Quantum Duels

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Abstract: Quantum Game Theory is the merging of Quantum Mechanics and Game Theory. In this article we are going to study the modeling of the struggle to survive against several agents, known as duels. We will present the classical version of the problem and we will formalize the quantum version for two players, studying the best strategies that can be adopted depending on the desired result. Since quantum entanglement allows players to revive, we will also analyze the suitability of different counter-intuitive strategies for players, such as shooting into the air or even shooting themselves. Finally we will see how to generalize this formalism for three players.

I. INTRODUCTION

In many situations, it is not easy to make the optimal decision, and it can be important when money is involved. During the 1930's, this motivated some mathematicians to model strategic decision making, and it was finally formalized by J. von Neumann and O. Morgenstern in [1]. Game Theory was born, and in the 1950's it would be extensively developed and applied in many different areas, such as economy, social sciences and biology.

The birth of quantum information theory and quantum computing at the end of the 20th century led David Meyer in 1999 to mix quantum physics and game theory. He proposed in [2] a quantum version of the penny flipover game: Two players, Alice and Bob, have a coin in a box that is heads up. For three consecutive turns starting with Alice, they have to flip the coin (or not), without the other player seeing it. Alice wins if the coin is still heads up after her second turn. In the classic version of the game, Alice has a 50% chance of winning. In the quantum version proposed by Meyer, the coin can be modeled as the spin of an electron, where heads and tails would equal the spin pointing up/down in the z-axis, and flipping the coin is equivalent to rotating the spin on the x-axis. If Alice can point the spin wherever she wants, she will always have a winning strategy. Indeed, if she points it on the x-axis on her first move, Bob's coin flip will have no effect on the state, and Alice will only have to reverse her first move to keep the coin heads up. Giving players access to the set of quantum strategies increases their chances of winning against classical players.

In the same year, Eisert et al. published in [3] a quantum version of the famous prisoner's dilemma, formalizing a general protocol for two-player quantum games. From then on, many articles proliferated, describing the most popular games in game theory from a quantum perspective, such as the Monty Hall Problem, the battle of the sexes or the game of chicken, among others. An exhaustive review of several of these games was made by Adrian Flitney and Derek Abbott in [4], where they conclude that a quantum player always has an advantage over a classical one.

These authors also formalized the quantum two and three person duels in [5]. In this dangerous game, players will shoot each other in turns, where quantum shooting means flipping the spin of a $1/2-$spin particle. This game is richer, since both players have access to quantum strategies, and they also study the quantum entanglement and decoherence of the states. Some years later, Alexandre Schmidt and Milena Paiva revisited quantum duels in [6], and expand on some of the results found by Flitney and Abbott.

In this article we are going to study quantum duels, replicating the formulation of [5] and [6] and extending some results for different strategies. But first we are going to study duels from the perspective of classical game theory.

II. CLASSICAL DUELS & TRUELS

Two gunfighters, Alice and Bob, duel for honor. What strategy should they use to maximize their chances of victory? From a game theory point of view the answer is trivial: The best strategy is to shoot their opponent. Suppose our duelists have one shot per turn, starting with Alice. We will define marksmanship as the probability of hitting the shot, and suppose that Alice's is $\bar{a} = 1/3$ and Bob's is $\bar{b} = 1/2$. After one round, both have a $1/3$ chance to be dead. In fact, if we consider that the game ends when one dies, the chances of victory are the same for both. Alice makes up for her poorer marksmanship by being the first person to shoot.

From a classical perspective, the game is much more interesting if we add a third contender, whom we will call Charlie. Suppose that in this three-person duel, also known as truel, players can shoot each other sequentially in alphabetical order, and that they have the option of shooting into the air. If all of them were perfect shooters, they would all end up shooting into the air. Indeed, if one player shoots another player, the third will always eliminate him. More generally, if the marksmanships of Alice, Bob and Charlie are $0 < \bar{a} \leq \bar{b} < \bar{c} \leq 1$, respectively, the best strategy for Alice is always to shoot into the air, since if she hit another player she would automatically be the target. However, the strategies for both Bob and Charlie is to shoot each other as both want to take out the best shooter.

