Research Article

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# Rate of convergence of uniform transport processes to a Brownian sheet

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**Abstract:** We give the rate of convergence to a Brownian sheet from a family of processes constructed starting from a set of independent standard Poisson processes. These processes have realizations that converge almost surely to the Brownian sheet, uniformly in the unit square.

**Keywords:** rate of convergence, Brownian sheet, almost sure convergence

**MSC 2010:** 60G15, 60B10

## 1 Introduction

Let $W = \{W(s, t) : (s, t) \in [0, 1]^2\}$ be a Brownian sheet, i.e., a zero mean real continuous Gaussian process with covariance function $E[W(s_1, t_1)W(s_2, t_2)] = (s_1 \wedge s_2)(t_1 \wedge t_2)$ for any $(s_1, t_1), (s_2, t_2) \in [0, 1]^2$. The aim of this study is to obtain the rate of convergence of strong approximations of the Brownian sheet. This result is not only interesting from a purely mathematical point of view but also of great interest to provide sound approximation strategies to solutions of stochastic partial differential equations which arise in many fields such as physics, biology or finance.

The study of the approximations of the Brownian sheet by uniform transport processes or processes constructed from a Poisson process begins with the proof of the weak convergence. Bardina and Jolis [1] prove that the process

$$\frac{1}{n}\int_0^{tn}\int_0^{sn}\sqrt{xy}\,(-1)^{N(x,y)}\mathrm{d}x\mathrm{d}y,$$

where $\{N(x, y), x \geq 0, y \geq 0\}$ is a Poisson process in the plane, converges in law to a Brownian sheet when $n$ goes to infinity, and Bardina et al. [2] extended this result to the $d$-parameter Wiener processes.

On the other hand, Bardina et al. [3] constructed a family of processes, starting from a set of independent standard Poisson processes, that has realizations that convergence almost surely to a Brownian sheet. Our purpose is to give the rate of convergence of such approximations. As far as we know, our work is the first rate of convergence for the multiparameter case for this family of approximations.

There exist several literature studies about strong convergence of uniform transport processes and the study of the corresponding rate of convergence. In the seminal paper of Griego et al. [4], the authors presented realizations of a sequence of uniform transform processes that converges almost surely to the standard Brownian motion, uniformly on the unit time interval. In [5], Gorostiza and Griego extended the result of [4] to the case of diffusions. Again Gorostiza and Griego [6] and Csörgo and Horváth [7] obtained the rate of convergence of the approximation sequence. More recently, Garzón et al. [8] defined a sequence of processes that converges strongly

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to fractional Brownian motion uniformly on bounded intervals, for any Hurst parameter $H \in (0,1)$ and computed the rate of convergence. In [9,10], the same authors deal with subfractional Brownian motion and fractional stochastic differential equations. Bardina et al. [11] proved the strong convergence to a complex Brownian motion and obtained the corresponding rate of convergence.

The structure of the article is as follows. In Section 2, we recall the approximations that converge almost surely to the Brownian sheet and we present our theorem. In Section 3, we give the proof of our result. It is based on a combination of the properties of the Brownian sheet and the use of the rate of convergence for the Brownian motion given by Gorostiza and Griego in [6].

## 2 Approximations and main result

Let us recall the approximation processes introduced in [3]. For any $n$ and $\lambda > 0$, consider the partition of the unit square $[0,1]^2$ in disjoint rectangles

$$\left(\left[0, \frac{1}{n^\lambda}\right] \times [0,1]\right) \cup \left(\bigcup_{k=2}^{\lfloor n^\lambda \rfloor} \left(\frac{k-1}{n^\lambda}, \frac{k}{n^\lambda}\right] \times [0,1]\right) \cup \left(\left(\frac{\lfloor n^\lambda \rfloor}{n^\lambda}, 1\right] \times [0,1]\right),$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

If $W = \{W(s,t) : (s,t) \in [0,1]^2\}$ is a Brownian sheet on the unit square, let $W^k$ denote its restriction to each of the above-defined rectangles $\left(\frac{k-1}{n^\lambda}, \frac{k}{n^\lambda}\right] \times [0,1]$. That is,

$$W^k(t) := W\left(\frac{k}{n^\lambda}, t\right) - W\left(\frac{k-1}{n^\lambda}, t\right),$$

for $k \in \{1, 2, \ldots, \lfloor n^\lambda \rfloor\}$. Thus, for any $l \in \{1, 2, \ldots, \lfloor n^\lambda \rfloor\}$ and $t \in [0,1]$

$$W\left(\frac{l}{n^\lambda}, t\right) = \sum_{k=1}^{l} W^k(t).$$

Moreover, putting $\tilde{W}^k(t) := n^{\frac{\lambda}{2}} W^k(t)$, we obtain a family

$$\{\tilde{W}^k;\, k \in \{1, 2, \ldots, \lfloor n^\lambda \rfloor\}\}$$

of independent standard Brownian motions defined in $[0,1]$.

