ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Analytic capacity and singular integrals

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Abstract

In this project we introduce the notion of analytic capacity (γ) as well as some of its essential properties. Using this concept we identify the family of *removable* compact subsets of $\mathbb{C}$, which are those such that, for any bounded holomorphic function defined on their complementary, they allow to extend analytically such function to the whole complex plane. From this point on, we discuss a possible geometric characterization for removable subsets, popularly known as the Painlevé problem. The previous task is done in terms of the Hausdorff dimension of these subsets, obtaining a full classification for values different than 1. This remaining case, usually referred to as the *critical dimension associated to* γ, has to be dealt with apart. It is at this point that we invoke the theory of singular integrals in order to study a particular family of these subsets: those contained in graphs of Lipschitz functions. We end our project by tackling this case, introduced by Arnaud Denjoy in the early 1900's, and providing a proof of a characterization theorem in this particular setting.

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Chapter 1

Introduction

Let us start by recalling a basic result that is typically encountered in a first course in complex analysis. The details of the proof can be found in [14, Proposition 3.3.4].

Theorem. *Let $D \subset \mathbb{C}$ be open, $a \in D$ and $f \in \mathcal{H}(D \setminus \{a\})$. Then, f having a (unique) holomorphic extension over a is equivalent to f being continuously extendable over a.*

This result can be extended, by means of Morera's theorem, to the case where, instead of single point, the set where f is continuous but not holomorphic is a line segment.

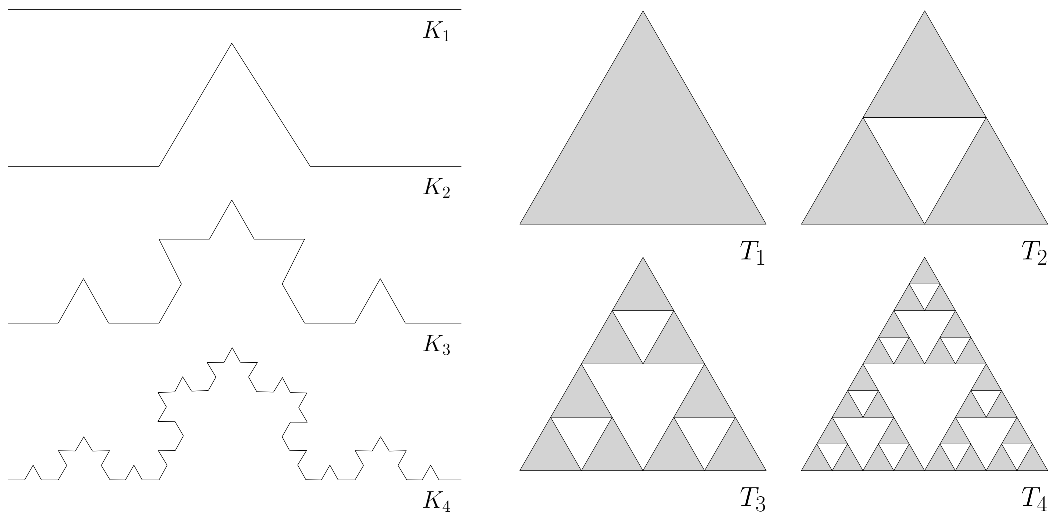
Bearing in mind such property, assume that we want to distinguish between compact subsets $E \subset \mathbb{C}$ depending on the kind of holomorphic functions we may define on their complementary. That is, depending on the properties of the family $\mathcal{H}(\mathbb{C} \setminus E)$. For example, if $E = \{a\}$ a single point it is clear that the elements of $\mathcal{H}(\mathbb{C} \setminus E)$ are either functions which are entire or that exhibit a pole or an essential singularity at $z = a$. On the other hand, if E is a line segment, such family does not admit a clear characterization in the same terms as with the previous example. And the reason for this relies, mainly, in the fact that we have imposed no condition on f other than being holomorphic on $\mathbb{C} \setminus E$. Hence, a first reasonable way to approach the problem of classifying compact subsets of $\mathbb{C}$ would be to restrict the properties of f in order to gain control over its extensions. So a question arises: what is the proper subfamily of $\mathcal{H}(\mathbb{C} \setminus E)$ suitable for this purpose?

The choice that will be made in this project is to restrict f to the family of *bounded* holomorphic functions on $\mathbb{C} \setminus E$. This way, it is possible to distinguish compact subsets of $\mathbb{C}$. Indeed, one can check, using Liouville's and Riemann mapping theorems [13, §5.1]; that the possible extensions for a bounded function $f \in \mathcal{H}(\mathbb{C} \setminus E)$ are different if E is a point or a line segment. Moreover, if one proceeds this way, a new concept emerges naturally: the *analytic capacity* of a compact subset. Its definition was first introduced by Ahlfors [1] (1947) and it is the following:

Definition (Analytic capacity)**.** The analytic capacity of $E \subset \mathbb{C}$ compact is

$$\gamma(E) := \sup |f'(\infty)|,$$

where $f'(\infty) := \lim_{z\to\infty} z\big(f(z) - f(\infty)\big)$ and the supremum is taken over all functions $f \in \mathcal{H}(\mathbb{C} \setminus E)$ satisfying $|f| \leq 1$.



(a) First four iterates involved in the definition the von Koch curve

(b) First four iterates involved in the definition of the Sierpinski triangle

Figure 1.1: Two examples of sets with Hausdorff dimension greater than 1.

Throughout the project we will see that the nature of the extensions of a bounded function in $\mathcal{H}(\mathbb{C} \setminus E)$ is closely related to the analytic capacity of E. In particular, we will prove that compact subsets with null analytic capacity, known as *removable subsets*, are exactly those such that for any $f \in \mathcal{H}(\mathbb{C} \setminus E)$ bounded, they admit an holomorphic extension of f to $\mathbb{C}$.

At this point, we may also wonder if there is a way, apart from the previous (analytic) point of view, to identify removable subsets in a pure geometric way. This perspective leads to the study of the Hausdorff dimension of compact sets of $\mathbb{C}$, that turns out to be connected to their analytic capacity. In fact, we will be able to prove the following fundamental result:

Theorem. *If $E \subset \mathbb{C}$, then $dim_{\mathcal{H}}(E) < 1$ implies that E is removable. On the other hand, if $dim_{\mathcal{H}}(E) > 1$ the set will not be removable.*

This means, for example, that the usual real Cantor set, that has Hausdorff dimension $\log_3(2) \approx 0.631$ is removable; but a greater variety of sets, precisely those with a well-defined positive length (maybe infinite), are not. Examples are a line-segment, the graph of a Lipschitz function or even more exotic sets like the von Koch curve, with Hausdorff dimension $\log_3(4) \approx 1.262$; or the Sierpinski triangle, with Hausdorff dimension $\log_2(3) \approx 1.585$ (see Figure 1.1). The reader may consult the book of Mattila [16, Chapter 4] for computational techniques of the Hausdorff dimension of *self-similar sets* such as the last two mentioned.

And what about the case $\dim_{\mathcal{H}}(E) = 1$*?* It is, precisely, in the study of this critical dimension that the theory of singular integrals becomes helpful. We will introduce the Calderón-Zygmund theory to prove a result – proposed by Arnaud Denjoy (1884-1974) near the first decade of the 20th century – related to the specific case where E is a finite rectifiable curve.

The previous approach clarifies the connection between analytic capacity and singular integrals; but, in fact, although the early work in analytic capacity relies on one complex variable methods, in recent articles such as the one from Verdera [28] (2007), γ is expressed as

$$\gamma(E) := \sup_T \Big\{ |\langle T, 1\rangle| \ : \ T \in \mathcal{C}_c^\infty(\mathbb{R}^2)' \ \text{ s.t. } \ \text{supp}(T) \subset E \ \text{ and } \ \|T * z^{-1}\|_\infty \leq 1 \Big\}.$$

Let us justify, rather informally, the reason for this alternative definition. First notice that since $(-\pi z)^{-1}$ is the fundamental solution to the $\overline{\partial}$-equation (Theorem A.2.1), then $\overline{\partial}\big(T * z^{-1}\big) \simeq T$, and so we can think T as represented by a function of the type $\overline{\partial} f$. Notice that since T is supported on E, then f becomes holomorphic in $\mathbb{C} \setminus E$ and, in fact, bounded, because $\big\|T * z^{-1}\big\|_\infty = \|f\|_\infty < \infty$. Consider now $R > 0$ so that $\text{supp}(\overline{\partial} f) \subset D(0,R)$ and pick $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, with $0 \leq \varphi \leq 1$, satisfying $\varphi \equiv 1$ for $|z| \leq 2R$ and $\varphi \equiv 0$ for $|z| \geq 3R$. This way, the Cauchy-Pompeiu formula (Theorem A.1.1, assuming ∂E regular enough) yields

$$\langle T, \varphi \rangle = \int_E \overline{\partial} f(z) \varphi d\mathcal{L}^2(z) = \int_E \overline{\partial} f(z) d\mathcal{L}^2(z) = \frac{1}{2i} \int_{\partial E} f(z) dz = \pi f'(\infty),$$

and since R is arbitrary, we may define $\langle T, 1\rangle$ as $\pi f'(\infty)$, so that it clarifies that both definitions of $\gamma(E)$ are equivalent. Hence it is natural to suggest that there will be a connection between the properties of analytic capacity and those of the singular integral given by the convolution against a specific kernel, precisely z^{-1}. In fact, it is by changing the nature of this kernel that we are able to define many other capacities (such as the harmonic, Lipschitz or $\mathcal{C}^1$ capacities, for example. See Mattila & Paramonov [17] for more details).

In any case, our project will focus on studying a particular capacity, the analytic capacity. As we have mentioned initially, we will present some of its properties as well as its relation to Hausdorff measure and the geometric characterization of removable subsets of $\mathbb{C}$. For this, we will need some basic results in measure theory in our proofs, found in the introductory chapter of Mattila [16]. From that point on, in order to study those sets of Hausdorff dimension 1, we will develop the necessary theory of singular integrals to give a partial proof of a conjecture proposed by Denjoy, that was first tackled by Calderón [3] in 1977 and fully answered by Coifman, McIntosh & Meyer [4] in 1982.

The geometric characterization of removable compact subsets of the complex plane with Hausdorff dimension 1 has been a challenging problem of the 20th century, and many mathemticians have worked on it (Pommerenke (1960) [22]; Garnett (1970) [11]; Calderón (1977) [3]; Coifman, McIntosh & Meyer (1982) [4]; David (1984) [5]; Mattila & Paramonov (1995) [17]; Garnett & Verdera (2003) [12]; Tolsa (2003) [25], (2005) [26]). Two of the most recent and relevant results are found in the last two papers by Xavier Tolsa, in which two fundamental tools were introduced. The first was the *semiadditivity* of γ [25], meaning that for any compact subsets $E, F \subset \mathbb{C}$ there exists an absolute constant $C > 0$ so that

$$\gamma(E \cup F) \leq C\big(\gamma(E) + \gamma(F)\big).$$

The second refers the *stability of analytic capacity with respect to homeomorphic bilipschitz transformations* [26]. That is, if $\varphi : \mathbb{C} \to \mathbb{C}$ is a bijective map satisfying that there exists $L > 0$ so that

$$L^{-1}|z - w| \leq |\varphi(z) - \varphi(w)| \leq L|z - w|, \qquad \forall z, w \in \mathbb{C},$$

then, there exists $C > 0$, depending only on φ, so that for any compact subset E

$$C^{-1}\gamma(E) \leq \gamma(E) \leq C\gamma(E).$$

Chapter 2

Analytic capacity

We would like to begin our project by defining, straightaway, the concept of analytic capacity, which will be one of the main pillars of our study. In order to do it, we review first some of the basic notation needed to define it, to carry on with the statement of some of its basic properties. To provide a proof of the latter, we will need some important results in complex analysis such as Montel's, Weierstrass' or the 1/4 Koebe theorems. These will lead us to a more geometric comprehension of the notion of analytic capacity. We will continue by giving a particular meaning to those compact sets $E \subset \mathbb{C}$ which have null analytic capacity. This goal will be achieved using the Cauchy transform, a tool that will allow us, eventually, to connect the previous notions with the theory of singular integrals using the Hausdorff measure as an intermediary.

2.1 Some preliminaries and definition of analytic capacity

Previous to presenting the notion of analytic capacity let us clarify some aspects of notation. Our general setting will be the complex plane $\mathbb{C}$ and holomorphic functions defined on open subsets of the form $\mathbb{C} \setminus E$, for E compact. Observe that these may have different connected components, since E might not be simply connected. This gives rise to a first definition.

Definition 2.1.1 (Outer boundary)**.** Let $E \subset \mathbb{C}$ be a compact set. Its *outer boundary*, denoted by $\partial_o E$, will be the boundary of the unbounded connected component of $\mathbb{C} \setminus E$. It is clear that $\partial_o E \subset \partial E$.

Also, given $f : \mathbb{C} \setminus E \to \mathbb{C}$ holomorphic function, we will use the following notation

$$f(\infty) := \lim_{z \to 0} (f \circ g)(w), \quad \text{where} \quad g(w) = \frac{1}{w},$$

whenever the previous limit exists. This way of writing is motivated by viewing the domain of f as an open subset of the Riemann sphere $\mathbb{C}_\infty$, and defining its value at ∞ via the Möbius transformation $g(w) = w^{-1}$, which is just the usual inversion.

This way of thinking allows us also to define, for example, the values of the derivatives of f at infinity just as

$$f^{(k)}(\infty) := (f\circ g)^{(k)}(0), \quad \text{so in particular} \quad f'(\infty) := \lim_{z\to\infty} z\big(f(z)-f(\infty)\big).$$

This motivates considering the Laurent expansion of f near ∞, which is the Laurent expansion of $f\circ g$ near 0. Since f is holomorphic in $\mathbb{C}\setminus E$, $f\circ g$ also is in a neighborhood of 0. Therefore, the Laurent expansion at this point lacks its principal part, obtaining

$$(f\circ g)(w) = a_0 + a_1 w + a_2 w^2 + \cdots$$

which holds for every $w\in \overline{D_r}\subset g(\mathbb{C}\setminus E)$, where $\overline{D_r} = \overline{D(0,r)}$ is the closed disk of radius $r>0$ centered at the origin. Recall that the coefficients a_k are given by

$$a_k = \frac{1}{2\pi i}\int_{\partial D_r} \frac{(f\circ g)(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{-\partial D_{1/r}} f(z)z^{k+1}\frac{dz}{-z^2} = \frac{1}{2\pi i}\int_{\partial D_{1/r}} f(z)z^{k-1}dz.$$

So, formally, we can state the definition of the Laurent expansion of f near infinity as:

Definition 2.1.2 (Laurent expansion near ∞)**.** Let $E\subset\mathbb{C}$ be compact and $f:\mathbb{C}\setminus E\to\mathbb{C}$ holomorphic. Its *Laurent expansion near* ∞ will be

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad \text{where} \quad a_k = \frac{1}{2\pi i}\int_\Gamma f(z)z^{k-1}dz,$$

and Γ is any curve (regular enough) enclosing E. Notice that

$$f^{(k)}(\infty) = \frac{k!}{2\pi i}\int_\Gamma f(z)z^{k-1}dz, \quad \text{so in particular} \quad f'(\infty) = \frac{1}{2\pi i}\int_\Gamma f(z)dz. \tag{2.1.1}$$

Now we are ready to present the notion of analytic capacity, that will possibly seem, at first, rather not intuitive. By this we mean that its definition might not be as transparent as it could be to convey, just with a first glance, what is its purpose as a mathematical object neither the meaning of its value. The motivation to conceive the previous concept was to try to understand better the so called *removable subsets*.

Definition 2.1.3 (Removable subset)**.** A compact subset $E\subset\mathbb{C}$ will be called *removable* (for bounded holomorphic functions) if for every open set Ω containing E, any bounded holomorphic function on $\Omega\setminus E$ has an analytic extension to Ω.

And it was in 1947 that Ahlfors [1] introduced the concept of analytic capacity, a positive quantity associated to each compact subset of $\mathbb{C}$ that becomes helpful when characterizing removability.

Definition 2.1.4 (Analytic capacity)**.** The analytic capacity of a compact subset $E\subset\mathbb{C}$ is

$$\gamma(E) := \sup|f'(\infty)|, \tag{2.1.2}$$

where the supremum is taken over all holomorphic functions $f:\mathbb{C}\setminus E\to\mathbb{C}$ with $|f|\leq 1$. Holomorphic functions defined on $\mathbb{C}\setminus E$ satisfying this last condition will be called *admissible*.

Example 1: one trivial example to consider is the case where E is a finite collection of points. In this case, notice that the family of admissible functions is the family of entire bounded functions, that is, the family of constant functions. Therefore $\gamma(E) = 0$.

Example 2: consider $E = \overline{D_r}$, the closed disk of radius r centered at the origin. In this case we deduce that its analytic capacity must be positive, since taking the admissible function $f(z) = rz^{-1}$, we get $\gamma(E) \geq r$. Later on we will see that, in fact, we have an equality.

Hence, the previous results may suggest that we should understand $\gamma(E)$ as "the size" of E as a removable singularity. In Section 2.3 we will formalize the connection between the value of $\gamma(E)$ and its removability with respect to the bounded functions of $\mathcal{H}(\mathbb{C} \setminus E)$.

2.2 Basic properties of analytic capacity

In this section we would like to give some of the most basic but yet fundamental properties of analytic capacity. In order to discuss and prove these features, we will need two important theorems that are typically encountered in an advanced course in complex analysis. These will provide us with information about converging sequences of holomorphic functions and the properties of their limits regarding holomorphicity.

2.2.1 The theorems of Weierstrass and Montel

Let us denote by $\Omega \subset \mathbb{C}$ a domain of the complex plane and $\mathcal{H}(\Omega)$ the class of holomorphic functions on Ω. The first result we will present, known as Weierstrass' theorem, gives a sufficient condition to ensure the holomorphicity of a function obtained as a limit of holomorphic functions. To obtain it, first we need to consider a specific kind of convergence.

Definition 2.2.1 (Uniform convergence on compact subsets)**.** We will say that a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}(\Omega)$ *converges uniformly on compact subsets of* Ω if there exists a function $f : \Omega \to \mathbb{C}$ such that for any $K \subset \Omega$ compact

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(K)} = \lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0.$$

Theorem 2.2.1. (Weierstrass, [14, Theorem 3.1.8])**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{H}(\Omega)$ that converges uniformly to f on compact subsets of Ω. Then*

1. *$f \in \mathcal{H}(\Omega)$.*
2. *For each $k \in \mathbb{N}$, the sequence of k-th derivatives $(f_n^{(k)})_{n\in\mathbb{N}}$ converges uniformly to $f^{(k)}$ on compact subsets of Ω.*

Proof. Let us begin by proving *1*. Since holomorphicity is a local property, it will be enough to fix an arbitrary $z_0 \in \Omega$ and a small disk $\overline{D(z_0, \varepsilon)} \subset \Omega$ and check the property there. By the hypothesis of uniform convergence, f is continuous in $\overline{D(z_0, \varepsilon)}$. Therefore, applying Morera's theorem in this disk and using that the uniform convergence allows us to take limits outside of the integral, we are done.

Concerning *2*, we again fix any $z_0 \in \Omega$, a disk $D(z_0, \varepsilon) \subset \Omega$ and a possibly bigger one of radius $R \geq \varepsilon > 0$ so that $\overline{D(z_0, R)} \subset \Omega$. The Cauchy formula applied to $f_n - f$ yields

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_{\partial D(z_0,R)} \frac{f_n(w)-f(w)}{(w-z)^{k+1}}dw, \qquad \forall z \in \overline{D(z_0,\varepsilon)}.$$

Then

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \sup_{w\in\overline{D(z_0,R)}} |f_n(w)-f(w)|\frac{k!}{|R-\varepsilon|^{k+1}}R.$$

Therefore taking the supremum over all $z \in \overline{D(z_0,\varepsilon)}$ and letting $n \to \infty$ we obtain the uniform convergence for closed disks. But since every compact subset can be covered by a finite number of open disks, we can deduce the result for general compact subsets. □

Turning to the second result, in some way, it will be similar to the well-known Arzelá-Ascoli theorem. That is, it will give a sufficient condition for a sequence of holomorphic functions defined on a compact subset to have a subsequence converging uniformly on compact subsets. As we know, the Arzelá-Ascoli theorem relies on two conditions that the family of functions must satisfy: uniform boundedness and equicontinuity. In our setting, the functions of any family we may consider will be holomorphic, and this property will relax both of the previously mentioned requirements. Montel's theorem (see [24, §2.2] for a proof) formalizes this fact, and in order to state it we will need the following definition.

Definition 2.2.2 (Normal family)**.** Let $\Phi \subset \mathcal{H}(\Omega)$ be a family of holomorphic functions. We will say that Φ is a *normal family* if every sequence in Φ has a subsequence converging uniformly on compact subsets of Ω.

Theorem 2.2.2 (Montel)**.** *Let $\Phi \subset \mathcal{H}(\Omega)$ be a family of holomorphic functions. Then the following are equivalent*

1. *Φ is normal.*
2. *Φ is uniformly bounded on compact subsets. That is, for each $K \subset \Omega$ compact, there exists $C_K > 0$ such that $\sup_{f\in\Phi}\sup_{z\in K}|f(z)| \leq C_K$.*

2.2.2 Properties of analytic capacity

Let us state some of the most basic properties concerning analytic capacity. Some of them will follow right away from its definition, just as the following ones.

Proposition 2.2.3. *Let $E, F \subset \mathbb{C}$ and $z, \lambda \in \mathbb{C}$. Then the following hold*

1. *If $E \subset F$, then $\gamma(E) \leq \gamma(F)$.*
2. *$\gamma(z+\lambda E) = |\lambda|\gamma(E)$.*
3. *If E is compact, then $\gamma(E) = \gamma(\partial_o E)$.*

Proof. Property *1* is just a consequence of the fact that if f is admissible for E, then it is also admissible for F. Regarding *2*, proving that γ is invariant under translations is just a matter of a change of variables given, precisely, by this same translation. Also, if we dilate the set E by λ, again by a change of variables and taking into account that $|f'(\lambda z)| = |\lambda||f'(z)|$, the result easily follows.

For *3* we apply *1* to get $\gamma(\partial_o E) \leq \gamma(E)$. The equality can be deduced arguing by contradiction, using the fact that we can always extend, if necessary, an holomorphic function in $\mathbb{C} \setminus E$ to $\mathbb{C} \setminus \partial_o E$ (by defining it to be constant with modulus less than 1, for example) as well as that E is bounded, so the possibly modified values of the extension do not affect $|f'(\infty)|$. □

Let us give now the first main theorem about analytic capacity that will assert that, for $E \subset \mathbb{C}$ compact, the supremum involved in the definition of analytic capacity is attained by some admissible function. That is, for every compact subset $E \subset \mathbb{C}$, there is an extremal function f for $\gamma(E)$. In fact, under an additional condition, this function will be unique.

Theorem 2.2.4. *Let $E \subset \mathbb{C}$ be compact. Then, the supremum defining $\gamma(E)$ is attained. Moreover, if $\gamma(E) > 0$, the extremal function is unique and it satisfies $f(\infty) = 0$.*

Proof. We begin by proving the existence of an admissible function attaining the supremum. Let Φ be the family of admissible functions for E. By property *3* of Proposition 2.2.3 we can assume that each connected component of E is simply connected, so that $\mathbb{C} \setminus E$ is connected; and by property *2* we also assume that there exists a connected component of E that contains the origin. Consider now the auxiliary family of functions

$$\Phi' := \big\{ f \circ g \;:\; f \in \Phi,\, g(z) = z^{-1} \big\},$$

whose elements are holomorphic in $g(\mathbb{C} \setminus E)$ with modulus bounded by 1. Notice also that $g(\mathbb{C} \setminus E)$ is a bounded subset of the complex plane that contains the origin. We pick a sequence $(f_n \circ g)_{n\in\mathbb{N}} \subset \Phi'$ so that at the origin it approximates the value $\sup_{f\in\Phi} |(f \circ g)'(0)|$. Applying Montel's theorem 2.2.2 we deduce that the family Φ' is normal in $g(\mathbb{C} \setminus E)$, so we may consider a subsequence $(f_{n_k} \circ g)_k$ convergent on compact subsets. This way, taking an exhaustion by compact subsets of $g(\mathbb{C} \setminus E)$ and applying Weierstrass' theorem 2.2.1, we may construct F holomorphic in $\mathbb{C} \setminus E$ with $|F| \leq 1$ so that $|(F \circ g)'(0)| = \sup_{f\in\Phi} |(f \circ g)'(0)|$, that is $|F'(\infty)| = \sup_{f\in\phi} |f'(\infty)|$, and we are done.

Before proving the uniqueness, we will see that if $\gamma(E) > 0$, then any extremal function f must satisfy $f(\infty) = 0$. To do it, we consider the following function on $\mathbb{C} \setminus E$

$$h(z) = \frac{f(z) - f(\infty)}{1 - \overline{f(\infty)} f(z)} = \frac{f(z) - (f \circ g)(0)}{1 - \overline{(f \circ g)(0)} f(z)}, \quad \text{where} \quad g(z) = \frac{1}{z}.$$

Since $|f(z)| \leq 1$ in $\mathbb{C} \setminus E$, we notice that $h(z)$ has the form of an automorphism of the unit disk, and so $|h(z)| \leq 1$ for every $z \in \mathbb{C} \setminus E$, meaning that h is admissible for E. Moreover, $h(\infty) = (h \circ g)(0) = 0$ and

$$h'(\infty) = (h \circ g)'(0) = \lim_{z\to 0} \frac{(h \circ g)(z)}{z} = \lim_{z\to\infty} z h(z) = \lim_{z\to\infty} \frac{z(f(z) - f(\infty))}{1 - \overline{f(\infty)} f(z)} = \frac{f'(\infty)}{1 - |f(\infty)|^2}.$$

Therefore, $|h'(\infty)| \geq |f'(\infty)|$. But since f attains the supremum, the inequality must be an equality and so necessarily $f(\infty) = 0$.

Finally, let us see now that if $\gamma(E) > 0$, such function f is unique (we will follow the proof of Fisher [10, Theorem 1]). Suppose that f_1, f_2 are admissible functions with $f_1'(\infty) = f_2'(\infty) = \gamma(E)$ and $f_1(\infty) = f_2(\infty) = 0$. Let

$$f=\frac{f_1+f_2}{2},\qquad g=\frac{f_2-f_1}{2}.$$

Notice that both f and g are admissible with $f(\infty)=g(\infty)=0$, and $f'(\infty)=\gamma(E)$. In addition, we have the identities $f_1=f-g$ and $f_2=f+g$. Therefore $|f\pm g|\leq 1$, which means, explicitly

$$|f\pm g|^2=|f|^2+|g|^2\pm 2\mathfrak{Re}(f\overline{g})\leq 1.$$

Adding both inequalities, we deduce $|f|^2+|g|^2\leq 1$, which at the same time implies

$$\frac{|g|^2}{2}\leq\frac{1-|f|^2}{2}=\frac{(1-|f|)(1+|f|)}{2}\leq 1-|f|\quad\Leftrightarrow\quad |f|+\frac{|g|^2}{2}\leq 1.$$

Now, let us assume that $g\neq 0$ in the unbounded component of $\mathbb{C}\setminus E$ and reach a contradiction. If this was the case, in particular $g\neq 0$ in a neighborhood of ∞ and we may consider the Laurent series of $g^2/2$ near ∞

$$\frac{g(z)^2}{2}=\frac{a_n}{z^n}+\frac{a_{n+1}}{z^{n+1}}+\cdots$$

with $n\geq 2$, because $g(\infty)=0$ and g is raised to the second power. Assume also that n was chosen so that $a_n\neq 0$. Now we choose $\varepsilon>0$ small enough and define the function

$$\widetilde{f}(z)=f(z)+\varepsilon\overline{a_n}z^{n-1}\frac{g(z)^2}{2},$$

so that $|\varepsilon\overline{a_n}z^{n-1}|\leq 1$ in a bounded neighborhood V of E. Then, in this neighborhood

$$|\widetilde{f}(z)|\leq|f(z)|+\left|\varepsilon\overline{a_n}z^{n-1}\frac{g(z)^2}{2}\right|\leq|f(z)|+\frac{|g(z)|^2}{2}\leq 1.$$

So in $(\mathbb{C}\setminus E)\cap V$ the function $|\widetilde{f}|$ is bounded by 1. On the other hand, since by construction $\widetilde{f}(\infty)=0$, we also have this bound for $|\widetilde{f}|$ in a neighborhood of ∞. This implies, by the maximum modulus principle, that the bound holds in the whole unbounded component of $\mathbb{C}\setminus E$, i.e. $\widetilde{f}$ is an admissible function. Finally, since $\widetilde{f}'(\infty)=f'(\infty)+\varepsilon|a_n|^2>\gamma(E)$, we reach the desired contradiction. From it we deduce that $g=0$, i.e. $f_1=f_2$, in the unbounded component of $\mathbb{C}\setminus E$, which we have assumed to be connected. Hence $f_1=f_2$. □

Definition 2.2.3 (Ahlfors function)**.** Let $E\subset\mathbb{C}$ be a compact set with $\gamma(E)>0$. Then, the unique admissible function $f:\mathbb{C}\setminus E\to\mathbb{C}$ attaining the supremum in $\gamma(E)$ is called the *Ahlfors function* of E, and it is such that $f(\infty)=0$.

The existence of functions defining the analytic capacity for compact subsets provides a first tool to compute this quantity by approximating methods, as the next result shows.

Corollary 2.2.5. (Outer regularity, [27, Proposition 1.7])**.** *Let $(E_n)_{n\geq 0}$ be a sequence of compact sets in $\mathbb{C}$ satisfying $E_{n+1}\subset E_n$ for each n. Then*

$$\gamma\Bigg(\bigcap_{n\geq 0}E_n\Bigg)=\lim_{n\to\infty}\gamma(E_n).$$

Proof. Set $E := \bigcap_{n\geq 0} E_n$ and notice that property *1* of Proposition 2.2.3 implies that the sequence $\{\gamma(E_n)\}_{n\geq 0}$ is non-increasing. So in particular $\lim_{n\to\infty}\gamma(E_n)$ makes sense. In addition, since $E\subset E_n$ for all n, we obtain $\gamma(E)\leq \lim_{n\to\infty}\gamma(E_n)$. To see the other inequality, we pick f_n admissible for E_n with $|f_n'(\infty)| = \gamma(E_n)$, for every n. By Montel's theorem 2.2.2 we get that $(f_n)_n$ is a normal family on $\mathbb{C}\setminus E$. Hence, we may choose $(f_{n_k})_k$ a subsequence that converges uniformly on compact subsets of $\mathbb{C}\setminus E$ to a function f, that also satisfies $|f|\leq 1$ in $\mathbb{C}\setminus E$. In addition, by Weierstrass' theorem 2.2.1, f is holomorphic in this domain, meaning that f is, in fact, admissible for E. Now, using the uniform convergence of the sequence $(f_{n_k})_k$ and the relation (2.1.1) for $k=1$, we get

$$f'(\infty) = \lim_{k\to\infty} f_{n_k}'(\infty).$$

On the other hand we know that $|f_{n_k}'(\infty)| = \gamma(E_{n_k})$, and so we obtain the identity

$$|f'(\infty)| = \lim_{k\to\infty}\gamma(E_{n_k}) = \lim_{n\to\infty}\gamma(E_n),$$

Therefore we conclude $\gamma(E)\geq |f'(\infty)| = \lim_{n\to\infty}\gamma(E_n)$ and we are done. $\square$

We continue by giving another helpful theorem to compute the analytic capacity of compact connected subsets of $\mathbb{C}$. The result involves a conformal transformation of the unbounded connected component of $\mathbb{C}\setminus E$ to $\mathbb{D}$, the unit disk.

Theorem 2.2.6. *Let $E\subset\mathbb{C}$ be a compact connected set with with at least two points. Let f be a conformal map from the unbounded connected component of $\mathbb{C}\setminus E$ to $\mathbb{D}$ satisfying $f(\infty)=0$. Then, $\gamma(E) = |f'(\infty)|$.*

Observe that the assumption of E being different from a single point is to avoid the case where this point is the origin, since in this context we would not be able to apply the Riemann mapping theorem [13, §5.1] to ensure the existence of the conformal map appearing in the statement (since the inverted domain would be the whole $\mathbb{C}$).