An exhaustive analysis of this problem was made by Kilgour in [7]. It can be seen that if the marksmanships are $\bar{a} = 1/3$, $\bar{b} = 2/3$ and $\bar{c} = 1$, the probability that Charlie is the only survivor is less than Alice's. Again, being the first to shoot and employing the best strategy gives you a better chance of winning than just having a good aim.

III. QUANTUM GAMES

Before formalizing the quantum version of the duel, we have to define some general concepts about quantum games. In analogy with quantum computing, the states are represented by qubits, a linear combination of the basis states $|0\rangle$ and $|1\rangle$ of a two-dimensional Hilbert space. Suppose we have two players, and each one is capable of manipulating his own qubit. A pure quantum strategy is a unitary quantum operator $\hat{U}$ acting on the qubit. Eisert et al. showed in [3] that pure quantum strategies gave the same results as mixed classical strategies (i.e., assigning a probability to each pure classical strategy) in two-player games. To unleash the full potential of quantum mechanics, they decided to produce quantum entanglement between the states of both players. Thus, if the initial state is $|0\rangle \otimes |0\rangle = |00\rangle$, after both players make their moves we will have a final state

$$|\psi_f\rangle = \hat{J}^\dagger \left(\hat{U}_1 \otimes \hat{U}_2\right) \hat{J} |00\rangle , \tag{1}$$

where $\hat{U}_1$ and $\hat{U}_2$ represent players 1 and 2 moves, respectively; $\hat{J}$ is an entangling operator and $\hat{J}^\dagger$ the corresponding disentangling operator in order to take a measure on the final state. Maximum entanglement can be obtained by choosing $\hat{J}$ such that $\hat{J}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

When taking a measurement, the final state will be one of four possible states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. If each player assigns a numerical value (payoff) P_{ij} to the utility of each of the possible states, the expected value of the payoff for that player will be

$$\langle \$ \rangle = \sum_{i,j=0,1} P_{ij} \left|\langle ij|\psi_f\rangle\right|^2 . \tag{2}$$

Pure quantum strategies $\hat{U}$ are elements of SU(2) and can be written as

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix} , \tag{3}$$

where $\theta \in [0, \pi]$ and $\alpha, \beta \in [-\pi, \pi]$. In particular, any strategy of the form $\hat{U}(\theta, 0, 0)$ is equivalent to a mixed classical move.

If both players have access to the quantum move set and are aware of the other's strategy, any move $\hat{U}_1(\theta, \alpha, \beta)$ can be undone by playing the so-called miracle move $\hat{U}_2(\theta, -\alpha, \frac{\pi}{2} - \beta)$. Thus, two-player quantum games only make sense to be played if the other player's moves are unknown. Based on this model, we are going to define quantum duels.

IV. QUANTUM DUELS

Alice and Bob duel again for honor, but now they decide to apply the rules of quantum mechanics. There are several ways to formalize the protocol of a two-player quantum duel based on the principles described in section III, and we will follow the formulation proposed by Flitney and Abbott in [5].

A. Quantum Duel Rules

To make the game realistic, we will state the rules as follows: Each player has control on the spin of an electron that is initially in state $|1\rangle$. A referee will operate on the system formed by both electrons, applying the strategies that the players had previously defined. Finally, the referee will take a measurement of the final state, which may be one of the four base states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of the 4-dimensional Hilbert space, where the first qubit is the state of Alice's electron and the second is Bob's, and will kill the player whose final state is $|0\rangle$. For simplicity, we will say that players are alive or dead if their state is $|1\rangle$ or $|0\rangle$, respectively; and that one player kills (or revives) the other by flipping their qubit. Neither player will have information about the strategies applied by the other and, unlike the classical duel, the final state will not be known until it is measured, so players could continue playing even if they are dead.

