ON STABILITY OF LOGARITHMIC TANGENT SHEAVES. SYMMETRIC AND GENERIC DETERMINANTS

DANIELE FAENZI, SIMONE MARCHESI

Abstract. We prove stability of logarithmic tangent sheaves of singular hypersurfaces D of the projective space with constraints on the dimension and degree of the singularities of D. As main application, we prove that determinants and symmetric determinants have simple (in characteristic zero, stable) logarithmic tangent sheaves and we describe an open dense piece of the associated moduli space.

Introduction

Given a hypersurface $D \subset \mathbb{P}^N$ defined by a homogeneous form F of degree d over a field $\mathbb{k}$, the vector fields on $\mathbb{P}^N$ which are tangent to D define the logarithmic tangent sheaf $\mathcal{T}_D$ of D. This sheaf is the first syzygy of the Jacobian ideal sheaf $\mathcal{J}_D$ of D, as the partial derivatives $\nabla(F)$ of F, that is, the generators of the Jacobian ideal J_D, express it as the kernel of the Jacobian matrix of F:

$$0 \to \mathcal{T}_D \to (N+1).\mathcal{O}_{\mathbb{P}^N} \xrightarrow{\nabla(F)} \mathcal{O}_{\mathbb{P}^N}(d-1). \tag{1}$$

If the characteristic of $\mathbb{k}$ does not divide d, the sheaf $\mathcal{T}_D(1)$ is a subsheaf of $\mathcal{T}_{\mathbb{P}^N}$, usually denoted by $\mathcal{T}_{\mathbb{P}^N}\langle D\rangle$, and the quotient of $\mathcal{T}_{\mathbb{P}^N}$ by $\mathcal{T}_{\mathbb{P}^N}\langle D\rangle$ is the equisingular normal sheaf of D. The sheaf $\mathcal{T}_{\mathbb{P}^N}\langle D\rangle$, or rather its dual, often denoted by $\Omega_{\mathbb{P}^N}(\log D)$, was studied in [Del1970] and [Sai1980] in connection with Hodge theory. All these sheaves play a major role in the deformation theory of the embedding $D \hookrightarrow \mathbb{P}^N$, see [Ser2006, Section 3.4]. The graded module of global sections of $\mathcal{T}_D$, which we denote by T_D, is called the module of logarithmic derivations, or of Jacobian syzygies of F. It has been also studied in detail, most notably for hyperplane arrangements, see for instance [OT1992].

For some noteworthy classes of hypersurface singularities the logarithmic tangent sheaf is locally free, and this plays an important role in the theory of discriminants and unfolding of singularities, see for instance [BEGvB2009]. For some remarkable classes of divisor, the module T_D is itself free, see for instance [Ter1981], so that $\mathcal{T}_D$ splits as a direct sum of line bundles.

In contrast to this, for some interesting classes of hypersurfaces the sheaf $\mathcal{T}_D$ is stable. This happens for instance for generic arrangements of at least $N+2$ hyperplanes [DK1993],

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but also for many highly non-generic arrangements see [FMV2013, AFV2016]. Recall that $\mathcal{T}_D$ is stable when D is smooth and, moreover, the stability of $\mathcal{T}_D$ for hypersurfaces with isolated singularities was studied in [Dim2017], in connection with the Torelli problem, on whether D can be reconstructed from $\mathcal{T}_D$. Stability of $\mathcal{T}_D$ is a fundamental preliminary step to connect the study of equisingular deformations of D to moduli problems of sheaves over $\mathbb{P}^N$. However, few methods for proving stability of $\mathcal{T}_D$ are available today, indeed very little seems to be known besides the case of arrangements and isolated singularities of curves and surfaces.

The goal of this paper is to propose some tools to prove stability in a wide range of situations. The general strategy is find a suitable closed subvariety $X \subset \mathbb{P}^N$ where we may prove that the restriction of $\mathcal{T}_D$ is stable and then argue that this implies stability of $\mathcal{T}_D$ itself.

The first possibility to explore is to take X to be a linear space disjoint from the singular locus $\operatorname{sing}(D)$ of D. In the first part of this paper (see §1) we show that $\mathcal{T}_D|_X$ is stable provided some vanishing of global sections of the reflexive hulls of exterior powers of $\mathcal{T}_D$ in terms of the codimension of singularities of D. More specifically, setting $s = \dim(\operatorname{sing}(D))$ and assuming $s \le N-2$, we obtain the following result.

Theorem A. *Assume that for all integers p with $1 \le p \le s+1$ we have:*

$$\mathrm{H}^0\left(\wedge^p \mathcal{T}_D(q)^{**}\right) = 0, \qquad \textit{with:} \qquad q = \left\lfloor \frac{(d-1)p}{N} \right\rfloor.$$

Then $\mathcal{T}_D$ is slope-stable.

We may also formulate the result in terms of the Hilbert function of $\mathcal{T}_D$ only.

Corollary B. *The sheaf $\mathcal{T}_D$ is slope-stable if:*

$$\mathrm{H}^0\left(\mathcal{T}_D(q)\right) = 0, \qquad \textit{with:} \qquad q = \left\lfloor \frac{(d-1)(s+1)}{N} \right\rfloor.$$

For isolated hypersurface singularities, this allows to generalize [Dim2017, Theorem 1.3] and [Dim2019, Theorem 3.3] to arbitrary dimension.

Theorem C. *Assume $\dim(\operatorname{sing}(D)) = 0$ and set $q = \left\lfloor \frac{d-1}{N} \right\rfloor$. Then $\mathcal{T}_D$ is stable if:*

$$\deg(\operatorname{sing}(D)) < (d-q-1)(d-1)^{N-1}.$$

In the second part of this paper, we consider some natural families of divisors, not covered by the previous results, where stability of $\mathcal{T}_D$ can be proved. Indeed, many interesting hypersurfaces tend to have singularities of small codimension, for instance many divisors coming from orbit closures, discriminants or from moduli theory are highly singular. In this case, one strategy we propose is to pick a subvariety $X \subset \mathbb{P}^N$ disjoint from $\operatorname{sing}(D)$, such that $\mathcal{T}_D$ restricts over X to a vector bundle of some special form, whose stability is under control, and deduce from this the stability of $\mathcal{T}_D$. A natural candidate for this is the bundle of principal parts $\mathcal{E}_{n-1}$. This is defined as kernel of the evaluation of sections $\mathrm{H}^0(\mathcal{O}_X(n-1)) \otimes \mathcal{O}_X \to \mathcal{O}_X(n-1)$. We contend that in some relevant situations $\mathcal{T}_D$ will restrict to a slope-stable bundle of principal parts $\mathcal{E}_{n-1}$ and that this suffices to prove stability of $\mathcal{T}_D$ itself. Let us point out that vector bundles of principal parts on projective spaces, and in particular their stability, is a matter of independent interest, see for example [Maa2005, Re2012]. We contribute to this by showing that the vector bundle of principal parts on a smooth quadric surface, namely on $\mathbb{P}^1 \times \mathbb{P}^1$, is slope-stable (see Proposition 3.10). For this we make use of representations of a quiver supported on a planar graph, rather than the tree appearing in [Re2012]. For this last point we need the base field to be of characteristic zero as the relevant representation theory is more tricky in positive characteristic. However the sheaf $\mathcal{T}_D$ is easily proved to be simple in any characteristic.

Going back to the main families of divisors where our strategy applies, let us first mention symmetric discriminants, see §2. In this case, we argue that the suitable subvariety X is a projective plane, where stability of vector bundles of principal parts is well-known. Also, in view of the Goto–Józefiak–Tachibana's resolution, see [Józ1978, GT1977] (see also [Wey2003, Section 6.3.8]), we get that $\mathcal{T}_D$ is a *Steiner sheaf*, that is, it has a linear resolution of length two. Altogether, the result is the following.

Theorem D. *Let $n \geq 2$ be an integer and let D be the determinant divisor of symmetric $n \times n$ matrices in $\mathbb{P}^{\binom{n+1}{2}-1}$. Then the logarithmic sheaf $\mathcal{T}_D$ satisfies:*

i) the sheafified minimal graded free resolution of $\mathcal{T}_D$ takes the form:

$$0 \to \binom{n}{2}.\mathcal{O}_{\mathbb{P}^N}(-2) \to (n^2-1).\mathcal{O}_{\mathbb{P}^N}(-1) \to \mathcal{T}_D \to 0;$$

ii) the restriction of $\mathcal{T}_D$ to a generic plane $P \subset \mathbb{P}^N$ is isomorphic to the bundle of principal parts $\mathcal{E}_{n-1}$ defined as kernel of the evaluation map:

$$\binom{n+1}{2}.\mathcal{O}_P \to \mathcal{O}_P(n-1);$$

iii) if $\mathrm{char}(\mathbb{k}) = 0$*, the logarithmic sheaf $\mathcal{T}_D$ is slope-stable.*

The next family we wish to mention is one of the main characters of this paper, namely the *generic determinant*, namely the divisor D defined as determinant of an $n \times n$ matrix of variables $(x_{i,j})_{1\leq i,j\leq n}$ in $\mathbb{P}^N = \mathbb{P}^{n^2-1}$. This time, the suitable subvariety $X \subset \mathbb{P}^N$ is a smooth quadric surface. As we mentioned above, the bundle of principal parts $\mathcal{E}_{n-1}$ on X is slope-stable, so the main point is to prove that $\mathcal{T}_D$ restricts over X to the bundle $\mathcal{E}_{n-1}$. To do this, we analyze the Artinian reduction A_L of the Jacobian algebra of D over the linear span $L \simeq \mathbb{P}^3 \subset \mathbb{P}^N$ of X. In particular, we prove a quadratic Lefschetz property of A_L, which in turn is obtained by specializing L to a well-chosen linear section which we call *semigeneric*. It will turn out that the intersection $D \cap L$ is a singular surface which is resolved by a projective plane, blown-up at $n(n-1)$ complete intersection points. Studying carefully the divisors on this blow-up we are able to prove the next result for all $n \geq 2$.

Theorem E. *Let $n \geq 2$ be an integer and let D be the generic determinant divisor of $n \times n$ matrices in $\mathbb{P}^{n^2-1}$. Then the logarithmic sheaf $\mathcal{T}_D$ satisfies:*

i) the sheafified minimal graded free resolution of $\mathcal{T}_D$ takes the form:

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-n-1) \to n^2.\mathcal{O}_{\mathbb{P}^N}(-2) \to 2(n^2-1).\mathcal{O}_{\mathbb{P}^N}(-1) \to \mathcal{T}_D \to 0;$$

ii) the restriction of $\mathcal{T}_D$ *to a generic quadric surface* $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^N$ *is isomorphic to the bundle of principal parts* $\mathcal{E}_{n-1}$ *defined as kernel of the evaluation map:*

$$n^2.\mathcal{O}_X \to \mathcal{O}_X(n-1);$$

iii) the sheaf $\mathcal{E}_{n-1}$ *is simple and, if* $\mathrm{char}(\Bbbk) = 0$*,* $\mathcal{E}_{n-1}$ *and* $\mathcal{T}_D$ *are slope-stable.*

In light of the last item, it is natural to investigate the moduli space of simple sheaves – or, if $\mathrm{char}(\Bbbk) = 0$, of stable sheaves – which we denote by $\mathfrak{M}_n$, that contains the sheaf $\mathcal{T}_D$. This is done in §4. Let us set up the framework needed to state our result in this direction. Let $n \geq 3$ be an integer. Consider two n-dimensional vector spaces U and V and the group $\mathrm{SL}(U) \times \mathrm{SL}(V)$. Put $\mathbf{A} = \mathrm{Hom}_{\Bbbk}(V, U) \simeq V^* \otimes U$ and consider the standard representation of $\mathrm{SL}(U)$ on U, tensored with id_{V^*}, so that $\mathrm{SL}(U)$ acts linearly on $\mathbf{A}$. This action commutes with the $\mathrm{SL}(V)$-action on $\mathbf{A}$ obtained via the dual representation on V^* tensored with id_U. So $\mathbf{A}$ is a representation of $\mathrm{SL}(U) \times \mathrm{SL}(V)$, which is faithful. We get an injective map $\mathrm{SL}(U) \times \mathrm{SL}(V) \to \mathrm{GL}(\mathbf{A})$ and an induced injective morphism $\mathrm{SL}(U) \times \mathrm{SL}(V) \to \mathrm{PGL}(\mathbf{A})$ which identifies $\mathrm{SL}(U) \times \mathrm{SL}(V)$ to a closed subgroup of $\mathrm{PGL}(\mathbf{A})$.

For any $\mathbf{f} \in \mathrm{End}_{\Bbbk}(\mathbf{A})$, we consider a map $M_{\mathbf{f}} : U \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}$ canonically associated with $\mathbf{f}$. When $\det(M_{\mathbf{f}}) \neq 0$, it turns out that, setting $D_{\mathbf{f}} = \mathbb{V}(\det(M_{\mathbf{f}}))$, we have:

$$\mathcal{T}_{D_{\mathbf{f}}} \text{ is simple} \quad \Leftrightarrow \quad [\mathbf{f}] \in \mathrm{PGL}(\mathbf{A}) \quad \Leftrightarrow \quad \mathcal{T}_{D_{\mathbf{f}}} \text{ is slope-stable (for } \mathrm{char}(\Bbbk) = 0).$$

The subgroup $\mathrm{SL}(U) \times \mathrm{SL}(V)$ acts on the matrices $M_{\mathbf{f}}$ by two-sided multiplication and this does not alter the isomorphism class of $\mathcal{T}_{D_{\mathbf{f}}}$. Hence, denoting by $\mathfrak{M}_n$ the moduli space of simple (or, in characteristic zero, stable) sheaves having the same Hilbert polynomial as $\mathcal{T}_D$, we see that the assignment $\Psi : [\mathbf{f}] \mapsto \mathcal{T}_{D_{\mathbf{f}}}$ defines a morphism:

$$\Psi : \mathrm{PGL}(\mathbf{A})/\mathrm{SL}(U) \times \mathrm{SL}(V) \to \mathfrak{M}_n$$

Of course, the transpose ${}^{\mathrm{t}}M_{\mathbf{f}}$ of $M_{\mathbf{f}}$ lands on the same divisor $D_{\mathbf{f}}$. Our main result concerning $\mathfrak{M}_n$ is that, up to the $2:1$ cover arising from transposition, the map Ψ captures essentially the whole geometry of the open dense piece of $\mathfrak{M}_n$ consisting of logarithmic sheaves.

Theorem F. *The map* Ψ *is an étale* $2:1$ *cover onto its image. The image of* Ψ *is a smooth open affine piece of an irreducible component of* $\mathfrak{M}_n$*, of dimension* $(n^2-1)^2$.

This is related to classical work of Frobenius, [Fro1897, §7.1], equivalent to the fact that the determinant is a $2:1$ map from $\mathrm{PGL}(\mathbf{A})/\mathrm{SL}(U) \times \mathrm{SL}(V)$ to the locus $\mathfrak{D}_n$ of determinantal hypersurfaces. The étale nature of this map is proved in [RV2017], see Remark (1) after Theorem 1.3 of that paper for a discussion of the interplay with Frobenius' theorem and related literature. So Theorem F can be though of as an interpretation of [RV2017] in terms of moduli spaces of simple or stable sheaves. Our method is characteristic-free (except for stability of bundles of principal parts on $\mathbb{P}^1 \times \mathbb{P}^1$) and relies on a direct proof of the Torelli theorem (in the sense of Dolgachev-Kapranov) for determinantal hypersurfaces and the divisor class group of determinantal hypersurfaces.

An analogous description as an algebraic group quotient, that is, a homogeneous space, could be obtained as well for the case of hypersurfaces defined by determinants of symmetric matrices. Nevertheless, recall that the generic element of the moduli space of semistable Steiner sheaves is locally free, therefore the image of such quotient would sit as a closed subscheme of the relevant moduli space. So there is no direct analogue of Theorem F for symmetric determinants.

Notation. Let us fix some notation that will be used throughout this paper. Denote by $\Bbbk$ a field. The assumptions on $\Bbbk$ may change in different sections. We use the polynomial ring $R = \Bbbk[x_0, \dots, x_N]$ and, if A is a graded R-module, we denote by A_p its degree-p summand.

