UNIVERSITAT DE BARCELONA

**ADVANCED MATHEMATICS**
**MASTER'S FINAL PROJECT**

# MODELING THE HOMOTOPY THEORY OF SPACES VIA POSETS


*Author:*
Alba Sendón Blanco

*Supervisor:*
Javier J. Gutiérrez Marín


**Facultat de Matemàtiques i Informàtica**

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# Abstract


## Abstract

The aim of this project is to study the basics of Quillen model structures as an essential tool in algebraic topology and abstract homotopy theory.

In the first part, we will focus on the necessary background on category theory and homotopy theory in order to understand the notion of model structure and some fundamental constructions and tools within this framework.

The second part will deal with particular examples of model structures. Namely, we will study Thomason's model structure on the category of small categories and how it relates to Kan-Quillen's model structure on simplicial sets via an equivalence of homotopy categories, providing a model for the homotopy theory of topological spaces.

Finally, we will describe how the category of partially ordered sets inherits this model structure, offering yet another model for the homotopy theory of spaces. Moreover, we will analyze the relation between this structure and $T_0$ Alexandroff spaces.

## Resumen

El objetivo de este proyecto es estudiar las nociones básicas de las estructuras de modelos de Quillen como una herramienta esencial en topología algebraica y teoría de homotopía abstracta.

En la primera parte, nos centraremos en los conceptos necesarios en teoría de categorías y teoría de homotopía para entender la noción de estructura de modelos y algunas construcciones y herramientas fundamentales en este encuadre.

La segunda parte tratará con ejemplos particulares de estructuras de modelos. Concretamente, estudiaremos la estructura de modelos de Thomason en la categoría de categorías pequeñas y cómo se relaciona con la estructura de modelos de Kan-Quillen en la categoría de conjuntos simpliciales vía una equivalencia de categorías homotópicas, proporcionando un modelo para la teoría de homotopía de los espacios topológicos.

Finalmente, describiremos cómo la categoría de conjuntos parcialmente ordenados hereda esta estructura de modelos, ofreciendo aún otro modelo para la teoría de homotopía de los espacios. Además, analizaremos la relación entre esta estructura y los espacios $T_0$ de Alexandroff.

## Resum

L'objectiu d'aquest projecte és estudiar les nocions bàsiques de les estructures de models de Quillen com una eina essencial en topologia algebraica i teoria d'homotopia abstracta.

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La segona part tractarà amb exemples particulars d'estructures de models. Concretament, estudiarem l'estructura de models de Thomason en la categoria de categories petites i com es relaciona amb l'estructura de models de Kan-Quillen en la categoria de conjunts simplicials via una equivalència de categories homotòpiques, proveint un model per a la teoria d'homotopia dels espais topològics.

Finalment, descriurem com la categoria de conjunts parcialment ordenats hereta aquesta estructura de models, oferint encara un altre model per a la teoria d'homotopia dels espais. A més, analitzarem la relació entre aquesta estructura i la dels espais $T_0$ d'Alexandroff.

## Resumo

O obxectivo deste proxecto é estudar as nocións básicas das estruturas de modelos de Quillen coma unha ferramenta esencial en topoloxía alxébrica e teoría de homotopía abstracta.

Na primeira parte, centrarémonos nos conceptos necesarios en teoría de categorías e teoría de homotopía para entender a noción de estrutura de modelos e algunhas construcións e ferramentas fundamentais neste encadre.

A segunda parte tratará con exemplos particulares de estruturas de modelos. Concretamente, estudaremos a estrutura de modelos de Thomason na categoría de categorías pequenas e como se relaciona coa estrutura de modelos de Kan-Quillen na categoría de conxuntos simpliciais vía unha equivalencia de categorías homotópicas, aportando un modelo para a teoría de homotopía dos espazos topolóxicos.

Finalmente, describiremos como a categoría de conxuntos parcialmente ordenados herda esta estrutura de modelos, ofrecendo aínda outro modelo para a teoría de homotopía dos espazos. Ademais, analizaremos a relación entre esta estrutura e a dos espazos $T_0$ de Alexandroff.

# Introduction

In our early years of the Bachelor, we learn algebraic structures: groups, rings, modules... For all of them, we establish, for example, that the inverse of an isomorphism is unique, that there are some isomorphism theorems... and once we prove it for one of them, the proof for the rest of them is analogous (and often left as an exercise for the student). This not only happens in Algebra: also in Topology, the inverse of a homeomorphism is unique. Once we have noticed this, a natural question that can arise is: isn't there a way to define groups, rings, modules, topological spaces... all together and prove the results just once for all of them? Here is where Category Theory enters: a category (1.1.1) would be a class of objects together with sets of morphisms verifying some properties. Groups (with group homomorphisms), rings (with ring homomorphisms), modules (with module homomorphisms) and topological spaces (with continuous maps) fit in this definition and hence we have solved our apparently naive first question. But we can go far beyond that: Category Theory helps us to generalize a lot of things, even up to a point that some people call "abstract nonsense". The first chapter of this memoir will be devoted to explain the most elementary notions on Category Theory (categories, functors, natural transformations), as well as constructions that will be useful for us (limits, colimits, adjoint functors) and the basic properties of the categories we will deal with: topological spaces, simplicial sets, small categories, posets... The main references used for this chapter are [ML98], [DS95], [Fri12] and [Hof].

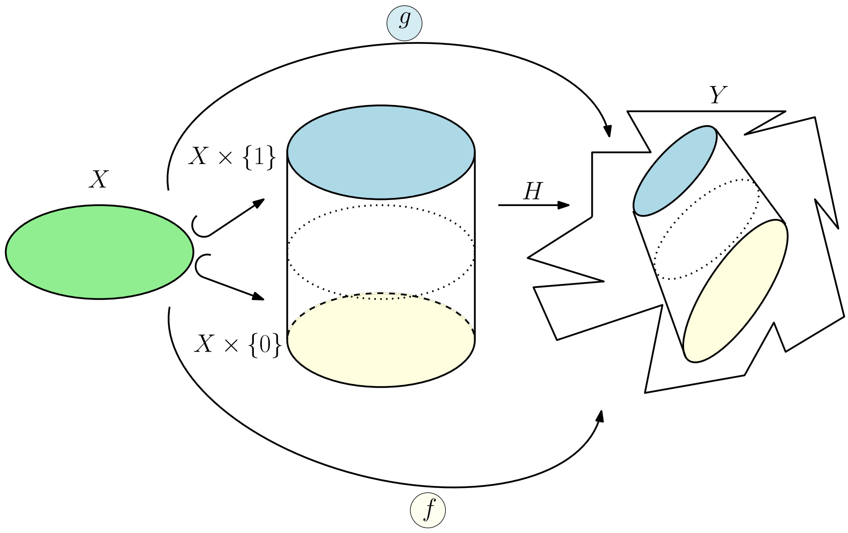



Figure 1: Sketch of a homotopy

In our case, we will deal with the generalization of a very well known topic in Algebraic Topology: the concept of homotopy. In our fist lessons in General Topology, we deal with topological spaces and continuous maps between them, outlining those continuous maps which have a continuous inverse, the homeomorphisms. From that moment, we treat homeomorphic topological spaces as if they where the same, but this might not be enough for us: for example, an $n$-dimensional disc cannot be homeomorphic to a single point (since we cannot even establish a bijection between them), but we can think about a point as an $n$-disc of infinitely small radius. That is where the notion of homotopy enters. If $X$ and $Y$ are two topological spaces, we say that two continuous maps $f, g\colon X \to Y$ between them are *homotopic* if there is a continuous map (*homotopy*) $H\colon X \times I \to Y$ between the *cylinder* of $X$ ($X \times I = X \times [0,1]$) and $Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. Therefore, the set of homotopies between $X$ and $Y$ will be the set of continuous maps from the cylinder of $X$, $X \times I$, to $Y$. A sketch of this notion can be seen in Figure 1.

Equivalently, by the Exponential Law, the set of homotopies between $X$ and $Y$ will be the set of continuous maps from $X$ to the topological space of continuous maps between $I$ and $Y$, that is, the topological space of *paths* in $Y$; see [Fox45, Theorem 2].

$$\begin{aligned}
\Phi\colon Y^{X\times[0,1]} &\longrightarrow (Y^{[0,1]})^X \\
H\colon X\times[0,1]\to Y &\longmapsto \Phi(H)\colon X\to Y^{[0,1]} \\
(x,t)\mapsto H(x,t) &\qquad\quad x\mapsto \Phi(H)(x)\colon [0,1]\to Y \\
&\qquad\qquad\qquad t\mapsto \Phi(H)(x)(t) = H(x,t).
\end{aligned}$$

From this point of view, a homotopy between $X$ and $Y$ would be a continuous map $H\colon X \to Y^I$ such that $H(x)(0) = f(x)$ for every $x \in X$ and $H(x)(1) = g(x)$ for every $x \in X$. An illustrative sketch can be seen in Figure 2.

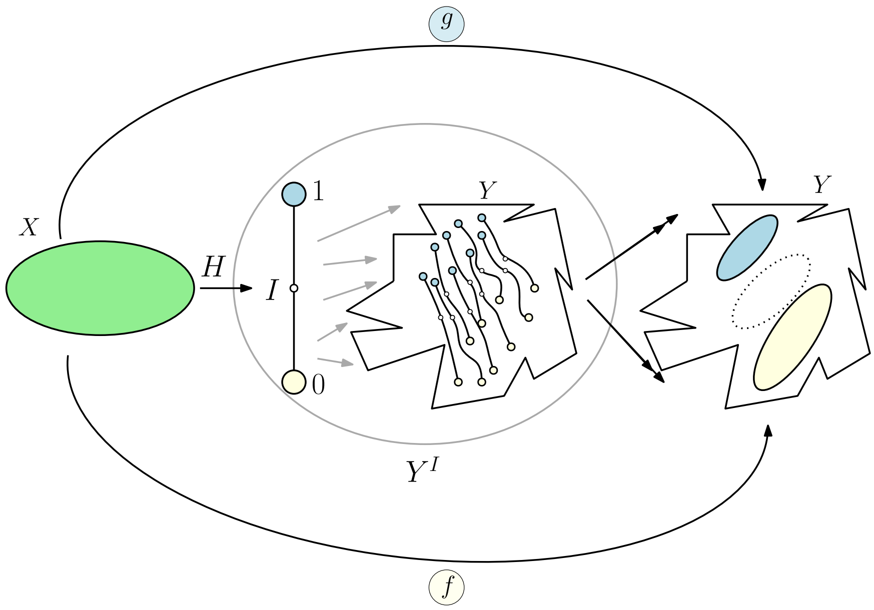



Figure 2: Sketch of a homotopy

We will say that two topological spaces $X, Y$ are *homotopy equivalent* if there exists a *homotopy equivalence* between them, that is, a continuous map $f\colon X \to Y$ such that there is another continuous map (*homotopy inverse*) $g\colon Y \to X$ with $g \circ f$ and $f \circ g$ homotopic to the corresponding identities. Once we have learned this, we consider homotopy equivalent topological spaces as if they were the same. But again, this might not be sufficent for us and we can weaken our request by considering *weak equivalences* (3.1.6) between topological spaces, that is, continuous maps that induce isomorphisms between their homotopy groups. After that, we can consider weakly homotopy equivalent topological spaces as if they were the same. But for being able to work with topological spaces in this contexts, we need more tools than just the (weak) homotopy equivalences, and it turns out that we also have fibrations and cofibrations, some special kind of continuous maps between topological spaces which are related among them and with (weak) homotopy equivalences and which help us to study topology within our new setting.