Proof. Notice that f is admissible, so $\gamma(E)\geq |f'(\infty)| > 0$. The second inequality can be assumed, without loss of generality, by the Riemann mapping theorem. Let g be any other admissible function that, by Theorem 2.2.4, is such that $g(\infty)=0$. We want to see that $|g'(\infty)|\leq |f'(\infty)|$. Observe that $g\circ f^{-1}:\mathbb{D}\to\mathbb{D}$ is an holomorphic map fixing the origin. Hence, by Schwarz's lemma [13, §3.1, p. 130] we get $|(g\circ f^{-1})(z)|\leq |z|$, $\forall z\in\mathbb{D}$. Defining now $w := f^{-1}(z)\in\mathbb{C}\setminus E$, we have

$$|g(w)|\leq |f(w)|, \qquad \forall w\in\mathbb{C}\setminus E.$$

In addition, using the genuine definition of derivative at ∞ (a method that was already used, implicitly, in the proof of Theorem 2.2.4)

$$f'(\infty) = \lim_{z\to\infty} z(f(z)-f(\infty)) = \lim_{z\to\infty} zf(z), \qquad \text{and} \qquad g'(\infty) = \lim_{z\to\infty} zg(z),$$

so we deduce straightaway that $|g'(\infty)|\leq |f'(\infty)|$, that was what we wanted to prove. $\square$

Using the previous theorem we can compute explicitly the analytic capacities of a disk and a segment of finite length.

Corollary 2.2.7. *The following hold:*

1. *Let $D_r \subset \mathbb{C}$ be any disk of radius r. Then $\gamma(D_r) = r$.*
2. *Let $[z,w] \subset \mathbb{C}$ be the line segment joining $z, w \in \mathbb{C}$. Then $\gamma([z,w]) = \ell([z,w])/4$, where $\ell([z,w])$ denotes the length of $[z,w]$.*

Proof. For the first statement, if $D_r = D(z_0, r)$ we pick the transformation

$$f(z) = \frac{r}{z - z_0},$$

that maps conformally $\mathbb{C} \setminus \overline{D(z_0,r)}$ to $\mathbb{D}$ with $f(\infty) = 0$. Hence, by the Theorem 2.2.6, we deduce $\gamma(D_r) = |f'(\infty)| = r$.

For the second statement we assume, without loss of generality, that the line segment is the real interval $[-L/2, L/2]$. Now, we consider the transformation

$$f(z) = \left(z + \frac{1}{z}\right)\frac{L}{4},$$

that maps conformally the unit disk to $\mathbb{C} \setminus [-L/2, L/2]$ and is such that $f(0) = \infty$. Hence

$$\gamma\left(\left[-\frac{L}{2}, \frac{L}{2}\right]\right) = \lim_{z\to\infty} \left|z\big(f^{-1}(z) - f^{-1}(\infty)\big)\right| = \lim_{w\to 0} |f(w)w| = \frac{L}{4}.$$

□

To end this section let us present a bound for the values of the analytic capacity for compact connected sets in terms of their size, more precisely in terms of their diameter. To prove this bound we will need the following result conjectured by Koebe and proved by Bieberbach.

Theorem 2.2.8 (1/4 Koebe)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be an holomorphic and univalent (injective) function with $f(0) = 0$ and $|f'(0)| = 1$. Then, $D(0, 1/4) \subset f(\mathbb{D})$.*

A proof of this result can be found in [23, Theorem 14.14]. We present now the theorem we have anticipated above:

Theorem 2.2.9. *Let $E \subset \mathbb{C}$ be compact and connected. Then,*

$$\frac{\operatorname{diam}(E)}{4} \leq \gamma(E) \leq \operatorname{diam}(E).$$

Proof. The right inequality follows from the monotonicity of analytic capacity and the fact that E is contained in a closed disk with radius $\operatorname{diam}(E)$. To prove the left inequality let $V \subset \mathbb{C}$ the unbounded component of $\mathbb{C} \setminus E$ and consider the conformal map $f : V \to \mathbb{D}$ with $f(\infty) = 0$ (here, to obtain this conformal map, we do not take into account the case where E is a single point, for which the result is trivial). By Theorem 2.2.6, $\gamma(E) = |f'(\infty)|$. Since E is compact we may pick $z_1, z_2 \in E$ so that $|z_1 - z_2| = \operatorname{diam}(E)$, and consider the function

$$g(z) = \frac{\gamma(E)}{f^{-1}(z) - z_1}.$$

As f^{-1} is, in particular, univalent, g is also a univalent map defined on $\mathbb{D}$ satisfying $g(0) = 0$. Moreover

$$|g'(0)| = \lim_{z\to 0}\left|\frac{\gamma(E)}{z(f^{-1}(z) - z_1)}\right| = \lim_{z\to 0}\left|\frac{\gamma(E)}{zf^{-1}(z)}\right| = \lim_{w\to\infty}\left|\frac{\gamma(E)}{f(w)w}\right| = \left|\frac{\gamma(E)}{f'(\infty)}\right| = 1.$$

Also, by construction, $z_2 \notin f^{-1}(\mathbb{D})$, and so $\frac{\gamma(E)}{z_2 - z_1} \notin g(\mathbb{D})$. Therefore, by Theorem 2.2.8 we conclude

$$\frac{1}{4} \leq \frac{\gamma(E)}{|z_2 - z_1|},$$

and so the inequality follows. □

Corollary 2.2.10. *Let $E \subset \mathbb{C}$ be compact with $\gamma(E) = 0$. Then, each connected component of E contains exactly a single point.*

Proof. Since analytic capacity is a monotone function, every connected component of E also has null analytic capacity. Then, by Theorem 2.2.9, their diameter is 0. But recall that for any $A \subset \mathbb{C}$ the definition of diameter is $\mathrm{diam}(A) := \sup\{|z_1 - z_2| \,:\, z_1, z_2 \in A\}$, and so the result follows. □

2.3 The Painlevé problem. An approach using the Cauchy transform

The main aim of this section is to relate the analytic capacity of a compact subset $E \subset \mathbb{C}$ with the kind of bounded holomorphic function that can be defined on $\mathbb{C} \setminus E$. More precisely, we will prove that if E is any compact set with null analytic capacity, then E will be removable (Definition 2.1.3), and vice versa.

Theorem 2.3.1 (Ahlfors [1])**.** *Let $\Omega \subset \mathbb{C}$ be open and $E \subset \mathbb{C}$ compact. Then, $\gamma(E) = 0$ if and only if every bounded analytic function on $\Omega \setminus E$ can be extended analytically to the whole set Ω, i.e. E is removable.*

The characterization of removable subsets for bounded holomorphic functions does not only admit this way of proceeding. Another natural approach could be purely geometrical. That is, given a compact subset of $\mathbb{C}$, are we able to determine if its removable or not just with geometric techniques? The problem of typifying removable sets by geometric terms is known as the *Painlevé problem* and for us it will be the key concept that will allow us, eventually, to connect analytic capacity and the theory of singular integrals.

2.3.1 The proof of the characterization theorem

In order to proof Theorem 2.3.1, we will introduce another pillar of our project: the *Cauchy transform*. Basically, we have to understand it as an operator acting on complex measures. Its properties are really helpful in order to express, in a more compact way, the attributes that are essential about holomorphic functions. It makes its first appearance here, allowing us to prove a characterization theorem for removable subsets, but it will also play a crucial role when trying to understand the meaning of having null analytic capacity in terms of another parameter such as the *Hausdorff dimension* of the compact subset itself. This methodology will be the center of our work in posterior sections.

But for now, let us start by presenting the definition of Cauchy transform:

Definition 2.3.1 (Cauchy transform)**.** Let ν be a finite measure (possibly with complex values) on $\mathbb{C}$. We define its *Cauchy transform* as

$$\mathscr{C}\nu(z) = \int_{\mathbb{C}} \frac{1}{\xi - z} d\nu(\xi).$$

The integral involved in $\mathscr{C}\nu$ is well-defined in the sense that it defines a locally integrable function in $\mathbb{C}$ (with respect to the planar Lebesgue measure $d\mathcal{L}^2$) and thus it is well-defined $\mathcal{L}^2$-a.e. Indeed, for any compact subset $K \subset \mathbb{C}$, taking $R > 0$ so that $K \subset \overline{D(0,R)}$, we have

$$\begin{aligned}\int_K |\mathscr{C}\nu(z)| d\mathcal{L}^2(z) &\leq \int_{\mathbb{C}} \left(\int_K \frac{1}{|\xi - z|} d\mathcal{L}^2(z) \right) d\nu(\xi) \leq \int_{\mathbb{C}} \left(\int_0^R \int_{S^1} \frac{1}{r} r dr d\theta \right) d\nu(\xi) \\ &= 2\pi R \nu(\mathbb{C}) < \infty.\end{aligned}$$

Let us make precise the above statement in which we claimed that the Cauchy transform *compactifies the notation for holomorphic functions.* Indeed, this operator appears naturally in complex analysis, just observe that for any holomorphic function defined on an open simply connected set $\Omega \subset \mathbb{C}$ and $\Gamma \subset \Omega$ any closed Jordan curve (with no self-intersections and regular enough), if z is a point that belongs to the interior of Γ (and hence $z \in \Omega$), then

$$f(z) = \mathscr{C}\nu(z), \quad \text{where} \quad \nu = \frac{1}{2\pi i} f(z) dz_\Gamma,$$

that is just the usual Cauchy integral formula.

Apart from this, the Cauchy transform has some genuine and essential properties that we summarize in the following theorem:

Theorem 2.3.2. *Let ν be a finite complex measure on $\mathbb{C}$. Then:*

1. *$\mathscr{C}\nu \in L^1_{\text{loc}}(\mathbb{C})$ with respect to the measure $d\mathcal{L}^2$. So $\mathscr{C}\nu$ defines a distribution.*
2. *$\mathscr{C}\nu$ is holomorphic in $\mathbb{C} \setminus \operatorname{supp}(\nu)$.*
3. *If $\operatorname{supp}(\nu)$ is compact, then $\mathscr{C}\nu(\infty) = 0$ and $(\mathscr{C}\nu)'(\infty) = -\nu(\mathbb{C})$.*

Proof. The first property has already been proved, just after the definition of the Cauchy transform, so let us focus on *2*. Recall that the support of a measure is the smallest closed set F such that $\nu(\mathbb{C} \setminus F) = 0$. So we want to prove that $\mathscr{C}\nu$ is holomorphic in $\mathbb{C} \setminus F$. Let us check the property locally for each point of this last open set. So fix $z_0 \in \mathbb{C} \setminus F$ and $\varepsilon > 0$ so that $D(z_0, \varepsilon) \subset \mathbb{C} \setminus F$. For any $w \in \overline{D(z_0, \varepsilon/2)}$ we have

$$\mathscr{C}\nu(w) = \int_{\mathbb{C}} \frac{1}{\xi - w} d\nu(\xi) = \int_F \frac{1}{\xi - w} d\nu(\xi),$$

and since $|\xi - w| > \varepsilon/2$ we get that $\mathscr{C}\nu$ is continuous in $\overline{D(z_0, \varepsilon/2)}$ (applying, for example, the dominated convergence theorem and that ν is a finite measure). We may apply Morera's theorem, so consider $\triangle$ any triangle in $D(z_0, \varepsilon/2)$ and compute

$$\int_{\triangle} \mathscr{C}\nu(w) dw_{\triangle} = \int_{\mathbb{C}} \left(\int_{\triangle} \frac{1}{\xi - w} dw_{\triangle} \right) d\nu(\xi) = 0,$$

where the inner integral is 0 since the integrand is holomorphic in $D(z_0, \varepsilon/2)$. Also, to apply Fubini's theorem we have used, again, the bound $|\xi - w| > \varepsilon/2$ and the fact the ν is a finite measure. Hence, by Morera's theorem we get the result.

Finally, let us check the identities of *3*. If $F := \operatorname{supp}(\nu)$, pick $R > 0$ so that $F \subset D_R := D(0, R)$. Then, if we fix $0 < \varepsilon < R^{-1}$ and any $w \in D(0, \varepsilon)$ we have

$$|\mathscr{C}\nu(w^{-1})| = \left| \int_F \frac{w}{w\xi - 1} d\nu(\xi) \right| \leq \int_{D_R} \frac{|w|}{|w\xi - 1|} d\nu(\xi) \leq \frac{\varepsilon}{1 - \varepsilon R} \nu(\mathbb{C}),$$

yielding the first equality taking the limit as $\varepsilon \to 0$. Turning to $(\mathscr{C}\nu)'(\infty)$, by definition we know

$$(\mathscr{C}\nu)'(\infty) = \lim_{w \to \infty} w\big(\mathscr{C}\nu(w) - \mathscr{C}\nu(\infty)\big) = \lim_{w \to \infty} w\mathscr{C}\nu(w) = \lim_{\eta \to 0} \int_F \frac{1}{\eta\xi - 1} d\nu(\xi),$$

where $\eta := w^{-1}$. By the same argument as before, defining the corresponding disk $D(0, \varepsilon)$ we can justify the application of the dominated convergence theorem, take the limit inside the integral and obtain the second equality of *3*. $\square$

For $w \in \operatorname{supp}(\nu)$, the value of $\mathscr{C}\nu(w)$ does not need to be defined ν-a.e. since ν may be singular with respect to $d\mathcal{L}^2$. This is because the domain of integration involved in the definition of $\mathscr{C}\nu(z)$ contains a point where the integrand is not defined. Hence, to assign a value in this case we work with a truncated operator

$$\mathscr{C}_\varepsilon\nu(z) := \int_{\xi \in F : |\xi - z| > \varepsilon} \frac{1}{\xi - z} d\nu(\xi)$$

and define $\mathscr{C}\nu(z) := \lim_{\varepsilon \to 0} \mathscr{C}_\varepsilon\nu(z)$ whenever this last expression makes sense.

The definition of the Cauchy transform can be extended to the whole family of compactly supported distributions.

Definition 2.3.2 (Cauchy transform of a distribution)**.** Let ν be a compactly supported (complex) distribution (by support of a distribution we mean the largest $K \subset \mathbb{C}$ so that for every $U \subset K$ open, we have $\nu|_{\mathcal{C}_c^\infty(U)} \neq 0$). We define its *Cauchy transform* as

$$\mathscr{C}\nu(z) = \Big\langle \nu, \tau_z\big(-\xi^{-1}\big) \Big\rangle = \nu * \big(-\xi^{-1}\big)(z),$$

where τ_z is the translation $\tau_z f(\,\cdot\,) = f(\,\cdot - z)$ for $f : \mathbb{C} \to \mathbb{C}$.

Let us clarify the notation we have used. Let ν be any distribution given by $f \in L^1_{\text{loc}}(\mathbb{C})$ compactly supported (imlpying, by definition, that ν is compactly supported). Then

$$\mathscr{C}\nu(z) = \int_{\mathbb{C}} \frac{f(\xi)}{\xi - z} d\mathcal{L}^2(\xi).$$

For this particular setting it is clear that $\mathscr{C}\nu \in L^1_{\text{loc}}(\mathbb{C})$, and so it also defines a distribution. In fact, there is an analogous version of Theorem 2.3.2 for compactly supported distributions, obtaining the same results in a distributional sense (we will not prove it, see [27, Proposition 1.15] for more details).

A key property that explains why the Cauchy transform is so important in complex analysis is a direct consequence of the Definition 2.3.2 and Theorem A.2.1.

Theorem 2.3.3. *If ν is a compactly supported distribution on $\mathbb{C}$, then*

$$\overline{\partial}(\mathscr{C}\nu) = -\pi\nu.$$

As a consequence, if $f \in L^1_{\text{loc}}(\mathbb{C})$ is analytic in a neighborhood of ∞ and $f(\infty) = 0$, then

$$\mathscr{C}(\overline{\partial} f) = -\pi f.$$

Proof. The (distributional) equality $\overline{\partial}(\mathscr{C}\nu) = -\pi\nu$ is a direct consequence of Theorem A.2.1, that gives us the fundamental solution to the $\overline{\partial}$-equation. For the second equality, observe that since f is holomorphic in a neighborhood of ∞, we shall think $\overline{\partial} f$ as a compactly supported distribution on $\mathbb{C}$. Hence, by the first equality we obtain

$$\overline{\partial}\big(\mathscr{C}(\overline{\partial} f)\big) = -\pi\overline{\partial} f \quad \Leftrightarrow \quad \overline{\partial}\big[\mathscr{C}(\overline{\partial} f) + \pi f\big] = 0.$$

So $\mathscr{C}(\overline{\partial} f) + \pi f$ is holomorphic on $\mathbb{C}$ and null at ∞ (by the distributional version of Theorem 2.3.2), and so it is bounded and therefore constant, yielding the desired equality. □

Next, we will present the fundamental operator that will do the task of relating the concepts of compact sets with null analytic capacity and hence their removability.

Definition 2.3.3 (Vitushkin's localization operator)**.** Let $f \in L^1_{\text{loc}}(\mathbb{C})$ and $\varphi \in \mathcal{C}^\infty_c(\mathbb{C})$. We define the *Vitushkin's localization operator* (associated to φ) as

$$V_\varphi f := \varphi f + \frac{1}{\pi}\mathscr{C}(f\overline{\partial}\varphi).$$

The principal property of V_φ (that will be responsible for the reason we refer to V_φ as a *localization* operator) is presented in the following lemma

Lemma 2.3.4. $f \in L^1_{\text{loc}}(\mathbb{C})$ *and* $\varphi \in \mathcal{C}^\infty_c(\mathbb{C})$. *Then, we have the following equality between distributions*

$$V_\varphi f = -\frac{1}{\pi}\mathscr{C}(\varphi\overline{\partial} f),$$

where $\overline{\partial} f$ should be also understood in the distributional sense.

Proof. Using the definition of V_φ as well as Theorem 2.3.3

$$\overline{\partial}(V_\varphi f) = f\overline{\partial}\varphi + \varphi\overline{\partial}f + \frac{1}{\pi}\overline{\partial}\mathscr{C}(f\overline{\partial}\varphi) = \varphi\overline{\partial}f = \overline{\partial}\bigg(-\frac{1}{\pi}\mathscr{C}(\varphi\overline{\partial}f)\bigg).$$

Since φf and $f\overline{\partial}\varphi$ are compactly supported, $V_\varphi f$ also satisfies this property; and as $\varphi\overline{\partial}f$ has compact support, $-\frac{1}{\pi}\mathscr{C}(\varphi\overline{\partial}f)$ also has. Therefore $V_\varphi f$ and $-\frac{1}{\pi}\mathscr{C}(\varphi\overline{\partial}f)$ are both analytic in a neighborhood of ∞ and are 0 at ∞. Therefore, applying $\mathscr{C}(\,\cdot\,)$ and the second part of Theorem 2.3.3 to the above equality, we get the desired relation. $\square$

Observe that, with the notation of the previous lemma, if we fix f to be $\mathscr{C}\nu$, where ν is a compactly supported complex measure or distribution; we have

$$V_\varphi(\mathscr{C}\nu) = \mathscr{C}(\varphi\nu).$$

We know that $\mathscr{C}\nu$ is analytic in $\mathbb{C}\setminus\text{supp}(\nu)$, so $\text{supp}(\nu)$ can be understood as the set of singularities of $\mathscr{C}\nu$. Hence, by the previous equality, we have that $V_\varphi(\mathscr{C}\nu)$ is analytic in the (larger) set $\mathbb{C}\setminus\text{supp}(\varphi\nu)$. That is why we say that V_φ is a *localization* operator, because now the singularities of $V_\varphi(\mathscr{C}\nu)$ are localized in $\text{supp}(\nu)\cap\text{supp}(\varphi)$.

Lemma 2.3.5. *Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{C})$ be supported in $D_r = D(0,r)$, the open disk of radius r centered at the origin. Then, for any $f \in L^1_{\text{loc}}(\mathbb{C})$, the following hold*

1. $\|V_\varphi f\|_\infty \leq C\|f\chi_{D_r}\|_\infty$, *for some $C>0$.*
2. $V_\varphi f$ *is holomorphic in* $\mathbb{C}\setminus\big(\text{supp}(\overline{\partial}f)\cap\text{supp}(\varphi)\big)$

Proof. By the definition of V_φ we have

$$\|V_\varphi f\|_\infty \leq \|\varphi f\|_\infty + \frac{1}{\pi}\|\mathscr{C}(f\overline{\partial}\varphi)\|_\infty.$$

It is clear that $\|\varphi f\|_\infty \lesssim \|\chi_{D_r}f\|_\infty$. For the second term notice that for any $z\in\mathbb{C}$

$$\begin{aligned}|\mathscr{C}(f\overline{\partial}\varphi)(z)| &\leq \int_{D_r}\frac{1}{|\xi - z|}|f(\xi)||\overline{\partial}\varphi(\xi)|d\mathcal{L}^2(\xi)\\ &\leq \|f\chi_{D_r}\|_\infty\|\nabla\varphi\|_\infty\int_{D_r}\frac{1}{|\xi-z|}d\mathcal{L}^2(\xi) \lesssim \|f\chi_{D_r}\|_\infty,\end{aligned}$$

and so we have *1.* Property *2* is just a consequence of Lemma 2.3.4 and the distributional version of Theorem 2.3.2 [27, Proposition 1.15]. $\square$

Now we are finally ready to prove Theorem 2.3.1.

Proof (Theorem 2.3.1). Proving that $\gamma(E)=0$ assuming that E is removable is almost trivial. To do it, notice that the assumption implies that every function analytic and bounded in $\mathbb{C}\setminus E$ is constant. Indeed, given $f:\mathbb{C}\setminus E\to\mathbb{C}$ holomorphic, we may consider its restriction to $\Omega\setminus E$. Then, by hypothesis, we can extend it to an entire bounded function, and thus constant. Therefore, by the definition of analytic capacity we conclude that, necessarily, $\gamma(E)=0$.

Let us see now the reverse implication. We may assume, without loss of generality, that Ω is bounded (if not, just consider a open bounded subset of Ω containing E and construct the analytic extension using the argument that will follow). Consider a grid of squares $\{Q_i\}_{i\in I}$ in $\mathbb{C}$ with side length ℓ, the same for every Q_i, and $\{\varphi_i\}_{i\in I}$ a family of smooth functions satisfying, for each i

1. $0 \leq \varphi \leq 1$.
2. $\varphi_i|_{Q_i} \equiv 1$ and $\varphi_i|_{\mathbb{C}\setminus 2Q_i} \equiv 0$.
3. $\sum_{i\in I} \varphi_i \equiv 1$.

In other words, $\{\varphi_i\}_i$ is a partition of unity subordinated to the family of squares $\{Q_i\}_i$.

Let us begin by extending f by 0 to $(\Omega \setminus E)^c$. As Ω is bounded, we have that f vanishes out of a bounded set. Therefore, since f is analytic in a neighborhood of ∞ and $f(\infty) = 0$, by Theorem 2.3.3 we have $f = -\frac{1}{\pi}\mathscr{C}(\overline{\partial} f)$. We also observe that once we restrict ourselves to a square Q_i satisfying that $2Q_i$ does not intersect the support of $\overline{\partial} f$, by Lemma 2.3.4 we have $V_{\varphi_i} f \equiv 0$. In all, we deduce

$$f = -\frac{1}{\pi}\mathscr{C}(\overline{\partial} f) = -\frac{1}{\pi}\mathscr{C}\left(\sum_{i\in I}\varphi_i\overline{\partial} f\right) = -\frac{1}{\pi}\sum_{i\in I}\mathscr{C}(\varphi_i\overline{\partial} f) = \sum_{i\in I} V_{\varphi_i} f,$$

where the sums taken over I are well-defined by the last observation. Let us see first that, if $2Q_i \cap \partial\Omega = \varnothing$, then $V_{\varphi_i} f \equiv 0$. We begin by noticing that $\operatorname{supp}(\overline{\partial} f) \subset E \cup \partial\Omega$, so Lemma 2.3.4 yields

$$\operatorname{supp}\big(\overline{\partial} V_{\varphi_i} f\big) = \operatorname{supp}\big(\overline{\partial}\mathscr{C}(\varphi\overline{\partial} f)\big) \subset 2Q_i \cap (E \cup \partial\Omega).$$

Therefore, if $2Q_i \cap \partial\Omega = \varnothing$, then $V_{\varphi_i} f$ is analytic out of $2Q_i \cap E$. As $V_{\varphi_i} f$ vanishes at ∞ and the extension of f by 0 to the whole $\mathbb{C}$ is bounded, we can apply the first point of Lemma 2.3.5 to deduce $\|V_{\varphi_i} f\|_\infty < \infty$. Therefore $V_{\varphi_i} f$ is admissible for $2Q_i \cap E$. On the other hand

$$\gamma(2Q_i \cap E) \leq \gamma(E) = 0 \qquad \Rightarrow \qquad \gamma(2Q_i \cap E) = 0$$

Let us see that the previous condition implies that the function $V_{\varphi_i} f$ needs to be constant. Assume not and let us reach a contradiction. Under the previous hypothesis, there exists $z_0 \in \mathbb{C} \setminus (2Q_i \cap E)$ so that $V_{\varphi_i} f(z_0) \neq V_{\varphi_i} f(\infty) = 0$. Now, we may define the function

$$g(z) = \frac{V_{\varphi_i} f(z) - V_{\varphi_i} f(z_0)}{z - z_0}, \qquad \text{for} \qquad z \neq z_0$$

and $g(z_0) := (V_{\varphi_i} f)'(z_0)$. It is clear that g is admissible for $\mathbb{C} \setminus (2Q_i \cap E)$, $g(\infty) = 0$ and

$$g'(\infty) = \lim_{z\to\infty} z(g(z) - g(\infty)) = V_{\varphi_i} f(\infty) - V_{\varphi_i} f(z_0) \neq 0$$

which contradicts $\gamma(E) = 0$. Hence, necessarily, $V_{\varphi_i} f \equiv 0$ if $2Q_i \cap \partial\Omega = \varnothing$. So we conclude

$$f = \sum_{\substack{i\in I \\ 2Q_i\cap\partial\Omega\neq 0}} V_{\varphi_i} f.$$

Considering the open neighborhood of $\partial\Omega$: $U_{4\ell}(\partial\Omega) := \big\{z \in \mathbb{C} \,:\, d(z, \partial\Omega) < 4\ell\big\}$, we have that in $\Omega \setminus \overline{U_{4\ell}(\partial\Omega)}$ the function f is a finite sum of holomorphic functions (recall that $V_{\varphi_i} f$ is analytic out of $2Q_i \cap E$) and hence is holomorphic. But ℓ can be chosen to be arbitrarily small, so we conclude that f is analytic in the whole Ω and we are done, since we have found the desired extension. $\square$

2.4 Connection between analytic capacity and Hausdorff measure

In this new section we return to a problem we have already presented: the characterization of removable subsets in terms of purely geometric properties. To do this, we will introduce the concept of *Hausdorff measure*, that we may understand, intuitively, as a continuous generalization of the usual Lebesgue measure.

2.4.1 Definition of Hausdorff measure. The Hausdorff dimension.

The concept of Hausdorff measure is a particular case of a more general family of measures introduced by Carathéodory (1914) [2]. First, we will present this general construction and some of its fundamental properties, to follow with a more complete study of the Hausdorff case.

Definition 2.4.1 (Carathéodory's δ-measure)**.** Let X be a metric space, $\mathcal{F} \subset \mathcal{P}(X)$ a collection of subsets of X and ζ a non-negative function on $\mathcal{F}$. Assume that

1. For every $\delta > 0$ there exist $E_1, E_2, \dots \in \mathcal{F}$ such that

$$X = \bigcup_{i=1}^{\infty} E_i \quad \text{with} \quad \operatorname{diam}(E) \leq \delta.$$

2. For every $\delta > 0$ there exists $E \in \mathcal{F}$ so that $\zeta(E) \leq \delta$ and $\operatorname{diam}(E) \leq \delta$.

Then for $0 < \delta \leq \infty$ we define the *Carathéodory's δ-measure* $\psi_\delta : X \to [0, \infty]$ to be

$$\psi_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \zeta(E_i) \,:\, A \subset \bigcup_{i=1}^{\infty} E_i,\, \operatorname{diam}(E_i) \leq \delta,\, E_i \in \mathcal{F} \right\}.$$

Observe that the first hypothesis of the definition was just added to ensure that the coverings appearing in the definition of ψ_δ exist for each δ. On the other hand, one of the reasons to add the second condition is to have $\psi_\delta(\varnothing) = 0$ for every δ. Indeed, this property allows us to construct a sequence $(E_n)_{n\geq 0} \subset \mathcal{F}$ with $\zeta(E_n) \to 0$ and $\operatorname{diam}(E_n) \to 0$ as $n \to \infty$.

Proposition 2.4.1. *The Carathéodory's δ-measure is monotonic and subadditive.*

Proof. Let $A, B \in \mathcal{F}$ with $A \subset B$. Let us also fix $0 < \delta \leq \infty$. Assume $(E_n)_{n\geq 0} \subset \mathcal{F}$ is such that $B \subset \bigcup_{n=1}^{\infty} E_n$ and $\operatorname{diam}(E_n) \leq \delta$. Then $A \subset \bigcup_{n=1}^{\infty} E_n$, implying

$$\left\{ (E_n)_n \subset \mathcal{F} \,:\, B \subset \bigcup_{i=1}^{\infty} E_i,\, \operatorname{diam}(E_i) \leq \delta \right\} \subset \left\{ (E_n)_n \subset \mathcal{F} \,:\, A \subset \bigcup_{i=1}^{\infty} E_i,\, \operatorname{diam}(E_i) \leq \delta \right\},$$

and so $\psi_\delta(A) \leq \psi_\delta(B)$, since the infimum of $\psi_\delta(A)$ is computed taking into account a larger family of sets.

Let us see now that is subadditive. We will treat explicitly the case $\psi_\delta(A \cup B) \leq \psi_\delta(A) + \psi_\delta(B)$ for any $A, B \in \mathcal{F}$. Fix $\varepsilon > 0$ and pick $(E_{n,A})_{n\geq 0} \subset \mathcal{F}$ and $(E_{n,B})_{n\geq 0} \subset \mathcal{F}$ such that $A \subset \bigcup_{n=1}^{\infty} E_{n,A}$ and $B \subset \bigcup_{n=1}^{\infty} E_{n,B}$, with $\operatorname{diam}(E_{n,A}) \leq \delta$, $\operatorname{diam}(E_{n,B}) \leq \delta$ and also satisfying

$$\sum_{n=0}^{\infty} \zeta(E_{n,A}) = \psi_\delta(A) + \varepsilon, \qquad \sum_{n=0}^{\infty} \zeta(E_{n,B}) = \psi_\delta(B) + \varepsilon.$$

Now define the alternating sequence $(\widetilde{E}_n)_{n\geq 0}$ with $\widetilde{E}_{2n} = E_{n,A}$ and $\widetilde{E}_{2n+1} = E_{n,B}$. This way we obtain

$$\psi_\delta(A \cup B) \leq \sum_{n=0}^{\infty} \zeta(\widetilde{E}_n) = \sum_{n=0}^{\infty} \zeta(E_{n,A}) + \sum_{n=0}^{\infty} \zeta(E_{n,B}) = \psi_\delta(A) + \psi_\delta(B) + 2\varepsilon,$$

and by the arbitrariness of ε we are done. By induction we get the result for finite unions and by a limiting argument we obtain the general result. $\square$

Another natural property that follows from the definition of the Carathéodory's δ-measure is that

$$\text{if} \quad 0 < \delta_1 < \delta_2 \leq \infty, \qquad \psi_{\delta_2}(A) \leq \psi_{\delta_1}(A),$$

since the infimum in the definition of ψ_{δ_2} includes a larger family of sets. This property justifies the following definition:

Definition 2.4.2 (Carathéodory's measure)**.** The *Carathéodory's measure*, denoted by $\psi = \psi(\mathcal{F}, \zeta)$, associated to the family of sets $\mathcal{F}$ with function ζ is

$$\psi(A) = \lim_{\delta \downarrow 0} \psi_\delta(A) = \sup_{\delta > 0} \psi_\delta(A), \qquad \text{for} \quad A \subset X$$

Theorem 2.4.2. *Let X be a metric space, $\mathcal{F} \subset \mathcal{P}(X)$ a collection of subsets of X and ζ a non-negative function on $\mathcal{F}$. The following properties hold:*

1. *$\psi = \psi(\mathcal{F}, \zeta)$ is a Borel measure.*
2. *If $\mathcal{F} \subset \mathscr{B}(X)$, the family of Borel sets, then ψ is Borel regular.*

Proof. Proving that ψ is indeed an (outer) measure is just a matter of checking the defining properties, which follow straightforward from the definition of ψ. To see that is a Borel measure we will apply Carathéodory's theorem [16, Theorem 1.7], that states that ψ is a Borel measure if and only if

$$\mu(A \cup B) = \mu(A) + \mu(B), \qquad \forall A, B \subset X \ \text{ with } \ d(A, B) > 0.$$

So let $A, B \subset X$ with $d(A, B) > 0$. We will be done if we prove $\psi(A \cup B) = \psi(A) + \psi(B)$. Choose $0 < \delta < d(A, B)/2$ so that if $E_1, E_2, \dots \in \mathcal{F}$ cover $A \cup B$ and are such that $d(E_n) \leq \delta$, then no E_n intersects both A and B. Therefore

$$\sum_{n=1}^{\infty} \zeta(E_n) = \sum_{A \cap E_n \neq \varnothing} \zeta(E_n) + \sum_{B \cap E_n \neq \varnothing} \zeta(E_n) \geq \psi_\delta(A) + \psi_\delta(B),$$

and so taking the infimum over this family of coverings we get $\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B)$. But as the other inequality follows from subadditivity, we have $\psi_\delta(A \cup B) = \psi_\delta(A) + \psi_\delta(B)$, and letting $\delta \downarrow 0$ we obtain the desired result.

