma dv(z) \to 0\end{aligned}$$

as $r \to 1^-$. Thus we have

$$\begin{aligned}&\operatorname{dist}(b, \mathcal{CM}^p_{\gamma,0}) \le \limsup_{r\to 1^-} \|b - b_r\|^{\frac{1}{p}}_{\mathcal{CM}^p_\gamma} \\ &\lesssim \limsup_{r\to1^-}\sup_{|a|>\delta}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} |Rb(z) - Rb(rz)|^p (1-|z|^2)^\gamma dv(z)\right)^{\frac{1}{p}} \\ &\le \sup_{|a|>\delta}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_{b,p,\gamma}(z)\right)^{\frac{1}{p}} + \\ &\qquad \limsup_{r\to1^-}\sup_{|a|>\delta}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_{b_r,p,\gamma}(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Let $\delta \to 1^-$, and we get

$$\operatorname{dist}(b, \mathcal{CM}^p_{\gamma,0}) \lesssim \limsup_{|a|\to 1^-}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_{b,p,\gamma}(z)\right)^{\frac{1}{p}},$$

which establishes the estimate

$$\operatorname{dist}(b, \mathcal{CM}^p_{\gamma,0}) \asymp \limsup_{|a|\to 1^-}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} d\mu_{b,p,\gamma}(z)\right)^{\frac{1}{p}}.$$

Noting that a positive Borel measure μ is a Carleson measure if and only if $\mu(Q(a)) \le C(1-|a|^2)^n$ for all $a \in \mathbb{B}_n$ and some absolute constant $C > 0$, and

$$\|\mu\|_{CM} = \sup_{a\in\mathbb{B}_n} \frac{\mu(Q(a))}{(1-|a|^2)^n},$$

the equivalent relation

$$\operatorname{dist}(b, \mathcal{CM}^p_{\gamma,0}) \asymp \limsup_{|a|\to 1^-} \left(\frac{\mu_{b,p,\gamma}(Q(a))}{(1-|a|^2)^n} \right)^{\frac{1}{p}}$$

can be proven by similar methods. □

Remark 4.3. The spaces $\mathcal{CM}^p_\gamma$ are closely related to the so-called Hardy-Carleson type spaces $\mathcal{CT}_{q,\alpha}$ studied in [17]. In fact, we have $b \in \mathcal{CM}^p_\gamma$ if and only if $Rb \in \mathcal{CT}_{p,\gamma-n}$, and this relation also holds for the little versions of these spaces. Therefore, the distance formulas similar to Lemma 4.2 can be obtained for $\mathcal{CT}_{q,\alpha}$. Let $1 \le q < \infty$, $\alpha > -n-1$ and $b \in \mathcal{CT}_{q,\alpha}$. Then

$$\begin{aligned}\operatorname{dist}(b, \mathcal{CT}^0_{q,\alpha}) &\asymp \limsup_{|a|\to 1^-} \left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}} |b(z)|^q (1-|z|^2)^{\alpha+n} dv(z) \right)^{\frac{1}{q}} \\ &\asymp \limsup_{|a|\to 1^-} \left(\frac{1}{(1-|a|^2)^n} \int_{Q(a)} |b(z)|^q (1-|z|^2)^{\alpha+n} dv(z) \right)^{\frac{1}{q}}.\end{aligned}$$

In the rest part of this section, we agree that

$$f_a(z) = \frac{(1-|a|^2)^{s-(n+1+\alpha)/p}}{(1-\langle z,a\rangle)^s}$$

for sufficiently large s and $a \in \mathbb{B}_n$. It is obvious that $\|f_a\|_{A^p_\alpha} \asymp 1$ and $f_a \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $|a| \to 1^-$. For any $a \in \mathbb{B}_n$, let

$$S(a) = \{\zeta \in \mathbb{S}_n : |1-\langle \zeta, a\rangle| < (1-|a|^2)^{1/3}\}.$$

Lemma 4.4. *Let $0 < p \le q < \infty$, $\alpha > -1$ and $b \in \mathcal{H}(\mathbb{B}_n)$ such that $J_b : A^p_\alpha \to H^q$ is bounded. Then*

$$\lim_{|a|\to 1} \int_{\mathbb{S}_n \setminus S(a)} |J_b f_a|^q d\sigma = 0.$$

Proof. Let $\sqrt{63}/8 < |a| < 1$. We claim that

$$|1-\langle r\zeta, a\rangle| > \frac{1}{4}(1-|a|^2)^{2/3}$$

for any $\zeta \in \mathbb{S}_n \setminus S(a)$ and $0 \le r \le 1$. In fact, if $0 \le r \le 1-(1-|a|^2)^{2/3}$, then

$$|1-\langle r\zeta, a\rangle| \ge 1 - r|\langle \zeta, a\rangle| \ge (1-|a|^2)^{2/3} > \frac{1}{4}(1-|a|^2)^{2/3}.$$

If $1-(1-|a|^2)^{2/3} < r \le 1$, then

$$\begin{aligned}|1-\langle r\zeta, a\rangle| &\ge r|1-\langle \zeta, a\rangle| - (1-r) \\ &\ge r(1-|a|^2)^{1/3} - (1-|a|^2)^{2/3} \\ &> (1-|a|^2)^{1/3}\left(\frac{1}{2} - (1-|a|^2)^{1/3}\right) \\ &> \frac{1}{4}(1-|a|^2)^{2/3}.\end{aligned}$$

Consequently,

$$|f_a(\zeta)| = \frac{(1-|a|^2)^{s-(n+1+\alpha)/p}}{|1-\langle\zeta,a\rangle|^s} \leq 4^s(1-|a|^2)^{\frac{s}{3}-\frac{n+1+\alpha}{p}}$$

and

$$|Rf_a(r\zeta)| = \frac{s|\langle r\zeta,a\rangle|(1-|a|^2)^{s-(n+1+\alpha)/p}}{|1-\langle r\zeta,a\rangle|^{s+1}} \leq 4^s sr(1-|a|^2)^{\frac{s-2}{3}-\frac{n+1+\alpha}{p}}$$

for any $\zeta \in \mathbb{S}_n \setminus S(a)$ and $0 \leq r \leq 1$. Therefore, for almost every $\zeta \in \mathbb{S}_n \setminus S(a)$, integration by parts yields

$$\begin{aligned}|J_bf_a(\zeta)|^q &\lesssim |f_a(\zeta)b(\zeta)|^q + \left(\int_0^1 |Rf_a(t\zeta)||b(t\zeta)|\frac{dt}{t}\right)^q\\ &\lesssim (1-|a|^2)^{\frac{q(s-2)}{3}-\frac{(n+1+\alpha)q}{p}}\left(|b(\zeta)|^q + \left(\int_0^1 |b(t\zeta)|dt\right)^q\right).\end{aligned}$$