If U is a $\Bbbk$-vector space, we write $\mathbb{P}(U)$ for the set of hyperplanes of U. For an integer m, if $\mathcal{E}$ is a vector space, or module, or a sheaf, we write $m.\mathcal{E}$ for the direct sum of m copies of $\mathcal{E}$. Put $\mathbb{P}^N = \mathbb{P}((N+1).\Bbbk) = \mathrm{Proj}(R)$. Given a non-zero homogeneous polynomial $F \in R$ of degree d, write $D = \mathbb{V}(F)$ for the hypersurface of $\mathbb{P}^N$ defined by F. Denoting by $\nabla(F)$ its Jacobian matrix, the Jacobian ideal J_D is the ideal generated by $\nabla(F)$ and $\mathcal{J}_D$ is the Jacobian ideal sheaf. The *logarithmic tangent sheaf* $\mathcal{T}_D$ associated to D is defined as the kernel of the gradient of F:

$$0 \to \mathcal{T}_D \to (N+1).\mathcal{O}_{\mathbb{P}^N} \xrightarrow{\nabla(F)} \mathcal{J}_D(d-1) \to 0.$$

Given a coherent sheaf $\mathcal{F}$ and $i \in \mathbb{N}$, we write $\mathrm{H}^i_*(\mathcal{F})$ for the i-th cohomology module of $\mathcal{F}$, namely $\mathrm{H}^i_*(\mathcal{F}) = \bigoplus_{t\in\mathbb{Z}} \mathrm{H}^i(\mathcal{F}(t))$. The *module of logarithmic derivations* of D is defined as $T_D = \mathrm{H}^0_*(\mathcal{T}_D)$. Moreover, $\mathrm{sing}(D)$ will denote the singular locus of D, equipped with its natural scheme structure, which is to say $\mathrm{sing}(D) = \mathbb{V}(J_D)$. We write $s = \dim(\mathrm{sing}(D))$.

We will say that a coherent sheaf on a subvariety $X \subset \mathbb{P}^N$ is stable or semistable if it is so in the sense of Gieseker, with respect to the hyperplane divisor on X. We will use the notion of slope-stability, again with respect to the hyperplane divisor, and use that slope-stability implies stability while semistability implies slope-semistability. We refer to [HL1997] for basic material on semistability of sheaves.

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1. Stability for low dimensional singularities

In this section, we study the general case, that is hypersurfaces D inside $\mathbb{P}^N$ of degree $d \geq 2$. Specifically, §1.1 is devoted to prove Theorems A and C and Corollary B. Sharpness of these results is also discussed briefly. Finally, in §1.2 we provide some further remarks and comments about the Torelli problem for logarithmic derivations, namely the question of whether the hypersurface D can be reconstructed from the sheaf $\mathcal{T}_D$. In this section $\Bbbk$ is an arbitrary field.

1.1. Stability of sheaves of logarithmic derivations.

We first prove Theorem A. Our strategy is to exclude the existence of destabilizing subsheaves having rank up to $s+1$ making use of the vanishing assumptions of spaces of global sections of reflexive hulls of exterior powers with a refinement of Hoppe's criterion.

Then, we take care of potentially destabilizing subsheaves of rank between $s+2$ and $n-1$ by restricting $\mathcal{T}_D$ to a linear space L of codimension $s+1$ disjoint from the singular locus $\mathrm{sing}(D)$ and working on the resulting Koszul complex of $\nabla(F)|_L$.

Proof of Theorem A. We assume that $\mathcal{T}_D$ is unstable despite satisfying the assumptions and we seek a contradiction. Without loss of generality, we may assume that the field $\Bbbk$ is algebraically closed.

Consider a destabilizing subsheaf $\mathcal{K}$ of $\mathcal{T}_D$, set $r = \mathrm{rk}(\mathcal{K})$ and put $c = \deg(c_1(\mathcal{K}))$ so that $r < N$ and:

$$\frac{c}{r} \geq \frac{1-d}{N}. \tag{2}$$

Without loss of generality, we may assume that $\mathcal{T}_D/\mathcal{K}$ is torsion-free. The embedding $j : \mathcal{K} \hookrightarrow \mathcal{T}_D$ gives a non-trivial map $\wedge^r\mathcal{K} \to \wedge^r\mathcal{T}_D$ and, applying the bi-duality functor, we get a non-trivial map:

$$j_r : \mathcal{O}_{\mathbb{P}^N}(c) \simeq (\wedge^r\mathcal{K})^{**} \to (\wedge^r\mathcal{T}_D)^{**}.$$

The image of j_r is a quotient of $\mathcal{O}_{\mathbb{P}^N}(c)$, hence it is a torsion sheaf unless j_c is injective. The former case is excluded since this image sits in $(\wedge^r\mathcal{T}_D)^{**}$, so j_r is injective and

$$\mathrm{H}^0(\mathbb{P}^N, \wedge^r\mathcal{T}_D(-c)^{**}) \neq 0. \tag{3}$$

This non-vanishing contradicts our vanishing assumptions if $r \leq s+1$. Hence we must have $s+2 \leq r \leq N-1$. In other words, we have to prove that $\mathcal{T}_D$ has no destabilizing subsheaf of rank r with $s+2 \leq r \leq N-1$. So the proof is finished for $s = N-2$ but needs further argumentation for $s < N-2$.

To comply with this, we consider a linear subspace L of $\mathbb{P}^N$ of codimension $s+1$ which is skew to $\mathrm{sing}(D)$ and meets transversely the locus where $\mathcal{T}_D/\mathcal{K}$ is not locally free. Observe that $\dim(L) = N-s-1 \geq 2$. Denote $\mathcal{F} = (\mathcal{T}_D)|_L$. Then the exact sequence defining $\mathcal{T}_D$ restricts to:

$$0 \to \mathcal{F} \to (N+1).\mathcal{O}_L \to \mathcal{O}_L(d-1) \to 0 \tag{4}$$

and therefore the sheaf $\mathcal{F}$ is locally free. Moreover, the map j restricts to an injective map $j_L : \mathcal{K}|_L \to \mathcal{F}$ and, taking exterior powers of j_L, we get:

$$\mathrm{H}^0(L, \wedge^r\mathcal{F}(-c)) \neq 0.$$

Using the natural isomorphism $\wedge^r\mathcal{F} \simeq \wedge^{N-r}\mathcal{F}^*(1-d)$, this amounts to:

$$\mathrm{H}^0(L, \wedge^{N-r}\mathcal{F}^*(1-d-c)) \neq 0. \tag{5}$$

Now, since $d \geq 2$ and $r < N$, the inequality (2) gives:

$$c \geq \frac{r(1-d)}{N} > 1-d,$$

or, equivalently, $1-d-c < 0$. Also, $s+2 \leq r \leq N-1$ gives $1 \leq N-r \leq N-s-2 = \dim(L)-1$. Therefore, to reach the desired contradiction, it suffices to show:

$$\mathrm{H}^0(L, \wedge^p\mathcal{F}^*(-1)) = 0, \qquad \text{for all integers } p \text{ with } 1 \leq p \leq \dim(L)-1. \tag{6}$$

To get this, we dualize (4) and take p-th exterior power to write an exact complex:

$$\bigwedge^p \Big(\mathcal{O}_L(1-d) \to (N+1).\mathcal{O}_L\Big) \longrightarrow \wedge^p \mathcal{F}^* \to 0.$$

Tensoring with $\mathcal{O}_L(-1)$ and taking cohomology, since $p \le \dim(L) - 1$ we get $\mathrm{H}^0(L, \wedge^p \mathcal{F}^*(-1)) = 0$. So (6) is proved and the theorem as well. □

Proof of Corollary B. Given an integer p with $1 \le p \le s+1$, the p-th exterior power of the injection $i : \mathcal{T}_D \to (N+1).\mathcal{O}_{\mathbb{P}^N}$ gives maps:

$$\bigwedge^p \mathcal{T}_D \xrightarrow{i_1} (N+1).\bigwedge^{p-1} \mathcal{T}_D \to \cdots \xrightarrow{i_{p-1}} (N+1)^{(p-1)}.\mathcal{T}_D,$$

and taking reflexive hulls the composition $i_{p-1} \circ \cdots \circ i_1$ gives:

$$\begin{array}{ccc} \bigwedge^p \mathcal{T}_D & \xrightarrow{i_{p-1}\circ\cdots\circ i_1} & (N+1)^{(p-1)}.\mathcal{T}_D \\ \downarrow & & \| \\ (\bigwedge^p \mathcal{T}_D)^{**} & \longrightarrow & (N+1)^{(p-1)}.\mathcal{T}_D^{**} \end{array}$$

Since the kernel of each of the maps $i_1, \ldots, i_{p-1}$ is a torsion sheaf, we get that $i_{p-1} \circ \cdots \circ i_1$ induces an injective map:

$$\wedge^p \mathcal{T}_D(q)^{**} \hookrightarrow (N+1)^{(p-1)}.\mathcal{T}_D(q).$$

Therefore, for all p with $1 \le p \le s+1$, setting $q_p = \left\lfloor \frac{(d-1)p}{N} \right\rfloor$ and assuming $\mathrm{H}^0(\mathcal{T}_D(q)) = 0$ we get that $\mathrm{H}^0(\wedge^p \mathcal{T}_D(q_p)^{**}) = 0$ for all p so Theorem A gives stability of $\mathcal{T}_D$. □

Proof of Theorem C. We assume $\deg(\mathrm{sing}(D)) < (d-q-1)(d-1)^{N-1}$ and prove that $\mathcal{T}_D$ is slope-stable. Since the sheaf $\mathcal{T}_D$ is reflexive and $\dim(\mathrm{sing}(D)) = 0$, in view of Corollary B, we only have to check:

$$\mathrm{H}^0(\mathcal{T}_D(q)) = 0.$$

The degree (which is to say, the length) of the 0-dimensional subscheme $\mathrm{sing}(D) \subset \mathbb{P}^N$ is the total Tjurina number of $\mathrm{sing}(D)$, obtained as the sum of the length of the localization of $\mathrm{sing}(D)$ at the points of the set-theoretic support of $\mathrm{sing}(D)$.

Consider the minimal degree relation of J_D, that is, the smallest integer r such that $\mathrm{H}^0(\mathcal{T}_D(r)) \neq 0$. If $\mathcal{T}_D$ was not slope-stable we would have $r \le q$.

According to [dPW2001, Theorem 5.3], the integer r satisfies $(d-r-1)(d-1)^{N-1} \le \deg(\mathrm{sing}(D))$. Hence, if $\mathcal{T}_D$ was not slope-stable then $r \le q$, so $(d-q-1)(d-1)^{N-1} \le \deg(\mathrm{sing}(D))$, which contradicts our assumption. □

Remark 1.1. An obvious obstruction to stability of $\mathcal{T}_D$ is that a partial derivative of the equation f defining D vanishes identically, in a suitable system of coordinates. Indeed, if this happens then the sheaf $\mathcal{T}_D$ admits a decomposition of the following type:

$$\mathcal{T}_D \simeq \mathcal{T}_{\tilde{D}} \oplus r.\mathcal{O}_{\mathbb{P}^N},$$

where r denotes the number of vanishing derivatives. This excludes that $\mathcal{T}_D$ is slope-semistable, or simple.

In characteristic zero this is equivalent to the fact that D is a cone, where $r-1$ equals the dimension of the linear (projective) subspace which is the apex of the cone.

Remark 1.2. If no partial derivative of F vanishes identically (up to a coordinate change) and $(d-1)(s+1) < N$, then the sheaf $\mathcal{T}_D$ is slope-stable by Corollary B. Notice that this numerical condition is sharp, as the following example shows. Assume $\mathrm{char}(\Bbbk)$ does not divide d nor $d-1$ and consider the hypersurface D defined by the homogeneous polynomial:

$$F = x_0 x_1^{d-1} + \sum_{j=2}^{N} x_j^d, \quad \text{which gives} \quad \nabla(F) = [x_1^{d-1},\ (d-1)x_0x_1^{d-2},\ dx_2^{d-1},\ \dots,\ dx_N^{d-1}].$$

Observe that D is singular only at the point $(1:0:\ldots:0)$. The associated sheaf $\mathcal{T}_D$ has $\mathrm{H}^0(\mathcal{T}_D)=0$ and $\mathrm{H}^0(\mathcal{T}_D(1)) \neq 0$. If $d \geq N+1$, we get $c_1(\mathcal{T}_D) \leq -\mathrm{rk}(\mathcal{T}_D)$, so the sheaf $\mathcal{T}_D$ is not slope-stable because $\mathcal{O}_{\mathbb{P}^N}(-1) \subset \mathcal{T}_D$ is a destabilizing subsheaf.

1.2. **A Torelli-type result.** In this section we will focus on some results of "Torelli type". Recall that such nomenclature is used in general for results on the embeddings between moduli spaces. In particular, this type of problems for logarithmic tangent sheaves has been proposed by Dolgachev and Kapranov in [DK1993], followed by many others.

In our case case we are interested in the morphism which associates to a hypersurface $D \subset \mathbb{P}^N$ its logarithmic tangent sheaf $\mathcal{T}_D$. More specifically we are interested in the following question: *Does the logarithmic tangent sheaf $\mathcal{T}_D$ determine the hypersurface D?* These hypersurfaces have been called *DK-Torelli* in [Dim2017]. Keeping this definition, we provide an extension of [Dim2017, Theorem 1.5] to the case of non-isolated singularities, with a similar proof. For terminology about Sebastiani-Thom hypersurfaces and multiplicity of singularities we refer to [Wan2015].

Before stating the result, recall that, by [Ser2014], if D is singular there is an identification of R-modules:

$$\mathrm{H}^1_*(\mathcal{T}_D) \simeq \frac{J_D^{\mathrm{sat}}}{J_D},$$

where J_D^{sat} denotes the saturation of J_D. In particular we have isomorphisms:

$$\mathrm{H}^1(\mathcal{T}_D(t-d)) \simeq \left(\frac{J_D^{\mathrm{sat}}}{J_D}\right)_t, \qquad \text{for all } t \in \mathbb{Z},$$

and these commute with multiplication maps in R.

Proposition 1.3. *Suppose that there exists an integer $m < d-1$ such that:*

- $\mathrm{H}^0(\mathcal{T}_D(2m)) = 0$*;*
- *there exist two elements in $\mathrm{H}^1(\mathcal{T}_D(m-d))$ that admit two non zero representatives $h_1, h_2 \in (J_D^{\mathrm{sat}})_m$ with no common factor.*

Then, $\mathcal{T}_D$ determines the Jacobian ideal of F.
Furthermore, if the hypersurface D is not DK-Torelli, then D has a singularity of multiplicity $d-1$ or the polynomial F is of Sebastiani-Thom type.

Remark 1.4. Notice that having a singularity of multiplicity $d-1$ is not a sufficient condition for the hypersurface D to fail the DK-Torelli property. Indeed, as we will see in §4.2, the hypersurfaces defined by determinants satisfy the DK-Torelli property but do admit such singularities – we will actually exploit these singularities in §3.4.

On the contrary, being of Sebastiani-Thom type is a sufficient condition for failing the DK-Torelli property, at least when D is not a cone. This can be seen taking the following description of the defining equation for a Sebastiani-Thom divisor:

$$F = G(x_0, \dots, x_i) + H(x_{i+1}, \dots, x_N),$$

with G and H non zero homogeneous polynomials of degree d, and consider the infinite family

$$F_\alpha = \alpha G(x_0, \dots, x_i) + H(x_{i+1}, \dots, x_N) \quad \text{with} \quad \alpha \in \Bbbk.$$

Having that the Jacobian ideals J_{D_α} coincide, all the hypersurfaces $D_\alpha = \mathbb{V}(F_\alpha)$ induce the same logarithmic sheaf and therefore are not DK-Torelli.

Proof of Proposition 1.3. Our first goal is to characterize when a homogeneous polynomial g of degree $d-1$ belongs to the Jacobian ideal J_F. In order to do so, consider, for any $k \in \mathbb{N}$, the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-d+1+k) \xrightarrow{\cdot g} \mathcal{O}_{\mathbb{P}^N}(k) \to \mathcal{O}_Y(k) \to 0,$$

with $Y = \mathbb{V}(g)$. Let us tensor it by $\mathcal{T}_D$ and note that the first map remains injective, since $\mathcal{T}_D$ is torsion free, and consider the induced exact sequence in cohomology:

$$\begin{aligned} 0 \quad &\to \mathrm{H}^0(\mathcal{T}_D(-d+1+k)) \to \mathrm{H}^0(\mathcal{T}_D(k)) \to \mathrm{H}^0(\mathcal{T}_D(k)|_Y) \to \\ &\to \mathrm{H}^1(\mathcal{T}_D(-d+1+k)) \to \mathrm{H}^1(\mathcal{T}_D(k)) \to \cdots \end{aligned}$$

We know that $\mathrm{H}^0(\mathcal{T}_D(k))$ describes the homogeneous syzygies of degree k of the Jacobian ideal, that is, it is given by all the $(N+1)$-tuples $(a_0, \dots, a_N)$ of homogeneous polynomials of degree k such that $\sum_{i=0}^N a_i \frac{\partial F}{\partial x_i} = 0$. This implies that the first linear map of the previous diagram does not depend on the choice of g.