Let us prove now *2*, that is, we have to prove that for any $A \subset X$ there is $B \in \mathscr{B}(X)$ so that $\psi(A) = \psi(B)$. To do it, for each $n = 1, 2, \dots$ we consider a sequence $E_{n,1}, E_{n,2}, \dots \in \mathcal{F} \subset \mathscr{B}(X)$ satisfying

$$A \subset \bigcup_{m=1}^{\infty} E_{n,m}, \quad \operatorname{diam}(E_{n,m}) \leq \frac{1}{n} \quad \text{and} \quad \sum_{m=1}^{\infty} \zeta(E_{n,m}) \leq \psi_{\frac{1}{n}}(A) + \frac{1}{n}.$$

This sequence exists due to the hypothesis of Definition 2.4.1. Then,

$$B := \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{n,m}$$

is a Borel set, since it belongs to the σ-algebra of measurable sets, and satisfies, by construction, $A \subset B$. Moreover, for every n we have $B \subset \bigcup_{m=1}^{\infty} E_{n,m}$, so $\psi(B) \leq \psi_{\frac{1}{n}}(A) + \frac{1}{n}$. Therefore $\psi(B) \leq \psi(A)$, and by the monotonicity of ψ we obtain the equality $\psi(A) = \psi(B)$. □

Given this construction by Carathéodory, we obtain the definition of the Hausdorff measure for a particular choice of $\mathcal{F}$ and ζ (under the additional assumption that X is separable).

Definition 2.4.3 (Hausdorff measure)**.** Let X be a separable metric space and let $s \in [0, \infty)$. Choose

$$\mathcal{F} := \mathcal{P}(X), \quad \zeta(E) := \operatorname{diam}(E)^s,$$

with the conventions $0^0 := 1$ and $\operatorname{diam}(\varnothing) := 0$. The corresponding measure ψ for this case is called the s-dimensional *Hausdorff measure* and is denoted by $\mathcal{H}^s$. Notice that by Theorem 2.4.2 $\mathcal{H}^s$ is a Borel regular measure. More explicitly

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A),$$

where

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(E_n)^s \, : \, A \subset \bigcup_{n=1}^{\infty} E_n, \, \operatorname{diam}(E_n) \leq \delta \right\}.$$

Let us try to give a better intuition about the role of this measure in each dimension. We will briefly comment some particular cases and state the most relevant results. If the reader seeks a more detailed discussion about the topic we suggest to consult the books of Evans & Gariepy [8, Chapter 2] or Mattila [16, Chapter 4].

1. For the case $s = 0$ it is not difficult to prove that the Hausdorff measure $\mathcal{H}^0$ coincides with the counting measure, i.e.

$$\mathcal{H}^0(A) = \operatorname{card}(A) = \text{number of points in } A.$$

2. For $s = 1$, the value of $\mathcal{H}^1$ also has a concrete meaning as a generalized length measure. Let us recall first the following definition:

   Definition 2.4.4 (Rectifiable curve)**.** Let $\Gamma : [0, 1] \to \mathbb{R}^n$ be a curve. We will say that Γ is *rectifiable* if it is a continuous curve that admits a Lipschitz parametrization.

We know that a Lipschitz function is differentiable at almost every point with bounded derivative [8, Theorem 3.2], implying, in particular, that rectifiable curves have finite (Lebesgue) length. Then, if one wants to consider a line integral along a certain curve (such as in Definition 2.1.2) the choice of requiring, at least, the curve to be rectifiable is a reasonable hypothesis to ask for, much less restrictive than demanding, for example, continuous differentiability. For a rectifiable curve Γ in $\mathbb{R}^n$ it can be shown that $\mathcal{H}^1(\Gamma)$ equals the length of Γ. For unrectifiable curves one gets $\mathcal{H}^1(\Gamma) = \infty$.

3. In the setting $X = \mathbb{R}^n$, if $1 < s < n$ and $s \in \mathbb{Z}$; it also can be proved that the restriction of $\mathcal{H}^s$ to any s-dimensional $\mathcal{C}^1$-submanifold is proportional to the Lebesgue measure of $\mathbb{R}^n$ restricted to the same submanifold. In fact, one has that there exists a constant $C = C(s,n)$ so that following inequality holds for any $x \in \mathbb{R}^n$ and $r > 0$

$$\mathcal{H}^s(B(x,r)) = c(s,n)r^s. \tag{2.4.1}$$

This condition on a measure is usually referred as having *s-dimensional growth.*

4. Again in $\mathbb{R}^n$, for the case $s = n$, the following fundamental equality between measures can be proved

$$\mathcal{H}^n = \left(2\pi^{-\frac{1}{2}}\right)^n \Gamma\left(\frac{n}{2}+1\right)\mathcal{L}^n, \tag{2.4.2}$$

where here Γ denotes the usual gamma function. Hence, in particular, we obtain

$$\mathcal{H}^n(B(x,r)) = (2r)^n, \quad \forall x \in \mathbb{R}^n,\ r \in (0,\infty).$$

5. Finally, for any $s > n$ in $\mathbb{R}^n$, we have $\mathcal{H}^s(\mathbb{R}^n) = 0$.

Continuing our discussion about the general properties of the Hausdorff measure, in practice one often is only interested in determining which sets have null $\mathcal{H}^s$-measure. Observe that in this context, since for $\delta_1 < \delta_2$ we have $H^s_{\delta_2} < H^s_{\delta_1}$, it is enough to study the approximating measures H^s_δ. In fact, the following result shows that we do not need any measure at all.

Proposition 2.4.3. *Let X be a separable metric space, $s \in [0,\infty)$ and $\delta \in (0,\infty]$. Then, for any $A \subset X$ the following conditions are equivalent*

1. $\mathcal{H}^s(A) = 0$.
2. $\mathcal{H}^s_\delta(A) = 0$.
3. *For each $\varepsilon > 0$ there exist $E_1, E_2, \dots \subset X$ such that*

$$A \subset \bigcup_{n=1}^\infty E_n, \quad \textit{and} \quad \sum_{n=1}^\infty \operatorname{diam}(E_n)^s < \varepsilon.$$

Proof. The implication *1*⇒*2* is trivial. To prove *2*⇒*3* begin by fixing $\varepsilon > 0$ and by observing that if $\mathcal{H}^s_\delta(A) = 0$, then for every $\eta > 0$ there exists $\{E_{n,\eta}\}_n$ so that $A \subset \bigcup_{n=1}^\infty E_{n,\eta}$ and $\sum_{n=1}^\infty \operatorname{diam}(E_n)^s < \eta$. Hence choosing $\eta := \varepsilon$ we are done. Finally, let us check *3*⇒*1*. Fix $\varepsilon := \delta^s$ so that

$$A \subset \bigcup_{n=1}^\infty E_n, \quad \text{and} \quad \sum_{n=1}^\infty \operatorname{diam}(E_n)^s < \delta^s.$$

Then, $\operatorname{diam}(E_n) < \delta$ for each n. Hence $\mathcal{H}^s_\delta < \delta^s$, and so taking the limit as $\delta \downarrow 0$ we obtain the result for $0 < s < \infty$. For the remaining case $s = 0$ notice that if *3* holds, picking $\varepsilon < 1$ we obtain that $A = \varnothing$, meaning $\mathcal{H}^0(A) = \operatorname{card}(\varnothing) = 0$, that is what we wanted to prove. $\square$

Observe that the previous theorem holds, in particular, for the case $\delta = \infty$, meaning that there is no restriction with respect to the diameters of the chosen coverings. This special case receives a name of its own.

Definition 2.4.5 (Hausdorff content)**.** Let X be a separable metric space and $s \in [0, \infty)$. We define the s-dimensional *Hausdorff content* of $A \subset X$, $\mathcal{H}^s_\infty$ as

$$\mathcal{H}^s_\infty(A) = \inf \left\{ \sum_{n=1}^{\infty} \operatorname{diam}(E_n)^s \,:\, A \subset \bigcup_{n=1}^{\infty} E_n \right\}.$$

By the previous result we have that $\mathcal{H}^s_\infty(A) = 0 \;\Leftrightarrow\; \mathcal{H}(A) = 0$.

We will finish this introduction to $\mathcal{H}$ by presenting one of its most essential and elementary properties, which will motivate the definition of the *Hausdorff dimension*.

Theorem 2.4.4. *Let X be a separable metric space, $0 \leq s < t < \infty$ and $A \subset X$. Then*

1. *$\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$.*
2. *$\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = \infty$.*

Proof. Notice that *2* is just a restatement of *1*, so as soon as we prove the first property we will be done. By the infimum involved in the definition of $\mathcal{H}^s_\delta$ we are able to consider $A \subset \bigcup_{n=1}^\infty E_n$ with $\operatorname{diam}(E_n) \leq \delta$ and $\sum_{n=1}^\infty \operatorname{diam}(E_n)^s \leq \mathcal{H}^s_\delta(A) + 1$. Therefore

$$\mathcal{H}^t_\delta(A) \leq \operatorname{diam}(E_n)^{t-s} \sum_{n=1}^{\infty} \operatorname{diam}(E_n)^s \leq \delta^{t-s} \sum_{n=1}^{\infty} \operatorname{diam}(E_n)^s \leq \delta^{t-s}\big(\mathcal{H}^s_\delta(A) + 1\big),$$

and taking the limit as $\delta \downarrow 0$ and using the hypothesis $\mathcal{H}^s(A) < \infty$ we obtain the result. $\square$

Notice that Theorem 2.4.4 implies that when calculating the Hausdorff measure of a certain set, there exists a borderline value of s (what we will understand as *dimension*) for which if we consider smaller values than that, the Hausdorff measure is ∞, and for bigger values the Hausdorff measure is 0. So according to this observation we may define:

Definition 2.4.6 (Hausdorff dimension)**.** Let X be a separable metric space and $A \subset X$. The *Hausdorff dimension* of A is

$$\begin{aligned}\dim_{\mathcal{H}}(A) &= \sup\{s \,:\, \mathcal{H}^s(A) > 0\} = \sup\{s \,:\, \mathcal{H}^s(A) = \infty\} \\ &= \inf\{t \,:\, \mathcal{H}^t(A) < \infty\} = \inf\{t \,:\, \mathcal{H}^t(A) = 0\}.\end{aligned}$$

In other words, $\dim_{\mathcal{H}}(A)$ is the unique value (that might be ∞) for which $s < \dim_{\mathcal{H}}(A)$ implies $\mathcal{H}^s(A) = \infty$; or, equivalently, $t > \dim_{\mathcal{H}}(A)$ implies $\mathcal{H}^t(A) = 0$.

For the case $s = \dim_{\mathcal{H}}(A)$ we do not have any general information about the value of $\mathcal{H}^s(A)$. It may happen to be 0, ∞ or finite. However, what we know is that if for a given A we can find some s so that $0 < \mathcal{H}^s(A) < \infty$, then s must be the Hausdorff dimension of A.

2.4.2 Estimating analytic capacity using the Hausdorff measure

We are ready to give some of the most important theorems of our project. These will connect the concepts of the analytic capacity of a compact subset of $\mathbb{C}$ with its Hausdorff measure, giving a first geometric understanding of γ. More specifically, the first (old) result, due to Painlevé, yields an estimate for $\gamma(E)$, where $E \subset \mathbb{C}$ is compact, in terms of the one-dimensional Hausdorff content of E.

Theorem 2.4.5 (Painlevé, 1888 [21])**.** *For every compact set $E \subset \mathbb{C}$, the following estimate holds*

$$\gamma(E) \leq \mathcal{H}^1_\infty(E).$$

Therefore, if $\mathcal{H}^1(E) = 0$, then E is removable.

Proof. Let us begin by fixing $\varepsilon > 0$ and cover E with a family of sets $\{A_n\}_n$ so that

$$\sum_{n=1}^{\infty} \operatorname{diam}(A_n) \leq \mathcal{H}^1_\infty(E) + \varepsilon.$$

Without loss of generality, we can assume that $\{A_n\}_n$ is a finite family of balls $\{B_n\}_{n=1}^N$ with respective radii r_n. Indeed, take if necessary r_n slightly bigger than $\operatorname{diam}(A_n)$ so that $E \subset \bigcup_n B_n$ and $\sum_n r_n \leq \mathcal{H}^1_\infty(E) + 2\varepsilon$, and use that E is compact to obtain a finite open subcover.

Consider now the curve $\Gamma = \partial_o\Big(\bigcup_{n=1}^N B_n\Big)$ and apply relation (2.1.1) so that, if f is the Ahlfors function of E (that recall that satisfies $|f| \leq 1$ in $\mathbb{C} \setminus E$), we obtain

$$|f'(\infty)| = \left|\frac{1}{2\pi i}\int_\Gamma f(z)dz\right| \leq \frac{1}{2\pi}\sum_{n=1}^{N}\int_{\Gamma\cap\partial B_i}|f(z)||dz| \leq \frac{1}{2\pi}\sum_{n=1}^{N} 2\pi r_n \leq \mathcal{H}^1_\infty(E) + 2\varepsilon,$$

and letting $\varepsilon \to 0$ we obtain the result. □

We also have the following estimate for the particular case in which $E \subset \mathbb{C}$ is a compact set contained in $\mathbb{R}$.

Theorem 2.4.6. *Let $E \subset \mathbb{C}$ be a compact set contained in $\mathbb{R}$. Then*

$$\frac{1}{4}\mathcal{H}^1(E) \leq \gamma(E) \leq \frac{1}{\pi}\mathcal{H}^1(E).$$

Proof. Since we can express any compact set of $\mathbb{R}$ as the limit of a sequence $\bigcap_{j=1}^n K_j$, where K_j is a finite collection of disjoint intervals; by Corollary 2.2.5 (outer regularity), it is sufficient to prove the result for E a finite collection of disjoint intervals.

So let us assume that E is of this form and consider $\varepsilon > 0$ and a rectifiable curve Γ surrounding E (that we may think as the sides of a rectangle) with $\mathcal{H}^1(\Gamma) = 2\mathcal{H}^1(E) + 2\varepsilon$ (recall that, for this type of curves, $\mathcal{H}^1(\Gamma)$ coincides with the length of the curve). By an analogous argument to the one used in the proof of Theorem 2.4.5, for any f admissible function for E we get

$$|f'(\infty)| = \left|\frac{1}{2\pi i}\int_\Gamma f(z)dz\right| \leq \frac{1}{\pi}\Big(\mathcal{H}^1(E) + \varepsilon\Big),$$

and so we deduce the first estimate $\gamma(E) \leq \frac{1}{\pi}\mathcal{H}^1(E)$.

To prove the remaining inequality we define the function $f := \frac{1}{2}\mathscr{C}(\mathcal{H}^1|_E)$. Notice that if $w = x + iy \notin E$, then

$$\mathfrak{Im} f(w) = \frac{1}{2}\int_E \mathfrak{Im}\frac{1}{t-w} d\mathcal{H}^1(t) = \frac{1}{2}\int_E \frac{y}{(t-x)^2 + y^2} d\mathcal{H}^1(t).$$

Hence, if $y = 0$ we have $\mathfrak{Im} f(w) = 0$, and if not

$$|\mathfrak{Im} f(w)| < \frac{1}{2}\int_{\mathbb{R}} \frac{|y|}{t^2 + y^2} d\mathcal{H}^1(t) = \frac{\pi}{2}.$$

So, in general, we deduce that f maps $\mathbb{C} \setminus E$ to the strip $|\mathfrak{Im}(z)| < \frac{\pi}{2}$. We now consider the conformal map

$$\varphi(z) = \frac{e^z - 1}{e^z + 1},$$

that maps the strip $|\mathfrak{Im}(z)| < \frac{\pi}{2}$ to the unit disk. This way $g := \varphi \circ f$ is an admissible function for E and, as $f(\infty) = 0$ and $g(0) = \varphi(0) = 0$, we have

$$\gamma(E) \geq |g'(\infty)| = \lim_{z\to\infty} |zg(z)| = \lim_{z\to\infty} |zf(z)| \left|\frac{\varphi(f(z))}{f(z)}\right| = |f'(\infty)\varphi'(0)|,$$

and using that $f'(\infty) = -\frac{1}{2}\mathcal{H}^1(E)$ and $\varphi'(0) = \frac{1}{2}$ we get the desired estimate. □

In fact, there is a much stronger result concerning the analytic capacity of compact subsets of the real line due to Pommerenke [22, Theorem 3], that we will not prove, that asserts

$$\gamma(E) = \frac{1}{4}\mathcal{H}^1(E). \tag{2.4.3}$$

Now we turn our attention to sets of Hausdorff dimension bigger than one. The result we will prove is the following essential estimate:

Theorem 2.4.7. *Let $s > 1$. Then, for every compact subset $E \subset \mathbb{C}$ there exists $c > 0$ such that the following estimate holds*

$$\gamma(E) \geq c\left(\frac{s-1}{s}\right)\mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

The proof of the previous theorem relies on the following lemma:

Lemma 2.4.8. (Frostman, [16, Theorem 8.8])**.** *Let $0 < s \leq d$ and $E \subset \mathbb{R}^d$ compact. Then $\mathcal{H}^s_\infty(E) > 0$ if and only if exists a non-trivial Borel measure ν supported on E so that*

$$\nu(B) \lesssim r(B)^s \qquad \text{and} \qquad \nu(E) \geq c\mathcal{H}^s_\infty(E),$$

for any ball B, where $r(B)$ is the radius of B; and for some constant $c > 0$.

Proof. Let us assume first that such measure ν exists and let us check that $\mathcal{H}^s_\infty(E) > 0$. Consider any covering $E \subset \bigcup_n A_n$ (that, by compactness, can be assumed to be finite) and take, for each n, a point $x_n \in A_n$. Since the union of the balls $B(x_i, \operatorname{diam}(A_i))$ is a covering of E, we obtain

$$\sum_n \operatorname{diam}(A_n)^s = \sum_n r\big(B(x_n, \operatorname{diam}(A_n))\big)^s \gtrsim \sum_n \nu\big(B(x_n, \operatorname{diam}(A_n))\big) \geq \nu(E).$$

So taking the infimum over all coverings we deduce $\mathcal{H}^s_\infty(E) \gtrsim \nu(E) > 0$, since ν is non-trivial and supported on E, and we are done with this implication.

Let us check now the converse. Assume that E is contained in some dyadic cube (by translating it, if necessary). Also, recall that the definition of $\mathcal{H}^s_\infty(E)$ was

$$\mathcal{H}^s_\infty(E) = \inf\left\{ \sum_{n=1}^\infty \operatorname{diam}(E_n)^s \,:\, E \subset \bigcup_{n=1}^\infty E_n \right\}.$$

Therefore, if $Q_1, Q_2, \dots$ is any family of cubes that cover E, we get that there exists a constant $c > 0$ so that

$$\sum_i \operatorname{diam}(Q_i)^s \geq c\mathcal{H}^s_\infty(E). \tag{2.4.4}$$

For every $m \in \mathbb{Z}_+$ we consider the family of dyadic cubes $\mathcal{Q}_m$ with side length 2^{-m}. We define a measure μ^m_m (depending on E) on $\mathbb{R}^d$ by requiring that for all $Q \in \mathcal{Q}_m$,

$$\begin{aligned} \mu^m_m|_Q &= 2^{-ms}\frac{1}{\mathcal{L}^d(Q)}\mathcal{L}^d|_Q, &&\text{if} \quad E\cap Q \neq \varnothing,\\ \mu^m_m|_Q &= 0, &&\text{if} \quad E \cap Q = \varnothing. \end{aligned}$$

This indeed defines a measure in $\mathbb{R}^d$, since $\mathcal{Q}_m$ covers the whole space implying that we can decompose any $\mathcal{L}^d$-measurable subset as a disjoint union of subsets of $\mathcal{Q}_m$ and use the σ-additivity of the Lebesgue measure.

Now we modify μ^m_m by defining another measure μ^m_{m-1} requiring that for every $Q \in \mathcal{Q}_{m-1}$ (the *dyadic parents* of the cubes in $\mathcal{Q}_m$) we have

$$\begin{aligned} \mu^m_{m-1}|_Q &= 2^{-(m-1)s}\frac{1}{\mu^m_m(Q)}\mu^m_m|_Q, &&\text{if} \quad \mu^m_m(Q) > 2^{-(m-1)s},\\ \mu^m_{m-1}|_Q &= \mu^m_m|_Q, &&\text{if} \quad \mu^m_m(Q) \leq 2^{-(m-1)s}. \end{aligned}$$

Notice that the μ^m_{m-1}-measure of a dyadic cube belonging to $\mathcal{Q}_{m-1}$ does not increase with respect to its μ^m_m-measure. We continue this recursive definition, setting μ^m_{m-k-1} to be, for each $Q \in \mathcal{Q}_{m-k-1}$,

$$\mu^m_{m-k-1}|_Q = \min\left\{1, 2^{-(m-k-1)s}\frac{1}{\mu^m_{m-k}(Q)}\right\}\mu^m_{m-k}|_Q.$$

We stop this process as soon as $E \subset Q$ for some $Q \in \mathcal{Q}_{m-k_0}$ and put $\mu^m := \mu^m_{m-k_0}$. Observe that by construction we have the following estimate

$$\mu^m(Q) \leq 2^{-(m-k)s} \quad \text{for} \quad Q \in \mathcal{Q}_{m-k},\ k = 0,1,2,\dots \tag{2.4.5}$$

In addition, observe that any set in $\mathbb{R}^d$ not intersecting E has μ^m-measure 0. That is because the μ^m_m-measure of such set would be 0 and the recursive definition implies, eventually, that it has also μ^m-measure 0.

Moreover, observe that for every $x \in E$ we will always be able to find $k \in \{0, 1, \dots, k_0\}$ and $Q \in \mathcal{Q}_{m-k}$ so that $x \in Q$ and if $\ell(Q)$ denotes the side length of Q,

$$\mu^m(Q) = 2^{-(m-k)s} = (\ell(Q))^s = d^{-\frac{s}{2}} \cdot \left(\sqrt{d} \cdot \ell(Q)\right)^s = d^{-\frac{s}{2}} \cdot \operatorname{diam}(Q)^s. \tag{2.4.6}$$

Let us clarify this. Assume we fix m and $x \in E$. Let us pick the unique $Q \in \mathcal{Q}_{m-k_0}$ so that $x \in Q$. If $\mu^m_{m-k_0+1}(Q) > 2^{-(m-k_0)s}$ we are done, choosing $k := k_0$. If not, that means that $\mu^m_{m-k_0+1}(Q) \leq 2^{-(m-k_0)s}$ and therefore, by definition, $\mu^m(Q) = \mu^m_{m-k_0+1}(Q)$. This process might continue, in the worst scenario, until we reach that $\mu^m(Q) = \mu^m_m(Q)$, but even for this case we would have $\mu^m(Q) = 2^{-ms}$, so by choosing $k := 0$ we would be done.

Picking for each x the largest Q satisfying the property (2.4.6), we obtain disjoint cubes $Q_1, \dots, Q_N$ that cover E. Now, using that if any $A \subset \mathbb{R}^d$ is such that $A \cap E = \varnothing$, then $\mu^m(A) = 0$, we have

$$\mu^m(\mathbb{R}^n) = \mu^m(E) = \sum_{i=1}^N \mu^m(Q_i) = d^{-\frac{s}{2}} \sum_{i=1}^N \operatorname{diam}(Q_i)^s \geq d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E).$$

where we have applied relation (2.4.4). Let, for each $m \in \mathbb{Z}_+$

$$\nu^m := \frac{1}{\mu^m(\mathbb{R}^d)} \mu^m.$$

It is clear that $\nu^m(E) = \nu^m(\mathbb{R}^d) = 1$ and also that for every $k = 0, 1, 2, \dots$ and $Q \in \mathcal{Q}_{m-k}$

$$\nu^m(Q) \leq \frac{1}{d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E)} 2^{-(m-k)s}$$

because of relation (2.4.5). Therefore, we have obtained a sequence $(\nu^m)_m$ of measures in $\mathbb{R}^d$ that satisfy $\sup_{m \in \mathbb{Z}_+} \nu^m(K) < \infty$ for every compact set $K \subset \mathbb{R}^d$. Hence, there is a weakly convergent subsequence $\nu^{m_k} \to \nu$ as $k \to \infty$ [16, Theorem 1.23], ν being a Borel measure and $\nu(E) = \nu(\mathbb{R}^d) = 1$.

Now, for any $x \in \mathbb{R}^n$ and $0 < r < \infty$ we can find $p \in \mathbb{Z}_+$ so that $B(x, r)$ is contained in the interior U of a union $\bigcup_{i=1}^{2^d} Q_i$ of 2^d cubes $Q_i \in \mathcal{Q}_p$ with

$$\operatorname{diam}(Q) = 2^{-p} d^{\frac{1}{2}} \leq 4 r d^{\frac{1}{2}}.$$

That is, p has been chosen so that the interior of 2^d cubes of $\mathcal{Q}_p$ cover $B(x, r)$ and do not have side length bigger than twice its diameter. Hence for $m \geq p$,

$$\nu^m(U) \leq 2^d \frac{1}{d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E)} 2^{-ps} \leq 2^d \frac{1}{d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E)} (4r)^s = 2^{d+2s} \frac{1}{d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E)} r^s.$$

Now, using [16, Theorem 1.24] we conclude

$$\nu(B(x, r)) \leq \nu(U) \leq \liminf_{k \to \infty} \nu^{m_k}(U) \leq 2^{d+2s} \frac{1}{d^{-\frac{s}{2}} c\mathcal{H}^s_\infty(E)} r^s,$$

and so $\nu(B(x, r)) \lesssim r^s$. Also, as $\nu(E) = 1$, choosing c small enough so that $c\mathcal{H}^s_\infty(E) \leq 1$, we are done, since we obtain the second estimate $\nu(E) \geq c\mathcal{H}^s_\infty(E)$. $\square$

Proof (*Theorem 2.4.7*). Use Frostman's lemma 2.4.8 to pick ν a Borel measure supported on E with $\nu(E) \geq c\mathcal{H}^s_\infty(E)$ and $\nu(B(z,r)) \lesssim r^s$ for each $z \in \mathbb{C}$ and $r > 0$. Consider the function $f := \mathscr{C}\nu$ and observe that $f(\infty) = 0$ and $|f'(\infty)| = \nu(E)$, by Theorem 2.3.2. Moreover

$$|\mathscr{C}\nu(z)| \leq \int \frac{1}{|w-z|} d\nu(w) = \int \bigg(\int_0^{|w-z|^{-1}} dt \bigg) d\nu(w) = \int_0^\infty \bigg(\int_{\{w\,:\,|w-z|<t^{-1}\}} d\nu(w) \bigg) dt$$
$$= \int_0^\infty \nu\big(\{w \,:\, |w-z| < t^{-1}\}\big) dt = \int_0^\infty \nu\big(B(z,t^{-1})\big) dt \lesssim \int_0^\infty \min\big(\nu(E), t^{-s}\big) dt.$$

And computing this last integral we obtain

$$\int_0^{\nu(E)^{-\frac{1}{s}}} \nu(E) dt + \int_{\nu(E)^{-\frac{1}{s}}}^\infty t^{-s} dt = \nu(E)^{1-\frac{1}{s}} + \frac{1}{s-1}\big(\nu(E)^{\frac{1}{s}}\big)^{s-1} = \frac{s}{s-1}\nu(E)^{1-\frac{1}{s}}.$$

Therefore, $\|f\|_\infty \lesssim \frac{s}{s-1}\nu(E)^{1-\frac{1}{s}}$. Now, normalizing f to be $f/\|f\|_\infty$ in order for f to be admissible, we finally conclude

$$\gamma(E) \geq \frac{|f'(\infty)|}{\|f\|_\infty} \gtrsim \bigg(\frac{s-1}{s}\bigg)\nu(E)^{\frac{1}{s}} \gtrsim \bigg(\frac{s-1}{s}\bigg)\mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

□

2.5 The critical dimension and Denjoy's conjecture

The two main results of the previous section that allow us to estimate the analytic capacity of a compact subset via its Hausdorff measure/dimension are Theorems 2.4.5 and 2.4.7. The first yields

- ⋆ $\gamma(E) \leq \mathcal{H}_\infty(E) \leq \mathcal{H}^1(E)$. *So in particular if* $\dim_{\mathcal{H}}(E) < 1$, *then* $\gamma(E) = 0$,

and the second, by Proposition 2.4.3, implies

- ⋆ *If* $\dim_{\mathcal{H}}(E) > 1$, *then* $\gamma(E) > 0$.

So by the above statements we deduce that **dimension 1 is the critical dimension** for the analytic capacity. A natural question arises: if a compact set E has positive analytic capacity, does it mean that its one-dimensional Hausdorff measure, $\mathcal{H}^1(E)$, is non-zero (maybe infinite)? That is, do the reciprocal implications hold?

Finding the answer to the previous question is not an easy task, and it was first achieved by Vitushkin in the 1960's, when he constructed a compact set with positive one-dimensional Hausdorff measure and null analytic capacity, concluding that the answer to the problem was *negative*. A paradigmatic example of such set is the so called *four-corner Cantor set*. This set is constructed in similar way as the usual Cantor set of $\mathbb{R}$. We show the first four iterates of its construction in Figure 2.1 (let us remark that the set of points we are considering in each iterate is formed by the union of the *boundaries* of the different squares of each generation). If E denotes the previous singular set, in Tolsa [27] we can find a rather simple proof of the fact that $0 < \mathcal{H}^1(E) < \sqrt{2}$, implying that $\dim_{\mathcal{H}}(E) = 1$. On the other hand, the proof of $\gamma(E) = 0$ turns out to be more intricate, and it was found by Garnett [11].

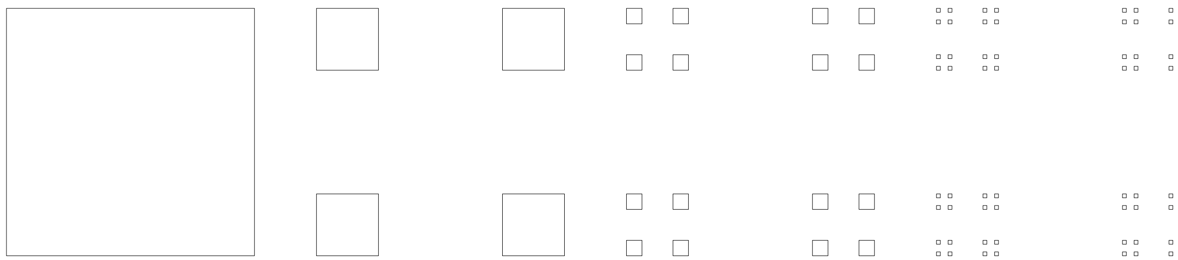


Figure 2.1: First four iterates involved in the definition of the four-corner Cantor set.

A full characterization of removable subsets of positive one-dimensional Hausdorff measure is not a trivial task. So from this point on, we will restrict ourselves to a more particular setting, focusing our attention on the following question:

> *given $E \subset \mathbb{C}$ compact, connected and with $\mathcal{H}^1(E) < \infty$; if $\gamma(E) = 0$, what does this imply about the geometric structure of E?*

Eventually, the type of compact sets E we will study will be those of Hausdorff dimension 1 with finite measure, so we can understand them as usual sets of Lebesgue dimension 1, with null area and finite length measure. Using [9, Exercise 3.5] we deduce that E being compact, connected and with finite $\mathcal{H}^1$-measure is equivalent to being the image of a rectifiable curve. So our case of study can be rephrased, in a slightly more general way, as follows:

> *given $\Gamma \subset \mathbb{C}$ rectifiable curve and $E \subset \Gamma$ compact, what are the implications of $\gamma(E) = 0$ on the geometric structure of E?*

The previous problem was first outlined by Arnaud Denjoy (1874-1974), and he proposed the following conjecture:

Conjecture 2.5.1 (Denjoy)**.** If a rectifiable curve contains a compact set with positive length, then this compact set has positive analytic capacity (and thus it is not removable).