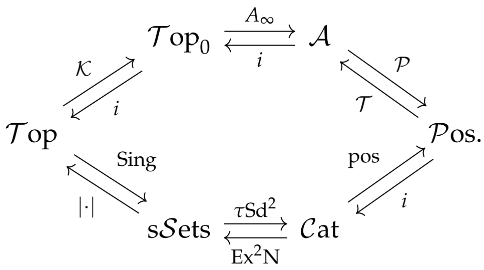
Then, for generalizing these notions to general categories, we first need some structure on them. Therefore, we will devote the second chapter of this memoir to study Quillen model categories (2.1.1), which are categories with some distinguished morphisms (weak equivalences, fibrations and cofibrations) satisfying certain properties. In this framework, we will be able to define cylinder objects and path objects for our categories, for later developing the notions of left and right homotopy and finally arriving to our goal of defining the concept of homotopy category, where our initial weak equivalences will be isomorphisms. Moreover, we will see that if we are able to relate two model categories with some special kind of functors called Quillen equivalences (2.5.3), we will get that their homotopy categories will be equivalent. This will be very useful for us since it will give us the freedom to study the same homotopy category from the point of view of different model categories, which can arise more naturally or be easier to understand in some cases, hence multiplying our possibilities when proving conjectures in this situation. These structures were first studied by the American mathematician Dan Quillen in [Qui67], but since they become a very important topic in Mathematics, a lot of topologists, such as Mark Hovey or Philip Hirschhorn, summarized and completed this theory; see [Hov99] or [Hir03]. Our main reference for this subject will be the survey by the American William Dwyer and one of its students, the Polish Jan Spaliński [DS95].

Of course, the category of topological spaces admits a model structure (and hence a homotopy category) with weak equivalences the weak homotopy equivalences. Moreover, the category of simplicial sets will admit a model structure which will be Quillen-equivalent to the model category of topological spaces. These facts were first studied by Quillen in [Qui67], and later, given its importance, by a lot of more mathematicians, such as Philip Hirschhorn, Paul Goerss and Rick Jardine or Edward B. Curtis; see [Hir03], [GJ09] or [Cur71]. We will give an overview of this in the first section of the third chapter.

Later on, the American mathematician Robert Wayne Thomason proved in [Tho80] that Quillen's model structure on simplicial sets can be lifted to the category of small categories with the help of two adjoint functors, namely $\tau\mathrm{Sd}^2\colon \mathrm{s}\mathcal{S}\mathrm{ets} \rightleftarrows \mathcal{C}\mathrm{at}\colon \mathrm{Ex}^2\mathrm{N}$. Some of the axioms for model categories are directly derived from this lifting movement, but others require a bit of more work and the development of a new kind of morphisms, the *Dwyer morphisms* (3.2.12), which will be very useful for us. We will devote the second section of the third chapter to analyze in depth this construction and to prove that the category of small categories admits indeed the aforesaid model structure.

But small categories, even though they are more simple than general categories, can have a lot of morphisms and still be a bit difficult to understand. A handy kind of small categories are the ones that come from partially ordered sets, since two objects in this kind of categories can only be linked by at most one morphism. In his article, Thomason mentions that for every simplicial set $X$, $\tau\mathrm{Sd}^2X$ is one of such categories, as well as the main objects used in his constructions. With this in mind, the Greek mathematician George Raptis proved in [Rap10] that the category of posets inherits the model structure of small categories in such a way that it is also Quillen-equivalent with Quillen's model structure for topological spaces. Thus, the first part of the third section of the third chapter of this memoir will be devoted to state this clearly.

In this way, we have up to four different ways of interpreting weak homotopies for spaces and of manipulating them in order to make new discoverings in the area. Nevertheless, in our long way we lost a bit the intuition of the relation between topological spaces and posets. For example, we created the notion of Dwyer morphisms for small categories, but we do not know what meaning they might have in topological spaces. Therefore, in the last part of this memoir we will give a characterization of Dwyer maps in topological spaces, also due to Raptis, making use of an equivalence between posets and Alexandroff $T_0$ spaces, as well as a chain of adjoint pairs between $A$-spaces and topological spaces.




After all this performance, there is still a lot of work to be done. As Raptis mentions in his paper, this equivalence between $A$-spaces and posets can be restricted to finite spaces and finite posets and is related to a notion of simple homotopy type, developed in [BM08] by the Argentinian mathematicians Jonathan Barmak and Gabriel Minian, which stands strictly between homotopy type and weak homotopy type and which is equivalent to a notion of simple homotopy type for finite simplicial complexes. This fact makes us wonder if there is a model structure on (finite) posets such that in the homotopy category posets with the same simple homotopy type are isomorphic. This and other lines of further work are discussed on the conclusion chapter of the memoir.

# Chapter 1

# Preliminaries on category theory

This first chapter is devoted, besides to set some notation, to introduce the basic notions on category theory and the most useful constructions for us. Furthermore, we will present the particular categories that we will deal with and some of their fundamental properties.

## 1.1. Categories, functors and natural transformations

The main objective of category theory is to set up an environment where we can frame a lot of mathematical features. In consequence, our first definitions will not be very demanding. Some references about this subject are [ML98], [AHS06] or [HS07].

**Definition 1.1.1.** A **category** $\mathcal{C}$ consists of:

1. A class $\mathrm{Ob}(\mathcal{C})$ of **objects**.
2. Given objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of **morphisms** (or **arrows**) $\mathcal{C}(A, B)$ with morphisms $f \in \mathcal{C}(A, B)$ (or $f\colon A \to B$). All these sets are disjoint.
3. Given objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a **composition law**

$$\begin{aligned} \mathcal{C}(A,B) \times \mathcal{C}(B,C) &\to \mathcal{C}(A,C) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

   such that:

   *a*) It is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
   *b*) For any object $A \in \mathrm{Ob}(\mathcal{C})$ there exists $\mathrm{Id}_A \in \mathcal{C}(A, A)$ such that $\mathrm{Id}_A \circ f = f$ and $g \circ \mathrm{Id}_A = g$ for any morphisms $f$ and $g$. We call such a morphism the **identity** of $A$.

A category $\mathcal{C}$ is said to be **small** if $\mathrm{Ob}(\mathcal{C})$ is a set and **finite** if $\mathrm{Ob}(\mathcal{C})$ is a finite set and it has only a finite number of morphisms between two objects.

**Definition 1.1.2.** Given a category $\mathcal{C}$ and two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we say that they are **isomorphic** if there exist morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, A)$ (**isomorphisms**) such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$. We write $g = f^{-1}$ and $A \cong B$.

**Definition 1.1.3.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{C}$ is a **subcategory** of $\mathcal{D}$ if:

1. $\mathrm{Ob}(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{D})$.

2. For any $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(A,B) \subseteq \mathcal{D}(A,B)$.

3. The composition law in $\mathcal{C}$ is the restriction of the composition law in $\mathcal{D}$.

**Definition 1.1.4.** Let $\mathcal{C}$, $\mathcal{D}$ be two categories. We define the **product category** $\mathcal{C} \times \mathcal{D}$ in the following way:

1. $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$.

2. For any $(C,D),(C',D') \in \mathrm{Ob}(\mathcal{C} \times \mathcal{D})$, $\mathcal{C} \times \mathcal{D}((C,D),(C',D')) = \mathcal{C}(C,C') \times \mathcal{D}(D,D')$.

3. The composition law is given by $(f,g) \circ (f',g') = (f \circ f', g \circ g')$.

**Definition 1.1.5.** Let $\mathcal{C}$ be a category. We define the **opposite category** $\mathcal{C}^{\mathrm{op}}$ in the following way:

1. $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$.

2. For any $A, B \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}^{\mathrm{op}}(A,B) = \mathcal{C}(B,A)$.

3. The composition law is given by the composition law in $\mathcal{C}$.

**Definition 1.1.6.** Let $\mathcal{C}$ be a category.

1. We say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is **initial** in $\mathcal{C}$ if for any object $B \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(A,B)$ has only one element.

2. We say that an object $B \in \mathrm{Ob}(\mathcal{C})$ is **terminal** in $\mathcal{C}$ if for any object $A \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(A,B)$ has only element.

3. We say that an object $A \in \mathrm{Ob}(\mathcal{C})$ is **zero** in $\mathcal{C}$ if it is both initial and terminal.

**Definition 1.1.7.** Given categories $\mathcal{C}$ and $\mathcal{D}$, a (covariant) **functor** $F$ between $\mathcal{C}$ and $\mathcal{D}$ ($F\colon \mathcal{C} \to \mathcal{D}$) is:

1. A law that assigns to each object $A \in \mathrm{Ob}(\mathcal{C})$ an object $F(A) \in \mathrm{Ob}(\mathcal{D})$.

2. For any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set map

$$\begin{aligned} F\colon \mathcal{C}(A,B) &\to \mathcal{D}(F(A),F(B)) \\ f &\mapsto F(f) \end{aligned}$$

   which satisfies:

   *a*) Compatibility with the composition law: $F(g \circ f) = F(g) \circ F(f)$.
   *b*) Preservation of the identities: $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ for all $A \in \mathrm{Ob}(\mathcal{C})$.

We say that a functor $F$ is:

1. **faithful** if for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the map $F\colon \mathcal{C}(A,B) \to \mathcal{D}(F(A),F(B))$ is injective.

2. **full** if for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the map $F\colon \mathcal{C}(A,B) \to \mathcal{D}(F(A),F(B))$ is surjective.

3. **fully faithful** if it is faithful and full.

**Definition 1.1.8.** Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be **full** if the inclusion $i\colon \mathcal{D} \to \mathcal{C}$ is a full functor, that is, if for any $A, B \in \mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ we have that $\mathcal{D}(A,B) = \mathcal{C}(A,B)$.

**Example 1.1.9.** Let $\mathcal{C}$ be a category. The following assignation is a functor for every $X \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{aligned} \mathcal{C}(X,\cdot)\colon \mathcal{C} &\to \mathcal{S}\text{ets}. \\ Y \in \mathrm{Ob}(\mathcal{C}) &\mapsto \mathcal{C}(X,Y) \\ f \in \mathcal{C}(Y_1,Y_2) &\mapsto \mathcal{C}(X,\cdot)(f)\colon \mathcal{C}(X,Y_1) \to \mathcal{C}(X,Y_2) \\ & \qquad\qquad\qquad\qquad g \mapsto f\circ g \end{aligned}$$

We will sometimes denote $f_* = \mathcal{C}(X,\cdot)(f)$.

Notice that if $f \in \mathcal{C}(Y_1,Y_2)$ is an isomorphism (with inverse $f^{-1}$), then $f_*$ will be a bijection with inverse $(f^{-1})_*$.

**Example 1.1.10.** Let $\mathcal{C}$ be a category. The following assignation is a functor for every $X \in \mathrm{Ob}(\mathcal{C})$:

$$\begin{aligned} \mathcal{C}(\cdot,X)\colon \mathcal{C}^{\mathrm{op}} &\to \mathcal{S}\text{ets}. \\ Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}) &\mapsto \mathcal{C}(Y,X) \\ f \in \mathcal{C}^{\mathrm{op}}(Y_1,Y_2) = \mathcal{C}(Y_2,Y_1) &\mapsto \mathcal{C}(\cdot,X)(f)\colon \mathcal{C}(Y_1,X) \to \mathcal{C}(Y_2,X) \\ & \qquad\qquad\qquad\qquad g \mapsto g\circ f \end{aligned}$$

We will sometimes denote $f^* = \mathcal{C}(\cdot,X)(f)$.

Notice that if $f \in \mathcal{C}^{\mathrm{op}}(Y_1,Y_2)$ is an isomorphism (with inverse $f^{-1}$), then $f^*$ will be a bijection with inverse $(f^{-1})^*$.