Consider the multiplication map by g:

$$\left(\frac{J_D^{\mathrm{sat}}}{J_D}\right)_m \xrightarrow{(\cdot g)_m} \left(\frac{J_D^{\mathrm{sat}}}{J_D}\right)_{m+d-1}.$$

It is straightforward to observe that, if $g \in J_D$, then $(\cdot g)_m = 0$. Let us prove the converse implication. Suppose thus that $(\cdot g)_m = 0$ and note that that both $g \cdot h_1$ and $g \cdot h_2$ belong to $(J_D)_{m+d-1}$. This means that there are $(N+1)$-tuples of homogeneous polynomials $(a_0, \dots, a_N)$ and $(b_0, \dots, b_N)$ of degree m, such that:

$$g \cdot h_1 = \sum_{i=0}^N a_i \frac{\partial F}{\partial x_i} \quad \text{and} \quad g \cdot h_2 = \sum_{j=0}^N b_j \frac{\partial F}{\partial x_j}.$$

Therefore we get:

$$0 = (g \cdot h_1)h_2 - (g \cdot h_2)h_1 = \sum_{j=0}^N (a_j h_2 - b_j h_1) \frac{\partial F}{\partial x_j}.$$

In view of the assumption $\mathrm{H}^0(\mathcal{T}_D(2m)) = 0$, we have thus:

$$a_j h_2 - b_j h_1 = 0, \quad \text{for all} \quad j = 0, \dots, N.$$

Since h_1 and h_2 have no common factor, we have that $h_1|a_j$ and $h_2|b_j$, for $j = 0,\dots,N$. In turn, this implies that $g \in J_D$.

Summing up, a polynomial g of degree $d-1$ lies in J_D if and only if $(\cdot g)_m = 0$. Since J_D is generated in degree $d-1$, this says that J_D is recovered by the R-module structure of $\mathrm{H}^1_*(\mathcal{T}_D)$, so that J_D is determined by $\mathcal{T}_D$. For the last part of the statement, once we have proven that $\mathcal{T}_D$ determines the Jacobian ideal, we apply [Wan2015, Theorem 1.1]. □

2. Symmetric determinants

In this section we suppose that the field $\Bbbk$ is of characteristic different from 2. Fixing an integer $n \geq 2$, we describe the ring R as $R = \Bbbk[x_{i,j} \mid 1 \leq i \leq j \leq n]$, hence $N = \binom{n+1}{2} - 1$. For $1 \leq i \leq j \leq n$, put $x_{j,i} = x_{i,j}$ and let M be the matrix $(x_{i,j})_{1\leq i,j\leq n}$. Consider $F = \det(M)$. The *generic* symmetric determinant is the degree-n hypersurface $D = \mathbb{V}(F) \subset \mathbb{P}^N$. It is singular along the subscheme $\mathrm{sing}(D)$ cut by the $N+1$ minors of order $n-1$ of M obtained by removing from M the i-th line and j-th column, with $1 \leq i \leq j \leq n$. Moreover, $\mathrm{sing}(D)$ has codimension 3 in $\mathbb{P}^N$.

Consider now a projective plane $P \subset \mathbb{P}^N$. The vector bundle of k-th principal parts $\mathcal{E}_k$ is defined as kernel of the natural evaluation of sections $\mathrm{H}^0(\mathcal{O}_P(k)) \otimes \mathcal{O}_P \to \mathcal{O}_P(k)$. The main goal of this section is to prove Theorem D, which establishes a link between $\mathcal{T}_D$ and $\mathcal{E}_{n-1}$ that yields the stability of $\mathcal{T}_D$.

2.1. Proof of Theorem D.

Define the graded algebra A as quotient of R by the homogeneous ideal generated by the minors of order $n-1$ of M. The minimal graded free resolution of A is given by the Goto–Józefiak–Tachibana complex, see [Józ1978, GT1977], see also [Wey2003, §6.3.8]. This takes the form:

$$0 \leftarrow A \leftarrow R \leftarrow \binom{n+1}{2}.R(1-n) \leftarrow (n^2-1).R(-n) \leftarrow \binom{n}{2}.R(-1-n) \leftarrow 0, \tag{7}$$

where the kernel of $A \leftarrow R$ is generated by the partial derivatives of F. The module $T_D(1-n)$ is the kernel of the resulting map $R \leftarrow \binom{n+1}{2}.R(1-n)$ so its resolution is the truncation of the above resolution at the middle step. Upon sheafification, this gives item i).

Next, note that iii) follows from ii). Indeed, by [Re2012], the vector bundle $\mathcal{E}_{n-1}$ is slope-stable. Now, if $\mathcal{T}_D$ had a destabilizing subsheaf $\mathcal{K}$, then choosing P to be a generic plane, transverse to the locus where $\mathcal{T}_D/\mathcal{K}$ fails to be locally free, we would get a subsheaf $\mathcal{K}|_P \subset \mathcal{T}_D|_P$ with the same rank and slope as $\mathcal{K}$, so that $\mathcal{K}|_P$ would destabilize $\mathcal{E}_{n-1}$, a contradiction.

So it remains to prove ii). Note that, since $\mathrm{sing}(D)$ has codimension 3 in $\mathbb{P}^N$, we may choose P disjoint from $\mathrm{sing}(D)$ so that $\mathcal{T}_D|_P$ fits into:

$$0 \to \mathcal{T}_D|_P \to (N+1).\mathcal{O}_P \to \mathcal{O}_P(n-1) \to 0. \tag{8}$$

Note that $N+1 = \mathrm{h}^0(\mathcal{O}_P(n-1))$ and observe that precomposing the evaluation of sections $\mathrm{H}^0(\mathcal{O}_P(n-1)) \otimes \mathcal{O}_P \to \mathcal{O}_P(n-1)$ with an automorphism of $(N+1).\Bbbk$ we get a kernel bundle which is isomorphic to $\mathcal{E}_{n-1}$. So it suffices to prove that, for generic P, the map $(N+1).\mathcal{O}_P \to \mathcal{O}_P(n-1)$ appearing in (8) is the evaluation of global sections, up to precomposing with an isomorphism. But all such maps are the same up to precomposing with an isomorphism provided that they have maximal rank, so it is enough to prove that for generic P we have $\mathrm{H}^0(\mathcal{T}_D|_P) = 0$, or equivalently $\mathrm{H}^1(\mathcal{T}_D|_P) = 0$.

To achieve this, consider the coordinate ring R_P of P. Taking the quotient by the homogeneous ideal generated by partial derivatives of F we obtain a graded algebra of dimension $N-2$. Passing to the quotient modulo the ideal of P we get thus an Artinian algebra A, whose resolution is just obtained by specialization of (7), hence:

$$0 \leftarrow A \leftarrow R_P \leftarrow \binom{n+1}{2}.R_P(1-n) \leftarrow (n^2-1).R_P(-n) \leftarrow \binom{n}{2}.R_P(-1-n) \leftarrow 0.$$

We get, for all $t \in \mathbb{Z}$, $\mathrm{H}^1(\mathcal{T}_D|_P(t-n+1)) \simeq A_t$. Computing dimension in the above display gives $A_t = 0$ for all $t \geq n-1$ so $\mathrm{H}^1(\mathcal{T}_D|_P) = 0$ and we are done.

3. Determinants

This section is devoted to the proof of stability of the logarithmic tangent sheaf of the determinant divisor of a matrix of indeterminates. We work over an arbitrary field $\Bbbk$.

3.1. **Basic setup.** Let us fix an integer $n \geq 2$. Consider the graded ring $R = \Bbbk[x_{i,j} \mid 1 \leq i,j \leq n]$ as coordinate ring of $\mathbb{P}^N$ with $N = n^2 - 1$. We call *tautological determinant* the form:

$$F = \det((x_{i,j})_{1 \leq i,j \leq n}).$$

The corresponding tautological determinantal hypersurface of degree n is the divisor:

$$D = \mathbb{V}(F).$$

The hypersurface $D \subset \mathbb{P}^N$ is singular along the subscheme $\mathrm{sing}(D)$ defined by Jacobian ideal of F, which in turn is generated by the n^2 minors of order $n-1$ of the tautological matrix of variables $(x_{i,j})_{1 \leq i,j \leq n}$. The subscheme $\mathrm{sing}(D)$ has codimension 4 in $\mathbb{P}^N$.

In light of the following remark, all the constructions proposed in the remaining of this section can be considered for $n \geq 3$. Moreover, it explains why we have to suppose $n \geq 3$ for §4 as well.

Remark 3.1. Consider the case $n = 2$, for which D is a smooth quadric in $\mathbb{P}^3$. It is known (reported for example in [Ang2014]) that the logarithmic vector bundle associated to any smooth quadric in $\mathbb{P}^3$ is $\Omega_{\mathbb{P}^3}(1)$. Hence, in this case, the DK-Torelli property does not hold.

The same happens when considering the symmetric case, in which the obtained divisors are smooth conics. Moreover, their defining equation is of Sebastiani-Thom type, that ensures once more that they are not DK-Torelli.

3.1.1. *Section outline.* Our goal is to prove Theorem E. Here are our main steps:

i) find a resolution of the module of global sections T_D;
ii) prove that the logarithmic sheaf restricts to a quadric surface X, with $X \cap \mathrm{sing}(D) = \varnothing$, to the bundle of principal parts $\mathcal{E}_{n-1} = \ker(\mathrm{H}^0(\mathcal{O}_X(n-1))) \otimes \mathcal{O}_X \to \mathcal{O}_X(n-1)$;
iii) prove that the principal part bundle $\mathcal{E}_{n-1}$ of the quadric surface is slope-stable.

§3.2 is devoted to prove item i). In §3.4, we introduce the concept of *semigeneric matrix* which allows us, through a quadratic Lefschetz property described in §3.5, to prove that $(\mathcal{T}_D)_{|X} \simeq \mathcal{E}_{n-1}$. §3.6 is devoted to prove that $\mathcal{E}_{n-1}$ is slope-stable. Finally, in §3.7, we combine all of the previous results to prove that $\mathcal{T}_D$ is slope-stable as well.

3.1.2. *An intrinsic setup.* Let U, V be two n-dimensional $\Bbbk$-vector spaces and set:

$$\mathbf{A} = \mathrm{Hom}_{\Bbbk}(V, U) \simeq V^* \otimes U.$$

We identify $\mathbb{P}^N$ with $\mathbb{P}(\mathbf{A})$, so an element $[\mathbf{a}]$ of $\mathbb{P}(\mathbf{A})$ is the proportionality class of $\mathbf{a} \in \mathbf{A}^* \simeq V \otimes U^*$, that is, of a non-zero linear map $\mathbf{a} : U \to V$.

An element of $[\mathbf{f}]$ of $\mathbb{P}(\mathrm{End}_{\Bbbk}(\mathbf{A}))$ is thus the proportionality class of an element $\mathbf{f} \in \mathrm{End}_{\Bbbk}(\mathbf{A})$, which under the identification $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}(\mathbf{A})}(1)) = \mathbf{A}$ can be seen as a map:

$$M_{\mathbf{f}} : U \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}.$$

We denote by $\mathbf{i}$ the special element $\mathbf{i} = \mathrm{id}_{\mathbf{A}} \in \mathrm{End}_{\Bbbk}(\mathbf{A})$. In any given basis $(u_i \mid 1 \le i \le n)$ and $(v_i \mid 1 \le i \le n)$ of U and V^*, setting $(x_{i,j} \mid 1 \le i, j \le n)$ for the dual basis of $(u_i \otimes v_j \mid i, j \le n)$, the matrix of $M_{\mathbf{i}}$ is $(x_{i,j})_{1 \le i,j \le n}$, so the tautological determinant is $D = D_{\mathbf{i}}$.

3.2. **The resolution of T_D.** Here we give a minimal graded free resolution of the graded module T_D associated with the sheaf $\mathcal{T}_D$. The resolution is obtained directly as a truncation of the Gulliksen-Negård's complex. The upshot is that the projective dimension of T_D is 2, one less than the codimension in D of the singular locus of D, in analogously with free divisors.

The divisor $D = \mathbb{V}(\det(M_{\mathbf{i}})) \subset \mathbb{P}(\mathbf{A})$ is invariant with respect to the action of the group $G = \mathrm{SL}(U) \times \mathrm{SL}(V)$ on $\mathbb{P}(\mathbf{A})$. We seek a resolution of T_D which is equivariant for the induced action of G on the polynomial ring R seen as the symmetric algebra of $\mathbf{A}$.

Proposition 3.2. *There is a minimal graded free G-equivariant resolution of T_D of the form:*

$$0 \leftarrow T_D \leftarrow (\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)) \otimes R(-n) \xleftarrow{\varphi} \mathbf{A} \otimes R(-n-1) \leftarrow R(-2n) \leftarrow 0. \tag{9}$$

Looking only at the homogeneous Betti numbers, the resolution reads:

$$0 \leftarrow T_D \leftarrow 2(n^2-1).R(-n) \leftarrow n^2.R(-n-1) \leftarrow R(-2n) \leftarrow 0.$$

Proof. Recall that the homogeneous Jacobian ideal J_D is defined by the partial derivatives of $F = \det(M_{\mathbf{i}})$, where the matrix of the map $M_{\mathbf{i}}$ is the matrix of indeterminates $(x_{i,j})_{1 \le i,j \le n}$. This ideal is generated by the n^2 minors of order $n-1$ of $M_{\mathbf{i}}$. Namely, there is a natural surjective map :

$$\mathbf{A}^* \otimes R(1-n) \to J_D. \tag{10}$$

This map is the first differential of the resolution of J_D given by the Gulliksen-Negård complex, see [GN1972] or [Wey2003, §6.1.8]. This is a G-equivariant resolution that reads:

$$0 \leftarrow J_D \leftarrow \mathbf{A}^* \otimes R(1-n) \leftarrow \begin{matrix} \mathfrak{sl}(U) \otimes R(-n) \\ \oplus \\ \mathfrak{sl}(V) \otimes R(-n) \end{matrix} \leftarrow \mathbf{A} \otimes R(-n-1) \leftarrow R(-2n) \leftarrow 0.$$

The resolution of T_D is obtained by truncation of the resolution of J_D, since T_D is the kernel of the map (10). □

3.3. **Simplicity of $\mathcal{T}_D$.** We consider the determinant hypersurface D. We would like to prove the following result.

Lemma 3.3. *The sheaf $\mathcal{T}_D$ is simple.*

Proof. Consider the singular locus $Z = \mathrm{sing}(D)$ and the ideal sheaf $\mathcal{I}_Z = \mathcal{I}_{Z/\mathbb{P}(\mathbf{A})}$. Sheafifying the surjection (10) we get the exact sequence:

$$0 \to \mathcal{T}_D \to \mathbf{A} \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})} \to \mathcal{I}_Z(n-1) \to 0, \tag{11}$$

where the surjection onto $\mathcal{I}_Z(n-1)$ is the natural evaluation of global sections, so that:

$$\mathrm{H}^0(\mathcal{T}_D) = \mathrm{H}^1(\mathcal{T}_D) = 0. \tag{12}$$

Since the surjection in (11) is the sheafification of the epimorphism of graded R-modules $\mathbf{A} \otimes R \to I_Z(n-1)$, actually we have $\mathrm{H}^1_*(\mathcal{T}_D) = 0$. Next, recall that $R_Z = R/I_Z$ is a graded Cohen-Macaulay ring of dimension $N-4$. This implies:

$$\mathrm{H}^p_*(\mathcal{I}_Z) = 0, \qquad \text{for } p \in \mathbb{Z} \smallsetminus \{0, N-3, N\}. \tag{13}$$

Together with $\mathrm{H}^1_*(\mathcal{T}_D) = 0$, this gives:

$$\mathrm{H}^p_*(\mathcal{T}_D) = 0, \qquad \text{for } p \in \mathbb{Z} \smallsetminus \{0, N-2, N\}. \tag{14}$$

Next, note that Serre duality gives a natural isomorphism:

$$\mathrm{H}^{N-p}(\mathcal{I}_Z(t-N-1))^* \simeq \mathrm{Ext}^p(\mathcal{I}_Z(t), \mathcal{O}_{\mathbb{P}(\mathbf{A})}), \qquad \text{for all } p, t \in \mathbb{Z}. \tag{15}$$

This implies $\mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})}) = 0$. Also, we have $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})}) = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}(\mathbf{A})}(1-n)) = 0$. Hence, applying $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), -)$ to (11), we get:

$$\Bbbk \simeq \mathrm{End}_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z) \simeq \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{T}_D).$$

Applying $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(-, \mathcal{T}_D)$ to (11) and using (12), we get:

$$\mathrm{End}_{\mathbb{P}(\mathbf{A})}(\mathcal{T}_D) \simeq \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{T}_D) \simeq \Bbbk.$$

The lemma is proved. ☐

3.4. **Semigeneric matrices.** The next step is to choose a linear section $L \simeq \mathbb{P}^3$ of $\mathbb{P}^N$ which is semigeneric, in a sense that we will make more precise in the next paragraph. The goal of this partial genericity will be to ensure that, for a honestly generic choice of L, the resulting quotient algebra A is Artinian and satisfies the quadratic Lefschetz property, as we will see in §3.5.