Since $J_b : A^p_\alpha \to H^q$ is bounded, we have

$$M := \sup_{z\in\mathbb{B}_n} |Rb(z)|(1-|z|^2)^{\frac{n}{q}+1-\frac{n+1+\alpha}{p}} < \infty,$$

which implies that b belongs to the Bloch space since $0 < p \leq q < \infty$ and $\alpha > -1$, and subsequently $|b(z)| \lesssim M\log\frac{1}{1-|z|}$ for $z \in \mathbb{B}_n$. Moreover, $b = b(0) + J_b1 \in H^q$. Then, noting that s is large enough, we establish

$$\begin{aligned}\int_{\mathbb{S}_n\setminus S(a)} |J_bf_a|^q d\sigma &\lesssim (1-|a|^2)^{\frac{q(s-2)}{3}-\frac{(n+1+\alpha)q}{p}}\left(\int_{\mathbb{S}_n}|b|^q d\sigma + M^q\left(\int_0^1 \log\frac{1}{1-t}dt\right)^q\right)\\ &= (1-|a|^2)^{\frac{q(s-2)}{3}-\frac{(n+1+\alpha)q}{p}}(\|b\|^q_{H^q} + M^q) \to 0\end{aligned}$$

as $|a| \to 1$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) When $n/q + 1 - (n+1+\alpha)/p < 0$, the boundedness of $J_b : A^p_\alpha \to H^q$ implies that b is a constant, and then there is nothing to prove. Suppose now $\gamma = n/q + 1 - (n+1+\alpha)/p \geq 0$.

The upper estimate can be deduced easily by Theorem 1.1, [13, Theorem 4] and Lemma 4.1. In fact, for any $g \in \mathcal{B}_{\gamma,0}$, by Theorem 1.1, $J_g : A^p_\alpha \to H^q$ is compact. Therefore, by the norm estimate for J_b in [13, Theorem 4], we have

$$\|J_b\|_e \leq \|J_b - J_g\| \asymp \|b-g\|_{\mathcal{B}_\gamma}.$$

Since $g \in \mathcal{B}_{\gamma,0}$ is arbitrary, the upper estimate follows from Lemma 4.1.

We now take care of the lower estimate. Suppose $K : A^p_\alpha \to H^q$ is a compact operator. It is easy to see $\int_{S(a)} |Kf_a|^q d\sigma \to 0$ as $|a| \to 1$. Hence we have

$$\begin{aligned}\|J_b - K\| &\gtrsim \limsup_{|a|\to1^-} \|(J_b-K)f_a\|_{H^q}\\ &\geq \limsup_{|a|\to1^-}\left(\min\{1,2^{1-\frac{1}{q}}\}\left(\int_{S(a)}|J_bf_a|^q d\sigma\right)^{1/q} - \left(\int_{S(a)}|Kf_a|d\sigma\right)^{1/q}\right)\end{aligned}$$

$$\asymp \limsup_{|a|\to 1^-}\left(\int_{S(a)} |J_b f_a|^q d\sigma\right)^{1/q}$$
$$= \limsup_{|a|\to 1^-} \|J_b f_a\|_{H^q},$$

where the last equality follows from Lemma 4.4. Since $K : A^p_\alpha \to H^q$ is an arbitrary compact operator, we obtain

$$\|J_b\|_e \gtrsim \limsup_{|a|\to 1^-} \|J_b f_a\|_{H^q}. \tag{4.1}$$

On the other hand, by the pointwise estimate for derivatives of Hardy space functions, we have

$$\frac{|Rb(a)|}{(1-|a|^2)^{(n+1+\alpha)/p}} = |R(J_b f_a)(a)| \lesssim \frac{\|J_b f_a\|_{H^q}}{(1-|a|^2)^{1+n/q}}. \tag{4.2}$$

Combining (4.1) and (4.2) gives the desired lower estimate.

(2) The boundedness of $J_b : A^p_\alpha \to H^p$ implies that b is a constant if $\frac{p-2\alpha}{p-2} \le -1$. Suppose now $\frac{p-2\alpha}{p-2} > -1$. The upper estimate for $\|J_b\|_e$ can be obtained by Theorem 1.1, Lemma 4.2 and the norm estimate for J_b in [13, Theorem 5], which completes the proof. □

5. Schatten(-Herz) classes and decay of singular values

5.1. Schatten class. Recall that A^2_α is a reproducing kernel Hilbert space with the reproducing kernel function given by

$$K_z(w) = \frac{1}{(1-\langle w, z\rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n$$

with norm $\|K_z\|_{A^2_\alpha} = K_z(z)^{1/2} = (1-|z|^2)^{-(n+1+\alpha)/2}$. The normalized kernel functions are denoted by $k_z = \frac{K_z}{\|K_z\|_{A^2_\alpha}}$. We also need to introduce some "fractional derivatives" of the kernel functions. For $z, w \in \mathbb{B}_n$ and $t \ge 0$, define

$$K^t_z(w) = \frac{1}{(1-\langle w, z\rangle)^{n+1+\alpha+t}}$$

and let $k^t_z = \frac{K^t_z}{\|K^t_z\|_{A^2_\alpha}}$.

The following lemma is a generalization of [21, Theorem 6.6], which can be found in [14].