**Definition 1.1.11.** Two categories $\mathcal{C}, \mathcal{D}$ are **isomorphic** if there exist fuctors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$, $F \circ G = \mathrm{Id}_{\mathcal{D}}$.

**Definition 1.1.12.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ two functors between them. A **natural transformation** $t\colon F \Rightarrow G$ between $F$ and $G$ consists on a law which assigns to any object $A \in \mathrm{Ob}(\mathcal{C})$ a morphism $t_A\colon F(A) \to G(A)$ such that for any morphism $f\colon A \to B$ in $\mathcal{C}$ we have that $G(f) \circ t_A = t_B \circ F(f)$:

$$\begin{array}{ccc} F(A) & \xrightarrow{t_A} & G(A) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{t_B} & G(B). \end{array}$$

A natural transformation is a **natural equivalence** if $t_A$ is an isomorphism in $\mathcal{D}$ for every $A \in \mathrm{Ob}(\mathcal{C})$.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be an **equivalence of categories** if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is naturally equivalent to $\mathrm{Id}_{\mathcal{C}}$ and $F \circ G$ is naturally equivalent to $\mathrm{Id}_{\mathcal{D}}$.

**Remark 1.1.13.** Let $\mathcal{I} = \{0 \to 1\}$ be the category with two objects and only one non-identity morphism. Having a natural transformation $t\colon F \Rightarrow G$ is equivalent to having a functor $H\colon \mathcal{C} \times \mathcal{I} \to \mathcal{D}$ such that $H(C,0) = F(C)$ and $H(C,1) = G(C)$ for every object $C \in \mathrm{Ob}(\mathcal{C})$ and $H(f, \mathrm{Id}_0) = F(f)$ and $H(f, \mathrm{Id}_1) = G(f)$ for every morphism $f$ in $\mathcal{C}$.

## 1.2. Limits and colimits

Once we know what categories are, we can start playing with them. Here we present the most handy constructions for us, following the guidelines of the second section of [DS95].

**Definition 1.2.1.** If $\mathcal{C}$ is a category and $\mathcal{D}$ is a small category, then there is a **functor category** (or **category of diagrams in $\mathcal{C}$ with the shape of $\mathcal{D}$**) $\mathcal{C}^{\mathcal{D}}$ in which the objects are the functors $F\colon \mathcal{D} \to \mathcal{C}$ and the morphisms are the natural transformations between them.

**Example 1.2.2.** In particular, if $\mathcal{D} = \mathcal{I} = \{0 \to 1\}$, then the objects of $\mathcal{C}^{\mathcal{I}}$ are exactly the morphisms $f\colon X(0) \to X(1)$ of $\mathcal{C}$ and an arrow $t\colon f \to g$ in $\mathcal{C}^{\mathcal{I}}$ is a commutative diagram

$$\begin{array}{ccc} X(0) & \xrightarrow{t_0} & Y(0) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X(1) & \xrightarrow{t_1} & Y(1). \end{array}$$

In this case, $\mathcal{C}^{\mathcal{I}}$ is called the **category of morphisms** of $\mathcal{C}$ and denoted by $\mathrm{Mor}(\mathcal{C})$.

**Definition 1.2.3.** An object $A$ of a category $\mathcal{C}$ is said to be a **retract** of an object $X$ if there exist morphisms $i\colon A \to X$ and $r\colon X \to A$ such that $r \circ i = \mathrm{Id}_A$.

If $f\colon A \to B$ and $g\colon X \to Y$ are morphisms of $\mathcal{C}$, we say that $f$ is a retract of $g$ if the object of $\mathrm{Mor}(\mathcal{C})$ represented by $f$ is a retract of the object of $\mathrm{Mor}(C)$ represented by $g$, that is, if there is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{t_0} & X & \xrightarrow{s_0} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{t_1} & Y & \xrightarrow{s_1} & B \end{array}$$

such that $s_0 \circ t_0 = \mathrm{Id}_A$ and $s_1 \circ t_1 = \mathrm{Id}_B$.

**Definition 1.2.4.** Let $\mathcal{C}$ be a category and $\mathcal{D}$ a small category. We define the **constant diagram** functor $\mathrm{Const}\colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ as the one that carries an object $X \in \mathrm{Ob}(\mathcal{C})$ to the constant functor $\mathrm{Const}(X)\colon \mathcal{D} \to \mathcal{C}$ (which sends each object $Y \in \mathrm{Ob}(\mathcal{D})$ to $X$ and each morphism in $\mathcal{D}$ to the identity morphism $\mathrm{Id}_X$) and a morphism $f \in \mathcal{C}(X, X')$ to the constant natural transformation $\mathrm{Const}(f)\colon \mathrm{Const}(X) \to \mathrm{Const}(X')$ (such that for each object $Y \in \mathrm{Ob}(\mathcal{D})$ we have the morphism $\mathrm{Const}(f)_Y = f$).

**Definition 1.2.5.** Let $\mathcal{C}$ be a category, $\mathcal{D}$ a small category and $F\colon \mathcal{D} \to \mathcal{C}$ a functor. A **colimit** for $F$ is an object $C = \mathrm{colim}(F) \in \mathrm{Ob}(\mathcal{C})$ together with a natural transformation $t\colon F \Rightarrow \mathrm{Const}(C)$ such that for every object $X \in \mathrm{Ob}(\mathcal{C})$ and every natural transformation $s\colon F \Rightarrow \mathrm{Const}(X)$, there exists a unique morphism $s'\colon C \to X$ in $\mathcal{C}$ such that $\mathrm{Const}(s') \circ t = s$.

**Remark 1.2.6.** If $\mathcal{D}$ has a terminal object $*$, then $\mathrm{colim}(F) = F(*)$.

**Example 1.2.7.** If $\mathcal{D}$ is a category with a set $I$ of objects and no nonidentity morphisms, a functor $X\colon \mathcal{D} \to \mathcal{C}$ is just a collection $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$. The colimit of $X$ is called the **coproduct** of the collection and written $\amalg_{i \in I} X_i$. The natural transformation $t\colon X \Rightarrow \mathrm{Const}(\amalg_{i \in I} X_i)$ gives inclusion morphisms $in_i\colon X_i \to \amalg_{i \in I} X_i$, $i \in I$. Moreover, given morphisms $f_i\colon X_i \to Y$, $i \in I$ (that is, a natural transformation $s\colon X \Rightarrow \mathrm{Const}(Y)$), there is a unique morphism $f\colon \amalg_{i \in I} X_i \to Y$ such that $f \circ in_i = f_i$ for $i \in I$. Furthermore, if we have $\{A_i\}_{i \in I} \subseteq \mathrm{Ob}(\mathcal{C})$ and $\{B_i\}_{i \in I} \subseteq \mathrm{Ob}(\mathcal{C})$ such that $\amalg_{i \in I} A_i$ and $\amalg_{i \in I} B_i$ exist and $f_i\colon A_i \to B_i$ for every $i \in I$, composing with the inclusions we get $in_i \circ f_i\colon A_i \to \amalg_{i:\, I} B_i$ for $i \in I$, so by the universal property of the coproduct there exists a unique morphism $\amalg_{i \in I} f_i\colon \amalg_{i \in I} A_i \to \amalg_{i \in I} B_i$ such that $(\amalg_{i \in I} f_i) \circ in_i = in_i \circ f_i$ for every $i \in I$. For $I = \{0, 1\}$, we simplify the notation to $X_0 \amalg X_1$, $f_0 + f_1$ and $f_0 \amalg f_1$.

**Example 1.2.8.** If $\mathcal{D} = \{b \leftarrow a \to c\}$ is a category with three objects and the two indicated nonidentity morphisms, a functor $X\colon \mathcal{D} \to \mathcal{C}$ is a diagram $X(b) \leftarrow X(a) \to X(c)$. The colimit of $X$ is called the **pushout** $P = X(b) \amalg_{X(a)} X(c)$ of the diagram, and it gives a natural commutative diagram (**pushout diagram**)

$$\begin{array}{ccc} X(a) & \xrightarrow{i} & X(c) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ X(b) & \xrightarrow{i'} & P. \end{array}$$

The morphism $i'$ is called the **cobase change** of $i$ along $j$ and the morphism $j'$ is called the cobase change of $j$ along $i$. Due to the definition of colimit, we also have that given an object $Y \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f_b\colon X(b) \to Y$ and $f_c\colon X(c) \to Y$ such that $f_b \circ j = f_c \circ i$ (or equivalently, a natural transformation $s\colon X \Rightarrow \mathrm{Const}(Y)$), there is a unique morphism $f\colon P \to Y$ such that $f \circ j' = f_c$ and $f \circ i' = f_b$. Furthermore, if we have $A_1, B_1, C_1, A_2, B_2\ C_2 \in \mathrm{Ob}(\mathcal{C})$ such that $B_1 \amalg_{A_1} C_1$ and $B_2 \amalg_{A_2} C_2$ exist and $f\colon A_1 \to A_2$, $g\colon B_1 \to B_2$, $h\colon C_1 \to C_2$ such that $g \circ j_1 = j_2 \circ f$ and $h \circ i_1 = i_2 \circ f$, composing with the canonical morphisms we get $i'_2 \circ g\colon B_1 \to B_2 \amalg_{A_2} C_2$, $j'_2 \circ h\colon C_1 \to B_2 \amalg_{A_2} C_2$ such that

$$i'_2 \circ g \circ j_1 = i'_2 \circ j_2 \circ f = j'_2 \circ i_2 \circ f = j'_2 \circ h \circ i_1,$$

and in consequence by the universal property of the pushout there exists a unique morphism

$$g \amalg_f h\colon B_1 \amalg_{A_1} C_1 \to B_2 \amalg_{A_2} C_2$$

such that $(g \amalg_f h) \circ j'_1 = j'_2 \circ h$ and $(g \amalg_f h) \circ i'_1 = i'_2 \circ g$.

**Remark 1.2.9.** If $\mathcal{C}$ has an initial object $\varnothing$, then the coproduct $X_0 \amalg X_1$ is the pushout of the diagram $X_0 \overset{g_0}{\leftarrow} \varnothing \overset{g_1}{\rightarrow} X_1$ (with $g_0$, $g_1$ the unique morphisms that there are).

**Remark 1.2.10.** For any $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f\colon A \to B$, $B = A \amalg_A B$ is the pushout of the diagram $A \overset{\mathrm{Id}_A}{\leftarrow} A \overset{f}{\rightarrow} B$. Similarly, given any diagram $B \leftarrow A \to C$ and any morphism $C \to D$, we have that $(B \amalg_A C) \amalg_C D = B \amalg_A D$.

**Example 1.2.11.** If $\mathcal{D} = \{0 \to 1 \to 2 \to 3 \to \dots\}$ is a category with objects the nonnegative integers and a single morphism $i \to j$ for $i \le j$, a functor $X\colon \mathcal{D} \to \mathcal{C}$ is a diagram of the form

$$X(0) \xrightarrow{f_{0,1}} X(1) \xrightarrow{f_{1,2}} \dots \to X(n) \to \dots$$

in $\mathcal{C}$. The colimit of $X$ is called the **sequential colimit** of the objects $X(n)$ and denoted by $\mathrm{colim}_n X(n)$. By definition of colimit, the natural transformation $t\colon X \Rightarrow \mathrm{Const}(\mathrm{colim}_n X(n))$ gives us the natural morphisms $in_n\colon X(n) \to \mathrm{colim}_n X(n)$ verifying $in_n = in_{n+1} \circ f_{n,n+1}$ for every $n$. Moreover, given morphisms $f_n\colon X(n) \to Y$ such that $f_n = f_{n+1} \circ f_{n,n+1}$ for every $n$ (or equivalently, a natural transformation $s\colon X \Rightarrow \mathrm{Const}(Y)$), there is a unique morphism $\mathrm{colim}_n f_n\colon \mathrm{colim}_n X(n) \to Y$ such that $\mathrm{colim}_n \circ in_n = f_n$ for every $n$.