Denjoy first thought of it as an extension of Pommerenke's result 2.4.3 in $\mathbb{R}$. This conjecture was first partially answered (positively) by Calderón [3] by means of the L^2-boundedness of the Cauchy transform. In his response he covered the case where the slope of Γ was small enough, connecting the subjects of analytic capacity and the theory of singular integrals.

For us this will be the objective from this point on. That is, we will develop the necessary theory concerning singular integrals in order to understand Calderón's partial proof of Denjoy's conjecture.

Chapter 3

Maximal operators and the Hilbert transform

Our main goal in the forthcoming sections is to present, eventually, the definition of a Calderón-Zygmund operator. We will see that the Cauchy transform is particular case of this type of operator and its properties will be the key to tackle Denjoy's conjecture in a similar way as Calderón did, covering a partial proof of it. More particularly, we will prove that if the slope of the graph of the curve Γ is small enough, Denjoy's conjecture holds.

But in order to achieve the previous goal as rigorously as possible, we need to do some groundwork. Indeed, to be able to cover the definitions and the proofs of some of the most important results, we will need to present a series of concepts which are necessary to give a coherent and significant meaning to the theory of singular integrals. In fact, in this chapter we will introduce some theorems that will be essential in order to establish, eventually, the boundedness of singular integral operators. More precisely, the main results will be the Marcinkiewicz interpolation theorem (Theorem 3.1.3) and a result concerning maximal operators (Theorem 3.1.2). Both will ease the task of checking the boundedness property of operators by checking it for just some specific cases so that, if they are proved to hold, then the boundedness will also follow for the rest.

3.1 The Marcinkiewicz interpolation theorem and maximal operators

We begin our discussion by introducing a weaker condition than usual boundedness for operators. We will find this necessary when studying maximal operators defined in $L^p(\mathbb{R}^n)$. Indeed, during this process we will see that for the case $p = 1$ such operators satisfy a weaker condition rather than being bounded in an ordinary way. So let us first introduce this less restrictive notion in a general setting for measurable spaces (although we will mainly work with $\mathbb{R}^n$ endowed with the Lebesgue measure).

Definition 3.1.1 (Weak type (p,q) operator)**.** Let $(X,\mu),(Y,\nu)$ be measurable spaces, $1\leq p\leq\infty$ and $T:L^p(X,\mu)\to\{g:Y\to\mathbb{C},\text{ measurable}\}$ an operator. We will say that T is of *weak type* (p,q), $1\leq q<\infty$, if for every $f\in L^p(X,\mu)$ there exists $C>0$ such that

$$\nu\big(\{y\in Y:\ |Tf(y)|>\lambda\}\big)\leq\left(\frac{C\|f\|_p}{\lambda}\right)^q,\qquad\forall\lambda>0$$

This last inequality will be usually referred as the *weak type* (p,q) *inequality.* We will also say that T is of *weak type* (p,∞) if T is a bounded operator from $L^p(X,\mu)$ to $L^\infty(Y,\nu)$.

The next lemma proves that the previous notion is less restrictive than being bounded.

Lemma 3.1.1. *Let $T:L^p(X,\mu)\to L^q(Y,\nu)$ be a bounded operator. Then T is of weak type (p,q).*

Proof. Let us fix $f\in L^p(X,\mu)$ and put $E_\lambda:=\{y\in Y:\ |Tf(y)|>\lambda\}$. Then

$$\nu(E_\lambda)=\int_{E_\lambda}d\nu(x)\leq\int_{E_\lambda}\left|\frac{Tf(x)}{\lambda}\right|^q d\nu(x)\leq\frac{\|Tf\|_q^q}{\lambda^q}\leq\left(\frac{\|T\|\|f\|_q}{\lambda}\right)^q.$$

□

For the sake of completeness we introduce the following definition just to fix notation.

Definition 3.1.2 (Strong type (p,q) operator)**.** Let $(X,\mu),(Y,\nu)$ be measurable spaces and let $T:L^p(X,\mu)\to L^q(Y,\nu)$ be an operator. We will say that T is of *strong type* (p,q) if it is a bounded operator. Notice that, for us, if $q=\infty$, weak and strong boundedness will refer to the same property.

Let us introduce now the concept of maximal operator associated to a family of operators, one of the central notions of this section.

Definition 3.1.3 (Maximal operator)**.** Let $\{T_t\}_t$ be a family of operators defined from $L^p(X,\mu)$ to the space of measurable functions from X to $\mathbb{C}$. We define the *maximal operator T^* associated to the family $\{T_t\}_t$* as follows

$$T^*f(x):=\sup_t|T_tf(x)|.$$

The interest in considering the maximal operator of a certain family is to infer properties of the latter just by studying the former. This is the case, for example, of the following theorem.

Theorem 3.1.2. *If T^* is of weak type (p,q), then for every $-\infty\leq t_0\leq\infty$*

1. *The set $\{f\in L^p(X,\mu):\ \lim_{t\to t_0}T_tf(x)=f(x)\ \ \mu\text{-a.e.}\}$ is closed.*
2. *The set $\{f\in L^p(X,\mu):\ \lim_{t\to t_0}T_tf(x)\ \text{ exists }\ \mu\text{-a.e.}\}$ is closed.*

Proof. We will begin by proving *1.* Let us fix t_0 and consider $(f_n)_n$ a sequence of functions converging to f in $L^p(X,\mu)$ and such that $T_tf_n(x)$ converges to $f_n(x)$ μ-a.e. for every n as $t\to t_0$. We first observe the for almost every x we have

$$\begin{aligned}\limsup_{t\to t_0} |T_t f(x) - f(x)| &= \limsup_{t\to t_0} \big(|T_t f(x) - f(x)| - |T_t f_n(x) - f_n(x)|\big)\\ &\le \limsup_{t\to t_0} |T_t(f-f_n)(x) - (f-f_n)(x)|\\ &\le \limsup_{t\to t_0} |T_t(f-f_n)(x)| + |(f-f_n)(x)|.\end{aligned}$$

Fix $\lambda > 0$ and use the above inequality, the weak type (p,q) inequality and Chebyshev's inequality to obtain

$$\begin{aligned}&\mu\Bigg(\bigg\{x\in X:\ \limsup_{t\to t_0}|T_t f(x)-f(x)|>\lambda\bigg\}\Bigg)\\ &\quad\le \mu\left(\left\{x\in X:\ \limsup_{t\to t_0}|T_t(f-f_n)(x)|+|(f-f_n)(x)|>\lambda\right\}\right)\\ &\quad\le \mu\left(\left\{x\in X:\ \limsup_{t\to t_0}|T_t(f-f_n)(x)|>\frac{\lambda}{2}\right\}\right)+\mu\left(\left\{x\in X:\ |(f-f_n)(x)|>\frac{\lambda}{2}\right\}\right)\\ &\quad\le \mu\left(\left\{x\in X:\ T^*(f-f_n)(x)>\frac{\lambda}{2}\right\}\right)+\mu\left(\left\{x\in X:\ |(f-f_n)(x)|>\frac{\lambda}{2}\right\}\right)\\ &\quad\le \left(\frac{2C}{\lambda}\|f-f_n\|_p\right)^q+\left(\frac{2}{\lambda}\|f-f_n\|_p\right)^p \xrightarrow[n\to\infty]{} 0\end{aligned}$$

proving the first result. To prove *2* notice that it is enough to check that

$$\mu\Bigg(\bigg\{x\in X:\ \limsup_{t\to t_0} T_t f(x) - \liminf_{t\to t_0} T_t f(x) > \lambda\bigg\}\Bigg) = 0, \qquad \forall\lambda>0.$$

Again, take $(f_n)_n$ converging to f in $L^p(X,\mu)$ such that $T_t f_n(x)$ converges to a certain value $\ell_n(x)$ μ-a.e. for each n as $t\to t_0$. Adding and subtracting $\ell_n(x)$ and using that $\ell_n(x) = \limsup_{t\to t_0} T_t f_n(x) = \liminf_{t\to t_0} T_t f_n(x)$ μ-a.e. we get for every $\lambda>0$

$$\begin{aligned}&\mu\Bigg(\bigg\{x\in X:\ \limsup_{t\to t_0} T_t f(x) - \liminf_{t\to t_0} T_t f(x) > \lambda\bigg\}\Bigg)\\ &\quad= \mu\Bigg(\bigg\{x\in X:\ \limsup_{t\to t_0} T_t(f-f_n)(x) - \liminf_{t\to t_0} T_t(f-f_n)(x) > \lambda\bigg\}\Bigg)\\ &\quad\le \mu\Bigg(\bigg\{x\in X:\ 2T^*(f-f_n)(x)>\lambda\bigg\}\Bigg) \le \left(\frac{2C}{\lambda}\|f-f_n\|_p\right)^q \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

□

One of the main applications of this result is the pointwise a.e. convergence of approximate identities in L^p spaces, $1\le p<\infty$. The way to proceed will be to prove it for the functions of the Schwartz class and use their density in the L^p spaces along with the previous theorem to deduce the general result.

As we can see, although the study of a maximal operator tells us a lot of information about its original family of operators, we are left to check an inequality that may be hard to prove. However the following result will ease significantly the previous task:

Theorem 3.1.3. (Marcinkiewicz interpolation, [7, Theorem 2.4])**.** *Let $(X,\mu),(Y,\nu)$ be measurable spaces, $1 \leq p_0 < p_1 \leq \infty$ and T a sublinear operator from $L^{p_0}(X,\mu)+L^{p_1}(X,\mu)$ to the space of measurable functions from Y to $\mathbb{C}$. Assume T is of weak type (p_0,p_0) and of weak type (p_1,p_1). Then, T is of strong type (p,p) for $p_0<p<p_1$.*

Previous to the proof of the theorem let us clarify the notation used in its statement:

- First, by the space $L^{p_0}(X,\mu)+L^{p_1}(X,\mu)$ we mean the set formed by the $\mathbb{C}$-valued functions defined on X which can be written as a sum f_0+f_1 where $f_0 \in L^{p_0}(X,\mu)$ and $f_1 \in L^{p_1}(X,\mu)$.
- And second, when we say that T is a *sublinear* operator we mean that it satisfies the following properties: for every $f,g \in L^{p_0}(X,\mu)+L^{p_1}(X,\mu)$ and $y \in Y$
  1. $|T(f+g)(y)| \leq |Tf(y)|+|Tg(y)|$.
  2. For every $\lambda \in \mathbb{C}$ we have $|T(\lambda f)(y)| \leq |\lambda||Tf(y)|$.

In addition, let us introduce a specific function that will be helpful in the proof of the theorem as well as one of its basic properties.

Definition 3.1.4 (Distribution function)**.** Let (X,μ) be a measure space and let $f: X \to \mathbb{C}$ be a measurable function. The *distribution function of f* (associated to μ), $a_f:(0,\infty)\to(0,\infty]$ is defined as

$$a_f(\lambda) := \mu\left(\{x \in X:\ |f(x)|>\lambda\}\right).$$

Proposition 3.1.4. *Let $\phi:[0,\infty)\to[0,\infty)$ be differentiable, increasing and such that $\phi(0)=0$. Then the following identity holds for every measurable $f: X\to\mathbb{C}$*

$$\int_X \phi\left(|f(x)|\right)d\mu = \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda.$$

Proof.

$$\begin{aligned}\int_X \phi\left(|f(x)|\right)d\mu &= \int_X\int_0^{|f(x)|}\phi'(\lambda)d\lambda d\mu = \int_0^\infty \phi'(\lambda)\left(\int_{\{x\in X:\ |f(x)|\geq\lambda\}}d\mu\right)d\lambda \\ &= \int_0^\infty \phi'(\lambda)a_f(\lambda)d\lambda,\end{aligned}$$

where in the first equality we have used that $\phi(0)=0$ and in the second that $\phi'(\lambda)\geq 0$, so that we can apply Tonelli's theorem. □

One particular case in which we will be interested in is when $\phi(\lambda)=\lambda^p$. The result reads as follows:

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1}a_f(\lambda)d\lambda. \tag{3.1.1}$$

Proof (*Theorem 3.1.3*). Let us fix $p\in(p_0,p_1)$ and prove that T is of strong type (p,p). First, we choose $f\in L^p(X,\mu)$ arbitrary and we prove that f can be written as $f=f_0+f_1$, where $f_0\in L^{p_0}(X,\mu)$ and $f_1\in L^{p_1}(X,\mu)$. Indeed, let $c>0$ and $\lambda>0$, and set

$$f_0 := f\chi_{\{|f|>c\lambda\}}, \qquad f_1 := f\chi_{\{|f|\leq c\lambda\}}.$$

Applying Hölder's inequality (with the exponents p/p_0 and $p/(p-p_0)$) and Chebyshev's inequality, we obtain

$$\begin{aligned}\|f_0\|_{p_0}^{p_0} &= \int_X |f(x)\chi_{\{|f|>c\lambda\}}(x)|^{p_0} d\mu(x) \leq \|f\|_p^{p_0} \mu\left(\{x : |f(x)| > c\lambda\}\right)^{1-p_0/p} \\ &\leq \|f\|_p^{p_0} \frac{1}{(c\lambda)^p} \|f\|_p^{p-p_0} = (c\lambda)^{-p}\|f\|_p^p < \infty.\end{aligned}$$

And similarly for f_1, applying Chebyshev's inequality we get

$$\|f_1\|_{p_1}^{p_1} = \int_X |f(x)\chi_{\{|f|\leq c\lambda\}}(y)|^{p_1} d\mu(x) \leq (c\lambda)^{p_1}\mu\left(\{x : |f(x)| \leq c\lambda\}\right) \leq (c\lambda)^{p_1-p}\|f\|_p^p < \infty.$$

Moreover, since T is sublinear we have for every $y \in Y$ the estimate $|Tf(y)| \leq |Tf_0(y)| + |Tf_1(y)|$. This implies, in particular

$$\begin{aligned}a_{Tf}(\lambda) = \mu\left(\{x : |Tf(x)| > \lambda\}\right) &\leq \mu\left(\left\{x : |Tf_0(x)| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x : |Tf_1(x)| > \frac{\lambda}{2}\right\}\right) \\ &= a_{Tf_0}\left(\frac{\lambda}{2}\right) + a_{Tf_1}\left(\frac{\lambda}{2}\right). \end{aligned} \tag{3.1.2}$$

Now we proceed with the proof by assuming first that $p_1 = \infty$. Since T is of weak type (p_1, p_1), for every $g \in L^\infty(X,\mu)$ we have that $Tg \in L^\infty(Y,\nu)$ and $\|Tg\|_\infty \leq A_1\|g\|_\infty$ for some $A_1 > 0$. Then, by fixing $c := (2A_1)^{-1}$ we get that $\|f_1\|_\infty \leq \lambda/(2A_1)$ and so $\|Tf_1\|_\infty \leq \lambda/2$, which implies $a_{Tf_1}(\lambda/2) = 0$. Now, regarding f_0, the weak type (p_0, p_0) inequality yields

$$a_{Tf_0}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0}.$$

So for this case, the decomposition (3.1.2) as well as the relation (3.1.1) imply the result, namely

$$\begin{aligned}\|Tf\|_p^p = p\int_0^\infty \lambda^{p-1} a_{Tf}(\lambda)d\lambda &\leq p\int_0^\infty \lambda^{p-1} a_{Tf_0}\left(\frac{\lambda}{2}\right)d\lambda \\ &\leq p(2A_0)^{p_0}\int_0^\infty \lambda^{p-1-p_0}\left(\int_{\{2A_1|f|>\lambda\}} |f(x)|^{p_0}d\mu(x)\right)d\lambda \\ &= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\left(\int_0^{2A_1|f(x)|}\lambda^{p-1-p_0}d\lambda\right)d\mu(x) = \frac{p2^p}{p-p_0}A_0^{p_0}A_1^{p-p_0}\|f\|_p^p.\end{aligned}$$

For the remaining case, $p_1 < \infty$, we will use both weak type inequalities

$$a_{Tf_0}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0}, \quad a_{Tf_1}\left(\frac{\lambda}{2}\right) \leq \left(\frac{2A_1}{\lambda}\|f_1\|_{p_1}\right)^{p_1},$$

and by the exact same argument as the previous one (now done also for f_1, since $a_{Tf_1}(\lambda/2)$ may not be 0) we obtain

$$\|Tf\|_p^p \leq p\left(\frac{(2A_0)^{p_0}}{(p-p_0)c^{p-p_0}} + \frac{(2A_1)^{p_1}}{(p_1-p)c^{p_1-p}}\right)\|f\|_p^p.$$

$\square$

3.1.1 The Hardy-Littlewood maximal function

We will now study an specific maximal operator whose properties will be closely related to those of certain families of functions which define an approximate identity. Before introducing the operator itself, as a motivation, let us recall what we mean by approximate identity and some of its properties.

Definition 3.1.5 (Approximate identity)**.** Let $\phi \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $t > 0$ we define

$$\phi_t(x) := \frac{1}{t^n}\phi\left(\frac{x}{t}\right),$$

which clearly satisfies $\phi_t \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi_t(x)dx = 1$. We will call the collection $\{\phi_t\}_{t>0}$ an *approximate identity*.

We know that we can identify every integrable function in $\mathbb{R}^n$ as an element of $\mathcal{S}'(\mathbb{R}^n)$ – that is, the space of linear continuous functionals defined in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. More specifically, and in the case of ϕ_t, that functional is

$$T_{\phi_t} f := \int_{\mathbb{R}^n} f(x)\phi_t(x)dx = \int_{\mathbb{R}^n} f(tx)\phi(x)dx, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{3.1.3}$$

Later on we will obtain a result that will allow us to state that the association $\phi_t \mapsto T_{\phi_t}$, which is clearly linear, is injective. Observe that for almost every $x \in \mathbb{R}^n$ we have $\lim_{t\to 0} \phi(x)f(tx) = \phi(x)f(0)$ and also $|\phi(x)f(tx)| \le \|f\|_\infty \phi(x)$, with $\phi \in L^1(\mathbb{R}^n)$. So by the dominated convergence theorem we get

$$\lim_{t\to 0} T_{\phi_t} f = \left(\int_{\mathbb{R}^n} \phi(x)dx\right) f(0) = f(0) = \int_{\mathbb{R}^n} f(x)d\delta_0 = \langle \delta_0, f\rangle, \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

being δ_0 the Dirac measure at $\{0\}$. An immediate consequence of the previous argument is that if $\{\phi_t\}_{t>0}$ is an approximate identity and $f \in \mathcal{S}(\mathbb{R}^n)$, then $\phi_t * f$ converges pointwise to f as $t \to 0$. In fact, there is a well-known refinement of the previous statement concerning the convolution with approximate identities that we will not prove [7, Theorem 2.1]. It reads as follows,

Theorem 3.1.5. *Let $\{\phi_t\}_{t>0}$ be an approximate identity. Then*

1. *If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, we have $\|\phi_t * f - f\|_p \xrightarrow[t\to 0]{} 0$.*
2. *If $f \in \mathcal{C}_0(\mathbb{R}^n)$ (continuous and tending to 0 as $|x| \to \infty$), we have $\|\phi_t * f - f\|_\infty \xrightarrow[t\to 0]{} 0$.*

In any case, suppose that we want to determine for $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, if there is also pointwise convergence $\mathcal{L}^n$-a.e. of $\phi_t * f$ to f as $t \to 0$. Since this result is true for $f \in \mathcal{S}(\mathbb{R}^n)$, if we manage to prove that the maximal operator

$$f \mapsto \sup_{t>0} |\phi_t * f|, \qquad f \in L^p(\mathbb{R}^n)$$

is weakly bounded for a certain class of approximate identities $\{\phi_t\}_{t>0}$, by the density of Schwartz class in the spaces $L^p(\mathbb{R}^n)$ (with respect to each of their norms) and Theorem 3.1.2, we can deduce the result.

Now, we are left to study a certain class of integral operators depending on an approximate identity. Writing explicitly the maximal operator we want to study, it is of the form

$$f(x) \mapsto \sup_{t>0} \left| \int_{\mathbb{R}^n} \phi_t(y) f(x-y)dy \right|. \tag{3.1.4}$$

A particular case of function ϕ, that gives rise to the definition of the Hardy-Littlewood maximal function, is obtained if we choose $\phi(x) := |B(0,1)|^{-1}\chi_{B(0,1)}(x)$.

Definition 3.1.6 (Hardy-Littlewood maximal function)**.** Let $B_r := B(0,r) \subseteq \mathbb{R}^n$ be the euclidean ball of radius r centered at the origin and let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a locally integrable function. The *Hardy-Littlewood maximal function of* f is given by

$$Mf(x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)|dy.$$

Notice that for non-negative functions we have that Mf coincides with the maximal operator in (3.1.4). The fundamental property of the M operator is given in the next theorem:

Theorem 3.1.6. *The operator M is of weak type $(1,1)$ and of strong type (p,p) for $1 < p \leq \infty$.*

We will give a proof of this result for the case $n = 1$. The reader may find in the book of Duoandikoetxea [7, Chapters 4 & 7], the proofs for the remaining cases. In any case, returning to the one-dimensional setting, first we will need the following lemma:

Lemma 3.1.7. *Let $\{I_\alpha\}_{\alpha\in A}$ be a collection of open intervals in $\mathbb{R}$ and let $K \subset \mathbb{R}$ be compact and contained in $\bigcup_{\alpha\in A} I_\alpha$. Then, there exists a finite subcollection of intervals $\{I_j\}_{j=1}^N \subset \{I_\alpha\}_{\alpha\in A}$ satisfying*

1. *$K \subseteq \bigcup_{j=1}^N I_j$*
2. *The intersection of three intervals of the subcollection is empty, i.e.*

$$\sum_{j=1}^N \chi_{I_j}(x) \leq 2, \qquad x \in \mathbb{R}.$$

Proof. Since K is compact we can pick a finite subcovering $\{I_{\alpha_j}\}_{j=1}^N$ from the initial collection of intervals. In addition, we will assume that $I_{\alpha_j} \cap K \neq 0$ for every $j = 1, \ldots, N$ and, moreover, that there is $x_j \in I_{\alpha_j}$ with $x \notin \bigcup_{k=1, k\neq j}^N I_{\alpha_k}$ and $x_j \in K$, for every $j = 1, \ldots, N$. Our final assumption will be that K is a closed interval that we name $[a,b]$ (the general case would be treated analogously using the argument that follows).

Let us begin by choosing $i_1 \in \{\alpha_1, \ldots, \alpha_N\}$ so that $a \in I_{i_1} = (x_{i_1}, y_{i_1})$ and we define $I_1 := I_{i_1}$. Notice that if $y_{i_1} \notin K$, then we are done. On the other hand, if $y_{i_1} \in K$ we pick i_2 so that $y_{i_1} \in I_{i_2} = (x_{i_2}, y_{i_2})$ and we define I_2 to be

$$I_2 := I_{i_2} \cap \left(\frac{x_{i_1} + y_{i_1}}{2}, y_{i_2} \right),$$

so that we do not have $I_1 \subset I_2$. Again, if $y_{i_2} \notin K$ we would be done. If not, we pick i_3 so that $y_{i_2} \in I_{i_3} = (x_{i_3}, y_{i_3})$ and we define I_3 to be

$$I_3 := I_{i_3} \cap \left(\frac{y_{i_1} + y_{i_2}}{2}, y_{i_3} \right).$$

This last construction illustrates the general pattern we need to repeat – a finite number of times, since $K \subset \bigcup_{j=1}^N I_j$ – to obtain the finite subcollection we are interested in. □

Proof (*Theorem 3.1.6*). Begin by observing that if $f \in L^\infty(\mathbb{R}^n)$, it is immediate that $\|Mf\|_\infty \leq \|f\|_\infty$. Thus, by Marcinkiewicz interpolation theorem 3.1.3 it is enough to prove that M is of weak type $(1,1)$.

So pick $f \in L^1(\mathbb{R})$ and for each $\lambda > 0$ consider $E_\lambda := \{x \in \mathbb{R} : Mf(x) > \lambda\}$. If $x \in E_\lambda$, then, by definition of M, there is an interval I_x centered at x such that

$$\frac{1}{|I_x|} \int_{I_x} |f(y)| dy > \lambda \quad \Leftrightarrow \quad |I_x| < \frac{1}{\lambda} \int_{I_x} |f(y)| dy. \tag{3.1.5}$$

Let $K \subset E_\lambda$ be compact and apply Lemma 3.1.7 to this set and the initial covering $\{I_x\}_{x \in E_\lambda}$. Then, we obtain $\{I_{x_j}\}_{j=1}^N$ a finite covering of K with $\sum_{j=1}^N \chi_{I_{x_j}} \leq 2$. Therefore by (3.1.5)

$$|K| \leq \sum_{j=1}^N |I_{x_j}| < \sum_{j=1}^N \frac{1}{\lambda} \int_{I_{x_j}} |f(y)| dy \leq \sum_{j=1}^N \frac{1}{\lambda} \int_{\mathbb{R}} \chi_{I_{x_j}}(y) |f(y)| dy \leq \frac{2}{\lambda} \|f\|_1.$$

Since this inequality is true for every compact $K \subset E_\lambda$, we obtain the desired weak type $(1,1)$ inequality, i.e. $|E_\lambda| \leq \frac{2}{\lambda}\|f\|_1$ (that is because the Lebesgue measure is a Radon measure, so in particular the measure of any open set can be obtained as the supremum over the measures of the compact sets contained in it). □

Hence, now we can state the following result, which will be a consequence of the previous Theorem 3.1.6 (we present it in its one-dimensional form, but recall that it is still valid in the multidimensional case).

Corollary 3.1.8. *If $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$ and $\phi := 2^{-1}\chi_{(-1,1)}$, then*

$$\lim_{t \to 0} (\phi_t * f)(x) = f(x), \qquad \mathcal{L}\text{-a.e.}$$

In addition, the result is still true also if $f \in \mathcal{C}_0(\mathbb{R}^n)$.

The last assertion is just a consequence of the second point of Theorem 3.1.5, and notice that for this case the convergence is more restrictive since it becomes uniform, i.e. with respect to the norm $\|\cdot\|_\infty$.

We may wonder if this last corollary could be also deduced if we chose another positive integrable function ϕ with similar properties to $\chi_{(-1,1)}$. And, indeed, this is the case, as it is shown in Duoandikoetxea [7, Corollary 2.9], where the result is proved just assuming that a general function ϕ would be a proper choice if $|\phi|$ is bounded $\mathcal{L}$-a.e. by ψ, a positive, integrable, radial and decreasing function.

Returning to Corollary 3.1.8, we would like to stress a particular case of the multidimensional version for $p = 1$, and we state it in a slightly more general setting. This important result has a name of its own:

Theorem 3.1.9. (Lebesgue differentiation theorem, [7, Corollary 2.13]). *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then*

$$\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r} f(x-y)dy = f(x), \qquad \mathcal{L}^n\text{-a.e.}$$

Proof. We already know that the result is true if $f \in L^1(\mathbb{R}^n)$. In the case $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we consider $\widetilde{f} := f\chi_{B_r}$, which belongs to $L^1(\mathbb{R}^n)$. Hence we get the result for almost every point $x \in B_r$ with $r > 0$. So by this same argument applied to every B_n with $n \in \mathbb{N}$ and making $n \to \infty$ we get the result for almost every $x \in \mathbb{R}^n$. □

As a consequence of this last theorem we are able to prove that the identification we did in (3.1.3), between an integrable function and a linear functional of $\mathcal{S}(\mathbb{R}^n)$, is injective.

Theorem 3.1.10. *If $\phi \in L^1(\mathbb{R}^n)$ is such that $T_\phi = 0$, then $\phi = 0$ $\mathcal{L}^n$-a.e.*

Proof. We know that for every $f \in \mathcal{S}(\mathbb{R}^n)$ we have $0 = T_\phi f = \int_{\mathbb{R}^n} \phi f$. Now, since the class of smooth functions with compact support (test functions) is dense in $L^1(\mathbb{R}^n)$ and it is contained in $\mathcal{S}(\mathbb{R}^n)$, we are able to consider a sequence $(f_n)_n$ of test functions that converges in $L^1(\mathbb{R}^n)$ to χ_{B_r}. In fact, we may assume, by restricting ourselves to a subsequence, that it converges pointwise $\mathcal{L}^n$-a.e. Therefore, for every $a \in \mathbb{R}^n$, the dominated convergence theorem yields

$$\int_{B_r} \phi(x-a)dx = \int_{\mathbb{R}^n} \phi(x-a)\chi_{B_r} dx = \lim_{n \to \infty} \int_{\mathbb{R}^n} \phi(x-a) f_n(x) dx = 0.$$

So multiplying both sides by $|B_r|^{-1}$, taking the limit as $r \to 0^+$ and applying Theorem 3.1.9, we obtain the result. □

Finally, to end this section on the Hardy-Littlewood maximal function we remark that, although we know that M is of weak type $(1,1)$, it is not of strong type $(1,1)$. In fact the following result holds:

Proposition 3.1.11. *If $f \in L^1(\mathbb{R}^n)$ and f is not identically 0, then $Mf \notin L^1(\mathbb{R}^n)$.*

Proof. There exists $R > 0$ such that $\int_{B_R} |f| \geq \varepsilon > 0$. Now, if $|x| > R$ we have $B_R \subset B(x, 2|x|)$ and so

$$Mf(x) \geq \frac{1}{(2|x|)^2} \int_{B_R} |f(x)|dx \geq \frac{\varepsilon}{2^n |x|^n}.$$

Therefore, if we were to compute the $L^1(\mathbb{R}^n)$ norm we would get (using polar coordinates)

$$\int_{\mathbb{R}^n} |Mf(x)|dx \geq \frac{\varepsilon}{2^n} \int_{\mathbb{R}^n} \frac{dx}{|x|^n} \simeq \frac{\varepsilon}{2^n} \int_0^\infty \frac{dr}{r} = \infty.$$

□

3.1.2 The dyadic maximal function

Our goal for this last point about maximal functions is to introduce another relevant example. Its interest not only relies in the fact that its definition, similar to the one of the Hardy-Littlewood maximal operator, gives rise to a result analogous to the Lebesgue differentiation theorem; but also relies in the tools needed to prove its boundedness. Indeed, we will introduce the concept of *Calderón-Zygmund decomposition* that will be useful when studying the Hilbert operator and, more generally, Calderón-Zygmund operators.

First, we begin fixing some notation (that was already used, inadvertently, in the proof of Lemma 2.4.8):

- In $\mathbb{R}^n$ we will understand the *unit cube*, open on the right, as the set $[0,1)^n$; and we let $\mathcal{Q}_0$ be the collection of unit cubes congruent to $[0,1)^n$ and such that its vertices lie on the lattice $\mathbb{Z}^n$.
- If we dilate $\mathcal{Q}_0$ by a factor of 2^{-k}, $k \in \mathbb{Z}$, we obtain the family $\mathcal{Q}_k$, formed by cubes, open on the right, with vertices lying at the lattice $(2^{-k}\mathbb{Z})^n$ and volume 2^{-nk}.

Definition 3.1.7 (Dyadic cube)**.** We call *dyadic cube* any element of any family $\mathcal{Q}_k$, $k \in \mathbb{Z}$.

The collection of dyadic cubes satisfies the following basic properties:

1. Let $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Then, there is a unique dyadic cube in $\mathcal{Q}_k$ that contains x.
2. Two dyadic cubes are either disjoint or one is contained in the other.
3. A dyadic cube $Q \in \mathcal{Q}_k$ is contained in a unique dyadic cube of $\mathcal{Q}_j$, $j < k$. Moreover, Q contains 2^n dyadic cubes of $\mathcal{Q}_{k+1}$.

Based on these collection of dyadic cubes it is natural to define the following approximation of f in terms of every $\mathcal{Q}_k$, since $\mathcal{Q}_k$ defines a partition in $\mathbb{R}^n$.

Definition 3.1.8 (Conditional expectation of f with respect to $\mathcal{Q}_k$)**.** Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define its *conditional expectation with respect to* $\mathcal{Q}_k$ as

$$E_k f(x) := \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q(x).$$

It is clear that if $f \in L^1(\mathbb{R}^n)$, then $E_k f$ satisfies

$$\int_{\mathbb{R}^n} E_k f = \int_{\mathbb{R}^n} f.$$

Also observe that as $k \to -\infty$ the size of dyadic cubes of $\mathcal{Q}_k$ gets larger. Therefore, since $f \in L^1(\mathbb{R}^n)$ we have, for almost every $x \in \mathbb{R}^n$,

$$|E_k f(x)| \le \|f\|_1 \sum_{Q \in \mathcal{Q}_k} 2^{nk} \chi_Q(x) = \|f\|_1 2^{nk} \xrightarrow[k \to -\infty]{} 0. \tag{3.1.6}$$

Definition 3.1.9 (Dyadic maximal function)**.** The *dyadic maximal function* M_d is defined for each $f \in L^1(\mathbb{R}^n)$ as

$$M_d f(x) := \sup_{k \in \mathbb{Z}} |E_k f(x)|.$$

Again, as with the Hardy-Littlewood maximal function, we aim to prove an analogous version of Theorem 3.1.6. Nevertheless, we will first obtain a result – which will be important by itself and useful in the sequel – that will imply a certain weak boundedness of M_d.