Lemma D. *Let $T : A^2_\alpha(\mathbb{B}_n) \to A^2_\alpha(\mathbb{B}_n)$ be a positive operator and $t \ge 0$. Set*

$$\widetilde{T^{(t)}}(z) = \langle T k^t_z, k^t_z\rangle_{A^2_\alpha}, \quad z \in \mathbb{B}_n.$$

(a) *For $0 < p \le 1$, if $\widetilde{T^{(t)}} \in L^p(\mathbb{B}_n, d\lambda_n)$, then $T \in S_p(A^2_\alpha)$.*

(b) *For $p \ge 1$, if $T \in S_p(A^2_\alpha)$, then $\widetilde{T^{(t)}} \in L^p(\mathbb{B}_n, d\lambda_n)$.*

Proposition 5.1. *Suppose $0 < p < \infty$, $\alpha > -1$ and $J_b \in S_p(A^2_\alpha, H^2)$, then*

$$\int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{\frac{p}{2}(1-\alpha)} d\lambda_n(z) < \infty. \tag{5.1}$$

Proof. Since J_b is in $S_p(A^2_\alpha, H^2)$, the positive operator $J_b^* J_b$ belongs to $S_{p/2}(A^2_\alpha)$.

We first consider the case $p \geq 2$. By Lemma D, the fact $J_b^* J_b \in S_{p/2}(A^2_\alpha)$ implies $\widetilde{J_b^* J_b} := \widetilde{J_b^* J_b}^{(0)}$ is in $L^{p/2}(\mathbb{B}_n, d\lambda_n)$. However, by Theorem A and subharmonic property of $|Rb|^2$, we have

$$\begin{aligned}\widetilde{J_b^* J_b}(z) &= \langle J_b^* J_b k_z, k_z\rangle_{A^2_\alpha} = \|J_b k_z\|^2_{H^2} \\ &\asymp \int_{\mathbb{B}_n} |k_z(w)|^2 |Rb(w)|^2 (1-|w|^2) dv(w) \\ &\gtrsim \frac{1}{(1-|z|^2)^{n+\alpha}} \int_{D(z,\delta)} |Rb(w)|^2 dv(w) \\ &\gtrsim |Rb(z)|^2 (1-|z|^2)^{1-\alpha}\end{aligned} \tag{5.2}$$

for any $z \in \mathbb{B}_n$. Thus, (5.1) follows.

We next consider the case $0 < p < 2$. For any $f, g \in A^2_\alpha$, by Theorem A and Fubini's theorem, we have

$$\begin{aligned}\langle J_b^* J_b f, g\rangle_{A^2_\alpha} &= \langle J_b f, J_b g\rangle_{H^2} \\ &= \frac{2}{n} \int_{\mathbb{B}_n} f(z)\overline{g(z)} |Rb(z)|^2 |z|^{-2n} \log \frac{1}{|z|} dv(z) \\ &= \langle T_{\nu_b} f, g\rangle_{A^2_\alpha},\end{aligned}$$

where $d\nu_b(z) = \frac{2}{n}|Rb(z)|^2|z|^{-2n}\log\frac{1}{|z|}dv(z)$. Therefore, we have $T_{\nu_b} = J_b^* J_b$ is in the Schatten class $S_{p/2}(A^2_\alpha)$. Let $\{a_k\}$ be a δ-lattice and

$$\hat{\nu}_{b,\delta}(z) = \frac{\nu_b(D(z,\delta))}{v_\alpha(D(z,\delta))}, \quad z \in \mathbb{B}_n,$$

then $\{\hat{\nu}_{b,\delta}(a_k)\} \in l^{p/2}$, see [22, Theorem 3]. It is well-known that $v_\alpha(D(z,\delta)) \asymp (1-|z|^2)^{n+1+\alpha}$. Consequently, by Hölder's inequality, we have

$$\begin{aligned}\infty &> \sum_k \Big(\hat{\nu}_{b,\delta}(a_k)\Big)^{\frac{p}{2}} \\ &\asymp \sum_k \left(\frac{1}{(1-|a_k|^2)^{n+1+\alpha}} \int_{D(a_k,\delta)} |Rb(z)|^2 |z|^{-2n} \log\frac{1}{|z|} dv(z)\right)^{\frac{p}{2}} \\ &\asymp \sum_k \left(\int_{D(a_k,\delta)} |Rb(z)|^2 |z|^{-2n} (1-|z|^2)^{-\alpha} \log\frac{1}{|z|} d\lambda_n(z)\right)^{\frac{p}{2}} \\ &\gtrsim \sum_k \int_{D(a_k,\delta)} |Rb(z)|^p |z|^{-np} (1-|z|^2)^{-\frac{\alpha}{2}p} \left(\log\frac{1}{|z|}\right)^{\frac{p}{2}} d\lambda_n(z) \\ &\gtrsim \int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{\frac{p}{2}(1-\alpha)} d\lambda_n(z)\end{aligned}$$

since $p/2 < 1$. Thus the proof is complete. ∎

Proposition 5.2. *Suppose $\alpha > -1$, $p \geq 2$ and*

$$\int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{\frac{p}{2}(1-\alpha)} d\lambda_n(z) < \infty,$$

then $J_b \in S_p(A^2_\alpha, H^2)$.

Proof. If $p = \infty$, that is, $\sup_{z\in\mathbb{B}_n} |Rb(z)|(1-|z|^2)^{(1-\alpha)/2} < \infty$, then $J_b : A^2_\alpha \to H^2$ is bounded and we have

$$\|J_b\|_{A^2_\alpha \to H^2} \lesssim \sup_{z\in\mathbb{B}_n} |Rb(z)|(1-|z|^2)^{(1-\alpha)/2}.$$

See [13, Theorem 1].