From the paper of Griego et al. [4], it is known that there exist realizations of uniform transport processes that converge strongly and uniformly on bounded time intervals to Brownian motions. So, we can get an approximation sequence $\{\tilde{W}^{(n)k};\, n \geq 1\}$ for each one of the standard Brownian motions $\tilde{W}^k;\, k \in \{1, 2, \ldots, \lfloor n^\lambda \rfloor\}$. We can state Theorem 1 in [6] for such an approximation sequence for any $k$.

**Theorem 2.1.** *There exists a version* $\{\tilde{W}^{(n)k}(t),\, t \geq 0\}$ *of the uniform transport processes on the same probability space as a Brownian motion process* $\{\tilde{W}^k(t),\, t \geq 0\}$, $\tilde{W}^k(0) = 0$ *so that*

$$\lim_{n \to \infty} \max_{0 \leq t \leq 1} |\tilde{W}^{(n)k}(t) - \tilde{W}^k(t)| = 0, \quad a.s.,$$

*and such that for all* $q > 0$

$$P\left(\max_{0 \leq t \leq 1} |\tilde{W}^{(n)k}(t) - \tilde{W}^k(t)| > an^{-\frac{1}{2}}(\log n)^{\frac{5}{2}}\right) = o(n^{-q}), \quad as\ n \to \infty,$$

*where* $\alpha$ *is a positive constant depending on* $q$.

Then, the Brownian sheet is approximated by a process $W_n$ such that for any $l \in \{1, 2, \ldots, [n^\lambda]\}$ and $t \in [0, 1]$

$$W_n\left(\frac{l}{n^\lambda}, t\right) = \sum_{k=1}^{l} W^{(n)k}(t) = \sum_{k=1}^{l} \frac{1}{n^{\frac{1+\lambda}{2}}} (-1)^{A_k} \int_0^{nt} (-1)^{N_k(u)} \, du,$$

where

$$W^{(n)k}(t) = \frac{1}{n^{\frac{\lambda}{2}}} \tilde{W}^{(n)k}(t),$$

and $\{N_k, k \geq 1\}$ is a family of independent standard Poisson processes and $\{A_k, k \geq 1\}$ is a sequence of independent random variables with law Bernoulli $\left(\frac{1}{2}\right)$, independent of the Poisson processes. Using linear interpolation, define $W_n(s, t)$ on the whole unit square as follows:

$$W_n(s, t) = W_n\left(\frac{\lfloor sn^\lambda \rfloor}{n^\lambda}, t\right) + (sn^\lambda - \lfloor sn^\lambda \rfloor) W_n\left(\frac{\lfloor sn^\lambda \rfloor + 1}{n^\lambda}, t\right),$$

for any $(s, t) \in \left[0, \frac{\lfloor n^\lambda \rfloor}{n^\lambda}\right] \times [0, 1]$ and $W_n(s, t) = W_n\left(\frac{\lfloor n^\lambda \rfloor}{n^\lambda}, t\right)$ for any $(s, t) \in \left[\frac{\lfloor n^\lambda \rfloor}{n^\lambda}, 1\right] \times [0, 1]$.

In the following theorem, we give our main result, the rate of convergence of these processes.

**Theorem 2.2.** *There exist realizations of the process $\{W_n(s, t), (s, t) \in [0, 1]^2\}$ with $\lambda \in \left(0, \frac{1}{5}\right)$ on the same probability space as a Brownian sheet $\{W(s, t), (s, t) \in [0, 1]^2\}$ such that*

$$\lim_{n \to \infty} \max_{0 \leq s, t \leq 1} |W_n(s, t) - W(s, t)| = 0 \; a.s.$$

*and such that for all $\beta < \frac{\lambda}{2}$ and $q > 0$*

$$P\left(\max_{0 \leq s, t \leq 1} |W_n(s, t) - W(s, t)| \; > \alpha n^{-\beta}\right) = o(n^{-q}), \quad \text{as } n \to \infty,$$

*where $\alpha$ is a positive constant depending on $q$.*

The first part of the theorem has been proved in Theorem 2.1 in [3]. In Section 3, we give the proof of the rate of convergence.