Intuitively, given a non-negative $f \in L^1(\mathbb{R}^n)$, we know that its integrability must have an effect on its growth as $|x| \to \infty$. So the idea is to construct a sequence of measurable subsets of $\mathbb{R}^n$ (which in the end will be cubes), depending on f, so that f is essentially bounded outside of them and so that the measure of the union of all the subsets forming the sequence is finite. Moreover, as f is also finite $\mathcal{L}^n$-a.e., we might ask for its expected value in every subset of the sequence to be finite. The existence of such sequence is given by the next theorem.

Theorem 3.1.12. (Calderón-Zygmund decomposition, [7, Theorem 2.11]). *Given a non-negative $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, there exists a sequence $(Q_j)_j$ of disjoint dyadic cubes such that:*

1. *$f(x) \leq \lambda$ for almost every $x \notin \bigcup_j Q_j$.*
2. *$|\bigcup_j Q_j| \leq \frac{1}{\lambda}\|f\|_1$.*
3. *For every Q_j we have: $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f \leq 2^n\lambda$.*

Proof. Let us begin by defining for each $k \in \mathbb{Z}$

$$\Omega_k := \{x \in \mathbb{R}^n \,:\, E_k f(x) > \lambda \text{ and } E_j f(x) \leq \lambda \text{ if } j < k\}.$$

That is, $x \in \Omega_k$ if $E_k f(x)$ is the first conditional expectation of f bigger than λ. Notice that this definition makes sense because of (3.1.6), which ensures the existence of this first value of k. Observe that if $Q_{k,x}$ is the unique dyadic cube of $\mathcal{Q}_k$ containing x, then for every $y \in Q_{x,k}$ we have also $y \in \Omega_k$, by definition of $E_k f$. So Ω_k can be expressed as the disjoint union of dyadic cubes of $\mathcal{Q}_k$. Moreover, by definition of Ω_k, if $j < k$, then $\Omega_j \cap \Omega_k = \varnothing$. Therefore, the family $\{\Omega_k\}_{k\in\mathbb{Z}}$ is a collection of disjoint dyadic cubes of different families $\mathcal{Q}_k$, $k \in \mathbb{Z}$. We define the sequence $(Q_j)_j$ as these precise disjoint dyadic cubes.

Let us prove that the previous sequence satisfies *2*. We claim that the following identity holds:

$$\{x \in \mathbb{R}^n \,:\, M_d f(x) > \lambda\} = \bigsqcup_{k\in\mathbb{Z}} \Omega_k = \bigsqcup_j Q_j, \tag{3.1.7}$$

(notice that the second equality follows just by construction) where the notation $\bigsqcup$ stands for *disjoint union*. Let us prove the first. The inclusion $\supseteq$ is clear, so we focus on the other. Let $x \in \mathbb{R}^n$ be such that $M_d f(x) > \lambda$. By definition, there exist $\varepsilon > 0$ and $k \in \mathbb{Z}$ satisfying $E_k f(x) \geq \lambda + \varepsilon$ (indeed, we can argue this by contradiction) and so $x \in \bigsqcup_{j\leq k} \Omega_j$ and we are done. So we obtain the following estimate

$$\begin{aligned}|\{x \in \mathbb{R}^n \,:\, M_d f(x) > \lambda\}| &= \sum_{k\in\mathbb{Z}} |\Omega_k| \leq \sum_{k\in\mathbb{Z}} \frac{1}{\lambda}\int_{\Omega_k} E_k f(x)dx \\ &= \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k} \frac{|\Omega_k \cap Q|}{|Q|}\int_Q f(y)dy \leq \frac{1}{\lambda}\sum_{Q\in\mathcal{Q}_k}\int_Q f(y)dy \leq \frac{1}{\lambda}\|f\|_1,\end{aligned}$$

which proves *2* as well as the fact that M_d is of weak type $(1,1)$.

Now, concerning *1*, if $x \notin \bigcup_j Q_j$, then $E_k f(x) \leq \lambda$ for all $k \in \mathbb{Z}$. If we proved that $\lim_{k\to\infty} E_k f(x) = f(x)$ for almost every point, that would imply the result by a limitting argument. Let us see first that this is the case if f is continuous. Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose K_x a compact neighbourhood of x with diameter small enough so that by uniform continuity $|f(x_1)-f(x_2)| < \varepsilon$, for any $x_1, x_2 \in K_x$. Pick also k large enough so that $Q_{k,x} \in \mathcal{Q}_k$ (the unique dyadic cube of diameter $\sqrt{n}2^{-k}$ containing x) satisfies $Q_{k,x} \subset K_x$. Then,

$$|E_k f(x) - f(x)| = \left| \frac{1}{|Q_{k,x}|} \int_{Q_{k,x}} f(y)dy - f(x) \right| \leq \frac{1}{|Q_{k,x}|} \int_{Q_{k,x}} |f(y) - f(x)|dy < \varepsilon, \quad (3.1.8)$$

obtaining the result, since x is arbitrary. If now $f \in L^1(\mathbb{R}^n)$, use the density of continuous functions and that M_d satisfies a weak type $(1,1)$ inequality together with Theorem 3.1.2.

Finally, regarding *3*, the first inequality is clear given the definition of the cubes Q_j. On the other hand, if $\widetilde{Q}_j$ is the dyadic cube containing Q_j with sides twice as long, by the definitions of Ω_k and $E_{k-1}f$, we get the desired estimate:

$$\frac{1}{|Q_j|} \int_{Q_j} f(x)dx \leq \frac{|\widetilde{Q}_j|}{|Q_j|} \frac{1}{|\widetilde{Q}_j|} \int_{\widetilde{Q}_j} f(x)dx \leq 2^n \lambda$$

□

Corollary 3.1.13. *The dyadic maximal operator M_d is of weak type $(1,1)$ and of strong type (p,p), $1 < p \leq \infty$. Also, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $\lim_{k\to\infty} E_k f(x) = f(x)$ for almost every point.*

Proof. We have already seen in the proof of the previous theorem that M_d is of weak type $(1,1)$. Notice that, although we have only checked the result for non-negative functions, since every real integrable function f can be decomposed as $f = f^+ - f^-$, the result still holds (and if f took complex values, we may decompose $f = \mathfrak{Re}(f) + i\mathfrak{Im}(f)$ and apply the same argument to its real and imaginary parts). Moreover, if $f \in L^\infty(\mathbb{R}^n)$, then

$$\|M_d f\|_\infty = \sup_{x\in\mathbb{R}^n} |M_d f(x)| = \sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left| \sum_{Q\in\mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f(y)dy \right) \chi_Q(x) \right| \leq \|f\|_\infty$$

and so M_d is of weak/strong type (∞,∞), and by the Marcinkiewicz interpolation theorem 3.1.3 we get that M_d if of strong type (p,p) for $1 < p < \infty$.

Now, let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Since for every $Q \in \mathcal{Q}_0$ we have $f\chi_Q \in L^1(\mathbb{R}^n)$, by the argument of (3.1.8) we deduce $\lim_{k\to\infty} E_k f\chi_Q(x) = f\chi_Q(x)$ for almost every point of $\mathbb{R}^n$. Equivalently, $\lim_{k\to\infty} E_k f(x) = f(x)$ for almost every point in Q. But since Q is arbitrary and $\mathcal{Q}_0$ is countable, we obtain the result for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. □

To end this section about maximal functions we will present a technical lemma that will be useful in the sequel, proving properties of the maximal operator of a family of Calderón-Zygmund operators.

Lemma 3.1.14. *If $f \in L^1(\mathbb{R}^n)$ is non-negative and $\lambda > 0$, then, if $C := \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)$, we have*

$$|\{x \in \mathbb{R}^n : Mf(x) > C8^n\lambda\}| \leq 2^n |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|.$$

Proof. Begin by considering $(Q_j)_j$ the Calderón-Zygmund decomposition associated to f and λ. We know by (3.1.7) that $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \bigsqcup_j Q_j$. If we define $2Q_j$ to be the cube centered at same point as Q_j with double side length, since the cubes are disjoint, $|\bigsqcup_j 2Q_j| = 2^n|\bigsqcup_j Q_j| = 2^n|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|$. So it is enough to check $\{x \in \mathbb{R}^n : Mf(x) > C8^n\lambda\} \subseteq \bigsqcup_j 2Q_j$ to obtain the desired estimate.

Let us fix then $x \notin \bigsqcup_j 2Q_j$ and let Q be any cube centered at x. If $\ell(Q)$ is the side length of Q, we choose $k \in \mathbb{Z}$ so that $2^{k-1} \leq \ell(Q) < 2^k$. Then, Q could intersect, at most, 2^n dyadic cubes of the collection $\mathcal{Q}_k$. Let $R_1, \ldots, R_m$ be those precise cubes, for $m \leq 2^n$. Notice that for every $s = 1, \ldots, m$, $R_s \not\subseteq Q_j$ for any j, because if not, since $Q \subseteq 2R_s$, we would have $x \in Q \subseteq 2R_s \subset \bigsqcup_j 2Q_j$, which contradicts the initial choice of x. Hence, for every R_s there exists a certain $x_s \in R_s$ such that $\sup_{k\in\mathbb{Z}} |E_k f(x_s)| \leq \lambda$. This means, by the definition of M_d, that the mean of f in each R_s is, at most, λ. Therefore

$$\frac{1}{|Q|}\int_Q f = \frac{1}{|Q|}\sum_{s=1}^{m}\int_{Q\cap R_s} f \leq \frac{2^{kn}}{|Q|}\sum_{s=1}^{m}\frac{1}{|R_i|}\int_{R_i} f \leq \frac{2^{nk}}{2^{n(k-1)}}m\lambda \leq 4^n\lambda.$$

On the other hand, we know that if B is the biggest ball centered at x and contained in Q, then, as f is non-negative

$$\frac{1}{|B|}\int_B f \leq \frac{|Q|}{|B|}\frac{1}{|Q|}\int_Q f = \frac{2^n\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}}\int_Q f \quad\Rightarrow\quad \frac{1}{|B|}\int_B f \leq \left(\frac{8}{\sqrt{\pi}}\right)^n\Gamma\left(\frac{n}{2}+1\right)\lambda,$$

and since Q (and hence B) is arbitrary, we are done. □

3.2 The Hilbert transform

Now we will turn our attention to a specific operator that will motivate the forthcoming study of the Calderón-Zygmund theory. Such operator is the *Hilbert transform* and, as in the previous section, our main goal will be to prove its $L^p(\mathbb{R})$-boundedness (in Chapter 4 we will treat the $\mathbb{R}^n$ situation).

The key result of the sequel will be the Kolmogorov-Riesz theorem (Theorem 3.2.4) and even more importantly its proof, that will justify, eventually, the definition of a Calderón-Zygmund operator. Indeed, we will establish its definition regarding, essentially, at the properties of the Hilbert transform that imply its L^p-boundedness.

3.2.1 Motivation and definition

Informally, the typical example when one is presented with the concept of *singular integral* is

$$\lim_{a,b\to 0}\int_{-b}^{a}\frac{1}{x}dx, \qquad a,b>0,$$

which is not well-defined if we do not specify how one approaches 0 both from the right and the left. It is well-known that one convention to assign a value to integrals of the last kind is to tend to 0 from both sides at the same rate, defining the so called *Cauchy's principal value* of the singular integral. In the previous example we would have

$$\text{p.v.} \int_{-a}^{a} \frac{1}{x} dx = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < a} \frac{1}{x} dx = 0.$$

Notice that in the last computation, if the numerator of the integrand would have been any even (locally integrable) function defined on $(-\delta, \delta) \setminus \{0\}$ for some $\delta > 0$; we would have obtained the same result. This motivates the study of a possible generalization of this particular principal value and a natural way to do it is via tempered distributions.

Definition 3.2.1 (Principal value of $\frac{1}{x}$)**.** We define the tempered distribution p.v.$\frac{1}{x}$ called the *principal value of* $\frac{1}{x}$ as

$$\text{p.v.}\frac{1}{x}(\phi) := \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\phi(x)}{x} dx, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

Proposition 3.2.1. *The principal value of* $\frac{1}{x}$ *is, indeed, a tempered distribution.*

Proof. Let us fix $\phi \in \mathcal{S}(\mathbb{R})$ and notice that

$$\left|\text{p.v.}\frac{1}{x}(\phi)\right| \leq \lim_{\varepsilon \to 0} \left| \int_{\varepsilon < |x| < 1} \frac{\phi(x) - \phi(0)}{x} dx \right| + \left| \int_{|x| > 1} \frac{\phi(x)}{x} dx \right| \lesssim \|\phi'\|_\infty + \|x\phi\|_\infty < \infty,$$

and hence p.v.$\frac{1}{x}$ is well-defined. To check that it is also a continuous functional, since it is clearly linear, it is enough to prove that if we have $(\phi_n)_n \subset \mathcal{S}(\mathbb{R})$ tending to 0 in $\mathcal{S}(\mathbb{R})$, then p.v.$\frac{1}{x}(\phi)$ tends to 0 in $\mathbb{C}$ (recall that $(\phi_n)_n \to 0$ in $\mathcal{S}(\mathbb{R})$ means that the sequence tends to 0 with respect to all the seminorms that are defined, for any $\alpha \geq 0$ and $\beta \in \mathbb{N} \cup \{0\}$, as the quantity $\sup_{x \in \mathbb{R}} |x^\alpha \phi^{(\beta)}(x)|$). But observe that this is immediate because of the estimate we have found at the beginning of the proof (which depends on two specific seminorms). $\square$

Observe that the previous proof shows that the principle value of $\frac{1}{x}$ can be defined also for any even locally integrable function or any Lipschitz function f for which there exists some $a > 0$ so that $x^{-1} f \in L^1(a, \infty)$. An example of the first is the cardinal sine, i.e.

$$\operatorname{sinc}(x) = \begin{cases} (\pi x)^{-1} \sin \pi x & x \neq 0, \\ 1 & x = 0, \end{cases}$$

with principal value 0, since it is an even function.

Thinking of the principal value of $\frac{1}{x}$ as a tempered distribution, it is reasonable to be interested in computing its Fourier transform. In order to do it, we prove first the following lemma:

Lemma 3.2.2. *Let* $Q_t := \frac{1}{\pi} \frac{x}{t^2 + x^2}$. *Then, the following equality between tempered distributions holds*

$$\lim_{t \to 0} Q_t = \frac{1}{\pi} p.v. \frac{1}{x}.$$

Proof. Let us express first the principal value of $\frac{1}{x}$ as the limit of another tempered distribution: $\psi_t(x) := \frac{1}{x} \chi_{\{|x| > t\}}$. Indeed, if $\phi \in \mathcal{S}(\mathbb{R})$ then

$$\left| \left(\text{p.v.}\frac{1}{x} - \psi_t \right)(\phi) \right| = \left| \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < t} \frac{\phi(x) - \phi(0)}{x} dx \right| \leq 2t \|\phi'\|_\infty \xrightarrow[t \to 0]{} 0.$$

Thus, it is enough to prove that $\lim_{t\to 0}(Q_t - \frac{\psi_t}{\pi})(\phi) = 0,\ \forall \phi \in \mathcal{S}(\mathbb{R})$. So we fix ϕ and compute

$$\begin{aligned}(\pi Q_t - \psi_t)(\phi) &= \int_{\mathbb{R}} \frac{x\phi(x)}{t^2+x^2}dx - \int_{|x|>t} \frac{\phi(x)}{x}dx \\ &= \int_{|x|<t} \frac{x\phi(x)}{t^2+x^2}dx - \int_{|x|>t} \left(\frac{x}{t^2+x^2} - \frac{1}{x}\right)\phi(x)dx \\ &= \int_{|x|<1} \frac{x\phi(tx)}{1+x^2}dx - \int_{|x|>1} \frac{\phi(tx)}{x(1+x^2)}dx.\end{aligned}$$

It is clear that we can apply the dominated convergence theorem in both integrals when considering the limit $t \to 0$, obtaining two integrals of odd functions in symmetric domains. Hence the limit is 0. $\square$

As a direct consequence of this lemma, we obtain, for any $f \in \mathcal{S}(\mathbb{R})$

$$\left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right) * f\right](x) = \left[\left(\lim_{t\to 0} Q_t\right) * f\right](x). \tag{3.2.1}$$

Now we wish to take the Fourier transform at both sides of the previous equality and apply its multiplicative property with respect to the convolution. Let us check that we are able to do it in the case where one of the terms is the tempered distribution p.v.$\frac{1}{x}$. Consider the explicit expression of the Fourier transform

$$\left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right) * f\right]^{\wedge}(\xi) = \frac{1}{\pi}\int_{\mathbb{R}} \lim_{\varepsilon\to 0}\left(\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy\right)e^{-2\pi i\xi x}dx. \tag{3.2.2}$$

Let us prove that we can take the limit outside the outer integral and we can interchange the order of integration. With respect to the first matter, notice that the pointwise limit is well-defined and it is $\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y} * f\right)(x)$. On the other hand notice that

$$\begin{aligned}\left|\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy\right| &\le \left|\int_{\varepsilon<|y|<1} \frac{f(x-y)}{y}dy\right| + \left|\int_{|y|>1} \frac{f(x-y)}{y}dy\right| \\ &\le \int_0^1 |f'(x-t_+)|dy + \int_{-1}^0 |f'(x+t_-)|dy + \int_{|y|>1} \frac{|f(x-y)|}{|y|}dy,\end{aligned}$$

where $t_+, t_- \in (0,1)$ and depend on y. Integrating with respect to x and applying Tonelli's theorem, we get that the first and second integrals are bounded by $\|f'\|_1$. Regarding the third, since $|y|>1$ we can estimate it as follows

$$\int_{\mathbb{R}}\int_{|y|>1} |f(x-y)|dydx \le \int_{\mathbb{R}\setminus\mathbb{D}} |f(x-y)|dydx = \sqrt{2}\int_0^{2\pi}\int_1^{\infty} |f(re^{i\theta})|rdrd\theta \le 2\pi\sqrt{2}\|xf\|_1$$

Therefore, we can apply the dominated convergence theorem and put the limit outside. Notice also that the previous computations also prove that we can apply Fubini's theorem in (3.2.2), so we finally obtain:

$$\left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right)*f\right]^\wedge(\xi)=\frac{1}{\pi}\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\left(\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy\right)e^{-2\pi i\xi x}dx$$
$$=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{1}{y}e^{-2\pi i\xi y}\left(\int_{\mathbb{R}}f(x-y)e^{-2\pi i\xi(x-y)}dx\right)dy$$
$$=\left(\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{1}{y}e^{-2\pi i\xi y}dy\right)\widehat{f}(\xi).$$

So indeed the multiplicative property holds. Hence, taking the Fourier transform at both sides of (3.2.1), recalling that it is a continuous operator from $\mathcal{S}'(\mathbb{R})$ to itself, and using that $\widehat{Q}_t(\xi)=-i\cdot\text{sgn}(\xi)e^{-2\pi t|\xi|}$ (this can be proved by a direct computation) we obtain

$$\left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right)*f\right]^\wedge(\xi)=\lim_{t\to 0}\widehat{Q}_t(\xi)\widehat{f}(\xi)=-i\cdot\text{sgn}(\xi)\widehat{f}(\xi),$$

so we conclude

$$\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right)^\wedge(\xi)=-i\cdot\text{sgn}(\xi).$$

Now we are ready to state the definition of the Hilbert transform, that, in fact, has already been presented.

Definition 3.2.2 (Hilbert transform)**.** Given $f\in\mathcal{S}(\mathbb{R})$, we define its *Hilbert transform* as

$$Hf(x)=\left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right)*f\right](x).$$

It can be equivalently defined through its Fourier transform, that is, H is the only operator satisfying

$$\widehat{Hf}(\xi)=-i\cdot\text{sgn}(\xi)\widehat{f}(\xi). \tag{3.2.3}$$

Let us make a series of observations regarding H:

1. Recall that p.v.$\frac{1}{x}$ is a tempered distribution that acts on functions ϕ of the Schwartz class as the integral operator: p.v.$\frac{1}{x}(\phi)=\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx$. So notice that the convolution of this operator against ϕ is just its action to a translate of ϕ.

2. Relation (3.2.3) yields $\widehat{Hf}\in L^2(\mathbb{R})$, therefore $\|Hf\|_2=\|\widehat{Hf}\|_2=\|\widehat{f}\|_2=\|f\|_2$. This determines, in particular, the Hilbert transform of a function $f\in L^2(\mathbb{R})$. Indeed, if $(f_n)_n\subset\mathcal{S}(\mathbb{R})$ is a sequence approximating f in $L^2(\mathbb{R})$, then

$$\|Hf_n-Hf_m\|_2=\|f_n-f_m\|_2\xrightarrow[n,m\to\infty]{}0.$$

So $(Hf_n)_n$ is a Cauchy sequence in $L^2(\mathbb{R})$ and hence convergent to a certain limit, defining the Hilbert transform of f (strictly, we would still have to check that this definition does not depend on the approximating sequence).

3. The previous point motivates considering the iteration $H(Hf)$, for any $f \in \mathcal{S}(\mathbb{R})$. This makes sense since $Hf \in L^2(\mathbb{R})$. By doing it, notice that if we write the explicit expression of the Fourier transform of $H(Hf)$ we obtain

$$[H(Hf)]^\wedge(\xi) = \left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right) * \left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right) * f\right]\right]^\wedge(\xi) = [-i \cdot \operatorname{sgn}(\xi)]^2 \widehat{f}(\xi) = -\widehat{f}(\xi),$$

and so we conclude that $H(Hf) = -f$.

4. The following identity holds for any $f, g \in \mathcal{S}(\mathbb{R})$ (or $L^2(\mathbb{R})$)

$$\int_{\mathbb{R}} Hf(x)g(x)dx = -\int_{\mathbb{R}} f(x)Hg(x)dx. \tag{3.2.4}$$

This is just a consequence of the previous point, the polarization identity in $L^2(\mathbb{R})$ and the fact that $\|Hf\|_2 = \|f\|_2$.

3.2.2 The Kolmogorov-Riesz theorem

The main goal of this section is to extend H as a bounded linear operator on $L^p(\mathbb{R})$ spaces, $1 < p < \infty$ (Riesz's theorem). On the other hand, for $p = 1$, similarly as with the Hardy-Littlewood maximal function, we will prove that H satisfies a weak type $(1,1)$ inequality (Kolmogorov's theorem). Nevertheless, let us fix first some notation before moving on to the result itself.

Definition 3.2.3 (Good and bad parts of f)**.** Let $f \in \mathcal{S}(\mathbb{R})$ and $\lambda > 0$. Let $(I_j)_j$ be a sequence of disjoint dyadic intervals associated to a Calderón-Zygmund decomposition of f in $\mathbb{R}$ with respect to λ. We define the *good part* of f as

$$g_f(x) = \begin{cases} f(x) & x \notin \bigcup_j I_j, \\ \frac{1}{|I_j|}\int_{I_j} f(y)dy & x \in I_j. \end{cases}$$

On the other hand, the *bad part* of f is

$$b_f(x) = \sum_j b_{f,j}, \qquad \text{where} \qquad b_{f,j}(x) = \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)dy\right)\chi_{I_j}(x).$$

Observe that by construction of the Calderón-Zygmund decomposition we get $g_f(x) \le 2\lambda$. Also, notice that $b_{f,j}$ is supported on I_j and so the sum that defines b_f makes sense pointwise. We will also need the following basic properties regarding g_f and b_f:

Lemma 3.2.3. *The good and bad parts of f satisfy:*

1. *For each j, $b_{f,j}$ has null integral.*
2. *The sum $\sum_j b_{f,j}$ converges to b_f in $L^2(\mathbb{R})$.*
3. *$\|g_f\|_1 \le \|f\|_1$ and, in fact, $\int_{\mathbb{R}} g_f(x)dx = \int_{\mathbb{R}} f(x)dx$.*
4. *It makes sense to consider Hg_f and Hb_f.*

Proof. The first assertion is trivial, so let us begin by focusing on the second. Recall that, by definition, for every interval I_j there exists $k \in \mathbb{Z}$ so that if $x \in I_j$, then $E_k f(x) > \lambda$ and $E_s f(x) \leq \lambda$ if $s \leq k$. Hence $|I_j| = |I_j(k)| = 2^{-k}$. This, combined with the conditions $|\bigcup_j I_j| \leq \lambda^{-1}\|f\|_1$ and $I_j \cap I_j = \varnothing$ for $i \neq j$, implies, necessarily, $\lim_{j\to\infty} |I_j| = 0$. Now, using the almost sure convergence given in Corollary 3.1.13

$$\lim_{j\to\infty}\left(f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)dy\right)\chi_{I_j}(x) = 0, \quad \text{for } \mathcal{L}\text{-a.e. } x \in \mathbb{R}.$$

Hence, for all $\varepsilon > 0$ there exists j_0 so that if $j \geq j_0$ we have

$$\left|\left\{x \in I_j \,:\, \left|f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)dy\right| \geq \frac{\varepsilon}{\sqrt{2^j |I_j|}}\right\}\right| = 0.$$

Then, we can conclude

$$\begin{aligned}\left\|b_f - \sum_{j=0}^{j_0} b_{f,j}\right\|_2 &= \left(\int_{\mathbb{R}}\left|\sum_{j=j_0+1}^{\infty}\left(f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)dy\right)\chi_{I_j}(x)\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=j_0+1}^{\infty}\int_{I_j}\left|f(x) - \frac{1}{|I_j|}\int_{I_j} f(y)dy\right|^2 dx\right)^{\frac{1}{2}} < \varepsilon\left(\sum_{j=j_0+1}^{\infty}\frac{1}{2^j}\right)^{\frac{1}{2}} < \varepsilon.\end{aligned}$$

To prove *3* we compute

$$\begin{aligned}\int_{\mathbb{R}} |g_f(x)|dx &= \sum_j \int_{I_j} |g_f(x)|dx + \int_{\mathbb{R}\cap(\bigcup_j I_j)^c} |g_f(x)|dx \\ &\leq \sum_j \int_{I_j} |f(x)|dx + \int_{\mathbb{R}\cap(\bigcup_j I_j)^c} |f(x)|dx = \int_{\mathbb{R}} |f(x)|dx.\end{aligned}$$

The equality between integrals follows also from the same argument but without taking the absolute value of the integrand.

In order to prove that Hg_f makes sense we will prove that $g_f \in L^2(\mathbb{R})$. If we do this, since $b_f = f - g_f$, Hb_f will be also well-defined. But this follows from Cauchy-Schwarz's inequality:

$$\begin{aligned}\int_{\mathbb{R}} |g_f(x)|^2 dx &= \sum_j \frac{1}{|I_j|}\left(\int_{I_j} |f(x)|dx\right)^2 + \int_{\mathbb{R}\cap(\bigcup_j I_j)^c} |f(x)|^2 dx \\ &\leq \sum_j \int_{I_j} |f(x)|^2 dx + \int_{\mathbb{R}\cap(\bigcup_j I_j)^c} |f(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx < \infty.\end{aligned}$$

□

Theorem 3.2.4. (Kolmogorov-Riesz, [7, Theorem 3.2]). *Let $f \in \mathcal{S}(\mathbb{R})$. The following hold:*

1. *H satisfies the weak type $(1,1)$ inequality: $|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1$.*
2. *H satisfies the strong type (p,p) inequality for $1 < p < \infty$: $\|Hf\|_p \leq C_p\|f\|_p$.*

Proof. We will start by proving *1.* Let us fix $\lambda > 0$, $f \in \mathcal{S}(\mathbb{R})$ non-negative and $(I_j)_j$ a sequence of intervals in $\mathbb{R}$ associated to a Calderón-Zygmund decomposition of f with respect to λ. By the last statement in Lemma 3.2.3 we have $Hf = Hg_f + Hb_f$ and therefore

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \left|\left\{x \in \mathbb{R} : |Hg_f(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R} : |Hb_f(x)| > \frac{\lambda}{2}\right\}\right|. \tag{3.2.5}$$

Regarding the first term, since $\|Hf\|_2 = \|f\|_2$ and $0 \leq g_f(x) \leq 2\lambda$,

$$\begin{aligned}\left|\left\{x \in \mathbb{R} : |Hg_f(x)| > \frac{\lambda}{2}\right\}\right| &\leq \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |Hg_f(x)|^2 dx = \frac{4}{\lambda^2}\int_{\mathbb{R}} g_f(x)^2 dx \leq \frac{8}{\lambda}\int_{\mathbb{R}} g_f(x)dx \\ &= \frac{8}{\lambda}\int_{\mathbb{R}} f(x)dx = \frac{8}{\lambda}\|f\|_1,\end{aligned}$$

where we have used the third statement of Lemma 3.2.3 in the second equality. On the other hand, in order to estimate the second term in (3.2.5), let $2I_j$ be the interval with the same center as I_j and with double side length. Observe that $|\bigcup_j 2I_j| \leq 2|\bigcup_j I_j| \leq \frac{2}{\lambda}\|f\|_1$, by construction of the Calderón-Zygmund decomposition. So

$$\begin{aligned}\left|\left\{x \in \mathbb{R} : |Hb_f(x)| > \frac{\lambda}{2}\right\}\right| &\leq \left|\bigcup_j 2I_j\right| + \left|\left\{x \notin \bigcup_j 2I_j : |Hb_f(x)| > \frac{\lambda}{2}\right\}\right| \\ &\leq \frac{2}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}\cap(\bigcup_j 2I_j)^c} |Hb_f(x)|dx.\end{aligned}$$

Observe that $|Hb_f(x)| \leq \sum_k |Hb_{f,k}(x)|$ for $\mathcal{L}$-a.e. x. This is immediate if the sum is finite. If it is not finite, by the second point of Lemma 3.2.3 and the continuity H in $L^2(\mathbb{R})$, the estimate still holds. So we can bound the last integral expression as follows

$$\int_{\mathbb{R}\cap(\bigcup_j 2I_j)^c} |Hb_f(x)|dx \leq \sum_k \int_{\mathbb{R}\cap(\bigcup_j 2I_j)^c} |Hb_{f,k}(x)|dx \leq \sum_k \int_{\mathbb{R}\setminus 2I_k} |Hb_{f,k}(x)|dx,$$

Notice that $Hb_{f,k}(x)$ is well-defined for $x \in \mathbb{R}\setminus 2I_k$. Indeed, since $|x-y| \geq |I_k|/2$ for every $y \in I_k$, we have

$$|Hb_{f,k}(x)| = \lim_{\varepsilon\to 0}\left|\int_{\{|x|>\varepsilon\}\cap I_k}\left(f(y) - \frac{1}{|I_k|}\int_{I_k} f(u)du\right)\frac{1}{x-y}dy\right| \leq \frac{4}{|I_k|}\int_{I_k}|f(y)|dy < \infty.$$

Also, if we denote by c_k the center of I_k, since $b_{f,k}$ has null integral, we get the following estimate

$$\begin{aligned}\int_{\mathbb{R}\setminus 2I_k} |Hb_{f,k}(x)|dx &= \int_{\mathbb{R}\setminus 2I_k}\left|\int_{I_k}\frac{b_{f,k}(y)}{x-y}dy\right|dx = \int_{\mathbb{R}\setminus 2I_k}\left|\int_{I_k} b_{f,k}(y)\left(\frac{1}{x-y} - \frac{1}{x-c_k}\right)dy\right|dx \\ &\leq \int_{I_k}|b_{f,k}(y)|\left(\int_{\mathbb{R}\setminus 2I_k}\frac{|y-c_k|}{|x-y||x-c_k|}dx\right)dy.\end{aligned}$$

Now, using that $|y-c_k| \leq |I_k|/2$ and $|x-y| > |x-c_k|/2$,

$$\int_{I_k} |b_{f,k}(y)| \left(\int_{\mathbb{R}\setminus 2I_k} \frac{|y-c_k|}{|x-y||x-c_k|} dx \right) dy \leq \int_{I_k} |b_{f,k}(y)| \left(\int_{\mathbb{R}\setminus 2I_k} \frac{|I_k|}{|x-c_k|^2} dx \right) dy$$
$$= \int_{I_k} |b_{f,k}(y)| \left(\int_{\mathbb{R}\setminus(-1,1)} \frac{1}{|u|^2} du \right) dy = 2 \int_{I_k} |b_{f,k}(y)| dy.$$

Therefore we conclude

$$\sum_k \int_{\mathbb{R}\setminus 2I_k} |Hb_{f,k}(x)| dx \leq 2 \sum_k \int_{I_k} |b_{f,k}(y)| dy \leq 4 \sum_k \int_{I_k} |f(y)| dy \leq 4\|f\|_1,$$

and combining these estimates we deduce the weak type $(1,1)$ inequality:

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \left(\frac{8}{\lambda} + \frac{2}{\lambda} + \frac{8}{\lambda} \right) \|f\|_1 = \frac{18}{\lambda} \|f\|_1, \tag{3.2.6}$$

which is valid for non-negative $f \in \mathcal{S}(\mathbb{R})$. But since any other function in the space can be decomposed in its positive and negative parts, we also get the result for a general $f \in \mathcal{S}(\mathbb{R})$ (and if f takes complex values we apply this same argument to its real and imaginary parts). So we have proved *1*.