If $p = 2$, by (5.2), Fubini's theorem and [20, Theorem 1.12], for the positive operator $J_b^* J_b$, we have

$$\begin{aligned}
\|J_b^* J_b\|_{S_1} = tr(J_b^* J_b) &\asymp \int_{\mathbb{B}_n} \widetilde{J_b^* J_b}(z) d\lambda_n(z) \\
&\asymp \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} |Rb(w)|^2 \frac{(1-|z|^2)^{n+1+\alpha}(1-|w|^2)}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} dv(w) d\lambda_n(z) \\
&= \int_{\mathbb{B}_n} |Rb(w)|^2 (1-|w|^2) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} dv(z) dv(w) \\
&\lesssim \int_{\mathbb{B}_n} |Rb(w)|^2 (1-|w|^2)^{-n-\alpha} dv(w) \\
&= \int_{\mathbb{B}_n} |Rb(z)|^2 (1-|z|^2)^{1-\alpha} d\lambda_n(z) < \infty.
\end{aligned}$$

Thus, by complex interpolation, for any $2 \leq p < \infty$, $Rb(z)(1-|z|^2)^{(1-\alpha)/2}$ belongs to $L^p(\mathbb{B}_n, d\lambda_n)$ implies $J_b^* J_b \in S_{p/2}(A^2_\alpha)$, which is, $J_b \in S_p(A^2_\alpha, H^2)$. The proof is complete. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. If $p(1-\alpha)/2 \leq n$, then it is sufficient to prove the necessity part. In fact, due to Proposition 5.1, we have the integrable condition (5.1), which implies that b is constant and gives the desired necessity.

If $n \geq 2$ and $p(1-\alpha)/2 > n$, then $p > \frac{4}{1-\alpha} > 2$ and the sufficiency part follows from Proposition 5.2.

We now consider the sufficiency part in the case $n = 1$. We only need to prove

$$\int_{\mathbb{B}_1} |Rb(z)|^p (1-|z|^2)^{p(1-\alpha)/2} d\lambda_1(z) < \infty$$

implies $J_b^* J_b \in S_{p/2}(A^2_\alpha(\mathbb{B}_1))$ when $p(1-\alpha)/2 > 1$ and $p < 2$ by Proposition 5.2. Choosing $t > 0$ large enough, we only need to prove $\widetilde{(J_b^* J_b)}^{(t)} \in L^{p/2}(\mathbb{B}_1, d\lambda_1)$ by Lemma D. By Theorem A we have

$$\begin{aligned}
\widetilde{(J_b^* J_b)}^{(t)}(z) &= \langle J_b^* J_b k_z^t, k_z^t\rangle_{A^2_\alpha} = \|J_b k_z^t\|^2_{H^2} \\
&\asymp \int_{\mathbb{B}_n} |k_z^t(w)|^2 |Rb(w)|^2 (1-|w|^2) dv(w)
\end{aligned}$$

$$\asymp \int_{\mathbb{B}_n} \frac{|Rb(w)|^2(1-|z|^2)^{n+1+\alpha+2t}(1-|w|^2)}{|1-\langle z,w\rangle|^{2(n+1+\alpha+t)}} dv(w).$$

Thus we only need to show

$$I := \int_{\mathbb{B}_1} \left(\int_{\mathbb{B}_1} \frac{|Rb(w)|^2(1-|w|^2)}{|1-\langle z,w\rangle|^{2(2+\alpha+t)}} dv(w) \right)^{p/2} (1-|z|^2)^{\frac{p}{2}(2+\alpha+2t)-2} dv(z) < \infty.$$

Suppose that $\{a_j\}$ is a δ-lattice of the unit disk. Then by subharmonic property and Hölder's inequality we have

$$\begin{aligned}
&\left(\int_{\mathbb{B}_1} \frac{|Rb(w)|^2(1-|w|^2)}{|1-\langle z,w\rangle|^{2(2+\alpha+t)}} dv(w) \right)^{p/2} \\
&\lesssim \left(\sum_j \frac{1-|a_j|^2}{|1-\langle z,a_j\rangle|^{2(2+\alpha+t)}} \int_{D(a_j,\delta)} \left(\frac{1}{(1-|w|^2)^2} \int_{D(w,\delta)} |Rb(u)|^p dv(u) \right)^{\frac{2}{p}} dv(w) \right)^{\frac{p}{2}} \\
&\lesssim \left(\sum_j \frac{(1-|a_j|^2)^{3-4/p}}{|1-\langle z,a_j\rangle|^{2(2+\alpha+t)}} \left(\int_{D(a_j,2\delta)} |Rb(u)|^p dv(u) \right)^{2/p} \right)^{p/2} \\
&\le \sum_j \frac{(1-|a_j|^2)^{3p/2-2}}{|1-\langle z,a_j\rangle|^{p(2+\alpha+t)}} \int_{D(a_j,2\delta)} |Rb(u)|^p dv(u) \\
&\lesssim \int_{\mathbb{B}_1} \frac{|Rb(u)|^p(1-|u|^2)^{3p/2-2}}{|1-\langle z,u\rangle|^{p(2+\alpha+t)}} dv(u).
\end{aligned}$$

Since $t > 0$ is large enough, by Fubini's theorem and [20, Theorem 1.12] we establish

$$\begin{aligned}
I &\lesssim \int_{\mathbb{B}_1} |Rb(u)|^p(1-|u|^2)^{3p/2-2} \int_{\mathbb{B}_1} \frac{(1-|z|^2)^{\frac{p}{2}(2+\alpha+2t)-2}}{|1-\langle z,u\rangle|^{p(2+\alpha+t)}} dv(z) dv(u) \\
&\lesssim \int_{\mathbb{B}_1} |Rb(u)|^p (1-|u|^2)^{p(1-\alpha)/2} d\lambda_1(u) < \infty.
\end{aligned}$$

The necessary part for $p(1-\alpha)/2 > n$ is a direct consequence of Proposition 5.1, which completes the proof. □

5.2. **Schatten-Herz class.** In order to characterize the membership in Schatten-Herz classes of $J_b : A^2_\alpha \to H^2$, we use the following inner product of the Hardy space H^2:

$$\langle f, g\rangle_{*,H^2} = f(0)\overline{g(0)} + \int_{\mathbb{B}_n} Rf(z)\overline{Rg(z)}(1-|z|^2) dv(z),$$

and let J_b^* denote the Hilbert adjoint of $J_b : A^2_\alpha \to H^2$ with respect to the standard inner product of A^2_α and the inner product $\langle \cdot, \cdot \rangle_{*,H^2}$ of H^2. It is easy to see

$$J_b^* J_b = T_{\nu_b}, \tag{5.3}$$

where $d\nu_b(z) = |Rb(z)|^2(1-|z|^2)dv(z)$.