Restricting $M_{\mathbf{i}}$ to L, we get an $n \times n$ matrix M_L of linear forms on L, whose n^2 minors of order $n-1$ generate the ideal $I_L \subset R_L = \Bbbk[x_0, x_1, x_2, x_3]$ defining A. Set $\mathfrak{m}_0 = (x_1, x_2, x_3)$ and $\mathfrak{m} = (x_0, x_1, x_2, x_3)$. In the next definition, we choose a basis of U and V.

Definition 3.4. We say that L is a semigeneric section if:

$$M_L : n.\mathcal{O}_L(-1) \to n.\mathcal{O}_L \qquad \text{satisfies} \qquad M_L = M_0 + x_0 E_{1,1},$$

where M_0 is generic in $R_0 = \Bbbk[x_1, x_2, x_3]$ and $E_{1,1}$ is the elementary matrix $(E_{1,1})_{i,j} = \delta_{i,1}\delta_{j,1}$, namely the forms $(M_L)_{i,j}$ lie outside a Zariski closed subset of the set of all n^2-tuples of

1-forms in R_0, except for $(M_L)_{1,1}$ which also involves x_0. In such case, we say that M_L is a linear semigeneric matrix of size n.

The goal of this subsection is to prove the following result.

Proposition 3.5. *Let M_L be a linear semigeneric matrix of size n. Then:*

$$I_L = x_0 \mathfrak{m}_0^{n-2} + \mathfrak{m}_0^{n-1}.$$

3.4.1. *Semigeneric matrices and the blown-up plane.* Our first observation aimed at proving Proposition 3.5 is that the determinant of a semigeneric matrix defines a model of the blown-up plane at $n(n-1)$ points.

Put $p_0 = (1:0:0:0)$. Define the threefold $T = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$ and the natural maps $\pi : T \to \mathbb{P}^2$ and $\sigma : T \to \mathbb{P}^3$ so that π is the tautological $\mathbb{P}^1$-bundle and σ is blow-up of $\mathbb{P}^3$ at p_0. Set $\mathfrak{l}$ (resp. $\mathfrak{h}$) for the pull-back to T of a hyperplane in $\mathbb{P}^2$ (resp. in $\mathbb{P}^3$).

Lemma 3.6. *If M_L is semigeneric, the degree-n surface $S = \mathbb{V}(\det(M_L)) \subset L \simeq \mathbb{P}^3$ is the image of $\mathbb{P}^2$ by the linear system of curves of degree n passing through a smooth complete intersection of $n(n-1)$ points. The surface S is smooth away from the $(n-1)$-tuple p_0 and has a natural desingularization $\hat{S}$ which is an element of the linear system $|\mathcal{O}_T((n-1)\mathfrak{l}+\mathfrak{h}))|$.*

Proof. The shape of M implies, by multilinearity of the determinant:

$$\det(M) = \det(M_0) + x_0 \det(M_1), \qquad \text{with:} \qquad M_1 = (M_0)_{2 \le i,j \le n}.$$

Therefore p_0 is a point of multiplicity $n-1$ of S. Working over $\mathbb{P}^2 = \mathrm{Proj}(R_0)$ we note that, if M_0 is general enough, we may assume that the curves $G_0 = \mathbb{V}(\det(M_0))$ and $G_1 = \mathbb{V}(\det(M_1))$ in $\mathbb{P}^2$ are smooth of degree n and $n-1$ and meet transversely at a subscheme $W \subset \mathbb{P}^2$ consisting of $n(n-1)$ reduced points. We have:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1-n) \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \to I_{W/\mathbb{P}^2}(n) \to 0. \tag{16}$$

Therefore S is smooth away from p_0 and the projection away from p_0 sends S birationally to $\mathbb{P}^2$. The inverse map is defined by the complete linear system $|\mathcal{I}_{W/\mathbb{P}^2}(n)|$.

Define the surface $\hat{S}$ as the blow-up of $\mathbb{P}^2$ at W. We have $\hat{S} \simeq \mathbb{P}(\mathcal{I}_{W/\mathbb{P}^2}(n))$, where the linear system associated to the tautological relatively ample line bundle sends $\hat{S}$ to $S \subset \mathbb{P}^3$. The smooth surface $\hat{S}$ sits canonically in T, the embedding being defined by the projectivization of the surjection in (16). □

The restriction of σ, π, $\mathfrak{l}$ and $\mathfrak{h}$ to $\hat{S}$ define objects which we denote by the same letters. We set $\mathfrak{e}_\pi$ for the exceptional divisor of $\pi : \hat{S} \to \mathbb{P}^2$, hence we have:

$$\mathfrak{h} = n\mathfrak{l} - \mathfrak{e}_\pi. \tag{17}$$

For all $i, j \in \mathbb{Z}$, the cohomology of $\mathcal{O}_T(i\mathfrak{h} + j\mathfrak{l})$ is:

$$\mathrm{H}^k(\mathcal{O}_T(i\mathfrak{h}+j\mathfrak{l})) \simeq \begin{cases} \bigoplus_{0 \le u \le i} \mathrm{H}^k(\mathcal{O}_{\mathbb{P}^2}(j+u)), & \text{if } i \ge 0, \\ \bigoplus_{i+1 \le u \le -1} \mathrm{H}^{k+1}(\mathcal{O}_{\mathbb{P}^2}(j+u)), & \text{if } i \le -2, \\ 0, & \text{if } i = -1. \end{cases} \tag{18}$$

Also, the cohomology of $\mathcal{O}_{\hat{S}}(i\mathfrak{h}+j\mathfrak{l})$ is computed via (18) by induction on i and j from the exact sequence:

$$0 \to \mathcal{O}_T((1-n)\mathfrak{l}-\mathfrak{h}) \oplus \mathcal{O}_T(\mathfrak{l}-\mathfrak{h}) \to \mathcal{O}_T(\mathfrak{l}) \oplus \mathcal{O}_T \to \mathcal{O}_{\hat{S}}(\mathfrak{h}) \to 0. \tag{19}$$

3.4.2. *Linear determinantal representation of the blown-up plane.* Set $\mathcal{L} = \operatorname{coker}(M_L)$ and write:

$$0 \to n.\mathcal{O}_L(-1) \xrightarrow{M_L} n.\mathcal{O}_L \to \mathcal{L} \to 0. \tag{20}$$

Here, $\mathcal{L}$ is a reflexive sheaf of rank 1 on S which is not locally free. The next lemma allows to determine an exact sequence on $\hat{S}$, where the rightmost term is a line bundle $\hat{\mathcal{L}}$ on $\hat{S}$. Moreover, it is possible to lift such sequence to the threefold T, in order to define $\hat{\mathcal{L}}$ as the determinant of a map of bundles or rank-n over T, whose push-forward to $\mathbb{P}^3$ is the semigeneric matrix M_L we started with.

Lemma 3.7. *There is a subset $\mathfrak{e}_1,\dots,\mathfrak{e}_m$, with $m = n(n-1)/2$, of the components of $\mathfrak{e}_1 + \dots + \mathfrak{e}_{n(n-1)} = \mathfrak{e}_\pi$, such that $\hat{\mathcal{L}} = (n-1)\mathfrak{l} - \mathfrak{e}_1 - \dots - \mathfrak{e}_m$ fits into:*

$$0 \to \mathcal{O}_T(-\mathfrak{h}) \oplus (n-1).\mathcal{O}_T(-\mathfrak{l}) \to n.\mathcal{O}_T \to \hat{\mathcal{L}} \to 0. \tag{21}$$

Moreover, the push-forward to $L \simeq \mathbb{P}^3$ of the above sequence is (20).

Proof. Let us use the notation described in §3.1.2 and also in the proof of Lemma 3.6. We therefore consider the matrices M_L and M_0 as morphisms:

$$M_L : U \otimes \mathcal{O}_L(-1) \to V \otimes \mathcal{O}_L, \qquad M_0 : U \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^2}.$$

The sheaf $\mathcal{L}_0 = \operatorname{coker}(M_0)$ is a line bundle supported on the curve $G_0 \subset \mathbb{P}^2$. Transposing M_0, by Grothendieck duality we get:

$$0 \to V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to U^* \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{L}_0^*(n-1) \to 0.$$

We observe that a non-zero global section u of $\mathcal{L}_0^*(n-1)$ is given uniquely by a non-zero element $u \in U^*$ and provides thus a 1-codimensional quotient $U_u^* = U^*/\Bbbk u$. The section u vanishes along a subscheme W_u of $\mathbb{P}^2$ of length $m = n(n-1)/2$ which is contained in G_0 and we have a resolution:

$$0 \to U_u \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_{W_u/\mathbb{P}^2}(n-1) \to 0. \tag{22}$$

Since M_L is semigeneric, there are two 1-dimensional marked subspaces of V and U^*, corresponding to the first row and column of M. We choose $u \in U^*$ to lie in this space, so that W_u is a subscheme of half the length of $G_0 \cap G_1 = W$.

Set $\mathfrak{e}_u$ for the union of the m components $(\mathfrak{e}_1,\dots,\mathfrak{e}_m)$ of $\mathfrak{e}_1 + \dots + \mathfrak{e}_{n(n-1)} = \mathfrak{e}_\pi$ which are contracted to W_u by π and write $\mathfrak{e}_{\bar{u}} = \mathfrak{e}_\pi - \mathfrak{e}_u$. Put $\hat{\mathcal{L}} = \mathcal{O}_{\hat{S}}((n-1)\mathfrak{l} - \mathfrak{e}_u)$. Pulling back (22) to $\hat{S}$ via π and removing the torsion part $\mathcal{O}_{\mathfrak{e}_u}(-1)$ of $\pi^*(\mathcal{I}_{W_u/\mathbb{P}^2}(n-1))$ we get the exact sequences:

$$0 \to \mathcal{K} \to V \otimes \mathcal{O}_{\hat{S}} \to \hat{\mathcal{L}} \to 0, \tag{23}$$

$$0 \to U_u \otimes \mathcal{O}_{\hat{S}}(-\mathfrak{l}) \to \mathcal{K} \to \mathcal{O}_{\mathfrak{e}_u}(-1) \to 0. \tag{24}$$

where the surjection in the first sequence is the natural evaluation of global sections of $\hat{\mathcal{L}}$ and $\mathcal{K}$ is defined as the kernel of this map. Since $\mathcal{O}_{\mathfrak{e}_u}(\mathfrak{h}) \simeq \mathcal{O}_{\mathfrak{e}_u}(1)$, we have:

$$0 \to \mathcal{O}_{\hat{S}}(-\mathfrak{h} - \mathfrak{e}_u) \to \mathcal{O}_{\hat{S}}(-\mathfrak{h}) \to \mathcal{O}_{\mathfrak{e}_u}(-1) \to 0.$$

Observe that $\mathcal{O}_{\hat{S}}(-\mathfrak{h} - \mathfrak{e}_u) \simeq \hat{\mathcal{L}}((1-n)\mathfrak{l} - \mathfrak{h})$. By (18) we have $\mathrm{H}^1(\mathcal{O}_{\hat{S}}(\mathfrak{h} - \mathfrak{l})) = 0$, hence the above surjection lifts to $\mathcal{O}_{\hat{S}}(-\mathfrak{h}) \to \mathcal{K}$. Patching this with (24) we get:

$$0 \to \hat{\mathcal{L}}((1-n)\mathfrak{l} - \mathfrak{h}) \to \mathcal{O}_{\hat{S}}(-\mathfrak{h}) \oplus U_u \otimes \mathcal{O}_{\hat{S}}(-\mathfrak{l}) \to \mathcal{K} \to 0. \tag{25}$$

We rewrite this as a long exact sequence:

$$0 \to \hat{\mathcal{L}}((1-n)\mathfrak{l} - \mathfrak{h}) \to \mathcal{O}_{\hat{S}}(-\mathfrak{h}) \oplus U_u \otimes \mathcal{O}_{\hat{S}}(-\mathfrak{l}) \to V \otimes \mathcal{O}_{\hat{S}} \to \hat{\mathcal{L}} \to 0, \tag{26}$$

where the sheaf $\mathcal{K}$ is the image of the middle map. Such map lifts to the threefold T and we have the determinantal representation of $\hat{\mathcal{L}}$:

$$0 \to \mathcal{O}_T(-\mathfrak{h}) \oplus U_u \otimes \mathcal{O}_T(-\mathfrak{l}) \to V \otimes \mathcal{O}_T \to \hat{\mathcal{L}} \to 0. \tag{27}$$

This is precisely (21). Also, we have $\sigma_*(\mathcal{O}_T(-\mathfrak{l})) \simeq \sigma_*(\mathcal{O}_T(-\mathfrak{h})) \simeq \mathcal{O}_L(-1)$ and $\sigma_*(\mathcal{O}_T) \simeq \mathcal{O}_L$. The functor σ_* sends maps $\mathcal{O}_T(-\mathfrak{h}) \to \mathcal{O}_T$ to linear forms and maps $\mathcal{O}_T(-\mathfrak{l}) \to \mathcal{O}_T$ to linear forms which vanish at p_0 and each coefficient of the matrix appearing in (21) is mapped via σ_* to the corresponding coefficient of M. Therefore, σ_* sends $\hat{\mathcal{L}}$ to $\mathcal{L}$ and the lemma is proved. □

3.4.3. *The rigid curve.* Set $\mathfrak{g}$ for the strict transform of G_1 in $\hat{S}$, so that $\mathfrak{g}$ is smooth and:

$$\mathfrak{g} \in |\mathcal{O}_{\hat{S}}((n-1)\mathfrak{l} - \mathfrak{e}_\pi)|, \qquad \mathfrak{g}^2 = 1-n, \qquad \mathfrak{g}\cdot\mathfrak{h} = 0, \qquad \mathrm{H}^0(\mathcal{O}_{\hat{S}}(\mathfrak{g})) \simeq \Bbbk.$$

More precisely note that $\mathfrak{g} + \mathfrak{l} \equiv \mathfrak{h}$, moreover $\deg(\mathfrak{h}|_{\mathfrak{g}}) = 0$ and $\mathrm{h}^0(\hat{S}, \mathfrak{h}) = 4 > 3 = \mathrm{h}^0(\hat{S}, \mathfrak{l})$, so that $\mathrm{h}^0(\mathfrak{g}, \mathfrak{h}|_{\mathfrak{g}}) \neq 0$. Therefore we have:

$$\mathfrak{g}|_{\mathfrak{g}} \equiv -\mathfrak{l}|_{\mathfrak{g}}. \tag{28}$$

We call $\mathfrak{g}$ the *rigid curve* of $\hat{S}$. We analyze the restriction of $\mathcal{L}$ and $\mathcal{K}$ to the rigid curve. We would like to prove :

$$\mathrm{H}^0(\mathfrak{g}, \mathcal{K}^* \otimes \hat{\mathcal{L}}(\mathfrak{g} - \mathfrak{l})|_{\mathfrak{g}}) = 0.$$

Write $\mathcal{N} = \hat{\mathcal{L}}|_{\mathfrak{g}}$, so $\mathcal{N} \simeq \mathcal{O}_{\mathfrak{g}}((n-1)\mathfrak{l} - e_u)$. Since $\mathfrak{g}|_{\mathfrak{g}} \equiv -\mathfrak{l}|_{\mathfrak{g}}$, it suffices to prove the following lemma.

