UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Weighted Fourier inequalities and Uncertainty Principle Relations

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Introduction and Summary

The main actor of this project is the Fourier transform, which, for f integrable is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \xi} dx.$$

This object, and its discrete counterpart, the Fourier series, are extremely important, not only in Mathematics but also in Physics and Engineering. In spite of its simple definition, the relation between f and $\hat{f}$ is not at all easy to understand. To this end, we can use norm inequalities, like the Hausdorff-Young inequality, which, for $1 \leq p \leq 2$ states that

$$\left\| \hat{f} \right\|_{p^*} \leq K \left\| f \right\|_p \tag{1}$$

or Uncertainty Principle relations, which, roughly speaking, assert that the Fourier transform of a localized function is not localized. The most famous of these is the Heisenberg Uncertainty Principle:

$$\left\| x f \right\|_2 \left\| \xi \hat{f} \right\|_2 \geq K \left\| f \right\|_2^2 . \tag{2}$$

From a qualitative point of view, these inequalities tell us that that the Fourier transform of an integrable function can not have important blow-ups and that the transform of a concentrated function can not be concentrated around one point.

The goal of this thesis is to present generalizations of the aforementioned inequalities. First, to obtain relations between the distribution of f and $\hat{f}$ in their domain, we study Weighted Fourier inequalities, that is, inequalities of the form

$$\left\| \hat{f} \right\|_{q,u} \leq K \left\| f \right\|_{p,v} , \tag{3}$$

where u, v are weights, that is, non-negative measurable functions. Observe that inequality (3) is clearly a generalization of (1).

Second, there are many ways in which the idea behind the Uncertainty Principle, that is, transforms of localized functions must be spread over their domain, can be quantified. For instance, we can measure the degree of localization of a function by studying its rate of decay, by computing the fraction of its mass which lies outside of some region, or by studying generalizations of inequality (2), namely,

$$\left\| f \right\|_{p,u} \left\| \hat{f} \right\|_{q,v} \geq K \left\| f \right\|_r . \tag{4}$$

The work is devoted to surveying known results and obtaining new ones on the previously mentioned problems. The thesis is organized as follows. Chapter 1 is devoted to introducing some preliminary results and concepts which are used in this work. In Chapter 2 we review the conditions on u and v obtained in [4] which guarantee that inequality (3) holds. Next, we review classical necessary conditions and obtain new ones (Theorems 2.3.6 and 2.3.7), thereby showing that the conditions in [4] are necessary when u and v satisfy a natural monotonicity condition. To conclude this chapter, we further explore the topic by studying the case of non-monotonous u, v and obtain new results. Finally, in Chapter 3 we survey several forms of the Uncertainty Principle (UP): the Hardy UP, the Amrein-Berthier UP and the Nazarov UP. We also study UP of the type (4), extending the results obtained in [28] for the whole range of parameters, (see

Theorem 3.4.2). Moreover, we fully characterize a symmetric Heisenberg type UP with broken power weights, see Theorem 3.5.1.

Finally, I would like to thank my supervisor, Sergey Tikhonov, for his guidance and also Kristina Oganesyan for numerous helpful remarks.

Contents

Chapter 1

Preliminaries

1.1 Notation

Throughout the whole text we shall make use of the following notation and terminology

- A weight is a non-negative measurable function;
- For a weight v, a function f and $p > 0$,

$$\|f\|_{p,v} := \left(\int_{\mathbb{R}} |f|^p v\right)^{\frac{1}{p}};$$

- L^p_v is the weighted Lebesgue space of measurable functions f for which $\|f\|_{p,v} < \infty$;
- $p^* = \frac{p}{p-1}$;
- $\mathbb{1}_E$ is the indicator function of the measurable set E.

Finally, we say that two expressions F_1 and F_2 are *equivalent* (we write $F_1 \approx F_2$) if there exists a constant $K > 0$ only dependent on p and q such that $K^{-1}F_1 \leq F_2 \leq KF_1$.

1.2 Some auxiliary inequalities

The first result is the well-known Chebyshev inequality

Lemma 1.2.1. *Let $f \geq 0$ be measurable, then for $\lambda > 0$,*

$$|\{x : f(x) > \lambda\}| \, \lambda \leq \|f\|_1 .$$

The following two results are classical inequalities, see for instance [6] [13], [18] or [19].

Lemma 1.2.2 (Hardy's Lemma)**.** *Let a, b be two non-negative functions and assume that for any $x > 0$*

$$\int_0^x a(s)ds \leq \int_0^x b(s)ds.$$

Then, for any non-increasing function f,

$$\int_0^\infty f(s)a(s)ds \leq \int_0^\infty f(s)b(s)ds.$$

Theorem 1.2.3 (Continuous Hardy's Inequality)**.** *Let $1 < p, q < \infty$ and let u, v be weights. Then, the inequality*

$$\left(\int_0^\infty u(x) \left(\int_0^x f(s)ds\right)^q dx\right)^{\frac{1}{q}} \leq K \left\|f\right\|_{p,v} \tag{1.1}$$

holds for any f if and only if $K < \infty$, where

1. *if $q \geq p$,*
$$K \approx \sup_{s>0} \left(\int_s^\infty u\right)^{\frac{1}{q}} \left(\int_0^s v^{1-p^*}\right)^{\frac{1}{p^*}}; \tag{1.2}$$

2. *if $q < p$,*
$$K \approx \left(\int_0^\infty v(x)^{1-p^*} \left(\int_x^\infty u\right)^{\frac{r}{q}} \left(\int_0^x v^{1-p^*}\right)^{\frac{r}{q^*}} dx\right)^{\frac{1}{r}}, \tag{1.3}$$
with $r^{-1} = q^{-1} - p^{-1}$.

Theorem 1.2.4 (Discrete Hardy inequality)**.**

1. *Let $r > 1$ and $p > 1$. Then, there exists a constant $K(r,p)$ such that for any non-negative sequence $(x_n)_{n\in\mathbb{Z}}$ and $p > 1$*
$$\sum_{n=-\infty}^{\infty} r^n x_n^p \leq \sum_{n=-\infty}^{\infty} r^n \left(\sum_{j=n}^{\infty} x_j\right)^p \leq K(r,p) \sum_{n=-\infty}^{\infty} r^n x_n^p.$$

2. *Assume that $q < 1$ and let u, v be weights with v non-increasing. Then, the inequality*
$$\left(\sum_{n=-\infty}^{\infty} u_n \left(\sum_{j=-\infty}^{n} x_j\right)^q\right)^{\frac{1}{q}} \leq K \sum_{n=-\infty}^{\infty} v_n x_n$$
holds for any non-negative x_n if and only if
$$\left(\sum_{n=-\infty}^{\infty} u_n \left(\frac{\sum_{j=n}^{\infty} u_j}{v_n}\right)^{-q^*}\right)^{-\frac{1}{q^*}} < \infty.$$
Moreover, the best constant K is equivalent to the previous expression.

We also need the following result about the boundedness of the Riesz potential:

Theorem 1.2.5 (Theorem 1 in [25])**.** *Let u be a non-negative measurable function. If for some $r > 1$, the supremum over all intervals I satisfies*

$$\sup_I |I|^\alpha \left(|I|^{-1} \int_I u^r\right)^{\frac{1}{2r}} < \infty,$$

then, there exists K such that for any g

$$\left(\int_{\mathbb{R}} u(x) \left(\int_{\mathbb{R}} \frac{g(y)}{|x-y|^{1-\alpha}} dy\right)^2 dx\right)^{\frac{1}{2}} \leq K \left\|g\right\|_2.$$

1.3 Decreasing rearrangement function

In this work we will need some facts about the rearrangement function. The interested reader can learn more about this function in [6].

Definition 1.3.1. *Let (X, μ) be a measure space and let $f : X \to \mathbb{C}$ be a measurable function. The distribution function of f is given by*

$$D_f(s) = \mu \{x \in X : |f(x)| > s\},$$

and the decreasing rearrangement of f, by

$$f^*(t) = \inf\{s \geq 0 : D_f(s) \leq t\}.$$

Proposition 1.3.2. *The following properties hold:*

1. *D_f and f^* are non-increasing;*
2. *if $(t_n)_{n=1}^\infty \subset \mathbb{R}_+$ is a decreasing sequence with limit s, then $D_f(s) = \lim_n D_f(t_n)$;*
3. *$D_f(s) = D_{f^*}(s)$;*
4. *for any $p > 0$,*
$$\int_X |f(s)|^p d\mu(s) = p \int_0^\infty s^{p-1} D_f(s) ds = \int_0^\infty f^*(t)^p dt;$$
5. *for any $p > 0$*
$$(f^p)^* = (f^*)^p.$$

Proof. (1) and (5) are clear. (2) is the Monotone Convergence Theorem for measures. To prove (3), unpacking definitions, we obtain

$$D_{f^*}(s) = |\{x \geq 0 : f^*(x) > s\}| = |\{x \geq 0 : \inf\{t \geq 0 : D_f(t) \leq x\} > s\}|.$$

Next, we show that

$$\{x \geq 0 : \inf\{t \geq 0 : D_f(t) \leq x\} > s\} = \{x \geq 0 : D_f(s) > x\} = [0, D_f(s)),$$

whence the result follows. Clearly,

$$\{x \geq 0 : \inf\{t \geq 0 : D_f(t) \leq x\} > s\} \subset \{x \geq 0 : D_f(s) > x\};$$

for the reverse inclusion, for the sake of contradiction assume that for some x with $D_f(s) > x$, $\inf\{t \geq 0 : D_f(t) \leq x\} = s$. Then, there exists a decreasing sequence $(t_n)_{n=1}^\infty$ such that $\lim_n t_n = s$ and $D_f(t_n) \leq x < D_f(s)$, which contradicts (2).
For (4) an application of Fubini's Theorem yields

$$\int_X |f(s)|^p d\mu(s) = p \int_X \int_0^{|f(s)|} x^{p-1} dx d\mu(s) = p \int_0^\infty \mu\{s : |f(s)| > x\} x^{p-1} dx$$
$$= p \int_0^\infty D_f(x) x^{p-1} ds,$$

and the result follows by noting that by (3), f and f^* have the same distribution function. □

Lemma 1.3.3 (Hardy-Littlewood rearrangament, [4], [6])**.** *Let f, g be non-negative functions. Then, the following inequalities hold*

$$\int_0^\infty f^* \left[(1/g)^*\right]^{-1} \leq \int_X fg \leq \int_0^\infty f^* g^*.$$

1.4 The Khintchine inequality

The last preliminary result is the Khintchine inequality.

Theorem 1.4.1 ([1]). *For any $p > 0$, there exist $0 < A_p, B_p < \infty >$ such that for all n and all $a_1, \dots, a_n \in \mathbb{C}$,*

$$A_p \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}} \le \left(\mathbb{E}\left[\left| \sum_{j=1}^{n} a_j \varepsilon_j \right|^p \right] \right)^{\frac{1}{p}} \le B_p \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}, \tag{1.4}$$

where $\varepsilon_1, \dots, \varepsilon_n$ are independent random variables which take the values ± 1 which probability $\frac{1}{2}$. We remark that, here

$$\mathbb{E}\left[\left| \sum_{j=1}^{n} a_j \varepsilon_j \right|^p \right] = 2^{-n} \sum_{\varepsilon_1 = \pm 1, \dots, \varepsilon_n = \pm 1} \left| \sum_{j=1}^{n} a_j \varepsilon_j \right|^p.$$

Proof. Before we begin, note that by considering separately the real and the imaginary part, it suffices to prove the theorem for $a_k \in \mathbb{R}$. First, assume that $p > 2$. In this case, the LHS is follows from Hölder's inequality by noting that

$$\left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\left| \sum_{j=1}^{n} a_j \varepsilon_j \right|^2 \right] \right)^{\frac{1}{2}}.$$

For the RHS, put $S_n = \sum_{j=1}^{n} a_j \varepsilon_j$. Then, for $\lambda > 0$ and using the independence of the ε_i, we obtain

$$\mathbb{E}[e^{\lambda S_n}] = \prod_{j=1}^{n} \mathbb{E}[e^{\lambda a_j \varepsilon_i}] = \prod_{j=1}^{n} \cosh(\lambda a_j).$$

Next, expanding in power series, and using that $(2i)! > 2^i i!$

$$\cosh(\lambda a_j) = \sum_{i=0}^{\infty} \frac{(\lambda a_j)^{2i}}{(2i)!} \le \sum_{i=0}^{\infty} \frac{(\lambda a_j)^{2i}}{2^i i!} = e^{\frac{\lambda^2 a_j^2}{2}}.$$

Thus, from Chebyshev's inequality,

$$\mathbb{P}(S_n > t) e^{\lambda t} \le \mathbb{E}[e^{\lambda S_n}] \le e^{\frac{\lambda^2 \|a\|_2^2}{2}}.$$

Here, setting $\lambda = \frac{t}{\|a\|_2^2}$, we get,

$$\mathbb{P}(S_n > t) \le e^{-\frac{t^2}{2\|a\|_2^2}}.$$

Finally, using that S_n is symmetric, we deduce that

$$\mathbb{P}(|S_n| > t) \le 2e^{-\frac{t^2}{2\|a\|_2^2}}$$

and

$$\mathbb{E}[|S_n|^p] = p \int_0^{\infty} t^{p-1} \mathbb{P}(|S_n| > t) dt \le 2p \int_0^{\infty} t^{p-1} e^{-\frac{t^2}{2\|a\|_2^2}} dt = \|a\|_2^p \, 2p \int_0^{\infty} t^{p-1} e^{-\frac{t^2}{2}} =: B_p^p \, \|a\|_2^p.$$

Second, if $p < 2$, the RHS is trivial. For the LHS, by Hölder's inequality,

$$\|a\|_2^2 = \mathbb{E}[|S_n|^2] \leq (\mathbb{E}[|S_n|^p])^{\frac{1}{p}} \left(\mathbb{E}[|S_n|^{p^*}]\right)^{\frac{1}{p^*}} \leq (\mathbb{E}[|S_n|^p])^{\frac{1}{p}} B_{p^*} \|a\|_2 ,$$

so that the result follows with $A_p = B_{p^*}^{-1}$. □

This result allows us to construct various counterexamples in harmonic analysis. For instance,

Corollary 1.4.2. *The Hausdorff-Young inequality*

$$\left\|\hat{f}\right\|_{p^*} \lesssim \|f\|_p ,$$

does not hold for $p > 2$.

Proof. Let ϕ be a smooth function with support in $[-\frac{1}{2}, \frac{1}{2}]$. Let $a_1, \ldots, a_n \in \mathbb{C}$ and consider

$$f(x) = \sum_{j=1}^{n} a_j \phi(x - j).$$

Clearly,

$$\hat{f}(\xi) = \hat{\phi}(\xi) \sum_{j=1}^{n} e^{2\pi i \xi j} a_j;$$

$$\|f\|_p^p = \|\phi\|_p^p \sum_{j=1}^{n} |a_j|^p;$$

$$\left\|\hat{f}\right\|_{p^*}^{p^*} = \int_{\mathbb{R}} |\hat{\phi}(\xi)|^{p^*} |\sum_{j=1}^{n} a_j e^{2\pi i j \xi}|^{p^*}.$$

Hence, if the Hausdorff-Young inequality holds, for any choice of signs ε_j,

$$\int_{\mathbb{R}} |\hat{\phi}(\xi)|^{p^*} |\sum_{j=1}^{n} \varepsilon_j a_j e^{2\pi i j \xi}|^{p^*} d\xi \lesssim \left(\|\phi\|_p^p \sum_{j=1}^{n} |a_j|^p\right)^{\frac{p^*}{p}} ,$$

and by taking expected values, applying Fubini's Theorem we deduce that

$$\int_{\mathbb{R}} |\hat{\phi}(\xi)|^{p^*} \mathbb{E}\left[|\sum_{j=1}^{n} \varepsilon_j a_j e^{2\pi i j \xi}|^{p^*}\right] d\xi \lesssim \left(\|\phi\|_p^p \sum_{j=1}^{n} |a_j|^p\right)^{\frac{p^*}{p}} ,$$

so that using the Khintchine inequality, we deduce that

$$\|a\|_2^{\frac{p^*}{2}} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^{p^*} \lesssim \left(\|\phi\|_p^p \sum_{j=1}^{n} |a_j|^p\right)^{\frac{p^*}{p}} .$$

Taking $a_j = 1$, we obtain that the LHS behaves like $N^{\frac{p^*}{2}}$ and the RHS like $N^{\frac{p^*}{p}}$, which is a contradiction since $p > 2$. □

Chapter 2

Weighted Fourier inequalities

Given an integrable function f, its Fourier transform $\hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \xi} dx.$$

It is well known that for f integrable, we have that

$$\left\| \hat{f} \right\|_{\infty} \leq \|f\|_1 . \tag{2.1}$$

Moreover, if f is also square-integrable, the Parseval identity,

$$\left\| \hat{f} \right\|_2 = \|f\|_2 \tag{2.2}$$

holds. Since $L^1 \cap L^2$ is a dense subspace of L^2, we can extend by continuity the Fourier transform to the whole L^2. In this case, for $f \in L^2$, it is true that $\hat{f} \in L^2$ and the Fourier inversion formula, $\hat{\hat{f}}(x) = f(-x)$, holds.
In order to define the Fourier transform in other spaces by using the same method, for instance, between the weighted Lebesgue spaces L^p_v and L^q_u, one needs to first establish a *Pitt inequality* (after Pitt, who obtained analogous inequalities for Fourier Series and power weights in [23], see also [26]), that is, an inequality of the form

$$\left\| \hat{f} \right\|_{q,u} \leq K \|f\|_{p,v} , \tag{2.3}$$

for f belonging to a suitable dense subspace of L^p_v, what motivates the study of (2.3), the main object of this chapter. It must also be mentioned that the Pitt inequality is a crucial tool in the study of Uncertainty Principles, see also [11], the theme of the third chapter of this work, as well as in many other areas of Analysis and PDE's, see [3], [8] and [9].

This chapter is divided in four sections. In the first one, we explain the method of rearrangements due to Heinig and Benedetto ([4]), in which sufficient conditions on u, v in terms of their non-increasing rearrangements for the inequality (2.3) to hold are obtained. It is interesting to note that the only properties of the Fourier transform which are used are equations (2.1) and (2.2). This, in particular, implies that any inequality obtained by this method must also be satisfied by any rearrangement of the Fourier

transform. Therefore, it is not expected that, in general, sufficient conditions thus obtained are necessary. However, when the weights satisfy some monotonicity conditions, it is possible to show that these conditions are indeed necessary. In the second section, we comment briefly on classical necessary conditions and obtain new ones, thereby showing that the conditions in [4] are sufficient and necessary for monotonous weights.
The third section illustrates what is lost by only using the aforementioned boundedness properties of the Fourier transform. We obtain instances in which the sufficient conditions of [4] do not hold, yet for which (2.3) is satisfied.

2.1 Definition and duality

As mentioned before, inequality (2.3) is needed to define the Fourier Transform as an operator from L^p_v to L^q_v. More precisely,

Proposition 2.1.1. *Assume that for given p, q and non-negative u, v the inequality* (2.3) *holds for $f \in L^1$. Then, there exists a unique bounded linear operator which extends the Fourier transform as an operator from L^p_v to L^q_u, i.e, there exists a linear bounded operator $T : L^p_v \to L^q_u$ such that $T(f) = \hat{f}$ whenever $f \in L^1 \cap L^p_v$.*

Proof. We show that $L^1 \cap L^p_v$ is dense in L^p_v, whence the result will follow. To this end, let f be such that $\|f\|_{p,v} < \infty$ and, for each N, define

$$f_N(x) = f(x)\mathbb{1}_{|x|\leq N}\mathbb{1}_{|f(x)|\leq N}.$$

Clearly, $f_N \in L^1$ and, by the Dominated Convergence Theorem, $\lim_{N\to\infty} \|f - f_N\|_{p,v} = 0$.

Lemma 2.1.2 (Duality)**.** *Inequality* (2.3) *holds for any $f \in L^1$ (and a posteriori for any $f \in L^p_v$) if and only if, for any $\phi \in L^1$ (and a posteriori for any $\phi \in L^{q^*}_{u^{\frac{1}{1-q}}}$) the following holds*

$$\left\|\hat{\phi}\right\|_{p^*,v^{\frac{1}{1-p}}} \leq K \|\phi\|_{q^*,u^{\frac{1}{1-q}}}. \tag{2.4}$$

Observe that the previous statement can be rephrased in a more compact way as

$$\sup_f \frac{\left\|\hat{f}\right\|_{q,u}}{\|f\|_{p,v}} = \sup_\phi \frac{\left\|\hat{\phi}\right\|_{p^*,v^{\frac{1}{1-p}}}}{\|\phi\|_{q^*,u^{\frac{1}{1-q}}}}.$$

Proof. First, observe that if we let $V = v^{\frac{1}{1-p}}$ and $U = u^{\frac{1}{1-q}}$, we have that $V^{\frac{1}{1-p^*}} = v$ and $U^{\frac{1}{1-q^*}} = U$, so applying twice the transformation takes us back to the original inequality. Hence, it suffices to prove one implication.
Assume that inequality (2.4) holds and that u is finite everywhere. Let

$$A_N = \{\phi \in L^{q^*} : \|\phi\|_{q^*} = 1, \left\|\phi u^{\frac{1}{q}}\right\|_1 \leq N\}.$$

Then, using Hölder's inequality, we obtain that for $f \in L^1$, and since u is finite everywhere,

$$\int_{\mathbb{R}} |\hat{f}|^q u = \sup_{\phi\in L_{q^*};\|\phi\|_{q^*}=1} \int_{\mathbb{R}} \hat{f} u^{\frac{1}{q}}\phi = \sup_{N,\phi\in A_N} \int_{\mathbb{R}} \hat{f} u^{\frac{1}{q}}\phi =: I \tag{2.5}$$

Here, since for each N, f and $u^{\frac{1}{q}}\phi$ belong to L^1, an application of Fubini's Theorem and then Hölder's inequality yield

$$I = \sup_{N,\phi\in A_N} \int_{\mathbb{R}} f(u^{\frac{1}{q}}\phi)^\wedge \leq \sup_{N,\phi\in A_N} \|f\|_{p,v} \left\|(u^{\frac{1}{q}}\phi)^\wedge\right\|_{p^*,v^{\frac{1}{1-p}}}. \tag{2.6}$$

Finally, the assumption implies that

$$\sup_{N,\phi\in A_N} \|f\|_{p,v} \left\|(u^{\frac{1}{q}}\phi)^\wedge\right\|_{p^*,v^{\frac{1}{1-p}}} \leq K \sup_{N,\phi\in A_N} \|f\|_{p,v} \left\|(u^{\frac{1}{q}}\phi)\right\|_{q^*,u^{\frac{1}{1-q}}} = K \sup_{N,\phi\in A_N} \|f\|_{p,v} \|\phi\|_{q^*},$$

and since $\|\phi\|_{q^*} = 1$ we obtain that (2.3) holds.
For general u, let $u_N = \min(N, u)$. Since $u \geq u_N$, we have that inequality (2.4) holds for u_N. Thus, for $f \in L^1$ we have

$$\left\|\hat{f}\right\|_{q,u_N} \leq K \|f\|_{p,v},$$

and the result follows by applying the Monotone Convergence Theorem. □

2.2 Sufficient conditions

In this section we describe the method used in [4] to obtain conditions on u, v for (2.3) to hold. The main tool is the following Calderón-type result, which was obtained in [16].

Lemma 2.2.1. *Let T be a linear operator of type $(1,\infty)$ and $(2,2)$ with norm ≤ 1. That is, for any $f \in L^1$,*

$$\|Tf\|_\infty \leq \|f\|_1$$

and for any $f \in L^2$,

$$\|Tf\|_2 \leq \|f\|_2.$$

Then, for any $f \in L^1 + L^2$ and any $x \geq 0$,

$$\int_0^x (Tf)^*(t)^2 dt \leq 4 \int_0^x \left(\int_0^{t^{-1}} f^*(s)ds\right)^2 dt. \tag{2.7}$$

Proof. For $u \geq 0$, define $f_u = f\mathbb{1}_{|f|\leq u} + u(1 - \mathbb{1}_{|f|\leq u})$ and $f^u = f - f_u$. Observe that $(f_u)^* = \min(f^*, u) = (f^*)_u$.
Then, by the triangle inequality and the $(1,\infty)$ boundedness of T,

$$|Tf|(x) \leq |Tf_u|(x) + |Tf^u|(x) \leq |Tf_u|(x) + \|f^u\|_1.$$

Hence,

$$(Tf)^*(x) \leq (Tf_u)^*(x) + \|f^u\|_1$$

and

$$I := \left(\int_0^x (Tf)^*(s)^2 ds\right)^{1/2} \leq \left(\int_0^x (Tf_u)^*(s)^2 ds\right)^{1/2} + x^{1/2} \|f^u\|_1$$

$$\leq \left(\int_0^\infty (Tf_u)^*(s)^2 ds\right)^{1/2} + x^{1/2} \|f^u\|_1 \leq \left(\int_0^\infty (f_u)^*(s)^2 ds\right)^{1/2} + x^{1/2} \|f^u\|_1,$$

where the last inequality follows from T being of type $(2,2)$. Set $u = f^*(x^{-1})$, then

$$\int_0^\infty (f_u)^*(s)^2 ds = \int_0^\infty (f^*)_u(s)^2 ds = x^{-1} f^*(x^{-1})^2 + \int_{x^{-1}}^\infty f^*(s)^2 ds$$

and

$$\|f^u\|_1 = \int_0^{x^{-1}} f^*(s) - f^*(x^{-1}) ds.$$

Putting everything together and applying the change of variable $y = x^{-1}$, we obtain

$$I \leq y^{1/2} f^*(y) + \left(\int_0^x f^*(s^{-1})^2 s^{-2} ds \right)^{\frac{1}{2}} + x^{1/2} \left(\int_0^{x^{-1}} f^*(s) ds \right) - y^{1/2} f^*(y) =$$

$$\left(\int_0^x f^*(s^{-1})^2 s^{-2} ds \right)^{\frac{1}{2}} + x^{1/2} \left(\int_0^{x^{-1}} f^*(s) ds \right).$$

Finally, since f^* is non-increasing, we deduce that

$$\int_0^x \left(s^{-1} f(s^{-1}) \right)^2 ds \leq \int_0^x \left(\int_0^{s^{-1}} f^*(t) dt \right)^2 ds$$

and

$$x \left(\int_0^{x^{-1}} f^*(s) ds \right)^2 \leq \int_0^x \left(\int_0^{s^{-1}} f^*(t) dt \right)^2,$$

whence the result follows. □

Next, we want to replace the power by a different $q \geq 2$. For this, the following lemma is fundamental.