Players will fire in turns starting with Alice, where in this case firing means applying a unitary quantum operator on the state, as we have defined in Eq. (3). With maximum generality, we will define the operator Alice-shoots-Bob $\hat{A}_B = \hat{A}_B(\theta_1, \alpha_1, \beta_1)$ as

$$\begin{aligned} \hat{A}_B = &\left[e^{-i\alpha_1}\cos(\theta_1/2)|11\rangle + ie^{i\beta_1}\sin(\theta_1/2)|10\rangle\right]\langle 11| \\ &+ \left[e^{i\alpha_1}\cos(\theta_1/2)|10\rangle + ie^{-i\beta_1}\sin(\theta_1/2)|11\rangle\right]\langle 10| \\ &+ |00\rangle\langle 00| + |01\rangle\langle 01| , \end{aligned} \tag{4}$$

where $\theta_1 \in [0, \pi]$ is related with her shooting skills, and $\alpha_1, \beta_1 \in [-\pi, \pi]$ are arbitrary phase factors. The operator Bob-shoots-Alice $\hat{B}_A = \hat{B}_A(\theta_2, \alpha_2, \beta_2)$ is constructed by permuting the position of the qubits, and changing the respective subscripts. By analogy with the classical duel, the terms $|00\rangle\langle 00|$ and $|01\rangle\langle 01|$ indicate that Alice cannot alter the state if her qubit is $|0\rangle$ (she cannot shoot if she is dead). However, note that the term $|11\rangle\langle 10|$ is counter-intuitive from a classical point of view, since it implies that Alice can revive Bob.

The game starts with both players alive in the state $|11\rangle$. Alice shoots first, bringing the state to $\hat{A}_B|11\rangle$, and then is Bob's turn, ending the first round with the entangled state $|\psi_1\rangle = \hat{B}_A\hat{A}_B|11\rangle$ equals to

$$\begin{aligned}|\psi_1\rangle =& e^{-i(\alpha_1+\alpha_2)}c_1c_2|11\rangle + ie^{i(\beta_2-\alpha_1)}c_1s_2|01\rangle \\ &+ ie^{i\beta_1}s_1|10\rangle\,;\end{aligned} \tag{5}$$

where, from now on, we will write $s_i = \sin(\theta_i/2)$ and $c_i = \cos(\theta_i/2)$, for $i = 1, 2$.

The probability that Alice is the only survivor is given by $|\langle 10|\psi_1\rangle|^2 = s_1^2$, which has to be equivalent to the probability that she hits her shot. Thus, we will define Alice's (Bob's) marksmanship as $\bar{a} = s_1^2$ ($\bar{b} = s_2^2$), so the probability that she (he) misses the shot is $a = 1-\bar{a} = c_1^2$ ($b = 1-\bar{b} = c_2^2$).

Note that the α_i, β_i phase factors do not affect these probabilities. In particular, any strategy $\hat{A}_B(\theta_1, \alpha_1, \beta_1)$ is equivalent to $\hat{A}_B(\theta_1, 0, 0)$ which, as we saw in section III, corresponds to a classical mixed strategy. The result of this quantum duel does not differ at all from the classical one after only one round.

After n rounds, the final state is obtained by repeatedly applying these operators, $|\psi_n\rangle = (\hat{B}_A\hat{A}_B)^n|11\rangle$. If the initial state is $|11\rangle$, the final state can never be $|00\rangle$, since a dead person cannot shoot. Therefore, after two rounds the final state will be

$$|\psi_2\rangle = (\hat{B}_A\hat{A}_B)^2|11\rangle = K_1|11\rangle + K_2|10\rangle + K_3|01\rangle\,. \tag{6}$$

Explicit expressions of these $K_j(\theta_i, \alpha_i, \beta_i)$ can be found in Eq. (8), (9), (10) and (11) of [6]. It can be seen that, for a two rounds duel, quantum entanglement takes effect, making players able to flip back (revive) a qubit that was already in state $|0\rangle$. The probabilities $|\langle jk|\psi_2\rangle|^2$ of each potential final state are calculated in Eq. (10) of [5], and they are a function of θ_i and α_i, but not of β_i.