Proof of Theorem 1.4. Fix $\delta > 0$ small enough. By [7, Theorem 4.1] and (5.3), we have that $J_b \in S_{p,q}(A^2_\alpha, H^2)$ if and only if

$$\sum_{k=0}^{\infty}\left(\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} d\lambda_n(z)\right)^{q/p} < \infty.$$

We now show that

$$\sum_{k=0}^{\infty}\left(\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} d\lambda_n(z)\right)^{q/p} \asymp \int_0^1 M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1} dr. \tag{5.4}$$

For simplicity, write $r_k = 1 - \frac{1}{2^k},\ k = 0, 1, 2, \cdots$. For $k \geq 1$, by the subharmonic property of $|Rb|^2$ and the monotonicity of $M_p(Rb, r)$, we obtain

$$\begin{aligned}\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} d\lambda_n(z) &\gtrsim \int_{A_k} |Rb(z)|^p (1-|z|^2)^{\frac{p(1-\alpha)}{2}} d\lambda_n(z) \\ &\gtrsim 2^{k(n+1-\frac{p(1-\alpha)}{2}-1)} M_p^p(Rb, r_k).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} d\lambda_n(z)\right)^{q/p} &\gtrsim \sum_{k=1}^{\infty} 2^{\frac{qk}{p}(n-\frac{p(1-\alpha)}{2})} M_p^q(Rb, r_k) \\ &\gtrsim \sum_{k=1}^{\infty}\int_{r_{k-1}}^{r_k} M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1} dr \\ &= \int_0^1 M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1} dr.\end{aligned}$$

To prove the converse inequality, note that

$$\left(\int_{D(z,\delta)} |Rb(w)|^2 dv(w)\right)^{p/2} \lesssim (1-|z|^2)^{(n+1)(\frac{p}{2}-1)} \int_{D(z,2\delta)} |Rb(w)|^p dv(w)$$

by subharmonic property and

$$\tilde{A}_k = \bigcup_{z\in A_k} D(z, 2\delta) \subset A_{k-1} \cup A_k \cup A_{k+1}$$

for any $k \geq 0$ since $\delta > 0$ is small enough. Here $A_{-1} = \emptyset$. Then by Fubini's theorem we have

$$\begin{aligned}\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} dv(z) &\lesssim \int_{A_k} (1-|z|^2)^{-p(n+\alpha)/2} \left(\int_{D(z,\delta)} |Rb(w)|^2 dv(w)\right)^{p/2} dv(z) \\ &\lesssim \int_{A_k} (1-|z|^2)^{\frac{p}{2}(1-\alpha)-n-1} \int_{D(z,2\delta)} |Rb(w)|^p dv(w) dv(z) \\ &\lesssim \int_{\tilde{A}_k} (1-|w|^2)^{p(1-\alpha)/2} |Rb(w)|^p dv(w),\end{aligned}$$

which implies

$$\left(\int_{A_k} \hat{\nu}_{b,\delta}(z)^{p/2} d\lambda_n(z)\right)^{q/p}$$

$$\begin{aligned}
&\lesssim (1-r_k)^{-q(n+1)/p}\left(\int_{\tilde{A}_k}(1-|w|^2)^{p(1-\alpha)/2}|Rb(w)|^p dv(w)\right)^{q/p}\\
&\lesssim (1-r_k)^{-q(n+1)/p}\left(\int_{r_{k-1}}^{r_{k+2}}(1-r)^{p(1-\alpha)/2}M_p^p(Rb,r)dr\right)^{q/p}\\
&\lesssim (1-r_k)^{-n\frac{q}{p}+\frac{q}{2}(1-\alpha)}M_p^q(Rb,r_{k+2})
\end{aligned}$$

for $k \geq 1$. It is easy to see

$$\left(\int_{A_0}\hat{\nu}_{b,\delta}(z)^{p/2}d\lambda_n(z)\right)^{q/p} \lesssim M_p^q(Rb,r_3).$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(\int_{A_k}\hat{\nu}_{b,\delta}(z)^{p/2}d\lambda_n(z)\right)^{q/p} &\lesssim \sum_{k=1}^{\infty}(1-r_k)^{-n\frac{q}{p}+\frac{q}{2}(1-\alpha)}M_p^q(Rb,r_{k+2})\\
&\lesssim \sum_{k=1}^{\infty}\int_{r_{k+2}}^{r_{k+3}}M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1}dr\\
&\leq \int_0^1 M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1}dr.
\end{aligned}$$

Hence (5.4) is established and $J_b \in S_{p,q}(A^p_\alpha, H^2)$ if and only if

$$\int_0^1 M_p^q(Rb,r)(1-r)^{\frac{q}{2}(1-\alpha)-n\frac{q}{p}-1}dr < \infty. \tag{5.5}$$

Since (5.5) implies that b is a constant if $p(1-\alpha)/2 \leq n$, the proof is complete. □

5.3. **Decay of singular values.** In the rest part of this section, we consider asymptotic property of singular values of $J_b : A^2_\alpha \to H^2$. Recall that, if H_1 and H_2 are separable Hilbert spaces and $T : H_1 \to H_2$ is a compact operator, then the kth singular value $s_k(T)$ of T is the square root of the kth eigenvalue of the positive operator T^*T if we rearrange the eigenvalues in nonincreasing order.

Let H be a separable Hilbert space and T be a compact operator on H, and let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function such that $h(0) = 0$. We say that $T \in S_h(H)$ if there exists $C > 0$ such that

$$\sum_k h(Cs_k(T)) < \infty.$$

The class $S_h(H)$ is a generalization of Schatten class, which was introduced in [5].

If μ is a positive Borel measure on $\mathbb{B}_n$, we use $\tilde{\mu}$ denoting the Berezin transform of μ, which is defined by

$$\tilde{\mu}(z) = \widetilde{T_\mu}(z) = \langle T_\mu k_z, k_z\rangle_{A^2_\alpha} = \int_{\mathbb{B}_n}|k_z(w)|^2 d\mu(w), \quad z \in \mathbb{B}_n.$$

Let $\{a_j\}$ be a δ-lattice and let $\hat{\mu}_\delta$ denote the average function of μ, that is,

$$\hat{\mu}_\delta(z) = \frac{\mu(D(z,\delta))}{v_\alpha(D(z,\delta))}, \quad z \in \mathbb{B}_n.$$

Then we have the following description of the membership in $S_h(A^2_\alpha)$ of Toeplitz operators, which was proved in [5] in the setting of weighted Bergman spaces A^2_ω on a domain $\Omega \subset \mathbb{C}$ for some more general weights ω.