**Remark 2.3.** Note that the rate of convergence obtained in this article is worse than the rate of convergence for the Brownian motion case. It is due to the fact that in the approximations given in [3] we decompose the Brownian sheet as a sum of $[n^\lambda]$ independent Brownian motions. We are able to apply Theorem 2.1 and obtain the rate of convergence of Theorem 2.1 for each one of these $[n^\lambda]$ Brownian motions, but we cannot keep the same rate with their sum.

# 3 Proof

The proof uses some well-known properties about the Brownian sheet $\{W(s, t) : (s, t) \in [0, 1]^2\}$. Let us recall, for instance, that for fixed $t > 0$, the process $\{t^{-\frac{1}{2}} W(s, t), s \geq 0\}$ is a standard Brownian motion or that for fixed $a > 0$, the process $\{a^{-\frac{1}{2}} W(as, t), s \geq 0, t \geq 0\}$ is also a Brownian sheet. We refer the reader to Khoshnevisan [12] for a detailed description of the Brownian sheet.

Furthermore, we will begin recalling technical results about submartingales (Theorem 1, p. 74 from Imkeller [13]) that we will apply to the submartingale $e^W$.

**Theorem 3.1.** *Let $M$ be a non-negative submartingale and set $\psi(t) = t \log^+ t$, for any $t \geq 0$, where $\log^+ t = \max(0, \log t)$. Then, there is a constant $c$ which does not depend on $M$ such that, for any $\beta > 0$*

$$P\left(\max_{(s,t)\in[0,1]^2}|M(s,t)|>\beta\right)\le\frac{c}{\beta}\|M(1,1)\|_\psi,$$

*where*

$$\|X\|_\psi=\inf\left\{\mu>0:E\left(\psi\left(\frac{|X|}{\mu}\right)\right)\le 1\right\}.$$

We will apply this theorem to the process

$$M(s,t):=\exp(B(s,t)),\quad (s,t)\in[0,1]^2,$$

where $\{B(s,t):(s,t)\in[0,1]^2\}$ is a Brownian sheet. Considering the family of $\sigma$-algebras $\mathcal{F}_{s,t}=\sigma<B(u,v);(u,v)\in[0,s]\times[0,t]>$, $M(s,t)$ is clearly nonnegative, integrable, $\mathcal{F}_{s,t}$-measurable and it is a submartingale. Indeed, if $0<s'<s$ and $0<t'<t$, then $s't'<st$ and

$$E(e^{B(s,t)}|\mathcal{F}_{s',t'})=E(e^{B(s,t)-B(s',t')}e^{B(s',t')}|\mathcal{F}_{s',t'})=E(e^{B(s,t)-B(s',t')})e^{B(s',t')}=e^{\frac{st-s't'}{2}}e^{B(s',t')}\ge e^{B(s',t')}.$$

On the other hand, $\|M(1,1)\|_\psi=\|\exp(B(1,1))\|_\psi$ is a finite constant. Indeed, using that $B(1,1)$ is a $N(0,1)$,

$$\begin{aligned}E\left(\frac{\exp(B(1,1))}{\mu}\log^+\left(\frac{\exp(B(1,1))}{\mu}\right)\right)&=\frac{1}{(2\pi)^{\frac12}}\int_{\log\mu}^{\infty}\frac{e^x}{\mu}(x-\log\mu)e^{-\frac12x^2}dx\\&=\frac{1}{(2\pi)^{\frac12}}\int_0^\infty\frac{e^{y+\log\mu}}{\mu}ye^{-\frac12(y+\log\mu)^2}dy\\&=\frac{1}{(2\pi)^{\frac12}}\frac{e^{\frac12}}{\mu}\int_0^\infty ye^{-\frac12(y+(\log\mu-1))^2}dy\\&=\frac{1}{(2\pi)^{\frac12}}\frac{e^{\frac12}}{\mu}\left(e^{-\frac12(\log\mu-1)^2}+\left(\frac{\pi}{2}\right)^{\frac12}(1-\log\mu)\right),\end{aligned}$$

and this quantity is bigger than 1 for $\mu$ sufficiently small and smaller than 1 for $\mu$ big enough. We have used that

$$\int_0^\infty xe^{-\frac12(x-m)^2}dx=e^{-\frac12m^2}+m\left(\frac{\pi}{2}\right)^{\frac12}.$$

Let us give now the proof of our theorem.