Lemma 2.2.2 ([17]). *Let h be a positive function and define $\Phi(x) = \int_0^x \int_0^s h(u) du ds$. Then if f, g are non-increasing positive functions such that for any x*

$$\int_0^x f(s) ds \lesssim \int_0^x g(s) ds$$

holds, we have that

$$\int_0^\infty \Phi(f(s)) ds \lesssim \int_0^\infty \Phi(g(s)) ds.$$

Proof. Observe that an application of Fubini's Theorem yields

$$\Phi(x) = \int_0^x h(u)(x-u) du = \int_0^\infty h(u)(x-u)_+ du.$$

Thus,

$$\int_0^\infty \Phi(f(x)) dx = \int_0^\infty \int_0^\infty h(u)(f(x)-u)_+ du dx = \int_0^\infty h(u) \int_0^\infty (f(x)-u)_+ dx du,$$

and the result follows because for any u

$$\int_0^\infty (f(x)-u)_+ dx \lesssim \int_0^\infty (g(x)-u)_+ dx.$$

Indeed, since f is non-increasing, there exists a $\infty \geq x^* > 0$ such that

$$\int_0^\infty (f(x)-u)_+ dx = \int_0^{x^*} (f(x)-u) dx \lesssim \int_0^{x^*} (g(x)-u) dx \leq \int_0^\infty (g(x)-u)_+ dx.$$

□

Corollary 2.2.3. *Let T be a linear operator of type $(1,\infty)$ and $(2,2)$ with norm ≤ 1. Let $q \geq 2$. Then, for any $f \in L^1 + L^2$ and any $x \geq 0$,*

$$\int_0^x (Tf)^*(t)^q dt \lesssim \int_0^x \left(\int_0^{t^{-1}} f(s)^* ds \right)^q dt. \tag{2.8}$$

Proof. From Lemma 2.2.1 we know that

$$\int_0^x (Tf)^*(t)^2 dt \lesssim \int_0^x \left(\int_0^{t^{-1}} f(s)^* ds \right)^2 dt,$$

and note that for $h(x) = (\frac{q}{2} - 1)\frac{q}{2} x^{\frac{q}{2}-2} \geq 0$, $\Phi(x) = x^{\frac{q}{2}}$. Hence, an application of Lemma 2.2.2 with $\mathbb{1}_{[0,x]}(Tf)^*(t)^2$ and $\mathbb{1}_{[0,x]} \left(\int_0^{t^{-1}} f(s)^* ds \right)^2$ yields the result. □

Using the previous results, we proceed to state and prove the central results of this section.

Theorem 2.2.4. *Let $1 < p, q < \infty$ with $q \geq 2$ and u, v be weights and T as before. Then, the inequality*

$$\left(\int_0^\infty (Tf)^*(t)^q u^*(t) dt \right)^{\frac{1}{q}} \lesssim K \left(\int_0^\infty (1/v)^*(t)^{-1} f^*(t)^p dt \right)^{\frac{1}{p}} \tag{2.9}$$

holds with:

1. *If $q \geq p$,*

$$K = \sup_s \left(\int_0^s u^* \right)^{\frac{1}{q}} \left(\int_0^{1/s} \left(\frac{1}{v} \right)^{* \; p^*-1} \right)^{\frac{1}{p^*}} < \infty; \tag{2.10}$$

2. *If $q \leq p$,*

$$K = \left(\int_0^\infty \left(\int_0^{1/s} u^* \right)^{\frac{r}{q}} \left(\int_0^s \left(\frac{1}{v} \right)^{* \; p^*-1} \right)^{\frac{r}{q^*}} \left(\frac{1}{v} \right)^* (s)^{p^*-1} ds \right)^{\frac{1}{r}} < \infty. \tag{2.11}$$

Proof. From Corollary 2.2.3 we deduce that

$$\int_0^x (Tf)^*(t)^q dt \lesssim \int_0^x \left(\int_0^{t^{-1}} f(s)^* ds \right)^q dt.$$

and using Hardy's Lemma (Lemma 1.2.2) and the change of variable formula, we obtain

$$\int_0^\infty u^*(t)(Tf)^*(t)^q dt \lesssim \int_0^\infty u^*(t) \left(\int_0^{t^{-1}} f(s)^* ds \right)^q dt = \int_0^\infty u^*(t^{-1}) t^{-2} \left(\int_0^t f(s)^* ds \right)^q dt.$$

Finally, from Hardy's Inequality (Theorem 1.2.3) and the conditions on $u^*.v^*$ we conclude that

$$\left(\int_0^\infty u^*(t^{-1}) t^{-2} \left(\int_0^t f(s)^* ds \right)^q dt \right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty (1/v)^*(t)^{-1} f^*(t)^p dt \right)^{\frac{1}{p}},$$

whence the result follows. □

Theorem 2.2.5 (Pitt inequality, [4]). *Let $1 < p, q < \infty$ and u, v be weights. Then, inequality* (2.3) *holds with:*

1. *If $q \geq p$,*

$$K \approx \sup_s \left(\int_0^s u^* \right)^{\frac{1}{q}} \left(\int_0^{1/s} \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{1}{p^*}} < \infty; \tag{2.12}$$

2. *If $q \leq p$ and either $2 \leq p, q$ or $2 \geq p, q$,*

$$K \approx \left(\int_0^\infty \left(\int_0^{1/s} u^* \right)^{\frac{r}{q}} \left(\int_0^s \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{q^*}} \left(\frac{1}{v} \right)^* (s)^{p^*-1} ds \right)^{\frac{1}{r}} < \infty, \tag{2.13}$$

equivalently,

$$\left(\int_0^\infty u^*(t) \left(\int_0^t u^* \right)^{\frac{r}{p}} \left(\int_0^{\frac{1}{t}} \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{p^*}} dt \right)^{\frac{1}{r}} < \infty, \tag{2.14}$$

where $r^{-1} = q^{-1} - p^{-1}$.

Proof. First, if $q \geq 2$, we obtain the result from Theorem 2.2.4 with $Tf = \hat{f}$ and the following chain of rearrangement inequalities:

$$\left\| \hat{f} \right\|_{q,u} \leq \left(\int_0^\infty u^*(t) (Tf)^*(t)^q dt \right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty (1/v)^*(t)^{-1} f^*(t)^p dt \right)^{\frac{1}{p}} \leq \|f\|_{p,v} .$$

If $q < 2$, by Lemma 2.1.2, we obtain that

$$\sup_f \frac{\left\| \hat{f} \right\|_{q,u}}{\|f\|_{p,v}} = \sup_f \frac{\left\| \hat{f} \right\|_{p^*, v^{-p^*/p}}}{\|f\|_{q^*, u^{-q^*/q}}}.$$

Since in this case $q^* > 2$, we deduce the result by observing that the transformation $(p, q, u, v) \mapsto (q^*, p^*, v^{-p^*/p}, u^{-q^*/q})$ does not transform the previous estimates for K. Indeed,

1. First, since $-p^*/p = 1 - p^*$ and $q^*/q = \frac{1}{q-1}$,

$$\sup_s \left(\int_0^s \left[v^{-p^*/p} \right]^* \right)^{\frac{1}{p^*}} \left(\int_0^{1/s} \left(\frac{1}{u^{-q^*/q}} \right)^{* \, q-1} \right)^{\frac{1}{q}} = \sup_s \left(\int_0^s u^* \right)^{\frac{1}{q}} \left(\int_0^{1/s} \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{1}{p^*}} .$$

2. Second, since $q < 2$, $r/q = r/p + 1$, applying Fubini's Theorem, we obtain

$$\begin{aligned} &\int_0^\infty \left(\int_0^{1/s} u^* \right)^{\frac{r}{q}} \left(\int_0^s \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{q^*}} \left(\frac{1}{v} \right)^* (s)^{p^*-1} ds \approx \\ &\quad \int_0^\infty \left[\int_0^{1/s} u^*(t) \left(\int_0^t u^* \right)^{\frac{r}{p}} dt \right] \left(\int_0^s \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{q^*}} \left(\frac{1}{v} \right)^* (s)^{p^*-1} ds = \\ &\quad \int_0^\infty u^*(t) \left(\int_0^t u^* \right)^{\frac{r}{p}} \left[\int_0^{\frac{1}{t}} \left(\int_0^s \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{q^*}} \left(\frac{1}{v} \right)^* (s)^{p^*-1} ds \right] \approx \\ &\qquad \int_0^\infty u^*(t) \left(\int_0^t u^* \right)^{\frac{r}{p}} \left(\int_0^{\frac{1}{t}} \left(\frac{1}{v} \right)^{* \, p^*-1} \right)^{\frac{r}{p^*}} dt. \end{aligned}$$

□

Remark 2.2.6. *In [4], it was stated that the conclusions of Theorem 2.2.5 hold in the case $q < 2 < p$ too. However, as it was noted in [24] this is not true. In [24], the authors obtained extra conditions which, together with the condition of Theorem 2.2.5 are sufficient for* (2.3) *to hold.*

As a counterexample, we also have

Proposition 2.2.7. *Let $p > 2$ and $v = x^\alpha$ with $0 < \alpha = \frac{p}{2} - 1$. Then, the inequality*

$$\left(\int_0^1 |\hat{f}|^{p^*}\right)^{\frac{1}{p^*}} \lesssim \|f\|_{p,v} \tag{2.15}$$

does not hold. However, condition 2 of Theorem 2.2.5 holds.

Proof. Condition 2 of Theorem 2.2.5 is here

$$\int_0^1 t^{\frac{r}{p}} \left(\int_0^{\frac{1}{t}} s^{-\alpha(p^*-1)}\right)^{\frac{r}{p^*}} dt \approx \int_0^1 t^{\frac{r}{p}} t^{-\frac{r}{p^*}+\frac{r\alpha}{p}} dt < \infty,$$

where we have used that $(1 - \frac{p}{2})(p^* - 1) > -1$. Since $\frac{1}{r} = \frac{1}{p^*} - \frac{1}{p}$, we deduce that this condition holds.
However, using the same method we used in Corollary 1.4.2, we deduce that inequality (2.15) implies that there exists a constant K such that for any sequence $(a_n)_{n=1}^\infty$

$$\|a\|_2 \le K \left(\sum_{n=1}^\infty |a_n|^p n^\alpha\right)^{\frac{1}{p}},$$

which, by Hölder's inequality, implies that

$$\sum_{n=1}^\infty n^{-\frac{\alpha}{p/2-1}} = \sum_{n=1}^\infty n^{-1} < \infty,$$

which is not true.

□

2.3 Necessary conditions

Now we set on to find necessary conditions for inequality (2.3) to hold. We shall show that if $q \ge p$, u is non-increasing and v is non-decreasing, the conditions obtained in 2.2.5 are necessary. We also obtain, by using results in Banach Space theory, new necessary conditions for the case $p > q$, which match conditions 2 in Theorem 2.2.5 when u and v are monotonic.
The following result is well-known:

Proposition 2.3.1. *Assume that inequality* (2.3) *holds. Then,*

$$\sup_{I,J \text{ intervals, } |I||J|=(2\pi)^{-1}} \left(\int_I u\right)^{\frac{1}{q}} \left(\int_J v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \lesssim K, \tag{2.16}$$

equivalently,

$$\sup_{I,J \text{ intervals, } |I||J|\le(2\pi)^{-1}} \left(\int_I u\right)^{\frac{1}{q}} \left(\int_J v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \lesssim K.$$

In particular, if u is non-increasing and v is non-decreasing

$$\sup_{s>0} \left(\int_0^{\frac{1}{s}} u\right)^{\frac{1}{q}} \left(\int_0^s v^{\frac{1}{1-p}} dx\right)^{\frac{1}{p^*}} \lesssim K.$$

Hence, in this case, and if $q \ge p$, Theorem 2.2.5 is sharp.

Proof. Assume first that $v > 0$. For $s > 0$ and $M > 0$, define $f_{s,M}(x) = v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} \mathbb{1}_{[0,s]}$. Observe that $f_{s,M}$ is integrable and $\|f_{s,M}\|_{p,v} < \infty$, Then,

$$\hat{f}_{s,M}(\xi) = \int_0^s v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} e^{2\pi\xi x} dx.$$

In particular, if $|2\pi s\xi| \le 1$, the real part of the Fourier transform satisfies

$$\operatorname{Re} \hat{f}_{s,M}(\xi) \gtrsim \int_0^s v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} dx.$$

Hence, inequality (2.3) implies

$$\left(\int_0^{\frac{1}{2\pi s}} u(\xi) \left(\int_0^s v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} dx\right)^q d\xi\right)^{\frac{1}{q}} \lesssim K \left(\int_0^s v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} dx\right)^{\frac{1}{p}},$$

whence we deduce that, independently of M,

$$\sup_{s>0} \left(\int_0^{\frac{1}{2\pi s}} u(\xi)\right)^{\frac{1}{q}} \left(\int_0^s v(x)^{\frac{1}{1-p}} \mathbb{1}_{v>M} dx\right)^{\frac{1}{p^*}} \lesssim K,$$

and by letting $M \to 0$ and using the Monotone Convergence Theorem, we deduce that, since $v > 0$

$$\sup_{s>0} \left(\int_0^{\frac{1}{2\pi s}} u(\xi)\right)^{\frac{1}{q}} \left(\int_0^s v(x)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p^*}} \lesssim K.$$

Now, if v vanishes at some points, inequality (2.3) clearly holds with the same constant if we replace v by $v + \varepsilon$ for any $\varepsilon > 0$. Then, we have that for any $\varepsilon > 0$

$$\sup_{s>0} \left(\int_0^{\frac{1}{2\pi s}} u(\xi)\right)^{\frac{1}{q}} \left(\int_0^s (v(x) + \varepsilon)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p^*}} \lesssim K,$$

and by letting $\varepsilon \to 0$ and applying the Monotone Convergence Theorem, we deduce that also in this case

$$\sup_{s>0} \left(\int_0^{\frac{1}{2\pi s}} u(\xi)\right)^{\frac{1}{q}} \left(\int_0^s v(x)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p^*}} \lesssim K.$$

Next, the behaviour of the Fourier Transform with respect to translations implies that (2.3) holds if and only if for u and v replaced by any of its translates. Therefore, we have

$$\sup_{I,J \text{ intervals}, |I||J|=(2\pi)^{-1}} \left(\int_I u\right)^{\frac{1}{q}} \left(\int_J v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \lesssim K.$$

□

Remark 2.3.2. *Observe that condition 1 of Theorem 2.2.5 may be rephrased as*

$$\sup_{E,F \text{ measurable}, |E||F|=(2\pi)^{-1}} \left(\int_E u\right)^{\frac{1}{q}} \left(\int_F v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \gtrsim K,$$

while the necessary condition we have just obtained is

$$\sup_{I,J \text{ intervals}, |I||J|=(2\pi)^{-1}} \left(\int_I u\right)^{\frac{1}{q}} \left(\int_J v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \lesssim K.$$

This is not surprising, since in the proof of Theorem 2.2.5 we used rearrangements, so that any "geometrical structure" of the Fourier Transform is neglected.

Lemma 2.3.3 and Theorems 2.3.4 and 2.3.5 are classical Banach Space results and are related to the notions of type and cotype, see for instance [1] and also [22].

Lemma 2.3.3. *Let $p \geq 2$. Then, there exists a constant K, which only depends on p, such that for any N and any sequence of functions $f_1, \ldots, f_N \in L^p_v$ there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_N$ such that the following holds:*

$$\left(\sum_{n=1}^N \|f_n\|_{p,v}^p\right)^{\frac{1}{p}} \leq K \left(\int v \left|\sum_{n=1}^N \varepsilon_n f_n\right|^p\right)^{\frac{1}{p}}. \tag{2.17}$$

Proof. First, observe that since $p \geq 2$,

$$\left(\sum_{n=1}^N \|f_n\|_{p,v}^p\right)^{\frac{1}{p}} = \left(\int v \sum_{n=1}^N |f_n|^p\right)^{\frac{1}{p}} \leq \left(\int v \left(\sum_{n=1}^N |f_n|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

Next, the Khintchine inequality (Theorem 1.4.1) and Fubini's Theorem imply that

$$A_p^p \left(v \int \left(\sum_{n=1}^N |f_n|^2\right)^{\frac{p}{2}}\right) \leq \int v \, \mathbb{E}\left[\left|\sum_{n=1}^N \varepsilon_n f_n\right|^p\right] = \mathbb{E}\left[\int v \left|\sum_{n=1}^N \varepsilon_n f_n\right|^p\right].$$

Thus, there exists a choice of signs ε_n for which

$$\left(\sum_{n=1}^N \|f_n\|_{p,v}^p\right)^{\frac{1}{p}} \lesssim \left(\int v \left|\sum_{n=1}^N \varepsilon_n f_n\right|^p\right)^{\frac{1}{p}}.$$

□

Theorem 2.3.4. *Let* $p \leq 2$ *and let* $T : L^p_v \to L^q_u$ *be a bounded linear operator with norm 1. Then there exits a constant* K*, which only depends on* p*, such that for any sequence of functions* $f_1, \dots, f_N \in L^p_v$ *the following holds:*

$$\left\| \max_n |T(f_n)| \right\|_{q,u} \leq K \left(\sum_{n=1}^{N} \|f_n\|^p_{p,v} \right)^{\frac{1}{p}}. \tag{2.18}$$

Moreover, by the Monotone we can remove the assumption on finiteness, that is, we also have

$$\left\| \sup_n |T(f_n)| \right\|_{q,u} \leq K \left(\sum_{n=1}^{\infty} \|f_n\|^p_{p,v} \right)^{\frac{1}{p}}, \tag{2.19}$$

for infinite sequences of functions.

Proof. Let $E_1, \dots, E_n$ be a disjoint partition of the domain for which $\max_n |T(f_n)| = \left| \sum_{n=1}^N \mathbb{1}_{E_n} T(f_n) \right|$. Then, we have

$$\left\| \max_n |T(f_n)| \right\|_{q,u} = \left\| \sum_{n=1}^{N} \mathbb{1}_{E_n} T(f_n) \right\|_{q,u}.$$

Next, by Hölder's inequality, it suffices to show that for any $g \in L^{q^*}_{u^{\frac{1}{1-q}}}$ with norm 1, we have

$$\int g \sum_{n=1}^{N} \mathbb{1}_{E_n} T(f_n) \leq K \left(\sum_{n=1}^{N} \|f_n\|^p_{p,v} \right)^{\frac{1}{p}}.$$

By definition of the adjoint T^* and Hölder's inequality, we have

$$\begin{aligned} \int g \sum_{n=1}^{N} \mathbb{1}_{E_n} T(f_n) &= \sum_{n=1}^{N} \int T^*(g\mathbb{1}_{E_n}) f_n \leq \sum_{n=1}^{N} \|T^*(g\mathbb{1}_{E_n})\|_{p^*, v^{\frac{1}{1-p}}} \|f_n\|_{p,v} \\ &\leq \left(\sum_{n=1}^{N} \|f_n\|^p_{p,v} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} \|T^*(g\mathbb{1}_{E_n})\|^{p^*}_{p^*, v^{\frac{1}{1-p}}} \right)^{\frac{1}{p^*}}. \end{aligned}$$

Here, since $p^* \geq 2$, by Lemma 2.3.3 we know that there exists a choice of signs for which

$$\left(\sum_{n=1}^{N} \|T^*(g\mathbb{1}_{E_n})\|^{p^*}_{p^*, v^{\frac{1}{1-p}}} \right)^{\frac{1}{p^*}} \leq K \left(\int v^{\frac{1}{1-p}} \left| \sum_{n=1}^{N} \varepsilon_n T^*(g\mathbb{1}_{E_n}) \right|^{p^*} \right)^{\frac{1}{p^*}}.$$

Finally, since for $h \in L^p_v$ with norm 1, we have that

$$\int h \sum_{n=1}^{N} \varepsilon_n T^*(g\mathbb{1}_{E_n}) = \sum_{n=1}^{N} \varepsilon_n \int g\mathbb{1}_{E_n} T(h) \leq \sum_{n=1}^{N} \int |g|\mathbb{1}_{E_n} |T(h)|;$$

using that the E_n are disjoint and that g, h and T have norm 1, we conclude that

$$\sum_{n=1}^{N} \int |g|\mathbb{1}_{E_n} |T(h)| \leq \int |g||T(h)| \leq \|g\|_{q^*, u^{\frac{1}{1-q}}} \|T(h)\|_{q,u} \leq 1.$$

The result follows by noting that, by Hölder's inequality,

$$\left(\int v^{\frac{1}{1-p}} \left|\sum_{n=1}^{N} \varepsilon_n T^*(g\mathbb{1}_{E_n})\right|^{p^*}\right)^{\frac{1}{p^*}} = \sup_{\|h\|_{p,v}=1} \int h \sum_{n=1}^{N} \varepsilon_n T^*(g\mathbb{1}_{E_n}) \leq 1.$$

□

For $p \geq 2$, the analogous result reads as follows

Theorem 2.3.5 (Theorem 7.13 in [1])**.** *Let $p \geq 2$ and let $T : L^p_v \to L^q_u$ be a bounded linear operator with norm 1. Then there exits a constant K, which only depends on p, such that for any sequence of functions $f_1, \dots, f_N \in L^p_v$ the following holds:*

$$\left\|\left(\sum_{n=1}^{N} |T(f_n)|^2\right)^{\frac{1}{2}}\right\|_{q,u} \leq K \left(\sum_{n=1}^{N} \|f_n\|^2_{p,v}\right)^{\frac{1}{2}}. \tag{2.20}$$

Moreover, by the Monotone Convergence Theorem, we can remove the assumption on finiteness, that is, we also have

$$\left\|\left(\sum_{n=-\infty}^{\infty} |T(f_n)|^2\right)^{\frac{1}{2}}\right\|_{q,u} \leq K \left(\sum_{n=-\infty}^{\infty} \|f_n\|^2_{p,v}\right)^{\frac{1}{2}}. \tag{2.21}$$

Now, we are going to show that a condition analogous to condition 2 of Theorem 2.2.5 is indeed necessary.

Theorem 2.3.6. *Let $1 < q < p \leq 2$ and assume that inequality (2.3) holds. Then*

$$K \gtrsim \left(\int_0^\infty v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}}\right)^{\frac{r}{q^*}} \left(\int_0^{\frac{1}{2\pi s}} u\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}}.$$

Proof. Since inequality (2.3) holds, applying Theorem 2.3.4, we have that for any sequence f_i in L^p_v we have

$$\left(\int_{\mathbb{R}} u \sup_i |\hat{f}_i|^q\right)^{\frac{1}{q}} \lesssim K \left(\sum_i \|f_i\|^p_{p,v}\right)^{\frac{1}{p}}. \tag{2.22}$$

Now, by Lemma 2.3.1, we know that

$$\sup_{I,J \text{ intervals, } |I||J|=(2\pi)^{-1}} \left(\int_I u\right)^{\frac{1}{q}} \left(\int_J v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} \lesssim K,$$

in particular, u and $v^{\frac{1}{1-p}}$ are locally integrable. Hence, for every $M > 0$ there exists a partition of the interval $[0, M]$, $(\alpha_n)_{n=-\infty}^{n_M}$ with $\alpha_{n_M} = M$ such that

$$V_n := \int_0^{\alpha_n} v^{\frac{1}{1-p}} \approx 2^n.$$

Let $\lambda_n \in \mathbb{R}_+$ be arbitrary. Consider the sequence $(f_n)_{n=-\infty}^{n_M}$, where $f_n = \lambda_n v^{\frac{1}{1-p}} \mathbb{1}_{[0,\alpha_n]}$. Recall, that, as we showed in Lemma 2.3.1, for $0 \le \xi \le \frac{1}{2\pi\alpha_n}$,

$$|\hat{f}_n(\xi)| \gtrsim \lambda_n \int_0^{\alpha_n} v^{\frac{1}{1-p}} = \lambda_n V_n.$$

Hence, inequality (2.22) implies that for any λ_n

$$\lambda_{n_M}^q V_{n_M}^q \int_0^{\frac{1}{2\pi M}} u + \sum_{n=-\infty}^{n_M-1} \lambda_n^q V_n^q \int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \lesssim K^q \left(\sum_{n=-\infty}^{n_M} \lambda_n^p V_n \right)^{\frac{q}{p}}.$$

Here, by Hölder's inequality, we deduce that, by formally setting $\alpha_{n_M+1} = \infty$ in order to simplify the notation,

$$\left(\sum_{n=-\infty}^{n_M} V_n^{\frac{r}{p^*}} \left(\int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \lesssim K.$$

Next, using that $V_n^{\frac{r}{p^*}} \approx 2^{\frac{nr}{p^*}}$, and setting $\beta_n = \left(\int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \right)$ we use the discrete Hardy inequality (Theorem 1.2.4)

$$\sum_{n=-\infty}^{n_M} 2^{\frac{nr}{p^*}} \beta_n^{\frac{r}{q}} \gtrsim \sum_{n=-\infty}^{n_M} 2^{\frac{nr}{p^*}} \left(\sum_{j=n}^{n_M} \beta_j \right)^{\frac{r}{q}}$$

to deduce that

$$\left(\sum_{n=-\infty}^{\alpha_{n_M}} V_n^{\frac{r}{p^*}} \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \lesssim K.$$

Finally, since $V_n \approx 2^n$

$$\sum_{n=-\infty}^{n_M-1} V_{n+1}^{\frac{r}{p^*}} \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \lesssim \sum_{n=-\infty}^{n_M-1} V_n^{\frac{r}{p^*}} \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \lesssim K^r,$$

and

$$\begin{aligned}
K^r &\gtrsim \sum_{n=-\infty}^{n_M-1} V_{n+1}^{\frac{r}{p^*}} \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \\
&\approx \sum_{n=-\infty}^{n_M-1} \int_0^{\alpha_{n+1}} v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}} \right)^{\frac{r}{q^*}} ds \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \\
&\ge \sum_{n=-\infty}^{n_M-1} \int_{\alpha_n}^{\alpha_{n+1}} v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}} \right)^{\frac{r}{q^*}} ds \left(\int_0^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} \\
&\ge \sum_{n=-\infty}^{n_M-1} \int_{\alpha_n}^{\alpha_{n+1}} v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}} \right)^{\frac{r}{q^*}} \left(\int_0^{\frac{1}{2\pi s}} u \right)^{\frac{r}{q}} ds \\
&= \int_0^M v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}} \right)^{\frac{r}{q^*}} \left(\int_0^{\frac{1}{2\pi s}} u \right)^{\frac{r}{q}} ds,
\end{aligned}$$

and the result follows by letting $M \to \infty$.