Now the strategy of the players is not only probabilistic, because although the marksmanships (θ_i) are fixed, they can decide a strategy by choosing the parameters α_i and β_i. For this reason, it is worth asking what is the best strategy that a player, let's say Alice, can take to maximize their chances of victory. However, first of all we have to define what "victory" means, that is, we have to assign a payoff to each possible final state.

B. Best Strategy for a Variable Payoff

We can assign, without loss of generality, a payoff of 1 if Alice is the only survivor, $P_{10} = 1$; and a payoff of 0 if Alice dies, $P_{01} = 0$. In articles [5] and [6], they assign a payoff of 1/2 to both players ending up alive, considering that it is not the ideal result for Alice, but that she prefers it rather than dying. Since assigning an average value seems arbitrary, in this section we will study if the best strategy may be different when choosing a variable payoff $P_{11} = \lambda_1 \in [0, 1]$, where the extreme value $\lambda_1 = 0$ means that Alice's priority is to kill Bob; and $\lambda_1 = 1$ means that her priority is just to survive.

Substituting these values into Eq. (2), we can write the expectation value for Alice's payoff after two rounds as

$$\langle \$_A\rangle = |\langle 10|\psi_2\rangle|^2 + \lambda_1|\langle 11|\psi_2\rangle|^2\,. \tag{7}$$

This expected value for the payoff is a function of five variables, $\langle \$_A\rangle = \langle \$_A\rangle(\theta_1, \theta_2, \alpha_1, \alpha_2, \lambda_1)$, so studying its maximums is a very complicated task. However, since the marksmanships are fixed and the players can know each other's, we can study some particular cases.

In [5], the authors showed that for $\bar{a} = 1/3$ and $\bar{b} = 1/2$ (and $\lambda_1 = 1/2$, as stated), Alice's Payoff will reach a global maximum for $(\alpha_1, \alpha_2) = (\pm\pi/3, \mp 2\pi/3)$ and $(\alpha_1, \alpha_2) = (\pm\pi, 0)$. We are going to study the particular case $\alpha_2 = 0$, that is, if Bob is restricted to $\hat{B}_A(\theta_2, 0, \beta_2)$ strategies. Since the probabilities do not depend on β_i, we can write the payoff as a function of $\alpha_1, \lambda_1, a = 1-\bar{a}$ and $b = 1-\bar{b}$,

$$\begin{aligned}\langle \$_A\rangle =& \lambda_1 + (1-\lambda_1)a(1-a)\left[1 + b + 2\sqrt{b}\cos(2\alpha_1)\right] \\ &- \lambda_1(1-b)\Big[(1-a)^2 - 2a(1-a)\sqrt{b}\cos(2\alpha_1) \\ &+ ab(1+a) - 2\sqrt{a}\cos(\alpha_1)\left(ab - (1-a)\sqrt{ab}\right)\Big]\,.\end{aligned} \tag{8}$$

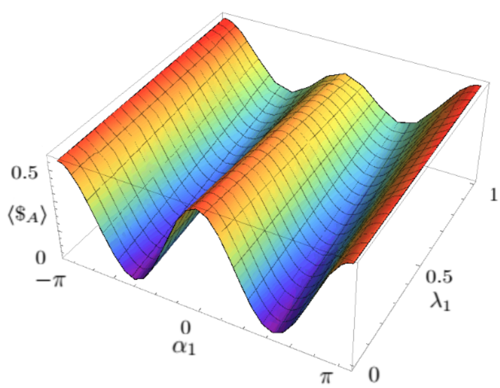

FIG. 1: Plot of the expectation value of Alice's payoff in a 2 round quantum duel for $\bar{a} = 1/3$, $\bar{b} = 1/2$ and $\alpha_2 = 0$, as a function of α_1 and λ_1.