In order to prove *2*, since we already know that H is of strong type $(2,2)$, combining *1* and the Marcinkiewicz interpolation theorem 3.1.3 we get the strong type (p,p) inequality for $1 < p < 2$, allowing to define the Hilbert transform in the spaces $L^p(\mathbb{R})$, $1 < p < 2$ (we will give the details on how to do it after the proof). We also observe that relation (3.2.4) still holds for $f \in \mathcal{S}(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $1 < p < 2$ (this follows from the density of $\mathcal{S}(\mathbb{R})$ in $L^p(\mathbb{R})$, the already proved continuity of H in this space, and the dominated convergence theorem, that we can apply taking a subsequence of functions in $\mathcal{S}(\mathbb{R})$ converging $\mathcal{L}$-a.e. to g).

The case $p > 2$ will follow by duality. Indeed, observe that if p' is such that $p^{-1} + (p')^{-1} = 1$, that is, $p' = \frac{p}{p-1}$; then $p' \in (1,2)$. Hence, by strong type inequality for this last case, we get

$$\left| \int_{\mathbb{R}} Hf(x)g(x)dx \right| = \left| \int_{\mathbb{R}} f(x)Hg(x)dx \right| \leq C_{p'} \|f\|_p \|g\|_{p'}, \quad \forall g \in L^{p'}(\mathbb{R}),$$

which proves that Hf is a bounded linear functional in $L^{p'}(\mathbb{R})$ with norm at most $C_{p'}\|f\|_p$. But we know that there is an isometric isomorphism between $L^p(\mathbb{R})$ and $(L^{p'}(\mathbb{R}))'$, implying that $\|Hf\|_p \leq C_{p'}\|f\|_p$, and so we are done. □

In the previous proof we have claimed that the strong type (p,p) inequality of the Kolmogorov-Riesz theorem allows to extend the definition of the Hilbert operator to $L^p(\mathbb{R})$ spaces, $1 < p < \infty$. Indeed, if $f \in L^p(\mathbb{R})$ and $(f_n)_n \subset \mathcal{S}(\mathbb{R})$ is a sequence that approximates f in the $L^p(\mathbb{R})$ norm, the strong type (p,p) inequality implies that $(Hf_n)_n$ is also a Cauchy sequence in $L^p(\mathbb{R})$, so by completeness it is convergent. Hence, we define Hf to be the limit of the previous sequence.

Corollary 3.2.5. *The Hilbert transform is a bounded linear functional in $L^p(\mathbb{R}), 1 < p < \infty$.*

If $p = 1$, in the previous context the weak type $(1,1)$ inequality would yield

$$\lim_{m,n\to\infty} |\{x \in \mathbb{R} : |(Hf_n - Hf_m)(x)| > \varepsilon\}| = 0, \qquad \forall\varepsilon > 0,$$

which means that the sequence $(Hf_n)_n$ is Cauchy in measure, and so also convergent in measure (see [18, Theorem 5] for details). We will call the limit (measurable) function the Hilbert transform of f.

So in particular we may wonder what is the Hilbert transform of a simple integrable function that is not smooth, such as a characteristic function. If we compute Hf for $f = \chi_{[0,1]}$ we obtain

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{\varepsilon<y<1} \frac{dy}{x-y} = \frac{1}{\pi} \lim_{\varepsilon\to 0} \ln \frac{|x-\varepsilon|}{|x-1|} = \frac{1}{\pi} \ln \frac{|x|}{|x-1|}.$$

Observe that Hf is neither bounded nor integrable, which shows, at the same time, that H does not satisfy (in general) a strong type (p,p) inequality for $p = 1$ or $p = \infty$. In fact, we can easily prove the following result that characterizes the integrability of Hf.

Proposition 3.2.6. *If $f \in \mathcal{S}(\mathbb{R})$, then $Hf \in L^1(\mathbb{R}) \Leftrightarrow \int_{\mathbb{R}} f(x)dx = 0$.*

Proof. ($\Leftarrow$) The case f non-negative is trivial. The general case is deduced from the decomposition $f = f^+ - f^-$.
($\Rightarrow$) We know that $\widehat{Hf}(\xi) = -i \cdot \mathrm{sgn}(\xi)\widehat{f}(\xi)$. Since Hf is integrable, its Fourier transform is, in particular, (uniformly) continuous, which implies necessarily that $\widehat{f}(0) = 0$. But this means, by definition, $\int_{\mathbb{R}} f(x)dx = 0$, so we are done. □

Chapter 4

Singular integrals

One of the motivations to study singular integrals is the Hilbert transform, presented in the previous section. Recall that given $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, we defined H to be

$$Hf(x) = \left[\left(\frac{1}{\pi}\text{p.v.}\frac{1}{y}\right) * f\right](x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy.$$

Now, observe that we can rewrite this last expression as

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\operatorname{sgn}(y)}{|y|}f(x-y)dy.$$

This writing motivates an extension of the definition of H to $\mathbb{R}^n$ in the following sense. Observe that the function $\operatorname{sgn}(\,\cdot\,)$ is homogeneous of degree 0, i.e., for every $x \in \mathbb{R}$ and $\lambda > 0$ we have $\operatorname{sgn}(\lambda x) = \operatorname{sgn}(x)$. Then, we can think of it as a function defined on the unit sphere of $\mathbb{R}$, which is just the pair of points $\{-1, 1\}$. Then, if we aim to define an analogous version of the Hilbert transform in $\mathbb{R}^n$, we need to take into account its possible dependence on a certain function Ω with domain S^{n-1}, integrable and with null integral over S^{n-1} (so that its properties resemble the function $\operatorname{sgn}(\,\cdot\,)$ of the one-dimensional case).

4.1 Definition and basic properties

Definition 4.1.1 (Singular integral)**.** A *singular integral* in $\mathbb{R}^n$ will be an operator on $\mathcal{S}(\mathbb{R}^n)$ of the form

$$Tf(x) := \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y')}{|y|^n}f(x-y)dy, \tag{4.1.1}$$

where $y' := y/|y|$ and where Ω is a function defined on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, integrable and with null integral over S^{n-1}.

Now we make two important observations that justify the definition given in (4.1.1):

1. The reason to raise the denominator of the integrand to the n-th power is to obtain that the singular integral operator is well-defined for $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y')}{|y|^n}\phi(x-y)dy = \int_{|y|>1}\frac{\Omega(y')}{|y|^n}\phi(x-y)dy + \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{\Omega(y')}{|y|^n}\phi(x-y)dy.$$

For the first term on the right-hand side, taking absolute values and using polar coordinates (denoting the Lebesgue measure of S^{n-1} as $d\sigma$) we obtain

$$\left|\int_1^\infty \frac{1}{r}\int_{S^{n-1}}\Omega(y')\phi(r(x'-y'))d\sigma(y')dr\right| \leq \sup_{x\in\mathbb{R}^n}|x\phi(x)|\cdot\|\Omega\|_{L^1(S^{n-1})}\int_1^\infty\frac{1}{r^2}dr < \infty.$$

Whereas for the second term, since Ω has null integral in S^{n-1}, we may use the mean value theorem for convex domains and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\left|\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{\Omega(y')}{|y|^n}\Big(\phi(x-y)-\phi(x)\Big)dy\right| &\leq \int_{|y|<1}\frac{|\Omega(y')|}{|y|^n}|\langle\phi'(\widetilde{x}),y\rangle|dy\\ &\leq \int_{|y|<1}\frac{|\Omega(y')|}{|y|^{n-1}}|\phi'(\widetilde{x})|dy = \int_0^1\int_{S^{n-1}}|\Omega(y')||\phi'(\widetilde{x})|d\sigma(x')dr\\ &\leq \sup_{x\in\mathbb{R}^n}|\phi'(x)|\cdot\|\Omega\|_{L^1(S^{n-1})} < \infty,\end{aligned}$$

where $\widetilde{x}$ is a point, dependent of y, in the segment that joins x and y. So the singular integral operator is well-defined. Moreover, notice that by the exact same argument we used in the proof of Proposition 3.2.1, we get that T is given by the following convolution

$$T(\,\cdot\,) = \text{p.v.}\frac{\Omega(x')}{|x|^n} * (\,\cdot\,) \in \mathcal{S}'(\mathbb{R}^n).$$

2. Let us see that the condition of Ω having null integral over S^{n-1} is necessary for T to make sense. Indeed, choose $f \in \mathcal{S}(\mathbb{R}^n)$ so that $f \equiv 1$ if $|x| \leq R$. If we split the singular integral as we have done at the beginning of the first observation (now with respect to the radius of reference R), the first term remains bounded, but concerning the second, for any $|x| < R/2$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<\frac{R}{2}}\frac{\Omega(y')}{|y|^n}f(x-y)dy &= \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<\frac{R}{2}}\frac{\Omega(y')}{|y|^n}dy = \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{R}{2}}\frac{dr}{r}\int_{S^{n-1}}\Omega(y')d\sigma(y')\\ &= \lim_{\varepsilon\to 0}\ln\left(\frac{R}{2\varepsilon}\right)\int_{S^{n-1}}\Omega(y')d\sigma(y'),\end{aligned}$$

that diverges unless the integral of Ω over S^{n-1} is 0.

In order to understand better singular integral operators, as they are tempered distributions, we are interested in studying their Fourier transform. To do it, let us introduce the concept of *homogeneous distribution*. To motivate its definition, notice that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$, we may consider $\phi_\lambda := \lambda^{-n}\phi(\lambda^{-1}x)$. Then, for any function f homogeneous of degree a determining a tempered distribution (via integration against it), we have

$$\int_{\mathbb{R}^n} f(x)\phi_\lambda(x)dx = \lambda^a\int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

Therefore, we can define

Definition 4.1.2 (Homogeneous distribution of degree a)**.** A (tempered) distribution is said to be *homogeneous of degree a* if for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ ($\mathcal{S}(\mathbb{R}^n)$) and $\lambda > 0$ we have

$$T(\phi_\lambda) = \lambda^a T(\phi), \quad \text{where} \quad \phi_\lambda = \lambda^{-n}\phi(\lambda^{-1}x).$$

Given a homogeneous tempered distribution of a certain degree, its Fourier transform will be also homogeneous with a degree related to the one of the original distribution:

Lemma 4.1.1. *If $T \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a, then its Fourier transform is homogeneous of degree $-n-a$.*

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda > 0$ we have

$$\widehat{T}(\phi\lambda) = T(\widehat{\phi_\lambda}) = T(\widehat{\phi}(\lambda\cdot\,)) = \lambda^{-n}T(\widehat{\phi}_{\lambda^{-1}}) = \lambda^{-n-a}T(\widehat{\phi}) = \lambda^{-n-a}\widehat{T}(\phi).$$

□

For the particular case $T = \text{p.v.}\frac{\Omega(x')}{|x|^n}$, we obtain

$$T(\phi_\lambda) = \frac{1}{\lambda^n}\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{\Omega(y')}{|y|^n}\phi_\lambda\left(x - \frac{y}{\lambda}\right)dy = \frac{1}{\lambda^n}\lim_{\varepsilon\to 0}\int_{|u|>\frac{\varepsilon}{\lambda}}\frac{\Omega(u')}{|u|^n}\phi(x-u)du = \lambda^{-n}T(\phi),$$

and since singular integral operators are homogeneous of degree $-n$, their Fourier transform is homogeneous of degree 0.

The second piece of information that we know about the Fourier transform of a singular integral operator is that it will be the Fourier transform of a convolution against $\text{p.v.}\frac{\Omega(x')}{|x|^n}$. Therefore, by definition, given any function $f \in \mathcal{S}(\mathbb{R}^n)$

$$\left[\left(\text{p.v.}\frac{\Omega(x')}{|x|^n}\right) * f\right]^\wedge(\xi) = \int_{\mathbb{R}^n}\lim_{\varepsilon\to 0}\left(\int_{|y|>\varepsilon}\frac{\Omega(y')}{|y|^n}f(x-y)dy\right)e^{-2\pi i\langle\xi,x\rangle}dx.$$

Following an analogous argument as the one we did for (3.2.2), we can take the limit outside the outer integral and change the order of integration. This way, if for any $x \in \mathbb{R}^n$ we are able to compute the quantity

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<\frac{1}{\varepsilon}}\frac{\Omega(y')}{|y|^n}e^{-2\pi i\langle y,\xi\rangle}dy,$$

we will obtain a function (thought as a distribution) corresponding to the Fourier transform of the singular integral operator. The following theorem gives the explicit expression:

Theorem 4.1.2. *Let Ω be an integrable function on S^{n-1} with null integral. Then, the Fourier transform of the tempered distribution* $\text{p.v.}\frac{\Omega(x')}{|x|^n}$ *is the homogeneous function of degree* 0 *given by*

$$m(\xi) = \int_{S^{n-1}}\Omega(u)\left[\frac{\pi}{2i}\text{sgn}\langle u,\xi'\rangle - \ln|\langle u,\xi'\rangle|\right]d\sigma(u). \tag{4.1.2}$$

Proof. Without loss of generality we can assume $|\xi| = 1$. Since Ω has null integral, we rewrite m as follows:

$$\begin{aligned} m(\xi) &:= \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1/\varepsilon}\frac{\Omega(y')}{|y|^n}e^{-2\pi i\langle y,\xi\rangle}dy = \lim_{\varepsilon\to 0}\int_{S^{n-1}}\Omega(u)\left[\int_{\varepsilon}^{1/\varepsilon}e^{-2\pi i r\langle u,\xi\rangle}\frac{dr}{r}\right]d\sigma(u)\\ &= \lim_{\varepsilon\to 0}\int_{S^{n-1}}\Omega(u)\left[\int_{\varepsilon}^{1}\Big(e^{-2\pi i r\langle u,\xi\rangle}-1\Big)\frac{dr}{r}+\int_{1}^{1/\varepsilon}e^{-2\pi i r\langle u,\xi\rangle}\frac{dr}{r}\right]d\sigma(u)\\ &= \lim_{\varepsilon\to 0}\int_{S^{n-1}}\Omega(u)\left[\int_{\varepsilon}^{1}\Big(\cos\left(2\pi r\langle u,\xi\rangle\right)-1\Big)\frac{dr}{r}+\int_{1}^{1/\varepsilon}\cos\left(2\pi r\langle u,\xi\rangle\right)\frac{dr}{r}\right]d\sigma(u)\\ &\qquad - i\Big(\lim_{\varepsilon\to 0}\int_{S^{n-1}}\Omega(u)\left[\int_{\varepsilon}^{1/\varepsilon}\sin\left(2\pi r\langle u,\xi\rangle\right)\frac{dr}{r}\right]d\sigma(u)\Big). \end{aligned} \tag{4.1.3}$$

Notice that since the integrals $\int_0^\infty\frac{\sin(x)}{x}$, $\int_0^1\frac{\cos(x)-1}{x}$ and $\int_1^\infty\frac{\cos(x)}{x}$ are finite, and Ω is integrable over S^{n-1}, we can apply the dominated convergence theorem to enter the limit inside the outer integrals. Also, assuming that $\langle u,\xi\rangle\neq 0$ (for this case it is clear that $m(0)=0$), we do (at the inner integral) the change of variables $s := 2\pi r\langle u,\xi\rangle$, for each fixed u. Hence, for the inner integral of the imaginary part of (4.1.3) we have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\sin\left(2\pi r\langle u,\xi\rangle\right)\frac{dr}{r} = \lim_{\varepsilon\to 0}\int_{2\pi\varepsilon\langle u,\xi\rangle}^{2\pi\langle u,\xi\rangle/\varepsilon}\frac{\sin\left(s\right)}{s}\cdot \operatorname{sgn}\langle u,\xi\rangle ds = \frac{\pi}{2}\operatorname{sgn}\langle u,\xi\rangle.$$

On the other hand, for the inner integrals of the real part of (4.1.3) we get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left[\int_{2\pi\varepsilon\langle u,\xi\rangle}^{2\pi\langle u,\xi\rangle}\Big(\cos\left(s\right)-1\Big)\frac{ds}{s}+\int_{2\pi\langle u,\xi\rangle}^{2\pi\langle u,\xi\rangle/\varepsilon}\cos\left(s\right)\frac{ds}{s}\right]\\ &\qquad = \lim_{\varepsilon\to 0}\left[\int_{2\pi\varepsilon\langle u,\xi\rangle}^{1}\Big(\cos\left(s\right)-1\Big)\frac{ds}{s}+\int_{1}^{2\pi\langle u,\xi\rangle/\varepsilon}\cos\left(s\right)\frac{ds}{s}-\int_{1}^{2\pi\langle u,\xi\rangle}\frac{ds}{s}\right]\\ &\qquad = \int_0^1\Big(\cos\left(s\right)-1\Big)\frac{ds}{s}+\int_0^\infty\frac{\cos\left(s\right)}{s}ds-\ln\left(2\pi\right)-\ln|\langle u,\xi\rangle|. \end{aligned}$$

Integrating now this expressions against $\Omega(u)$ over S^{n-1} the constant terms disappear, obtaining the desired result. $\square$

This result is coherent with the expression we obtained for the Fourier transform of the Hilbert transform, which was $-i\cdot\operatorname{sgn}(\xi)$. Indeed, for this particular one-dimensional case, since $\Omega(u) = \pi^{-1}\operatorname{sgn}(u)$, the expression (4.1.2) reads as

$$\begin{aligned} m(\xi) &= \int_{\{-1,1\}}\operatorname{sgn}(u)\left[\frac{1}{2i}\operatorname{sgn}(u\xi')-\ln|u\xi'|\right]d\sigma(u) = -\frac{i}{2}\Big(\operatorname{sgn}(\xi')-\operatorname{sgn}(-\xi')\Big)\\ &= -i\cdot\operatorname{sgn}(\xi') = -i\cdot\operatorname{sgn}(\xi), \end{aligned}$$

as expected. Notice that the parity of the two terms multiplying Ω in (4.1.2) is important, since they might not contribute to the value of the integral depending on the parity of Ω. Indeed, observe that

- The first term is odd and bounded, so it will have no contribution if Ω is even.

- The second term is even, so it will not contribute to the integral if Ω is odd. Although it is not bounded, any (positive) power of it is integrable in S^{n-1}. Indeed, if $a \in \mathbb{R}$ and we assume $\xi = (1, 0, \ldots, 0)$ without loss of generality (by the symmetry of the integral), expressing the integral in polar coordinates of $\mathbb{R}^n$ we get

$$\int_{S^{n-1}} |\ln|\langle u, \xi\rangle||^a d\sigma(u) \lesssim \int_0^\pi |\ln|\cos(\varphi)||^a d\varphi = -2\int_0^1 \frac{\ln(x)^a}{\sqrt{1-x^2}} dx$$

being this last integral finite for all $a > 0$.

Therefore, since every function Ω in S^{n-1} can be decomposed in its odd and even parts as follows

$$\Omega_e(u) := \frac{\Omega(u) + \Omega(-u)}{2}, \quad \Omega_o(u) := \frac{\Omega(u) - \Omega(-u)}{2},$$

we obtain immediately the following corollary (by Hölder's inequality):

Corollary 4.1.3. *If Ω is an integrable function on S^{n-1} with null integral such that*

1. $\Omega_o \in L^1(S^{n-1})$,
2. $\Omega_e \in L^q(S^{n-1})$ *for some* $q > 1$,

then the Fourier transform of p.v.$\frac{\Omega(x')}{|x|^n}$ *is bounded.*

In fact, it is possible to carry out a study of the boundedness of singular integral operators in terms of their parity. The reader may find details of this approach in [7, Chapter 4], which involves techniques such as the *method of rotations* or the *Riesz transform.*

4.2 The Calderón-Zygmund theorem

If one follows the book of Duoandikoetxea [7], the approach commented above regarding the study of boundedness of singular integral operators in terms of their parity, relies essentially, in properties about the Hilbert transform. In fact, the arguments used to prove the boundedness of such operators try to encapsulate the necessary information to retrieve an analogous kernel to that of the Hilbert case, and try to resemble the expression of a singular integral to a *directional* Hilbert transform; an approach that is possible once working with polar coordinates and applying H, independently, in each line passing through the origin.

The conditions for boundedness that are presented this way seem to rely excessively on the properties of the one-dimensional case. Far from this point view being not appropriate, we seek conditions rather more general and intrinsic to singular integral operators that allow us to formulate theorems which are independent of the nature of the particular kernel associated to the operator. A fact that suggests that such development could be necessary is that in [7, Chapter 4] one does not obtain information about the case $p = 1$.

In order to start our study, we wish to designate the necessary properties that makes us think of an operator K as a singular integral operator. We may start imposing a condition that characterizes the kernel $1/|x|^n$ in $\mathbb{R}^n$, which does not rely excessively in its form. The condition we choose is that

K is a tempered distribution corresponding to a locally integrable function on $\mathbb{R}^n \setminus \{0\}$.

This opens new possibilities such as having a kernel of the form $\ln|x|$. But now, we wish to demand additional properties that allow us to prove its boundedness (either weak or strong) as an operator. We may find some pieces of information in the proof Theorem 3.2.4, where we studied the boundedness of the Hilbert transform. To carry out the proof, notice that we had covered in advance a specific case ($p = 2$) that could be shown rather easily. To study this case we strongly relied on properties of the Fourier transform of the operator. By the observations done after Definition 3.2.2, notice that it will be sufficient to ask for

the Fourier transform of K to be essentially bounded.

Later on, this last condition will be improved in the sense that we will deduce the same boundedness results just relying on properties of K.

Finally, we observe that the main argument to extend the boundedness to every other case was based on the Marcinkiewicz interpolation theorem 3.1.3, and we reduced ourselves to prove the weak type inequality for the case $p = 1$ using a Calderón-Zygmund decomposition and considering the study of the good and bad parts of $f \in \mathcal{S}(\mathbb{R})$. In fact, checking the proof, the argument only relied on the precise definition of the Hilbert transform when proving

$$\int_{\mathbb{R}\setminus 2I_k} |Hb_{f,k}(x)|dx \leq C \int_{I_k} |b_{f,k}(x)|dx,$$

which implied the weak type $(1,1)$ inequality. Now, this condition could be written as

$$\int_{\mathbb{R}\setminus 2Q_k} |Tb_{f,k}(x)|dx \leq C \int_{Q_k} |b_{f,k}(x)|dx,$$

where $(Q_j)_j$ is a sequence of disjoint dyadic cubes associated to a Calderón-Zygmund decomposition of f in $\mathbb{R}^n$ for some $\lambda > 0$. To achieve the previous bound, we will require K to satisfy the so called *Hörmander condition*:

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \leq C, \qquad \forall y \in \mathbb{R}^n.$$

This assumption, in practice, is often obtained from another (stronger) property, that is such that if K satisfies it, then also verifies Hörmander's condition. It is called the *gradient condition*, which holds if

$$|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \qquad \forall x \neq 0.$$

If this is true, by the mean value theorem we get that for every x there exists $\widetilde{y} = \widetilde{y}(x)$ in the segment joining x and $x - y$ (and thus satisfying $|\widetilde{y}| > x/2$) such that

$$\begin{aligned}\int_{|x|>2|y|} |K(x-y) - K(x)|dx &\leq \int_{|x|>2|y|} |\nabla K(\widetilde{y})||y|dx \leq C \int_{|x|>2|y|} \frac{|y|}{|\widetilde{y}|^{n+1}} dx \\ &\leq 2^{n+1}C|y| \int_{|x|>2|y|} \frac{1}{|x|^{n+1}} dx = 2^{n+1}C|S^{n-1}||y| \int_{2|y|}^{\infty} \frac{dr}{r^2} = 2^n C|S^{n-1}| < \infty,\end{aligned}$$

and so the Hörmander condition holds.

Combining all the properties of the operator K we can state and prove the result we were looking for:

Theorem 4.2.1. (Calderón-Zygmund, [7, Theorem 5.1]). *Let K be a tempered distribution that corresponds to a locally integrable function on $\mathbb{R}^n \backslash \{0\}$ and that satisfies for some $A, B > 0$*

1. *$|\widehat{K}(\xi)| \leq A$ for almost every $\xi \in \mathbb{R}^n$.*
2. *The Hörmander condition:*

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq B, \qquad \forall y \in \mathbb{R}^n.$$

Then we have:

(*i*) *For $1 < p < \infty$: $\|K * f\|_p \leq C_p \|f\|_p$, for some $C_p > 0$.*

(*ii*) *There exists $C > 0$ such that for every $\lambda > 0$: $|\{x \in \mathbb{R}^n : |K * f(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1$.*

Proof. As we have been arguing before, it will be sufficient to prove for $f \in \mathcal{S}(\mathbb{R}^n)$ that $Tf := K * f$ is of weak type $(1,1)$. This is because the strong type inequality for $p = 2$ is already satisfied, so by the Marcinkiewicz interpolation theorem 3.1.3 the cases $1 < p < 2$ are covered, as well as those for which $2 < p < \infty$, by a duality argument. In this case what we mean precisely by the latter consists of taking $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ with $p > 2$ and p' conjugate of p (and hence such that $1 < p' < 2$), and observing that

$$\left| \int_{\mathbb{R}^n} (K * f)(x) g(x) dx \right| = \left| \int_{\mathbb{R}^n} f(y)(K(-\,\cdot\,) * g)(y) dx \right| \leq C_{p'} \|f\|_p \|g\|_{p'},$$

where in the first equality we have been able to apply Fubini due to the boundedness assumption for $1 < p < 2$ and Hölder's inequality. Then, $K * f$ is a bounded linear functional in $L^{p'}(\mathbb{R}^n)$ with norm at most $C_{p'} \|f\|_p$, and applying the isometry between $L^p(\mathbb{R}^n)$ and $(L^{p'}(\mathbb{R}^n))'$ we are done.

Returning now to the proof of the weak type $(1,1)$ case, we consider, as in Theorem 3.2.4, a Calderón-Zygmund decomposition of f in $\mathbb{R}^n$ with value $\lambda > 0$. We denote the associated sequence of disjoint dyadic cubes as $(Q_j)_j$. As previously mentioned, the key point to deduce the result was to check if

$$\int_{\mathbb{R}^n \backslash 2Q_k} |Tb_{f,k}(x)| dx \leq C \int_{Q_j} |b_{f,k}(x)| dx.$$

Let c_k be the center of Q_k and recall that each $b_{f,k}$ has null integral. Hence, if $x \notin 2Q_k$

$$Tb_{f,k}(x) = \int_{Q_k} K(x-y) b_{f,k}(y) dy = \int_{Q_k} \Big[K(x-y) - K(x-c_k) \Big] b_{f,k}(y) dy,$$

and so we obtain

$$\int_{\mathbb{R}^n \backslash 2Q_j} |Tb_{f,j}(x)| dx \leq \int_{Q_j} |b_{f,j}(y)| \bigg(\int_{\mathbb{R}^n \backslash 2Q_j} \Big| K(x-y) - K(x-c_j) \Big| dx \bigg) dy,$$

where the inner integral of the right-hand side is bounded, since $\mathbb{R}^n \setminus 2Q_j \subset \{x \in \mathbb{R}^n : |x - c_j| > 2|y - c_j|\}$, and so we can apply Hörmander's condition. $\square$

4.2.1 Refining the Calderón-Zygmund theorem using truncated integrals

In the discussion previous to Theorem 4.2.1 it arouse the necessity to obtain, in a rather direct manner, the boundedness for the case $p = 2$. We chose to approach this problem by assuming a certain property about the Fourier transform of the operator K. The goal of this section is to determine conditions concerning the properties of K itself that ensure its $L^p(\mathbb{R}^n)$-boundedness, for $1 < p < \infty$. We begin by stating the following result that will cover our goal not for the operator, but for a truncation of it.

Theorem 4.2.2. *Let $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be a function satisfying*

1. *There exists $A > 0$ such that for every $0 < a < b < \infty$*

$$\left| \int_{a<|x|<b} K(x)dx \right| \leq A. \tag{4.2.1}$$

2. *There exists $B > 0$ such that for every $a > 0$*

$$\int_{a<|x|<2a} |K(x)|dx \leq B. \tag{4.2.2}$$

3. *There exists $C > 0$ such that for every $y \in \mathbb{R}^n$*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \leq C. \tag{4.2.3}$$

Then, for any $\varepsilon > 0$ and $R > 0$ the function $K_{\varepsilon,R}(x) := K(x)\chi_{\{\varepsilon<|x|<R\}}$ satisfies

$$|\widehat{K_{\varepsilon,R}}| \leq C.$$

Proof. Let us fix $\xi \in \mathbb{R}^n$ and ε, R positive constants. Observe that for $\varepsilon < |\xi|^{-1} < R$ we have

$$\begin{aligned} \widehat{K_{\varepsilon,R}}(\xi) &= \int_{\varepsilon<|x|<R} K(x)e^{-2\pi i\langle x,\xi\rangle}dx \\ &= \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)e^{-2\pi i\langle x,\xi\rangle}dx + \int_{|\xi|^{-1}<|x|<R} K(x)e^{-2\pi i\langle x,\xi\rangle}dx \end{aligned} \tag{4.2.4}$$

(in any other case, i.e. $|\xi|^{-1}$ or $|\xi|^{-1} > R$, we would only need to study one of the two integrals). Let us focus on the first integral. We rewrite it as

$$\int_{\varepsilon<|x|<|\xi|^{-1}} K(x)dx + \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)\Big(e^{-2\pi i\langle x,\xi\rangle} - 1\Big)dx.$$

Taking absolute values and applying the mean value theorem we obtain the estimate

$$\begin{aligned} \left| \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)dx \right| &+ \int_{\varepsilon<|x|<|\xi|^{-1}} |K(x)| \left| \nabla_\xi \Big(e^{-2\pi i\langle x,\xi\rangle}\Big)(\widetilde{\xi}) \right| |\xi|dx \\ &\leq A + 2\pi|\xi| \int_{\varepsilon<|x|<|\xi|^{-1}} |x||K(x)|dx. \end{aligned} \tag{4.2.5}$$

To bound the second term we claim that relation (4.2.2) is equivalent to

$$\int_{|x|<a} |x||K(x)|dx \leq B'a, \tag{4.2.6}$$

for some constant $B' > 0$ and for every $a > 0$. Indeed, observe that if we assume (4.2.2)

$$\int_{|x|<a} |x||K(x)|dx = \sum_{k=0}^{\infty} \int_{\frac{a}{2^{k+1}}<|x|<\frac{a}{2^k}} |x||K(x)|dx \leq \sum_{k=0}^{\infty} \frac{a}{2^k} B = 2Ba,$$

and on the other hand, if (4.2.6) holds

$$\int_{a<|x|<2a} |K(x)|dx \leq \int_{a<|x|<2a} \frac{|x|}{a}|K(x)|dx \leq \int_{|x|<2a} \frac{|x|}{a}|K(x)|dx \leq 2B',$$

so the claim is proved. Hence, returning to (4.2.5) we get the bound $A + 2\pi B'$.