□

Likewise, we obtain

Theorem 2.3.7. *Let* $1 < q < 2 < p$ *and assume that inequality* (2.3) *holds. Then*

$$K \gtrsim \left(\int_0^\infty v(s)^{\frac{1}{1-p}} \left(\int_0^s v^{\frac{1}{1-p}} \right)^{\frac{r}{q^*}} \left(\int_0^{\frac{1}{2\pi s}} u \right)^{\frac{r}{q}} ds \right)^{\frac{1}{r}}.$$

Proof. In a similar way, let $f_n = \lambda_n v^{\frac{1}{1-p}} \mathbb{1}_{[0,\alpha_n]}$, where α_n are such that (assuming by simplicity that $\int v^{\frac{1}{1-p}} = \infty$, otherwise a limiting argument like the one used int he previous result yields the result)

$$V_n := \int_0^{\alpha_n} v^{\frac{1}{1-p}} \approx 2^n.$$

Then, if $0 \le \xi \le \frac{1}{2\pi\alpha_n}$

$$|\hat{f}_n(\xi)| \gtrsim \lambda_n \int_0^{\alpha_n} v^{\frac{1}{1-p}} = \lambda_n V_n,$$

and

$$\left(\sum_{j=-\infty}^{\infty} |\hat{f}_j(\xi)|^2 \right)^{\frac{1}{2}} \gtrsim \left(\sum_{j=-\infty}^{n} \lambda_j^2 V_j^2 \right)^{\frac{1}{2}}.$$

Thus, if inequality (2.3) holds, we also have that for any sequence of λ,

$$\left(\sum_{n=-\infty}^{\infty} \int_{\frac{1}{2\pi\alpha_n}}^{\frac{1}{2\pi\alpha_{n-1}}} u \left(\sum_{j=-\infty}^{n} \lambda_j^2 V_j^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \lesssim \left(\sum_{n=-\infty}^{\infty} \lambda_n^2 V_n^{\frac{2}{p}} \right)^{\frac{1}{2}}.$$

Hence, the characterization of the discrete Hardy inequality (Theorem 1.2.4) implies that

$$\sum_{n=-\infty}^{\infty} \int_{\frac{1}{2\pi\alpha_n}}^{\frac{1}{2\pi\alpha_{n-1}}} u \left(\int_0^{\frac{1}{2\pi\alpha_{n-1}}} u \right)^{\frac{R}{2}} V_n^{\frac{R}{p^*}} < \infty,$$

where $R^{-1} = q^{-1} - 2^{-1}$. Thus, using that $V_n \approx 2^n$ and that $R/2 + 1 = R/q$, we deduce that

$$\sum_{n=-\infty}^{\infty} V_n^{\frac{R}{p^*}} \left(\int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{R}{q}} < \infty.$$

Finally, since by Lemma 2.3.1 we know that

$$\sup_n V_n^{\frac{1}{p^*}} \left(\int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{1}{q}} < \infty,$$

we deduce that

$$\sum_{n=-\infty}^{\infty} V_n^{\frac{r}{p^*}} \left(\int_{\frac{1}{2\pi\alpha_{n+1}}}^{\frac{1}{2\pi\alpha_n}} u \right)^{\frac{r}{q}} < \infty,$$

and the result follows as in the previous theorem. □

Corollary 2.3.8. *Let $1 < p, q < \infty$ and u, v be weights. Assume that u and v^{-1} are non-increasing. Then, inequality* (2.3) *holds if and only if*

1. *if $q \geq p$,*
$$\sup_s \left(\int_0^s u\right)^{\frac{1}{q}} \left(\int_0^{1/s} v^{\frac{1}{1-p}}\right)^{\frac{1}{p^*}} < \infty; \tag{2.23}$$

2. *if $q < p$ and either $2 \leq p, q$ or $2 \geq p, q$,*
$$\left(\int_0^\infty \left(\int_0^{1/s} u\right)^{\frac{r}{q}} \left(\int_0^s v^{\frac{1}{1-p}}\right)^{\frac{r}{q^*}} v^{\frac{1}{1-p}} ds\right)^{\frac{1}{r}} < \infty \tag{2.24}$$
equivalently,
$$\left(\int_0^\infty u \left(\int_0^t u\right)^{\frac{r}{p}} \left(\int_0^{\frac{1}{t}} v^{\frac{1}{1-p}}\right)^{\frac{r}{p^*}} dt\right)^{\frac{1}{r}} < \infty, \tag{2.25}$$
where $r^{-1} = q^{-1} - p^{-1}$.

3. *if $q < 2 < p$, condition* (2.24) *is necessary but not sufficient.*

Moreover, the best constant in (2.3) *is equivalent to the corresponding expression above.*

Proof. The sufficiency is Theorem 2.2.5. For the necessity, the case $1 < q < p < 2$ is Theorem 2.3.6 and using duality (Lemma 2.1.2), we deduce the result for $2 < q < p$, just like we did in Theorem 2.2.5. If $q \geq p$, we obtain the result from Lemma 2.3.1. Finally, the necessity of case $q < 2 < p$ follows from Theorem 2.3.7 and the non-sufficiency, from Proposition 2.2.7. □

As an easy consequence, we characterize inequality (2.3) for power weights.

Corollary 2.3.9. *Let $\alpha, \beta \geq 0$ and $u(\xi) = \xi^{-q\alpha}$ and $v(x) = x^{p\beta}$. Then, inequality* (2.3), *that is,*
$$\left\|\hat{f}\right\|_{\xi^{-q\alpha},q} \lesssim \|f\|_{x^{p\beta},p},$$
holds if and only if $q \geq p$, $\alpha < \frac{1}{q}$, $\beta < \frac{1}{p^*}$ and $\beta - \alpha = 1 - \frac{1}{p} - \frac{1}{q}$.

2.4 Inequalities without rearrangements

In the previous sections we observed the "gap" (Remark 2.3.2) existing between necessary and sufficient conditions for non-monotonic weights in even the simplest case $p = 2 = q$. Here we describe instances in which the Pitt inequality holds but the conditions of Theorem 2.2.5 do not. Proposition 2.4.2 is specially illustrative because it shows that, for u the indicator function of a measurable set E and $v = \sqrt{x}$, a type of density of E determines whether the Pitt inequality holds. It is obvious that any information of this kind is lost in the method of rearrangements of Heinig and Benedetto, thereby demonstrating an important weakness of this method.

Proposition 2.4.1. *Set $v = (1+x^2)^2$ and $p = q = 2$. Then, for any u, condition* (2.16) *is also sufficient for inequality* (2.3) *to hold. Moreover, condition* (1) *in Theorem 2.2.5 is not necessary.*

Proof. Let $v = (1+x^2)^2$ and u arbitrary. Observe that $f \in \mathcal{S}$ if and only if $(1+x^2)f \in \mathcal{S}$. Hence, inequality (2.3) holds for any $f \in \mathcal{S}$, and a posteriori for any $f \in L^2_v$, if and only if for any $f \in \mathcal{S}$

$$\int_{\mathbb{R}} u(\xi) \left| \left(f(1+x^2)^{-1} \right)^{\wedge} (\xi) \right|^2 d\xi \lesssim \|f\|_2$$

holds.
Next, using the Parseval relation, and that

$$\left(f(1+x^2)^{-1} \right)^{\wedge} (\xi) = \pi \left(\hat{f} \star e^{-|x|} \right) (\xi),$$

we conclude that our original inequality is equivalent to

$$\int_{\mathbb{R}} u(x) \left| \int_{\mathbb{R}} e^{-|x-y|} g(y) dy \right|^2 dx \lesssim \|g\|_2^2$$

for any g. This last inequality can be dealt with by using the Hardy inequality (Theorem 1.2.3) as follows (it is clear that we may assume that $g \geq 0$)

$$\int_{\mathbb{R}} u(x) \left| \int_{\mathbb{R}} e^{-|x-y|} g(y) dy \right|^2 dx$$

$$\approx \int_{\mathbb{R}} u(x) e^{-2x} \left| \int_{-\infty}^{x} e^{y} g(y) dy \right|^2 dx + \int_{\mathbb{R}} u(x) e^{2x} \left| \int_{x}^{\infty} e^{-y} g(y) dy \right|^2 dx.$$

Now, using Hardy's inequality (Theorem 1.2.3), we conclude that

$$\int_{\mathbb{R}} u(x) e^{-2x} \left| \int_{-\infty}^{x} e^{y} g(y) dy \right|^2 dx \lesssim \|g\|_2^2$$

and

$$\int_{\mathbb{R}} u(x) e^{2x} \left| \int_{x}^{\infty} e^{-y} g(y) dy \right|^2 dx \lesssim \|g\|_2^2$$

hold for any g if and only if both

$$\sup_y e^{2y} \int_y^{\infty} u(x) e^{-2x} dx$$

and

$$\sup_y e^{-2y} \int_{-\infty}^{y} u(x) e^{2x} dx$$

are finite. That is, the latter conditions hold if and only if

$$\sup_y \int_{\mathbb{R}} u(x) e^{-2|x-y|} dx < \infty. \tag{2.26}$$

Finally, all that remains to be shown is that condition (2.26) is implied by

$$\sup_{I,J \text{ intervals, } |I||J|=(2\pi)^{-1}} \left(\int_I u \right)^{\frac{1}{2}} \left(\int_J v^{-1} \right)^{\frac{1}{2}} < \infty.$$

To begin with, observe that since v^{-1} is decreasing, the supremum over J is attained when $J = [0, s]$ for $s > 0$. Hence, we have that there exists a constant K such that for all I interval

$$\int_I u \leq K \left(\int_0^{(2\pi)^{-1}|I|^{-1}} (1+x^2)^{-2} dx \right)^{-1}.$$

Thus, for any y,

$$\int_{\mathbb{R}} u(x) e^{-2|x-y|} dx = \int_{\mathbb{R}} u(x+y) e^{-2|x|} dx = \sum_{n\in\mathbb{Z}} \int_n^{n+1} u(x+y) e^{-2|x|} dx$$
$$\leq \sum_{n\in\mathbb{Z}} e^{-2|n|} \int_n^{n+1} u(x+y) dx \lesssim \sum_{n\in\mathbb{Z}} e^{-2|n|} \left(\int_0^{(2\pi)^{-1}} (1+x^2)^{-2} dx \right)^{-1} < \infty.$$

For the second part, consider $u(\xi) = \sum_{n=1}^{\infty} n \mathbb{1}_{[n, n+n^{-1}]}$. A simple computation shows that condition (2.26) holds but $u^* = \infty$. $\square$

Proposition 2.4.2. *Let $E \subset \mathbb{R}$ measurable and $p = q = 2$. Set $u = \mathbb{1}_E$ and $v = |x|^{\frac{1}{2}}$. Then, if there exists $r > 1$ such that*

$$|E \cap I| \leq |I|^{1-\frac{r}{2}}$$

inequality (2.3) *holds.*

Proof. Observe that by a limiting argument it suffices to consider $f \in C_c^\infty$ supported away from zero. Indeed, assume that inequality (2.3) holds for $f \in C_c^\infty$ supported away from zero. Then, by density, there exists an operator T defined on L_v^2 which coincides with the Fourier Transform on the previous subspace of functions and which satisfies

$$\|T(g)\|_{2,u} \leq K \|g\|_{2,v}.$$

It remains to show that if $g \in L^1 \cap L_v^2$, $T(g) = \hat{g}$ in L_u^2. To do so, take a sequence $(f_n)_n \subset C_c^\infty$ supported away from zero such that $\lim_n \|f_n - g\|_1 = \lim_n \|f_n - g\|_{2,v} = 0$. Then, $T(f_n)$ converges to $T(g)$ in L_u^2, so there exists a subsequence of $T(f_n) = \hat{f}_n$ that converges u-almost everywhere to $T(g)$; besides, since $f_n \to g$ in L^1, $\hat{f}_n$ converges almost everywhere to $\hat{g}$, so $T(g) = \hat{g}$ in L_u^2.
Now, analogously to the previous proposition and if we let $\Phi(\xi) = \int_{\mathbb{R}} |x|^{-\frac{1}{4}} e^{-2\pi i x\xi} dx$ inequality (2.3) holds if and only if for any g

$$\int_{\mathbb{R}} u(x) \left| \int_{\mathbb{R}} \Phi(x-y) g(y) dy \right|^2 dx \lesssim \|g\|_2^2.$$

Observe that a change of variable yields that for any $|\xi| > 0$,

$$\Phi(\xi) = |\xi|^{-\frac{3}{4}} \Phi(1).$$

Hence, the inequality that needs to be dealt with is

$$\int_{\mathbb{R}} u(x) \left| \int_{\mathbb{R}} \frac{g(y)}{|x-y|^{\frac{3}{4}}} dy \right|^2 dx \lesssim \|g\|_2^2.$$

By Theorem 1.2.5, it suffices to show that there exists a $r > 1$ for which

$$\sup_I |I|^{\frac{1}{4}-\frac{1}{2r}}|E\cap I|^{\frac{1}{2r}} < \infty,$$

that is,

$$|E\cap I| \le |I|^{1-\frac{r}{2}}.$$

If we dualize the previous result (Lemma 2.1.2) we obtain the following uncertainty type result:

Corollary 2.4.3. *Let E be a measurable set. Then, for all f such that $\hat{f}$ is supported in E*

$$\left(\int_{\mathbb{R}} |f(x)|^2|x|^{-\frac{1}{2}}\right)^{\frac{1}{2}} \lesssim \left\|\hat{f}\right\|_2 \tag{2.27}$$

if there exists $r > 1$ such that

$$|E\cap I| \lesssim |I|^{1-\frac{r}{2}}. \tag{2.28}$$

Chapter 3

Uncertainty Principles

Roughly speaking, the Uncertanty Principle (UP) asserts that it is not possible to simultaneously localize a function and its Fourier Transform. The most famous instances of this phenomenon are the following:

Theorem A (Heisenberg Uncertainty Principle)**.** *Let f be a square-integrable function. Then, the following holds:*

$$\|f\|_2^2 \lesssim \|fx\|_2 \left\|\hat{f}\xi\right\|_2 .$$

Theorem B. *Let f be a compactly supported integrable function such that $\hat{f}$ is also compactly supported. Then, f is the zero function.*

Theorem B' (Benedicks Theorem, [5])**.** *Let f be a square integrable function such that f and $\hat{f}$ are supported in sets of finite measure. Then, f is the zero function.*

The first three sections of this chapter are devoted to surveying generalizations of Theorems B and B', using the book [14] as the main source. First, in Section 3.1, we shall discuss the Hardy UP (see [12] and Theorem 3.1.2), which restricts the simultaneous decay of a function and its Fourier Transform.
Then, in Section 3.2, we explain and prove the Amrein-Berthier Theorems, which can also be found in [2], (Theorems 3.2.11 and 3.2.12). Theorem 3.2.11 can be understood as a more robust version of Theorem B', in which the conclusion that f is identically zero is not drawn from the vanishing of both f and $\hat{f}$ outside of a small set but just from their "smallness" outside of a suitable set. Theorem 3.2.12 shows that there are no restrictions on the simultaneous behaviour of f and $\hat{f}$ in small sets, thus limiting the type of UP which can be obtained.
The main topic of Section 3.3 is a multidimensional version of the Nazarov UP (see [15] and [21]), which is a sharpened version of Theorem 3.2.11.

The last two sections deal with generalizations of Theorem A. In Section 3.4 we complete and generalize the characterization of the inequality

$$\|f\|_q^{\alpha+\beta-\frac{1}{q^*}+\frac{1}{p}} \lesssim \|fx^\alpha\|_q^{\beta-\frac{1}{q^*}+\frac{1}{p}} \left\|\hat{f}\xi^\beta\right\|_p^\alpha$$

for $1 < p, q < \infty$ and $\alpha, \beta > 0$, which was started in [28]; and, in Section 3.5, we characterize the parameters for which the inequality

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left\|x^A f\right\|_p \left\|\xi^B \hat{f}\right\|_{p^*}$$

holds, where x^A and ξ^B are a generalization of power weights.

3.1 Hardy uncertainty principle

As mentioned in the introduction, the Hardy UP restricts the simultaneous decay of f and $\hat{f}$; more precisely, it asserts that it is impossible for both f and $\hat{f}$ to decay more rapidly than the gaussian function $e^{-\pi x^2}$. This result was obtained by Hardy in 1933 and published in [12]. The proof we give is close to the original and is based on Complex Analysis, more exactly, on the Lindelöf-Phragmen principle. There are also proofs using real-variable methods; for instance, in [7], the result is obtained by using estimates for the norms of the solutions of the Schrödinger equation.

Lemma 3.1.1. *Let F be an entire function, and $a, C \in \mathbb{R}_+$. Assume that*

1. $|F(x)| \leq Ce^{-ax}$ *for* $0 < x \in \mathbb{R}$;
2. $|F(z)| \leq Ce^{a|z|}$ *for* $z \in \mathbb{C}$.

Then there exists a C' such that

$$F(z) = C'e^{-az}.$$

Proof. To begin with, if $F = 0$ we are done. From now on, assume that $F \neq 0$.

Let

$$\Phi_\delta(z) = F(z)e^{(a+ia\tan\frac{\delta}{2})z}$$

for $\delta > 0$ and small.
Then, if $0 < x \in \mathbb{R}$,

$$|\Phi_\delta(x)| = |F(x)e^{(a+ia\tan\frac{\delta}{2})x}| \leq |F(x)|e^{ax} \leq C.$$

Likewise, if $z = xe^{i(\pi-\delta)}$ for $0 < x \in \mathbb{R}$,

$$\mathrm{Re}\left[a(1 + i\tan\frac{\delta}{2})z\right] = -ax(\cos\delta + \sin\delta\tan\frac{\delta}{2}) = -ax,$$

so that

$$|\Phi_\delta(z)| = |F(z)e^{(a+ia\tan\frac{\delta}{2})z}| \leq Ce^{ax}e^{-ax} = C.$$

Next, we apply a version of the Lindelöf-Phragmen principle to conclude that $|\Phi_\delta| \leq C$ in the sector $S_\delta := \{re^{i\theta} : 0 \leq \theta \leq \pi - \delta\}$. Choose $1 < \beta < \frac{\pi-\delta/4}{\pi-\delta/2}$, $\varepsilon > 0$ and for $z \in S_\delta$, define

$$G(z) := \Phi_\delta(z)\exp\left(i\varepsilon(ze^{i\frac{\delta}{2}})^\beta\right).$$

Note that since $ze^{i\frac{\delta}{2}}$ is in the upper-half-plane, we can take an holomorphic branch of the function $z \mapsto z^\beta$.

Observe that if we put $z = re^{i\theta}$ for $0 \le \theta \le \pi - \delta$, then

$$\operatorname{Re}\left(i\varepsilon(ze^{i\frac{\delta}{2}})^{\beta}\right) = -r^{\beta}\varepsilon\sin(\beta(\theta+\delta/2)) \le -r^{\beta}\varepsilon\sin(\beta(\pi-\delta/2)) \le -r^{\beta}\varepsilon\sin(\pi-\delta/4) < 0.$$

Therefore, the non-zero holomorphic function G satisfies

$$\lim_{|z|\to\infty} G(z) = 0,$$

and consequently there exists a $z^* \in S_\delta$ such that $\sup_{z\in S_\delta} G(z) = G(z^*)$ and, by the Maximum Modulus principle, $z^* \in \partial S_\delta$. Since for $z \in \partial S_\delta$, $|G(z)| \le |\Phi_\delta(z)| \le C$, we conclude that for any $\varepsilon > 0$ and $z \in S_\delta$

$$\left|\Phi_\delta(z)\exp\left(i\varepsilon(ze^{i\frac{\delta}{2}})^{\beta}\right)\right| \le C.$$

Then, letting $\varepsilon \to 0$, we deduce that for any $\delta > 0$ and $z \in S_\delta$

$$\left|F(z)e^{(a+ia\tan\frac{\delta}{2})z}\right| = |\Phi_\delta(z)| \le C.$$

Thus, letting $\delta \to 0$, we have that for any $z \in \{re^{i\theta} : 0 \le \theta < \pi\}$,

$$|F(z)e^{az}| \le C.$$

Finally, by continuity we extend the previous inequality to the whole upper-half-plane and repeating the same argumentation for $\bar{F}(\bar{z})$, we obtain

$$|F(z)e^{az}| \le C$$

in the whole complex plane, and by Liouville's Theorem,

$$F(z) = C'e^{-az}.$$

□

Theorem 3.1.2 (Hardy uncertainty principle, [12]). *Let $f \in L^1(\mathbb{R})$ be such that for $A, C \in \mathbb{R}_+$, the following hold*

1. $|f(x)| \le Ce^{-\pi Ax^2}$;
2. $|\hat{f}(\xi)| \le Ce^{-\pi\frac{\xi^2}{A}}$.

Then, $f(x)$ is a multiple of $e^{-\pi Ax^2}$.

Proof. First, the decay of f implies that

$$\hat{f}(z) = \int_{\mathbb{R}} f(x)e^{-2\pi ixz}dx$$

is defined for any $z \in \mathbb{C}$ and defines an entire function such that $|\hat{f}(z)| \le C'e^{\frac{|z|^2}{A}}$. Indeed,

$$\int_{\mathbb{R}}\left|f(x)e^{-2\pi ixz}\right|dx \le C\int_{\mathbb{R}} e^{-A\pi x^2+2\pi x\operatorname{Im} z}dx \le C'e^{\frac{\pi|z|^2}{A}}.$$

Now, assume that f is even. Then, $\hat{f}$ is even and $\hat{f}(z) = F(z^2)$ for some holomorphic function F, which satisfies

1. $|F(x)| \leq Ce^{-\pi\frac{x}{A}}$ for $0 < x \in \mathbb{R}$;

2. $|F(z)| \leq C'e^{\pi\frac{|z|}{A}}$ for $z \in \mathbb{C}$.

Using Lemma 3.1.1 we conclude that

$$\hat{f}(z) = F(z^2) = C''e^{-\pi\frac{z^2}{A}}.$$

If f is odd, $\hat{f}(0) = 0$. Therefore, $G(z) := \hat{f}(z)z^{-1}$ is holomorphic and even. Hence $G(z) = F(z^2)$ for some holomorphic function which satisfies

1. $|F(x)| \leq C'e^{-\pi\frac{x}{A}}$ for $0 < x \in \mathbb{R}$;

2. $|F(z)| \leq C'e^{\pi\frac{|z|}{A}}$ for $z \in \mathbb{C}$.

Once again, this implies that $\hat{f}(z^2)z^{-2} = C''e^{-\pi\frac{z^2}{A}}$. However, if $C'' \neq 0$, $|\hat{f}(z)| = ze^{-\pi\frac{z^2}{A}} \not\leq Ce^{-\pi\frac{z^2}{A}}$, a contradiction. Hence, $f = 0$.

In the general case, put $f = f_e + f_o$ with f_e even and f_0 odd. It is clear that the bounds for f transform into suitable bounds for f_e, f_o so that we can conclude that

$$\hat{f}(\xi) = \hat{f}_e(\xi) + \hat{f}_0(\xi) = C''e^{-\pi\frac{z^2}{A}},$$

and, by the Fourier inversion formula,

$$f(x) = C'''e^{-\pi Ax^2}.$$

□

Corollary 3.1.3. *Let $f \in L^1(\mathbb{R})$ be such that for $A, B, C \in \mathbb{R}_+$, the following hold*

1. $|f(x)| \leq Ce^{-\pi Ax^2}$;

2. $|\hat{f}(\xi)| \leq Ce^{-\pi\frac{\xi^2}{B}}$.

Then, if $B < A$, $f = 0$.

Proof. Observe that, since $B < A$

$$|f(x)| \leq Ce^{-\pi Ax^2} \leq Ce^{-\pi Bx^2}.$$

Thus, an application of Theorem 3.1.2 implies that $f(x) = C'e^{-\pi Bx^2}$. This contradicts the hypothesis on f unless $C' = 0$. □

3.2 Amrein-Berthier Theorems

The main results of this section, Theorems 3.2.11 and 3.2.12, were obtained in 1977 by the quantum physicists Amrein and Berthier. In this subfield of physics, for a suitable normalized f, the probability density for the position of a particle is given by $|f|^2$, and the probability of finding a particle in a given zone of space C is computed as $\int_C |f|^2$; while the probability density for the momentum is $|\hat{f}|^2$. It is thus not surprising that these results are of interest to physicists, for whom the integral $\int_{\mathbb{R}\setminus S} |f|^2$ found in Theorem 3.2.11 describes the probability of finding a particle outside of S, and the fact, proved in Theorem 3.2.12, that for S and Σ with $|S| + |\Sigma| < \infty$ the restriction of f to S and the restriction of $\hat{f}$ to Σ are independent limits the conclusions about the momentum of a particle which can be drawn by observing its behaviour in a small zone of the space.