Lemma 3.8. *We have:*

$$\mathrm{H}^0(\mathfrak{g}, \mathcal{K}^*|_{\mathfrak{g}} \otimes \mathcal{N}(-2\mathfrak{l})) = 0. \tag{29}$$

Proof. The divisor $e_u|_{\mathfrak{g}}$ has degree $n(n-1) - m = m$ and consists of m generic points of $\mathfrak{g}$, so $\mathrm{h}^0(\mathfrak{g}, \mathcal{N}) = n-1$ and $\mathrm{h}^1(\mathfrak{g}, \mathcal{N}) = 0$. Note that the defining equation of the curve G_0 corresponds to the first element v of the chosen basis of the space of curves V of degree $n-1$ through W_u. Hence, setting $V_v = V/\Bbbk v$, we get an identification $\mathrm{H}^0(\mathfrak{g}, \mathcal{N}) = V_v$ and restricting (23) we get:

$$0 \to \mathcal{K}_0 \to V_v \otimes \mathcal{O}_{\mathfrak{g}} \to \mathcal{N} \to 0, \qquad \mathcal{K}|_{\mathfrak{g}} \simeq \mathcal{K}_0 \oplus \mathcal{O}_{\mathfrak{g}}.$$

Here, the sheaf $\mathcal{K}_0$ is defined by the sequence and the copy of $\mathcal{O}_{\mathfrak{g}}$ sits in $V \otimes \mathcal{O}_{\mathfrak{g}}$ as the line spanned by v. Since $\mathrm{H}^0(\mathfrak{g}, \mathcal{N}(-2\mathfrak{l})) = 0$, we have to show:

$$\mathrm{H}^0(\mathfrak{g}, \mathcal{K}_0^* \otimes \mathcal{N}(-2\mathfrak{l})) = 0.$$

Restricting (25) to $\mathfrak{g}$ and using $\mathcal{O}_{\mathfrak{g}}(-\mathfrak{h}) \simeq \mathcal{O}_{\mathfrak{g}}$ we get:

$$0 \to \mathcal{N}((1-n)\mathfrak{l}) \to U_u \otimes \mathcal{O}_{\mathfrak{g}}(-\mathfrak{l}) \to \mathcal{K}_0 \to 0. \tag{30}$$

We may summarize this in the following long exact sequence:

$$0 \to \mathcal{N}((1-n)\mathfrak{l}) \to U_u \otimes \mathcal{O}_{\mathfrak{g}}(-\mathfrak{l}) \to V_v \otimes \mathcal{O}_{\mathfrak{g}} \to \mathcal{N} \to 0.$$

This gives a presentation:

$$0 \to U_u \otimes \mathcal{O}_{\hat{S}}(-\mathfrak{l}) \to V_v \otimes \mathcal{O}_{\hat{S}} \to \mathcal{N} \to 0.$$

In particular $\mathrm{H}^0(\mathfrak{g}, \mathcal{N}(-\mathfrak{l})) = 0$. We also note that $\mathcal{N} \simeq \pi^*(\mathrm{coker}(M_1))$ and that the above sequence is the pull-back to $\hat{S}$ via π of:

$$0 \to (n-1).\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{M_1} (n-1).\mathcal{O}_{\mathbb{P}^2} \to \mathrm{coker}(M_1) \to 0.$$

Applying $\mathcal{H}om_{\mathfrak{g}}(-, \mathcal{N}_{\mathfrak{g}}(-2\mathfrak{l}))$ to (30) we get:

$$0 \to \mathcal{K}_0^* \otimes \mathcal{N}(-2\mathfrak{l}) \to U_u^* \otimes \mathcal{N}(-\mathfrak{l}) \to \mathcal{O}_{\mathfrak{g}}((n-3)\mathfrak{l}) \to 0.$$

Since $\mathrm{H}^0(\mathfrak{g}, \mathcal{N}(-\mathfrak{l})) = 0$, we get $\mathrm{H}^0(\mathfrak{g}, \mathcal{K}_0^* \otimes \mathcal{N}(-2\mathfrak{l})) = 0$ and we are done. □

3.4.4. *Matrix factorization and the proof of Proposition 3.5.* Restricting M_L to S, by matrix factorization we get:

$$0 \to \mathcal{L}(-n) \to U \otimes \mathcal{O}_S(-1) \to V \otimes \mathcal{O}_S \to \mathcal{L} \to 0. \tag{31}$$

Recall that $\deg(S) = n$ and use [KMR2005, Lemma 3.2] to get:

$$\mathrm{Hom}_S(\mathcal{L}(-n), \mathcal{L}(-1)) \simeq \mathrm{H}^0(S, \mathcal{O}_S(n-1)) \simeq \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-1)).$$

Therefore, applying $\mathrm{Hom}_S(-, \mathcal{L}(-1))$ to the inclusion $\mathcal{L}(-n) \to U \otimes \mathcal{O}_S(-1)$ appearing in (31) and recalling that we set $V = \mathrm{H}^0(S, \mathcal{L})$, we get a linear map:

$$\mathrm{Hom}(U, V) = \mathbf{A}^* \to \mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-1)). \tag{32}$$

Proof of Proposition 3.5. First observe that, since the degree of x_0 in any of the minors defining I_L is at most 1, we have $I_L \subset x_0\mathfrak{m}_0^{n-2} + \mathfrak{m}_0^{n-1}$. Also, both I_L and $x_0\mathfrak{m}_0^{n-2} + \mathfrak{m}_0^{n-1}$ are generated by n^2 forms of degree $n-1$, those of $x_0\mathfrak{m}_0^{n-2} + \mathfrak{m}_0^{n-1}$ being linearly independent. Hence it suffices to prove that the n^2 generators of I_L are also linearly independent.

The linear span of the n^2 minors under consideration is the image of the map (32), so we have to prove that this map is injective.

Set $\mathcal{L}' = \hat{\mathcal{L}}((n-2)\mathfrak{h} + (1-n)\mathfrak{l})$. Using (18), we deduce from (27):

$$\mathrm{H}^0(\mathcal{L}'(\mathfrak{h})) \simeq \mathrm{H}^0(\mathcal{L}'(\mathfrak{l})) \simeq V.$$

Therefore, we get an identification:

$$\mathrm{Hom}_{\hat{S}}(\mathcal{O}_{\hat{S}}(-\mathfrak{h}) \oplus U_u \otimes \mathcal{O}_{\hat{S}}(-\mathfrak{l}), \mathcal{L}') \simeq U^* \otimes V = \mathbf{A}^*.$$

Also, we have:

$$\mathrm{Hom}_{\hat{S}}(\hat{\mathcal{L}}((1-n)\mathfrak{l}-\mathfrak{h}),\mathcal{L}') \simeq \mathrm{H}^0(\mathcal{O}_{\hat{S}}((n-1)\mathfrak{h})) \simeq \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(n-1)).$$

Now, applying $\mathrm{Hom}_{\hat{S}}(-,\mathcal{L}')$ to (25) we get an exact sequence:

$$0 \to \mathrm{Hom}_{\hat{S}}(\mathcal{K},\mathcal{L}') \to \mathbf{A}^* \to \mathrm{H}^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(n-1)) \to \mathrm{Ext}^1_{\hat{S}}(\mathcal{K},\mathcal{L}').$$

In view of Lemma 3.7, the sequence (31) is the image via σ_* of (26), in particular the middle map of the above sequence is identified via σ_* with the map (32). Thus we are reduced to prove $\mathrm{Hom}_{\hat{S}}(\mathcal{K},\mathcal{L}')=0$, that is:

$$\mathrm{H}^0(\hat{S},\mathcal{K}^*\otimes\mathcal{L}') = 0. \tag{33}$$

To do this, we use the rigid curve $\mathfrak{g} \equiv \mathfrak{h}-\mathfrak{l}$. Note that:

$$\mathcal{L}' \simeq \hat{\mathcal{L}}((n-2)\mathfrak{g}-\mathfrak{l}).$$

By (28), for all integer j we have:

$$0 \to \mathcal{O}_{\hat{S}}((j-1)\mathfrak{g}-\mathfrak{l}) \to \mathcal{O}_{\hat{S}}(j\mathfrak{g}-\mathfrak{l}) \to \mathcal{O}_{\mathfrak{g}}(-(j+1)\mathfrak{l}) \to 0.$$

Since $\mathfrak{l}|_{\mathfrak{g}}$ is effective, using induction on j with $1\le j\le n-2$, to show (33) it suffices to prove:

$$\mathrm{H}^0(\hat{S},\mathcal{K}^*\otimes\hat{\mathcal{L}}(-\mathfrak{l}))=0, \qquad \mathrm{H}^0(\mathfrak{g},\mathcal{K}^*\otimes\hat{\mathcal{L}}(-2\mathfrak{l})|_{\mathfrak{g}})=0.$$

The second vanishing is precisely Lemma 3.8 so we only need to show the first one. But this follows by looking at the dual of (23), tensored with $\hat{\mathcal{L}}(-\mathfrak{l})$, which reads:

$$0 \to \mathcal{O}_{\hat{S}}(-\mathfrak{l}) \to V^*\otimes\hat{\mathcal{L}}(-\mathfrak{l}) \to \mathcal{K}^*\otimes\hat{\mathcal{L}}(-\mathfrak{l}) \to 0,$$

so since $\mathrm{H}^0(\hat{S},\mathcal{L}(-\mathfrak{l})) = \mathrm{H}^1(\hat{S},\mathcal{O}_{\hat{S}}(-\mathfrak{l}))=0$, we get $\mathrm{H}^0(\hat{S},\mathcal{K}^*\otimes\hat{\mathcal{L}}(-\mathfrak{l}))=0$. This completes the proof of Proposition 3.5. □

3.5. **Quadratic Lefschetz property.** Consider a linear subspace $L\simeq\mathbb{P}^3\subset\mathbb{P}^N$. We get a projection $R\to R_L$ onto a polynomial ring R_L in 4 variables, denoting as before $R_L\simeq \Bbbk[x_0,\dots,x_3]$.

Choose an integer $n\ge 3$. Since the singular locus $\mathrm{sing}(D)$ has codimension 4, generically L will not meet $\mathrm{sing}(D)$. In this case the image of J_D in R_L defines an Artinian Gorenstein algebra A_L as quotient of $R/(J_D+I_L)$. The algebra $A=A_L$, called the Artinian reduction of R/J_D, inherits a minimal graded free resolution:

$$0\leftarrow A\leftarrow R_L\leftarrow n^2.R_L(1-n)\leftarrow 2(n^2-1).R_L(-n)\leftarrow n^2.R_L(-n-1)\leftarrow R_L(-2n)\leftarrow 0. \tag{34}$$

We say that the algebra A has the degree-k Lefschetz property if, for each graded piece A_t of A, there is an element $h\in A_1$ such that $\cdot h^k : A_t\to A_{t+k}$ has maximal rank.

If $L\cap Z\neq\varnothing$, the algebra $A=A_L$ is no longer Artinian. In the next paragraph we will see how, choosing L in a semigeneric way, the resulting non-Artinian algebra allows to establish the quadratic Lefschetz property for the Artinian algebras $A'=A_{L'}$ given by the generic choice $L'\simeq\mathbb{P}^3$.

Lemma 3.9. *Assume that there is a linear subspace $L=\mathbb{P}^3\subset\mathbb{P}^N$ and a linear form h on L such that $\cdot h^2 : A_{n-3}\to A_{n-1}$ is an isomorphism. Then, for generic choice of $L'\simeq\mathbb{P}^3\subset\mathbb{P}^N$, the Artinian algebra $A'=A_{L'}$ has the quadratic Lefschetz property.*

Proof. Note that, since for a generic choice of L' the Artinian algebra A' has a graded resolution of the form (34), the graded algebra structure of A' and R_L coincide up to degree $n-1 \geq 1$. Therefore, $\cdot h^2 : A'_t \to A'_{t+2}$ has maximal rank for any choice of $0 \neq h \in A'_1$ and all $t \in \{0, \dots, n-4\}$. By Gorenstein duality, the same happens for $t \in \{n-2, \dots, 2n-6\}$, indeed A' has socle degree $2n-4$. Therefore, A' has the quadratic Lefschetz property if there is $h \in A'_1$ such that $\cdot h^2 : A_{n-3} \to A_{n-1}$ has maximal rank. Because $\dim(A_{n-1}) = \binom{n+2}{3} - n^2 = \binom{n}{3} = \dim(A_{n-3})$, this amounts to ask that $\cdot h^2 : A_{n-3} \to A_{n-1}$ is an isomorphism. Since, by our assumption, this holds for a special choice of the linear space L and the element $h \in A_1$, it also holds for a generic choice of the linear space L' and the element $h \in A'_1$. Therefore A' has the quadratic Lefschetz property, as required.

□

3.6. **Vector bundle of principal parts on a quadric surface.** In this section, we assume $\mathrm{char}(\Bbbk) = 0$. Consider the Segre product $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and, for any $(a_1, a_2) \in \mathbb{Z}^2$, put $\mathcal{O}_X(a_1, a_1) = p_1^* \mathcal{O}_{\mathbb{P}^1}(a_1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(a_2)$, where p_1 and p_2 are the two projections of X onto its two $\mathbb{P}^1$ factors. Write $U_1 = \mathrm{H}^0(X, \mathcal{O}_X(1,0))$ and $U_2 = \mathrm{H}^0(X, \mathcal{O}_X(0,1))$ and consider, for $n \in \mathbb{N}$, the sheaf of principal parts $\mathcal{E}_n$ defined as kernel of the natural evaluation:

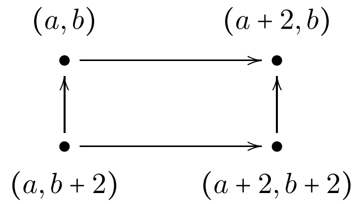
$$\mathcal{E}_n = \ker\left(S^n U_1 \otimes S^n U_2 \otimes \mathcal{O}_X \to \mathcal{O}_X(n,n)\right).$$

The goal of this subsection is to prove the following result.

Proposition 3.10. *For any integer $n \geq 3$, the sheaf $\mathcal{E} = \mathcal{E}_n$ is slope-stable.*

Set $G = \mathrm{SL}_2(\Bbbk) \times \mathrm{SL}_2(\Bbbk)$ and let P be the subgroup of G consisting of pairs of upper triangular matrices. Then $X \simeq G/P$. We will use a special case of the equivalence of categories of linearized G-equivariant vector bundles on $X = G/P$ and of finite-dimensional representations of P. In turn, following [BK1990, Hil1998, OR2006], these categories are equivalent to the category of representations of the quiver with relations $\mathcal{Q}_X$ that we describe below. For the case under consideration of $X = \mathbb{P}^1 \times \mathbb{P}^1$, this takes place in spite of the fact that the group G is not strictu sensu of type A, D, E.

Indeed, consider the quiver $\mathcal{Q}_X$, whose vertices are defined by the irreducible representations of the semisimple part of P, isomorphic to $\Bbbk^* \times \Bbbk^*$. The weight function gives an identification of the vertice $\mathcal{Q}_X$ and points of the lattice $\mathbb{Z}^2$, independently of $\mathrm{char}(\Bbbk)$. In terms of sheaves over X, a vertex $\lambda = (a,b) \in \mathbb{Z}^2$ of $\mathcal{Q}_X$ is given by $\mathcal{O}_X(\lambda)$. The arrows of $\mathcal{Q}_X$ are determined by the invariant part of the extensions between representations. Namely, there is an arrow from $\lambda = (a,b) \in \mathbb{Z}^2$ to $\mu = (c,d) \in \mathbb{Z}^2$ if $\mathrm{Ext}^1_X(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu))^G \neq 0$, which happens if and only if $a - c = d - b + 2$. Note that in this case we have $\mathrm{Ext}^1_X(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu))^G = \Bbbk$. The infinite quiver $\mathcal{Q}_X$ has four connected components, characterized by the fact of containing $\mathcal{O}_X$, or $\mathcal{O}_X(1,0)$, or $\mathcal{O}_X(0,1)$, or $\mathcal{O}_X(1,1)$. There relations of the quiver $\mathcal{Q}_X$ are given by imposing commutativity of all the square diagrams of the following form:



Given a G-homogeneous bundle E, there exists a G-equivariant filtration:

$$0 \subset E_1 \subset \cdots \subset E_k = E$$

such that E_i/E_{i-1} is a line bundle. The *associated graded bundle* is defined as:

$$\mathrm{gr}(E) = \bigoplus_i E_i/E_{i-1},$$

and does not depend on the chosen filtration. Write the graded bundle as

$$\mathrm{gr}(E) = \bigoplus_{\lambda \in \mathbb{Z}^2} V_\lambda . \mathcal{O}_X(\lambda),$$

where V_λ is a $\Bbbk$-vector space whose rank is the number of copies of $\mathcal{O}_X(\lambda)$ in $\mathrm{gr}(E)$. The portion of $\mathcal{Q}_X$ whose vertices λ satisfy $V_\lambda \neq 0$ is called the *support* of E and denoted by $\mathrm{supp}(E)$. The G-action on E determines a linear map $V_\lambda \to V_\mu$ for all λ, μ in $\mathrm{supp}(E)$ satisfying $\mathrm{Ext}^1_X(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu))^G = \Bbbk$.