**Proof of Theorem 2.2.** Our aim is to bound

$$P:=P\left(\max_{(s,t)\in[0,1]^2}|W_n(s,t)-W(s,t)|>K_n\right),$$

where $K_n=cn^{-\beta}$, with $\beta\in\left(0,\frac{\lambda}{2}\right)$.

Fixed $s\in[0,1]$, $n\ge1$, there exists $k\in\{0,\dots,\lfloor n^\lambda\rfloor\}$ such that

$$\frac{k}{n^\lambda}\le s<\frac{k+1}{n^\lambda},$$

that is, $k=\lfloor sn^\lambda\rfloor$. So, we can write

$$\begin{aligned}
P &\le P\left(\max_{(s,t)\in[0,1]^2}\left|W_n(s,t)-W_n\left(\frac{\lfloor sn^\lambda\rfloor}{n^\lambda},t\right)\right|>\frac{K_n}{3}\right)\\
&\quad+P\left(\max_{(s,t)\in[0,1]^2}\left|W_n\left(\frac{\lfloor sn^\lambda\rfloor}{n^\lambda},t\right)-W\left(\frac{\lfloor sn^\lambda\rfloor}{n^\lambda},t\right)\right|>\frac{K_n}{3}\right)\\
&\quad+P\left(\max_{(s,t)\in[0,1]^2}\left|W\left(\frac{\lfloor sn^\lambda\rfloor}{n^\lambda},t\right)-W(s,t)\right|>\frac{K_n}{3}\right)\\
&:=P_1+P_2+P_3.
\end{aligned}$$

Let us study first $P_2$. Using the definitions of $W^{(n)k}$ and $W^k$, for any $k$, we get

$$\begin{aligned}
P_2 &= P\left(\max_{1\le l\le\lfloor n^\lambda\rfloor}\max_{t\in[0,1]}\left|\sum_{k=1}^{l}W^{(n)k}(t)-\sum_{k=1}^{l}W^k(t)\right|>\frac{K_n}{3}\right)\\
&= P\left(\max_{1\le l\le\lfloor n^\lambda\rfloor}\max_{t\in[0,1]}\left|\sum_{k=1}^{l}\tilde W^{(n)k}(t)-\sum_{k=1}^{l}\tilde W^k(t)\right|>n^{\frac{\lambda}{2}}\frac{K_n}{3}\right)\\
&\le P\left(\sum_{k=1}^{\lfloor n^\lambda\rfloor}\max_{t\in[0,1]}|\tilde W^{(n)k}(t)-\tilde W^k(t)|>n^{\frac{\lambda}{2}}\frac{K_n}{3}\right)\\
&\le \sum_{k=1}^{\lfloor n^\lambda\rfloor}P\left(\max_{t\in[0,1]}|\tilde W^{(n)k}(t)-\tilde W^k(t)|>n^{-\frac{\lambda}{2}}\frac{K_n}{3}\right).
\end{aligned}$$

Finally, from the rate of convergence from $\tilde W^{(n)k}$ to $\tilde W^k$, for any $k$, in Theorem 2.1, we get

$$P_2\le n^\lambda P\left(\max_{t\in[0,1]}|\tilde W^{(n)1}(t)-\tilde W^1(t)|>n^{-\frac{\lambda}{2}}\frac{K_n}{3}\right)=o(n^{-q}),$$

for any $q>0$.

Let us consider now $P_3$. We can write, using the well-known properties of the Brownian sheet

$$\begin{aligned}
P_3 &= P\left(\max_{0\le l\le\lfloor n^\lambda\rfloor}\max_{t\in[0,1]}\max_{r\in[0,n^{-\lambda}]}\left|W\left(\frac{l}{n^\lambda}+r,t\right)-W\left(\frac{l}{n^\lambda},t\right)\right|>\frac{K_n}{3}\right)\\
&\le \sum_{l=0}^{\lfloor n^\lambda\rfloor}P\left(\max_{t\in[0,1]}\max_{r\in[0,n^{-\lambda}]}\left|W\left(\frac{l}{n^\lambda}+r,t\right)-W\left(\frac{l}{n^\lambda},t\right)\right|>\frac{K_n}{3}\right)\\
&\le (n^\lambda+1)P\left(\max_{t\in[0,1]}\max_{r\in[0,n^{-\lambda}]}|W(r,t)|>\frac{K_n}{3}\right)\\
&= (n^\lambda+1)P\left(\max_{(s,t)\in[0,1]^2}\left|W\left(\frac{s}{n^\lambda},t\right)\right|>\frac{K_n}{3}\right).
\end{aligned}$$