**Definition 1.2.12.** Let $\mathcal{D}$ be a small category and $F\colon \mathcal{D} \to \mathcal{C}$ a functor. A **limit** for $F$ is an object $L = \lim(F) \in \mathrm{Ob}(\mathcal{C})$ together with a natural transformation $t\colon \mathrm{Const}(L) \Rightarrow F$ such that for every object $X \in \mathrm{Ob}(\mathcal{C})$ and every natural transformation $s\colon \mathrm{Const}(X) \Rightarrow F$, there exists a unique morphism $s'\colon X \to L$ in $\mathcal{C}$ such that $t \circ \mathrm{Const}(s') = s$.

**Remark 1.2.13.** If $\mathcal{D}$ has an initial object $\varnothing$, then $\lim(F) = F(\varnothing)$.

**Example 1.2.14.** If $\mathcal{D}$ is a category with a set $I$ of objects and no nonidentity morphisms, a functor $X\colon \mathcal{D} \to \mathcal{C}$ is just a collection $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$. The limit of $X$ is called the **product** of the collection and written $\prod_{i\in I} X_i$. The natural transformation $t\colon \mathrm{Const}(\prod_{i\in I} X_i) \Rightarrow X$ gives the projection morphisms $pr_i\colon \prod_{i\in I} X_i \to X_i$, $i \in I$. Moreover, given morphisms $f_i\colon Y \to X_i$, $i \in I$ (that is, a natural transformation $s\colon \mathrm{Const}(Y) \Rightarrow X$), there is a unique morphism $f\colon Y \to \prod_{i\in I} X_i$ such that $pr_i \circ f = f_i$ for $i \in I$. Furthermore, if we have $\{A_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$ and $\{B_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$ such that $\prod_{i\in I} A_i$ and $\prod_{i\in I} B_i$ exist and $f_i\colon A_i \to B_i$ for every $i \in I$, composing with the projections we get $f_i \circ pr_i\colon \prod_{i\in I} A_i \to B_i$ for $i \in I$, so by the universal property of the product there exists a unique morphism $\prod_{i\in I} f_i\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ such that $pr_i \circ (\prod_{i\in I} f_i) = f_i \circ pr_i$ for every $i \in I$. For $I = \{0, 1\}$, we simplify the notation to $X_0 \times X_1$, $(f_0, f_1)$ and $f_0 \times f_1$.

**Example 1.2.15.** If $\mathcal{D} = \{b \to a \leftarrow c\}$ is a category with three objects and the two indicated nonidentity morphisms, a functor $X\colon \mathcal{D} \to \mathcal{C}$ is a diagram $X(b) \to X(a) \leftarrow X(c)$. The limit of $X$ is called the

**pullback** $P = X(b)\prod_{X(a)} X(c)$ of the diagram, and it gives a natural commutative diagram (**pullback diagram**)

$$\begin{array}{ccc} P & \xrightarrow{i'} & X(c) \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ X(b) & \xrightarrow{i} & X(a). \end{array}$$

The morphism $i'$ is called the **base change** of $i$ along $j$ and the morphism $j'$ is called the base change of $j$ along $i$. By the definition of limit, we also have that given an object $Y \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f_b \colon Y \to X(b)$ and $f_c \colon Y \to X(c)$ such that $i \circ f_b = j \circ f_c$ (or equivalently, a natural transformation $s \colon \mathrm{Const}(Y) \Rightarrow X$), there is a unique morphism $f \colon Y \to P$ such that $i' \circ f = f_c$ and $j' \circ f = f_b$. Furthermore, if we have $A_1, B_1, C_1, A_2, B_2, C_2 \in \mathrm{Ob}(\mathcal{C})$ such that $B_1 \prod_{A_1} C_1$ and $B_2 \prod_{A_2} C_2$ exist and $f \colon A_1 \to A_2$, $g \colon B_1 \to B_2$, $h \colon C_1 \to C_2$ such that $j_2 \circ h = f \circ j_1$ and $i_2 \circ g = f \circ i_1$, composing with the canonical morphisms we get $g \circ j_1' \colon B_1 \prod_{A_1} C_1 \to B_2$, $h \circ i_1' \colon B_1 \prod_{A_1} C_1 \to C_2$ such that

$$i_2 \circ j \circ j_1' = f \circ i_1 \circ j_1' = f \circ j_1 \circ i_1' = j_2 \circ h \circ i_1',$$

and in consequence by the universal property of the pushout there exists a unique morphism

$$g \textstyle\prod_f h \colon B_1 \prod_{A_1} C_1 \to B_2 \prod_{A_2} C_2$$

such that $i_2' \circ (g \prod_f h) = h \circ i_1'$ and $j_2' \circ (g \prod_f h) = g \circ j_1'$.

**Remark 1.2.16.** If $\mathcal{C}$ has a terminal object $*$, then the product $X_0 \times X_1$ is the pullback of the diagram $X_0 \overset{g_0}{\to} * \overset{g_1}{\leftarrow} X_1$ (with $g_0, g_1$ the only morphisms that there are).

**Remark 1.2.17.** For any $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f \colon A \to B$, $A = A \prod_B B$ is the pullback of the diagram $A \overset{f}{\to} B \overset{\mathrm{Id}_B}{\leftarrow} B$. Similarly, given any diagram $B \to A \leftarrow C$ and any morphism $C \leftarrow D$, we have that $(B \prod_A C) \prod_C D = B \prod_A D$.

**Example 1.2.18.** If $\mathcal{D} = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \ldots\}$ is a category with objects the nonnegative integers and a single morphism $i \leftarrow j$ for $i \leq j$, a functor $X \colon \mathcal{D} \to \mathcal{C}$ is a diagram of the form

$$X(0) \overset{f_{1,0}}{\longleftarrow} X(1) \overset{f_{2,1}}{\longleftarrow} \ldots \longleftarrow X(n) \longleftarrow \ldots$$

in $\mathcal{C}$. The limit of $X$ is called the **sequential limit** of the objects $X(n)$ and denoted by $\lim_n X(n)$. By definition of limit, the natural transformation $t \colon \mathrm{Const}(\lim_n X(n)) \to X$ gives us the natural morphisms $pr_n \colon \lim_n X(n) \to X(n)$ verifying $f_{n+1,n} \circ pr_{n+1} = pr_n$ for every $n$. In addition, given morphisms $f_n \colon Y \to X(n)$ such that $f_{n+1,n} \circ f_{n+1} = f_n$ for every $n$ (or equivalently, a natural transformation $s \colon \mathrm{Const}(Y) \to X$), there is a unique morphism $\lim_n f_n \colon Y \to \lim_n X(n)$ such that $pr_n \circ \lim_n = f_n$ for every $n$.

**Proposition 1.2.19.** *Let $\mathcal{C}$ be a category, $\mathcal{D}$ the empty category (that is, the category with no objects nor morphisms) and $F \colon \mathcal{D} \to \mathcal{C}$ the unique functor. Then,* $\mathrm{colim}(F)$*, if it exists, is an initial object of $\mathcal{C}$ and* $\lim(F)$*, if it exists, is a terminal object of $\mathcal{C}$.*

*Proof.* Let us prove that $\mathrm{colim}(F)$ (consisting on an object $C \in \mathrm{Ob}(\mathcal{C})$ and a natural transformation $t \colon F \Rightarrow \mathrm{Const}(C)$ satisfying all the premises) is an initial object, that is, that given $X \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{C}(C, X)$ has a unique element. It is clear that we have a unique natural transformation $s \colon F \Rightarrow \mathrm{Const}(X)$ (since $\mathcal{D}$ is the empty category, $\mathrm{Const}(X) \colon \mathcal{D} \to \mathcal{C}$ must be also the unique functor $F$, and then $s$ is the identity natural transformation), so by definition there exists a unique morphism $s' \colon C \to X$ in $\mathcal{C}$ such that $\mathrm{Const}(s') \circ t = s$. This gives us $\mathcal{C}(C, X) \neq \emptyset$. Assume there exists a morphism $s'' \colon C \to X$. Then, $\mathrm{Const}(s'') \circ t = s$ because $s$ is the only natural transformation that there is. By the uniqueness of $s'$, we get $s'' = s'$ and $\mathcal{C}(C, X)$ has only one element.

The proof for the terminal object is analogous. □

**Example 1.2.20.** Limits and colimits exist in the category $\mathcal{S}$ets. Indeed, let $\mathcal{D}$ be a small category and $F\colon \mathcal{D} \to \mathcal{S}$ets a functor.

The colimit of $F$ is described as follows. Let $C = \amalg_{d\in \mathrm{Ob}(\mathcal{D})} F(d)/\sim \in \mathrm{Ob}(\mathcal{S}\mathrm{ets})$ be the quotient set of the disjoint union of the set of images of the objects of $\mathcal{D}$ under $F$ with $\sim$ the equivalence relation generated by:

$$x \in F(d) \sim x' \in F(d') \text{ if there exists } f \in \mathcal{D}(d,d') \text{ such that } F(f)(x) = x'.$$

Consider also $t\colon F \Rightarrow \mathrm{Const}(C)$ the natural transformation that assigns to every $X \in \mathrm{Ob}(\mathcal{D})$ the map which is the composition of the inclusion with the quotient map $t_X\colon F(X) \to \amalg_{d\in \mathrm{Ob}(\mathcal{D})} F(d) \to C$.

The limit of $F$ is described as follows. Let $C \in \mathrm{Ob}(\mathcal{D})$ be the following subset of the cartesian product of the set of images of the objects of $\mathcal{D}$ under $F$:

$$C = \left\{ (x_d)_{d\in \mathrm{Ob}(\mathcal{D})} \in \prod_{d\in \mathrm{Ob}(\mathcal{D})} F(d) \colon F(f)(x_d) = x_{d'} \ \forall f \in \mathcal{D}(d,d') \right\}.$$

Consider also $t\colon \mathrm{Const}(C) \Rightarrow F$ the natural transformation that assigns to every $X \in \mathrm{Ob}(\mathcal{D})$ the map which is the composition of the inclusion with the projection map $t_X\colon C \to \prod_{d\in \mathrm{Ob}(\mathcal{D})} F(d) \to F(X)$.

## 1.3. Adjoint functors

Now, we will present a special kind of functors between categories which will have many nice properties. As a consequence, we will always try to relate our categories with this kind of functors.

**Definition 1.3.1.** Let $\mathcal{C}$, $\mathcal{D}$ be two categories and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ a pair of functors. We say that $(F, G)$ is an **adjoint pair** if for any objects $C \in \mathrm{Ob}(\mathcal{C})$, $D \in \mathrm{Ob}(\mathcal{D})$ there exists a bijection

$$\tau_{CD}\colon \mathcal{D}(F(C), D) \to \mathcal{C}(C, G(D))$$

providing the following two natural equivalences of functors:

$$\mathcal{D}(\cdot, D) \circ F \simeq \mathcal{C}(\cdot, G(D))\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{ets},$$
$$\mathcal{D}(F(C), \cdot) \simeq \mathcal{C}(C, \cdot) \circ G\colon \mathcal{D} \to \mathcal{S}\mathrm{ets}.$$

We will denote this by $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$, and say that $F$ is **left adjoint** and $G$ is **right adjoint**.