For the second term of (4.2.4), we introduce the new variable $\omega := \frac{1}{2}\xi|\xi|^{-2}$, that satisfies $e^{2\pi i\langle\omega,\xi\rangle} = -1$. Changing the variable $x \mapsto x - \omega$, the integral, that we will call I, reads as

$$I := -\int_{|\xi|^{-1}<|x-\omega|<R} K(x-\omega)e^{-2\pi i\langle x,\xi\rangle}dx.$$

Hence, summing this expression with the equivalent one of (4.2.4) we get

$$2I = \int_{|\xi|^{-1}<|x|<R} K(x)e^{-2\pi i\langle x,\xi\rangle}dx - \int_{|\xi|^{-1}<|x-\omega|<R} K(x-\omega)e^{-2\pi i\langle x,\xi\rangle}dx.$$

Observe that $2|\omega| = |\xi|^{-1}$, which means that $|x-\omega| = \big|x - \frac{1}{2}|\xi|^{-1}\big|$. Then, the domain of integration of the second integral is contained in $\big(\frac{1}{2}|\xi|^{-1}, R + \frac{1}{2}|\xi|^{-1}\big)$ and, at the same time, contains $\big(\frac{3}{2}|\xi|^{-1}, R - \frac{1}{2}|\xi|^{-1}\big)$. Therefore

$$\begin{aligned}
2I = &\int_{\frac{3}{2}|\xi|^{-1}<|x|<R-\frac{1}{2}|\xi|^{-1}} \big[K(x) - K(x-\omega)\big]e^{-2\pi i\langle x,\xi\rangle}dx + \int_{|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} K(x)e^{-2\pi i\langle x,\xi\rangle}dx \\
&+ \int_{R-\frac{1}{2}|\xi|^{-1}<|x|<R} K(x)e^{-2\pi i\langle x,\xi\rangle}dx - \int_{|\xi|^{-1}<|x-\omega|<\frac{3}{2}|\xi|^{-1}} K(x-\omega)e^{-2\pi i\langle x,\xi\rangle}dx \\
&- \int_{R-\frac{1}{2}|\xi|^{-1}<|x-\omega|<R} K(x-\omega)e^{-2\pi i\langle x,\xi\rangle}dx.
\end{aligned}$$

Now, taking the modulus and enlarging the domains of integration, we can bound $2I$ in the following way

$$\begin{aligned}
2|I| \leq \int_{|\xi|^{-1}<|x|} \big|K(x) - K(x-\omega)\big|dx &+ 2\int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |K(x)|dx \\
&+ 2\int_{R-\frac{1}{2}|\xi|^{-1}<|x|<R+\frac{1}{2}|\xi|^{-1}} |K(x)|dx.
\end{aligned}$$

Since $|\xi|^{-1} = 2|\omega|$, we can bound the first integral by (4.2.3); the second one can be bounded as follows

$$\begin{aligned} 2\int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |K(x)|dx &= 2\int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{4}|\xi|^{-1}} |K(x)|dx + 2\int_{\frac{3}{4}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |K(x)|dx \\ &\leq 2\int_{\frac{1}{2}|\xi|^{-1}<|x|<|\xi|^{-1}} |K(x)|dx + 2B \leq 4B, \end{aligned}$$

and the third can also be bounded it by $4B$ by a similar argument, so we are done. □

An immediate consequence of the previous theorem is the following:

Corollary 4.2.3. *If K satisfies the hypothesis of Theorem 4.2.2, then for every $\varepsilon > 0$ and $R > 0$ the operator $K_{\varepsilon,R} * \cdot$ is of weak type $(1,1)$ and of strong type (p,p) for $1 < p < \infty$.*

Proof. Notice that is just a mere application of Theorem 4.2.1, since if the conditions of Theorem 4.2.2 are satisfied, then so are those of the former theorem. □

It is clear that our next goal should be to try to transfer the properties of the truncated operators to the original one. That is, we wish to define the singular integral operator T as the limit, in some sense, of $K_{\varepsilon,R} * f$ as $\varepsilon \to 0$ and $R \to \infty$ so that the properties are preserved.

Begin by noticing that condition (4.2.2) implies $K_{\varepsilon,R} \in L^1(\mathbb{R}^n)$, so by Minkowski's integral inequality [23, Exercise 16], for any $f \in L^p(\mathbb{R}^n)$ the convolution $K_{\varepsilon,R} * f$ is well-defined in $L^p(\mathbb{R}^n)$. Also, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, if $N = N(R,\varepsilon)$ is the smallest positive integer satisfying $2^N > R\varepsilon^{-1}$, we have

$$\begin{aligned} \lim_{R\to\infty} \left|K_{\varepsilon,R} * \phi(0)\right| &\leq \lim_{R\to\infty} \int_{\varepsilon<|x|<R} |K(x)||\phi(x)|dx < \lim_{R\to\infty} \sum_{k=0}^{N} \int_{2^k\varepsilon<|x|<2^{k+1}\varepsilon} |K(x)||\phi(x)|dx \\ &\leq \lim_{R\to\infty} B\||x|\phi\|_\infty \sum_{k=0}^{N} \frac{1}{2^k\varepsilon} = \frac{2B}{\varepsilon}\|x\phi\|_\infty < \infty, \end{aligned}$$

where the constant B appears in relation (4.2.2). Therefore, $K_{\varepsilon,\infty} * \phi$ is well-defined for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ (the previous computation has been done at the point 0, but an analogous argument can be used for any other point considering a translation of ϕ). In fact, we can extend the previous definition to any $f \in L^p(\mathbb{R}^n)$ as follows: if $(R_n)_n$ is an increasing sequence of radii tending to ∞, and $(\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ approximates f in $L^p(\mathbb{R}^n)$, then

$$\begin{aligned} &\|K_{\varepsilon,R_i} * f - K_{\varepsilon,R_j} * f\|_p \\ &\qquad \leq \|K_{\varepsilon,R_i} * (f-\phi_n)\|_p + \|K_{\varepsilon,R_i} * \phi_n - K_{\varepsilon,R_j} * \phi_n\|_p + \|K_{\varepsilon,R_j} * (f-\phi_n)\|_p \\ &\qquad \leq \|K_{\varepsilon,R_i}\|\|f-\phi_n\|_p + \|(K_{\varepsilon,R_i} - K_{\varepsilon,R_j}) * \phi_n\| + \|K_{\varepsilon,R_j}\|\|f-\phi_n\|_p, \end{aligned}$$

where the first and third terms tend to 0 as $n \to \infty$ by construction, and the second also tends to zero as $i,j \to \infty$, since we can apply again (4.2.2) and the fact that it involves an integral of a rapid decaying function in a domain near infinity. Hence we have found that the sequence $(K_{\varepsilon,R_n} * f)_n$ is Cauchy in $L^p(\mathbb{R}^n)$, and hence convergent, i.e. $K_{\varepsilon,\infty} * f$ is indeed well-defined for $f \in L^p(\mathbb{R}^n)$.

Therefore, we have reduced the problem of giving meaning to the limit $\lim_{\varepsilon\to 0, R\to\infty} K_{\varepsilon,R} * f$ to just studying the existence of

$$\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} K(x)\phi(x)dx, \qquad \phi\in\mathcal{S}(\mathbb{R}^n),$$

that is just the principal value of K applied to ϕ. The next result answers this question:

Proposition 4.2.4. *Let $K\in L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ be a function satisfying (4.2.2). Then, the tempered distribution* p.v. K *exists if and only if the following limit exists*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)dx.$$

Proof. Let us assume first that the tempered distribution exists. We pick $\phi\in\mathcal{S}(\mathbb{R}^n)$ so that $\phi\equiv 1$ in $B(0,1)$. Then,

$$\mathrm{p.v.}\, K(\phi) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)dx + \int_{|x|>1} K(x)\phi(x)dx.$$

Observe that the second integral is bounded by

$$\begin{aligned}\int_{|x|>1}|K(x)\phi(x)|dx &= \sum_{k=0}^{\infty}\int_{2^k<|x|<2^{k+1}}\frac{|x|}{|x|}|K(x)\phi(x)|dx\\ &\le \||x|\phi\|_\infty\sum_{k=0}^{\infty}\frac{1}{2^k}\int_{2^k<|x|<2^{k+1}}|K(x)|dx \le 2B\||x|\phi\|_\infty<\infty,\end{aligned}$$

and therefore the first integral must exist too. We prove now the remaining implication. Assume that such limit exists, call it ℓ. Then

$$\begin{aligned}\mathrm{p.v.}\, K(\phi) &= \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)\phi(x)dx + \int_{|x|>1} K(x)\phi(x)dx\\ &= \phi(0)\ell + \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)\big[\phi(x)-\phi(0)\big]dx + \int_{|x|>1} K(x)\phi(x)dx.\end{aligned}$$

The second integral exists by the same argument of the previous implication. For first integral, using that $|\phi(x)-\phi(0)|\le\|\nabla\phi\|_\infty|x|$ as well as property (4.2.6), which is equivalent to (4.2.2), we can bound it by

$$\int_{|x|<1}|K(x)||\phi(x)-\phi(0)|dx \le \|\nabla\phi\|_\infty\int_{|x|<1}|x||K(x)|dx \le B'\|\nabla\phi\|_\infty,$$

and so the principal value exists. □

Corollary 4.2.5. *If we add to the hypothesis of Theorem 4.2.2 the assumption*

4. *The limit $\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)dx$ exists,*

then the tempered distribution T defined as

$$Tf(x) := \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} K(y)f(x-y)dy$$

is extended to be a bounded operator in $L^p(\mathbb{R}^n)$, $1<p<\infty$, which is also of weak type $(1,1)$.

Proof. Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $(\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ a sequence that approximates f in $L^p(\mathbb{R}^n)$. Then, if $(\varepsilon_n)_n$ is any sequence of positive real numbers approaching 0,

$$\begin{aligned}
&\|K_{\varepsilon_i,\infty} * f - K_{\varepsilon_j,\infty} * f\|_p \\
&\quad \leq \|K_{\varepsilon_i,\infty} * f - K_{\varepsilon_i,\infty} * \phi_n\|_p + \|K_{\varepsilon_i,\infty} * \phi_n - K_{\varepsilon_j,\infty} * \phi_n\|_p + \|K_{\varepsilon_j,\infty} * \phi_n - K_{\varepsilon_j,\infty} * f\|_p \\
&\quad \leq \|K_{\varepsilon_i,\infty}\|\|f - \phi_n\|_p + \|(K_{\varepsilon_i,\infty} - K_{\varepsilon_j,\infty}) * \phi_n\|_p + \|K_{\varepsilon_j,\infty}\|\|f - \phi_n\|_p.
\end{aligned}$$

By the comments previous to Proposition 4.2.4, the first and third terms tend to 0 as $n \to \infty$. For the second term we use that the tempered distribution of the principal value of K exists, and so as $i, j \to \infty$ it also tends to 0. Therefore, the sequence $(K_{\varepsilon_n,\infty} * f)_n$ is Cauchy in $L^p(\mathbb{R}^n)$, and by completeness it converges. Such limit will define p.v. $K * f$. The process of extension for the weak case is treated similarly but with the measure topology (see the comments after Corollary 3.2.5). □

4.3 Calderón-Zygmund operators

The conditions we have been giving so far in order to apply, ultimately, the Calderón-Zygmund theorem 4.2.1, take advantage of the base case $p = 2$ and the fact that the operator involved is the convolution with a tempered distribution.

Now we will proceed by assuming from the start the $L^2(\mathbb{R}^n)$-boundedness, so that the study of the rest of $L^p(\mathbb{R}^n)$ spaces only relies on the Hörmander condition, for which the operator needs not to be initially of the convolution-type.

Following the same reasoning as in the beginning of the previous section, we wish to find sufficient conditions to follow a similar proof as the one given for the Kolmogorov-Riesz theorem 3.2.4.

We begin by noticing that at the very end of the proof of Theorem 4.2.1, to verify the weak type $(1, 1)$ inequality, we reached an integral of the form

$$\int_{Q_k} |b_{f,k}(y)| \left(\int_{\mathbb{R}^n \setminus 2Q_k} \Big| K(x - y) - K(x - c_k) \Big| dx \right) dy,$$

and used that $\mathbb{R}^n \setminus 2Q_k \subset \{x \in \mathbb{R}^n : |x - c_k| > 2|y - c_k|\}$ to apply Hörmander's condition. In the present case, if we want to generalize the last argument, we should start by understanding our operator T (bounded in $L^2(\mathbb{R}^n)$) as represented by K, a function of two variables in $\mathbb{R}^n \times \mathbb{R}^n$. In fact, observe that we only need to use this kind of representation when applying T to $b_{f,j}$, which belongs to the family of functions in $L^2(\mathbb{R}^n)$ compactly supported. Moreover, the domain of K needs not to be $\mathbb{R}^n \times \mathbb{R}^n$, but we can discard the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ due to the nature of the domains of integration. Overall, we require that there exists K defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ such that, if $f \in L^2(\mathbb{R}^n)$ has compact support, then T is given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \qquad \forall x \notin \operatorname{supp}(f).$$

Moreover, it has to satisfy an analogous Hörmander condition, that now is written as

$$\int_{|x-y|>2|y-z|} |K(x, y) - K(x, z)| dx \leq C.$$

Also, there is a detail when generalizing the property for the case $p > 2$ using a duality argument. If we look at the proof of Theorem 4.2.1, there is a step where we do a change of variables induced by the convolution itself. To follow the same argument we impose the same *Hörmander-like* condition but with respect to the first variable, i.e.

$$\int_{|x-y|>2|x-w|} |K(x,y) - K(w,y)| dy \leq C.$$

Now we are ready to state the main theorem of this section.

Theorem 4.3.1. *Let T be a bounded operator in $L^2(\mathbb{R}^n)$ and K a function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ such that if $f \in L^2(\mathbb{R}^n)$ has compact support then T is given by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad \forall x \notin \operatorname{supp}(f).$$

Assume also that K satisfies

$$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)| dx \leq C, \tag{4.3.1}$$

$$\int_{|x-y|>2|x-w|} |K(x,y) - K(w,y)| dy \leq C. \tag{4.3.2}$$

Then T is of strong type (p,p) for $1 < p < \infty$ and of weak type $(1,1)$.

Proof. It is analogous to the proof of Theorem 4.2.1. □

We may wonder if conditions (4.3.1) and (4.3.2) are too restrictive and do not give rise to kernels of different nature. But, in fact, there exists a whole family of functions that satisfy the previous Hörmander conditions:

Definition 4.3.1 (Standard kernel)**.** A function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ is called a *standard kernel* if there exists $\delta > 0$ such that

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{4.3.3}$$

$$|K(x,y) - K(x,z)| \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}, \quad \text{if} \quad |x-y| > 2|y-z|, \tag{4.3.4}$$

$$|K(x,y) - K(w,y)| \leq C \frac{|x-w|^\delta}{|x-y|^{n+\delta}}, \quad \text{if} \quad |x-y| > 2|x-w|. \tag{4.3.5}$$

We remark that standard kernels are functions such that its properties are clearly designed to satisfy the Hörmander conditions (4.3.1) and (4.3.2). These kernels motivate the definition of a general type of integral operators that are continuous in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and satisfy a weak type $(1,1)$ inequality:

Definition 4.3.2 (Calderón-Zygmund operator)**.** We will say that an operator T is a *Calderón-Zygmund operator* if

1. T is bounded in $L^2(\mathbb{R}^n)$.
2. There is a standard kernel K so that for any $f \in L^2(\mathbb{R}^n)$ with compact support

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad \forall x \notin \operatorname{supp}(f).$$

Therefore, if we are given an operator defined through a standard kernel, if we show that it is bounded in $L^2(\mathbb{R}^n)$, by Theorem 4.3.1, it is enough to conclude that it is also bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. We will tackle precisely this question in the next section.

4.4 BMO and the statement of the $T1$ theorem

The main goal of this section is to introduce a fundamental tool that will be helpful to check the $L^2(\mathbb{R}^n)$-boundedness of an operator defined through a standard kernel, so that it has all the properties of a Calderón-Zygmund operator. It will be the $T1$ *theorem* 4.4.5, and we will state it without proof, since we find it rather technical and beyond the purpose of our project. One may consult a complete proof in the book of Duoandikoetxea [7, §9.4] as well as a more general version of it (that involves non-doubling measures) in the article of Nazarov, Treil & Volberg [20, Theorem 1.2].

So we will begin our discussion by presenting a specific subspace of locally integrable functions that will play a fundamental role in the $T1$ theorem.

4.4.1 The space BMO

Definition 4.4.1 (Sharp maximal function)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$ a cube. Let us denote f_Q the average of f in Q, i.e.

$$f_Q := \frac{1}{|Q|} \int_Q f(y)dy.$$

We define the *sharp maximal function* as

$$M^{\#}f(x) = \sup_{Q \ni x} \frac{1}{Q} \int_Q |f(y) - f_Q|dy,$$

where we have to understand the supremum as taken over all the cubes containing x.

Definition 4.4.2 (Bounded mean oscillation)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be so that $M^{\#}f$ is bounded. In this case we say that f has *bounded mean oscillation*, and we denote the space of functions with this property as BMO, that is

$$BMO := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) \, : \, M^{\#}f \in L^\infty(\mathbb{R}^n) \right\}. \tag{4.4.1}$$

We wish to define now a norm in BMO and the first natural choice is

$$\|f\|_{BMO} := \|M^{\#}f\|_\infty.$$

However this is not a proper norm, since any function which is constant almost everywhere has zero oscillation. Nevertheless, these are the only functions with this property, and so we will understand (as it is customary also in L^p spaces) BMO as the quotient of the original space (4.4.1) by the constant functions. This way, $(BMO, \|\cdot\|_{BMO})$ not only becomes a normed space, but a Banach space [7, §6.2].

Proposition 4.4.1. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, there exists a constant $C_n > 0$ such that for every $x \in \mathbb{R}^n$*

$$M^{\#}f(x) \leq C_n Mf(x),$$

where M is the Hardy-Littlewood maximal function.

Proof. Let us begin by proving *1.* Fix $x \in \mathbb{R}^n$ us recall that M was defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy.$$

Our first observation is that computing M using balls or cubes is essentially equivalent. Indeed, for every $Q(0,r) = Q_r \subset \mathbb{R}^n$ (cube of side length $2r$ centered at the origin) we can find $r_1 > 0$ and $r_2 > 0$ so that $B_{r_1} \subset Q_r \subset B_{r_2}$, and therefore for every $r > 0$ there exist constants $c_1 > 0$ and $c_2 > 0$ so that

$$\frac{c_1}{|B_r|} \int_{B_r} |f(x-y)| dy \leq \frac{1}{|Q_r|} \int_{Q_r} |f(x-y)| dy \leq \frac{c_2}{|B_r|} \int_{B_r} |f(x-y)| dy \tag{4.4.2}$$

We will denote the maximal operator computed through cubes as M' and it is such that $Mf(x) \approx M'f(x)$, where $\approx$ refers to a relation such as (4.4.2). Hence, if Q denotes a cube of side length $2r$ containing x, we have

$$\begin{aligned} M^{\#}f(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \leq \sup_{Q \ni x} \frac{2}{|Q|} \int_Q |f(y)| dy \\ &\leq \sup_{Q \ni x} \frac{2}{|Q|} \frac{|Q(x, 2\sqrt{n}r)|}{|Q(x, 2\sqrt{n}r)|} \int_{Q(x,2\sqrt{n}r)} |f(y)| dy \\ &= 2\big(2\sqrt{n}\big)^n \sup_{r>0} \frac{1}{|Q(x, \sqrt{n}r)|} \int_{Q(x,\sqrt{n}r)} |f(y)| dy = 2\big(2\sqrt{n}\big)^n Mf(x) \end{aligned}$$

□

Notice that if we combine the previous result with Theorem 3.1.6 for the case $p = \infty$, we have the inclusion

$$L^\infty \subset BMO.$$

Let us now prove an important result that connects BMO spaces with singular integrals:

Theorem 4.4.2. *Let T be a bounded operator such as in Theorem 4.3.1. Then, for any bounded function f compactly supported, the following are satisfied:*

1. *$Tf \in BMO$.*
2. *$\|Tf\|_{BMO} \leq C\|f\|_\infty$.*

Proof. Let us fix a cube $Q \subset \mathbb{R}^n$ and denote its center by c_Q. Set also $Q^\star$ the cube centered at c_Q whose side length is $4\sqrt{n}$ times that of Q (this dilation factor coincides with four times the length of the diagonal of the unit cube of $\mathbb{R}^n$). Now we decompose f as the sum $f_1 + f_2$ where $f_1 = f\chi_{Q^\star}$ and $f_2 = f - f_1$. Finally, we choose $a := Tf_2(c_Q)$ to obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tf(x) - a|dx &\leq \frac{1}{|Q|}\int_Q |Tf_1(x)|dx + \frac{1}{|Q|}\int_Q |Tf_2(x) - Tf_2(c_Q)|dx \\ &\leq \left(\frac{1}{|Q|}\int_Q |Tf_1(x)|^2 dx\right)^{\frac{1}{2}} + \frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n\setminus Q^\star} \big[K(x,y) - K(c_Q,y)\big] f(y)dy\right| dx \\ &\leq C\left(\frac{1}{|Q|}\int_{Q^\star} |f(x)|^2 dx\right)^{\frac{1}{2}} + \frac{1}{|Q|}\int_Q \int_{\mathbb{R}^n\setminus Q^\star} \big|K(x,y) - K(c_Q,y)\big| dydx \cdot \|f\|_\infty \\ &\leq C'\|f\|_\infty + C''\|f\|_\infty \lesssim \|f\|_\infty,\end{aligned}$$

where we have useded Hölder's inequality, the $L^2(\mathbb{R}^n)$-boundedness of T and the estimate (4.3.2), that can be applied since if $\ell(Q)$ is the side length of Q, for every $x \in Q$ and $y \in Q^\star$ we have $|y - c_Q| > 2\sqrt{n}\ell(Q) \geq 2|x - c_Q|$. □

One of the applications of the previous result is that it allows us to define an operator T as in Theorem 4.3.1 for any $f \in L^\infty(\mathbb{R}^n)$. We do it following a similar scheme as in the above proof: consider $f \in L^\infty(\mathbb{R}^n)$, a cube $Q \subset \mathbb{R}^n$ centered at the origin and its associated bigger cube $Q^\star$. We decompose $f = f_1 + f_2$ where $f_1 = f\chi_{Q^\star}$ is bounded and with compact support. Hence Tf_1 is well-defined thinking f_1 as an $L^2(\mathbb{R}^n)$ function, and so $Tf_1(x)$ exists for almost every x. Now we define for each $x \in Q$

$$Tf(x) := Tf_1(x) + \int_{\mathbb{R}^n} \big[K(x,y) - K(0,y)\big] f_2(y)dy. \tag{4.4.3}$$

The integral involved in the previous expression converges, since it can be bounded as

$$\left|\int_{\mathbb{R}^n} \big[K(x,y) - K(0,y)\big] f_2(y)dy\right| \leq \|f\|_\infty \int_{\mathbb{R}\setminus Q^\star} \big|K(x,y) - K(0,y)\big| dy \lesssim \|f\|_\infty,$$

by the definition of $Q^\star$ and the assumptions made about K (implied by the fact that T is an operator as in Theorem 4.3.1).

Observe that if $\widetilde{Q}$ is some other cube centered at the origin containing Q, we might encounter a problem, since we have two definitions for $Tf(x)$ if $x \in Q$. But defining $\widetilde{f}_1 := f\chi_{\widetilde{Q}^\star}$ and $\widetilde{f}_2 := f\chi_{\mathbb{R}^n\setminus\widetilde{Q}^\star}$ we get that the difference between both definitions at $x \in Q$ is

$$T(f_1 - \widetilde{f}_1)(x) + \int_{\mathbb{R}^n\setminus Q^\star} \big[K(x,y) - K(0,y)\big] f(y)dy - \int_{\mathbb{R}^n\setminus\widetilde{Q}^\star} \big[K(x,y) - K(0,y)\big] f(y)dy. \tag{4.4.4}$$

Since $f_1 - \widetilde{f}_1 \in L^2(\mathbb{R}^2)$ is compactly supported and $x \notin \operatorname{supp}(f_1 - \widetilde{f}_1) = \widetilde{Q}^\star\setminus Q^\star$, we have

$$T(f_1 - \widetilde{f}_1)(x) = \int_{\mathbb{R}^n} K(x,y)(f_1 - \widetilde{f}_1)(y)dy = \int_{\widetilde{Q}^\star\setminus Q^\star} K(x,y)f(y)dy,$$

so the difference computed in (4.4.4) now simply reads as

$$-\int_{\widetilde{Q}^\star \setminus Q^\star} K(0,y)f(y)dy,$$

which is constant and independent of x. Therefore, both definitions coincide in BMO, so we can take (4.4.3) as the proper definition of Tf for $f \in L^\infty$. In fact, by a similar argument as the one done in the proof of the previous theorem, we can show that

Theorem 4.4.3. *The results of Theorem 4.4.2 also hold if* $f \in L^\infty(\mathbb{R}^n)$.

4.4.2 The statement of the $T1$ theorem

Along this section we will consider operators T mapping the Schwartz class to the space of tempered distributions, i.e. $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. We will also assume that T is associated with a standard kernel K, meaning that for $f, g \in \mathcal{S}(\mathbb{R}^n)$ with disjoint compact supports, T is given by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)dxdy. \tag{4.4.5}$$

We will define its adjoint operator T^* as

$$\langle T^* f, g\rangle = \langle f, Tg\rangle \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

where we think f as a an element in $\mathcal{S}'(\mathbb{R}^n)$. T^* is associated with the standard kernel

$$K^*(x,y) = K(y,x).$$

We know that if T is bounded in $L^2(\mathbb{R}^n)$ it becomes a Calderón-Zygmund operator and so we can define Tf for $f \in L^\infty(\mathbb{R}^n)$ as an element in BMO. However, the boundedness condition in $L^2(\mathbb{R}^n)$ may not be satisfied *a priori*. Hence, we will give another argument that allows us to give meaning to Tf if $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$.

To do it we begin by fixing $g \in \mathcal{C}^\infty_{c,0}(\mathbb{R}^n)$, that, for us, will denote the space of smooth functions in $\mathbb{R}^n$ compactly supported and with null integral. Assume $\mathrm{supp}(g) \subset B(0,R)$ and choose also $\psi_1, \psi_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$ so that $\mathrm{supp}(\psi) \subset B(0,3R)$, $\psi = 1$ in $B(0,2R)$ and also $\psi_1 + \psi_2 = 1$.

Then, for any $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ we get $f\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and so $\langle T(f\psi_1), g\rangle$ is well-defined. In fact, if f had compact support, since $\psi_2 = 0$ in $\mathrm{supp}(g) \subset B(0,2R)$, we would have that $f\psi_2$ and g have disjoint supports and the following identity would hold

$$\langle T(f\psi_2), g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)\psi_2(y)g(x)dxdy.$$

However, as f may not be compactly supported, we define

$$\langle T(f\psi_2), g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [K(x,y) - K(0,y)]f(y)\psi_2(y)g(x)dxdy,$$

that generalizes the previous definition because g has null integral. This last expression makes sense, since the support of ψ_2 lies in $\mathbb{R}^n \setminus B(0,2R)$ and because of condition (4.3.5) on standard kernels, that reads

$$|K(x,y) - K(0,y)| \le C\frac{|x|^\delta}{|y|^{n+\delta}}, \qquad \text{if} \qquad |y| > 2|x|.$$

Indeed

$$|\langle T(f\psi_2), g\rangle| \leq C \int_{B(0,R)} \left(\int_{\mathbb{R}^n \setminus B(0,2R)} \frac{|f(y)|}{|y|^{n+\delta}} dy \right) |x|^\delta |g(x)| dx$$
$$\leq \|f\|_\infty \|x^\delta g\|_\infty |B(0,R)| |S^{n-1}| \left(\int_{2R}^\infty \frac{dr}{r^{1+\delta}} \right) < \infty.$$

Definition 4.4.3. We define Tf, for $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$, to be

$$\langle Tf, g\rangle = \langle T(f\psi_1), g\rangle + \langle T(f\psi_2), g\rangle, \qquad \forall g \in \mathcal{C}^\infty_{c,0}(\mathbb{R}^n).$$

Notice that the fact that g has null integral ensures that the definition does not depend on the choice of ψ_1 and ψ_2. To prove it choose ψ_1, ψ_2 and $\widetilde{\psi}_1, \widetilde{\psi}_2$. Since $\operatorname{supp}(g) \subset B(0,R)$ and the differences $\psi_1 - \widetilde{\psi}_1$ and $\psi_2 - \widetilde{\psi}_2$ are compactly supported in $B(0,3R) \setminus B(0,2R)$, we can apply the genuine definition (4.4.5) to justify that the expressions $\langle T(f\psi_1 - f\widetilde{\psi}_1), g\rangle$, $\langle T(f\psi_2 - f\widetilde{\psi}_2), g\rangle$ make sense. Moreover, since $1 = \psi_1 + \psi_2 = \widetilde{\psi}_1 + \widetilde{\psi}_2$, we deduce the equality $\langle T(f\psi_1 - f\widetilde{\psi}_1), g\rangle = \langle T(f\psi_2 - f\widetilde{\psi}_2), g\rangle$, and so indeed $\langle Tf, g\rangle$ is well-defined.

Let us present now two more definitions that are necessary to state the $T1$ theorem. First, for a given $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$, we will introduce what we will understand for $Tf \in BMO$; and second, we present a weaker condition than boundedness that will be sufficient to state the desired theorem.

Definition 4.4.4. Given $f \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$ we say that $Tf \in BMO$ if there exists a function $b \in BMO$ so that

$$\langle Tf, g\rangle = \langle b, g\rangle, \qquad \forall g \in \mathcal{C}^\infty_{c,0}(\mathbb{R}^n).$$

Notice that $\langle b, g\rangle$ is well-defined since $BMO \subset L^1_{\text{loc}}(\mathbb{R}^n)$ and the functions g have compact support.

Definition 4.4.5 (Weak boundedness property)**.** An operator T will have the *weak boundedness property* (WBP) if for every bounded subset $B \subset \mathcal{C}^\infty_c(\mathbb{R}^n)$, there exists a constant C_B so that for any $f_1, f_2 \in B$, $x \in \mathbb{R}^n$ and $R > 0$

$$\left|\langle Tf_1^{x,R}, f_2^{x,R}\rangle\right| \leq C_B R^n, \qquad \text{where} \quad f_j^{x,R}(y) = \varphi_j\left(\frac{y-x}{R}\right) \quad \text{for} \quad j = 1, 2.$$

Recall that a subset $B \subset \mathcal{C}^\infty_c(\mathbb{R}^n)$ is said to be bounded if for every functional $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)'$ we have $\sup\{|\langle \phi, f\rangle| : f \in B\} < \infty$.

Let us give an example of an operator that satisfies the weak boundedness property that will be important in the sequel.

Proposition 4.4.4. *Let K be a standard kernel which is anti-symmetric, i.e. $K(x,y) = -K(y,x)$; associated with the operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$ given by*

$$\langle Tf, g\rangle = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x,y) f(y) g(x) dy dx, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^n). \tag{4.4.6}$$

Then T is well-defined and has the WBP.

Proof. Observe that by the anti-symmetry, the mean value theorem and property(4.3.3)

$$
\begin{aligned}
|\langle Tf, g\rangle| &\leq \left|\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)\big[f(y)g(x) - f(x)g(y)\big]dydx\right| \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)||f(y)g(x) - f(y)g(y)|dydx \\
&\qquad\qquad\qquad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)||f(y)g(y) - f(x)g(y)|dydx \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(y)g'(\widetilde{x})|}{|x-y|^{n-1}}dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f'(\widetilde{y})g(x)|}{|x-y|^{n-1}}dydx,
\end{aligned}
$$

and if we apply Fubini's theorem and work in polar coordinates, it is clear that these last integrals are finite. Therefore, the integral defining $\langle Tf, g\rangle$ is absolutely convergent and hence well-defined.

Let us check now the WBP. Notice that for any bounded subset $B \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $f_1, f_2 \in B$ we have

$$
\begin{aligned}
\langle Tf_1^{z,R}, f_2^{z,R}\rangle &= \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x,y) f_1\left(\frac{y-z}{R}\right) f_2\left(\frac{x-z}{R}\right)dydx \\
&= \lim_{\varepsilon\to 0}\int_{|x-y|>\frac{\varepsilon}{R}} R^{2n} K(Ru+z, Rv+z) f_1(u) f_2(v)dudv.
\end{aligned}
$$

So $\langle Tf_1^{z,R}, f_2^{z,R}\rangle$ can be written in terms of a new kernel $T_{z,R} := R^n K(Ru+z, Rv+z)$, which is still standard and such that

$$
\langle Tf_1^{z,R}, f_2^{z,R}\rangle = R^n T_{z,R} f_1(f_2) < R^n \sup\{|\langle \phi, f\rangle| \,:\, f \in B\}.
$$

Hence, (4.4.6) satisfies the WBP. $\square$

We are now ready to state the $T1$ theorem:

Theorem 4.4.5. ($T1$ theorem, [7, Theorem 9.9])**.** *An operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, associated with a standard kernel K, extends to be a bounded operator on $L^2(\mathbb{R}^n)$ if and only if the following hold*

1. $T1 \in BMO$.
2. $T^*1 \in BMO$.
3. *T has the WBP.*

Observe that by Proposition 4.4.4, if K is an anti-symmetric standard kernel and T is its associated operator, since $T^* = -T$, we deduce that

$$
T \text{ will be bounded in } L^2(\mathbb{R}^n) \quad \Leftrightarrow \quad T1 \in BMO. \tag{4.4.7}
$$

Chapter 5

Returning to Denjoy's conjecture

We end our project by connecting the theory we have developed in the previous sections with Denjoy's conjecture 2.5.1, being able to provide a proof of it for a specific case. We recall its statement:

Conjecture (Denjoy)**.** If a rectifiable curve contains a compact set with positive length, then this compact set has positive analytic capacity (and thus it is not removable).