It is remarkable that the proof proceeds by studying a more general problem, in which the Fourier Transform does not play any role, namely, given M, N two closed subspaces of a Hilbert space H, with orthogonal projections P, Q, it is discussed when it is possible to obtain the inequality

$$\|v\| \lesssim \left\|P^{\perp}v\right\| + \left\|Q^{\perp}v\right\|,$$

for any $v \in H$; and when it is possible to find a solution f to the system of equations

1. $$Pf = f_1;$$

2. $$Qf = f_2.$$

After discussing this abstract setting, the results are applied to the Fourier Transform setting by taking $H = L^2(\mathbb{R}^d)$ and suitable P, Q. It is clear that for the previous results to be true it is necessary that $M \cap N = \{0\}$, and if H is finite dimensional it is also sufficient. However, in the infinite-dimensional case, the relevant setting for the Fourier Transform, more restrictive conditions are needed.

3.2.1 Hilbert-Schmidt operators

In this section we give some necessary background about Hilbert-Schmidt operators.

Definition 3.2.1. *We say that a continuous operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a Hilbert-Schmidt operator if*

$$\|T\|_{HS} := \left(\sum_{j\in I} \|Te_j\|^2\right)^{\frac{1}{2}} < \infty,$$

where $\{e_j, j \in I\}$ is an orthonormal basis.

Proposition 3.2.2. *Let T be defined as follows:*

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy,$$

with $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Then T is a Hilbert-Schmidt operator.

Proof. To begin with, an application of Hölder's inequality and Fubini's Theorem yield that the operator is continuous.
Next, let $\{e_j, j \in I\}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Then, the functions $f_{ij}(x,y) := e_i(x)\bar{e_j}(y)$ for $(i,j) \in I \times I$ are orthonormal.
Observe that

$$\langle f_{ij}, K\rangle = \int_{\mathbb{R}^d\times\mathbb{R}^d} K(x,y)e_j(y)\bar{e_i}(x)dxdy = \int_{\mathbb{R}^d} \bar{e_i}(x)\int_{\mathbb{R}^d} K(x,y)e_j(y)dy = \langle e_i, Te_j\rangle.$$

Thus, since $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, Parseval's Theorem yields

$$\infty > \sum_{i,j\in I} |\langle f_{ij}, K\rangle|^2 = \sum_{i,j\in I} |\langle e_i, Te_j\rangle|^2 = \sum_{j\in I} \|Te_j\|^2 .$$

□

Proposition 3.2.3. *Let T be a Hilbert-Schmidt operator. Then it is compact.*

Proof. First, we know that $L^2(\mathbb{R}^d)$ has a countable orthonormal basis $\{e_i : i \in \mathbb{N}\}$.
We are going to show that the finite-rank operators

$$T_n(x) := \sum_{k=1}^{n}\langle e_k, x\rangle T(e_k)$$

approximate T, whence the result will follow.
Observe that for any $x \in L^2(\mathbb{R})$ with $\|x\|$=1,

$$\|(T_n - T)(x)\| \le \sum_{k=n+1}^{\infty} \|T(e_k)\| \, |\langle e_k, x\rangle| \le \left(\sum_{k=n+1}^{\infty} \|T(e_k)\|^2\right)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} |\langle e_k, x\rangle|^2\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{k=n+1}^{\infty} \|T(e_k)\|^2\right)^{\frac{1}{2}} .$$

Hence, since $\left(\sum_{k=1}^{\infty} \|T(e_k)\|^2\right)^{\frac{1}{2}} < \infty$, $T_n \to T$ in the operator norm. □

3.2.2 Characterization of strong annihilating pars

Let M, N be two closed subspaces of a Hilbert space H and let P, Q be their corresponding projectors. As it was said in the introduction, a goal of this section was to find conditions on M and N which guarantee that the inequality

$$\|v\| \le c(\left\|P^{\perp}v\right\| + \left\|Q^{\perp}v\right\|)$$

holds for $v \in H$. Such pair M, N shall be called a strong annihilating Pair or strong a-Pair.

Definition 3.2.4. *We say that P, Q are an **a-Pair** whenever $M \cap N = \{0\}$.*

Definition 3.2.5. *We say that P, Q are a **strong a-Pair** whenever there is $c > 1$ such that for any $v \in H$*

$$\|v\| \leq c(\left\|P^{\perp}v\right\| + \left\|Q^{\perp}v\right\|). \tag{3.1}$$

Clearly, strong a-Pairs are a-Pairs.

Next we outline some equivalent definitions of strong a-Pairs.

Example. *The following are strong a-pairs:*

1. *for $H = \mathbb{R}^n$, if $M \cap N = \{0\}$, P, Q are a strong a-Pair;*
2. *for any H, if $P^{\perp} = Q$, P, Q are a strong a-Pair.*

Lemma 3.2.6. *Let I be the identity operator. Then, the following are equivalent*

1. $\|PQ\| < 1$;
2. *$R := I - PQ$ is invertible;*
3. *there is $c > 1$ such that for any v, $\|v\| \leq c(\left\|P^{\perp}v\right\| + \left\|Q^{\perp}v\right\|)$.*
4. *there is $c' > 1$ such that for any v, $\|Qv\| \leq c' \left\|P^{\perp}Qv\right\|$.*
5. $\|QP\| < 1$;
6. *$R^* := I - QP$ is invertible.*

Note that (3) is the definition of strong a-pair. Moreover, in (4) $\implies$ (3), we may take $c = c' + 1$.

Proof. (1) $\implies$ (2) is known. For the reverse, note that $RQ = (I - PQ)Q = Q - PQ = (I - P)Q = P^{\perp}Q$, which implies that

$$\|Qv\|^2 = \|PQv\|^2 + \left\|P^{\perp}Qv\right\|^2 = \|PQv\|^2 + \|RQv\|^2 \geq \|PQv\|^2 + \left\|R^{-1}\right\|^{-2} \|Qv\|^2,$$

so that

$$\|PQv\|^2 \leq (1 - \left\|R^{-1}\right\|^{-2}) \|Qv\|^2 \leq \|v\|^2 (1 - \left\|R^{-1}\right\|^{-2}),$$

whence the result follows, since $\|R^{-1}\|^{-1} \neq 0$.
If (4) holds then

$$\|PQv\|^2 + c'^{-2} \|Qv\|^2 \leq \|PQv\|^2 + \left\|P^{\perp}Qv\right\|^2 = \|Qv\|^2,$$

so that

$$\|PQv\|^2 \leq \|Qv\|^2 (1 - c'^{-2}) \leq \|Q\|^2 \|v\|^2 (1 - c'^{-2}) \leq \|v\|^2 (1 - c'^{-2}),$$

that is, (1) holds.
If (1) holds, we have (4) as follows:

$$0 < (1 - \|PQ\|^2) \|Qv\|^2 = \|Qv\|^2 - \|PQ\|^2 \|Qv\|^2 \leq \|Qv\|^2 - \|PQv\|^2 = \left\|P^{\perp}Qv\right\|^2.$$

Finally, (3) clearly implies (4) and we can obtain (3) from (4) as follows:

$$\|v\| \leq \|Qv\| + \left\|Q^\perp v\right\| \leq c\left\|P^\perp Qv\right\| + \left\|Q^\perp v\right\| = c\left\|P^\perp(I - Q^\perp)v\right\| + \left\|Q^\perp v\right\|$$

$$\leq c\left\|P^\perp Q^\perp v\right\| + c\left\|P^\perp v\right\| + \left\|Q^\perp v\right\| \leq (\left\|P^\perp\right\| + c)(\left\|Q^\perp v\right\| + \left\|P^\perp v\right\|),$$

whence the result follows (note that $\left\|P^\perp\right\| = 1$ unless it is zero).
To finish the proof, note that since $(PQ)^* = QP$, (1) and (5) are equivalent; and (5) and (6) are equivalent just like (1) and (2). □

Remark 3.2.7. *Observe that in general, $M \cap N = \{0\}$ does not imply that M, N are a strong a-pair, or equivalently, by the previous lemma, that $\|PQ\| < 1$. Indeed, let $M = \{x \in \ell_2(\mathbb{N}) : x_{2n+1} = 0, n \in \mathbb{N}\}$ and $N = \{x \in \ell_2(\mathbb{N}) : x_{2n} = (n+1)x_{2n+1}, n \in \mathbb{N}\}$. They are clearly closed subspaces and if $x \in M \cap N$, then $0 = x_{2n+1}(n+1) = x_{2n}$ for all n, so $x = 0$.*
However, if e_n is the sequence with zeros at every position except at the n-th, where it has a 1,

$$PQ\left(e_{2n} + \frac{e_{2n+1}}{n+1}\right) = P\left(e_{2n} + \frac{e_{2n+1}}{n+1}\right) = e_{2n},$$

since

$$\|e_{2n}\| = 1$$

and

$$\left\|e_{2n} + \frac{e_{2n+1}}{n+1}\right\| = \sqrt{1 + (n+1)^{-2}},$$

$$\|PQ\| = 1.$$

Hence, M, N is an a-Pair which is not strong.

Lemma 3.2.8. *Let K be a compact operator. Then, there exists $v \in H$ with $\|v\| = 1$ such that $\|Kv\| = \|K\|$.*

Proof. To begin with, if $\|K\| = 0$ the result is obvious, so we may assume that $\|K\| > 0$. Let $(x_n)_{n=1}^\infty$ be a sequence of elements with norm 1 such that $\|Kx_n\| \to \|K\|$. Since K is compact, by extracting a subsequence, we may assume that Kx_n converges to some $y \in H$ with $\|y\| = \|K\|$.
Next, we have that, for K^* the adjoint of K,

$$\|K^*y\| = \sup_{\|z\|=1} \langle z, K^*y\rangle = \sup_{\|z\|=1} \langle Kz, y\rangle = \|K\|^2,$$

where the last inequality is true because $\langle Kz, y\rangle \leq \|K\| \|y\| = \|K\|^2$ and the sequence x_n has norm 1 and Kx_n converges to y.
Finally,

$$\|K\| \|KK^*y\| = \|y\| \|KK^*y\| \geq \langle y, KK^*y\rangle = \langle K^*y, K^*y\rangle = \|K\|^4 \tag{3.2}$$

so that

$$\|K\|^3 \leq \|KK^*y\| \leq \|K\| \|K^*\| \|y\| = \|K\|^3.$$

In conclusion, since
$$\|KK^*y\| = \|K\|^3$$
and $\|K^*y\| = \|K\|^2$, we can take
$$v = K^*y\,\|K\|^{-2}.$$

□

Lemma 3.2.9. *Let P, Q be an a-Pair. If PQ is a compact operator, then it is a strong a-Pair.*

Proof. For the sake of contradiction, assume that $\|PQ\| = 1$. By Lemma 3.2.8 there exists a v with norm 1 such that $\|PQv\| = \|PQ\| = 1$. Then,
$$\left\|P^{\perp}Qv\right\|^2 + \|PQv\|^2 = \|Qv\|^2 \leq \|v\|^2 = 1.$$
Thus,
$$\left\|P^{\perp}Qv\right\|^2 \leq 1 - \|PQv\|^2 = 0,$$
whence we deduce that $Qv \in M \cap N$.
Since P, Q is an a-Pair, we have that $Qv = 0$, which contradicts the fact that $\|PQv\| = 1$. □

Lemma 3.2.10. *Let P, Q be a strong a-Pair. Then, for any $m \in M$ and $n \in N$, there exists a $v \in H$ such that $Pv = m$ and $Qv = n$.*

Proof. Since P, Q is a strong a-Pair, we know that $R = I - PQ$ and $R^* = I - QP$ are invertible. Put
$$v = Q^{\perp}R^{-1}m + P^{\perp}(R^*)^{-1}n.$$
Then,
$$Pv = PQ^{\perp}R^{-1}m = PR^{-1}m - PQR^{-1}m = P(I - PQ)R^{-1}m = PRR^{-1}m = m,$$
and by symmetry, $Qv = n$. □

3.2.3 Application to Fourier Analysis

We are now going to apply the previous abstract results to some special problems in Fourier Analysis. Here and in this whole subsection, $H = L^2(\mathbb{R}^d)$ and $S, \Sigma \subset \mathbb{R}^d$ are measurable sets of finite measure.
Let
$$M = \{f \in H : \operatorname{supp\,ess} f \subset S\};$$
$$N = \{f \in H : \operatorname{supp\,ess} \hat{f} \subset \Sigma\},$$
where supp ess denotes the essential support. Note that these are closed subspaces of H. We denote by P the projection onto M and by Q the projection onto N.
The goal of this section is to show that P, Q form a strong a-Pair (this will be shown in Lemma 3.2.16), what will imply that

Theorem 3.2.11 ([2]). *There exists $c > 0$ such that for any $f \in L^2(\mathbb{R}^d)$, the following holds:*

$$c\,\|f\|_2 \leq \left(\int_{\mathbb{R}^d\setminus S} |f|^2\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2\right)^{\frac{1}{2}}, \tag{3.3}$$

Theorem 3.2.12 ([2]). *Let $f_1 \in L^2(S)$ and $f_2 \in L^2(\Sigma)$, then there exists $f \in L^2(\mathbb{R}^d)$ such that the following hold:*

$$f|_S = f_1;$$

$$\hat{f}|_\Sigma = f_2.$$

Lemma 3.2.13. *Using the previous notation, PQ is a compact operator.*

Proof. For $f \in \mathcal{S}$, the Schwartz class, Q is given by

$$Qf(x) = \int_\Sigma e^{2\pi i\langle x,\xi\rangle}\left(\int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle y,\xi\rangle}dy\right)d\xi = \int_{\mathbb{R}^d} f(y)\left(\int_\Sigma e^{-2\pi i\langle y-x,\xi\rangle}d\xi\right)dy.$$

Hence, for $f \in \mathcal{S}$,

$$PQf(x) = \int_{\mathbb{R}^d} f(y)\hat{\mathbb{1}}_\Sigma(y-x)\mathbb{1}_S(x)dy.$$

Next, since

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(y)\hat{\mathbb{1}}_\Sigma(y-x)\mathbb{1}_S(x)dy\right)^2 dx = \int_S\left(\int_{\mathbb{R}^d} f(y)\hat{\mathbb{1}}_\Sigma(y-x)dy\right)^2 dx$$

$$\leq \int_S\left(\int_{\mathbb{R}^d}|f(y)|^2\right)\cdot\left(\int_{\mathbb{R}^d}|\hat{\mathbb{1}}_\Sigma(y-x)|^2dy\right)dx = |S||\Sigma|\,\|f\|_2^2,$$

by continuity, for any $f \in L^2(\mathbb{R}^d)$,

$$PQf(x) = \int_{\mathbb{R}^d} f(y)\hat{\mathbb{1}}_\Sigma(y-x)\mathbb{1}_S(x)dy.$$

Finally, PQ is a Hilbert-Schmidt operator, and thus, compact (see Section 3.2.1). □

Remark 3.2.14. *Note that if $|S||\Sigma| < 1$, the previous calculations show that $\|PQ\| < 1$, so that we already have that P, Q is a strong a-Pair.*
Besides, if S and Σ are bounded, it is a consequence of the Theorem B that P, Q is an a-Pair, so by Lemma 3.2.9 it is also a strong one.
To show the result in the general case, that is, the case covered by Theorem B' we still need to do some additional work.

Lemma 3.2.15. *Using the previous notation, P, Q is an a-Pair and, by Lemma 3.2.9, it is a strong a-Pair.*

Proof. Recall that we need to show that $M \cap N = \{0\}$, equivalently, that if f is such that $\operatorname{supp\,ess} f \subset S$ and $\operatorname{supp\,ess} \hat{f} \subset \Sigma$, then $f = 0$.
Put

$$\mathcal{E} := \ker(I - PQ) = \{v \in H : PQv = v\} = \{f \in L^2(\mathbb{R}^d) : f(x) = \int_{\mathbb{R}^d} f(y)\hat{\mathbb{1}}_\Sigma(y-x)\mathbb{1}_S(x)dy\}.$$

Note that if $v \in M \cap N$, then $PQv = v$, so that $M \cap N \subset \mathcal{E}$. We are going to show that $\dim \mathcal{E} = 0$, what implies $M \cap N = \{0\}$.

Step 1:
To begin with, we show that $\dim \mathcal{E} < \infty$. Let $f_1, \ldots, f_N$ be an orthonormal system of elements in $\mathcal{E}$. Then, if $g_i = f_i(x)\bar{f}_i(y)$, $g_1, \ldots, g_N$ is an orthogonal system of functions in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Indeed,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_i(x)\bar{f}_i(y)\bar{f}_j(x)f_j(y)dxdy = \int_{\mathbb{R}^d} f_i(x)\bar{f}_j(x)dx \int_{\mathbb{R}^d} \bar{f}_i(y)f_j(y)dy = \delta_{ij}^2 = \delta_{ij}.$$

Put $K(x,y) = \hat{\mathbb{1}}_\Sigma(y-x)_\Sigma \mathbb{1}_S(x)dy$. We have shown that $\|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \leq \sqrt{|S||\Sigma|}$. Hence, the Parseval inequality gives

$$|S||\Sigma| \geq \sum_{j=1}^{N} \langle g_j, K\rangle^2.$$

Finally, since $f_i \in \mathcal{E}$ we have that

$$\langle g_j, K\rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} f_i(y)\bar{f}_i(x)K(x,y)dxdy = \int_{\mathbb{R}^d} \bar{f}_i(x)\left(\int_{\mathbb{R}^d} f_i(y)K(x,y)dy\right)dx = \int_{\mathbb{R}^d} \bar{f}_i(x)f_i(x) = 1.$$

In conclusion, $N \leq |S||\Sigma|$, what implies that

$$\dim \mathcal{E} \leq |S||\Sigma| < \infty.$$

Step 2:
Next, we show that if for some S, Σ we have that $\dim \mathcal{E} > 0$, then we can find $S \subset S^*$ with $|S^*| \leq |S| + 1$, such that $\dim \mathcal{E}^* = \infty$ (where $\mathcal{E}^*$ is the obvious modification of $\mathcal{E}$). Since by doing this we obtain a contradiction with Step 1, we conclude the proof.
We shall make use of the following lemma:

Lemma 3.2.16. *Let $S_0 \subset T$ be measurable sets of $\mathbb{R}^d$. Further assume that $0 < |S_0| \leq |T| < \infty$. Then, for any $\varepsilon > 0$, there exists a $v \in \mathbb{R}^d$ such that $|T| + \varepsilon > |T \cup (S_0 - v)| > |T|$.*

Proof. Put

$$h(v) = |T| + |S_0| - |T \cup (S_0 - v)| = |T \cap (S_0 - v)|.$$

Clearly, $h(0) = S_0$.
Observe that $h(v) = \int_T \mathbb{1}_{S_0}(x+v)dx$, from where we deduce that h is a continuous function. Indeed,

$$|h(v_1) - h(v_2)| \leq \int_T |\mathbb{1}_{S_0}(x+v_1) - \mathbb{1}_{S_0}(x+v_2)|dx \leq \int_{T-v_1} |\mathbb{1}_{S_0}(x) - \mathbb{1}_{S_0}(x+v_2-v_1)|dx,$$

and it is known that as $v_2 \to v_1$, this last expression has limit zero, because $\mathbb{1}_{S_0} \in L^1(\mathbb{R}^d)$.
Finally, we show that $\lim_{v\to\infty} |T \cap (S_0 - v)| = 0$.

It is known that for any measurable T and any $\delta > 0$, there exists a compact set $K \subset T$ such that $|T \setminus K| < \delta$. Then,

$$h(v) = \int_T \mathbb{1}_{S_0}(x+v)dx \leq \delta + \int_{\mathbb{R}^d} \mathbb{1}_{S_0 \cap K}(x+v)dx.$$

Since K is bounded, for any $x \in K$, $\lim_{v\to\infty} \mathbb{1}_{S_0\cap K}(x+v) = 0$. Hence, the Dominated Convergence Theorem yields

$$\limsup h(v) \leq \delta + \lim \int_{\mathbb{R}^d} \mathbb{1}_{S_0\cap K}(x+v)dx = \delta$$

for any $\delta > 0$, whence the claim follows.
In conclusion, h is a continuous function for which $h(0) = |S_0| > 0$ and $\lim_{v\to\infty} h(v) = 0$. This implies that the function

$$g(v) = |T \cup (S_0 - v)|$$

is also continuous and satisfies $g(0) = |T|$ and $\lim_{v\to\infty} g(v) = |T| + |S_0|$. Thus, by the Mean Value Theorem, any value between $|T|$ and $|T|+|S_0|$ is attained, whence the result follows. $\square$

We resume the proof of Step 2.
Let ϕ_0 be a non-zero function in $\mathcal{E}$ and put $S_0 = \operatorname{supp\,ess} \phi_0$.
We proceed by induction as follows: set $T_0 = S_0$ and let $v_k \in \mathbb{R}^d$ be such that

$$|T_k| < |T_k \cup (S_0 - v_k)| < |T_k| + 2^{-k+1},$$

which exists by the previous lemma, with $T = T_k$ and $\varepsilon = 2^{-k+1}$. Define $T_{k+1} = T_k \cup (S_0 - v_k)$. Observe that $|T_{k+1}| \leq T_k + 2^{-k+1}$, so $|T_k| \leq |S_0| + 1$ for all k.
Put $S^* = \cup_{j=0}^{\infty} T_j$. Observe that since $T_k \subset T_{k+1}$, the Monotone Convergence Theorem yields

$$|S^*| \leq |S_0| + 1.$$

Finally, set $\phi_k(x) = \phi_0(x + v_k)$. Note that $\operatorname{supp\,ess} \phi_k = \operatorname{supp\,ess} \phi_0 - v_k = S_0 - v_k \subset T_{k+1} \subset S^*$. Besides, $\operatorname{supp\,ess} \hat{\phi}_k = \operatorname{supp\,ess} \hat{\phi}_0 \subset \Sigma$. Hence, $\phi_k \in \mathcal{E}^*$.
To finish the proof, we show that $\phi_0, \phi_1, \ldots$ are linearly independent, and consequently, $\mathcal{E}^*$ is infinite dimensional. For the sake of contradiction, let N be the minimal number such that there exist $\lambda_0, \lambda_1, \ldots \lambda_N$ for which

$$\lambda_0\phi_0 + \lambda_1\phi_1 + \cdots + \lambda_{N-1}\phi_{N-1} = \lambda_N\phi_N,$$

clearly, since $\phi_0 \neq 0$, $N \geq 1$ and, by minimality, $\lambda_N \neq 0$..
Recall that $\operatorname{supp\,ess} \phi_k \subset T_k$. Hence, the LHS is supported in T_{N-1}, whence we deduce that $\operatorname{supp\,ess} \lambda_N\phi_N \subset T_{N-1}$. However, if $\lambda_N \neq 0$, $\operatorname{supp\,ess} \lambda_N\phi_N = \operatorname{supp\,ess} \phi_N = S_0 - v_N$. This is a contradiction, because we defined v_N to satisfy $|T_{N-1} \cup (S_0 - v_N)| > |T_{N-1}|$, what implies that $(S_0 - v_N)$ is not contained in T_{N-1}. $\square$

3.3 Nazarov uncertainty principle

For applications, the main drawback of Theorem 3.2.11 is that the value of c is completely unspecified. In this section we reproduce the proof of the Nazarov UP, originally proved in the 1-dimensional case by Nazarov himself in [21] and later generalized to the multidimensional case in [15], which shows that c can be taken to be $A^{|\Sigma||S|}$ for A a constant, whose value can be traced dawn by going carefully over the proof; in particular, its value depends only on the sizes of S and Σ.

3.3.1 The Nazarov-Turán Lemma

A fundamental step in the proof of this UP is Nazarov's generalization of the Turán Lemma: the Nazarov-Turán Lemma (see Lemma 3.3.4 below). It allows us to estimate the Fourier coefficients of a 1-dimensional trigonometric polynomial by controlling the polynomial in a small subset of the torus $\mathbb{T}$ and the number of non-zero coefficients. Further, we will deal with a generalization of the lemma to the multidimensional case (see Lemma 3.3.6), due to Fontes-Merz.

Lemma 3.3.1 (Weak boundedness of the Hilbert transform, [27])**.** *Let $f \in C^\infty(\mathbb{T})$. For $z \in \mathbb{T}$, define*

$$H(f)(z) = \frac{1}{2\pi i} \lim_{r \nearrow 1} \int_{\mathbb{T}} \frac{f(w)}{w - rz} \, \mathrm{d}w.$$

Then,

1. *For any $z \in \mathbb{T}$ the previous limit exists. Moreover, if f is real-valued then $Hf(z) = \frac{f(z)+\hat{f}(0)}{2} + i\tilde{f}(z)$, for $\tilde{f}(z)$ real-valued.*
2. *The following weak-type inequality holds:*

$$\left|\left\{|\tilde{f}(z)| > t\right\}\right| \lesssim \frac{\|f\|_1}{t},$$

or, equivalently,

$$|\{|Hf(z)| > t\}| \lesssim \frac{\|f\|_1}{t}. \tag{3.4}$$

Proof. First, we show that the limit does actually exist. For $r < 1$, define

$$H_r(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w)}{w - rz} \, \mathrm{d}w = \frac{1}{2\pi i} \int_{\mathbb{T}} w^{-1} f(w) \sum_{j=0}^{\infty} \left(\frac{rz}{w}\right)^j \mathrm{d}w =$$

$$\frac{1}{2\pi i} \sum_{j=0}^{\infty} \int_{\mathbb{T}} f(w) w^{-1} \left(\frac{rz}{w}\right)^j \mathrm{d}w = \sum_{j=0}^{\infty} (rz)^j \hat{f}(j),$$

where as usual

$$\hat{f}(j) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} \, \mathrm{d}\theta.$$

Hence, as the smoothness of f implies that $\sum_{j=0}^{\infty} |\hat{f}(j)| < \infty$, the Dominated Convergence Theorem yields

$$Hf(z) = \sum_{j=0}^{\infty} z^j \hat{f}(j).$$

In the particular case where f is real valued, the Fourier Inversion formula yields

$$f(z) = \sum_{j\in\mathbb{Z}} f(j)z^j = \sum_{j=0}^{\infty} f(j)z^j + \overline{\sum_{j=1}^{\infty} f(j)z^j} = Hf(z) + \overline{H(f)(z)} - \hat{f}(0),$$

whence we deduce that

$$\operatorname{Re}(Hf(z)) = \frac{f(z)+\hat{f}(0)}{2}.$$

Second, we show that if $f \geq 0$, then for $r<1$, $\operatorname{Re} H_r(f) \geq 0$. For this, note that

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(w)}{w-rz}\,\mathrm{d}w = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{1-rze^{-i\theta}}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{|1-rze^{-i\theta}|^2}(1-r\bar{z}e^{i\theta})\,\mathrm{d}\theta,$$

and since $\operatorname{Re}(1-r\bar{z}e^{i\theta})>0$, the result follows.