**Remark 1.3.2.** Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjoint pair and $C \in \mathrm{Ob}(\mathcal{C})$, $D \in \mathrm{Ob}(\mathcal{D})$. By definition of adjointness, given the commutative diagrams (the first one in $\mathcal{C}$, the second one in $\mathcal{D}$):

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow{v} & G(Y), \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{w} & X \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow{z} & Y, \end{array}$$

we also have the following commutative diagrams (the first one in $\mathcal{D}$, the second one in $\mathcal{C}$):

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_{AX}^{-1}(u)} & X \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow[\tau_{BY}^{-1}(v)]{} & Y, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\tau_{AX}(w)} & G(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow[\tau_{BY}(z)]{} & G(Y). \end{array}$$

We will sometimes denote $u^\flat = \tau_{AX}^{-1}(u), v^\flat = \tau_{BY}^{-1}(v)$ and $w^\sharp = \tau_{AX}(w), z^\sharp = \tau_{BY}(z)$.

**Definition 1.3.3.** Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a pair of adjoint functors.

For any $C \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Id}^{\sharp}_{F(C)}\colon C \to G(F(C))$ is called the **unit morphism** of the adjunction at $C$.

For any $D \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Id}^{\flat}_{G(D)}\colon F(G(D)) \to D$ is called the **counit morphism** of the adjunction at $D$.

**Proposition 1.3.4.** *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a pair of adjoint functors. Then,*

1. $f^{\sharp} = G(f) \circ \mathrm{Id}^{\sharp}_{F(C)}$ *for every morphism* $f \in \mathcal{D}(F(C), D)$ *and* $f^{\flat} = \mathrm{Id}^{\flat}_{G(D)} \circ F(f)$ *for every morphism* $f \in \mathcal{C}(C, G(D))$.
2. *The unit morphisms give us a natural transformation* $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ *with* $\eta_C = \mathrm{Id}^{\sharp}_{F(C)}$ *for every* $C \in \mathrm{Ob}(\mathcal{C})$*; similarly, the counit morphisms give us a natural transformation* $\epsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ *with* $\epsilon_D = \mathrm{Id}^{\flat}_{G(D)}$ *for every* $D \in \mathrm{Ob}(\mathcal{D})$.
3. $\mathrm{Id}_{F(C)} = \mathrm{Id}^{\flat}_{G(F(C))} \circ F(\mathrm{Id}^{\sharp}_{F(C)})$ *for every object* $C \in \mathrm{Ob}(\mathcal{C})$,
   $\mathrm{Id}_{G(D)} = G(\mathrm{Id}^{\flat}_{G(D)}) \circ \mathrm{Id}^{\sharp}_{F(G(D))}$ *for every object* $D \in \mathrm{Ob}(\mathcal{D})$.

*Proof.* The first assertion follows from the fact that the commutative diagrams

$$\begin{array}{ccc} F(C) & \xrightarrow{\mathrm{Id}_{F(C)}} & F(C) \\ {\scriptstyle F(\mathrm{Id}_C)}\downarrow & & \downarrow{\scriptstyle f} \\ F(C) & \xrightarrow{f} & D, \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & G(D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(\mathrm{Id}_D)} \\ G(D) & \xrightarrow{\mathrm{Id}_{G(D)}} & G(D) \end{array}$$

yield by adjointness the commutative diagrams

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{Id}^{\sharp}_{F(C)}} & G(F(C)) \\ {\scriptstyle \mathrm{Id}_C}\downarrow & & \downarrow{\scriptstyle G(f)} \\ C & \xrightarrow{f^{\sharp}} & G(D), \end{array} \qquad \begin{array}{ccc} F(C) & \xrightarrow{f^{\flat}} & D \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_D} \\ F(G(D)) & \xrightarrow{\mathrm{Id}^{\flat}_{G(D)}} & D \end{array}$$

and these diagrams show exactly what we wanted.

For proving the second assertion, we need to show the naturality of $\eta$ and $\epsilon$, but this follows from the fact that the commutative diagrams

$$\begin{array}{ccc} F(A) & \xrightarrow{\mathrm{Id}_{F(A)}} & F(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ F(B) & \xrightarrow{\mathrm{Id}_{F(B)}} & F(B), \end{array} \qquad \begin{array}{ccc} G(X) & \xrightarrow{\mathrm{Id}_{G(X)}} & G(X) \\ {\scriptstyle G(g)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ G(Y) & \xrightarrow{\mathrm{Id}_{G(Y)}} & G(Y) \end{array}$$

yield by adjointness the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}^{\sharp}_{F(A)}} & G(F(A)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(F(f))} \\ B & \xrightarrow{\mathrm{Id}^{\sharp}_{F(B)}} & G(F(B)), \end{array} \qquad \begin{array}{ccc} F(G(X)) & \xrightarrow{\mathrm{Id}^{\flat}_{G(X)}} & X \\ {\scriptstyle F(G(g))}\downarrow & & \downarrow{\scriptstyle g} \\ F(G(Y)) & \xrightarrow{\mathrm{Id}^{\flat}_{G(Y)}} & Y \end{array}$$

and these diagrams show exactly what we wanted.

Finally, the third assertion follows from the computations below:

$$\mathrm{Id}_{F(C)} = (\mathrm{Id}^\sharp_{F(C)})^\flat \underbrace{=}_{(1)} \mathrm{Id}_{G(F(C))} \circ F(\mathrm{Id}^\sharp_{F(C)}),$$
$$\mathrm{Id}_{G(D)} = (\mathrm{Id}^\flat_{G(D)})^\sharp \underbrace{=}_{(1)} G(\mathrm{Id}^\flat_{G(D)}) \circ \mathrm{Id}^\sharp_{F(G(D))}.$$

□

**Proposition 1.3.5.** *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a pair of adjoint functors. Then,*

1. *$F$ preserves colimits: let $\mathcal{E}$ be a small category and $H\colon \mathcal{E} \to \mathcal{C}$ a functor. If $(C, t)$ is a colimit for $H$, then $(F(C), F(t))$ is a colimit for $F \circ H$.*
2. *$G$ preserves limits: let $\mathcal{E}$ be a small category and $H\colon \mathcal{E} \to \mathcal{D}$ a functor. If $(L, t)$ is a limit for $H$, then $(G(L), G(t))$ is a limit for $G \circ H$.*

*Proof.* Let us prove the first assertion. It is clear that $F(C) \in \mathrm{Ob}(\mathcal{D})$ and that $F(t)\colon F \circ H \Rightarrow \mathrm{Const}(F(C))$ is a natural transformation. Now, let $X \in \mathrm{Ob}(\mathcal{D})$ be an object and $s\colon F \circ H \Rightarrow \mathrm{Const}(X)$ a natural transformation. Then, we also have an object $G(X) \in \mathrm{Ob}(\mathcal{C})$ and $\widehat{s}\colon H \Rightarrow \mathrm{Const}(G(X))$ a natural transformation that assigns to every $E \in \mathrm{Ob}(\mathcal{E})$ the morphism $s^\sharp_E$. Indeed, let $f \in \mathcal{E}(E_1, E_2)$, then by the naturality of $s$ we have a commutative diagram

$$\begin{array}{ccc} F(H(E_1)) & \xrightarrow{s_{E_1}} & X \\ \downarrow{\scriptstyle (F\circ H)(f)} & & \downarrow{\scriptstyle \mathrm{Id}_X} \\ F(H(E_2)) & \xrightarrow{s_{E_2}} & X, \end{array}$$

which gives us the following commutative diagram by adjointness

$$\begin{array}{ccc} H(E_1) & \xrightarrow{s^\sharp_{E_1}} & G(X) \\ \downarrow{\scriptstyle H(f)} & & \downarrow{\scriptstyle \mathrm{Id}_{G(X)}} \\ H(E_2) & \xrightarrow{s^\sharp_{E_2}} & G(X), \end{array}$$

and this last diagram proves the naturality of $\widehat{s}$. Hence, by the universal property of the colimit, there is a unique $s'\colon C \to G(X)$ such that $\mathrm{Const}(s') \circ t = \widehat{s}$. In consequence, there is a unique $s'^\flat\colon F(C) \to X$ such that $\mathrm{Const}(s'^\flat) \circ F(t) = s$, which completes our proof.

The proof of the second assertion is analogous. □

## 1.4. Some examples of categories

In this section, we present the particular categories we will be dealing with in the next chapters, study their properties and see how they are related through some functors.

### 1.4.1. Topological spaces

Of course, as topologists, we need to take care of topological spaces.

**Definition 1.4.1.** The class of topological spaces together with continuous maps between them forms the **category of topological spaces** $\mathcal{T}\mathrm{op}$.

This category admits a lot of subcategories. The most important ones for us will be described below.

**Definition 1.4.2.** A topological space $X$ is said to be $T_0$ (or a **Kolmogorov** space) if for every pair of distinct points of $X$ there is an open neighborhood of one of them not containing the other one.

**Definition 1.4.3.** The class of $T_0$ topological spaces together with continuous maps between them forms the **category of $T_0$ topological spaces** $\mathcal{T}\mathrm{op}_0$, which is a full subcategory of $\mathcal{T}\mathrm{op}$.

There is an obvious inclusion functor $i\colon \mathcal{T}\mathrm{op}_0 \to \mathcal{T}\mathrm{op}$. Let us see that it has a left adjoint.

**Definition 1.4.4.** Let us define the **Kolmogorov functor** as

$$\begin{aligned} \mathcal{K}\colon \mathcal{T}\mathrm{op} &\to \mathcal{T}\mathrm{op}_0, \\ (X,\tau) &\mapsto \mathcal{K}(X) \\ f \in \mathcal{T}\mathrm{op}(X,Y) &\mapsto \mathcal{K}(f)\colon \mathcal{K}(X) \to \mathcal{K}(Y) \\ & \qquad\qquad [x] \mapsto \mathcal{K}(f)([x]) = [f(x)] \end{aligned}$$

with $\mathcal{K}(X) = X/\sim$ the quotient space (**Kolmogorov quotient**) given by:

$$x \sim x' \Leftrightarrow \{U \subseteq X\colon x \in U \in \tau\} = \{U \subseteq X\colon x' \in U \in \tau\}$$

(that is, two points will be equivalent if and only if they have the same open neighborhoods).

**Proposition 1.4.5.** $\mathcal{K}\colon \mathcal{T}\mathrm{op} \rightleftarrows \mathcal{T}\mathrm{op}_0 \colon i$ *form an adjoint pair.*

*Proof.* Given $X \in \mathrm{Ob}(\mathcal{T}\mathrm{op})$ and $Y \in \mathrm{Ob}(\mathcal{T}\mathrm{op}_0)$, the desired natural bijection is

$$\begin{aligned} \mathcal{T}\mathrm{op}_0(\mathcal{K}(X),Y) &\overset{\tau_{XY}}{\cong} \mathcal{T}\mathrm{op}(X,i(Y)). \\ f &\mapsto \tau_{XY}(f)\colon X \to i(Y) \\ & \qquad\qquad x \mapsto f([x]) \\ \tau_{XY}^{-1}(g)\colon \mathcal{K}(X) \to Y &\leftarrow g \\ [x] \mapsto g(x) & \end{aligned}$$

□

**Definition 1.4.6.** A topological space $X$ is called an **Alexandroff $T_0$ space** (or **$A$-space**) if it is $T_0$ and every intersection of open sets in $X$ is open.

**Definition 1.4.7.** The class of $A$-spaces together with continuous maps between them forms the **category of $A$-spaces** $\mathcal{A}$, which is a full subcategory of $\mathcal{T}\mathrm{op}_0$.

There is an obvious inclusion functor $i\colon \mathcal{A} \to \mathcal{T}\mathrm{op}_0$. Let us see that this time it has a right adjoint.

