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The representation of gappy sentences in four-valued semantics

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Abstract: Three-valued logics are standardly used to formalize gappy languages, i.e., interpreted languages in which sentences can be true, false or neither. A three-valued logic that assigns the same truth value to all gappy sentences is, in our view, insufficient to capture important semantic differences between them. In this paper we will argue that there are two different kinds of pathologies that should be treated separately and we defend the usefulness of a four-valued logic to represent adequately these two types of gappy sentences. Our purpose is to begin the formal exploration of the four-valued logics that could be used to represent the phenomena in question and to show that these phenomena are present in natural language, at least according to some semantic theories of natural language.

Keywords: four-valued logic; four-valued semantics; gappy proposition; meaningless sentence; truth value gap

1 Introduction

There are sentences in natural language that cannot be uncontroversially classified as either true or false: sentences that apply vague predicates to borderline cases ('Tim is thin'), some semantic paradoxes ('this sentence is false'), sentences that contain categorical mistakes ('triangularity is yellow'), sentences with non-denoting terms ('Vulcan is a planet,' 'the present king of France is bald'), sentences with a failed presupposition ('he has quit smoking' – said of someone who never smoked), sentences with partial predicates, etc. Although some of the theories proposed in the literature in logic and semantics classify those sentences as just true or false, other theoretical explanations consider them to be cases of sentences which are neither true nor false. We will use the term *gappy sentence* to designate any sentence that is neither true nor false. The formal representation of

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gappy sentences has usually been given in a three-valued logic, with a third semantic value added to the classical ones.

A three-valued logic that assigns the same truth value to all gappy sentences is, in our view, insufficient to capture important semantic differences between gappy sentences. In this paper, we will focus on the formal semantics of interpreted languages that contain two different types of gappy sentences. In particular, we will concentrate on the difference that arises according to certain semantic approaches between: (i) sentences that are arguably neither true nor false but have a truth conditional content (namely, they express a proposition) so they could be true or false in other indices of evaluation and (ii) meaningful sentences that, in spite of having a systematic use in the speaker's community and in spite of clearly having cognitive significance, fail to have truth conditional content (namely, they fail to express a proposition), hence they are neither true nor false in the actual world and in any other index of evaluation.[1] We will argue that (i) and (ii) are two different kinds of pathologies that should be treated separately and we will argue for the usefulness of a four-valued logic to represent adequately these two types of gappy sentences. Our purpose in this paper is to begin exploring formally the four-valued logics that could be used to represent the phenomena in question and to show that these phenomena are present in natural language, at least according to some semantic theories of natural language.

2 Several four-valued semantics

We will use as semantic values the elements of the set $E_4 = \{0, 1, 2, 3\}$, with 0 being identified with the value "false" and 1 with the value "true"; 2 and 3 will be reserved for gappy sentences. We will assign 3 to sentences that lack a classical truth value because they do not express a proposition and 2 to sentences that express a proposition, but one such that it is undetermined whether the proposition is true or false. The first question we will ask ourselves is: which is the four-valued scheme of interpretation that we need in order to give a formal semantics to a language of this sort? In this paper, we will restrict ourselves to sentential languages built from atomic sentences p, q, etc. with the usual connectives. We will concentrate this

1 To be more precise, we should say that the propositions expressed by uses of sentences, as encoders of truth conditional content, are the truth bearers. Here, and as it is customary in logic studies, we take the truth value that the proposition expressed by a sentence has at a given index to be the truth value of the sentence at that index. Moreover, nothing substantial changes if utterances or uses of sentences in context are considered to be the truth bearers. The formal representation of these truth bearers would require unnecessary complications, so in this paper we will stick to sentences as truth bearers.

	$\neg_B$
0	1
1	0
2	2
3	3

$\wedge_B$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

Figure 1: Belnap-Dunn operators.

discussion on the connectives of negation (¬) and conjunction (∧). Disjunction (∨) and the material conditional (⊃) will be given by the standard interdefinitions. It is natural to start our quest by considering the most important four-valued logic: Belnap-Dunn's logic (also called FDE; see Belnap 1977). Let us recall the Belnap-Dunn operators (Figure 1).

The Belnap-Dunn operators are monotonic on the order of information on E_4 represented by the vertical order on the bilattice $D4$ (the horizontal order corresponds to the order of truth; Figure 2).

Even though this scheme of interpretation for the connectives has important applications, as a representation of the specific languages we are analyzing the operators do not make sense. In Belnap-Dunn logic, $2 \wedge_B 3 = 3 \wedge_B 2 = 0$ (and, dually, $2 \vee_B 3 = 3 \vee_B 2 = 1$). But why should a conjunction of two gappy sentences be, in general, a false one? Why should the disjunction of two gappy sentences be a true one?

Once the main option available has been discarded, we need to ask ourselves what conditions should a good generalization of the classical operators satisfy. We certainly want four-valued operators to be normal, i.e., when the arguments are classical values, the value coincides with the value given by the classical operator. How about monotonicity properties? It seems clear that the information order $D4$ does not reflect correctly the degree of information about the classical value that a

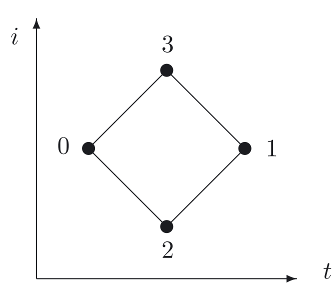

Figure 2: Bilattice *D*4.