Given a homogeneous bundle E, we denote by $[E]$ the corresponding representation and we talk indifferently of the support of E or of $[E]$.

Lemma 3.11. *We have that*

$$\mathrm{gr}\left(S^n U_1 \otimes S^n U_2 \otimes \mathcal{O}_X\right) = \bigoplus_{t,k \in [\![0,n]\!]} \mathcal{O}_X(-n+2k, -n+2t).$$

Proof. First start by computing $\mathrm{gr}(S^n U_1 \otimes \mathcal{O}_X)$ by induction, observing that, for $n = 1$, we have an $\mathrm{SL}_2(\Bbbk)$-equivariant exact sequence:

$$0 \to \mathcal{O}_X(-1,0) \to U_1 \otimes \mathcal{O}_X \to \mathcal{O}_X(1,0) \to 0. \tag{35}$$

This gives rise, for any n, to:

$$0 \to S^{n-1}U_1 \otimes \mathcal{O}_X(-1,0) \to S^n U_1 \otimes \mathcal{O}_X \to \mathcal{O}_X(n,0) \to 0$$

We get the following:

$$\mathrm{gr}(S^n U_1 \otimes \mathcal{O}_X) = \bigoplus_{k \in [\![0,n]\!]} \mathcal{O}_X(-n+2k, 0).$$

Analogously, we have that

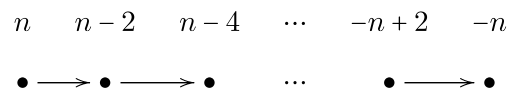
$$\mathrm{gr}(S^n U_2 \otimes \mathcal{O}_X) = \bigoplus_{t \in [\![0,n]\!]} \mathcal{O}_X(0, -n+2t)$$

The proof is achieved observing that, for any pair of G-homogeneous bundles E and F, we have $\mathrm{gr}(E \otimes F) = \mathrm{gr}(E) \otimes \mathrm{gr}(F)$. $\square$

The previous lemma accounts for the vertices in $\mathrm{supp}(\mathcal{E})$, which are:

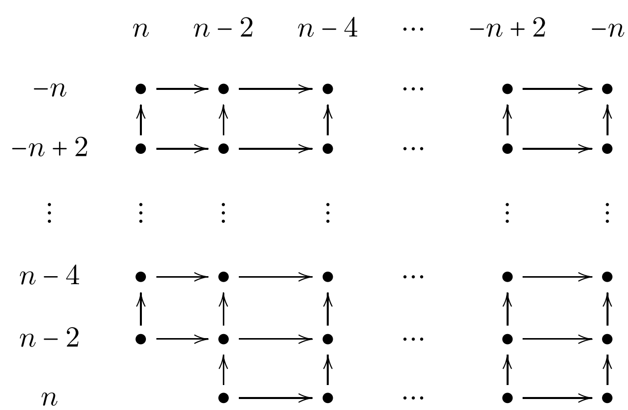
$$(n,n) - \{2(k,t) \mid (0,0) \neq (k,t) \in [\![0,n]\!] \times [\![0,n]\!]\}.$$

Let us look at the arrows of $[\mathcal{E}]$. We start by observing that the linear map in $\mathcal{Q}_X$ arising from (35) is non-zero. More generally, the support of $S^n U_1 \otimes \mathcal{O}_X$ is:



The arrows correspond to elements of $\mathrm{Ext}^1_X(\mathcal{O}_X(a+2,0), \mathcal{O}_X(a,0))$. Note that all maps in $\mathcal{Q}_X$ associated with $[S^n U_1 \otimes \mathcal{O}_X]$ are non-zero. We get the following picture for $\mathrm{supp}(\mathcal{E})$, where all the associated linear maps are non-zero.

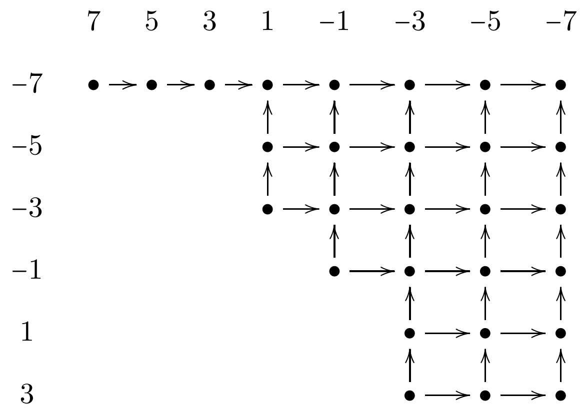
(36)



Here, the side labels denote the degrees of the summand $\mathcal{O}_X(a,b)$ in the associated graded bundle. Moreover, the horizontal (resp. vertical) arrows are determined by $\mathrm{Ext}^1(\mathcal{O}_X(a+2,b), \mathcal{O}_X(a,b))$ (resp. $\mathrm{Ext}^1(\mathcal{O}_X(a,b+2), \mathcal{O}_X(a,b))$). We will call *main diagonal* of the support the set of vertices of the form $(a,-a)$.

Consider a subrepresentation $[\mathcal{E}']$ of $[\mathcal{E}]$. Note that all arrows of $[\mathcal{E}]$ are isomorphisms and every vertex in $\lambda \in \mathrm{supp}(\mathcal{E})$ is connected to another vertex to the right of λ or above λ until reaching $(-n,-n)$. Then the main observation is that, if a vertex $\lambda_1 = (a_1,b_1)$ is in the support of $[\mathcal{E}']$, then every vertex of $\mathrm{supp}(\mathcal{E})$ to right of λ_1 or above λ_1 is also in the support of $[\mathcal{E}']$, that is $(a_2,b_2) \in \mathrm{supp}(\mathcal{E}')$ if $-n \leq a_2 \leq a_1$ and $-n \leq b_2 \leq b_1$. Therefore, $[\mathcal{E}']$ is completely described by means of its *boundary vertices*, namely the vertices (a,b) in $\mathrm{supp}(\mathcal{E}')$, such that neither $(a+2,b)$ nor $(a,b+2)$ is in $\mathrm{supp}(\mathcal{E}')$.

Example 3.12. Let us consider $n = 7$ and a homogeneous bundle $\mathcal{E}'$ whose representation has the following support.



If $[\mathcal{E}']$ is a subrepresentation of $[\mathcal{E}]$, then all arrows must be non-zero. The boundary vertices are given by the four vertices of the quiver indexed by $(7,-7),(1,-3),(-1,-1)$ and $(-3,3)$.

Let us introduce stability with respect to the line bundle $\mathcal{O}_X(1,1)$ in terms of representations of $\mathcal{Q}_X$ according to [Kin1994]. For $\lambda = (a,b) \in \mathbb{Z}^2$, put $c_1([V_\lambda \otimes \mathcal{O}_X(\lambda)]) = \mathrm{rk}(V_\lambda)(a+b)$ and $\mathrm{rk}([V_\lambda \otimes \mathcal{O}_X(\lambda)]) = \mathrm{rk}(V_\lambda)$. For a G-homogeneous bundle E on X, define $c_1([E]) = c_1([\mathrm{gr}(E)])$ and $\mathrm{rk}(E) = \mathrm{rk}([\mathrm{gr}(E)])$ by linearity. For every G-homogeneous bundle E' we put:

$$\mu_E([E']) = c_1([E])\mathrm{rk}([E']) - \mathrm{rk}([E])c_1([E']).$$

The representation $[E]$ is called *stable* if for all subrepresentations $[E']$ we have that $\mu_E([E']) \geq 0$ and the equality holds if only if $[E']$ is either $[E]$ or $[0]$.

For the G-homogeneous bundle $\mathcal{E}$, the stability of the representation $[\mathcal{E}]$ is equivalent to the slope-stability of $\mathcal{E}$ itself. Indeed, the proof of [OR2006, Theorem 7.2] can be adapted to the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$ to show that the representation $[\mathcal{E}]$ is slope-stable if and only if $\mathcal{E} \simeq W \otimes \mathcal{E}'$, where W is an irreducible G-module and $\mathcal{E}'$ is a slope-stable G-homogeneous bundle on X. Since $\mathrm{H}^1(\mathcal{E}(-n,-n)) \simeq \Bbbk$, we must then have $W \simeq \Bbbk$ and $\mathcal{E} \simeq \mathcal{E}'$.

To conclude that $\mathcal{E}$ is slope-stable, we need only show that $[\mathcal{E}]$ is stable, which we do in the next result.

Lemma 3.13. *For any subrepresentation $[\mathcal{E}']$, we have that $\mu_{\mathcal{E}}([\mathcal{E}']) \geq 0$. Moreover, $\mu_{\mathcal{E}}([\mathcal{E}']) = 0$ if only if either $[\mathcal{E}'] = [\mathcal{E}]$ or $[\mathcal{E}'] = [0]$.*

Proof. Let $[\mathcal{E}']$ be a non-zero subrepresentation of $[\mathcal{E}]$. We have:

$$\mu_{\mathcal{E}}([\mathcal{E}']) = \sum_{(a,b)\in \mathrm{supp}(\mathcal{E}')} (c_1([\mathcal{E}]) - \mathrm{rk}([\mathcal{E}])(a+b)) = \sum_{(a,b)\in \mathrm{supp}(\mathcal{E}')} \left(-2n + (1-n^2)(a+b)\right).$$

For a point $\lambda = (a,b) \in \mathbb{Z}^2$, write $\tau(\lambda) = (-b,-a)$, so that τ is the reflection along the main diagonal in (36). Any vertex $\lambda = (a,b) \in \mathrm{supp}(\mathcal{E}')$ satisfies $a+b = 2t$ for some $t \in [\![-n,n]\!]$. Write $\mathrm{supp}_t(\mathcal{E}') = \{(a,b) \in \mathrm{supp}(\mathcal{E}') \mid a+b = 2t\}$. We get:

$$\frac{1}{2}\mu_{\mathcal{E}}([\mathcal{E}']) = -n\left|\mathrm{supp}(\mathcal{E}')\right| + \sum_{t\in[\![-n,n]\!]} \sum_{\lambda\in \mathrm{supp}_t(\mathcal{E}')} (1-n^2)t. \tag{37}$$

Now recall that, for any $\lambda \in \operatorname{supp}(\mathcal{E}')$, the vertices of $\operatorname{supp}(\mathcal{E})$ to right of λ or above λ are also in $\operatorname{supp}(\mathcal{E}')$. Therefore, for any $\lambda \in \operatorname{supp}_t(\mathcal{E}')$ with $t \geq 0$, the vertex $\tau(\lambda)$ also lies in $\operatorname{supp}(\mathcal{E}')$, more precisely $\tau(\lambda) \in \operatorname{supp}_{-t}(\mathcal{E}')$. Note that the two terms $(1-n^2)t$ in the summation (37) arising from a pair $(\lambda, \tau(\lambda)) \in \operatorname{supp}_t(\mathcal{E}') \times \operatorname{supp}_{-t}(\mathcal{E}')$ add up to zero so we may restrict the summation to the vertices $\lambda \in \operatorname{supp}(\mathcal{E}')$ such that $\tau(\lambda)$ does not lie in $\operatorname{supp}(\mathcal{E}')$. In turn this can happen only if $\lambda \in \operatorname{supp}_{-t}(\mathcal{E}')$ with $t \geq 1$. Set $\mathcal{V}_t(\mathcal{E}')$ for the set of vertices $\lambda \in \operatorname{supp}_{-t}(\mathcal{E}')$ with $\tau(\lambda) \notin \operatorname{supp}(\mathcal{E}')$. Hence we rewrite (37) as:

$$\frac{1}{2}\mu_{\mathcal{E}}([\mathcal{E}']) = -n\left|\operatorname{supp}(\mathcal{E}')\right| - \sum_{t \in [\![1,n]\!]} \sum_{\lambda \in \mathcal{V}_t(\mathcal{E}')} (1-n^2)t.$$

We have $|\operatorname{supp}(\mathcal{E}')| \leq n^2 - 1$ so:

$$\frac{1}{2}\mu_{\mathcal{E}}([\mathcal{E}']) \geq -n(n^2-1) - \sum_{t \in [\![1,n]\!]} \sum_{\lambda \in \mathcal{V}_t(\mathcal{E}')} (1-n^2)t =$$

$$= (n^2-1) \sum_{t \in [\![1,n]\!]} \left(-1 + \sum_{\lambda \in \mathcal{V}_t(\mathcal{E}')} t \right).$$

Note that, since $[\mathcal{E}']$ is non-zero, we must have $(-n,-n) \in \mathcal{V}_n(\mathcal{E}')$, hence:

$$\frac{1}{2}\mu_{\mathcal{E}}([\mathcal{E}']) \geq (n^2-1)\left(n - 1 + \sum_{t \in [\![1,n-1]\!]} \left(-1 + \sum_{\lambda \in \mathcal{V}_t(\mathcal{E}')} t \right) \right) \geq$$

$$\geq (n^2-1)\left(n - 1 + (1-n)\right) = 0,$$

where the last inequality simply follows from the fact that $t \geq 0$.

We have thus proved that $[\mathcal{E}]$ is semistable. Moreover, if equality is attained in the above displays, then we must have $|\operatorname{supp}(\mathcal{E}')| = n^2 - 1$ which implies that $[\mathcal{E}']$ is equal to $[\mathcal{E}]$. □

3.7. **Proof of Theorem E.** This section is again in arbitrary characteristic. All the ingredients to prove Theorem E are ready. According to Proposition 3.10, the sheaf $\mathcal{E}_{n-1}$ is slope-stable in characteristic zero. So it suffices to see that $\mathcal{T}_D$ restricts over X to $\mathcal{E}_{n-1}$, for a generic quadric hypersurface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, of a linear 3-dimensional subspace of $\mathbb{P}^N$. We show next that this is indeed the case (in arbitrary characteristic).

By Lemma 3.9 we only need to show that there is a linear space $L \simeq \mathbb{P}^3 \subset \mathbb{P}^N$ and a linear form h over L such that, in the resulting algebra $A = A_L = R_L/I_L$, the multiplication $\cdot h^2 : A_{n-3} \to A_{n-1}$ is an isomorphism.

Choosing L to be semigeneric in the sense of §3.4 and $h = x_0$, by Proposition 3.5 we get that $\cdot x_0^2 : A_{n-3} \to A_{n-1}$ is injective since there is no polynomial involving x_0^2 in the graded piece of degree $n-1$ of I_L. Moreover, we observed that this graded piece has dimension n^2 so again $\dim(A_{n-3}) = \dim(A_{n-1}) = \binom{n}{3}$ and therefore $\cdot x_0^2 : A_{n-3} \to A_{n-1}$ is an isomorphism.

Since $\cdot h^2 : A_{n-3} \to A_{n-1}$ is an isomorphism for a given choice of a linear form h in R_L, then, by Lemma 3.9, we get an isomorphism $\cdot g : A_{n-3} \to A_{n-1}$ also for a generic choice of a quadric form g in R_L.

Next, considering $\mathcal{T}_D|_L$ we get the fundamental relation:

$$\mathrm{H}^1(\mathcal{T}_D|_L(t)) \simeq A_{t+n-1}, \qquad \text{for all } t \in \mathbb{Z},$$

and these isomorphisms are compatible with the R-module structure. Then, we compute the cohomology of the restriction of $\mathcal{T}_D$ to the quadric surface X defined in L by the form g, for $t \le 0$ by the diagram:

$$\begin{array}{ccccccccc} 0 \to \mathrm{H}^0(\mathcal{T}_D|_X(t)) & \to & \mathrm{H}^1(\mathcal{T}_D|_L(t-2)) & \xrightarrow{\cdot g} & \mathrm{H}^1(\mathcal{T}_D|_L(t)) & \to & \mathrm{H}^1(\mathcal{T}_D|_X(t)) \to 0 \\ & & \| & & \| & & \\ & & A_{t+n-3} & \xrightarrow{\cdot g} & A_{t+n-1}, & & \end{array}$$

For $t = 0$ we get, by our assumption, $\mathrm{H}^0(\mathcal{T}_D|_X) = \mathrm{H}^1(\mathcal{T}_D|_X) = 0$. It follows as in the proof of Theorem D, see the paragraph following (8), that $\mathcal{T}_D|_X$ is isomorphic to $\mathcal{E}_{n-1}$.