For example, for $\bar{a} = 1/3$ and $\bar{b} = 1/2$ we obtain that $\alpha_1 = \pm\pi$ is a strategy that maximizes Alice's payoff, for all $\lambda_1 \in [0, 1]$. So, the maximum found in [5] for $\alpha_2 = 0$ does not depend on λ_1. However, for $\lambda_1 = 0$ the global maximum is also reached at $\alpha_1 = 0$, as we can see in Fig. 1. That is, if Bob plays $\alpha_2 = 0$, it is better for Alice to play $\alpha_1 = 0$ the more she prioritizes killing Bob than surviving ($\lambda_1 \to 0$).

C. Different Strategies

Until now, we have only considered as a strategy to choose the parameters of the shot. However, since in the quantum version a player can be revived, or can revive another unintentionally, it might be a good strategy not to shoot the other.

In [5] it is suggested that if Alice has a bad marksmanship, her best strategy is to shoot into the air on her second shot. This strategy is studied in a more general way in [6], where some of the formulas and conclusions of [5] are revised. We are going to do the calculations for the $\alpha_2 = 0$ case and we will see if the results are different for a variable payoff λ_1.

Shooting into the air is equivalent to not shooting, so the unitary operator is the identity. If Alice avoids firing on the second turn, the final state after two rounds will thus be $\left|\psi_2^{\text{air}}\right\rangle = \hat{B}_A\hat{B}_A\hat{A}_B\left|11\right\rangle$,

$$\begin{aligned}\left|\psi_2^{\text{air}}\right\rangle =& ie^{i\beta_1}s_1\left|10\right\rangle + e^{-i\alpha_1}c_1\left(e^{-2i\alpha_2}c_2^2+s_2^2\right)\left|11\right\rangle \\ &+ 2ic_1c_2s_2\cos(\alpha_2)e^{i(\beta_2-\alpha_1)}\left|01\right\rangle.\end{aligned} \tag{9}$$

The probability that Alice dies is

$$\left|\left\langle 01\middle|\psi_2^{\text{air}}\right\rangle\right|^2 = 4ab(1-b)\cos^2(\alpha_2), \tag{10}$$

which is equivalent to Eq.(18) of [6] but is different to Eq. (13) of [5], so we conclude that there is a misprint in [5]. Calculating the other probabilities and substituting into Eq. (7), the expected value of Alice's payoff when she shoots into the air is:

$$\left\langle \$_A^{\text{air}}\right\rangle = 1+(\lambda_1-1)a-4\lambda_1 ab(1-b)\cos^2(\alpha_2). \tag{11}$$

As we can see, it does not depend on α_1 or β_i. However, what interests us is not to maximize the function $\langle \$_A^{\text{air}}\rangle$, but to compare these maximums with those of $\langle \$_A\rangle$. In other words, what we want is to maximize the difference function $\langle \$_A^{\text{dif}}\rangle = \langle \$_A^{\text{air}}\rangle - \langle \$_A\rangle$. Indeed, the parameters for which $\langle \$_A^{\text{dif}}\rangle$ is positive will be those for which shooting into the air is a better strategy.

This difference function, however, does depend on α_1. If we again restrict Bob to play $\alpha_2 = 0$ and we consider him an intermediate shooter ($\bar{b} = 1/2$), we can explicitly calculate the function by subtracting Eq. (11) from Eq. (8). In particular, if we study the values for which the function $\langle \$_A\rangle$ is maximal, $\alpha_1 = \pm\pi$, we obtain the functional dependency represented in Fig. 2.

It can be seen in Fig. 2 that the function is positive for low values of a and λ_1, and that the global maximum is reached for $a = 0$ and $\lambda_1 = 0$. Therefore, only if Alice is a good shooter and her priority is to kill Bob, should she avoid the second shot. This makes sense, because with the first shot it would be very likely that she would have already killed Bob, so by shooting into the air she avoids reviving him.