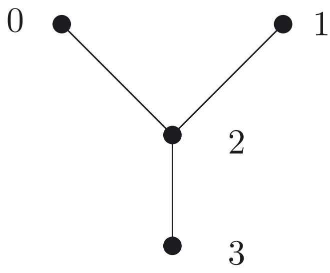

Figure 3: Order *Y*4.

sentence may have. We submit that the correct information order on E_4 (that we will call *Y*4) is given by the diagram presented in Figure 3.

This order reflects how much information we have about the connection between what a sentence expresses and the world: not expressing a proposition is the worst situation, since there is no full content expressed that in this situation, or in any other situation, could be true or false; expressing an indeterminate proposition means that the sentence expresses a content that is evaluable, but in the context of utterance does not establish an adequate connection to the world and, finally, expressing a true or false proposition gives a maximal amount of information: the sentence expresses a content that is either true or false. We contend that acceptable operators should be monotonic on this order.

If we add the condition that the double negation of a formula should not alter the truth value of the formula, normality plus monotonicity on *Y*4 determine a unique candidate for negation, the standard four-valued negation: $\neg 0 = 1$, $\neg 1 = 0$, $\neg 2 = 2$, $\neg 3 = 3$.

For conjunction, if we require normality, monotonicity on *Y*4 and the standard conditions of commutativity, associativity and idempotence, it is easy to determine that there are exactly the five operators presented in Figure 4.[2,3]

We will now characterize the logics determined by these operators. In the application we are discussing, consequence is to be taken as preservation of truth, so the designated value has to be only 1, since all the other values do not represent

2 The names of the operators $\wedge_{ij}$ are formed using the following rules: i expresses the three-valued operator obtained when the values are restricted to the set $\{0, 1, 2\}$ and j the operator obtained with the restriction to $\{0, 1, 3\}$. Then w represents the Weak Kleene operator, s the Strong Kleene one and $\star$ the case in which the restriction is not closed, so it is not a three-valued operator.

3 It is interesting to notice that the operators $\wedge_{ww}$ and $\wedge_{sw}$ are also monotonic on the order of information on *D*4, while $\wedge_{ss}$, $\wedge_{w\star}$ and $\wedge_{s\star}$ are not.

$\wedge_{ww}$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

$\wedge_{sw}$	0	1	2	3
0	0	0	0	3
1	0	1	2	3
2	0	2	2	3
3	3	3	3	3

$\wedge_{ss}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	3
3	0	3	3	3

$\wedge_{w\star}$	0	1	2	3
0	0	0	2	2
1	0	1	2	2
2	2	2	2	2
3	2	2	2	3

$\wedge_{s\star}$	0	1	2	3
0	0	0	0	0
1	0	1	2	2
2	0	2	2	2
3	0	2	2	3

Figure 4: Four-valued conjunctions.

truth.[4] Let us then call L_{ij} the propositional language interpreted with the standard four-valued negation and $\wedge_{ij}$ and $\mathbf{L_{ij}}$ the logic (L_{ij-ij}), where $\vDash_{ij}$ is the operator of logical consequence corresponding to the language L_{ij}, using as set of designated values the set $\{1\}$. Then it is easy to prove the following facts:

(i) $\mathbf{L_{ww}} = \mathbf{L_{w\star}} = \mathbf{L_{WK}}$.
(ii) $\mathbf{L_{ss}} = \mathbf{L_{s\star}} = \mathbf{L_{SK}}$.
(iii) $\mathbf{L_{sw}} = \mathbf{L_{SK}} \cap \mathbf{L_{WK}}$.

where $\mathbf{L_{WK}}$ and $\mathbf{L_{SK}}$ represent Weak and Strong Kleene logic, respectively. (i) and (ii) follow from the fact that identifying the values 2 and 3 in those logics yields the operators of either the weak or the strong Kleene logics. The left-to-right inclusion of the third line follows from the fact that the restrictions of the operators of L_{sw} to the sets $\{0, 1, 2\}$ and $\{0, 1, 3\}$ coincide with the Strong and Weak Kleene operators, respectively. For the right-to-left inclusion, take a set of sentences Γ, a sentence ϕ and any valuation v in E_4 such that (in L_{sw}) $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\phi) \neq 1$. Let us distinguish two cases: (i) if $v(\phi)$ is 0 or 2, it is trivial to show that then (restricting our attention to the atomic sentences that occur in the argument) v is a valuation on $\{0, 1, 2\}$ and, since the operators of L_{sw} coincide with the strong Kleene operators on that set, v is a counterexample to $\Gamma \vDash_{SK} \phi$. If $v(\phi) = 3$, then define the

4 In general, if one value is designated, it is so because it expresses something that we want our arguments to preserve. Also, if one value is less than another in the order of information, the latter expresses more informational content than the former. This can be applied not only to truth values in a strict sense, but also to epistemic interpretations of values. So, if either 2 or 3 were designated in $Y4$, it would follow that 0 would also be designated, which is unacceptable on our interpretation of the values. Notice that considering 1 and 3 as designated in the order of information on $D4$, as it is standardly done, is compatible with these principles.

following valuation v' on $\{0, 1, 3\}$: if $v(p) = 2$, then $v'(p) = 0$; $v'(p) = v(p)$ otherwise. Then $v \leq v'$ in the pointwise order on $Y4$ and, by monotonicity of the operators on $Y4$ and the contaminant character of the value 3, v' is a counterexample to $\Gamma \vDash_{WK} \phi$.[5]

Given these facts, one could wonder whether it is worth exploring those logics, except for the logic $\mathbf{L_{sw}}$. All the other logics coincide with one of the Kleene logics, which have a simpler semantics. If nothing changes in the logic when we consider three-valued or four-valued semantics for gappy sentences, should we not prefer the three-valued one and refuse to distinguish between the two classes of gappy sentences? If the distinction does not introduce a difference in logic, it is a useless complication.