**Definition 1.4.8.** Let us define the **Alexandroff functor** as

$$\begin{aligned} A_\infty\colon \mathcal{T}\mathrm{op}_0 &\to \mathcal{A}, \\ X &\mapsto A_\infty(X) \\ f \in \mathcal{T}\mathrm{op}_0(X,Y) &\mapsto A_\infty(f)\colon A_\infty(X) \to A_\infty(Y) \\ & \qquad\qquad x \mapsto f(x) \end{aligned}$$

with $A_\infty(X)$ the topological space whose underlying set is $X$ and whose topology is given by arbitrary intersections of open sets in $X$.

**Proposition 1.4.9.** $i\colon \mathcal{A} \rightleftarrows \mathcal{T}\mathrm{op}_0 \colon A_\infty$ *form an adjoint pair.*

*Proof.* Given $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{T}\mathrm{op}_0)$, the desired natural bijection

$$\mathcal{T}\mathrm{op}_0(i(X),Y) \cong \mathcal{A}(X, A_\infty(Y))$$

is given by the identity. □

### 1.4.2. Posets

**Definition 1.4.10.** An **order relation** over a set is a reflexive, antisymmetric and transitive relation defined on it. A **poset** (partially order set) is a set with an order relation.

We will normally use "$\leq$", "$\geq$" for denoting the order relation, although we wil sometimes employ "$\subseteq$", "$\supseteq$". We will make use of "$<$", "$>$", "$\subset$", "$\supset$" for indicating that the relation is strict.

A **totally ordered set** is a set with an order relation such that every pair of elements are comparable. A **chain** of a poset is any totally ordered subset of it.

**Definition 1.4.11.** A **morphism between posets** $f\colon P \to Q$ is an order-preserving set map, that is, such that if $p_1 \leq p_2$ in $P$, then $f(p_1) \leq f(p_2)$ in $Q$.

**Definition 1.4.12.** The class of posets together with order-preserving maps between them forms the **category of posets** $\mathcal{P}$os.

Let us see a result that will be useful later.

**Proposition 1.4.13.** *Limits and colimits exist in* $\mathcal{P}$*os.*

*Proof.* Let $\mathcal{D}$ be a small category and $F\colon \mathcal{D} \to \mathcal{P}$os a functor. Composing with the forgetful functor $U\colon \mathcal{P}\text{os} \to \mathcal{S}\text{ets}$ (which sends each poset to its underlying set and each order-preserving map to its underlying set map), we get a functor $U \circ F\colon \mathcal{D} \to \mathcal{S}$ets. Since limits and colimits exist in $\mathcal{S}$ets (see Example 1.2.20), we have $C = \operatorname{colim}(U \circ F) \in \operatorname{Ob}(\mathcal{S}\text{ets})$ and a universal natural transformation $t\colon U \circ F \Rightarrow \operatorname{Const}(C)$, as well as $L = \lim(U \circ F) \in \operatorname{Ob}(\mathcal{S}\text{ets})$ and a universal natural transformation $s\colon \operatorname{Const}(L) \Rightarrow U \circ F$.

Now, consider in $C$ the following order:

$$[x] \leq [x'] \Leftrightarrow \text{ there are } y \in [x],\ y' \in [x'] \text{ such that } y \leq y'.$$

With it, $C$ becomes a poset and $t_d\colon F(d) \to C$ becomes an order-preserving map for every $d \in \operatorname{Ob}(\mathcal{D})$, as well as the morphisms derived from the universal property of $t$. We have therefore constructed our colimit poset.

Similarly, consider in $L$ the following order:

$$(x_d)_{d\in \operatorname{Ob}(\mathcal{D})} \leq (x'_d)_{d\in \operatorname{Ob}(\mathcal{D})} \Leftrightarrow x_d \leq x'_d \ \forall d \in \operatorname{Ob}(\mathcal{D}).$$

With it, $L$ becomes a poset and $s_d\colon L \to F(d)$ becomes an order-preserving map for every $d \in \operatorname{Ob}(\mathcal{D})$, as well as the morphisms derived from the universal property of $s$. We have therefore constructed our limit poset. □

#### Isomorphism between posets and Alexandroff $T_0$ spaces

**Definition 1.4.14.** Let us define the following functor:

$$\begin{aligned} \mathcal{T}\colon \mathcal{P}\text{os} &\to \mathcal{A}, \\ P &\mapsto \mathcal{T}(P) \\ f \in \mathcal{P}\text{os}(P,Q) &\mapsto \mathcal{T}(f)\colon \mathcal{T}(P) \to \mathcal{T}(Q) \\ & \qquad\qquad x \mapsto f(x) \end{aligned}$$

with $\mathcal{T}(P)$ the topological space whose underlying set is $P$ and whose topology is given by the basis $\{U_p\}_{p\in P}$ with $U_p = \{q \in P\colon q \geq p\}$.

**Definition 1.4.15.** Let us define the following functor:

$$\begin{aligned} \mathcal{P}\colon \mathcal{A} &\to \mathcal{P}\text{os}, \\ X &\mapsto \mathcal{P}(X) \\ f \in \mathcal{T}\text{op}(X,Y) &\mapsto \mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y) \\ & \qquad\qquad x \mapsto f(x) \end{aligned}$$

with $\mathcal{P}(X)$ the poset whose underlying set is $X$ and the order relation is given by:

$$x \leq x' \leftrightarrow \cap\{U \subseteq X \text{ open} : x' \in U\} \subseteq \cap\{U \subseteq X \text{ open} : x \in U\}.$$

**Proposition 1.4.16.** $\mathcal{P} \circ \mathcal{T} = \text{Id}_{\mathcal{P}\text{os}}$, $\mathcal{T} \circ \mathcal{P} = \text{Id}_{\mathcal{A}}$. *That is, the category of Alexandroff $T_0$ spaces and the category of posets are isomorphic.*

### 1.4.3. Simplicial complexes

**Definition 1.4.17.** Given a set $V_K$ we can define a **(abstract) simplicial complex** $K$ over the **set of vertices** $V_K$ as a set $K \subseteq 2^{V_K} \setminus \{\emptyset\}$, with $2^{V_K}$ the power set of $V_K$, such that:

- $\{v\} \in K$ for every $v \in V_K$.
- if $\sigma \in K$ and $\emptyset \neq \sigma' \subseteq \sigma$, then $\sigma' \in K$.

The elements of $K$ are called its **simplices** and they will be called $n$-simplices if they have $n+1$ elements. In particular, the 0-simplices of $K$ are its vertices.

If $V_K$ is a totally ordered set, we will talk about an **ordered** abstract simplicial complex.

**Remark 1.4.18.** As we said, the set of vertices of a simplicial complex is completely determined by its set of 0-simplices, so we usually will not pay attention to $V_K$.

**Definition 1.4.19.** A **morphism between simplicial complexes** $f\colon K \to L$ is a set map $f\colon V_K \to V_L$ such that if $\sigma \in K$, then $f(\sigma) \in L$.

**Definition 1.4.20.** The class of simplicial complexes together with morphisms between them forms the **category of simplicial complexes** s$\mathcal{C}$omp. In the same way, the class of ordered simplicial complexes together with morphisms between them forms the **category of ordered simplicial complexes** os$\mathcal{C}$omp, which is a full subcategory of s$\mathcal{C}$omp.

#### Functors between simplicial complexes and posets

**Definition 1.4.21.** Let us define the functor

$$\begin{aligned} \mathcal{X}\colon \text{s}\mathcal{C}\text{omp} &\to \mathcal{P}\text{os}, \\ K \in \text{Ob}(\text{s}\mathcal{C}\text{omp}) &\mapsto \mathcal{X}(K) \\ f \in \text{s}\mathcal{C}\text{omp}(K,L) &\mapsto \mathcal{X}(f)\colon \mathcal{X}(K) \to \mathcal{X}(L) \\ & \qquad\qquad \sigma \mapsto f(\sigma) \end{aligned}$$

with $\mathcal{X}(K)$ the poset given by $K$ with the inclusion order, also called the **poset of faces** of $K$.

**Definition 1.4.22.** Let us define the functor

$$\begin{aligned} \mathcal{K}\colon \mathcal{P}\text{os} &\to \text{s}\mathcal{C}\text{omp}, \\ P \in \text{Ob}(\mathcal{P}\text{os}) &\mapsto \mathcal{K}(P) \\ f \in \mathcal{P}\text{os}(P,Q) &\mapsto \mathcal{K}(f)\colon \mathcal{K}(P) \to \mathcal{K}(Q) \\ &\qquad\qquad \{x\} \in V_{\mathcal{K}(P)} \mapsto \{f(x)\} \end{aligned}$$

with $\mathcal{K}(P)$ the simplicial complex of the chains of $P$.

**Example 1.4.23.** Let $[n]$ be the finite ordered set $0 \leq 1 \leq \ldots \leq n$. Then, its associated simplicial complex is $2^{[n]} \setminus \{\emptyset\}$.

In general $\mathcal{K} \circ \mathcal{X} \neq \text{Id}_{\text{s}\mathcal{C}\text{omp}}$ and $\mathcal{X} \circ \mathcal{K} \neq \text{Id}_{\mathcal{P}\text{os}}$, which motivates the following definitions.

**Definition 1.4.24.** Let $K \in \text{Ob}(\text{s}\mathcal{C}\text{omp})$ be a simplicial complex. The **barycentric subdivision** of $K$ is the simplicial complex $\mathcal{K}(\mathcal{X}(K))$.

**Definition 1.4.25.** Let $P \in \text{Ob}(\mathcal{P}\text{os})$ be a poset. The **barycentric subdivision** of $P$ is the poset $\mathcal{X}(\mathcal{K}(P))$.

### 1.4.4. Simplicial sets

Sometimes, simplicial complexes are too restrictive for us to work with them, so let us introduce the category of simplicial sets. A very good reference for a first approach to them is [Fri12].

**Definition 1.4.26.** A **simplicial set** $X$ consists of a sequence of sets $\{X_n\}_{n\geq 0}$ and, for each $n \geq 0$, maps $d_i\colon X_n \to X_{n-1}$ and $s_i\colon X_n \to X_{n+1}$ for each $0 \leq i \leq n$ such that:

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i,\ i < j, \\ d_i \circ s_j &= s_{j-1} \circ d_i,\ i < j, \\ d_j \circ s_j &= d_{j+1} \circ s_j = \text{Id}, \\ d_i \circ s_j &= s_j \circ d_{i-1},\ i > j+1, \\ s_i \circ s_j &= s_{j+1} \circ s_i,\ i \leq j. \end{aligned}$$

The elements of $X_n$ are called **$n$-simplices**, the 0-simplices are called the **vertices** of $X$, the $d_i$'s are called **face operators** and the $s_i$'s are called **degeneracy operators**. A simplex is called **degenerate** if it is in the image of some $s_i$, and **non-degenerate** otherwise.

**Definition 1.4.27.** A **morphism of simplicial sets** $f\colon X \to Y$ is a collection of set maps

$$\{f_n\colon X_n \to Y_n\}_{n\geq 0}$$

such that $d_i \circ f_n = f_{n-1} \circ d_i$ and $s_i \circ f_n = f_{n+1} \circ s_i$ for every $n \geq 0$ and every $0 \leq i \leq n$.

**Definition 1.4.28.** The class of simplicial sets together with morphisms between them forms the **category of simplicial sets** s$\mathcal{S}$ets.

**Example 1.4.29.** Recall that an ordered abstract simplicial complex $K$ is a set of subsets of some totally ordered vertex set $V_K$ such that all the singletons of $V_K$ belong to $K$ and such that any nonempty subset of a subset of $K$ also belongs to $K$. We can construct a simplicial set $X_K$ from it as follows:

- Each set of $n$-simplices will be constituted by the ordered sequences of elements of $V_K$ $(v_{k_0}, \ldots, v_{k_n})$ with $k_0 \leq \ldots \leq k_n$ (it may have repeated elements) such that the set $\{v_{k_0}, \ldots, v_{k_n}\}$ (deleting repeated elements) belongs to $K$.