The way we will proceed will be the following: we will focus on one particular operator we have already presented, the Cauchy transform (Definition 2.3.1), restricted to the graph of a Lipschitz function, the rectifiable curve that will contain the compact set we are interested in. We will assume that the graph, as a closed subset of $\mathbb{R}^2$, is bounded, since we are interested in studying a compact set contained in it. We will see that the previous operator is associated with an anti-symmetric standard kernel and, in the end, the result that will yield the boundedness of the Cauchy transform will be relation 4.4.7.

Having proved the latter, to tackle Denjoy's conjecture we will have to present three different lemmas, one without proof and the remaining two fully covered. The final proof will turn out to be a rather direct consequence of the previous results.

5.1 L^2 boundedness of the Cauchy transform on Lipschitz graphs

Let A be a Lipschitz function on $[a,b] \subset \mathbb{R}$ (hence A' exists almost everywhere with $A' \in L^\infty(\mathbb{R})$) and let Γ be the graph of rectifiable curve given by $\gamma(t) = t + iA(t)$, for $t \in [a,b]$. We define for every $f \in L^1(\Gamma)$ (that is $f \circ \gamma \in L^1[a,b]$) its *Cauchy integral along* Γ as

$$\mathscr{C}_\Gamma f(z) = \int_a^b \frac{f(t+iA(t))}{t+iA(t)-z}(1+iA'(t))dt = \int_a^b \frac{f(\gamma(t))}{\gamma(t)-z}d\gamma(t), \qquad \forall z \in \mathbb{C} \setminus \Gamma,$$

which makes sense since $|\gamma(t) - z| > 0$. Observe that the previous definition, extending f by 0 to $\mathbb{C} \setminus \Gamma$ if necessary, can be rewritten as

$$\mathscr{C}\nu(z) = \int_{\mathbb{C}} \frac{1}{w-z}d\nu(w), \qquad \text{where} \qquad d\nu(w) = f(w)d\mathcal{H}^1|_\Gamma(w).$$

By the properties of $\mathscr{C}$, we already now that $\mathscr{C}\nu$ defines an holomorphic function in $\mathbb{C} \setminus \Gamma$ (Theorem 2.3.2). We must understand $d\mathcal{H}^1|_\Gamma$ as the usual length measure for a curve in $\mathbb{C}$.

Our goal will be to give meaning to the previous expression when $z \in \Gamma$. This will lead to consider the Cauchy transform of the length measure of a bounded Lipschitz graph, restricted to the latter as a subset of $\mathbb{C}$. The main result we will prove is that, under some additional conditions, this last operator is a bounded operator in $L^2(\Gamma)$. So the operator we are interested in is the following:

Definition 5.1.1 (Cauchy singular integral operator)**.** Let $A : [a, b] \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\|A'\|_\infty$, and let $\Gamma = \{\gamma(t) = t + iA(t) : t \in [a, b]\} \subset \mathbb{C}$ be its graph. For $f \in \mathcal{S}(\mathbb{R})$, the *Cauchy singular integral operator* (over the graph of A) is given by

$$\mathscr{C}_\Gamma f(z) = \lim_{\varepsilon \to 0} \int_{w \in \Gamma:\, |z-w|>\varepsilon} \frac{f(w)}{z-w} d\mathcal{H}^1|_\Gamma(w), \qquad \text{for } \mathcal{H}^1|_\Gamma\text{-a.e. } z \in \Gamma.$$

To prove that it can be well-defined in $L^2(\Gamma)$, we begin by checking that the kernel associated with $\mathscr{C}_\Gamma$, that is

$$K(z, w) := \frac{1}{z-w}, \qquad z, w \in \Gamma,$$

is standard. Observe also that it is anti-symmetric, i.e. $K(z, w) = -K(z, w)$. Expressing $z = \gamma(x), w = \gamma(y)$ for $x, y \in [a, b]$, condition 4.3.3, with $n = 1$, is deduced as a direct consequence of the Lipschitz hypothesis:

$$|K(z, w)| \leq \frac{1}{|x-y|} \left| 1 - i\frac{A(x) - A(y)}{x-y} \right|^{-1} \leq \frac{1}{|x-y|}.$$

Regarding (4.3.4), if $\xi = \gamma(s)$ and $|x - y| > 2|y - s|$ we have

$$\begin{aligned}
|K(z, w) - K(z, \xi)| &= \left| \frac{y - s + i\big(A(y) - A(s)\big)}{\big[x - y + i\big(A(x) - A(y)\big)\big]\big[x - s + i\big(A(x) - A(s)\big)\big]} \right| \\
&\lesssim \frac{|y-s|}{|x-y|^2} \left| 1 + i\frac{A(x) - A(y)}{x-y} \right|^{-1} \left| 1 + i\frac{A(x) - A(y)}{x-y} + \frac{y-s}{x-y}\left(1 + \frac{A(y) - A(s)}{y-s}\right) \right|^{-1} \\
&\leq \frac{|y-s|}{|x-y|^2} \left| 1 - \frac{|y-s|}{|x-y|}\big(1 + \|A'\|_\infty\big) \right|^{-1} \lesssim \frac{|y-s|}{|x-y|^2}.
\end{aligned}$$

Thus, the condition is satisfied for $\delta = 1$. In an analogous way we can prove that (4.3.5) also holds with $\delta = 1$. Therefore K is an standard kernel.

Now we continue by rewriting K so that its dependence on Γ becomes explicit. By this we mean that instead of working with the complex variables z and w, we use that $z = \gamma(x)$ and $w = \gamma(y)$ to obtain the equivalent expression

$$K(x, y) = K(\gamma(x), \gamma(y)) = \frac{1}{\gamma(x) - \gamma(y)} = \frac{1}{x - y - i\big(A(y) - A(x)\big)}, \qquad \forall x, y \in [a, b].$$

Notice that if we assume $\|A'\|_\infty < 1$ we may expand the kernel K as a geometric series

$$K(x, y) = \sum_{k=0}^{\infty} i^k K_k(x, y), \qquad \text{where} \qquad K_k(x, y) = \frac{1}{x-y}\left(\frac{A(x) - A(y)}{x-y}\right)^k.$$

Definition 5.1.2 (Calderón commutators)**.** If $\|A'\|_\infty < 1$, the following anti-symmetric kernels

$$K_k(x,y) = \frac{1}{x-y}\left(\frac{A(x)-A(y)}{x-y}\right)^k, \qquad k = 0, 1, 2\dots$$

are called *Calderón commutators*. In a similar way as for K, they are also a standard kernel with $\delta = 1$ and constants proportional to $\|A'\|_\infty^k$.

By the $T1$ theorem 4.4.5, the kernels K_k satisfy the following essential property:

Theorem 5.1.1. *Let $A : [a,b] \to \mathbb{C}$ be a Lipschitz function, $k \geq 0$ an integer and $\varepsilon > 0$. The linear functional $T_k : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$*

$$T_k f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K_k(x,y) f(y) dy$$

is bounded on $L^2(\mathbb{R})$ and there exists a constant $C > 0$ such that

$$\|T_k\| \leq C^k \|A'\|_\infty^k.$$

Proof. First we notice that by Proposition 4.4.4, $T_k f$ is well-defined as an element of $\mathcal{S}(\mathbb{R})'$ and so is also T_k as a map from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})'$. Moreover, since T satisfies the WBP, by the $T1$ theorem 4.4.5 it will suffice to prove that $T_k 1 \in BMO$ to obtain the $L^2(\mathbb{R})$-boundedness of this operator.

In fact, we will prove that there is $C > 0$ so that for all k

$$\|T_k 1\|_{BMO} \leq C^{k+1} \|A'\|_\infty^k. \tag{5.1.1}$$

Once we prove this, the second part of the statement will follow as a consequence of an argument used in the proof of the $T1$ theorem [7, §9.4] that implies that

> *the norm of T_k depends linearly on the constants of K_k as a standard kernel and the BMO norm of $T_k 1$.*

Then, using that the first constants are proportional to $\|A'\|_\infty^k$ as well as (5.1.1), we have

$$\|T_k\|_{L^\infty \to BMO} \leq (C' + C^{k+1}) \|A'\|_\infty^k \approx C^{k+1} \|A'\|_\infty^k, \tag{5.1.2}$$

so the second statement of the theorem follows. Hence we focus on proving (5.1.1) and we will do it by induction. The case $k = 0$ is straightforward, since T_0 is the Hilbert transform and so $T_0 = 0$. Now let us assume that the inequality holds for a fixed k and prove it for $k+1$. Integration by parts yields

$$T_{k+1} 1 = T_k A'.$$

(although formally it is clear, the details to make this statement rigorous become cumbersome. See [7, Corollary 9.12] for a justification of the previous equality). Thus, by Theorem 4.4.3

$$\|T_{k+1} 1\|_{BMO} = \|T_k A'\|_{BMO} \leq \|T_k\|_{L^\infty \to BMO} \|A'\|_\infty.$$

Also, from the proof of Theorem 4.4.2 and recalling that the constants from the estimates that ensure that K is a standard kernel are proportional to $\|A'\|_\infty$, as well as (5.1.2) (that holds for the case k by induction hypothesis), we have the estimate

$$\|T_k\|_{L^\infty \to BMO} \leq C_2\big(\|T_k\|_{L^2\to L^2} + C_1\|A'\|_\infty^k\big) \leq C_2(C_3C^{k+1} + C_1)\|A'\|_\infty^k.$$

Then, choosing C large enough so that $C_2(C_3C^{k+1}+C_1) \leq C^{k+2}$ we are done. □

Therefore, if $\|A'\|_\infty < 1$ we can expand the kernel K as a geometric series and by Theorem 5.1.1 we deduce that T_k is a Calderón-Zygmund operator with standard kernel $K_k(x,y)$.

Finally, setting $d\xi_\Gamma := d\mathcal{H}^1|_\Gamma(\xi)$ to ease the notation, we carry out the following computation for $f \in \mathcal{S}(\mathbb{R})$ (a priori)

$$\begin{aligned}
\|\mathscr{C}_\Gamma f\|_{L^2(\Gamma)} &= \left(\int_\Gamma |\mathscr{C}_\Gamma f(z)|^2 dz_\Gamma\right)^{1/2} = \left(\int_\Gamma \left|\lim_{\varepsilon\to 0}\int_{w\in\Gamma:\,|z-w|>\varepsilon} K(z,w)f(w)dw_\Gamma\right|^2 dz_\Gamma\right)^{1/2}\\
&= \left(\int_\Gamma \left|\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\sum_{k=0}^\infty i^k K_k(\gamma(x),\gamma(y))f(\gamma(y))dw_\Gamma(y)\right|^2 dz_\Gamma(x)\right)^{1/2}\\
&= \left(\int_\Gamma \left|\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\sum_{k=0}^\infty i^k K_k(x,y)f(\gamma(y))(1+iA'(y))dy\right|^2 dz_\Gamma(x)\right)^{1/2}\\
&= \left(\int_\Gamma \left|\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\lim_{N\to\infty}\sum_{k=0}^N i^k K_k(x,y)f(\gamma(y))(1+iA'(y))dy\right|^2 dz_\Gamma(x)\right)^{1/2}.
\end{aligned} \tag{5.1.3}$$

Observe that the product of functions $g_f := (f\circ\gamma)(1+iA')$ is bounded and hence belongs to $L^2[a,b]$. Notice that it can also be thought as a function in $L^2(\mathbb{R})$ once we multiply it by the characteristic function $\chi_{[a,b]}$. Let us define

$$\mathscr{C}_{\Gamma,N}f(x) := \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\sum_{k=0}^N i^k K_k(x,y)g_f(y)dy = \sum_{k=0}^N i^k T_k g_f(x),$$

where now, because of Theorem 5.1.1, we can assume $f \in L^2(\mathbb{R}) \subset L^2[a,b]$. Notice that if the following estimate holds

$$C\|A'\|_\infty < 1, \tag{5.1.4}$$

we deduce

$$\|\mathscr{C}_{\Gamma,N}f - \mathscr{C}_{\Gamma,M}f\|_{L^2[a,b]} \leq \sum_{k=M+1}^N \|T_k g_f\|_{L^2[a,b]} \leq \left(\sum_{k=M+1}^N C^k\|A'\|_\infty^k\right)\|g_f\|_{L^2[a,b]} \xrightarrow{N,M\to\infty} 0.$$

Therefore, by completeness, the sequence $(\mathscr{C}_{\Gamma,N})_N$ is convergent to an element of $\mathscr{B}(L^2[a,b])$ (the space of bounded linear operators defined from $L^2[a,b]$ to itself) as $N\to\infty$. This means, in particular, that for every $f\in L^2[a,b]$, the sequence $(\mathscr{C}_{\Gamma,N}f)_N$ is convergent to an element of $L^2[a,b]$, that we will call $\widetilde{\mathscr{C}_\Gamma} f$. Taking a subsequence that converges for almost every point, that abusing notation we will equally denote by $(\mathscr{C}_{\Gamma,N}f)_N$, we have

$$\widetilde{\mathscr{C}_\Gamma}f(x) = \lim_{N\to\infty}\mathscr{C}_{\Gamma,N}f(x) = \lim_{N\to\infty}\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\sum_{k=0}^N i^k K_k(x,y)g_f(y)dy, \qquad \text{for } \mathcal{L}\text{-a.e. } x.$$

On the other hand, by construction, if we take the limit with respect to N inside, we recover pointwise almost everywhere $\mathscr{C}_\Gamma f$ (see expression (5.1.3)). Then, $\widetilde{\mathscr{C}_\Gamma} f$ and $\mathscr{C}_\Gamma f$ must coincide pointwise $\mathcal{L}$-a.e. on $[a, b]$.

So if we now return to (5.1.3), we are able to take the limit with respect to N outside the inner integral and obtain

$$\begin{aligned}
\|\mathscr{C}_\Gamma f\|_{L^2(\Gamma)} &= \left(\int_\Gamma \left| \sum_{k=0}^{\infty} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} i^k K_k(x,y) f(\gamma(y)) dw_\Gamma(y) \right|^2 dz_\Gamma(x) \right)^{1/2} \\
&= \left(\int_\Gamma \left| \sum_{k=0}^{\infty} i^k T_k(f \circ \gamma)(x) \right|^2 dz_\Gamma(x) \right)^{1/2} \leq \sum_{k=0}^{\infty} \|T_k(f\circ\gamma)\|_{L^2[a,b]} \\
&\leq \left(\sum_{k=0}^{\infty} C^k \|A'\|_\infty^k \right) \|f \circ \gamma\|_{L^2[a,b]} = \left(\sum_{k=0}^{\infty} C^k \|A'\|_\infty^k \right) \|f\|_{L^2(\Gamma)}.
\end{aligned}$$

Therefore, the previous argument together with assumption (5.1.4) finally yield:

Corollary 5.1.2. *There exists $\delta > 0$ small enough so that if $\|A'\|_\infty \leq \delta$, the operator $\mathscr{C}_\Gamma$ is bounded in $L^2(\Gamma)$ (and so it becomes a Calderón-Zygmund operator).*

In other words, the previous corollary proves the L^2-boundedness of the Cauchy transform of the length measure of Lipschitz graphs with small enough slope.

Brief comments on the general result

The way we have studied the L^2-boundedness using the $T1$ theorem was first tackled by Calderón [3] (1977). Those who first proved that the restriction $\|A'\|_\infty < \varepsilon$ was not necessary were Coifman, McIntosh & Meyer [4] (1982). Later on, David [5] (1984) characterized the curves on which the Cauchy integral defines a bounded operator on L^2: they are precisely those such that any circle of radius r contains in its interior a piece of the curve of length, at most, Cr with $C > 0$ fixed. These curves are referred to as *Ahlfors-David curves*.

In any case, the result that generalizes the one we have proved using the Calderón-Zygmund theory is the following:

Theorem 5.1.3. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $\Gamma \subset \mathbb{R}^2$ its graph. Consider the measure $\nu = \mathcal{H}^1|_\Gamma$. Then, the Cauchy transform $\mathscr{C}\nu$ is bounded in L^2 and its norm does not exceed a constant depending only on $\|A'\|_\infty$.*

One can find an exhaustive proof of the previous result in Tolsa [27, Theorem 3.11], where more advanced tools are used in the argument, such as the Calderón-Zygmund theory for non-doubling measures or the Menger curvature of a measure.

5.2 The Denjoy's conjecture

The last goal of our project is to combine the techniques developed for singular integrals as well as for analytic capacity to obtain an idea of the proof of Denjoy's conjecture, whose statement was foreshadowed at the end of Chapter 2. Let us recall it, now as a theorem:

Theorem 5.2.1 (Denjoy)**.** *Let $\Gamma \subset \mathbb{C}$ be a rectifiable curve and $E \subset \Gamma$ compact. Then $\gamma(E) > 0$ if and only if $\mathcal{H}^1(E) > 0$.*

The proof of the previous result will rely heavily on the L^2-boundedness of the Cauchy transform on Lipschitz graphs. However, we will need a technical yet fundamental result that we will not prove. That is the reason we mentioned before that we will just provide an *idea* of the proof, since we will present the type of result needed to rigorously deduce the conjecture. Previous to its statement, let us recall first that a measure ν is said to have *linear growth* if for every disk $D_r(z)$ – with center z and radius r – it satisfies $\nu(D_r(z)) \lesssim r$. Bearing in mind this definition, the result we are interested in asserts the following:

Lemma 5.2.2. *Assume that a compact set $E \subset \mathbb{C}$ supports a non-zero Radon measure ν with linear growth and such that $\mathscr{C}\nu$ is bounded in L^2. Then $\gamma(E) > 0$.*

The proof of this lemma can be found in Tolsa [27, Remark 4.8] and it needs concepts that are far from the scope of this project, such as weak Lebesgue spaces or the Calderón-Zygmund theory for non-doubling measures.

Apart from this result, we will need two more geometric lemmas (easier to prove).

Lemma 5.2.3. *Let $F \subset \mathbb{C}$ be so that for any $z, w \in F$ there is a constant $C_F > 0$ satisfying*

$$\left| \frac{\mathfrak{Im}(z) - \mathfrak{Im}(w)}{\mathfrak{Re}(z) - \mathfrak{Re}(w)} \right| \leq C_F$$

Then F is contained in the graph a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$.

Proof. From the property defining F we observe that if $z, w \in F$ are such that $\mathfrak{Re}(z) = \mathfrak{Re}(w)$, then $z = w$. Hence, we may define the map

$$\begin{aligned} \widetilde{A} : \mathfrak{Re}(F) &\longrightarrow \mathfrak{Im}(F) \\ \mathfrak{Re}(z) &\longmapsto \mathfrak{Im}(z) \end{aligned}$$

It is clear, by definition, that $F \subset \{(x, A(x)) \,:\, x \in \mathfrak{Re}(F)\}$. Moreover, the defining property of F makes $\widetilde{A}$ a Lipschitz map with constant at most C_F. In fact, we can extend $\widetilde{A}$ to a Lipschitz map $A : \mathbb{R} \to \mathbb{R}$ by the formula

$$A(x) = \inf \left\{ \widetilde{A}(y) + C_F|x - y| \,:\, y \in \mathfrak{Re}(F) \right\}.$$

It is clear that if $x \in \mathfrak{Re}(F)$, then $A(x) = \widetilde{A}(x)$. Moreover, for any $\varepsilon > 0$, if $x_1, x_2 \in \mathbb{R}$ we can find $y_1, y_2 \in \mathfrak{Re}(F)$ such that

$$\big|A(x_1) - \widetilde{A}(y_1) - C_F|x_1 - y_1|\big| < \frac{\varepsilon}{2}, \qquad \big|A(x_2) - \widetilde{A}(y_2) - C_F|x_2 - y_2|\big| < \frac{\varepsilon}{2}.$$

Using this property, we check the Lipschitz condition on A for different cases depending on whether if $x_1, x_2 \in F$ or not. It is clear that if both belong to F, the condition is satisfied. If $x_1 \in F$ and $x_2 \notin F$,

$$
\begin{aligned}
|A(x_1) - A(x_2)| < \big|\widetilde{A}(x_1) - \widetilde{A}(y_2) - C_F|x_2 - y_2|\big| &\leq C_F\big||x_1 - y_2| - |x_2 - y_2|\big| \\
= C_F\big||x_1 - x_2 + x_2 - y_2| - |x_2 - y_2|\big| &\leq C_F|x_1 - x_2|
\end{aligned}
$$

and the argument for the case $x_1, x_2 \notin F$ is analogous to the previous one, concluding that A is still a Lipschitz map with constant at most C_F and with graph containing F. $\square$

Lemma 5.2.4. *Let Γ be a rectifiable curve and $E \subset \Gamma$ with $\mathcal{H}^1(E) > 0$. Then there exists a compact subset $F \subset E$ with $\mathcal{H}^1(F) > 0$ which is contained in the (possibly rotated) graph of a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$.*

Proof. Let $g : [a, b] \to \mathbb{C}$ be the arc length parametrization of Γ, i.e. $g([a, b]) = \Gamma$ with g differentiable $\mathcal{L}$-a.e. in (a, b) with $|g'(t)| = 1$. Observe that since $g' \in L^1[a, b]$, by Theorem 3.1.9 we have that for $\mathcal{L}$-a.e. $t_0 \in (a, b)$

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} |g'(t_0 - t) - g'(t_0)| dt = 0.$$

Therefore, fixing t_0 one of the previous points, we can find an interval $I_0 \ni t_0$ so that

$$\int_{I_0} |g'(t) - g'(t_0)| dt \leq \frac{1}{20} \mathcal{L}(I_0). \tag{5.2.1}$$

We will also assume, without loss of generality, that $g'(t_0) = 1$ (by a rotation if necessary). Now we consider the following set:

$$G := \left\{ t \in I_0 \, : \, \exists g'(t) \quad \text{and} \quad |g'(t) - 1| \leq \frac{1}{10} \right\},$$

The set G has positive $\mathcal{L}$-measure due to condition (5.2.1). Indeed, recalling that g' exists almost everywhere we have

$$
\begin{aligned}
\frac{\mathcal{L}(I_0)}{20} &\geq \int_{I_0} |g'(t) - g'(t_0)| dt = \int_G |g'(t) - g'(t_0)| dt + \int_{I_0 \setminus G} |g'(t) - 1| dt \\
&> \int_G |g'(t) - g'(t_0)| dt + \frac{\mathcal{L}(I_0 \setminus G)}{10}.
\end{aligned}
$$

so if G had null measure we would reach a contradiction. Now for each $m \geq 1$ we set

$$G_m = \left\{ t \in G \, : \, \left| \frac{g(s) - g(t)}{s - t} - g'(t) \right| \leq \frac{1}{10} \quad \text{if} \quad |s - t| \leq \frac{1}{m} \right\}.$$

Since by definition $\bigcup_{m \geq 1} G_m = G$, we deduce that for m big enough $\mathcal{L}(G_m) > 0$. For such value of m, we pick an interval $J \subset I_0$ of length $(2m)^{-1}$ so that $\mathcal{L}(J \cap G_m) > 0$. Now we observe that if $s, t \in J \cap G_m$, then

$$\left| \frac{g(s) - g(t)}{s - t} - 1 \right| \leq \left| \frac{g(s) - g(t)}{s - t} - g'(t) \right| + |g'(t) - 1| \leq \frac{1}{10} + \frac{1}{10} = \frac{1}{5}.$$

Observe that calling $F(s,t) := \frac{g(s)-g(t)}{s-t}$ we have obtained

$$\big|F(s,t)-1\big| = \sqrt{[\mathfrak{Re}(F(s,t))-1]^2 + \mathfrak{Im}(F(s,t))^2} \leq \frac{1}{5},$$

so geometrically (thinking the last expression as the equation of a circumference), we have

$$\left|\frac{\mathfrak{Re}(g(s)) - \mathfrak{Re}(g(t))}{s-t}\right| \geq \frac{4}{5}, \qquad \left|\frac{\mathfrak{Im}(g(s)) - \mathfrak{Im}(g(t))}{s-t}\right| \leq \frac{1}{5}.$$

Hence, we finally obtain that for any $s,t \in J \cap G_m$

$$\left|\frac{\mathfrak{Im}(g(s)) - \mathfrak{Im}(g(t))}{\mathfrak{Re}(g(s)) - \mathfrak{Re}(g(t))}\right| \leq \frac{1}{4}.$$

So applying Lemma 5.2.3 we get that $g(J \cap G_m)$ is contained in the graph of a Lipschitz function. Finally, taking a compact subset $F_0 \subset J \cap G_m$ with $\mathcal{L}(F_0) > 0$ and setting $F = g(F_0)$ we are done. $\square$

Proof (*Denjoy*). Recall that we already know the inequalities $\gamma(E) \leq \mathcal{H}^1_\infty(E) \leq \mathcal{H}^1(E)$, given by Painlevé's theorem (2.4.5), so we are left to prove that for $E \subset \Gamma$ with $\mathcal{H}^1(E) > 0$, we have $\gamma(E) > 0$. By Lemma 5.2.4 we know that there exists a compact subset $F \subset E$ with $\mathcal{H}^1(F) > 0$ contained in a (possibly rotated) Lipschitz graph. By Theorem 5.1.3, we know that the Cauchy transform is bounded in L^2 with respect to the arc length of this Lipschitz graph. But it will also be bounded with respect to the measure $\mathcal{H}^1|_F$, which is Radon (since it is Borel regular and locally finite [16, Corollary 1.11]) and we know that satisfies a condition of the type (2.4.1) with $s = 1$. Hence, we can apply Lemma 5.2.2, obtaining $\gamma(E) > 0$ and we are done. $\square$

5.3 And what about other types of compact subsets?

The characterization of removable compact subsets has led to different conjectures apart from the one we have presented. One of the most well-known is *Vitushkin's conjecture*, proposed by A. G. Vitushkin in 1967 [30], that asserts that removable subsets are those with null Favard length. To define this notion, denote by p_θ the orthogonal projection onto the line through the origin at angle θ to the positive x-axis. Then, the *Favard length* of a compact subset $E \subset \mathbb{C}$ is the average length of its projections over all directions:

$$\mathrm{Fav}(E) := \int_0^\pi \mathcal{H}^1(p_\theta(E))d\theta$$

The conjecture is $\gamma(E) = 0 \;\Leftrightarrow\; \mathrm{Fav}(E) = 0$. In 1986 Mattila [15] proved this conjecture *wrong* by showing that having positive Favard length is not invariant under conformal mappings, while removability for bounded holomorphic functions is. Nevertheless, although the conjecture may not hold in full generality, it turns out to be true in the particular case where E has finite length. This was proved by David in 1998 [6], where he showed that a compact set $E \subset \mathbb{C}$ with finite length is removable if and only if is purely unrectifiable, i.e. if it intersects any rectifiable curve at most in a $\mathcal{H}^1$-null set. And applying a theorem of Besicovitch (see [16, Chapter 18] for more details) we deduce the result, since for subsets of finite length, being purely unrectifiable becomes equivalent to having null Favard length.

Appendix A

The fundamental solution of the $\overline{\partial}$-equation.

A.1 The Cauchy-Pompeiu formula

A well known basic result in complex analysis is Cauchy's integral formula, that says that if f is holomorphic on Ω, a bounded domain with piecewise regular boundary and positively oriented, then for $z \in \Omega$ we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(w)}{w-z}dw.$$

The formula we are about to present generalizes the previous integral representation for a bigger class of functions:

Theorem A.1.1 (Cauchy-Pompeiu)**.** *Let Ω be a bounded domain with piece-wise regular boundary and positively oriented. Then, if $f \in \mathcal{C}^1(\overline{\Omega})$, we have for every $z \in \Omega$*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(w)}{w-z}dw - \frac{1}{\pi}\int_{\Omega}\frac{\overline{\partial}f(w)}{w-z}d\mathcal{L}^2(w).$$

Proof. Consider $0 < \varepsilon < d(z,\partial\Omega)$ and define $\Omega_\varepsilon := \Omega \setminus D(z,\varepsilon)$. We name also $g(w) := \frac{f(w)}{w-z}$, which belongs to $\mathcal{C}^1(\overline{\Omega_\varepsilon})$. Since the boundary of Ω satisfies the necessary conditions to apply Stoke's theorem, we can apply it to obtain

$$\begin{aligned}
\int_{\partial\Omega_\varepsilon} g(w)dw &= \int_{\Omega_\varepsilon} d\big(g(w)dw\big) = \int_{\Omega_\varepsilon} dg\wedge dw = \int_{\Omega_\varepsilon}\big(\partial g dw + \overline{\partial}g d\overline{w}\big)\wedge dw \\
&= \int_{\Omega_\varepsilon}\overline{\partial}g d\overline{w}\wedge dw = \int_{\Omega_\varepsilon}\left[\frac{\overline{\partial}f(w)}{w-z} + f(w)\cancelto{0}{\overline{\partial}\left(\frac{1}{w-z}\right)}\right]d\overline{w}\wedge dw \\
&= \int_{\Omega_\varepsilon}\frac{\overline{\partial}f(w)}{w-z}(dx+idy)\wedge(dx-idy) = 2i\int_{\Omega_\varepsilon}\frac{\overline{\partial}f(w)}{w-z}d\mathcal{L}^2(w) \\
&= 2i\int_{\Omega}\frac{\overline{\partial}f(w)}{w-z}\chi_{\Omega_\varepsilon}(w)d\mathcal{L}^2(w).
\end{aligned}$$

Now recalling that $f \in \mathcal{C}^1(\overline{\Omega})$ we deduce, in particular, that $|\overline{\partial} f|$ is bounded by some constant $C > 0$ in Ω. Therefore

$$\left|2i \int_\Omega \frac{\overline{\partial} f(w)}{w-z} \chi_{\Omega_\varepsilon}(w) d\mathcal{L}^2(w)\right| \leq 2C \int_\Omega \frac{d\mathcal{L}^2(w)}{|w-z|} = 2C\left[\int_{\Omega_\varepsilon} \frac{d\mathcal{L}^2(w)}{|w-z|} + \int_{\overline{D(z,\varepsilon)}} \frac{d\mathcal{L}^2(w)}{|w-z|}\right]$$
$$\leq 2C\left[\frac{\mathcal{L}^2(\Omega_\varepsilon)}{\varepsilon} + 2\pi\varepsilon\right] < \infty,$$

since Ω is bounded. Hence, by the dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} g(w) dw = 2i \int_\Omega \frac{\overline{\partial} f(w)}{w-z} d\mathcal{L}^2(w).$$

On the other hand, if we do not apply Stokes theorem, we also have the identities

$$\int_{\partial\Omega_\varepsilon} g(w)dw = \int_{\partial\Omega} g(w)dw + \int_{\partial\overline{D(z,\varepsilon)}} g(w)dw = \int_{\partial\Omega} \frac{f(w)}{w-z}dw + \int_{\partial\overline{D(z,\varepsilon)}} \frac{f(w)}{w-z}dw.$$

The second integral can be rewritten, using polar coordinates, as follows

$$\int_{\partial\overline{D(z,\varepsilon)}} \frac{f(w)}{w-z}dw = \int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} d\theta = i\int_0^{2\pi} f(z+\varepsilon e^{i\theta})d\theta.$$

Using again the dominated convergence theorem we obtain that the integral converges to $2\pi i f(z)$ as $\varepsilon \to 0$, and equating both equivalent expressions obtained for $\int_{\partial\Omega_\varepsilon} g(w)dw$ once we have taken the limit as $\varepsilon \to 0$, we deduce the desired result. □

Corollary A.1.2. *If $f \in \mathcal{C}_c^1(\mathbb{C})$ (compactly supported continuously differentiable function on the whole $\mathbb{C}$), then for each $z \in \mathbb{C}$*

$$f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\overline{\partial} f(w)}{w-z} d\mathcal{L}^2(w).$$

A.2 The fundamental solution of the $\overline{\partial}$-equation

One of the most important consequences of the Cauchy-Pompeiu formula has to do with the fundamental solution of the $\overline{\partial}$-equation. By this we mean the following: suppose we are asked to find a certain function f (preferably compactly supported and continuously differentiable up to a certain order of derivatives) that satisfies

$$\overline{\partial}(f) = \delta_0,$$

where the equality is in the sense of distributions. To that end, notice that if $f \in \mathcal{C}_c^1(\mathbb{C})$ we have, by the previous corollary

$$f(0) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial} f(w)}{w} d\mathcal{L}^2(w) \quad \Leftrightarrow \quad \delta_0(f) = \Big(-\frac{1}{\pi z}\Big)(\overline{\partial} f) = \Big[\overline{\partial}\Big(-\frac{1}{\pi z}\Big)\Big](f)$$

implying that

Theorem A.2.1. *The fundamental solution of the $\overline{\partial}$-equation is $-\frac{1}{\pi z}$. That is,*

$$\delta_0 = \overline{\partial}\Big(-\frac{1}{\pi z}\Big).$$

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