We think that this not the case, and there are reasons to explore semantics that do make a distinction between the truth values. First of all, there are phenomena in natural languages that motivate the separation of gappy sentences in two different kinds. Second, the new semantic distinctions motivate the introduction of new operators that, when they are present in the language, give rise to differences in the logics of the expanded formal languages. In the rest of the paper we first examine the natural language phenomena and then we go on to discuss the impact of this semantics on logic.[6,7]

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5 Damian Szmuc (2016) introduces a map of logics that includes the logic $\mathbf{L_{sw}}$. The operators of the language L_{sw} were introduced in Fitting (1994). His designated values, in our notation, are the set $\{1, 2\}$, so his logic does not coincide with ours. The operators in L_{ww} are also studied in Szmuc (2016) and Da Ré et al. (2020), but again with different sets of designated values. The operators in L_{ss}, $L_{w\star}$ and $L_{s\star}$ have not been studied previously, as far as we know.

6 However, even if there were no direct impact of the semantics on logic, as a general principle, we think that it is important to be able to represent formally distinctions which are semantically relevant.

7 Even though, for simplicity, we are restricting ourselves in this paper to sentential languages, each language L_{ij} can be expanded to a first-order language by adding the distribution quantifiers defined in a canonical way as generalizations of conjunction and disjunction. A distribution quantifier is a function from non-empty subsets of truth values into truth values, so that the value of a sentence $Qx\phi(x)$ in a model is the function corresponding to the quantifier Q aplied to the set of values of $\phi(d)$, when d varies over the universe of the model. These are the universal quantifiers corresponding to our languages (for $A \subseteq E_4$):

$\forall_{ww}(A) = 3$, if $3 \in A$; $\forall_{ww}(A) = 2$, if $3 \notin A$ and $2 \in A$; $\forall_{ww}(\{1\}) = 1$; $\forall_{ww}(A) = 0$, otherwise.
$\forall_{sw}(A) = 3$, if $3 \in A$; $\forall_{sw}(A) = 0$, if $3 \notin A$ and $0 \in A$; $\forall_{sw}(\{1\}) = 1$; $\forall_{sw}(A) = 2$, otherwise.
$\forall_{ss}(A) = 0$, if $0 \in A$; $\forall_{ss}(A) = 3$, if $0 \notin A$ and $3 \in A$; $\forall_{ss}(\{1\}) = 1$; $\forall_{ss}(A) = 2$, otherwise.
$\forall_{w\star}(A) = 0$, if $0 \in A$, $2 \notin A$ and $3 \notin A$; $\forall_{w\star}(\{1\}) = 1$; $\forall_{w\star}(\{3\}) = 3$; $\forall_{w\star}(A) = 2$, otherwise.
$\forall_{s\star}(A) = 0$, if $0 \in A$; $\forall_{s\star}(\{1\}) = 1$; $\forall_{s\star}(\{3\}) = 3$; $\forall_{s\star}(A) = 2$, otherwise.

3 Two illustrations

The semantics we have explored in the previous section allows us to represent and distinguish formally two different kinds of pathologies that we find in natural language discourse, corresponding to two different types of gappy meaningful sentences: (i) sentences that do express a proposition but that, due to some semantic deficiency, or due to what we could characterize as lack of collaboration by the world, are, arguably, neither true nor false; (ii) sentences that lack a classical truth value because they fail to express a complete proposition. We will examine both phenomena separately.

3.1 Semantic content and indeterminate truth value: the case of denotationless definite descriptions

When it comes to definite descriptions that fail to denote, a Russellian would not see any need to appeal to anything over and above a classical two-valued logic. From a Russellian perspective (Russell 1956 [1905]), simple sentences with the grammatical form *The G is P*, where *The G* lacks a denotation are to be assigned the value 0. A recent utterance of 'The present King of France is bald' asserts the present existence of a King of France, and it is hence, false.[8]

The Russellian perspective is not universally accepted. Those who take inspiration in P. F. Strawson (1971 [1950]) would argue rather that the sentence 'The present King of France is bald' is neither true nor false. It is not our purpose here to adjudicate between the two approaches to definite descriptions, but simply to highlight that if one takes a Strawsonian perspective, sentences containing denotationless definite descriptions should be assigned 2, a third truth value that reflects their indeterminateness.[9]

8 Sentences containing other improper definite descriptions, i.e., descriptions whose matrix is satisfied by more than one individual, are also false, according to Russell, because the requirement of uniqueness is not met. For reasons that will become apparent later, we will focus here on sentences in which the requirement of existence is not satisfied. We are also focusing only on attributive uses of definite descriptions although we will address some issues related to the distinction between referential and attribute uses made by Keith Donnellan (1966) in section 4.