- The face and degeneracy maps are defined as follows:

$$d_i(v_{k_0}, \ldots, v_{k_n}) = (v_{k_0}, \ldots, \widehat{v_{k_i}}, \ldots, v_{k_n}) = (v_{k_0}, \ldots, v_{k_{i-1}}, v_{k_{i+1}}, \ldots, v_{k_n})$$
(deleting the $i$-th element)
$$s_i(v_{k_0}, \ldots, v_{k_n}) = (v_{k_0}, \ldots, v_{k_i}, v_{k_i}, \ldots, v_{k_n})$$
(repeating the $i$-th element)

It is clear that these maps verify the compatibility equations.

Moreover, given ordered simplicial complexes $K$ and $L$ such that $K \subseteq L$, we have an inclusion functor between their associated simplicial sets $i\colon X_K \to X_L$ with $i_n\colon (X_K)_n \to (X_L)_n$ the inclusion, which obviously commutes with the face and degeneracy maps.

Therefore, given the standard $n$-simplex $K = 2^{[n]} \setminus \{\varnothing\}$, there is an associated **standard $n$-simplicial set** $\Delta[n]$ whose $k$-simplices will be of the form

$$[0, \ldots, n]_k = \{(i_0, \ldots, i_k) \colon 0 \leq i_0 \leq \ldots \leq i_k \leq n\}\,.$$

Moreover, given the simplicial complex

$$K = 2^{[n]} \setminus \{\varnothing, \{0, \ldots, n\}\},$$

there is an associated simplicial set $\partial\Delta[n]$ and a morphism $i_n\colon \partial\Delta[n] \to \Delta[n]$.

Finally, given the simplicial complex

$$K = 2^{[n]} \setminus \{\varnothing, \{0, \ldots, n\}, \{0, \ldots, k-1, k+1, \ldots, n\}\},$$

there is an associated simplicial set $\Delta[n,k]$ and a morphism $i_{n,k}\colon \Delta[n,k] \to \Delta[n]$.

Now let us explore an equivalent definition, more "category theoretical" and more useful for us in some cases.

**Definition 1.4.30.** The **simplicial category** $\Delta$ is the category that has as objects the finite ordered sets $[n] = \{0 \leq 1 \leq \ldots \leq n\}$ and as morphisms the order-preserving maps between them.

**Remark 1.4.31.** Let us highlight two special types of morphisms in this category:

$$D_i\colon [n] \to [n+1], \qquad j \mapsto D_i(j) = \begin{cases} j, \text{ if } j < i \\ j+1, \text{ if } j \geq i \end{cases}$$

$$S_i\colon [n] \to [n-1]. \qquad j \mapsto S_i(j) = \begin{cases} j, \text{ if } j \leq i \\ j-1, \text{ if } j > i \end{cases}$$

Every order-preserving map $f \in \Delta([m],[n])$ is a composition of $D_i$'s and $S_i$'s: if it is the identity, we are done; otherwise, let $i_1 > \ldots > i_s$ the elements of $[n]$ not in $f([m])$ and $j_1 < \ldots < j_t$ the elements of $[m]$ such that $f(j) = f(j+1)$ and notice that

$$f = D_{i_1} \circ \ldots \circ D_{i_s} \circ S_{j_1} \circ \ldots \circ S_{j_t}.$$

Moreover, it is easy to see that:

$$D_j \circ D_i = D_i \circ D_{j-1},\ i < j,$$
$$S_j \circ D_i = D_i \circ S_{j-1},\ i < j,$$
$$S_j \circ D_j = S_j \circ D_{j+1} = \mathrm{Id},$$
$$S_j \circ D_i = D_{i-1} \circ S_j,\ i > j+1,$$
$$S_j \circ S_i = S_i \circ S_{j+1},\ i \leq j.$$

**Definition 1.4.32.** The **category of simplicial sets** will be $s\mathcal{S}\text{ets} = \mathcal{S}\text{ets}^{\Delta^{\text{op}}}$, whose elements are the functors $F\colon \Delta^{\text{op}} \to \mathcal{S}\text{ets}$, the **simplicial sets**, and whose morphisms are the natural transformations between those functors. For a simplicial set $X\colon \Delta^{\text{op}} \to \mathcal{S}\text{ets}$, the image $X([n])$ will be usually denoted by $X_n$ and called the set of $n$**-simplices**.

**Example 1.4.33.** In this setting, the $n$-simplicial set $\Delta[n]$ corresponds to the functor $\Delta(\cdot, [n])$.

### 1.4.5. Small categories

**Definition 1.4.34.** Let us consider the category $\mathcal{C}\text{at}$ of **small categories**, whose objects are the small categories and whose morphisms are the functors between them.

As we did for $\mathcal{P}\text{os}$, the existence of limits and colimits in $\mathcal{C}\text{at}$ can also be derived from their existence in $\mathcal{S}\text{ets}$.

**Proposition 1.4.35.** *Limits and colimits exist in* $\mathcal{C}\text{at}$.

*Proof.* Let $\mathcal{D}$ be a small category and $F\colon \mathcal{D} \to \mathcal{C}\text{at}$ a functor. Composing with the forgetful functor $U\colon \mathcal{C}\text{at} \to \mathcal{S}\text{ets}$ (which sends each small category to its set of objects and functor to its associated set map between the sets of objects of the domain and the codomain), we get a functor $U \circ F\colon \mathcal{D} \to \mathcal{S}\text{ets}$. Since limits and colimits exist in $\mathcal{S}\text{ets}$ (see Example 1.2.20), we have $C = \operatorname{colim}(U \circ F) \in \operatorname{Ob}(\mathcal{S}\text{ets})$ and a universal natural transformation $t\colon U \circ F \Rightarrow \operatorname{Const}(C)$, as well as $L = \lim(U \circ F) \in \operatorname{Ob}(\mathcal{S}\text{ets})$ and a universal natural transformation $s\colon \operatorname{Const}(L) \Rightarrow U \circ F$.

Now, consider the small category whose set of objects is $C$ and it has a morphism $[x] \to [x']$ for each $y \in [x]$, $y' \in [x']$ such that there is a morphism $y \to y'$. We will still call this category $C$. Then, given $d \in \operatorname{Ob}(\mathcal{D})$, we can construct the following functor from $t_d$:

$$\begin{aligned} T_d\colon F(d) &\to C, \\ x &\mapsto t_d(x) = [x] \\ (x \to x') &\mapsto ([x] \to [x']) \end{aligned}$$

and we can do the same with the set maps derived from the universal property of $t$. We have therefore constructed our colimit small category.

In a resembling way, consider the small category whose set of objects is $L$ and it has a morphism $(x_d)_{d\in\operatorname{Ob}(\mathcal{D})} \to (x'_d)_{d\in\operatorname{Ob}(\mathcal{D})}$ if and only if there is a morphism $x_d \to x'_d$ for all $d \in \operatorname{Ob}(\mathcal{D})$. We will still call this category $L$. Then, given $d \in \operatorname{Ob}(\mathcal{D})$, we can construct the following functor from $s_d$:

$$\begin{aligned} S_d\colon L &\to F(d), \\ (x_d)_{d\in\operatorname{Ob}(\mathcal{D})} &\mapsto s_d((x_d)_{d\in\operatorname{Ob}(\mathcal{D})}) = x_d \\ ((x_d)_{d\in\operatorname{Ob}(\mathcal{D})} \to (x'_d)_{d\in\operatorname{Ob}(\mathcal{D})}) &\mapsto (x_d \to x'_d) \end{aligned}$$

and we can do the same with the set maps derived from the universal property of $s$. We have therefore constructed our limit small category. □

**Functors between posets and small categories**

**Definition 1.4.36.** Let us define the following functor:

$$\begin{aligned} i\colon \mathcal{P}\text{os} &\to \mathcal{C}\text{at}, \\ P \in \text{Ob}(\mathcal{P}\text{os}) &\mapsto i(P) \\ f \in \mathcal{P}\text{os}(P,Q) &\mapsto i(f)\colon i(P) \to i(Q) \\ &\qquad p \in \text{Ob}(i(P)) = P \mapsto f(p) \in Q = \text{Ob}(i(Q)) \\ &\qquad (p_1 \to p_2) \mapsto (f(p_1) \to f(p_2)) \end{aligned}$$

with $i(P)$ the small category that has as set of objects $\text{Ob}(i(P)) = P$ and such that there is a morphism $r \to s$ if and only if $r \leq s$, and the composition law is given by the transitivity of $\leq$.

**Example 1.4.37.** In particular, for the full subcategory $\Delta$ of $\mathcal{P}$os we have a functor $i\colon \Delta \to \mathcal{C}\text{at}$ such that for $[n] \in \text{Ob}(\Delta)$,

$$i([n]) = \{0 \overset{f_0}{\to} 1 \overset{f_1}{\to} \dots \overset{f_{n-1}}{\to} n\},$$

$$\begin{aligned} i(D_i)\colon i([n]) &\to i([n+1]), \\ j &\mapsto \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases} \\ f_j &\mapsto \begin{cases} f_j, & \text{if } j < i-1 \\ f_{j+1} \circ f_j, & \text{if } j = i-1 \\ f_{j+1}, & \text{if } j \geq i \end{cases} \end{aligned} \qquad \begin{aligned} i(S_i)\colon i([n]) &\to i([n-1]). \\ j &\mapsto \begin{cases} j, & \text{if } j \leq i \\ j-1, & \text{if } j > i \end{cases} \\ f_j &\mapsto \begin{cases} f_j, & \text{if } j < i \\ \text{Id}_i, & \text{if } j = i \\ f_{j-1}, & \text{if } j > i \end{cases} \end{aligned}$$

**Definition 1.4.38.** Let us define the following functor:

$$\begin{aligned} \text{pos}\colon \mathcal{C}\text{at} &\to \mathcal{P}\text{os}, \\ C \in \text{Ob}(\mathcal{C}\text{at}) &\mapsto \text{pos}(C) \\ F \in \mathcal{C}\text{at}(C,D) &\mapsto \text{pos}(F)\colon \text{pos}(C) \to \text{pos}(D) \\ &\qquad\qquad p \mapsto F(p) \end{aligned}$$

with $\text{pos}(C)$ defined as follows: in the set $\text{Ob}(\mathcal{C})$ we put $a \leq b$ if there is a morphism in $\mathcal{C}(a,b)$. This gives us a reflexive and transitive relation. If in addition we put $a = b$ if $a \leq b$ and $b \leq a$ we get antisymmetry and therefore a poset $\text{pos}(C)$.

**Remark 1.4.39.** Notice that $\text{pos}(i(P)) = P$ for every poset $P$ and that a small category is the inclusion of some poset if and only if there is at most one morphism between every pair of objects.

**Proposition 1.4.40.** *The functors $i$ and* pos *form an adjoint pair* $\text{pos}\colon \mathcal{C}\text{at} \rightleftarrows \mathcal{P}\text{os}\colon i$.

*Proof.* Given $C \in \text{Ob}(\mathcal{C}\text{at})$ and $P \in \text{Ob}(\mathcal{P}\text{os})$, the desired natural bijection is

$$\begin{aligned} \mathcal{P}\text{os}(\text{pos}(C), P) &\overset{\tau_{CP}}{\cong} \mathcal{C}\text{at}(C, i(P)). \\ f &\mapsto \tau_{CP}(f)\colon C \to i(P) \\ &\qquad c \in \text{Ob}(C) \mapsto f(c) \\ &\qquad g \in C(c,d) \mapsto (f(c) \to f(d)) \\ \tau_{CP}^{-1}(F)\colon \text{pos}(C) \to P &\leftarrow F \\ c &\mapsto F(c) \end{aligned}$$

□

### 1.4.6. Yoneda extension

Next, we will see a construction which will set more functors between our categories. More information can be found in [Hof].