This concludes the proof of Theorem E.

4. Families of determinants

Let $n \ge 3$ be an integer. In view of Theorem E, we know that the logarithmic sheaf $\mathcal{T}_D$ associated to the tautological determinant $D = D_{\mathbf{i}}$ of the n-matrix of variables is a slope-stable reflexive sheaf on $\mathbb{P}^N$. Denote by $\mathfrak{M}_n$ the moduli space of simple sheaves on $\mathbb{P}^N$ containing $\mathcal{T}_D$ or, in characteristic zero, the moduli space of stable sheaves on $\mathbb{P}^N$ containing $\mathcal{T}_D$. Our goal is to describe a dense open piece of this moduli space as a certain group quotient.

4.1. **Moduli space and group quotient.** In view of the setup of §3.1.2, for any $\mathbf{f} \in \mathrm{End}_{\Bbbk}(\mathbf{A})$ we may consider $\det(M_{\mathbf{f}})$ as en element of $S^n\mathbf{A}$. We get a rational map:

$$\mathbf{det} : \mathbb{P}(\mathrm{End}_{\Bbbk}(\mathbf{A})) \dashrightarrow \mathbb{P}(S^n\mathbf{A}),$$

defined at the points where $\det(M_{\mathbf{f}}) \ne 0$. The image of $\mathbf{det}$ is the set of determinantal hypersurfaces of degree n. We denote it by $\mathfrak{D}_n$. Recall that $G = \mathrm{SL}(U) \times \mathrm{SL}(V)$ acts on $\mathbb{P}(\mathrm{End}_{\Bbbk}(\mathbf{A}))$ by left and right composition. In terms of the matrices $M_{\mathbf{f}}$, for $(\mathbf{g}, \mathbf{h}) \in G$, the actions is $M_{(\mathbf{g},\mathbf{h}).\mathbf{f}} = (\mathbf{g}, \mathbf{h}).M_{\mathbf{f}} = \mathbf{h}M_{\mathbf{f}}\mathbf{g}^{-1}$. The determinant is fixed by this action, so we have a map:

$$\underline{\mathbf{det}} : \mathbb{P}(\mathrm{End}_{\Bbbk}(\mathbf{A}))/G \dashrightarrow \mathbb{P}(S^n\mathbf{A}),$$

whose image is again $\mathfrak{D}_n$. Recall that we put $D_{\mathbf{f}} = \mathbb{V}(\det(M_{\mathbf{f}}))$.

Lemma 4.1. *The following are equivalent:*

i) $\mathbf{f} \in \mathrm{GL}(\mathbf{A})$,

ii) $\mathcal{T}_{D_{\mathbf{f}}}$ *has no trivial direct summand,*

iii) $\mathcal{T}_{D_{\mathbf{f}}}$ *is simple,*

and, if $\mathrm{char}(\Bbbk) = 0$ *these conditions are equivalent to:*

iv) $\mathcal{T}_{D_{\mathbf{f}}}$ *is semistable.*

v) $\mathcal{T}_{D_{\mathbf{f}}}$ *is slope-stable.*

Proof. In arbitrary characteristic, we have i) ⇒ iii) by Lemma 3.3. Also, iii) implies ii).

Let us check that ii) implies i). If $\mathbf{f} \in \mathrm{End}_{\Bbbk}(\mathbf{A}) \setminus \mathrm{GL}(\mathbf{A})$, then up to choosing a suitable basis of $\mathbf{A}$, the entries of the associated matrix $M_{\mathbf{f}}$ will not form a basis of $\mathbf{A}$. In other words there is a choice of coordinates or R such that not all the variables $(x_{i,j})_{(i,j)\in[\![1,n]\!]^2}$ occur in $M_{\mathbf{f}}$, which is to say that $M_{\mathbf{f}}$ is constant in some variable of R. Therefore, the equation of

$D_{\mathbf{f}} = \mathbb{V}(\det(M_{\mathbf{f}}))$ does not depend of this variable. Hence, the partial derivative of $\det(M_{\mathbf{f}})$ with respect to that variable is zero, which in turn implies that the sheaf $\mathcal{T}_{D_{\mathbf{f}}}$ has a trivial direct summand. We have proved ii) $\Rightarrow$ i).

In characteristic zero, the implication i) $\Rightarrow$ v) is essentially Theorem E. Indeed, if $\mathbf{f} \in \mathrm{GL}(\mathbf{A})$ then the entries of $M_{\mathbf{f}}$ form a basis of $\mathbf{A}$. Hence we can consider an appropriate change of coordinates to transform $M_{\mathbf{f}}$ into the matrix of indeterminates $M_{\mathbf{i}} = (x_{i,j})_{(i,j)\in [\![1,n]\!]^2}$, being $\mathbf{i}$ the identity in $\mathrm{GL}(\mathbf{A})$. This manipulation has no consequence on the stability of the associated sheaf and we know that $\mathcal{T}_{D_{\mathbf{i}}}$ is slope-stable, so the sheaf $\mathcal{T}_{D_{\mathbf{f}}}$ is slope-stable. Finally, v) implies ii), iii) and iv) (recall Remark 1.1), while ii) still implies i). ☐

Having this in mind, we note that, since $M_{\mathbf{f}}$ is canonically associated to $\mathbf{f}$ and the formation of $\mathcal{T}_{D_{\mathbf{f}}}$ is functorial, the sheaves $(\mathcal{T}_{D_{\mathbf{f}}} \mid [\mathbf{f}] \in \mathrm{PGL}(\mathbf{A}))$ glue to a coherent sheaf over $\mathrm{PGL}(\mathbf{A})$ and thus yield a moduli map $\mathrm{PGL}(\mathbf{A}) \to \mathfrak{M}_n$. This descends to a moduli map up to the action of the closed subgroup $G = \mathrm{SL}(U) \times \mathrm{SL}(V) \subset \mathrm{PGL}(\mathbf{A})$ and therefore Ψ factors through the map $\underline{\mathbf{det}}$. We write $\mathfrak{D}_n^\circ$ for the set of tautological determinantal hypersurfaces up to a change of basis, that is, the image of $\underline{\mathbf{det}}$ restricted to $\mathrm{PGL}(\mathbf{A})$. We obtain an induced map $\Phi : \mathfrak{D}_n^\circ \to \mathfrak{M}_n$ fitting in the following commutative diagram.

$$\begin{array}{ccc} \mathrm{PGL}(\mathbf{A})/G & \xrightarrow{\quad\Psi\quad} & \mathfrak{M}_n \\ & {\scriptstyle\underline{\mathbf{det}}}\searrow \quad \nearrow{\scriptstyle\Phi} & \\ & \mathfrak{D}_n^\circ & \end{array}$$

4.2. **The DK-Torelli property of the determinant.** We first analyze the map Φ of the above diagram via a Torelli-type result. Note that Proposition 1.3 fails for $D = D_{\mathbf{i}}$. Indeed, $\mathrm{H}^1_*(\mathcal{T}_D) = 0$ so of course we cannot find the elements h_1, h_2 required to apply Proposition 1.3. Moreover, D has singularities of multiplicity $n-1$, for example the point $(1:0:\ldots:0)$. Therefore, we couldn't use anyway [Wan2015] to recover D from the Jacobian ideal of D. In spite of this, the following result shows that D enjoys the DK-Torelli property.

Proposition 4.2. *The map Φ is injective.*

Proof. Consider the tautological determinant $D = D_{\mathbf{i}} = \det(M_{\mathbf{i}})$. We give a closer look to the Gulliksen-Negård complex considered in the proof of Proposition 3.2. Fixing a basis of U and V^* we identify $\mathbf{A} = \mathrm{Hom}_{\Bbbk}(U, V)$ with the vector space $\mathrm{M}_n(\Bbbk)$ of square matrices of size n. Following [BV1988], we write an explicit description of the presentation matrix of $\mathcal{T}_D$, that is, of the map φ appearing in (9). To do this, we consider a non-zero matrix $\mathbf{a} \in \mathrm{M}_n(\Bbbk)$ and we describe φ fibre-wisely over $\mathbf{a}$. Consider the complex:

$$\Bbbk \xhookrightarrow{\iota} \mathrm{M}_n(\Bbbk) \oplus \mathrm{M}_n(\Bbbk) \xrightarrow{\pi} \Bbbk$$

with $\iota(\lambda) = (\lambda \mathbb{1}_n, \lambda \mathbb{1}_n)$ for $\lambda \in \Bbbk$ and $\mathbb{1}_n \in \mathrm{M}_n(\Bbbk)$ the identity matrix and $\pi(\mathbf{a}, \mathbf{b}) = \mathrm{tr}(\mathbf{a} - \mathbf{b})$. The homology of this complex is a vector space of dimension $2(n^2 - 1)$. The map φ is induced at the point corresponding to a matrix $\mathbf{a}$ by:

$$\begin{array}{rccc} \psi_{\mathbf{a}} : & \mathrm{M}_n(\Bbbk) & \longrightarrow & \mathrm{M}_n(\Bbbk) \oplus \mathrm{M}_n(\Bbbk) \\ & \mathbf{b} & \mapsto & (\mathbf{ab}, \mathbf{ba}) \end{array}$$

Up to the choice of a new basis of U and V^*, we may suppose that $\mathbf{a}$ is diagonal. On the other hand, the rank of φ, and hence of $\mathcal{T}_D$ at a diagonal matrix $\mathbf{a}$ can be read off from the expression of $\psi_{\mathbf{a}}$. Indeed, if $\mathbf{a}$ is invertible then $\ker(\psi_{\mathbf{a}})$ is spanned by $\mathbf{a}^{-1}$, while for $\mathbf{a}$ of rank $n-k$, with $k \in [\![1, n-1]\!]$, writing $\mathbf{a} = \mathrm{diag}(\lambda_1, \dots, \lambda_{n-k}, 0, \dots, 0)$ with $\lambda_i \neq 0$ for all $i \in [\![1, n-k]\!]$ we see that $\ker(\psi_{\mathbf{a}})$ consists of matrices $\mathbf{b} = (b_{i,j})$ with $b_{i,j} = 0$ for $i \le n-k$ or $j \le n-k$. Summing up, for $\mathbf{a}$ of rank $n-k$, with $k \in [\![1, n-1]\!]$, we have

$$\mathrm{rk}(\mathcal{T}_D|_{\mathbf{a}}) = n^2 + k^2 - 2.$$

This gives:

$$\left\{\mathbf{a} \in \mathbb{P}^N \mid \mathrm{rk}(\mathbf{a}) = 1\right\} = \left\{\mathbf{a} \in \mathbb{P}^N \mid \mathrm{rk}(\mathcal{T}_D|_{\mathbf{a}}) = 2n^2 - 2n - 1\right\}.$$

In other words, the locus of rank-1 matrices is the support of the Fitting ideal of $\mathcal{T}_D$ defined by the minors of order $2n$ of φ.

Now, the hypersurface D is determined as the variety of $(n-1)$-secant subspaces of dimension $n-2$ to the locus of matrices of rank 1. This says in particular that $\mathcal{T}_D$ determines D as the $(n-1)$-secant variety to the locus where $\mathcal{T}_D$ has rank $2n^2 - 2n - 1$.

After an appropriate change of coordinates, as mentioned before, we get that for every hypersurface $D_{\mathbf{f}} \in \mathfrak{D}_n^\circ$, the associated reflexive sheaf $\mathcal{T}_{D_{\mathbf{f}}}$ determines $D_{\mathbf{f}}$. □

4.3. **The determinant as a $2:1$ cover.** Here we show that the fibre of the map

$$\underline{\mathbf{det}} : \mathrm{PGL}(\mathbf{A})/\mathrm{SL}(U) \times \mathrm{SL}(V) \to \mathfrak{D}_n^\circ$$

consists of 2 distinct points. This result goes back to Frobenius, [Fro1897, §7.1]. It has been extended and recasted in various ways, let us mention [Wat1995, §4], [BGL2014, §8], see also [Die1949, MM1959] We provide a proof for self-containedness and to point out a slightly different approach based on divisor class groups involving Ulrich sheaves, close to the methods of [Wat1995, RV2017].

Proposition 4.3. *The morphism* $\underline{\mathbf{det}}$ *is set-theoretically* $2:1$.

Proof. By the argument of Proposition 4.2, it is enough to prove that the set-theoretic fibre of $\underline{\mathbf{det}}$ at $D = D_{\mathbf{i}} = \det(M_{\mathbf{i}})$ consists of two distinct points. To do this, we look more closely at the geometry of a resolution of singularities $\sigma^+ : D^+ \to D$ and argue that, up to the G-action, the elements $\mathbf{f} \in \mathrm{PGL}(\mathbf{A})$ such that $D_{\mathbf{f}} = D$ are in bijection with effective divisor classes $\mathfrak{l}$ on $D^+_{\mathbf{f}}$ such that:

$$\mathfrak{l} \cdot \mathfrak{h}^{n^2-3} = \binom{n}{2},$$

where $\mathfrak{h}$ is the pull-back to D^+ of the hyperplane class of $D \subset \mathbb{P}^N$. We then show that there are precisely two such divisor classes.

To define D^+, we consider $\mathbb{P}(V)$ and the tautological quotient bundle $\mathcal{Q}_+$ of rank $(n-1)$ on $\mathbb{P}(V)$, defined by the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V^* \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{Q}_+ \to 0.$$

Put $D^+ = \mathbb{P}(U \otimes \mathcal{Q}_+)$. Note that $\mathrm{H}^0(U \otimes \mathcal{Q}_+) \simeq \mathbf{A}$. Geometrically, we have:

$$D^+ = \{([v], [\mathbf{a}]) \in \mathbb{P}(V) \times \mathbb{P}(\mathbf{A}) \mid v \circ \mathbf{a} = 0\}.$$

The linear system associated with the tautological relatively ample divisor $\mathfrak{h}$ defines a birational morphism $\sigma^+ : D^+ \to D$. Denote by $\mathfrak{l}^+$ the pull-back to D^+ of a hyperplane of $\mathbb{P}(V)$ via the bundle map $\pi^+ : D^+ \to \mathbb{P}(V)$. The map σ^+ is an isomorphism away from the singular locus $\operatorname{sing}(D)$ of D which consists of the matrices $\mathbf{a} : U \to V$ of rank at most $n-2$. This locus has codimension 3 in D. The maps π^+ and σ^+ are the restrictions to D^+ of the projections from $\mathbb{P}(V) \times \mathbb{P}(\mathbf{A})$ onto the first and second factor. Note that the generic fibre of σ^+ over $\operatorname{sing}(D)$ is a projective line, so the exceptional locus $\mathfrak{e}^+$ of σ^+ has codimension 2 in D^+. Therefore σ^+ induces an isomorphism:

$$\operatorname{Cl}(D) \simeq \operatorname{Pic}(D^+) \simeq \mathbb{Z}\mathfrak{h} \oplus \mathbb{Z}\mathfrak{l}^+.$$

Note that D^+ is cut in $\mathbb{P}(V) \times \mathbb{P}(\mathbf{A})$ by a linear section, whose Koszul complex reads:

$$0 \to \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})}(-n,-n) \to \cdots \to U \otimes \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})}(-1,-1) \to \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})} \to \mathcal{O}_{D^+} \to 0.$$

Set ${}^{\mathrm{t}}\mathfrak{l}^+ = (n-1)\mathfrak{h} - \mathfrak{l}^+$. From the above complex we compute:

$$\mathrm{H}^0(\mathcal{O}_{D^+}({}^{\mathrm{t}}\mathfrak{l}^+)) \simeq U^*.$$

To see this, for $i \in [\![1,n]\!]$, set $\mathcal{K}_j$ for the image of the j-th differential of the Koszul complex, taking the form:

$$\bigwedge^{j} U \otimes \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})}(-j,-j) \to \bigwedge^{j-1} U \otimes \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})}(1-j,1-j).$$

For all $j \in [\![1,n]\!]$, the Künneth formula gives $\mathrm{H}^p(\mathcal{K}_j(-1,n-1)) = 0$ for $p \in \mathbb{N} \setminus \{j\}$. We obtain:

$$\mathrm{H}^0(\mathcal{O}_{D^+}({}^{\mathrm{t}}\mathfrak{l}^+)) \simeq \mathrm{H}^1(\mathcal{K}_1(-1,n-1)) \simeq \cdots \simeq \mathrm{H}^{n-1}(\mathcal{K}_{n-1}(-1,n-1)) \simeq \bigwedge^{n-1} U \simeq U^*.$$

The linear system $|{}^{\mathrm{t}}\mathfrak{l}^+|$ gives a rational map $D^+ \dashrightarrow \mathbb{P}(U^*)$. Resolving the indeterminacies of this map we get a variety $\hat{D}$ and a morphism $\hat{D} \to \mathbb{P}(U^*)$. Geometrically:

$$\hat{D} = \{([v],[\mathbf{a}],[u]) \in \mathbb{P}(V) \times \mathbb{P}(\mathbf{A}) \times \mathbb{P}(U^*) \mid v \circ \mathbf{a} = 0 = \mathbf{a} \circ u\}.$$

Starting from $\mathbb{P}(U^*)$ and the quotient bundle $\mathcal{Q}_-$ over $\mathbb{P}(U^*)$ we get second a desingularization $D^- = \mathbb{P}(V^* \otimes \mathcal{Q}_-)$ with a birational map $\sigma^- : D^- \to D$. This is described as:

$$D^- = \{([\mathbf{a}],[u]) \in \mathbb{P}(\mathbf{A}) \times \mathbb{P}(U^*) \mid \mathbf{a} \circ u = 0\}.$$

The manifold $\hat{D}$ is the blow-up of D along $\operatorname{sing}(D)$ and the map $\hat{D} \to D$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over the smooth locus of $\operatorname{sing}(D)$.