9 In his 1950 paper Strawson himself does not use the term 'presupposition' (he talks about a form of non-logic entailment) but, arguably, the idea is implicit there. The notion of presupposition is explicitly introduced in chapter 6, section III, 7 of Strawson 1952. We leave the discussion of issues such as projection, cancellability and the kinds of three valued semantics that better allow to represent these phenomena aside here, since our purpose in this paper is to illustrate the usefulness of a semantics with not just three, but four truth values.

The apparatus of structured propositions used by Kaplan (1978) will prove useful for illustration. Kaplan represents propositions as n-tuples that contain the elements contributed to truth conditional content by expressions in sentences. The propositional contribution of a definite description such as *the P*, is general, an attributive complex that selects at each index of evaluation w the individual relevant for the computation of truth value at w. Thus, *The P is Q* expresses a proposition that can be represented as $<!\,P^{*}, Q^{*}>$ (where '$!P^{*}$' represents the condition 'being uniquely P' and 'Q^{*}' represents the property expressed by the predicate).

A use of a sentence such as 'The inventor of Kevlar was a chemist' expresses a proposition that can be represented as $<!\,KV^{*}, C^{*}>$. In the actual world the proposition has the value 1, since Stephanie Kwolek was the inventor of Kevlar, and she was a chemist. Had someone else, someone with a degree in another discipline, invented Kevlar, the truth value of the proposition would be 0. And in a world w where no one invents Kevlar, the attributive complex contributed by the definite description to the proposition $<!\,KV^{*}, C^{*}>$ selects no individual. Those are the conditions in which, taking a Strawsonian perspective, we would say that the proposition's value is 2, neither 0 nor 1. That distribution of truth values across possible worlds allows us to assign a truth conditional content to our use of the sentence 'The inventor of Kevlar was a chemist' a truth conditional content that can be represented as an intension, a function from possible worlds to truth values.

In the actual world, the sentence 'The present King of France is bald,' as used in 2020, is gappy, in that it lacks a classical truth value. It is nevertheless meaningful and, more importantly, the sentence clearly has truth conditions. It expresses a proposition that can be represented as $<!\,KF^{*}, B^{*}>$, and, hence, the sentence has truth conditional content. Since there is no present King of France the truth value of the sentence 2. Had France continued to be a monarchy, the truth value would be 1 (in worlds in which the individual satisfying the attributive complex '$!KV^{*}$ is bald) or 0 (in worlds in which such individual is not bald).

3.2 Empty names in direct reference theory

The situation as regards names that fail to refer may seem prima facie very similar to the pathological phenomenon involving denotationless definite descriptions. And, in fact, a proponent of classical descriptivism would argue that the situation is exactly the same, since from a descriptivist perspective, a sentence such as 'Vulcan is large' is equivalent to a sentence that contains a denotationless definite description. From a descriptivist point of view, 'Vulcan is large,' expresses a proposition, and hence it has truth conditions. Although, arguably, the proposition is neither true nor false in the actual world – and hence it should be assigned the truth value 2—the sentence has a classical truth value in worlds in which the definite description happens to be satisfied uniquely by an object.

But things look very different from the perspective of direct reference theory, the approach to reference that emerged after Kripke's strong arguments against classical descriptivism in his 1970 lectures (Kripke 1980). Among other things, Kripke argued that the truth conditions of a sentence such as 'Aristotle was a philosopher' depended on the referent of the name 'Aristotle' and not on a description nor a mode of presentation speakers associate with the name. The truth conditions of a sentence containing a definite description, for instance, the sentence 'the tutor of Alexander the Great was a philosopher,' depend on the satisfier of the definite description, which may be different in different possible worlds. But the truth conditions of 'Aristotle was a philosopher' depend directly on Aristotle: in a world in which Aristotle is a philosopher the sentence is true; in a world in which Aristotle is not a philosopher the sentence is false. This truth conditional dependence on the referent is captured in the claim that sentences containing proper names express singular propositions, propositions whose truth conditions depend on an object, its properties and relations. Resorting again to the apparatus of structured propositions proposed by David Kaplan we could say that the proposition expressed by 'Aristotle was a philosopher' can be represented as the singular proposition $<a, P^{*}>$, where a is Aristotle himself and P^{*} is the property of being a philosopher.

Direct reference theory postulates that the semantic behavior of definite descriptions and names is radically different. Whereas the truth conditional contribution of definite descriptions is an attributive complex, the truth conditional contribution of a name is the entity the name refers to; whereas the propositions expressed by sentences of the form *the G is H* are general, the propositions expressed by sentences of the form *n is H* are singular: an object is a constitutive element of the truth conditional content of the latter kind of sentences.[10]

Now, 'Vulcan is large' is a pathological sentence, for it contains a non-denoting term so, arguably, the sentence is neither true nor false.[11] But if we were to assign the truth value 2 to that sentence, we would be inattentive to an important trait, according to the theory of direct reference. For, on that view, sentences that contain an empty name are pathological in a different way than sentences that

10 And the differences hold even if the individual selected by *the G* is the same, say the referent of *n*, at all indices of evaluation (to use Kripke's terminology, even if *the G* is rigid).