We have considered that $W\left(\frac{l}{n^\lambda}+r,t\right)=W(1,t)$ if $\frac{l}{n^\lambda}+r>1$. Let us define $B(s,t):=n^{\frac{\lambda}{2}}W\left(\frac{s}{n^\lambda},t\right)$, for any $(s,t)\in[0,1]^2$. Clearly, $B$ is a Brownian sheet and $P_3$ can be bounded as

$$P_3\le(n^\lambda+1)P\left(\max_{(s,t)\in[0,1]^2}|B(s,t)|>\frac{K_n}{3}n^{\frac{\lambda}{2}}\right).$$

So, using Theorem 3.1

$$\begin{aligned}
P_3 &\le 2(n^\lambda+1)P\left(\max_{(s,t)\in[0,1]^2}B(s,t)>\frac{K_n}{3}n^{\frac{\lambda}{2}}\right)\\
&= 2(n^\lambda+1)P\left(\max_{(s,t)\in[0,1]^2}\exp(B(s,t))>\exp\left(\frac{K_n}{3}n^{\frac{\lambda}{2}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2(n^\lambda + 1)c \exp\left(-\frac{K_n}{3} n^{\frac{\lambda}{2}}\right) \| \exp(B(1,1)) \|_\psi \\
&= 2(n^\lambda + 1)c_2 \exp\left(-\frac{K_n}{3} n^{\frac{\lambda}{2}}\right) \\
&= 2(n^\lambda + 1)c_2 \exp(-c_3 n^{\frac{\lambda}{2}-\beta}),
\end{aligned}
$$

since $\| \exp(B(1,1)) \|_\psi$ is a finite positive constant that does not depend on $n$.

Finally, let us study $P_1$. Using that we have defined the intermediate points using linear interpolation we have that for any $s \in \left[\frac{l}{n^\lambda}, \frac{l+1}{n^\lambda}\right)$ we get that

$$
\left| W_n(s,t) - W_n\left(\frac{\lfloor s n^\lambda \rfloor}{n^\lambda}, t\right) \right| \leq \left| W_n\left(\frac{l}{n^\lambda}, t\right) - W_n\left(\frac{l+1}{n^\lambda}, t\right) \right|
$$

and we can write that

$$
\begin{aligned}
P_1 &= P\left( \max_{0 \le l \le \lfloor n^\lambda \rfloor} \max_{t \in [0,1]} \left| W_n\left(\frac{l}{n^\lambda}, t\right) - W_n\left(\frac{l+1}{n^\lambda}, t\right) \right| > \frac{K_n}{3} \right) \\
&\leq P\left( \max_{0 \le l \le \lfloor n^\lambda \rfloor} \max_{t \in [0,1]} \left| W_n\left(\frac{l}{n^\lambda}, t\right) - W\left(\frac{l}{n^\lambda}, t\right) \right| > \frac{K_n}{9} \right) \\
&\quad + P\left( \max_{0 \le l \le \lfloor n^\lambda \rfloor} \max_{t \in [0,1]} \left| W\left(\frac{l}{n^\lambda}, t\right) - W\left(\frac{l+1}{n^\lambda}, t\right) \right| > \frac{K_n}{9} \right) \\
&\quad + P\left( \max_{0 \le l \le \lfloor n^\lambda \rfloor - 1} \max_{t \in [0,1]} \left| W\left(\frac{l+1}{n^\lambda}, t\right) - W_n\left(\frac{l+1}{n^\lambda}, t\right) \right| > \frac{K_n}{9} \right) \\
&:= P_{1,1} + P_{1,2} + P_{1,3},
\end{aligned}
$$

where if $\frac{l+1}{n^\lambda} > 1$ we assume that $W\left(\frac{l+1}{n^\lambda}, t\right) = W_n\left(\frac{l+1}{n^\lambda}, t\right) = W(1,t)$.

Note that $P_{1,2}$ can be bounded as $P_3$ and on the other hand, $P_{1,1}$ and $P_{1,3}$ can be studied as $P_2$. Putting together the bounds for $P_1$, $P_2$ and $P_3$ we finish the proof. □

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