Motivated by the fact that a player can revive and be revived, it might be interesting to study if shooting yourself can be in some cases a good strategy to win a quantum duel. To build an operator Alice-shoots-Alice, $\hat{A}_A$, one can consider that it has to be equivalent to Bob shooting her but with probability of success 1. Thus, $\hat{A}_A = \hat{B}_A(\pm\pi/2, \alpha_1, \beta_1)$. However, letting Alice play

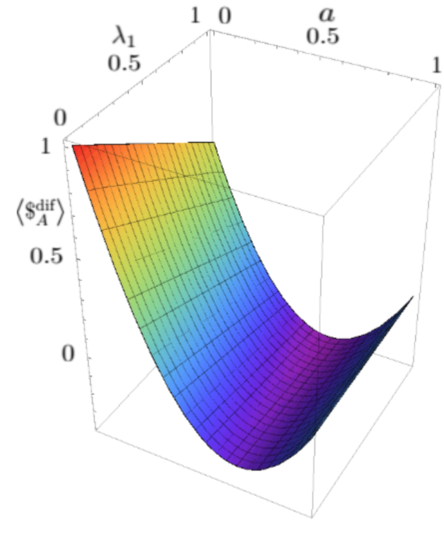

FIG. 2: Plot of $\langle \$_A^{\text{dif}}\rangle$ in a 2 round quantum duel when Alice fires into the air on his second turn, for $\bar{b} = 1/2$, $\alpha_1 = \pm\pi$ and $\alpha_2 = 0$, as a function of a and λ_1.

such a quantum strategy violates one of the dueling rules: that a player cannot shoot while dead. If we break this rule, a player could revive himself and the game would lose the analogy with the classical duel.

An Alice-shoots-Alice operator consistent with our quantum duel will flip Alice's spin if it is $|1\rangle$, and do nothing if it is $|0\rangle$. Such operator is equivalent to suicide in the classical duel, where it is an automatically losing strategy. This is not the case in the quantum version, as Bob can revive her.

Suppose that in the first round Alice shoots herself, bringing the state to $|01\rangle$, then Bob shoots and we have

$$\left|\psi_1^{\text{sui}}\right\rangle = \hat{B}_A\left|01\right\rangle = e^{i\alpha_2}c_2\left|01\right\rangle + ie^{-i\beta_2}s_2\left|11\right\rangle. \tag{12}$$

The probability that Alice will be revived after one round is $\left|\left\langle 11\middle|\psi_1^{\text{sui}}\right\rangle\right|^2 = s_2^2 = \bar{b}$. This implies that if Bob is a perfect shooter, Alice will always be revived. The function $\langle \$_A^{\text{dif}}\rangle = \langle \$_A^{\text{sui}}\rangle - \langle \$_A\rangle$ is

$$\langle \$_A^{\text{dif}}\rangle = \left[\bar{b} - (1-\bar{a})(1-\bar{b})\right]\lambda_1 - \bar{a}. \tag{13}$$

This function is positive when Alice is not a good shooter but Bob is, and is highest when $\lambda_1 \to 1$. For example, for $\bar{a} = 1/3$ and $\bar{b} = 2/3$, we get $\langle \$_A^{\text{dif}}\rangle > 0$ if $\lambda_1 > 3/4$. Counter-intuitively, it is a good strategy for Alice to commit quantum suicide the more she prioritizes surviving.

Many other strategies can be analyzed, such as letting players take measurements and vary their strategies between rounds, or introducing partial decoherence of the system after each move. We will see what happens if we add a third player to the game.