11 Of course, some would argue that the sentence is clearly false. But relying on a reasoning by Russell (for a non-Russellian conclusion), we are sensitive to the fact that there is no Vulcan among the large objects nor among the non-large objects.

contain a non-denoting definite description: although a non-denoting description *the G* makes a truth conditional contribution to the sentence *the G is H*, a sentence of the form *n is G* has a term that fails to make a truth conditional contribution. 'The planet causing anomalies in the orbit of Mercury is large' is a pathological sentence that expresses a proposition (has truth conditional content). 'Vulcan is large' contains a name that, from the direct reference perspective, has no referent. A fundamental truth conditional contribution in the sentence 'Vulcan is large' is simply missing.[12]

An advocate of direct reference theory may follow two paths as regards what to say about the truth conditional content of, or proposition expressed by, 'Vulcan is large.' On the one hand they may argue that 'Vulcan is large' does not express a proposition. On the other hand, and following David Kaplan (1989, footnote 23) and David Braun (1993), one may observe that 'Vulcan is large' still has a component, the predicate, that is not empty and argue that the sentence expresses a 'gappy' or incomplete proposition. On that view, the truth conditional content of 'Vulcan is large' can be represented as the structured proposition $<\_, L^*>$.

No matter which path a direct reference theorist decides to follow, it is important to assign a truth value to the neither true nor false 'Vulcan is large' that highlights how different the semantic behavior of that sentence is compared to 'the planet that causes anomalies in the orbit of Mercury is large.' If we do not want to obliterate the difference between these two sentences, assigning a fourth truth value to 'Vulcan is large' would appear to be a good course of action.

4 Further applications: empty demonstrations and empty referential uses of definite descriptions

The distinction between referential and attributive uses of definite descriptions was introduced by Keith Donnellan in his seminal article 'Reference and Definite Descriptions': "a speaker who uses a definite description attributively … states something about whoever or whatever is the so-and-so. A speaker who uses a definite description referentially … uses the description to enable his audience to

12 This is not to say that according to direct reference theory the sentence is a meaningless sound. The sentence is significant. There is a use, a chain of communication, that corresponds to 'Vulcan' even if, to use Keith Donnellan's terminology, that chain 'ends in a block' (Donnellan 1974). Moreover, the sentence is significant, in that competent speakers associate with 'Vulcan' the same kind of mental files they associate with other proper names.

pick out whom or what he is talking about and states something about that person or thing" (1966: 285).

In other words, the truth conditional contribution of an attributive use of *the G* in an utterance of *the G is H* is the condition *being the unique G*, whereas the truth conditional contribution of a referential use of that description is an object, say a. Truth value at each index in this case will depend on whether a is or is not H.

Appealing again to the instrument of structured propositions discussed in Section 3.1, we can say that attributive uses of definite descriptions result in general propositions of the form $<!G^*, H^*>$, whereas referential uses generate singular propositions, propositions that can be pictured as having the form $<a, H^*>$.

According to Donnellan a characteristic mark of referential uses is the fact that they can refer to objects that do not satisfy the description. So, in his famous example, 'the man drinking a martini' can be used referentially to pick out someone who is just drinking water. This characterization of an essential feature of referential uses has led many to postulate that Donnellan's distinction is purely pragmatic. From this, rather pervasive, point of view, sentences containing definite descriptions always express propositions of the form $<!G^*, H^*>$.

But some semanticists have afforded semantic import to Donnellan's distinction. Among those the most extended interpretation of the difference between referential and attributive uses is due to David Kaplan (1978) and Howard Wettstein (1981). According to Kaplan and Wettstein a referentially used definite description is akin to a demonstration. A speaker that utters, for instance, 'that is heavy,' while pointing with her finger and demonstrating a sack of potatoes, expresses a proposition in which the sack of potatoes in question is the truth conditional contribution of her utterance of 'that.' On that model, a referential use of the description in, say, 'the first person to orbit the Earth was brave' is tantamount to pointing at Yuri Gagarin while uttering 'he was brave.' The truth conditions of the claim expressed are determined, in this and in every possible world, by whether Gagarin was a brave person, even if, with respect to some indices of evaluation, he was not the first cosmonaut to orbit the Earth.

Now, Kaplan's and Wettstein's interpretation of the semantic import of referential uses is not quite faithful to what Donnellan identified as the characteristic mark of referential uses. A demonstration picks out the object demonstrated; similarly a description used as a demonstrative picks an object that satisfies the description. So, on Kaplan's and Wettstein's reading referentially used definite descriptions never refer to objects that do not satisfy the description. Nevertheless, there is, in their approach, a clear difference in the truth conditional content contributed by referential and attributive uses.

Interestingly, if we follow Kaplan and Wettstein in their interpretation of the semantic import of Donnellan's distinction, we may ask what happens if the definite description fails to denote. Because the description is treated, in Kaplan's and Wettstein's approach, as a demonstration, the fate of the definite description referentially used, in such a case, is the same as the fate of an utterance of a demonstrative with an accompanying failed demonstration.