Theorem 5.3. *Let $\alpha > -1$, μ be a positive Borel measure on $\mathbb{B}_n$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Then the following conditions are equivalent.*
(a) $T_\mu \in S_h(A^2_\alpha)$.
(b) There exists $C_1 > 0$ such that

$$\int_{\mathbb{B}_n} h(C_1\tilde{\mu}(z))d\lambda_n(z) < \infty.$$

(c) There exists $C_2 > 0$ such that

$$\sum_j h(C_2\hat{\mu}_\delta(a_j)) < \infty.$$

Recall that, $J_b^* J_b = T_{\nu_b}$ on A^2_α, where

$$d\nu_b(z) = \frac{2}{n}|Rb(z)|^2|z|^{-2n}\log\frac{1}{|z|}dv(z).$$

Therefore, we get

$$s_k(J_b) = \sqrt{s_k(J_b^* J_b)} = \sqrt{s_k(T_{\nu_b})}.$$

By (5.2), we have

$$\tilde{\nu}_b(z) = \widetilde{T_{\nu_b}}(z) = \widetilde{J_b^* J_b}(z) \asymp \int_{\mathbb{B}_n} \frac{|Rb(w)|^2(1-|z|^2)^{n+1+\alpha}(1-|w|^2)}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}dv(w).$$

It is easy to see

$$\begin{aligned}\hat{\nu}_{b,\delta}(a_j) &\asymp \int_{D(a_j,\delta)} |Rb(z)|^2|z|^{-2n}(1-|z|^2)^{-\alpha}\log\frac{1}{|z|}d\lambda_n(z)\\ &\asymp \int_{D(a_j,\delta)} |Rb(z)|^2(1-|z|^2)^{1-\alpha}d\lambda_n(z)\end{aligned}$$

if j is large. Thus, due to [5, Lemma 6.1] and Theorem 5.3, we have the following result about the asymptotic behavior of the singular values of $J_b : A^2_\alpha \to H^2$.

Theorem 5.4. *Suppose $\alpha > -1$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Let $\eta : [1,\infty) \to (0,\infty)$ be a decreasing convex function such that $\eta(\infty) = 0$ and satisfy*

$$\eta(x\log x) \asymp \eta(x), \quad x \to \infty.$$

Define $h_\eta : \mathbb{R}^+ \to \mathbb{R}^+$ by $h_\eta(\eta^2(x)) = 1/x$ and let $s_k(J_b)$ denote the kth singular value of $J_b : A^2_\alpha \to H^2$. Then the following conditions are equivalent.

(a) $s_k(J_b) = O(\eta(k))$.
(b) There exists $C_1 > 0$ *such that*

$$\int_{\mathbb{B}_n} h_\eta\left(C_1 \int_{\mathbb{B}_n} \frac{|Rb(w)|^2(1-|z|^2)^{n+1+\alpha}(1-|w|^2)}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}} dv(w)\right) d\lambda_n(z) < \infty.$$

(c) There exists $C_2 > 0$ *such that*

$$\sum_j h_\eta\left(C_2 \int_{D(a_j,\delta)} |Rb(z)|^2(1-|z|^2)^{1-\alpha} d\lambda_n(z)\right) < \infty.$$

6. Concluding remarks

6.1. **Schatten-Herz class of Toeplitz type operators on the Hardy space.** For a positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz type operator Q_μ is defined by

$$Q_\mu f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1-\langle z,w\rangle)^n} d\mu(w), \quad z \in \mathbb{B}_n.$$

The boundedness, compactness and membership in Schatten class of Q_μ on Hardy spaces were studied by Pau and Perälä in [16]. In particular, the Schatten class of $Q_\mu : H^2 \to H^2$ can be characterized as follows, which is a direct consequence of [16, Theorem 10].

Theorem E. *Let* $0 < p < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{B}_n$. *Then* $Q_\mu \in S_p(H^2)$ *if and only if* $\hat{\mu}_\delta \in L^p(\mathbb{B}_n, d\lambda_n)$ *for some (or any)* $\delta > 0$. *Moreover,* $\|Q_\mu\|_{S_p} \asymp \|\hat{\mu}_\delta\|_{L^p(\mathbb{B}_n,d\lambda_n)}$. *Here,*

$$\hat{\mu}_\delta(z) = \frac{\mu(D(z,\delta))}{(1-|z|^2)^n}, \quad z \in \mathbb{B}_n.$$

Following [10], for $0 < p,q < \infty$, we say Q_μ is in the Schatten-Herz class $S_{p,q}(H^2)$ if each $Q_{\mu\chi_k}$ is in $S_p(H^2)$ and the sequence $\{\|Q_{\mu\chi_k}\|_{S_p}\}_{k\geq 0}$ is in l^q. Using Theorem E, and the same method as in the proof of [10, Theorem 4.2], we obtain the following result characterizing the membership in $S_{p,q}(H^2)$ of Toeplitz type operator Q_μ.

Theorem 6.1. *Let* $0 < p,q < \infty$ *and* μ *be a positive Borel measure on* $\mathbb{B}_n$. *Then* $Q_\mu \in S_{p,q}(H^2)$ *if and only if*

$$\sum_{k=0}^\infty \left(\int_{A_k} \hat{\mu}_\delta(z)^p d\lambda_n(z)\right)^{q/p} < \infty$$

for some (or any) $\delta > 0$.

6.2. **Volterra operators from Hardy to Bergman spaces.** We consider the operator $J_b : H^p \to A^q_\alpha$ for $b \in \mathcal{H}(\mathbb{B}_n)$, $0 < p,q < \infty$ and $\alpha > -1$. The boundedness of $J_b : H^p \to A^q_\alpha$ was mentioned in [13]. The compactness of $J_b : H^p \to A^q_\alpha$ is trivial if we note that

$$\|J_b f\|^q_{A^q_\alpha} \asymp \|R(J_b f)\|^q_{A^q_{\alpha+q}} \asymp \int_{\mathbb{B}_n} |f(z)|^q |Rb(z)|^q (1-|z|^2)^{\alpha+q} dv(z)$$

(see [20, Theorem 2.16]). Thus, the compactness of $J_b : H^p \to A^q_\alpha$ is equivalent to the compactness of the embedding $I_d : H^p \to L^q(d\mu_{b,q})$, where $d\mu_{b,q}(z) = |Rb(z)|^q(1-|z|^2)^{\alpha+q}dv(z)$. More specifically, we have the following result.