Third, note that for $z\in\mathbb{D}$, the map

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(w)}{w-z}\,\mathrm{d}w$$

is holomorphic in $\mathbb{D}$ and, as we saw before, if $f \geq 0$ $\operatorname{Re} F(z) \geq 0$. Thus, for each s, $\log|1+sF(z)|$ is harmonic in $\mathbb{D}$. Hence, for each $r<1$,

$$\log\left|1+s\hat{f}(0)\right| = \log|1+sF(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|1+sF(re^{i\theta})\right|\mathrm{d}\theta.$$

Here, since

$$\left|1+sF(re^{i\theta})\right|^2 \geq \left(1+s\operatorname{Re}(F(re^{i\theta}))\right)^2 \geq 1$$

the integrand is non-negative, so we can apply Fatou's Lemma to conclude that

$$\log\left|1+s\hat{f}(0)\right| \geq \frac{1}{2\pi}\int_0^{2\pi}\log\left|1+sHf(e^{i\theta})\right|\mathrm{d}\theta.$$

Now, since

$$\left|1+sHf(e^{i\theta})\right|^2 \geq 1+s^2\tilde{f}^2(e^{i\theta}),$$

an application of Chebyshev's inequality yields

$$s\hat{f}(0) \geq \log\left|1+s\hat{f}(0)\right| \geq |\{|\tilde{f}|>t\}|\sqrt{\log(1+s^2t^2)},$$

whence the result follows by setting $s=t^{-1}$ and recalling that $\hat{f}(0) = \frac{1}{2\pi}\|f\|_1$.
Observe that for general smooth f and Φ_n a non-negative, smooth approximation of the identity,

$$H(\Phi_n\star f)(z) = \sum_{j=0}^{\infty}\hat{\Phi}_n(j)\hat{f}(j)z^j.$$

Thus, the Dominated convergence Lemma yields that

$$\lim_n H(\Phi_n\star f)(z) = \sum_{j=0}^{\infty}\hat{f}(j)z^j = Hf(z).$$

For a general smooth f, write $f = f_1 - f_2 + if_3 - if_4$, with each $f_j \geq 0$, then

$$|\{|Hf(z)| > t\}| = \lim_n |\{|H(\Phi_n \star f)(z)| > t\}| \leq \lim_n \sum_{j=1}^{4} |\{|H(\Phi_n \star f_j)(z)| > t/4\}|$$

$$\lesssim \lim_n \sum_{j=1}^{4} \frac{\|\Phi_n \star f_j\|_1}{t} = \sum_{j=1}^{4} \frac{\|f_j\|_1}{t} = \frac{\|f\|_1}{t}.$$

□

Lemma 3.3.2 ([14]). *Let P, Q be polynomials and let $R = P/Q$. Then there exists $B > 0$ such that for all $R \neq 0$, and for any t, the following holds:*

$$|\{z \in \mathbb{T} : |R'(z)| > t \cdot r \cdot |R(z)|\}| < \frac{B}{t}, \tag{3.5}$$

where $r = \deg P + \deg Q$.

Proof. Clearly, the form of $\frac{R(z)}{R'(z)}$ shows that it suffices to prove that for any $c_1, \dots, c_r \in \mathbb{C}$, possibly repeated,

$$\left|\left\{z \in \mathbb{T} : |\sum_{j=1}^{r} \frac{1}{z - c_j}| > t\right\}\right| \lesssim \frac{r}{t}.$$

First, assume that $|c_j| > 1$, then for $|z| < 1$,

$$I_j = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\bar{t}}{|t - c_j|^2} \frac{\mathrm{d}t}{t - z} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{1}{(t - c_j)(1 - t\bar{c_j})} \frac{\mathrm{d}t}{t - z}.$$

Now, since $\frac{1}{t - c_j}$ is holomorphic in the disk and

$$\frac{1}{1 - t\bar{c_j}} \frac{1}{t - z} = \frac{-1}{1 - \bar{c_j}z} \left(\frac{1}{t - \bar{c_j}^{-1}} + \frac{-1}{t - z} \right),$$

the Cauchy integral formula yields

$$I_j = \frac{-1}{1 - \bar{c_j}z} \left(\frac{1}{\bar{c_j}^{-1} - c_j} + \frac{-1}{z - c_j} \right) = \frac{1}{1 - |c_j|^2} \frac{1}{z - c_j}.$$

Hence, if

$$f_j(t) = \left(1 - |c_j|^2\right) \frac{\bar{t}}{|t - c_j|^2},$$

we have

$$H_r f_j(z) = \frac{1}{rz - c_j};$$

$$Hf_j(z) = \frac{1}{z - c_j}$$

and

$$\|f_j\|_1 = (|c_j|^2 - 1) \int_{\mathbb{T}} \frac{|dt|}{|t - c_j|^2} \approx 1.$$

Thus, using Lemma 3.3.1, we have

$$\left| \left\{ z \in \mathbb{T} : \left| \sum_{j=1, |c_j|>1}^{r} \frac{1}{z - c_j} \right| > t \right\} \right| \lesssim \frac{r}{t}.$$

Moreover, we can extend the previous result to the c_j with modulus 1. Indeed,

$$\left| \left\{ z \in \mathbb{T} : \left| \sum_{j=1, |c_j|=1}^{r} \frac{1}{z - c_j} \right| > t \right\} \right| = \lim_{r \searrow 1} \left| \left\{ z \in \mathbb{T} : \left| \sum_{j=1, |c_j|=1}^{r} \frac{1}{z - rc_j} \right| > t \right\} \right| \lesssim \frac{r}{t}.$$

Finally, if $|c_j| < 1$,

$$Hf_j(z) = \frac{1}{c_j - z} + f_j(z),$$

and since $\|f_j\|_1 \approx 1$, we obtain the result by repeating the previous considerations. □

Definition 3.3.3. *A trigonometric polynomial is a function of the form*

$$P = \sum_{k=1}^{N} c_k z^{n_k},$$

for $z \in \mathbb{T}$, $n_k \in \mathbb{Z}$, $n_1 < n_2 < \cdots < n_N$ *and* $c_k \in \mathbb{C} \setminus \{0\}$.

We say that $\operatorname{ord} P = N$ *and* $\left\| \hat{P} \right\|_1 = \sum_{k=1}^{N} |c_k|$.

Lemma 3.3.4 (One dimensional Nazarov-Turán Lemma, [20])**.** *Let P be a trigonometric polynomial. Then, there exists $A > 0$ such that for any measurable set $\Gamma \subset \mathbb{T}$, the following holds*

$$\left\| \hat{P} \right\|_1 \leq \left(\frac{A}{|\Gamma|} \right)^{\operatorname{ord} P - 1} \sup_{\Gamma} |P|. \tag{3.6}$$

Proof. Put $N = \operatorname{ord} P$. Note that in the trivial case $|\Gamma| = 1$, we have $\sup_\Gamma |P| = \sup_{\mathbb{T}} |P| \geq \max_k |c_k|$. Hence,

$$\left\| \hat{P} \right\|_1 \leq N \sup_{\Gamma} |P| \leq \left(\frac{A}{|\Gamma|} \right)^{\operatorname{ord} P - 1} \sup_{\Gamma} |P|,$$

if $A \geq 2$. From now on, we assume that $|\Gamma| < 1$.

Step 1: We construct a sequence of trigonometric polynomials $P_1, \ldots, P_N$ which satisfy the following:

1. $P_N = P$;
2. $\operatorname{ord} P_k = k$, for $1 \leq k \leq N$;
3. for $2 \leq k \leq N$, $\left\| \hat{P}_{k-1} \right\|_1 \geq \frac{1}{6B} \left\| \hat{P}_k \right\|_1$, (the same B as in Lemma 3.3.2) ;

4. $|\{z \in \mathbb{T} : |P_{k-1}| > t|P_k|\}| < \frac{1}{t}$, for $t > 0$ and $2 \leq k \leq N$.

Using the notation of Definition 3.3.3, we proceed as follows: put

$$Q_-(z) = \left(z^{-n_1}P\right)'(z) = \sum_{k=1}^{N} c_k(n_k - n_1)z^{n_k-n_1-1};$$

$$Q_+(z) = \left(z^{-n_N}P\right)'(z) = \sum_{k=1}^{N} c_k(n_k - n_N)z^{n_k-n_N-1}.$$

Clearly, both $\operatorname{ord} Q_- = \operatorname{ord} Q_+ = N - 1$ and

$$\left\|\hat{Q}_-\right\|_1 + \left\|\hat{Q}_+\right\|_1 = \sum_{k=1}^{N} |c_k|(n_N - n_1) = (n_N - n_1)\left\|\hat{P}\right\|_1.$$

Put $Q_{N-1} = Q_-$, if $2\left\|\hat{Q}_-\right\|_1 \geq (n_N - n_1)\left\|\hat{P}\right\|_1$ and $Q_{N-1} = Q_+$ otherwise.
Finally, for B as in Lemma 3.3.2, define

$$P_{N-1} = (3B(n_N - n_1))^{-1}Q_{N-1}.$$

Repeating the same procedure for P_{N-1} in place of P, we obtain P_{N-2} until we reach P_1 with order 1.
We now verify that the constructed sequence satisfies properties 1-4:

1. Clear.

2. Also clear.

3. By definition of Q_{k-1}, we have $\left\|\hat{P}_{k-1}\right\|_1 = (3B(n_N - n_1))^{-1}\left\|\hat{Q}_{k-1}\right\|_1 \geq \frac{1}{6B}\left\|\hat{P}_k\right\|_1$.

4. By construction, if n_s is the power of z chosen in the definition of Q_{k-1}, we have

$$|\{|P_{k-1}| > t|P_k|\}| = \left|\{|\left(z^{-n_s}P_k\right)'| > 3B(n_N - n_1)t|z^{-n_s}P_k|\}\right|.$$

   Here we use Lemma 3.3.2 with $R(z) = z^{-n_s}P_k(z)$. Observe that if $n_s = n_1$, then R is a polynomial with $r = n_N - n_1$; and if $n_s = n_N$,

$$R = \frac{\sum_{k=1}^{N} c_k z^{n_k-n_1}}{z^{n_N-n_1}},$$

   so that $r = 2(n_N - n_1)$. Thus, the estimate (3.5) gives

$$|\{|P_{k-1}| > t|P_k|\}| < \frac{2(n_N - n_1)B}{3B(n_N - n_1)}t < \frac{1}{t}.$$

Step 2: Since $P_1 = Cz^m$ for some $m \in \mathbb{Z}$, we have, iterating property (3),

$$|C| = \left\|\hat{P}_1\right\|_1 \geq \left\|\hat{P}\right\|_1 (6B)^{-N+1}.$$

Now we proceed to estimate C. Put $\phi_k = \log(|P_{k-1}|/|P_k|)$. Then, property (4) gives $|\{z : |\phi_k(z)| > t\}| < e^{-t}$. Also

$$\frac{|C|}{|P|} = \frac{|P_1|}{|P|} = \exp(\sum_{k=2}^{N} \phi_k).$$

Finally, for $\alpha \in \mathbb{R}_+$, put $\rho_k = \phi_k - \min(\phi_k, \alpha) \geq \phi_k - \alpha$. Then

$$X := \left|\left\{\sum_{k=2}^{N} \phi_k > (\alpha+1)(N-1)\right\}\right| = \left|\left\{\sum_{k=2}^{N} (\phi_k - \alpha) > (N-1)\right\}\right| \leq \left|\left\{\sum_{k=2}^{N} \rho_k > (N-1)\right\}\right|.$$

Using the Chebyshev inequality, we deduce that

$$X \leq \frac{1}{N-1}\left(\sum_{k=2}^{N} \int_{\mathbb{T}} \rho_k(z) dm(z)\right).$$

Now, for each k, and using Fubini's Theorem,

$$\int_{\mathbb{T}} \rho_k(z)dm(z) = \int_{\mathbb{T}\cap\{\phi_k>\alpha\}} \left(\int_{\alpha}^{\phi_k(z)} 1dt\right) dm(z) = \int_{\alpha}^{\infty} \int_{\{z:\phi_k(z)>t\}} 1dm(z)dt = \int_{\alpha}^{\infty} |\{\phi_k > t\}|dt.$$

So that using the estimate $|\{\phi_k > t\}| < e^{-t}$, we conclude that $X < e^{-\alpha}$. Setting $\alpha = -\log(|\Gamma|)$, we have that $X < |\Gamma| < 1$. Thus, we deduce that there must exist a $z \in \Gamma$ for which

$$\sum_{k=2}^{N} \phi_k(z) \leq (N-1)(-\log(|\Gamma|) + 1).$$

This implies that

$$C \leq \sup_{\Gamma} |P| e^{(N-1)(1-\log(|\Gamma|)} = \sup_{\Gamma} |P| \cdot (e/|\Gamma|)^{N-1}.$$

In conclusion,

$$\left\|\hat{P}\right\|_1 \leq \left(\frac{6Be}{|\Gamma|}\right)^{N-1}.$$

□

Definition 3.3.5. *A d-dimensional trigonometric polynomial is a function of the form*

$$P(z) = \sum_{k=1}^{N} c_k z_1^{n_{k,1}} \cdots z_n^{n_{k,d}},$$

for $z_i \in \mathbb{T}$, $n_{k,1}, \ldots, n_{k,d} \in \mathbb{Z}$ and $c_k \in \mathbb{C} \setminus \{0\}$ for $k = 1, \ldots N$. Note that we can write

$$P(z) = \sum_{k=1}^{N'} c'_k z_i^{n'_k} Q_k(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots z_d),$$

with $n'_1 < \ldots, n'_{N'}$, so that, if we "freeze" z_j for $j \neq i$, we obtain a trigonometric polynomial of order at most N'. We define $\operatorname{ord}_i P = N'$. Observe that $\operatorname{ord}_i P \geq 1$ unless $P = 0$.

Lemma 3.3.6 (Multidimensional Nazarov-Turan by Fontes-Merz, [10]). *Let P be a d-dimensional trigonometric polynomial and $E \subset \mathbb{T}^d$ a measurable set. Then, there exists $A > 0$ independent of d such that*

$$\sup_{\mathbb{T}^d} |P| \leq \left(\frac{Ad}{|E|}\right)^{-d+\sum_{i=1}^d \operatorname{ord}_i P} \sup_{z\in E} |P(z)|.$$

Proof. We proceed by induction on the dimension d. For $d = 1$ it has already been proved, because $\left\|\hat{P}\right\|_1 \geq \sup_{\mathbb{T}} |P|$. Now we assume the result for $d \geq 1$ and prove it for $d+1$. Let $P(z) = \sum_{k=1}^N c_k z_1^{n_{k,1}} \cdots z_n^{n_{k,d}}$ and $E \subset \mathbb{T}^{d+1}$. For any $z \in \mathbb{T}$, put

$$E_z = \{(z_1, \dots, z_d) : (z_1, \dots, z_d, z) \in E\} \subset \mathbb{T}^d$$

and let $B = \{z \in \mathbb{T} : |E_z| \geq |E|/C\}$, for $C = \frac{d+1}{d}$. At the end of the proof, we shall show that $|B| > 0$.
Let $w = (w_1, \dots, w_d, w_{d+1})$ be the point of $\mathbb{T}^{d+1}$ at which the maximum of P is attained. Then, an application of Lemma 3.3.4 with B in place of Γ and the univariate polynomial $Q(z) = P(w_1, \dots, w_d, z)$ yields

$$\sup_{\mathbb{T}^{d+1}} |P| = \sup_{\mathbb{T}} |Q| \leq \left(\frac{A}{|B|}\right)^{-1+\operatorname{ord}_{d+1} P} \sup_{z\in B} |Q(z)| = \left(\frac{A}{|B|}\right)^{-1+\operatorname{ord}_{d+1} P} \sup_{z\in B} |P(w_1, \dots, w_d, z)|.$$

This means that for any $\varepsilon > 0$, there exists $z(\varepsilon)$ in B such that

$$\sup_{\mathbb{T}^{d+1}} |P| \leq \left(\frac{A}{|B|}\right)^{-1+\operatorname{ord}_{d+1} P} \left(\varepsilon + |P(w_1, \dots, w_d, z(\varepsilon)|\right).$$

Next, we apply the induction hypothesis to the polynomial $(z_1, \dots, z_d) \to P(z_1, \dots, z_d, z(\varepsilon))$ and the set $E_{z(\varepsilon)}$. Then,

$$\sup_{(z_1,\dots z_d)\in\mathbb{T}^d} |P(z_1, \dots, z_d, z(\varepsilon))| \leq \left(\frac{Ad}{|E_{z(\varepsilon)}|}\right)^{-d+\sum_{i=1}^d \operatorname{ord}_i P} \sup_{(z_1,\dots,z_d)\in E_{z(\varepsilon)}} |P(z_1, \dots, z_d, z(\varepsilon))|.$$

Now, clearly

$$|P(w_1, \dots, w_d, z(\varepsilon)| \leq \sup_{(z_1,\dots z_d)\in\mathbb{T}^d} |P(z_1, \dots, z_d, z(\varepsilon))|$$

and

$$\sup_{(z_1,\dots,z_d)\in E_{z(\varepsilon)}} |P(z_1, \dots, z_d, z(\varepsilon))| \leq \sup_{z\in E} |P(z)|.$$

Besides, since $z(\varepsilon) \in B$, we have that $|E_{z(\varepsilon)}| \geq |E|/C$, so putting everything together, we have

$$\sup_{\mathbb{T}^{d+1}} |P| \leq \left(\frac{A}{|B|}\right)^{-1+\operatorname{ord}_{d+1} P} \left(\varepsilon + \left(\frac{CAd}{|E|}\right)^{-d+\sum_{i=1}^d \operatorname{ord}_i P} \sup_{z\in E} |P(z)|\right),$$

and letting $\varepsilon \to 0$,

$$\sup_{\mathbb{T}^{d+1}} |P| \leq \left(\frac{A}{|B|}\right)^{-1+\operatorname{ord}_{d+1} P} \left(\frac{CAd}{|E|}\right)^{-d+\sum_{i=1}^{d} \operatorname{ord}_i P} \sup_{z\in E} |P(z)|.$$

Finally, to estimate $|B|$, note that, by Fubini's Theorem,

$$|E| = \int_{\mathbb{T}} |E_z| dm(z) \leq |B| + (1-|B|)\frac{|E|}{C} \leq |B| + \frac{|E|}{C},$$

so that

$$|B| \geq |E|\frac{C-1}{C} = \frac{1}{d+1}.$$

In conclusion,

$$\sup_{\mathbb{T}^{d+1}} |P| \leq \left(\frac{A(d+1)}{|E|}\right)^{-(d+1)+\sum_{i=1}^{d+1} \operatorname{ord}_i P} \sup_{z\in E} |P(z)|.$$

□

Before proving Nazarov's Uncertainty Theorem, we first show a similar result which uses the same tools while avoiding some technical difficulties.

Theorem 3.3.7 (Nazarov's uncertainty for Fourier Series)**.** *Let* $f \in L^2([0,1])$ *and let* $S = \operatorname{supp\,ess} f$*. Then, if* I *is a finite subset of* $\mathbb{Z}$*,*

$$\sum_{n\in I} |\hat{f}(n)| \leq \left(\frac{A}{1-|S|}\right)^{|I|} \left(\frac{2}{1-|S|}\sum_{n\notin I} |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Proof. Put

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi ixn} = \sum_{n\in I} \hat{f}(n)e^{2\pi ixn} + \sum_{n\notin I} \hat{f}(n)e^{2\pi ixn} =: f_1(x) + f_2(x).$$

Observe that since f vanishes outside of S,

$$\int_{[0,1]\setminus S} |f_1|^2 = \int_{[0,1]\setminus S} |f_2|^2 \leq \sum_{n\notin I} |\hat{f}(n)|^2.$$

Then, by Chebyshev,

$$\left| x \in [0,1]\setminus S : \left\{ |f_1(x)|^2 > \frac{2}{1-|S|}\sum_{n\notin I} |\hat{f}(n)|^2 \right\} \right| \leq \frac{1-|S|}{2}.$$

Hence, in a set of measure at least $\frac{1-|S|}{2}$

$$|f_1|^2 \leq \frac{2}{1-|S|}\sum_{n\notin I} |\hat{f}(n)|^2,$$

so that by Lemma 3.3.4,

$$\sum_{n\notin I} |\hat{f}(n)| \leq \left(\frac{A}{1-|S|}\right)^{|I|} \left(\frac{2}{1-|S|}\sum_{n\notin I} |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

□

Finally, using Lemma 3.2.6 we obtain the following corollary:

Corollary 3.3.8. *There exists a B such that for $f \in L^2([0,1])$, $S \subset [0,1]$ and $I \subset \mathbb{Z}$ finite, the following holds:*

$$\|f\|_2 \leq \left(\frac{B}{1-|S|}\right)^{|I|+\frac{1}{2}} \left(\left(\int_{[0,1]\setminus S} |f|^2\right)^{\frac{1}{2}} + \left(\sum_{n\notin I} |\hat{f}(n)|^2\right)^{\frac{1}{2}}\right). \tag{3.7}$$

Proof. First, observe that from Theorem 3.3.7, we deduce that for f supported in S,

$$\left(\sum_{n\in I} |\hat{f}(n)|^2\right)^{\frac{1}{2}} \leq \left(\frac{2A}{1-|S|}\right)^{|I|+\frac{1}{2}} \left(\sum_{n\notin I} |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Clearly, this implies that for f supported in S,

$$\|f\|_2 \leq \left(\frac{4A}{1-|S|}\right)^{|I|+\frac{1}{2}} \left(\sum_{n\notin I} |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Now, if we define

$$Pf = \sum_{n\in I} \hat{f}(n) e^{2\pi i x n}$$

and

$$Qf = \mathbb{1}_S f,$$

we have that for any f,

$$\|Qf\| \leq \left(\frac{4A}{1-|S|}\right)^{|I|+\frac{1}{2}} \left\|P^{\perp} Q f\right\|.$$

This is item 4 in Lemma 3.2.6, which implies item 3 from the same Lemma, that is, that for any f,

$$\|f\| \leq \left[1 + \left(\frac{4A}{1-|S|}\right)^{|I|+\frac{1}{2}}\right] (\left\|P^{\perp} f\right\| + \left\|Q^{\perp} f\right\|),$$

so that the result follows with $B = \max(2, 10A)$.

□

Observe that if we want to prove an analogous theorem for the Fourier transform we have the problem that the sum of the values of a function at a discrete set of points need not be related to the integral of the function. What we are going to prove is that there exist some points for which it is true. More precisely,

Lemma 3.3.9 (Lattice averaging, [15])**.** *Let $\phi \in L^1(\mathbb{R}^d)$, $\phi \geq 0$. Then,*

$$\int_{SO^d} \int_1^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \phi(\rho(vk))\, \mathrm{d}v\, \mathrm{d}m(\rho) \approx \int_{\|x\|\geq 1} \phi(x)\, \mathrm{d}m(x). \tag{3.8}$$

We remark that the integral over SO^d is with respect to its Haar measure normalized so that the total measure is 1, since we are not going to use any deep result about this theory, the reader can simply think of a uniform measure on SO^d which is left-invariant under the action of the group.

Proof. The picture of the situation is the following: $\mathbb{Z}^d \setminus \{0\}$ is the (punctured) lattice with integer coordinates, multiplying by $v \in (1,2)$ dilate the lattice; and, applying ρ, we rotate the lattice.
Now we begin the proof.

For each k,

$$I_k := \int_{SO^d} \int_1^2 \phi(\rho(vk))\, \mathrm{d}v\, \mathrm{d}m(\rho) = \|k\|^{-1} \int_{SO^d} \int_{a\|k\|}^{b\|k\|} \phi(v\rho(\frac{k}{\|k\|}))\, \mathrm{d}v\, \mathrm{d}m(\rho).$$

Then, since for a fixed $n \in \mathbb{S}^{d-1}$, the orbit of n by SO^d is the whole $\mathbb{S}^{d-1}$ (with equivalent measure), we deduce that

$$\int_{SO^d} \phi(v\rho(\frac{k}{\|k\|}))\, \mathrm{d}m(\rho) \approx \int_{\mathbb{S}^{d-1}} \phi(vn)\, \mathrm{d}m(n).$$

Hence, a change of variable yields

$$I_k\, \|k\|^{-1} \int_{\left[\|k\|,2\|k\|\right] \times \mathbb{S}^{d-1}} \phi(vn)\, \mathrm{d}v\, \mathrm{d}m(n) \approx \|k\|^{-d} \int_{\|k\| \leq \|x\| \leq 2\|k\|} \phi(x) dm(x).$$

Finally,

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} I_k \approx \int_{\mathbb{R}^d} K(x)\phi(x) dm(x),$$

where

$$K(x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|k\|^{-d}\, \mathbb{1}_{\|k\| \leq \|x\| \leq 2\|k\|}(x).$$

Since for each x,

$$K(x) = \sum_{\left\{k \in \mathbb{Z}^d \setminus \{0\} : \frac{\|x\|}{2} \leq \|k\| \leq \|x\|\right\}} \|k\|^{-d},$$

which is zero if $\|x\| < 1$ and, if $\|x\| > 1$,

$$K(x) \approx \|x\|^{-d} \# \left\{k \in \mathbb{Z}^d \setminus \{0\} : \|k\| \leq \|x\| \leq 2\|k\|\right\} \approx 1,$$

as the number of integer points in a spherical region is approximately its volume, we conclude the proof. □

Definition 3.3.10 (Periodization)**.** *Given $f \in \mathcal{S}(\mathbb{R}^d)$, $\rho \in SO^d$ and $2 > v > 1$, we define, for $t \in [0,1]^d$,*

$$\Gamma_{\rho,v}(t) = \frac{1}{\sqrt{v}} \sum_{k \in \mathbb{Z}^d} f\left(\frac{\rho(k+t)}{v}\right).$$

Proposition 3.3.11. *The following hold:*

1. *For $m \in \mathbb{Z}^d$,*
$$\widehat{\Gamma(f)_{\rho,v}}(m) = v^{d-1/2} \hat{f}(v\rho(m));$$
2. *For $S \subset \mathbb{R}^d$ of finite measure and f supported in S*
$$|\{t \in (0,1) : \Gamma_{\rho,v}(t) \neq 0\}| \leq 2^d |S|;$$

Proof. For (1), assume first that $v = 1$ and ρ is the identity. Then, for $m \in \mathbb{Z}^d$, a computation shows that

$$\int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} f(k+t)e^{-2\pi i\langle t,m\rangle} dt = \sum_{k\in\mathbb{Z}^d} \int_{k+[0,1]^d} f(t)e^{-2\pi i\langle t,m\rangle} dt = \int_{\mathbb{R}^d} f(t)e^{-2\pi i\langle t,m\rangle} = \hat{f}(m).$$

Now, in the general case, given ρ and v, define $f_{\rho,v}(x) = f(\frac{\rho(x)}{v})$. Clearly,

$$\frac{1}{\sqrt{v}}\Gamma(f_{\rho,v})_{Id,1}(t) = \Gamma(f)_{\rho,v}(t),$$

and

$$\widehat{f_{\rho,v}}(m) = v^d \hat{f}(v\rho(m)),$$

whence the result follows.