Imagine that due to a distraction, my hand wanders and I point into empty space, meaning to demonstrate an apple, while I say: 'that is delicious!' According to an approach that postulates that the reference of a use of a demonstrative is the entity demonstrated by the demonstration that accompanies the utterance of the demonstrative, the demonstrative has no reference, and hence the sentence expresses no proposition or an incomplete proposition, as was the case with an empty proper name.[13]

The situation with empty referential uses of definite descriptions is, arguably, the same: imagine that the space program never took off and no human has ever orbited the Earth, and that our speaker has been misled into thinking that there was such a person. Perhaps our speaker, massively misled, has thought so much about the (non-existent) space program and the (non-existent) first person to orbit the Earth that, from her perspective, she can make a singular assertion about the individual in question, one whose truth conditions depend on the (alleged) person, using the definite description 'the first person to orbit the Earth' referentially. On the Kaplan-Wettstein approach, this case is similar to one in which a demonstration fails and there is nothing being pointed at. A sentence of the form *the G is H*, where the definite description is used referentially, expresses a proposition of the form $<a, H^*>$ if a satisfies 'being the unique G.' But if no object or individual is uniquely G the proposition simply lacks a constituent and there is no full proposition to be evaluated (or no proposition at all).

Yet, sentences such as 'that is delicious' or 'the first person to orbit the Earth was brave,' in the circumstances envisaged, are not meaningless noises. They are,

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13 Intentionalists would be inclined to say about this kind of case what Donnellan proposed in the case of referential uses of definite descriptions: the speaker refers to the object she has in mind, so it would seem that from an intentionalist perspective there are no empty uses of demonstratives, nor empty referential uses of definite descriptions. But it seems to us that, even from an intentionalist perspective, the possibility of failures of reference, both for uses of demonstratives and for referential uses of definite descriptions, should be contemplated: a speaker may mean to point at something (in the case of a demonstrative) or they may think they have in mind what they believe to be an object (in the case of a referential use of a description) and yet simply fall prey to some trick of the light.

arguably, neither true nor false, on the basis of arguments similar to those put forward for empty names. But assigning to them the truth value 2 would fail to highlight their peculiar form of pathology.[14]

5 The logic of the natural language phenomena

Let us now examine our five formal semantics introduced in Section 2 to see how they fit as formal representations of the natural phenomena of Section 3. As regards negation, the use of the standard four-valued negation seems well justified: it seems clear that if ‘Vulcan is a planet’ has a missing semantic constituent, ‘Vulcan is not a planet’ has the same problem, so it should be classified with the same value, 3.

We should concentrate on how much the interpretation of the values in the truth table for conjunction (and, accordingly, the other defined propositional operators) are justified when 2 marks a sentence that expresses a content that is neither true nor false and 3 marks a sentence which expresses an incomplete content. So in this section we will assume, for the sake of illustration, a Strawsonian approach to definite descriptions and a direct reference approach to propositional content.

We can immediately discard the languages $L_{s\star}$ and $L_{w\star}$. In both of them $1\wedge 3 = 2$ and there is no philosophical justification for that: if ‘Vulcan is a planet’ does not express a full content because ‘Vulcan’ does not refer, it is impossible that ‘Mercury is a planet and Vulcan is a planet’ expresses a full content.

Hence, we can focus on L_{ww}, L_{sw} and L_{ss}. The choice between $\wedge_{ww}$, $\wedge_{sw}$ and $\wedge_{ss}$ is determined once we decide whether the operators restricted to $\{0, 1, 2\}$ and $\{0, 1, 3\}$ correspond to the weak or to the strong Kleene operator. Since it is obvious that when two gappy sentences are of the same type their conjunction still is of the same

14 One could wonder how far these possible applications extend and whether any possible situation in which a sentence expresses an indeterminate proposition should be modeled in a four-valued logic like the ones explored here. This is a complex issue, but we think the answer is no. Consider, for instance, vagueness. Vague sentences have been considered by many authors to express propositions which are indeterminate. But few of them (see, for instance, Tye 1994) have considered that logic to be a three-valued truth-functional one. Supervaluational or fuzzy logics seem more promising candidates for an analysis of vagueness (see a general presentation in Keefe 2000). It is an open question for future study whether and how these logics could be integrated in a four-valued setting.

type[15] we should concentrate on the behavior of gappy sentences in combination with classical ones.

The feature that characterizes the values of the Weak Kleene operators is the fact that the non-classical value is contaminant: if any sentence has that value, the composite formula always has that value. On the contrary, the values of the Strong Kleene operators when one member of a formula has the non-classical value depend on the values of the other component. One way to justify the choice of truth values for the Strong Kleene conjunction when some sentence is indeterminate (a justification that can be equally applied to the other operators) is to check what would happen to the value of the total formula if the indeterminate sentence were true or false. Consider $p \wedge q$ such that the value of p is 2 and the value of q is 1. Then, if the value of p were 1 the value of $p \wedge q$ would be 1, but if the value of p were 0 the value of $p \wedge q$ would be 0. Since the value of $p \wedge q$ in indeterminate between 0 and 1, we get that $2 \wedge 1 = 2$. However, if the value of p is 2 and the value of q is 0, the value of $p \wedge q$ would be 0 irrespective of the classical value of p, hence $2 \wedge 0 = 0$.

We have to decide what behavior is more adequate for the two types of gappy sentences. In the case of gappy sentences that express an indeterminate proposition, like 'the King of France is bald,' there seems to be prima facie a good justification for the strong behavior: in the sentence 'the Prime Minister of Canada is bald and the King of France is bald,' the falsity of the first conjunct seems to entail that the whole sentence is false. Independently of the semantic value of the description 'the King of France,' there is no need to evaluate the second sentence to know that the conjunction has to be false.