We look at the effective cone of D^+. Tensoring the Koszul complex above with $\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(\mathbf{A})}(x,y)$, for some $(x,y) \in \mathbb{Z}^2$, we see $\mathrm{H}^0(\mathcal{O}_{D^+}(x\mathfrak{l}^+ + y\mathfrak{h})) = 0$ if $y < (1-n)x$ or if $y < 0$. So the effective cone of D^+ is spanned over $\mathbb{Q}$ by $\mathfrak{l}^+$ and ${}^{\mathrm{t}}\mathfrak{l}^+$. Therefore, an effective divisor on D^+ is of the form $x\mathfrak{l}^+ + y\mathfrak{h}$, with:

$$(x,y) \in \mathbb{Z} \times \mathbb{N}, \qquad \text{and:} \qquad y \geq (1-n)x. \tag{38}$$

We compute:

$$\mathfrak{l}^+ \cdot \mathfrak{h}^{n-3} = {}^{\mathrm{t}}\mathfrak{l}^+ \cdot \mathfrak{h}^{n-3} = \binom{n}{2}, \qquad \mathfrak{h}^{n-2} = n.$$

Choose now $[\mathbf{f}] \in \mathrm{PGL}(\mathbf{A})$ such that the determinant $D_{\mathbf{f}}$ of the matrix $M_{\mathbf{f}}$ satisfies $D_{\mathbf{f}} = D$. Then the coherent sheaf $\mathcal{L}_{\mathbf{f}} = \mathrm{coker}(M_{\mathbf{f}})$ is a rank-one reflexive sheaf over D, actually an Ulrich sheaf. Similarly we get a second Ulrich sheaf of rank 1 as:

$$ {}^{\mathrm{t}}\mathcal{L}_{\mathbf{f}} = \mathrm{coker}\left({}^{\mathrm{t}}M_{\mathbf{f}} : V^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}(-1) \to U^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}\right); \qquad \text{we have:} \qquad \mathcal{L}_{\mathbf{f}}(1-n)^* \simeq {}^{\mathrm{t}}\mathcal{L}_{\mathbf{f}}. $$

Each of these sheaves determines an element of $\mathrm{End}(\mathbf{A})$ up to G-action arising as the minimal presentation matrix of the module of global sections of the sheaf, in some basis.

Next, note that a non-zero global section of $\mathcal{L} = \mathcal{L}_{\mathbf{f}}$ vanishes along the Weil divisor B of D consisting of matrices of size $n \times (n-1)$ that have rank at most $n-2$. This pulls-back via σ^+ to an effective divisor of D^+ of the form $x\mathfrak{l}^+ + y\mathfrak{h}$, for some $(x,y) \in \mathbb{Z} \times \mathbb{N}$. The degree of B in $\mathbb{P}(\mathbf{A})$ is $\binom{n}{2}$ so:

$$ (x\mathfrak{l}^+ + y\mathfrak{h}) \cdot \mathfrak{h}^{n-2} = \binom{n}{2}, \qquad \text{so:} \qquad y = \frac{(n-1)(1-x)}{2}. $$

Together with (38), this gives two possibilities for (x,y), namely either $(x,y) = (1,0)$, in which case the divisor class is $\mathfrak{l}^+$, or $(x,y) = (-1, n-1)$ so that the divisor class is ${}^{\mathrm{t}}\mathfrak{l}^+$.

In turn, $|\mathfrak{l}^+|$ gives the rational projection $D \dashrightarrow \mathbb{P}(V)$ and the sheaf $\mathcal{L} = \sigma^+_*(\mathcal{O}_{D^+}(\mathfrak{l}^+))$, while $|{}^{\mathrm{t}}\mathfrak{l}^+| = |\mathfrak{l}^-|$ gives $D \dashrightarrow \mathbb{P}(U^*)$, and the sheaf ${}^{\mathrm{t}}\mathcal{L} = \sigma^-_*(\mathcal{O}_{D^-}(\mathfrak{l}^-))$, the indeterminacies of these maps being resolved by $\pi^+ : D \dashrightarrow \mathbb{P}(V)$ and $\pi^- : D \dashrightarrow \mathbb{P}(U^*)$ and simultaneously over $\hat{D}$. The two possible divisors of degree $\binom{n}{2}$ give precisely two points in the fibre of $\underline{\mathbf{det}}$ over D. □

4.4. **The Hilbert scheme.** Our next goal is to prove that Ψ is a local isomorphism. To do this, we consider a further space, which we denote by $\mathfrak{H}_n$. This is defined as the Hilbert scheme of subschemes of $\mathbb{P}(\mathbf{A})$ having the same Hilbert polynomial as $\mathrm{sing}(D)$. Given any $\mathbf{f} \in \mathrm{PGL}(\mathbf{A})$, the minors of order $(n-1)$ of the matrix $M_{\mathbf{f}}$ cut a subscheme lying in $\mathfrak{H}_n$, so the assignment $[\mathbf{f}] \mapsto Z_{\mathbf{f}} = \mathrm{sing}(D_{\mathbf{f}})$ defines a morphism:

$$ \Xi : \mathrm{PGL}(\mathbf{A}) \to \mathfrak{H}_n, $$

whose image we denote by $\mathfrak{H}_n^\circ$.

Given $Z = \mathrm{sing}(D_{\mathbf{f}}) \in \mathfrak{H}_n^\circ$, the ideal homogeneous ideal I_Z is minimally generated by the n^2 minors of degree $n-1$ and the kernel of this set of generators (namely the first syzygy) determines the module T_D up to isomorphism. After sheafification, this yields a morphism $\mathbf{syz} : \mathfrak{H}_n^\circ \to \mathfrak{M}_n$. Summing up, we get a different factorization of Ψ as:

$$ \begin{array}{ccc} \mathrm{PGL}(\mathbf{A})/G & \xrightarrow{\ \Psi\ } & \mathfrak{M}_n \\ & {\scriptstyle \Xi}\searrow \quad \nearrow{\scriptstyle \mathbf{syz}} & \\ & \mathfrak{H}_n^\circ & \end{array} $$

Proposition 4.4. *The morphisms* $\mathbf{syz}$ *and* Ψ *are submersions.*

Proof. Up to an appropriate change of coordinates, it is enough to prove the statement at the point $\mathbf{i}$ associated to the tautological matrix of indeterminates $(x_{i,j})_{1 \le i,j \le n}$, as mentioned before. We consider the sequence (11) and denote by $\xi \in \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{T}_D)$ the extension corresponding to that sequence. We proved in Lemma 3.3 that $\mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})}) = 0$

and $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})}) = 0$ so $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{T}_D(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})})$ is canonically identified with $\mathbf{A}^*$ and hence $\mathcal{I}_Z(n-1)$ is recovered as cokernel of the dual evaluation:

$$\mathcal{T}_D \to \mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{T}_D(n-1), \mathcal{O}_{\mathbb{P}(\mathbf{A})})^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{A})}.$$

The upshot is that the map **syz** is injective, for the ideal sheaf of the subscheme $Z \subset \mathbb{P}(\mathbf{A})$ is reconstructed by $\mathcal{T}_D$. Therefore **syz** is bijective as it is surjective by definition.

Taking $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), -)$ of (11) and using (13) gives a natural isomorphism:

$$\wedge\xi : \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z, \mathcal{I}_Z) \to \mathrm{Ext}^2_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{T}_D).$$

Next we observe that, applying $\mathrm{Hom}_{\mathbb{P}(\mathbf{A})}(-, \mathcal{T}_D)$ to (11) and using (14) we get a natural isomorphism:

$$\wedge\xi : \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{T}_D, \mathcal{T}_D) \to \mathrm{Ext}^2_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z(n-1), \mathcal{T}_D).$$

We get an isomorphism $\mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{I}_Z, \mathcal{I}_Z) \to \mathrm{Ext}^1_{\mathbb{P}(\mathbf{A})}(\mathcal{T}_D, \mathcal{T}_D)$ induced by $\wedge\xi$ and $(\wedge\xi)^{(-1)}$ which corresponds to the differential of **syz** at the point Z of $\mathfrak{H}_n^\circ$. We have showed that **syz** is a submersion. Finally, we use that Ξ is a submersion, see [KMR2020, Corollary 7.6]. This can also be obtained by explicit computation of the differential as in [RV2017, Lemma 4.1]. Summing up, we proved that Ψ is also a submersion. □

4.5. **Proof of Theorem F.** In order to prove Theorem F, we show the following more precise result.

Theorem 4.5. *The map Φ is an open immersion onto a smooth affine piece of an irreducible component of $\mathfrak{M}_n$ of dimension $(n^2-1)^2$. The map $\underline{\mathbf{det}}$ is an étale $2:1$ cover onto $\mathfrak{D}_n^\circ$.*

Proof. In order to prove this, in view of Proposition 4.2 and Proposition 4.3 it suffices to show that Ψ is a submersion and that the image of Ψ is affine. The first statement is proved in Proposition 4.4. The fact that the image of Ψ is affine follows from the fact that $\mathrm{PGL}(\mathbf{A})/G$ is affine, as $\mathrm{PGL}(\mathbf{A})$ is affine and $\mathrm{PGL}(\mathbf{A})/G$ is the spectrum of the ring of G-invariants of the coordinate ring of $\mathrm{PGL}(\mathbf{A})$, see [Hum1975, Chapter IV]. So the image of Ψ is affine as well as it is the quotient of $\mathrm{PGL}(\mathbf{A})/G$ by the free $\mathbb{Z}/2\mathbb{Z}$-action given by transposition. □

References

[AFV2016] Takuro Abe, Daniele Faenzi, and Jean Vallès, *Logarithmic bundles of deformed Weyl arrangements of type A_2*, Bull. Soc. Math. France **144** (2016), no. 4, 745–761.

[Ang2014] Elena Angelini, *Logarithmic bundles of hypersurface arrangements in $\mathbf{P}^n$*, Collect. Math. **65** (2014), no. 3, 285–302.

[BEGvB2009] Ragnar-Olaf Buchweitz, Wolfgang Ebeling, and Hans-Christian Graf von Bothmer, *Low-dimensional singularities with free divisors as discriminants*, J. Algebraic Geom. **18** (2009), no. 2, 371–406.

[BGL2014] Hernando Bermudez, Skip Garibaldi, and Victor Larsen, *Linear preservers and representations with a 1-dimensional ring of invariants*, Trans. Amer. Math. Soc. **366** (2014), no. 9, 4755–4780.

[BK1990] Alexei I. Bondal and Mikhail M. Kapranov, *Homogeneous bundles*, Helices and vector bundles, London Math. Soc. Lecture Note Ser., vol. 148, Cambridge Univ. Press, Cambridge, 1990, pp. 45–55.

[BV1988] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Math., vol. 1327, Springer-Verlag, Berlin, 1988.

[Del1970] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math., Vol. 163, Springer-Verlag, Berlin, 1970.

[Die1949] Jean Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*, Arch. Math. **1** (1949), 282–287.

[Dim2017] Alexandru Dimca, *Jacobian syzygies, stable reflexive sheaves, and Torelli properties for projective hypersurfaces with isolated singularities*, Algebr. Geom. **4** (2017), no. 3, 290–303.

[Dim2019] Alexandru Dimca, *Versality, bounds of global Tjurina numbers and logarithmic vector fields along hypersurfaces with isolated singularities*, Singularities and their interaction with geometry and low dimensional topology, Trends Math. (2021), 1–12.

[DK1993] Igor V. Dolgachev and Mikhail M. Kapranov, *Arrangements of hyperplanes and vector bundles on* $\mathbb{P}^n$, Duke Math. J. **71** (1993), no. 3, 633–664.

[dPW2001] Andrew A. du Plessis and C. T. C. Wall, *Discriminants, vector fields and singular hypersurfaces*, New developments in singularity theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 21, Kluwer Acad. Publ., Dordrecht, 2001, pp. 351–377.

[FMV2013] Daniele Faenzi, Daniel Matei, and Jean Vallès, *Hyperplane arrangements of Torelli type*, Compos. Math. **149** (2013), no. 2, 309–332.

[Fro1897] Georg Frobenius, *Ueber die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Berl. Ber. **1897** (1897), 994–1015 (German).

[GN1972] Tor Holtedahl Gulliksen and Odd Guttorm Negård, *Un complexe résolvant pour certains idéaux déterminantiels*, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A16–A18.

[GT1977] Shiro Goto and Sadao Tachibana, *A complex associated with a symmetric matrix*, J. Math. Kyoto Univ. **17** (1977), no. 1, 51–54.

[Hil1998] Lutz Hille, *Homogeneous vector bundles and Koszul algebras*, Math. Nachr. **191** (1998), 189–195.

[HL1997] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Hum1975] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975, Graduate Texts in Mathematics, No. 21.

[Józ1978] Tadeusz Józefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helv. **53** (1978), no. 4, 595–607.

[Kin1994] Alastair King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.

[KMR2005] Jan O. Kleppe and Rosa Maria Miró-Roig, *Dimension of families of determinantal schemes*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2871–2907 (electronic).

[KMR2020] Jan O. Kleppe and Rosa M. Miró-Roig, *Deformation and unobstructedness of determinantal schemes*, To appear in Mem. Amer. Math. Soc. (2020).

[Maa2005] Helge Maakestad, *A note on principal parts on projective space and linear representations*, Proc. Amer. Math. Soc. **133** (2005), no. 2, 349–355.

[MM1959] Marvin Marcus and Benjamin N. Moyls, *Linear transformations on algebras of matrices*, Canad. J. Math. **11** (1959), 61–66.

[OR2006] Giorgio Ottaviani and Elena Rubei, *Quivers and the cohomology of homogeneous vector bundles*, Duke Math. J. **132** (2006), no. 3, 459–508.

[OT1992] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.

[Re2012] Riccardo Re, *Principal parts bundles on projective spaces and quiver representations*, Rend. Circ. Mat. Palermo (2) **61** (2012), no. 2, 179–198.

[RV2017] Zinovy Reichstein and Angelo Vistoli, *On the dimension of the locus of determinantal hypersurfaces*, Canad. Math. Bull. **60** (2017), no. 3, 613–630.

[Sai1980] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[Ser2006] Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006.
[Ser2014] ———, *The local cohomology of the Jacobian ring*, Doc. Math. **19** (2014), 541–565.
[Ter1981] Hiroaki Terao, *Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula*, Invent. Math. **63** (1981), no. 1, 159–179.
[Wan2015] Zhenjian Wang, *On homogeneous polynomials determined by their Jacobian ideal*, Manuscripta Math. **146** (2015), no. 3-4, 559–574.
[Wat1995] William C. Waterhouse, *Automorphism group schemes of basic matrix invariants*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 3859–3872.
[Wey2003] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003.

Daniele Faenzi, Institut de Mathématiques de Bourgogne, Université de Bourgogne et Franche-Comté, 9 avenue Alain Savary – BP 47870, 21078 Dijon Cedex, France.
Email address: `daniele.faenzi@u-bourgogne.fr`

Simone Marchesi, Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585 – 08007 Barcelona, Spain.
Email address: `marchesi@ub.edu`