For (2), note that

$$|\{t \in (0,1) : \Gamma_{\rho,v}(t) \neq 0\}| \subset \left|\cup_{k\in\mathbb{Z}^d}(vS - k) \cap [0,1]^d\right| = \left|\cup_{k\in\mathbb{Z}^d} vS \cap (k + [0,1]^d)\right| = |vS| = v^d|S|,$$

whence the result follows from the fact that $v \leq 2$.

□

Proposition 3.3.12. *For $F : SO^d \times [1,2] \to \mathbb{R}$ put*

$$\mathbb{E}(F) := \int_{SO^d} \int_1^2 F(v,\rho) dv d\rho.$$

Then, there exists $C > 0$ such that for $0 \in \Sigma$ a measurable set of finite measure and

$$\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z} : v\rho(k) \in \Sigma\},$$

the following hold:

1.

$$\mathbb{E}(\#\mathcal{M}_{\rho,v} - 1) \leq C|\Sigma|;$$

2.

$$\mathbb{E}\left(\sum_{k\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}} |\hat{f}(v\rho(k)|^2\right) \leq C\int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2.$$

Proof. Using Lemma 3.3.9 and recalling that $0 \in \Sigma$,

$$\mathbb{E}(\#\mathcal{M}_{\rho,v} - 1) = \int_{SO^d} \int_1^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \mathbb{1}_\Sigma(v\rho(k)) dv d\rho \approx \int_{\|x\|\geq 1} \mathbb{1}_\Sigma \leq |\Sigma|,$$

whence (1) follows. For (2), similarly,

$$\mathbb{E}\left(\sum_{k\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}} |\hat{f}(v\rho(k)|^2\right) = \int_{SO^d} \int_1^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\hat{f}(v\rho(k)|^2 \mathbb{1}_{\mathbb{R}\setminus\Sigma}(v\rho(k))\, \mathrm{d}v\, \mathrm{d}\rho \approx \int_{\|x\|\geq 1} |\hat{f}|^2 \mathbb{1}_{\mathbb{R}\setminus\Sigma}.$$

□

Theorem 3.3.13 (Nazarov's uncertainty principle, [15]). *There exists $A > 0$ such that for any $f \in L^2(\mathbb{R}^d)$ and any $S, \Sigma \subset \mathbb{R}^d$ of finite measure, the following holds:*

$$\int_{\mathbb{R}^d} |f|^2 \leq A^{|\Sigma||S|} \left(\int_{\mathbb{R}^d \setminus S} |f|^2 + \int_{\mathbb{R}^d \setminus \Sigma} |\hat{f}|^2 \right). \tag{3.9}$$

Proof. Before we begin with the proof, some simplifications are in order.
First, if $|\Sigma||S|$ is small, the results of Section 3.2.2 suffice to obtain the result. Indeed, using the notation of that section,

$$f = R^{-1}(P(I-Q)f + (I-P)f),$$

so that

$$\|f\| \leq \left\|R^{-1}\right\| \left(\left\|Q^{\perp} f\right\| + \left\|P^{\perp} f\right\| \right).$$

Since by Lemma 3.2.13, we know that $\|PQ\| \leq |S||\Sigma|$, and

$$\left\|R^{-1}\right\| \leq \sum_{j=0}^{\infty} \|PQ\|^j \leq \frac{1}{1 - |\Sigma||S|} \leq 2^{|\Sigma||S|};$$

if $|\Sigma||S| \leq \frac{1}{2}$, we may assume that $|\Sigma||S| \geq \frac{1}{2}$.
Second, by a change of scale, we may assume that $|S| = 2^{-d-1}$. Indeed, given f consider $f_\lambda(x) = \lambda f(\lambda x)$. Then,

$$\begin{aligned} \lambda^{-1} \|f\|_2^2 = \|f_\lambda\|_2^2 &\leq A^{|\Sigma||S|} \left(\left\|f_\lambda \mathbb{1}_{\mathbb{R}^d \setminus \lambda S}\right\|_2^2 + \left\|\widehat{f_\lambda} \mathbb{1}_{\mathbb{R}^d \setminus \lambda^{-1}\Sigma}\right\|_2^2 \right) \\ &= \lambda^{-1} A^{|\Sigma||S|} \left(\left\|f \mathbb{1}_{\mathbb{R}^d \setminus S}\right\|_2^2 + \left\|\hat{f} \mathbb{1}_{\mathbb{R}^d \setminus \Sigma}\right\|_2^2 \right), \end{aligned}$$

so that it suffices to obtain the result for f_λ for a given λ.
Third, by a density argument, we may assume that $f \in \mathcal{S}(\mathbb{R}^d)$.
Fourth, it suffices to obtain the result for f supported in S. Indeed, this is implication (4) $\implies$ (3) of Lemma 3.2.6 (note that we are assuming that $|\Sigma||S| \geq \frac{1}{2}$, and this means that $A^{|\Sigma||S|} + 1 \approx A^{|\Sigma||S|}$).
Fifth, if we prove, that for f supported in S,

$$\left\|\hat{f} \mathbb{1}_{\Sigma}\right\|_2^2 \leq A^{|\Sigma|} \left\|\hat{f} \mathbb{1}_{\mathbb{R}^d \setminus \Sigma}\right\|_2^2, \tag{3.10}$$

we can conclude the proof. Indeed, if (3.10) holds,

$$\|f\|_2^2 = \left\|\hat{f} \mathbb{1}_{\mathbb{R}^d \setminus \Sigma}\right\|_2^2 + \left\|\hat{f} \mathbb{1}_{\Sigma}\right\|_2^2 \leq (A^{|\Sigma|} + 1) \left\|\hat{f} \mathbb{1}_{\mathbb{R}^d \setminus \Sigma}\right\|_2^2 \lesssim (A^{|\Sigma|}) \left\|\hat{f} \mathbb{1}_{\mathbb{R}^d \setminus \Sigma}\right\|_2^2,$$

where the last estimate follows from $|\Sigma| \geq \frac{1}{2|S|} = 2^d$.

Now we begin the proof. For each ρ, v, put

$$\Gamma_{\rho,v}(t) = \sum_{k \in \mathcal{M}_{p,v}} \widehat{\Gamma_{\rho,v}}(k) e^{2\pi i \langle k,t \rangle} + \sum_{k \notin \mathcal{M}_{p,v}} \widehat{\Gamma_{\rho,v}}(k) e^{2\pi i \langle k,t \rangle} =: \Gamma_1(t) + \Gamma_2(t).$$

We want to estimate $\widehat{\Gamma_{\rho,v}}(0) \approx \hat{f}(0)$. Clearly, $|\widehat{\Gamma_{\rho,v}}(0)| \leq \sup_{t\in[0,1]^d} |\Gamma_{\rho,v}|$. To estimate this last quantity we shall make use Lemma 3.3.6.

Using Proposition 3.3.12 and the Chebyshev inequality, we deduce that

$$\mathbb{P}(\#\mathcal{M}_{\rho,v} - 1 > 2C|\Sigma|) < \frac{1}{2};$$

$$\mathbb{P}\left(\sum_{k\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}} |\hat{f}(v\rho(k)|^2 > 2C\int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2\right) < \frac{1}{2};$$

so that there is a choice of v, ρ such that the following hold simultaneously

1. $$\#\mathcal{M}_{\rho,v} - 1 \leq 2C|\Sigma|;$$

2. $$\sum_{k\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}} |\hat{f}(v\rho(k)|^2 \leq 2C\int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2.$$

Now, by Proposition 3.3.11 and that $|S| = 2^{-d-1}$, we know that $\Gamma_{\rho,v}$ vanishes in a set F of measure at least 1/2. Thus $|\Gamma_1(t)| = |\Gamma_2(t)|$ whenever $t \in F$. Once again, since

$$\int_F |\Gamma_2(t)|^2 \leq \int_{[0,1]^d} |\Gamma_2(t)|^2 = C' \sum_{k\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}} |\hat{f}(v\rho(k)|^2 \leq 2C'C\int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2,$$

the Chebyshev inequality guarantees that $|\Gamma_2(t)|^2 \leq 8CC' \int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2$ in a set G of measure 1/4 inside F, consequently, the same holds for Γ_1. Finally, applying Lemma 3.3.6 to Γ_1 with

$$-d + \sum_{i=1}^d \operatorname{ord}_i \Gamma_i \leq \prod_i \operatorname{ord}_i \Gamma_1 = \#\mathcal{M}_{\rho,v} \leq 2C|\Sigma| + 1$$

and $E = G$, we obtain that

$$|\hat{f}(0)|^2 \lesssim A^{|\Sigma|} \int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2,$$

and repeating the proof by changing the role of 0 by any other point in Σ, we obtain

$$|\hat{f}(y)|^2 \leq A^{|\Sigma|} \int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2,$$

for $y \in \Sigma$, so that integrating over Σ we deduce that

$$\int_\Sigma |\hat{f}|^2 \leq |\Sigma| A^{|\Sigma|} \int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2 \leq A'^{|\Sigma|} \int_{\mathbb{R}^d\setminus\Sigma} |\hat{f}|^2,$$

that is, equation (3.10) holds. □

Finally, using Corollary 2.4.3 we can obtain similar results even in the case of $|\Sigma| = \infty$. Indeed,

Proposition 3.3.14. *Assume that Σ is a measurable set that satisfies $|\Sigma \cap I| \leq |I|^{1-\frac{r}{2}}$ for some $r > 1$ and every interval I. Then, there exists a $\delta > 0$ for which the inequality*

$$\|f\|_2^2 \lesssim \int_{\mathbb{R}\setminus I_\delta} |f|^2 + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2$$

holds whenever I_δ is an interval of length δ.

Proof. By Corollary 2.4.3 we know that there exists a K such that whenever $\hat{f}$ is supported in Σ,

$$\delta^{-\frac{1}{2}} \int_0^\delta |f|^2 \leq \int_0^1 |f|^2 x^{-\frac{1}{2}} \leq K \|f\|_2.$$

Let δ be such that

$$2\int_0^\delta |f|^2 \leq \|f\|_2^2 .$$

Then,

$$\|f\|_2^2 \lesssim \int_{\mathbb{R}\setminus[0,\delta]} |f|^2.$$

In the notation of Section 3.2.2, let Q be the projection onto Σ in the frequency side and P the projection onto $[0, \delta]$. Then, we have

$$\|Qv\| \lesssim \left\|P^\perp Qv\right\|$$

and by Lemma 3.2.6,

$$\|v\| \lesssim \left\|P^\perp v\right\| + \left\|Q^\perp v\right\| .$$

□

3.4 L^p Heisenberg type uncertainty inequalities

The Heisenberg UP

$$\|f\|_2^2 \lesssim \|fx\|_2 \left\|\hat{f}\xi\right\|_2$$

is a crucial result both in mathematics and physics, with numerous applications in both sciences. It is thus natural to consider the question of whether the L^2-norm or the weights x, ξ can be be replaced.

In [28], the following result was obtained

Theorem 3.4.1. *The inequality*

$$\|f\|_q^{\alpha+\beta} \lesssim \|fx^\alpha\|_q^\beta \left\|\hat{f}\xi^\beta\right\|_{q^*}^\alpha \tag{3.11}$$

holds in the following cases:

1. *$q < 2$ and $\beta > \frac{1}{q} - \frac{1}{2}$;*

2. *$q \geq 2$ and $0 < \alpha < \frac{1}{q}$.*

Observe that the Heisenberg UP is recovered by setting $p = q = 2$ and $\alpha = \beta = 1$.

The main theorem of this section is the following generalization of inequality (3.11):

Theorem 3.4.2. *Let $1 < p, q < \infty$ and $\alpha, \beta > 0$. Then, the inequality*

$$\|f\|_q^{\alpha+\beta-\frac{1}{q^*}+\frac{1}{p}} \lesssim \|fx^\alpha\|_q^{\beta-\frac{1}{q^*}+\frac{1}{p}} \left\|\hat{f}\xi^\beta\right\|_p^\alpha \tag{3.12}$$

holds if and only if $\beta > 1 - \frac{1}{q} - \frac{1}{p}$ and one of the following conditions holds:

1. *$q \geq 2$;*
2. *$p, q < 2$;*
3. *$q < 2 \leq p$ and $\beta > \frac{1}{2} - \frac{1}{p}$.*

Remark 3.4.3. *It is worth mentioning that $1 - \frac{1}{q} - \frac{1}{p} = \frac{1}{q^*} - \frac{1}{p} = \frac{1}{p^*} - \frac{1}{q}$.*

Remark 3.4.4. *Note that if $\beta - \frac{1}{q^*} + \frac{1}{p} < 0$, inequality (3.12) does not hold. Indeed, for any f, consider $f_N(x) = f(x+N)$ and let $N \to \infty$. If $\beta - \frac{1}{q^*} + \frac{1}{p} = 0$, the inequality becomes*

$$\|f\|_q \lesssim \left\|\hat{f}\xi^\beta\right\|_p, \tag{3.13}$$

which holds, assuming $\beta = 1 - \frac{1}{q} - \frac{1}{p}$, if and only if $q \geq p$ (see Corollary 2.3.9). In conclusion, in Theorem 3.4.2, we may always assume that $\beta > 1 - \frac{1}{q} - \frac{1}{p}$.

Remark 3.4.5. *The exponents of the norms which appear in inequality (3.12) are seen to be necessary by scaling arguments. Indeed, for a given f consider $f_\lambda(x) := Cf(\lambda x)$. Then, simple computations show that*

1. *$\|f_\lambda\|_q = C\lambda^{-\frac{1}{q}} \|f\|_q$;*
2. *$\|f_\lambda x^\alpha\|_q = C\lambda^{-\frac{1}{q}-\alpha} \|fx^\alpha\|_q$;*
3. *$\left\|\hat{f}_\lambda \xi^\beta\right\|_p = C\lambda^{-1+\frac{1}{p}+\beta} \left\|\hat{f}\xi^\beta\right\|_p$;*

and one can check that the exponents of the norm must be the ones appearing in inequality (3.12). Moreover, note that by choosing suitable C, λ we may always assume that two of the previous three norms in items 1-3 are equal to 1.

3.4.1 Proof of sufficiency

Lemma 3.4.6. *Let $1 < p, q < \infty$ and $\alpha, \beta > 0$. Assume that $\beta > \max(0, 1 - \frac{1}{q} - \frac{1}{p})$ Then, inequality (3.12) holds if one of the following conditions holds:*

1. *$q \geq 2$;*
2. *$p, q < 2$;*
3. *$q < 2 \leq p$ and $\beta > \frac{1}{2} - \frac{1}{p}$.*

Proof. **Step 1:**
First, assume that $\frac{1}{p^*} - \frac{1}{q} < \beta < \frac{1}{p^*}$, the inequality

$$\left(\int_{|\xi|\leq 1} |\hat{f}|^q\right)^{\frac{1}{q}} \lesssim \left\|f\xi^\beta\right\|_p$$

holds if $q \geq 2$ or if $p, q < 2$. Indeed, by Pitt's Theorem (Theorem 2.2.5) it suffices to check that

- For $q \geq p$,

$$\sup_s \left(\int_0^{1/s} \mathbb{1}^*_{|t|\leq 1} dt\right)^{\frac{1}{q}} s^{-\beta+\frac{1}{p^*}} < \infty.$$

- For $q < p$, (recall that $r^{-1} = q^{-1} - p^{-1}$)

$$\int_0^\infty \left(\int_0^{1/s} \mathbb{1}^*_{|t|\leq 1} dt\right)^{\frac{r}{q}} s^{\frac{r}{q^*}-r\beta} ds < \infty.$$

Since

$$\int_0^{1/s} \mathbb{1}^*_{|t|\leq 1} dt \leq \min(2, s^{-1}),$$

some computations show that both these conditions hold.
Second, if $q < 2 \leq p$, and $\frac{1}{p^*} > \beta > \frac{1}{p^*} - \frac{1}{2}$, an application of Hölder's inequality and the previous result shows that

$$\left(\int_{|\xi|\leq 1} |\hat{f}|^q\right)^{\frac{1}{q}} \lesssim \left(\int_{|\xi|\leq 1} |\hat{f}|^2\right)^{\frac{1}{2}} \lesssim \left\|f\xi^\beta\right\|_p$$

also holds.
Step 2: Continuing with the proof, by considering $f_\lambda(x) = Cf(\lambda x)$ for suitable C, λ instead of f, normalize $\hat{f}$ so that $\|f\|_q = \left\|\hat{f}\xi^\beta\right\|_p = 1$. After this normalization, inequality (3.12) becomes $\|fx^\alpha\|_q \gtrsim 1$. We show that $\|fx^\alpha\|_q$ can't be too small.
Let $\varepsilon > 0$ be such that $\frac{1}{p^*} - \frac{1}{q} < \beta - \varepsilon < \frac{1}{p^*}$, then for a fixed $N \in \mathbb{N}$, applying the inequalities proved in Step 1 to the function

$$\Delta_h^N f = \sum_{j=0}^N (-1)^j f(x + jh)\binom{N}{j}, \tag{3.14}$$

with the Fourier transform

$$(\Delta_h^N f)^\wedge(\xi) = \hat{f}(\xi)(1 - e^{-2\pi i\xi h})^N,$$

we deduce that

$$\left(\int_{|x|\leq 1} |\Delta_h^N f|^q\right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}} |\hat{f}|^p|\xi|^{p(\beta-\varepsilon)}(1 - e^{2\pi i\xi h})^{pN}\right)^{\frac{1}{p}} \lesssim \left(\int_{\mathbb{R}} |\hat{f}|^p|\xi|^{p(\beta-\varepsilon)} \min(1, |\xi h|)^{pN}\right)^{\frac{1}{p}}.$$

Further, using that if $N > \varepsilon$, then $\min(1, |h\xi|^N) \leq |h\xi|^\varepsilon$, we obtain

$$\left(\int_{|x|\leq 1} |\Delta_h^N f|^q\right)^{\frac{1}{q}} \lesssim h^\varepsilon \left\|\hat{f}\xi^\beta\right\|_p = h^\varepsilon.$$

Finally, let h_0 be small enough such that $\left(\int_{|x|\leq 1} |\Delta_h^N f|^q\right)^{\frac{1}{q}} < \frac{1}{4}$ and $4h_0 < 1$. If

$$\left(\int_{|x|\leq h_0} |f|^q\right)^{\frac{1}{q}} \leq \frac{3}{4},$$

then

$$\|fx^\alpha\|_q \geq \frac{h_0^\alpha}{4}$$

and the result follows.
Otherwise, we have that

$$\left(\int_{|x|\leq h_0} |f|^q\right)^{\frac{1}{q}} > \frac{3}{4}.$$

Then, by our choice of h_0, we have

$$\left(\int_{|\xi|\leq h_0} |\Delta_h^N f(x) - f(x)|^q\right)^{\frac{1}{q}} \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

Since

$$\frac{1}{2} \leq \left(\int_{|\xi|\leq h_0} |\Delta_{h_0}^N f(x) - f(x)|^q\right)^{\frac{1}{q}} = \left(\sum_{j=1}^N \binom{N}{j}^q \int_{|x-jh|\leq h_0} |f(x)|^q\right)^{\frac{1}{q}} \lesssim \left(\int_{|x|\geq h_0} |f|^q\right)^{\frac{1}{q}}$$
$$\leq h_0^{-\alpha} \|fx^\alpha\|_q,$$

we conclude the proof. □

3.4.2 Proof of necessity

Lemma 3.4.7. *Assume that inequality* (3.12) *holds for* $p > 2 > q$. *Then,* $\beta > \frac{1}{2} - \frac{1}{p}$.

Proof. Let $\hat{\Phi}$ be a C^∞ function with support contained in $[-1/2, 1/2]$. We know that Φ belongs to the Schwartz class of rapidly decreasing functions.
For $M \in \mathbb{N}$ and $c_j \in \mathbb{R}$ to be fixed, consider the following function:

$$f(x) = \Phi(x) \sum_{j=0}^M \varepsilon_j c_j e^{-2\pi i j},$$

where each ε is either $+1$ or -1.
Observe that

$$\int_{\mathbb{R}^n} |f(x)|^q dx = \int_{\mathbb{R}^n} \left|\Phi(x) \sum_{j=0}^M \varepsilon_j c_j e^{-2\pi i j}\right|^q.$$

Next, by Fubini and Khintchine's inequality (Theorem 1.4.1), averaging over all possible values for ε, we obtain

$$\mathbb{E}\left[\int_{\mathbb{R}}\left|\Phi(x)\sum_{j=0}^{M}\varepsilon_j c_j e^{-2\pi ij}\right|^q\right] = \int_{\mathbb{R}}\mathbb{E}\left|\Phi(x)\sum_{j=0}^{M}\varepsilon_j c_j e^{-2\pi ij}\right|^q dx$$

$$\approx \int_{\mathbb{R}}\left|\Phi(x)\left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{1}{2}}\right|^q dx \approx \left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{q}{2}}.$$

Similarly,

$$\mathbb{E}\left[\int_{\mathbb{R}}||x|^\alpha f(x)|^q\right] \approx \left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{q}{2}}.$$

Next, a straightforward computation shows that

$$\hat{f}(\xi) = \sum_{j=0}^{M}\varepsilon_j c_j \hat{\Phi}(\xi + j).$$

Thus,

$$\left\|\hat{f}\xi^\beta\right\|_p = \left(\int_{\mathbb{R}}\left(|\xi|^\beta\sum_{j=0}^{M}\varepsilon_j c_j\hat{\Phi}(\xi+j)\right)^p\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}}\sum_{j=0}^{M}|c_j|^p||\xi+j|^\beta\hat{\Phi}(\xi)|^p\right)^{\frac{1}{p}}$$

$$\lesssim \left(\sum_{j=0}^{M}(|j|+1)^{\beta p}|c_j|^p\right)^{\frac{1}{p}},$$

where the last equality is true because the supports of the translates are pairwise disjoint. Finally, if

$$\|f\|_q^q \lesssim \|fx^\alpha\|_q^{q\frac{\beta-1/q^*+1/p}{\alpha+\beta-1/q^*+1/p}}\left\|\hat{f}\xi^\beta\right\|_p^{q\frac{\alpha}{\alpha+\beta-1/q^*+1/p}},$$

taking expected values and noting that for any $X \geq 0$ random variable,

$$\mathbb{E}[X^{\frac{\beta-1/q^*+1/p}{\alpha+\beta-1/q^*+1/p}}] \leq \mathbb{E}[X]^{\frac{\beta-1/q^*+1/p}{\alpha+\beta-1/q^*+1/p}},$$

we obtain

$$\left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{q}{2}} \lesssim \left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{\beta-1/q^*+1/p}{\alpha+\beta-1/q^*+1/p}\frac{q}{2}}\left(\sum_{j=0}^{M}(j+1)^{\beta p}|c_j|^p\right)^{\frac{\alpha}{\alpha+\beta-1/q^*+1/p}\frac{q}{p}},$$

that is,

$$\left(\sum_{j=0}^{M}|c_j|^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{j=0}^{M}(j+1)^{\beta p}|c_j|^p\right)^{\frac{1}{p}},$$

equivalently,

$$\left(\sum_{j=0}^{\infty}|c_j|^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{j=0}^{\infty}(j+1)^{\beta p}|c_j|^p\right)^{\frac{1}{p}},$$

uniformly on c_j.
Since $2 < p$, by Hölder's inequality, this condition holds for all c_j if and only if

$$\sum_{j=0}^{\infty}(j+1)^{-\frac{\beta}{\frac{1}{2}-\frac{1}{p}}} < \infty,$$

which is known to converge if and only if $-\frac{\beta}{\frac{1}{2}-\frac{1}{p}} < -1$, which is equivalent to $\beta > \frac{1}{2}-\frac{1}{p}$. □

3.5 Broken power weights

Another possible generalization of the Heisenberg UP is the following family of inequalities:

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left\|x^A f\right\|_p \left\|\xi^B \hat{f}\right\|_{p^*}, \tag{3.15}$$

where $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are broken weights, that is,

$$x^A = \begin{cases} |x|^{A_1}, |x| < 1 \\ |x|^{A_2}, |x| \geq 1 \end{cases} \tag{3.16}$$

with $A_1, A_2, B_1, B_2 \geq 0$. Indeed, setting $p = p^* = 2$ and $A_1 = A_2 = B_1 = B_2 = 1$, we obtain

$$\|f\|_2^2 = \|f\|_2 \left\|\hat{f}\right\|_2 \lesssim \|xf\|_2 \left\|\xi\hat{f}\right\|_2,$$

the classical Heisenberg UP.
It is worth mentioning that in [8], the authors used the Pitt inequality to obtain sufficient conditions for inequalities similar to (3.15) to hold for general weights. Nevertheless, if one wants to fully characterize the previous inequality, it is necessary to use more delicate arguments.