As for the case of gappy sentences that do not express a full content, if we go back to our example 'Vulcan is a planet,' it seems to us that it does not matter which is the propositional connective that we apply to this sentence: as long as the denotationless name 'Vulcan' appears in it, the sentence will not express a full content, so the value 3 will be contaminant, and the weak interpretation of the operators should be preferred.[16]

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15 Strictly speaking, this is not obvious: supervaluationists, in the study of the semantics of vague predicates, have given reasons to accept that, when the value of p is 2, $p \wedge p$ should have value 2, but $p \wedge \neg p$ should have value 0, even though the value of $\neg p$ is also 2. However, it is obvious if one restricts the field of semantic considerations to truth-functional semantics, as we do here. If the interpretation of conjunction is truth-functional, any other choice would be arbitrary and would prevent the validity of idempotence, one of the basic properties of conjunction.

16 Against this point, it might be argued that sentences such as (*) 'If Vulcan is a planet, then Vulcan is a planet' should be classified as expressing a true proposition. We disagree: this sentence has the same semantic defect as 'Vulcan is a planet,' so neither of them expresses an evaluable content, and they should be assigned value 3. We can explain the tendency to see (*) as a tautology because the schema $p \rightarrow p$ is valid, so that any instance of it is true. But only instances in which p expresses a proposition can be true. If p does not, the sentence fails to express any proposition. We thank an anonymous referee for raising this point.

With these choices, in our view the most natural logic for the illustration given in Section 3 would be the logic $\mathbf{L_{sw}}$. However the other logics $\mathbf{L_{ww}}$ and $\mathbf{L_{ss}}$ are also interesting. Someone could argue that the best semantics for indeterminate sentences should be also weak or that both should be strong. Cases like this or other possible applications would justify exploring them more fully.[17]

6 New operators in the four-valued semantics

In the previous section we have seen the adequacy of the four-valued semantics to represent formally at least some of the aspects of the natural language phenomena presented in Sections 3 and 4, but we still need to justify how the four-valued semantics allows us to justify the introduction of new operators that increase the expressive power of the language.

In a formal language which represents a portion of natural language in which there are meaningful sentences that do not express propositions it makes sense to add an operator that formalizes the predicate 'this sentence expresses a proposition.' A way of doing that in our four-valued semantics is using what we will call *reflection operators*, i.e., unary operators that when applied to a sentence give the value 1 when the sentence has a specific semantic value, and give value 0 otherwise. They are the operators in Figure 5.

	$\downarrow_0$		$\downarrow_1$		$\downarrow_2$		$\downarrow_3$		$\downarrow_0^\star$		$\downarrow_1^\star$		$\downarrow_2^\star$
0	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**
1	**0**	**1**	**1**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**1**	**1**	**0**
2	**0**	**2**	**0**	**2**	**1**	**2**	**0**	**2**	**0**	**2**	**0**	**2**	**1**
3	**0**	**3**	**0**	**3**	**0**	**3**	**1**	**3**	**3**	**3**	**3**	**3**	**3**

Figure 5: Reflection operators.

17 A possible line of argument that could give evidence in favor of a weak reading of conjunction for indeterminate sentences could start by taking into account sentences like ($\star$) 'both the Prime Minister of Canada and the King of France are bald.' It is not obvious that ($\star$) has to be construed as the conjunction of two sentences. It is also possible to interpret it as attributing the property of baldness to a complex subject, a pair of individuals. In this latter case, the noun phrase 'both the Prime Minister of Canada and the King of France' would be interpreted as a generalized quantifier. Since the pair does not exist if one of the individuals is missing, the sentence would express an indeterminate proposition and not a false one. It would be interesting to study experimentally the reaction of non philosophers towards ($\star$). This analysis opens up interesting issues whose exploration is beyond the scope of this paper.

The operator $\downarrow_3$ expresses that a sentence does not express a proposition, therefore, $\neg\downarrow_3 p$ formalizes 'p expresses a proposition': it is true when p expresses a false, true or indeterminate proposition, and it is false when p does not express a proposition. The operators of semantic reflection for the values 0, 1 and 2 come in two varieties, an external and an internal one. The difference affects the value of the operator for the argument 3. Let us consider semantic reflection for the value 2. In the external version, $\downarrow_2 p$ is true when p expresses an indeterminate proposition and it is false if that is not the case, i.e., if p expresses a true or false proposition or if p does not express a proposition (as, in this case, it certainly does not express an indeterminate proposition). The internal version ($\downarrow_2^\star$) coincides with the external one except that $\downarrow_2^\star 3 = 3$. The rationale for this choice comes from the fact that, if a sentence p does not express a proposition because a constituent of the content is missing, $\downarrow_2^\star p$ also has the same defect, so it should not express a proposition. If 'Vulcan is a planet' does not express a proposition because 'Vulcan' does not contribute the referent that would be necessary to express a whole content, 'It is indeterminate whether Vulcan is a planet' does not express a proposition for the same reason. In the external reading, $\downarrow_2 p$ formalizes the metalinguistic operator 'p does express an indeterminate proposition.' If p is 'Vulcan is a planet,' the sentence 'p does express an indeterminate proposition,' being a metalinguistic analysis of p, does express a proposition, and a false one, since 'Vulcan is a planet' does not express a proposition.

This distinction can also be applied to negations. Let us consider two varieties of negation (see Figure 6).