Theorem 6.2. *Let $\alpha > -1$, $0 < p, q < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) For $0 < p < \infty$, $J_b : H^p \to A^p_\alpha$ is compact if and only if

$$|Rb(z)|^p(1-|z|^2)^{\alpha+p}dv(z)$$

is a vanishing Carleson measure.

(2) For $0 < p < q < \infty$, $J_b : H^p \to A^q_\alpha$ is compact if and only if

$$\lim_{|z|\to 1^-} Rb(z)(1-|z|^2)^{1+\frac{n+1+\alpha}{q}-\frac{n}{p}} = 0.$$

(3) For $0 < q < p < \infty$, $J_b : H^p \to A^q_\alpha$ is compact if and only if it is bounded, which is equivalent to

$$\xi \mapsto \int_{\Gamma(\xi)} |Rb(z)|^q(1-|z|^2)^{\alpha+q-n}dv(z)$$

belongs to $L^{p/(p-q)}(\mathbb{S}_n)$.

Applying Theorem 6.2, Lemma 4.1, Lemma 4.2 and the norm estimates for $J_b : H^p \to A^q_\alpha$ implied in [13, Theorem 10], we can obtain the following estimates for the essential norm of $J_b : H^p \to A^q_\alpha$. The proof is analogous to Theorem 1.2 and is left to the reader.

Theorem 6.3. *Let $1 < p \le q < \infty$, $\alpha > -1$ and $b \in \mathcal{H}(\mathbb{B}_n)$ such that $J_b : H^p \to A^q_\alpha$ is bounded.*

(1) If $1 < p = q < \infty$, then

$$\|J_b\|_e \asymp \limsup_{|a|\to 1^-}\left(\int_{\mathbb{B}_n} \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}}|Rb(z)|^p(1-|z|^2)^{p+\alpha}dv(z)\right)^{1/p}.$$

(2) If $1 < p < q < \infty$, then

$$\|J_b\|_e \asymp \limsup_{|a|\to 1^-}|Rb(a)|(1-|a|^2)^{1+\frac{n+1+\alpha}{q}-\frac{n}{p}}.$$

In order to characterize the membership in Schatten classes of $J_b : H^2 \to A^2_\alpha$, we consider the following inner product

$$\langle f, g\rangle_{*,A^2_\alpha} = f(0)\overline{g(0)} + \int_{\mathbb{B}_n} Rf(z)\overline{Rg(z)}(1-|z|^2)^{\alpha+2}dv(z)$$

of the Bergman space A^2_α, and let J_b^* denote the Hilbert adjoint of $J_b : H^2 \to A^2_\alpha$ with respect to the standard inner product of H^2 and the inner product $\langle\cdot,\cdot\rangle_{*,A^2_\alpha}$ of A^2_α. Then for any $f, g \in H^2$, we have

$$\begin{aligned}\langle J_b^*J_bf, g\rangle_{H^2} &= \langle J_bf, J_bg\rangle_{*,A^2_\alpha}\\ &= \int_{\mathbb{B}_n} f(z)\overline{g(z)}|Rb(z)|^2(1-|z|^2)^{\alpha+2}dv(z)\end{aligned}$$

$$= \langle Q_{\mu_{b,2}} f, g\rangle_{H^2}.$$

Thus we have $J_b^* J_b = Q_{\mu_{b,2}}$ and $J_b \in S_p(H^2, A^2_\alpha)$ if and only if $Q_{\mu_{b,2}} \in S_{p/2}(H^2)$. More precisely, we have the following characterization of the membership in Schatten classes of $J_b : H^2 \to A^2_\alpha$.

Theorem 6.4. *Let $\alpha > -1$, $0 < p < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) If $n < p(3+\alpha)/2 < \infty$, then J_b belongs to $S_p(H^2, A^2_\alpha)$ if and only if

$$\int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{\frac{p}{2}(3+\alpha)} d\lambda_n(z) < \infty. \tag{6.1}$$

(2) If $p(3+\alpha)/2 \le n$, then J_b is in $S_p(H^2, A^2_\alpha)$ if and only if b is constant.

Proof. Suppose $J_b \in S_p(H^2, A^2_\alpha)$, then $Q_{\mu_{b,2}} \in S_{p/2}(H^2)$. By [16, Theorem 10], we have

$$S_t\mu_{b,2}(w) = (1-|w|^2)^{n+t} \int_{\mathbb{B}_n} \frac{d\mu_{b,2}(z)}{|1-\langle w, z\rangle|^{2n+t}}$$

belongs to $L^{p/2}(\mathbb{B}_n, d\lambda_n)$ for some $t > 0$ large enough. By subharmonic property, we get

$$S_t\mu_{b,2}(w) \gtrsim |Rb(w)|^2 (1-|w|^2)^{3+\alpha}.$$

Since the integrable condition (6.1) implies b is constant when $p(3+\alpha)/2 \le n$, it completes the necessity.