The main results of this section is the following:

Theorem 3.5.1. *Let $p \geq 2$. Then, inequality (3.15) holds if and only if $B_2 \geq A_1, A_2 \geq B_1$ and one the following holds:*

1. $A_2 > \frac{1}{2}-\frac{1}{p}, A_1 < \frac{1}{2}-\frac{1}{p}$ *and* $B_2^2 \geq (\frac{1}{2}-\frac{1}{p})A_1$;
2. $A_2 > \frac{1}{2}-\frac{1}{p}, A_1 > \frac{1}{2}-\frac{1}{p}$;
3. $A_2 > \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p} = A_1 < B_2$;
4. $A_2 \leq \frac{1}{2}-\frac{1}{p}, A_1 = 0$ *and* $B_1 = 0$.

In particular, for $p = 2$, inequality (3.15) holds if and only if $A_1 \leq B_2$, $A_2 \geq B_1$.

Remark 3.5.2. *Note that replacing f by $\hat{f}$ in inequality (3.15) has the effect of exchanging the roles of p by p^* and A by B. Hence, the condition $p \geq 2$ is not restrictive.*

Remark 3.5.3. *Observe that the Hausdorff-Young inequality yields*

$$\|f\|_p^2 \leq \|f\|_p \left\|\hat{f}\right\|_{p^*} \leq \left\|\hat{f}\right\|_{p^*}^2,$$

so that inequality (3.15) *is in between the type of inequalities described by* (3.11) *for* $\alpha = \beta$.

Corollary 3.5.4. *For usual power weights, that is, $A = A_1 = A_2$ and $B = B_1 = B_2$, we have that the inequality* (3.15) *holds if and only if $A = B = 0$ or $A = B > \frac{1}{2} - \frac{1}{p}$.*

The proof of the theorem is given in the two following sections.

3.5.1 Proofs of sufficiency

Lemma 3.5.5. *Assume $p = 2$ and that that inequality* (3.15) *holds for A, B, then it also holds for $\lambda A, \lambda B$ for all $\lambda > 1$.*

Proof. We use Hölder's inequality to obtain

$$\left\|x^A f\right\|_2 \leq \|f\|_2^{\frac{1}{\lambda^*}} \left\|x^{\lambda A} f\right\|_2^{\frac{1}{\lambda}};$$

$$\left\|\xi^B \hat{f}\right\|_2 \leq \left\|\hat{f}\right\|_2^{\frac{1}{\lambda^*}} \left\|\xi^{\lambda B} \hat{f}\right\|_2^{\frac{1}{\lambda}},$$

where, as usual, $\frac{1}{\lambda} + \frac{1}{\lambda^*} = 1$.
So from formula (3.15) we deduce

$$\|f\|_2 \left\|\hat{f}\right\|_2 \lesssim \left\|x^A f\right\|_2 \left\|\xi^B \hat{f}\right\|_2 \leq \|f\|_2^{\frac{1}{\lambda^*}} \left\|x^{\lambda A} f\right\|_2^{\frac{1}{\lambda}} \left\|\hat{f}\right\|_2^{\frac{1}{\lambda^*}} \left\|\xi^{\lambda B} \hat{f}\right\|_2^{\frac{1}{\lambda}},$$

whence the result follows. □

Lemma 3.5.6. *Let $A = (A_1, A_2)$ and $A^t = (A_2, A_1)$ with $0 \leq A_1, A_2 < 1/2$, then*

$$\int_{\mathbb{R}} |\hat{f}|^2 |\xi|^{-2A} d\xi \lesssim \int_{\mathbb{R}} |f|^2 |x|^{2A^t} dx.$$

Proof. It follows from

$$\sup_s \left(\int_0^{1/s} \xi^{-2A} d\xi\right) \left(\int_0^s x^{-2A^t} dx\right) \lesssim 1$$

and Pitt's inequality (Theorem 2.2.5). □

Lemma 3.5.7. *For $p = 2$, if $A_1 \leq B_2$ and $A_2 \geq B_1$, then inequality* (3.15) *holds.*

Proof. Observe that by monotonicity it suffices to prove the result for $A_1 = B_2$ and $A_2 = B_1$, that is, for $B = A^t$. Besides, an application of Lemma 3.5.5 shows that it suffices to prove the result for $A_1, A_2 < 1/2$.
Then, by Hölder's inequality and Lemma 3.5.6,

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx \leq \left\|f x^A\right\|_2 \left\|f x^{-A}\right\|_2 \lesssim \left\|f x^A\right\|_2 \left\|\hat{f} \xi^{A^t}\right\|_2.$$

□

Lemma 3.5.8. *Let $p > 2$. Inequality* (3.15) *holds if $B_2 \geq A_1, A_2 \geq B_1$ and one the following conditions holds:*

1. $A_2 > \frac{1}{2} - \frac{1}{p}, A_1 < \frac{1}{2} - \frac{1}{p}$ *and* $B_2^2 \geq (\frac{1}{2} - \frac{1}{p})A_1$;
2. $A_2 > \frac{1}{2} - \frac{1}{p}, A_1 > \frac{1}{2} - \frac{1}{p}$;
3. $A_2 > \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p} = A_1 < B_2$;
4. $A_2 \leq \frac{1}{2} - \frac{1}{p}, A_1 = 0$ *and* $B_1 = 0$.

Proof. Before we begin, note that the fourth item is clear because in this case both weights are pointwise greater than 1; the third item follows from the second one as follows: let $\frac{1}{2} - \frac{1}{p} = A_1 < B_2$, then there exists $\varepsilon > 0$ such that $\frac{1}{2} - \frac{1}{p} < A_1 + \varepsilon =: A_1' \leq B_2$ and by item (2),

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left\|fx^{A'}\right\|_p \left\|\hat{f}x^B\right\|_{p^*} \leq \left\|fx^A\right\|_p \left\|\hat{f}x^B\right\|_{p^*}.$$

We use further the normalization $\|f\|_2 = 1$ and assume throughout that $A_2 > \frac{1}{2} - \frac{1}{p}$. We use several tools.

Tool 1:

From Hölder's inequality, we deduce that, for $t > 0$

$$\left(\int_{-t/2}^{t/2} |\hat{f}|^{p^*}\right)^{\frac{1}{p^*}} \leq t^{\frac{1}{p^*} - \frac{1}{2}} \left(\int_{-t/2}^{t/2} |\hat{f}|^2\right)^{\frac{1}{2}} \leq t^{\frac{1}{p^*} - \frac{1}{2}} \left\|\hat{f}\right\|_2 = t^{\frac{1}{p^*} - \frac{1}{2}}.$$

This means that (recall the definition of broken weights in equation (3.16))

$$\int_{|\xi| \geq t/2} |\hat{f}|^{p^*} |\xi|^{p^* B} \geq t^{p^* B} \left(\left\|\hat{f}\right\|_{p^*}^{p^*} - t^{1 - \frac{p^*}{2}}\right)$$

and setting $2t = \|f\|_{p^*}^{\frac{1}{p^*} - \frac{1}{2}}$, we establish that

$$\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*} \left\|\hat{f}\right\|_{p^*}^{\frac{B}{\frac{1}{p^*} - \frac{1}{2}}}. \tag{3.17}$$

Tool 2:

Next, once again by Hölder's inequality, we deduce that

$$\left(\int_{|x| \leq t} |f|^2\right)^{\frac{1}{2}} \lesssim \|f\|_p t^{\frac{1}{2} - \frac{1}{p}}$$

and using that $A_2 > \frac{1}{2} - \frac{1}{p}$,

$$\left(\int_{|x| \geq t} |f|^2\right)^{\frac{1}{2}} \leq \left\|fx^A\right\|_p \left(\int_{|x| \geq t} x^{-\frac{A}{\frac{1}{2} - \frac{1}{p}}}\right)^{\frac{1}{2} - \frac{1}{p}},$$

and combining these estimates, we obtain for $t > 0$ that

$$1 = \|f\|_2 \lesssim \|f\|_p t^{\frac{1}{2} - \frac{1}{p}} + \left\|fx^A\right\|_p \left(\int_{|x| \geq t} x^{-\frac{A}{\frac{1}{2} - \frac{1}{p}}}\right)^{\frac{1}{2} - \frac{1}{p}}. \tag{3.18}$$

Tool 3:
First, for $t < 1$, and $k \in \mathbb{N}$, the triangle inequality yields

$$\left(\int_0^t |f|^p\right)^{\frac{1}{p}} \le \left(\int_0^t |f(x) - f(x+t)|^p\right)^{\frac{1}{p}} + \left(\int_t^{2t} |f|^p\right)^{\frac{1}{p}} \le \left(\int_0^t |\Delta_t^k f|^p\right)^{\frac{1}{p}} + \left(\int_t^{\infty} |f|^p\right)^{\frac{1}{p}},$$

where $\Delta_h^N f$ is as in equation (3.14). Thus, using the inequality

$$\left(\int_t^{\infty} |f|^p\right)^{\frac{1}{p}} \le \left\|fx^A\right\|_p t^{-A_1},$$

we deduce that

$$\|f\|_p \lesssim \left(\int_{-t}^t |\Delta_t^k f|^p\right)^{\frac{1}{p}} + \left\|fx^A\right\|_p t^{-A_1}.$$

Now, Pitt's inequality (Theorem 2.2.5) implies that for $D = (0, D_2)$, if $D_2 < \frac{1}{p}$,

$$\left(\int_0^t |f|^p\right)^{\frac{1}{p}} \lesssim C(t) \left\|\hat{f}x^D\right\|_{p^*},$$

with

$$C(t) = \sup_s \left(\int_0^s \mathbb{1}_{[0,t]}\right)^{\frac{1}{p}} \left(\int_0^{s^{-1}} x^{-pD}\right)^{\frac{1}{p}} \approx t^{D_2}.$$

Therefore, just like we did in the previous section,

$$\left(\int_{-t}^t |\Delta_t^k f|^p\right)^{\frac{1}{p}} \lesssim t^{D_2} \left\|\hat{f}(1 - e^{2\pi i x t})^k x^{\beta}\right\|_{p^*}.$$

Hence, for $t < 1$,

$$\|f\|_p \lesssim t^{D_2} \left\|\hat{f}(\xi)(1 - e^{2\pi i x t})^k \xi^D\right\|_{p^*} + \left\|fx^A\right\|_p t^{-A_1}. \tag{3.19}$$

Now, we find it convenient to split the proof into four cases.
Case 1. Assume $\left\|\hat{f}\right\|_{p^*} \ge 1$ and $\left\|fx^A\right\|_p \ge \|f\|_p$. From (3.17) we have that $\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*}$, and consequently,

$$\left\|fx^A\right\|_p \left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \|f\|_p \left\|\hat{f}\right\|_{p^*} \left\|\hat{f}\right\|_{p^*}^{\frac{B}{\frac{1}{p^*} - \frac{1}{2}}} \ge \|f\|_p \left\|\hat{f}\right\|_{p^*}.$$

Case 2: Assume $\left\|\hat{f}\right\|_{p^*} < 1$ and $\left\|fx^A\right\|_p \ge \|f\|_p$. Let $t = \left(\frac{\|fx^A\|_p}{\|f\|_p}\right)^{\frac{1}{A_2}} \ge 1$, then equation (3.18) becomes

$$1 \lesssim \|f\|_p \left(\frac{\left\|fx^A\right\|_p}{\|f\|_p}\right)^{\frac{\frac{1}{2} - \frac{1}{p}}{A_2}},$$

and from (3.17) we deduce

$$\|f\|_p^{\frac{A_2}{\frac{1}{2} - \frac{1}{p}}} \left\|fx^A\right\|_p \left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \|f\|_p \left\|\hat{f}\right\|_{p^*} \left\|\hat{f}\right\|_{p^*}^{\frac{B}{\frac{1}{p^*} - \frac{1}{2}}}.$$

Finally, using that $A_2 \geq B_1$ and $\left\|\hat{f}\right\|_{p^*} < 1$, we have by using the Hausdorff-Young inequality that

$$\left\|\hat{f}\right\|_{p^*}^{\frac{B}{\frac{1}{p^*}-\frac{1}{2}}} = \left\|\hat{f}\right\|_{p^*}^{\frac{B_1}{\frac{1}{p^*}-\frac{1}{2}}} \geq \left\|\hat{f}\right\|_{p^*}^{\frac{A_2}{\frac{1}{p^*}-\frac{1}{2}}} \geq \|f\|_p^{\frac{A_2}{\frac{1}{2}-\frac{1}{p}}},$$

(note that $\frac{1}{2}-\frac{1}{p}=\frac{1}{p^*}-\frac{1}{2}$) whence the result follows.

Case 3.0: Assume $\left\|\hat{f}\right\|_{p^*} \geq 1$ and $\left\|fx^A\right\|_p \geq \left\|\hat{f}\xi^B\right\|_{p^*}$. From (3.17) we derive that

$$\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*} \left\|\hat{f}\right\|_{p^*}^{\frac{B}{\frac{1}{p^*}-\frac{1}{2}}} \gtrsim \left\|\hat{f}\right\|_{p^*}.$$

Finally, from the Hausdorff-Young inequality and the hypothesis $\left\|fx^A\right\|_p \gtrsim \left\|\hat{f}x^B\right\|_{p^*}$ we deduce

$$\left\|fx^A\right\|_p \left\|\hat{f}x^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*} \|f\|_p.$$

Case 3.1: Assume $\left\|\hat{f}\right\|_{p^*} \geq 1$, $\left\|fx^A\right\|_p < \|f\|_p$ and $A_1 > \frac{1}{2}-\frac{1}{p}$. Let $t = \left(\frac{\|fx^A\|_p}{\|f\|_p}\right)^{\frac{1}{A_1}} < 1$, then (3.18) implies that

$$1 \lesssim \left\|fx^A\right\|_p + \|f\|_p \left(\frac{\left\|fx^A\right\|_p}{\|f\|_p}\right)^{\frac{\frac{1}{2}-\frac{1}{p}}{A_1}}.$$

Indeed, since $t<1$,

$$1 \lesssim \|f\|_p t^{\frac{1}{2}-\frac{1}{p}} + \left\|fx^A\right\|_p \left(\int_{|x|\geq t} x^{-\frac{A}{\frac{1}{2}-\frac{1}{p}}}\right)^{\frac{1}{2}-\frac{1}{p}} \lesssim \|f\|_p t^{\frac{1}{2}-\frac{1}{p}} + \left\|fx^A\right\|_p t^{\frac{1}{2}-\frac{1}{p}-A_1} + \left\|fx^A\right\|_p,$$

and all that remains is to substitute $t = \left(\frac{\|fx^A\|_p}{\|f\|_p}\right)^{\frac{1}{A_1}}$. Continuing with the proof, from (3.17) we deduce

$$\left(\left\|fx^A\right\|_p^{\frac{A_1}{\frac{1}{2}-\frac{1}{p}}} + \frac{\left\|fx^A\right\|_p}{\|f\|_p}\|f\|_p^{\frac{A_1}{\frac{1}{2}-\frac{1}{p}}}\right)\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*}\left\|\hat{f}\right\|_{p^*}^{\frac{B_2}{\frac{1}{p^*}-\frac{1}{2}}},$$

equivalently,

$$\left(\left(\frac{\left\|fx^A\right\|_p}{\|f\|_p}\right)^{\frac{A_1}{\frac{1}{2}-\frac{1}{p}}} + \frac{\left\|fx^A\right\|_p}{\|f\|_p}\right)\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*}\left(\frac{\left\|\hat{f}\right\|_{p^*}^{B_2}}{\|f\|_p^{A_1}}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

Finally, since $B_2 \geq A_1$ and $\left\|\hat{f}\right\|_{p^*} \geq 1$, by the Hausdorff-Young inequality,

$$\frac{\left\|\hat{f}\right\|_{p^*}^{B_2}}{\|f\|_p^{A_1}} \geq \frac{\left\|\hat{f}\right\|_{p^*}^{A_1}}{\|f\|_p^{A_1}} \geq 1;$$

and since $A_1 > \frac{1}{2} - \frac{1}{p}$ and $\left\|fx^A\right\|_p < \|f\|_p$

$$\left(\frac{\left\|fx^A\right\|_p}{\|f\|_p}\right)^{\frac{A_1}{\frac{1}{2}-\frac{1}{p}}} + \frac{\left\|fx^A\right\|_p}{\|f\|_p} \approx \frac{\left\|fx^A\right\|_p}{\|f\|_p},$$

whence the result follows.

Case 3.2: Assume $\left\|\hat{f}\right\|_{p^*} \geq 1$, $\left\|fx^A\right\|_p < \left\|\hat{f}\xi^B\right\|_{p^*}$, $\left\|fx^A\right\|_p < \|f\|_p$, $A_2 > \frac{1}{2} - \frac{1}{p} > A_1$, $B_2^2 \geq A_1(\frac{1}{2} - \frac{1}{p})$.

Write $B_2 = D_2 + r$ with $r \geq 0$ and $D_2 < \frac{1}{p}$. Now, applying inequality (3.19) with $r \leq k \in \mathbb{N}$, and applying the inequality

$$(1 - e^{2\pi ixt})^k \lesssim \min(1, xt)^k \leq (xt)^r,$$

we deduce that, for $R = (r, B_2)$ (see (3.16)),

$$\|f\|_p \lesssim t^{B_2} \left\|\hat{f}\xi^R\right\|_{p^*} + \left\|fx^A\right\|_p t^{-A_1}. \tag{3.20}$$

Besides, since $\left\|\hat{f}\right\|_{p^*} \geq 1$, from inequality (3.17), we conclude that $\left\|\hat{f}\right\|_{p^*} \lesssim \left\|\hat{f}\xi^B\right\|_{p^*}$. Hence,

$$\left\|\hat{f}\xi^r \mathbb{1}_{[0,1]}\right\|_{p^*} \leq \left\|\hat{f}\right\|_{p^*} \lesssim \left\|\hat{f}\xi^B\right\|_{p^*},$$

so we have that

$$\left\|\hat{f}\xi^R\right\|_{p^*} \lesssim \left\|\hat{f}\xi^B\right\|_{p^*},$$

and equation (3.20) becomes for $t < 1$,

$$\|f\|_p \lesssim t^{B_2} \left\|\hat{f}\xi^B\right\|_{p^*} + \left\|fx^A\right\|_p t^{-A_1}.$$

Setting $1 > t = \left(\frac{\left\|fx^A\right\|_p}{\left\|\hat{f}x^B\right\|_{p^*}}\right)^{\frac{1}{A_1+B_2}}$, we get

$$\|f\|_p \lesssim \left\|fx^A\right\|_p^{\frac{B_2}{B_2+A_1}} \left\|\hat{f}\xi^B\right\|_{p^*}^{\frac{A_1}{B_2+A_1}}. \tag{3.21}$$

Finally, equation (3.17)

$$\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim \left\|\hat{f}\right\|_{p^*} \left\|\hat{f}\right\|_{p^*}^{\frac{B_2}{\frac{1}{p^*}-\frac{1}{2}}}$$

is equivalent to

$$\left\|\hat{f}\right\|_{p^*} \lesssim \left\|\hat{f}\xi^B\right\|_{p^*}^{\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}+B_2}},$$

and since $A_1 < \frac{1}{2} - \frac{1}{p} < A_2$,

$$1 = \|f\|_2 \lesssim \left\|fx^A\right\|_p$$

we arrive at

$$\left\|\hat{f}\right\|_{p^*} \lesssim \left\|\hat{f}\xi^B\right\|_{p^*}^{\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}+B_2}} \left\|fx^A\right\|_p^{\frac{B_2}{\frac{1}{2}-\frac{1}{p}+B_2}}.$$

Combining this estimate with (3.21), we obtain

$$\left\|\hat{f}\right\|_{p^*}\|f\|_p \lesssim \left\|fx^A\right\|_p^{\frac{B_2}{B_2+A_1}} \left\|\hat{f}\xi^B\right\|_{p^*}^{\frac{A_1}{B_2+A_1}} \left\|\hat{f}\xi^B\right\|_{p^*}^{\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}+B_2}} \left\|fx^A\right\|_p^{\frac{B_2}{\frac{1}{2}-\frac{1}{p}+B_2}}.$$

Finally, since $B_2^2 \geq A_1(\frac{1}{2}-\frac{1}{p})$, we have that (to simplify the computations, note that the sum of the fractions below is 2)

$$\frac{B_2}{B_2+A_1}+\frac{B_2}{\frac{1}{2}-\frac{1}{p}+B_2} \geq 1 \geq \frac{A_1}{B_2+A_1}+\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}+B_2},$$

and since $\left\|fx^A\right\|_p \leq \left\|\hat{f}\xi^B\right\|_{p^*}$, the result follows.

Case 4.1: Assume that $\left\|\hat{f}\right\|_{p^*} < 1$ and $\left\|fx^A\right\|_p < \|f\|_p$ and $A_1 < \frac{1}{2}-\frac{1}{p}$. By setting $t=0$ in equation (3.18) we obtain

$$1 \lesssim \left\|fx^A\right\|_p < \|f\|_p \lesssim \left\|\hat{f}\right\|_{p^*} < 1,$$

so that

$$1 \approx \left\|fx^A\right\|_p \approx \|f\|_p \approx \left\|\hat{f}\right\|_{p^*},$$

so that from equation (3.17) we deduce

$$\left\|\hat{f}\xi^B\right\|_{p^*} \gtrsim 1,$$

whence the result follows.

Case 4.2: Assume $\left\|\hat{f}\right\|_{p^*} < 1$, $\left\|fx^A\right\|_p < \|f\|_p$ and $A_1 > \frac{1}{2}-\frac{1}{p}$.

Here we have

$$\left(\int_{|x|\leq 1}|f|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{|x|\leq 1}|f|^p\right)^{\frac{1}{p}} \leq \|f\|_p$$

and

$$\left(\int_{|x|\geq 1}|f|^2\right)^{\frac{1}{2}} \lesssim \left(\int_{|x|\geq 1}|x^A f|^p\right)^{\frac{1}{p}} \leq \|f\|_p,$$

so

$$1 = \|f\|_2 \lesssim \|f\|_p \leq \left\|\hat{f}\right\|_{p^*},$$

and $\left\|\hat{f}\right\|_{p^*} \approx 1$, so that the result follows from Case 3.1.

□

3.5.2 Proofs of necessity

Lemma 3.5.9. *Assume that inequality* (3.15) *holds for* A, B. *Then, the following hold:*

1. *For any* $\lambda > 0$ *we have*

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left(\lambda^{-pA_1}\int_{|x|\leq\lambda}|f(x)|^p|x|^{pA_1}dx + \lambda^{-pA_2}\int_{|x|\geq\lambda}|f(x)|^p|x|^{pA_2}dx\right)^{\frac{1}{p}} \times$$
$$\left(\lambda^{p^*B_1}\int_{|\xi|\leq\lambda^{-1}}|\hat{f}(\xi)|^{p^*}|\xi|^{p^*B_1}d\xi + \lambda^{p^*B_2}\int_{|\xi|\geq\lambda^{-1}}|\hat{f}(\xi)|^{p^*}|\xi|^{p^*B_2}d\xi\right)^{\frac{1}{p^*}} \quad (3.22)$$

2. $A_1 \leq B_2$ *and* $B_1 \leq A_2$*;*

3. If $A_2 = B_1$*, then, inequality* (3.15) *also holds for* $A = (A_2, A_2)$ *and* $B = (B_1, B_1)$*;*

4. If $A_1 = B_2$*, then, inequality* (3.15) *also holds for* $A = (A_1, A_1)$ *and* $B = (B_2, B_2)$*.*

Proof. For a function f and $\lambda > 0$, define

$$f_\lambda(x) = \lambda f(\lambda x).$$

Then, $\hat{f}_\lambda(\xi) = \hat{f}(\xi/\lambda)$. Thus,

$$\|f_\lambda\|_p = \lambda^{1/p^*} \|f\|_p$$

and

$$\left\|\hat{f}_\lambda\right\|_{p^*} = \lambda^{1/p^*} \left\|\hat{f}\right\|_{p^*}.$$

Also,

$$\left\|f_\lambda x^A\right\|_p = \lambda^{1/p^*} \left(\lambda^{-pA_1} \int_{|x|\leq\lambda} |f(x)|^p |x|^{pA_1} dx + \lambda^{-pA_2} \int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx\right)^{\frac{1}{p}},$$

$$\left\|\hat{f}_\lambda \xi^B\right\|_{p^*} = \lambda^{1/p^*} \left(\lambda^{p^*B_1} \int_{|\xi|\leq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_1} d\xi + \lambda^{p^*B_2} \int_{|\xi|\geq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_2} d\xi\right)^{\frac{1}{p^*}},$$

whence (1) follows.
Next, taking f a Schwartz function, note that

$$\lambda^{-pA_1} \int_{|x|\leq\lambda} |f(x)|^p |x|^{pA_1} dx \leq \int_{|x|\leq\lambda} |f(x)|^p dx$$

and for any $n \in \mathbb{N}$

$$\lim_{\lambda\to\infty} \lambda^n \int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx = 0,$$

and similarly for $\hat{f}$.
Hence, since $\lambda^{-A_2} \int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx$ decays faster than any polynomial as $\lambda \to \infty$, we have that for λ large enough,

$$\lambda^{-A_2} \left(\int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx\right)^{\frac{1}{p}} \leq \lambda^{-A_1} \left(\int_{|x|\leq\lambda} |f(x)|^p |x|^{pA_1} dx\right)^{\frac{1}{p}}.$$

Thus,

$$\|f\|_p \frac{\left\|f_\lambda x^A\right\|_p}{\|f_\lambda\|_p} \approx \lambda^{-A_1} \left(\int_{|x|\leq\lambda} |f(x)|^p |x|^{pA_1} dx\right)^{\frac{1}{p}}$$

and similarly

$$\left\|\hat{f}\right\|_{p^*} \frac{\left\|\hat{f}_\lambda x^B\right\|_p}{\left\|\hat{f}_\lambda\right\|_{p^*}} \approx \lambda^{B_2} \left(\int_{|x|\geq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_2} dx\right)^{\frac{1}{p^*}},$$

whence we deduce that $B_2 \geq A_1$ and similarly when $\lambda \to 0$, $A_2 \geq B_1$, and (2) follows. Besides, note that if $B_2 = A_1$, multiplying the two last equations we deduce that

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left(\int_{|x|\leq\lambda} |f(x)|^p |x|^{pA_1} dx\right)^{\frac{1}{p}} \left(\int_{|x|\geq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^* B_2} dx\right)^{\frac{1}{p^*}},$$

whence (3) follows by letting $\lambda \to \infty$. (4) follows in the same fashion. □

Lemma 3.5.9 implies some easy restrictions on parameters. We now give an idea of the proof of necessity of the remaining conditions. The next three lemmas are variations of the following construction. For a smooth and compactly supported ϕ, we consider functions of the form

$$f = f_1 + f_2 := N\phi(Nx) + \sum_{n=1}^{M} \varepsilon_n c_n \phi(x-n),$$

for c_n to be chosen later and N, M large. This function consists of two parts: a sharp peak and a tail which ressembles a step function. Clearly, if N is substantially large,

$$\|f_1\|_p \gg \left\|f_1 x^A\right\|_p,$$

(by $\gg$ we mean "much greater than" in an informal way) and if we chose c_n for which for every M,

$$\left\|f_2 x^A\right\|_p \lesssim 1,$$

we will have

$$\|f\|_p \gg \left\|f x^A\right\|_p. \tag{3.23}$$

The Fourier transform of f,

$$\hat{f} = \hat{f}_1 + \hat{f}_2 := \hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i \xi n}\right),$$

also has two parts: a wide component and a component roughly concentrated around the origin. It is easy to see that if N is large $\left\|\hat{f}_1 \xi^B\right\|_{p^*} \gg \left\|\hat{f}_1\right\|_{p^*}$ in such a way that

$$\left\|\hat{f}_1\right\|_{p^*} \|f\|_p \lesssim \left\|f x^A\right\|_p \left\|\hat{f}_1 \xi^B\right\|_{p^*}.$$

The crucial point is to choose the c_n so that

$$\left\|\hat{f}_2\right\|_{p^*} \gg \left\|\hat{f}_1 \xi^B\right\|_{p^*}.$$

If we can accomplish that, since $\hat{f}_2$ is concentrated around the origin,

$$\left\|\hat{f}\right\|_{p^*} \approx \left\|\hat{f}_2\right\|_{p^*} \gtrsim \left\|\hat{f}\xi^B\right\|_{p^*}$$

and we obtain a violation of inequality (3.15) from (3.23).
As a final remark, our c_n must satisfy both

$$\|f_2\|_p \leq \left\|f_2 x^A\right\|_p \lesssim 1 \ll \|f_1\|_p$$

and

$$\left\|\hat{f}_2\right\|_{p^*} \gg \left\|\hat{f}_1\xi^B\right\|_{p^*} \gg \left\|\hat{f}_1\right\|_{p^*},$$

since it is easy to see that

$$\left\|\hat{f}_1\right\|_{p^*} \approx \|f_1\|_p,$$

we conclude (see Corollary 1.4.2) that

$$\|f_2\|_p \ll \left\|\hat{f}_2\right\|_{p^*}.$$

We are now ready to make the previous considerations precise.