The first one expresses the external version of a strong form of negation and corresponds to a metalinguistic reading of 'it is not the case that … is true': if p has a value other than true, then it is true that it is not the case that p is true. The second

	$\neg_S$
0	1
1	0
2	1
3	1

	$\neg_\star$
0	1
1	0
2	1
3	3

Figure 6: Varieties of negation.

corresponds to the internal reading of strong negation, justified with the same rationale explained in the internal reading of the operators of semantic reflection.[18]

One should notice that, even though all these operators (both internal and external) are non monotonic on the order $Y4$, there is a prima facie reason to want to incorporate them into the language precisely because they allow the object language to express distinctions that are introduced in the metalanguage when the object language is analyzed semantically. The commitment to the acceptability of these operators is then implicit in the acceptance of the original semantic analysis of the language.

The expressive power afforded by the reflection operators allows the expanded language to formalize semantic properties of the basic language. For instance, the compositional principle: 'if a conjunction expresses a proposition, both conjuncts express propositions,' can be represented as: $\vDash \neg \downarrow_3 (p \wedge q) \supset (\neg \downarrow_3 p \wedge \neg \downarrow_3 q)$. In our interpretation of the truth values, the principle is true, and its formalization valid in our preferred logic, $\mathbf{L_{sw}}$.[19]

We still have to return to the criticism we presented in Section 2: are there any specific logical advantages in using a four-valued semantics over a three-valued one that does not distinguish between gappy sentences? We would like to end up by pointing out an important advantage of a four-valued over a three-valued representation of gappy sentences: languages can represent more faithfully the semantic structure of propositions if we allow in the formal semantics the representation of sentences not expressing propositions. The advantage only shows up when the sentential language is made self-referential and a truth predicate is added to it.[20] If we use a three-valued language and lump together all gappy sentences, it is well known that a Strong Kleene language expanded with the operator that represents 'p is a gappy sentence' (i.e., the operator $\downarrow$ such that $\downarrow 0 = \downarrow 1 = 0$, $\downarrow 2 = 1$) cannot contain a Kripkean truth predicate, while a Weak

18 Notice that $\neg_s = \neg \downarrow_1$, $\downarrow_1 = \neg_s \neg_s$, $\neg_\star = \neg \downarrow_1^\star$ and $\downarrow_1^\star = \neg_\star \neg_\star$, so commitment to the reflection operators $\downarrow_1$ and $\downarrow_1^\star$ is equivalent to commitment to $\neg_s$ and $\neg_\star$. The logic $\mathbf{L_{sw}}$ expanded with $\neg_s$ is one of the logics of formal undeterminedness (LFUs) studied in Szmuc (2016).

19 Notice that $\supset$ represents the material conditional, so the principle is equivalent to: $\vDash \neg (\neg \downarrow_3 (p \wedge q) \wedge \neg (\neg \downarrow_3 p \wedge \neg \downarrow_3 q))$. The principle is also valid in $\mathbf{L_{ww}}$, and invalid in $\mathbf{L_{ss}}$, $\mathbf{L_{w\star}}$ and $\mathbf{L_{s\star}}$. Another example is given by the semantic principle: 'if $p \vee \neg p$ expresses a proposition, then it expresses a true or an indeterminate one,' formalized as: $\vDash \neg \downarrow_3 (p \vee \neg p) \supset (\downarrow_1 (p \vee \neg p) \vee \downarrow_2 (p \vee \neg p))$, which is valid in the five logics.

20 When modeling the semantics of natural language, a truth predicate is an essential tool. Also, the formal language has to be able to talk about its own sentences, as natural language does. So we think that these conditions are very natural. For details on how to create self-referential sentential languages, see the stipulation logic in Visser (1984). For Kripkean truth predicates, see Kripke (1975).

Kleene language does.[21] Hence, Weak Kleene languages are more capable of expressing their own semantics than Strong Kleene ones. However, a strong semantics for gappy sentences is usually considered to be philosophically well justified, so one has to choose between philosophical cogency and expressive power. If we move to the four-valued case and choose the language L_{sw}, a result of Martínez-Fernández (2014, section 4) shows that L_{sw} can be expanded with $\downarrow_1^*$, $\downarrow_2^*$ and $\downarrow_3$ and still contain a Kripkean truth predicate.[22] We can then accept a strong semantics for indeterminate sentences and yet have a powerful expressive language.[23]

7 Conclusion

In this paper we have presented some evidence that attention should be paid to the exploration of certain four-valued logics that have not been previously discussed in depth. On the one hand, we have shown that at least one of those four-valued logics ($\mathbf{L_{sw}}$) can be used to represent adequately certain phenomena that, according to some semantic theories, occur in natural language. On the other hand, we have motivated the interest of reflection operators which express semantic properties of four-valued languages and briefly explored the usefulness of expanding $\mathbf{L_{sw}}$ with the reflection operators.

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21 For a proof of this fact, see Gupta and Belnap (1993: section 2E).
22 As we explain in footnote 6, the operator $\neg_*$ can be defined in the expanded language.
23 As shown in Martínez-Fernández (2014), these languages can also add special conditionals which make true the Tarskian biconditionals: 'it is true that *p* if, and only if, *p*' for sentences which express a proposition, something that is impossible to do in three-valued logic for gappy sentences. For instance, it is possible to add a four-valued conditional that coincides with Łukasiewicz's conditional on the set {0, 1, 2} and is contaminant on the value 3.

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