We now consider the sufficiency part. In the case $p \ge 2$, suppose $\{a_k\}$ is a δ-lattice of the unit ball. Then we need to show

$$\sum_k \left(\frac{\mu_{b,2}(D(a_k,\delta))}{(1-|a_k|^2)^n} \right)^{p/2} < \infty$$

by [16, Theorem 10]. By Hölder's inequality, we have

$$\begin{aligned} \sum_k \left(\frac{\mu_{b,2}(D(a_k,\delta))}{(1-|a_k|^2)^n} \right)^{p/2} &\asymp \sum_k \left(\int_{D(a_k,\delta)} |Rb(z)|^2 (1-|z|^2)^{\alpha+3} d\lambda_n(z) \right)^{p/2} \\ &\lesssim \sum_k \int_{D(a_k,\delta)} |Rb(z)|^p (1-|z|^2)^{p(\alpha+3)/2} d\lambda_n(z) \\ &\lesssim \int_{\mathbb{B}_n} |Rb(z)|^p (1-|z|^2)^{p(\alpha+3)/2} d\lambda_n(z) < \infty. \end{aligned}$$

In the case $0 < p < 2$, it is enough to prove that

$$\int_{\mathbb{B}_n} \left((1-|w|^2)^{n+t} \int_{\mathbb{B}_n} \frac{|Rb(z)|^2 (1-|z|^2)^{\alpha+2}}{|1-\langle w,z\rangle|^{2n+t}} dv(z) \right)^{p/2} d\lambda_n(w) < \infty$$

for some $t > 0$ large enough by [16, Theorem 10]. This can be done by the same method as in the proof of Theorem 1.3. Therefore, the proof is finished. $\square$

For $0 < p, q < \infty$, we say J_b is in $S_{p,q}(H^2, A^2_\alpha)$ if $J_b^* J_b \in S_{\frac{p}{2},\frac{q}{2}}(H^2)$. Using Theorem 6.1, we have the following result. The proof is the same as the proof of Theorem 1.4 and so is omitted.

Theorem 6.5. *Let $\alpha > -1$, $0 < p, q < \infty$ and $b \in \mathcal{H}(\mathbb{B}_n)$. Then the following hold:*

(1) If $n < p(3+\alpha)/2 < \infty$, then J_b belongs to $S_{p,q}(H^2, A^2_\alpha)$ if and only if

$$\int_0^1 M_p^q(Rb, r)(1-r)^{\frac{q}{2}(3+\alpha)-\frac{q}{p}n-1} dr < \infty.$$

(2) If $p(3+\alpha)/2 \le n$, then J_b is in $S_{p,q}(H^2, A^2_\alpha)$ if and only if b is constant.

6.3. **Volterra companion integration operators.** It is also interesting to study the Volterra companion integration operator I_b, which is defined by

$$I_b f(z) = \int_0^1 Rf(tz) b(tz) \frac{dt}{t}, \quad z \in \mathbb{B}_n$$

for $b, f \in \mathcal{H}(\mathbb{B}_n)$. The operator I_b is closely related to Volterra operator J_b as follows:

$$I_b f + J_b f = M_b f - f(0)b(0),$$

and has been studied between various spaces of holomorphic functions, where $M_b f = bf$ is the multiplication operator induced by b. We consider the boundedness and compactness of $I_b : A^p_\alpha \to H^q$ and $I_b : H^p \to A^q_\alpha$ here.

Noting that $R(I_b f)(z) = Rf(z)b(z)$ for any $z \in \mathbb{B}_n$, by [20, Theorem 2.16], we have

$$\|I_b f\|^q_{A^q_\alpha} \asymp \|R(I_b f)\|^q_{A^q_{\alpha+q}} \asymp \int_{\mathbb{B}_n} |Rf(z)|^q |b(z)|^q (1-|z|^2)^{\alpha+q} dv(z).$$

Thus, the boundedness (resp. compactness) of $I_b : H^p \to A^q_\alpha$ is equivalent to the boundedness (resp. compactness) of the embedding derivative $R : H^p \to L^q(d\nu_{b,q})$, where $d\nu_{b,q}(z) = |b(z)|^q(1-|z|^2)^{\alpha+q} dv(z)$.

We next consider the boundedness and compactness of $I_b : A^p_\alpha \to H^q$. Using the standard pointwise estimate for the derivative of Hardy space functions $I_b f_z$, where

$$f_z(w) = \frac{(1-|z|^2)^{s-(n+1+\alpha)/p}}{(1-\langle w, z\rangle)^s}, \quad z, w \in \mathbb{B}_n$$

for large s, we can get the following necessary condition for $I_b : A^p_\alpha \to H^q$ to be bounded.

Proposition 6.6. *Let $0 < p, q < \infty$ and $\alpha > -1$. If $I_b : A^p_\alpha \to H^q$ is bounded, then*

$$\sup_{z \in \mathbb{B}_n} |b(z)|(1-|z|^2)^{\frac{n}{q}-\frac{n+1+\alpha}{p}} < \infty.$$

By Proposition 6.6, if $p < (1+\frac{1+\alpha}{n})q$ and $I_b : A^p_\alpha \to H^q$ is bounded, then $b = 0$. In the case $p \ge (1+\frac{1+\alpha}{n})q$, by the similar methods as in the proof of $J_b : A^p_\alpha \to H^q$ when $p > q$, we can get the following result.

Theorem 6.7. *Let $\alpha > -1$, $0 < p, q < \infty$ and $b \in (\mathbb{B}_n)$. Then the following hold:*

(1) If $p < (1+\frac{1+\alpha}{n})q$ or $(1+\frac{1+\alpha}{n})q \le p \le 2$, then $I_b : A^p_\alpha \to H^q$ is bounded if and only if $b = 0$.

(2) If $p \geq (1+\frac{1+\alpha}{n})q$ and $p>2$, then $I_b : A^p_\alpha \to H^q$ is bounded if and only if $I_b : A^p_\alpha \to H^q$ is compact, which in turn is equivalent to

$$\xi \mapsto \left(\int_{\Gamma(\xi)} |b(z)|^{\frac{2p}{p-2}} (1-|z|^2)^{-\frac{p+2\alpha}{p-2}-n} dv(z) \right)^{\frac{p-2}{2p}}$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{S}_n)$.

Declarations

Funding. J. Pau was partially supported by the grants MTM2017-83499-P (Ministerio de Educación y Ciencia) and 2017SGR358 (Generalitat de Catalunya). M. Wang was partially supported by NSFC (No. 11771340) of China.
Conflicts of interest/Competing interests. There are no conflicts of interest.
Availability of data and material. Not applicable.
Code availability. Not applicable.

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Jiale Chen, School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail address: `jialechen@whu.edu.cn`

Jordi Pau, Departament de Matemàtiques i Informàtica, Universitat de Barcelona, 08007 Barcelona, Catalonia, Spain.
E-mail address: `jordi.pau@ub.edu`

Maofa Wang, School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
E-mail address: `mfwang.math@whu.edu.cn`