Lemma 3.5.10. *Assume that inequality* (3.15) *holds for* $p > 2$ *and* $A_1 > 0$. *Then,* $A_2 > \frac{1}{2} - \frac{1}{p}$.

Proof. For the sake of contradiction, assume that $A_2 \le \frac{1}{2} - \frac{1}{p}$. Choose $M, N > 0$, $1 \ge \phi \ge 0$ smooth and supported in $(0,1)$, $c_1, \ldots, c_M \in \mathbb{R}$ and define

$$f(x) = f_1 + f_2 := N\phi(Nx) + \sum_{n=1}^{M} \varepsilon_n c_n \phi(x-n),$$

where ε_n is either 1 or -1. A computation shows that

$$\hat{f}(\xi) = \hat{f}_1 + \hat{f}_2 := \hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right).$$

Thus, raising inequality (3.15) to p^*, we have

$$\|f\|_p^{p^*} \int_{\mathbb{R}} \left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} d\xi \lesssim \left\|fx^A\right\|_p^{p^*} \int_{\mathbb{R}} \left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} \xi^{Bp^*} dx.$$

Taking expectations over all possible values of ε (see Theorem 1.4.1), we deduce that

$$\mathbb{E}\left[\|f\|_p^{p^*} \int_{\mathbb{R}} \left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} d\xi\right] \lesssim \mathbb{E}\left[\left\|fx^A\right\|_p^{p^*} \int_{\mathbb{R}} \left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} \xi^{Bp^*} d\xi\right]. \quad (3.24)$$

For the RHS of (3.24), we have

$$\mathbb{E}\left[\left\|fx^A\right\|_p^{p^*} \int_{\mathbb{R}} \left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} \xi^{Bp^*} d\xi\right] \lesssim \left\|fx^A\right\|_p^{p^*} \left(\int_{\mathbb{R}} |\hat{\phi}(\xi/N)x^B|^{p^*} d\xi + \mathbb{E}\left[\int_{\mathbb{R}} \left|\hat{\phi}(\xi)\left(\sum_{n=1}^{M} \varepsilon_n c_n e^{2\pi i\xi n}\right)\right|^{p^*} \xi^{Bp^*} d\xi\right]\right),$$

and applying Fubini's and Khintchine's (or simply Hölder's) inequality, we deduce that

$$RHS \lesssim \left\|fx^A\right\|_p^{p^*} \left(\left\|\hat{\phi}\xi^B\right\|_{p^*}^{p^*} \left(\sum_{n=1}^M |c_n|^2\right)^{\frac{p^*}{2}} + \left\|\hat{f}_1\xi^B\right\|_{p^*}^{p^*}\right). \tag{3.25}$$

For the LHS of (3.24), using the reverse triangle inequality for the norm $X \mapsto \mathbb{E}[|X|^{p^*}]^{\frac{1}{p^*}}$, we deduce that

$$\begin{aligned} &\|f\|_p^{p^*} \int_{\mathbb{R}} \mathbb{E}\left[\left|\hat{\phi}(\xi/N) + \hat{\phi}(\xi)\left(\sum_{n=1}^M \varepsilon_n c_n e^{2\pi i \xi n}\right)\right|^{p^*}\right]^{p^* \frac{1}{p^*}} d\xi \\ &\qquad \geq \|f\|_p^{p^*} \left\{\int_{\mathbb{R}} \left(\mathbb{E}\left[\left|\hat{\phi}(\xi)\left(\sum_{n=1}^M \varepsilon_n c_n e^{2\pi i \xi n}\right)\right|^{p^*}\right]^{\frac{1}{p^*}} - |\hat{\phi}(\xi/N)|\right)^{p^*} d\xi\right\}. \end{aligned} \tag{3.26}$$

Continuing the estimates for the LHS, we have that, by the Khintchine inequality (Theorem 1.4.1) and the reverse triangle inequality we conclude that

$$\begin{aligned} \mathbb{E}\left[\left\|\hat{f}\right\|_{p^*}^{p^*}\right]^{\frac{1}{p^*}} \|f\|_p \gtrsim \|f\|_p \left\{\int_{\mathbb{R}} \left(A_p|\hat{\phi}(\xi)| \left(\sum_{n=1}^M |c_n|^2\right)^{\frac{1}{2}} - |\hat{\phi}(\xi/N)|\right)^{p^*} d\xi\right\}^{\frac{1}{p^*}} \geq \\ \|f\|_p \left(A_p \left\|\hat{\phi}\right\|_{p^*} \left(\sum_{n=1}^M |c_n|^2\right)^{\frac{1}{2}} - \left\|\hat{f}_1\right\|_{p^*}\right). \end{aligned} \tag{3.27}$$

Hence, using both the estimates for the LHS and the RHS, we deduce that

$$\|f\|_p \left(A_p \left\|\hat{\phi}\right\|_{p^*} \left(\sum_{n=1}^M |c_n|^2\right)^{\frac{1}{2}} - \left\|\hat{f}_1\right\|_{p^*}\right) \lesssim \left\|fx^A\right\|_p \left(\left\|\hat{\phi}\xi^B\right\|_{p^*} \left(\sum_{n=1}^M |c_n|^2\right)^{\frac{1}{2}} + \left\|\hat{f}_1\xi^B\right\|_{p^*}\right). \tag{3.28}$$

Next, some easy computations show that (here $\approx$ is allowed to depend on ϕ)

$$\|f\|_p \approx \|f_1\|_p + \|f_2\|_p \approx N^{\frac{1}{p^*}} + \left(\sum_{n=1}^M |c_n|^p\right)^{\frac{1}{p}};$$

$$\left\|\hat{f}_1\right\|_{p^*} = \left\|\hat{\phi}\right\|_{p^*} N^{\frac{1}{p^*}}.$$

$$\left\|fx^A\right\|_p \leq \left\|f_1x^{A_1}\right\|_p + \left\|f_2x^{A_2}\right\|_p \approx N^{\frac{1}{p^*}-A_1} + \left(\sum_{n=1}^M |c_n|^p n^{pA_2}\right)^{\frac{1}{p}};$$

$$\left\|\hat{f}_1\xi^B\right\|_{p^*} \lesssim N^{\frac{1}{p^*}+B_1+B_2}.$$

Finally, define $c_n = \frac{1}{\sqrt{n\log(n)}}$. Note that $\sum_{n=1}^\infty c_n^2 = \infty$ but, since $A_2 \leq \frac{1}{2} - \frac{1}{p}$, $S := \left(\sum_{n=1}^\infty c_n^p n^{pA_2}\right)^{\frac{1}{p}} \leq \sum_{n=1}^\infty \frac{1}{n\log(n)^{\frac{p}{2}}} < \infty$. Fix N such that $N^{\frac{1}{p^*}} \approx \|f_1\|_p \geq S$, so that we have $\|f\|_p \approx N^{\frac{1}{p^*}}$, then letting $M \to \infty$, inequality (3.28) becomes

$$N^{\frac{1}{p^*}} \left\|\hat{\phi}\right\|_{p^*} \lesssim \left(N^{\frac{1}{p^*}-A_1} + S\right) \left\|\hat{\phi}\xi^B\right\|_{p^*},$$

and since $A_1 > 0$, we obtain a contradiction by letting $N \to \infty$. $\square$

Lemma 3.5.11. *Assume that inequality* (3.15) *holds for* $p > 2$. *If* $A_2 \leq \frac{1}{2} - \frac{1}{p}$ *it is necessary that* $B_1 = A_1 = 0$.

Proof. We already know that it is necessary that $A_1 = 0$. For the sake of contradiction assume that $B_1 > 0$. Then, let ϕ be a non-negative Schwartz function supported in $[0,1]$. Take $c_2, \dots, c_n \dots \in \mathbb{R}$. For each $M, N \geq \lambda^{-1} > 1$ define

$$f(x) = N\phi(Nx) + \sum_{n=2}^{M} \varepsilon_n c_n \phi(x-n).$$

Then,

$$\hat{f}(\xi) = \hat{\phi}(\xi/N) + \hat{\phi}(\xi) \sum_{n=2}^{M} \varepsilon_n c_n e^{2\pi i n x},$$

where $\varepsilon_i = \pm 1$. Clearly,

$$\|f\|_p \approx N^{\frac{1}{p^*}} + \left(\sum_{n=2}^{M} |c_n|^p\right)^{\frac{1}{p}}.$$

Besides, as we did in the previous proof (equation (3.27)),

$$\mathbb{E}\left[\left\|\hat{f}\right\|_{p^*}^{p^*}\right]^{\frac{1}{p^*}} \gtrsim \left(A_p \left\|\hat{\phi}\right\|_{p^*} \left(\sum_{n=1}^{M} |c_n|^2\right)^{\frac{1}{2}} - \left\|\hat{f}_1\right\|_{p^*}\right) = \left\|\hat{\phi}\right\|_{p^*} \left(A_p \left(\sum_{n=1}^{M} |c_n|^2\right)^{\frac{1}{2}} - N^{\frac{1}{p^*}}\right).$$

From equation (3.22) in Lemma 3.5.9 , we have

$$\|f\|_p \left\|\hat{f}\right\|_{p^*} \lesssim \left(\int_{|x|\leq\lambda} |f(x)|^p dx + \lambda^{-pA_2} \int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx\right)^{\frac{1}{p}} \cdot$$

$$\left(\lambda^{p^*B_1} \int_{|\xi|\leq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_1} d\xi + \lambda^{p^*B_2} \int_{|\xi|\geq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_2} d\xi\right)^{\frac{1}{p^*}}.$$

Because of our choice of f, (observe that $\phi(Nx)$ vanishes for $x \geq \lambda \geq N^{-1}$)

$$\left(\int_{|x|\leq\lambda} |f(x)|^p dx\right)^{\frac{1}{p}} \leq N^{\frac{1}{p^*}};$$

$$\left(\lambda^{-pA_2} \int_{|x|\geq\lambda} |f(x)|^p |x|^{pA_2} dx\right)^{\frac{1}{p}} \approx \left(\lambda^{-pA_2} \sum_{n=2}^{M} |c_n|^p n^{pA_2}\right)^{\frac{1}{p}};$$

$$\left(\lambda^{p^*B_1} \int_{|\xi|\leq\lambda^{-1}} |\hat{f}(\xi)|^{p^*} |\xi|^{p^*B_1} d\xi\right)^{\frac{1}{p^*}} \lesssim N^{\frac{1}{p^*}} + \lambda^{B_1} \left\|\hat{\phi} \sum_{n=2}^{M} \varepsilon_n c_n e^{2\pi i n x}\right\|_{p^*};$$

$$\left(\lambda^{p^*B_2}\int_{|\xi|\geq\lambda^{-1}}|\hat{f}(\xi)|^{p^*}|\xi|^{p^*B_2}d\xi\right)^{\frac{1}{p^*}}\lesssim\lambda^{B_2}\left(N^{B_2-\frac{1}{p^*}}+\left\|\mathbb{1}_{|\xi|\geq\lambda^{-1}}\hat{\phi}\sum_{n=2}^{M}\varepsilon_n c_n e^{2\pi inx}\right\|_{p^*}\right).$$

Analogously to the previous lemma, raising inequality (3.22) to the p^*-th power, taking expectations, using the Khintchine inequality and then taking p^*-th roots, we get

$$\left(N^{\frac{1}{p^*}}+\left(\sum_{n=2}^{M}|c_n|^p\right)^{\frac{1}{p}}\right)\left(\left\|\hat{\phi}\right\|_{p^*}A_p\left(\sum_{n=1}^{M}|c_n|^2\right)^{\frac{1}{2}}-\|\phi\|_{p^*}N^{\frac{1}{p^*}}\right)\lesssim$$
$$\left(N^{\frac{1}{p^*}}+\left(\lambda^{-pA_2}\sum_{n=2}^{M}|c_n|^p n^{pA_2}\right)^{\frac{1}{p}}\right)\times$$
$$\left[N^{\frac{1}{p^*}}+\lambda^{B_1}\left\|\hat{\phi}\right\|_{p^*}\left(\sum_{n=1}^{M}|c_n|^2\right)^{\frac{1}{2}}+\lambda^{B_2}\left(N^{B_2-\frac{1}{p^*}}+\left\|\mathbb{1}_{|\xi|\geq\lambda^{-1}}\hat{\phi}\right\|_{p^*}\left(\sum_{n=1}^{M}|c_n|^2\right)^{\frac{1}{2}}\right)\right] \quad (3.29)$$

Here, since $A_2\leq\frac{1}{2}-\frac{1}{p}$, there exists a sequence c_n such that $\sum_{n=2}^{\infty}c_n^p n^{pA_2}=S^p<\infty$ and $\sum_{n=2}^{\infty}|c_n|^2=\infty$.
Choosing N such that $N^{\frac{1}{p^*}}>S$ and letting $M\to\infty$, we deduce that

$$N^{\frac{1}{p^*}}\lesssim\left(N^{\frac{1}{p^*}}+\lambda^{-A_2}S\right)(\lambda^{B_1}+\lambda^{B_2}\left\|\mathbb{1}_{|\xi|\geq\lambda^{-1}}\hat{\phi}\right\|_{p^*}).$$

Letting $N\to\infty$ and then letting $\lambda\to 0$ we obtain the desired contradiction. □

Lemma 3.5.12. *Assume that inequality* (3.15) *holds for* $p>2$. *If* $A_1<\frac{1}{2}-\frac{1}{p}$, *then* $B_2^2\geq A_1(\frac{1}{2}-\frac{1}{p})$.

Proof. Before we begin, note that if $B_2=0$, Lemma 3.5.9 implies that $A_1=0$, so that the conclusion holds. From now on assume that $B_2>0$. Once again, for any $\lambda>0$,

$$\|f\|_p\left\|\hat{f}\right\|_{p^*}\lesssim\left(\lambda^{-pA_1}\int_{|x|\leq\lambda}|f(x)|^p|x|^{pA_1}dx+\lambda^{-pA_2}\int_{|x|\geq\lambda}|f(x)|^p|x|^{pA_2}dx\right)^{\frac{1}{p}}\cdot$$

$$\left(\lambda^{p^*B_1}\int_{|\xi|\leq\lambda^{-1}}|\hat{f}(\xi)|^{p^*}|\xi|^{p^*B_1}d\xi+\lambda^{p^*B_2}\int_{|\xi|\geq\lambda^{-1}}|\hat{f}(\xi)|^{p^*}|\xi|^{p^*B_2}d\xi\right)^{\frac{1}{p^*}}$$

We are going to use a modification of the previous argument.
Choose $\lambda,N>2$ integers, $\phi\geq 0$ smooth and supported in $(0,1)$. Let further $c_1,\dots,c_n,\dots\in\mathbb{R}$ and define

$$f(x)=f_1+f_2:=N\phi(Nx)+\sum_{n=1}^{\lambda-1}\varepsilon_n c_n\phi(x-n),$$

where ε_n is either 1 or -1. It is easy to see that

$$\hat{f}(\xi)=\hat{f}_1+\hat{f}_2:=\hat{\phi}(\xi/N)+\hat{\phi}(\xi)\left(\sum_{n=1}^{\lambda-1}\varepsilon_n c_n e^{2\pi i\xi n}\right).$$

Since f is supported in $[0,\lambda]$,

$$\left(\lambda^{-pA_1}\int_{|x|\le\lambda}|f(x)|^p|x|^{pA_1}dx+\lambda^{-pA_2}\int_{|x|\ge\lambda}|f(x)|^p|x|^{pA_2}dx\right)^{\frac{1}{p}}$$

$$=\left(\lambda^{-pA_1}\int_{|x|\le\lambda}|f(x)|^p|x|^{pA_1}dx\right)^{\frac{1}{p}}\approx\lambda^{-A_1}\left(N^{\frac{1}{p^*}-A_1}+\left(\sum_{n=1}^{\lambda-1}|c_n|^pn^{pA_1}\right)^{\frac{1}{p}}\right).$$

Likewise,

$$\|f\|_p\approx\|f_1\|_p+\|f_2\|_p\approx N^{\frac{1}{p^*}}+\left(\sum_{n=1}^{\lambda-1}|c_n|^p\right)^{\frac{1}{p}}.$$

By monotonicity and using Hölder's inequality,

$$\left(\lambda^{p^*B_1}\int_{|\xi|\le\lambda^{-1}}|\hat f(\xi)|^{p^*}|\xi|^{p^*B_1}d\xi\right)^{\frac{1}{p^*}}\lesssim\lambda^{-\frac{1}{p^*}+\frac{1}{2}}\|f\|_2\approx\lambda^{-\frac{1}{p^*}+\frac{1}{2}}\left(\sqrt{N}+\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right).$$

We also have

$$\int_{|\xi|\ge\lambda^{-1}}|\hat f(\xi)|^{p^*}|\xi|^{p^*B_2}d\xi\le\int_{\mathbb{R}}|\hat f(\xi)|^{p^*}|\xi|^{p^*B_2}d\xi$$

$$\lesssim N^{1+p^*B_2}+\int_{\mathbb{R}}|\hat\phi(\xi)x^{B_2}|^{p^*}\left|\sum_{n=1}^{\lambda-1}\varepsilon_nc_ne^{2\pi i\xi n}\right|^{p^*}dx;$$

and in the usual way (equation (3.27)),

$$\mathbb{E}\left[\left\|\hat f\right\|_{p^*}^{p^*}\right]^{\frac{1}{p^*}}\gtrsim\left\|\hat\phi\right\|_{p^*}\left(\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}-N^{\frac{1}{p^*}}\right).$$

So that raising equation (3.22) to p^*, applying Khitchine's inequality and taking p^*-th root, we deduce that

$$\left\|\hat\phi\right\|_{p^*}\left(A_p\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}-N^{\frac{1}{p^*}}\right)\left(N^{\frac{1}{p^*}}+\left(\sum_{n=1}^{\lambda-1}|c_n|^p\right)^{\frac{1}{p}}\right)\tag{3.30}$$

$$\lesssim\lambda^{-A_1}\left(N^{\frac{1}{p^*}-A_1}+\left(\sum_{n=1}^{\lambda-1}|c_n|^pn^{pA_1}\right)^{\frac{1}{p}}\right)\times$$

$$\left[\lambda^{-\frac{1}{p^*}+\frac{1}{2}}\left(N^{\frac{1}{2}}+\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right)+\lambda^{B_2}\left(N^{\frac{1}{p^*}+B_2}+\left\|\hat\phi\xi^{B_2}\right\|_{p^*}\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right)\right]$$

$$\lesssim\lambda^{-A_1}\left(N^{\frac{1}{p^*}-A_1}+\left(\sum_{n=1}^{\lambda-1}|c_n|^pn^{pA_1}\right)^{\frac{1}{p}}\right)\cdot\lambda^{B_2}\left(N^{\frac{1}{p^*}+B_2}+\left\|\hat\phi\xi^{B_2}\right\|_{p^*}\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right),$$

because as $p^*<2$,

$$\lambda^{-\frac{1}{p^*}+\frac{1}{2}}\left(N^{\frac{1}{2}}+\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right)\lesssim\lambda^{B_2}\left(N^{\frac{1}{p^*}+B_2}+\left\|\hat{\phi}x^{B_2}\right\|_{p^*}\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right).$$

Choose c_n such that

$$\left(\sum_{n=1}^{\lambda-1}|c_n|^p n^{pA_1}\right)^{\frac{1}{p}}=N^{\frac{1}{p^*}-A_1}$$

and such that the Hölder inequality

$$\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\leq\left(\sum_{n=1}^{\lambda-1}|c_n|^p n^{pA_1}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\lambda-1}n^{-\frac{A_1}{\frac{1}{2}-\frac{1}{p}}}\right)^{\frac{1}{2}-\frac{1}{p}}$$

becomes an equality. This implies that

$$\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\approx N^{\frac{1}{p^*}-A_1}\lambda^{-A_1+\frac{1}{2}-\frac{1}{p}}.$$

Finally, we take λ and N to be related by

$$N^{B_2+\frac{1}{p^*}}=N^{\frac{1}{p^*}-A_1}\lambda^{-A_1+\frac{1}{2}-\frac{1}{p}}\approx\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}},$$

that is, $\lambda=N^{\frac{A_1+B_2}{\frac{1}{2}-\frac{1}{p}-A_1}}$.

To finish the proof, let us examine the effect of the previous choices in each term of equation (3.30). Recall that, as stated in the beginning of the proof, $B_2>0$.

$$\left(N^{\frac{1}{p^*}+B_2}+\left\|\hat{\phi}\xi^{B_2}\right\|_{p^*}\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}\right)\approx N^{\frac{1}{p^*}+B_2};$$

$$\left\|\hat{\phi}\right\|_{p^*}\left(\left(\sum_{n=1}^{\lambda-1}|c_n|^2\right)^{\frac{1}{2}}-N^{\frac{1}{p^*}}\right)\approx N^{\frac{1}{p^*}+B_2};$$

$$\left(N^{\frac{1}{p^*}-A_1}+\left(\sum_{n=1}^{\lambda-1}|c_n|^p n^{pA_1}\right)^{\frac{1}{p}}\right)\approx N^{\frac{1}{p^*}-A_1};$$

$$\left(N^{\frac{1}{p^*}}+\left(\sum_{n=1}^{\lambda-1}c_n^p\right)^{\frac{1}{p}}\right)\approx N^{\frac{1}{p^*}}.$$

Hence, inequality (3.30) implies

$$1\lesssim\lambda^{B_2-A_1}N^{-A_1}=N^{\frac{(B_2-A_1)(A_1+B_2)}{\frac{1}{2}-\frac{1}{p}-A_1}-A_1},$$

and letting $N \to \infty$ we conclude that

$$0 \leq \frac{(B_2 - A_1)(A_1 + B_2)}{\frac{1}{2} - \frac{1}{p} - A_1} - A_1.$$

That is,

$$B_2^2 - A_1^2 \geq -A_1^2 + A_1(\frac{1}{2} - \frac{1}{p}),$$

whence the result follows. □

We are now in posicion to conclude the proof of the Theorem.

Proof of necessity in Theorem 3.5.1. To begin with, the conditions $B_2 \geq A_1$ and $A_2 \geq B_1$ follow from the second item of Lemma 3.5.9. Second, if $A_2 \leq \frac{1}{2} - \frac{1}{p}$, Lemmas 3.5.10 and 3.5.11 show that $B_1 = A_1 = 0$. Third, if $A_2 > \frac{1}{2} - \frac{1}{p}$ and $A_1 < \frac{1}{2} - \frac{1}{p}$, Lemma 3.5.12 shows that $B_2^2 \geq A_1(\frac{1}{2} - \frac{1}{p})$; if $B_2 = A_1 = \frac{1}{2} - \frac{1}{p}$, the third item of Lemma 3.5.9 together with Lemma 3.5.10 provide a contradiction, so that necessarily $B_2 > A_1$. □

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