V. QUANTUM TRUELS

Now Charlie joins the duel, and we will identify him with the subscript 3. The new initial state will be $|111\rangle$, and, as proposed in [5], the operator $\hat{A}_B$ of Eq. (4) can be generalized by adding Charlie as an observer,

$$\begin{aligned}\hat{A}_B = \sum_{j=0,1} \Big\{ & \left[e^{-i\alpha_1} c_1 |11j\rangle + i e^{i\beta_1} s_1 |10j\rangle\right] \langle 11j| \\ & + \left[e^{i\alpha_1} c_1 |10j\rangle + i e^{-i\beta_1} s_1 |11j\rangle\right] \langle 10j| \Big\} \\ & + \sum_{jk=0,1} |0jk\rangle \langle 0jk| .\end{aligned} \tag{14}$$

The other shooting operators $\hat{A}_C, \hat{B}_A, \hat{B}_C, \hat{C}_A$ and $\hat{C}_B$ can be constructed by permuting the corresponding qubits and changing the subscripts. For simplicity, in this section we will study the case $\alpha_i = \beta_i = 0$, for $i = 1, 2, 3$. The rules are the same as those defined for the classical truel in section II, but with the fundamental difference that players cannot take measurements until the game is over. Therefore, if their marksmanships are $0 \leq \bar{a} \leq \bar{b} < \bar{c} \leq 1$, Alice does not have any incentive to shoot into the air in the first round because no one is going to target her even if she kills another player. Thus, the best strategy for all of them will be trying to eliminate the best shooter, so after one round we will have

$$\begin{aligned}|\psi_1\rangle = & \hat{C}_B \hat{B}_C \hat{A}_C |111\rangle = (c_1 s_2 + c_2 s_1) |110\rangle \\ & + (c_1 c_2 - s_1 s_2)(c_3 |111\rangle + s_3 |101\rangle) .\end{aligned} \tag{15}$$

We see that there are only three possible final states, and that Charlie can only survive if Alice does. In fact, the probability that Charlie survives this round is $|\langle 111|\psi_1\rangle|^2 + |\langle 101|\psi_1\rangle|^2 = (c_1c_2 - s_1s_2)^2$. Note that this probability can be zero if $c_1c_2 = s_1s_2$, which is equivalent to the condition $\bar{a} = 1 - \bar{b}$. This implies that if, for example, Alice's marksmanship is $\bar{a} = 1/3$ and Bob's is $\bar{b} = 2/3$, then Charlie will always die regardless of his marksmanship, and the second round will be a Duel between Alice and Bob.

Duels can easily be generalized to an arbitrary number of players by adding observers to the shooting operator in Eq. (14). However, the number of different cases that appear makes the study of these games very complex.

VI. CONCLUSIONS

The objective of this article was to introduce the formalization of quantum duels and see how players could maximize their payoffs by applying different strategies.

First we have studied the duel of two players up to two rounds, and we have seen that if $\bar{a} = 1/3$, $\bar{b} = 1/2$ and Bob is restricted to play $\alpha_2 = 0$, Alice always maximizes her payoff by playing $\alpha_1 = \pm\pi$. Also, if the payoff of surviving with Bob is 0, she also maximizes with $\alpha_1 = 0$.

Secondly we have seen that in some cases it is a good strategy for Alice to shoot into the air. In particular, with the mentioned marksmanships and $\alpha_1 = \pm\pi, \alpha_2 = 0$, Alice gets a bigger payoff by shooting into the air on her second turn if she is a good shooter and prioritizes killing Bob over surviving. We have also seen that Alice can choose to shoot herself on her first turn for a one-round duel, since if Bob has a good aim he will revive her. In fact, if Alice does not have a good aim and her priority is to survive, she gets a bigger payoff by committing suicide.

Finally we present a formalization of the truels for $\alpha_i = \beta_i = 0$ and, unlike the classical case, here Alice has no incentive to shoot into the air, and both Alice and Bob prefer to shoot Charlie. If Alice's and Bob's marksmanships meet the condition $\bar{a} = 1 - \bar{b}$, Charlie always dies and his strategy is irrelevant.

A quantum duel is a paradigmatic game of quantum game theory, since the shooting operators allow us to see how quantum entanglement affects states, and counterintuitive results appear shortly after we start playing with them. For all this, quantum duels are more interesting than the classical ones. Moreover, playing with electron spins is much safer than playing with guns.

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