**Definition 1.4.41.** Let $\mathcal{A}$ be a small category. The **Yoneda embedding** will be the following functor:

$$\begin{aligned} Y\colon \mathcal{A} &\to \mathcal{S}\text{ets}^{\mathcal{A}^{\text{op}}}. \\ A \in \text{Ob}(\mathcal{A}) &\mapsto Y(A) = \mathcal{A}(\cdot, A) \\ h \in \mathcal{A}(A_1, A_2) &\mapsto Y(h)\colon \mathcal{A}(\cdot, A_1) \Rightarrow \mathcal{A}(\cdot, a_2) \\ & \qquad A_3 \in \text{Ob}(\mathcal{A}) \mapsto Y(h)_{A_3} = \mathcal{A}(A_3, \cdot)(h)\colon \mathcal{A}(A_3, A_1) \to \mathcal{A}(A_3, A_2) \\ & \qquad\qquad h_1 \mapsto h \circ h_1 \end{aligned}$$

A functor $F\colon \mathcal{A}^{\text{op}}$ is called **representable** if it is the image under the Yoneda embedding for some object of $\mathcal{A}$.

We will see that the Yoneda embedding is dense, but for that we need to define a new category first.

**Definition 1.4.42.** Let $\mathcal{A}$ be a category and $P\colon \mathcal{A}^{\text{op}} \to \mathcal{S}\text{ets}$ a functor. We define the **category of elements** of $P$ (or **Grothendieck construction**) $\int_{\mathcal{A}} P$ as follows:

- The objects will be

$$\text{Ob}\left(\int_{\mathcal{A}} P\right) = \{(A, x)\colon A \in \text{Ob}(\mathcal{A}), x \in P(C) \in \text{Ob}(\mathcal{S}\text{ets})\}.$$

- Given $(A, x), (B, y) \in \text{Ob}(\int_{\mathcal{A}} P)$, the set of morphisms between them will be

$$\int_{\mathcal{A}} P((A, x), (B, y)) = \{f \in \mathcal{A}(A, B)\colon P(f)(y) = x\}.$$

**Example 1.4.43.** Let $X\colon \Delta^{\text{op}} \to \mathcal{S}\text{ets}$ be a simplicial set. Its Grothendieck construction $\int_{\Delta} X$ is defined as follows:

- $\text{Ob}\left(\int_{\Delta} X\right) = \{([n], x)\colon [n] \in \text{Ob}(\Delta), x \in X([n]) = X_n\} = \{([n], x)\colon x \text{ is a } n-\text{simplex}\}$.
- $\int_{\Delta} X(([n], x), ([m], y)) = \{f \in \Delta([n], [m])\colon X(f)(y) = x\}$.

Since the morphisms in $\Delta$ are generated by the $S_i$'s and the $D_i$'s, the same will happen to the morphisms in $\int_{\Delta} X$.

**Remark 1.4.44.** There is an obvious functor relating $\int_{\mathcal{A}} P$ and $\mathcal{A}$:

$$\begin{aligned} \pi_{\mathcal{A}}\colon \int_{\mathcal{A}} P &\to \mathcal{A}. \\ (A, x) \in \text{Ob}\left(\int_{\mathcal{A}} P\right) &\mapsto A \\ f \in \int_{\mathcal{A}} ((A, x), (B, y)) &\mapsto f \end{aligned}$$

**Proposition 1.4.45** (Density of the Yoneda embedding)**.** *Let $\mathcal{A}$ be a small category, $P\colon \mathcal{A}^{\text{op}} \to \mathcal{S}\text{ets}$ a functor. Then, $P = \text{colim}(Y \circ \pi_{\mathcal{A}})$. That is, every functor $\mathcal{A}^{\text{op}} \to \mathcal{S}\text{ets}$ is a colimit of representable functors.*

**Proposition 1.4.46** (Fully faithfullness of the Yoneda embedding)**.** *Let $\mathcal{A}$ be a small category. Then, the Yoneda embedding $Y\colon \mathcal{A} \to \mathcal{S}\mathrm{ets}^{\mathcal{A}^{\mathrm{op}}}$ is fully faithfull.*

Now let us see a useful setting for us to construct adjoint functors.

**Proposition 1.4.47.** *Let $\mathcal{A}$ be a small category, $\mathcal{C}$ a category where colimits exist, and $F\colon \mathcal{A} \to \mathcal{C}$ a functor. Let us define:*

$$\begin{aligned}
&\mathrm{N}_F\colon \mathcal{C} \to \mathcal{S}\mathrm{ets}^{\mathcal{A}^{\mathrm{op}}},\\
&C \in \mathrm{Ob}(\mathcal{C}) \mapsto \mathrm{N}_F(C) = \mathcal{C}(\cdot, C) \circ F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}\mathrm{ets}\\
&\qquad A \in \mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A}) \mapsto \mathrm{N}_F(C)(A) = \mathcal{C}(F(A), C)\\
&\qquad g \in \mathcal{A}^{\mathrm{op}}(A_1, A_2) = \mathcal{A}(A_2, A_1) \mapsto \mathrm{N}_F(f)\colon \mathcal{C}(F(A_1), C) \to \mathcal{C}(F(A_2), C)\\
&\qquad\qquad h \mapsto h \circ F(g)\\
&f \in \mathcal{C}(C_1, C_2) \mapsto \mathrm{N}_F(f)\colon \mathrm{N}_F(C_1) \Rightarrow \mathrm{N}_F(C_2)\\
&\qquad X \in \mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A}) \mapsto \mathrm{N}_F(f)_X = \mathcal{C}(F(X), \cdot)(f)\colon \mathcal{C}(F(X), C_1) \to \mathcal{C}(F(X), C_2)\\
&\qquad\qquad k \mapsto f \circ k
\end{aligned}$$

$$\begin{aligned}
\tau_F\colon \mathcal{S}\mathrm{ets}^{\mathcal{A}^{\mathrm{op}}} &\to \mathcal{C}.\\
\mathcal{A}(\cdot, A) &\mapsto F(A)\\
t\colon \mathcal{A}(\cdot, A) \Rightarrow \mathcal{A}(\cdot, B) &\mapsto F(t_A(\mathrm{Id}_A))
\end{aligned}$$

*(notice that defining $\tau_F$ for representable functors is enough because by Proposition 1.4.45 for $P = \mathrm{colim}(Y \circ \pi_A)$ we can put $\tau_F(P) = \mathrm{colim}(F \circ \pi_A)$ since colimits exist in $\mathcal{C}$). Then $\tau_F\colon \mathcal{S}\mathrm{ets}^{\mathcal{A}^{\mathrm{op}}} \rightleftarrows \mathcal{C}\colon \mathrm{N}_F$ form an adjoint pair of functors.*

#### Geometric realization and singular complex functors

**Definition 1.4.48.** Let us define the **standard geometrical $n$-simplex** as

$$|\Delta^n| = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}\colon x_0 + \ldots + x_n = 1, x_i \geq 0, 0 \leq i \leq n\right\},$$

with the euclidean subspace topology.

Let us also highlight the following family of continuous maps between them:

$$\begin{aligned}
&S_i\colon |\Delta^{n+1}| \to |\Delta^n|,\\
&(x_0, \dots, x_{n+1}) \mapsto (x_0 + \frac{x_i}{n+1}, \dots, x_{i-1} + \frac{x_i}{n+1}, x_{i+1} + \frac{x_i}{n+1}, \dots, x_{n+1} + \frac{x_i}{n+1})\\
&D_i\colon |\Delta^{n-1}| \to |\Delta^n|.\\
&(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})
\end{aligned}$$

On the above construction, let $\mathcal{A} = \Delta$, $\mathcal{C} = \mathcal{T}\mathrm{op}$ and

$$\begin{aligned}
F\colon \Delta &\to \mathcal{T}\mathrm{op}\\
[n] \in \mathrm{Ob}(\Delta) &\mapsto |\Delta^n|\\
D_i &\mapsto D_i\\
S_i &\mapsto S_i
\end{aligned}$$

(remember that every order-preserving map is composition of $D_i$'s and $S_i$'s, so $F$ is indeed a functor).

Then we get the adjunction

$$|\cdot| = \tau_F \colon \mathcal{S}\text{ets}^{\Delta^{\text{op}}} = \text{s}\mathcal{S}\text{ets} \rightleftarrows \mathcal{T}\text{op} \colon \text{N}_F = \text{Sing}$$

We will call $|\cdot|$ the **geometric realization** functor, and Sing the **singular complex** functor.

Notice that the geometric realization functor sends the representable functors, that is, the $n$-simplicial sets $\Delta[n]$, to the standard geometrical $n$-simplices $|\Delta^n|$. For an arbitrary simplicial set $X$, we are considering a geometric $n$-simplex $|\Delta^n|$ for each $n$-simplex in $X_n$, and then we identify common faces and suppress degenerate simplices.

#### Fundamental category and nerve functors

On the above construction, let $\mathcal{A} = \Delta$, $\mathcal{C} = \mathcal{C}\text{at}$ and $F = i \colon \Delta \to \mathcal{C}\text{at}$ (see Example 1.4.37). Then, we get the adjunction

$$\tau = \tau_F \colon \mathcal{S}\text{ets}^{\Delta^{\text{op}}} = \text{s}\mathcal{S}\text{ets} \rightleftarrows \mathcal{C}\text{at} \colon \text{N}_F = \text{N}.$$

We will call $\tau$ the **fundamental category** functor, and N the **nerve** functor.

Notice that the fundamental category of a $n$-simplicial set (representable functor $\Delta[n]$) is the small category associated to $[n]$. For an arbitrary simplicial complex $X$, we are considering a sequence of $n$-composable arrows for each $n$-simplex in $X_n$, and then we identify common arrows.

On the other hand, for $C \in \text{Ob}(\mathcal{C}\text{at})$ a small category, its nerve N$C$ is the simplicial set that has as $n$-simplices the sequences of $n$-composable morphisms in $C$ and whose face and degeneracy maps are given by the composition in $C$ and the insertion of identities respectively.

For example, for a poset $P$, we have that the $m$-simplices of the nerve of its associated small category $i(P)$ are $\text{N}(i(P))_m = \mathcal{C}\text{at}(i([m]), i(P))$, that is, the sequences of $m$ composable arrows of $i(P)$, which come from $m$-chains in $P$. Then, $\text{N}(i(P))$ is the simplicial set corresponding to $\mathcal{K}(P)$.

#### Barycentric subdivision and extension functors

On the above construction, let $\mathcal{A} = \Delta$, $\mathcal{C} = \text{s}\mathcal{S}\text{ets}$ and $F = \text{N} \circ i \circ \mathcal{X} \circ \mathcal{K} \colon \Delta \to \text{s}\mathcal{S}\text{ets}$. Then, we get the adjunction:

$$\text{Sd} = \tau_F \colon \text{s}\mathcal{S}\text{ets} \rightleftarrows \text{s}\mathcal{S}\text{ets} \colon \text{N}_F = \text{Ex}.$$

We will call Sd the **barycentric subdivision** functor and Ex the **extension** functor.

Notice that the barycentric subdivision of a $n$-simplicial set $\Delta[n]$ $\text{Sd}(\Delta[n]) = \text{N}(i(\mathcal{X}(\mathcal{K}([n]))))$ is the simplicial set associated to the simplicial complex $\mathcal{K}(\mathcal{X}(\mathcal{K}([n])))$, that is, to the (old) barycentric subdivision of $2^{[n]} \setminus \{\emptyset\}$. Now, for $X$ an arbitrary simplicial set, we are considering the (old) barycentric subdivision of an $n$-simplicial set for each $n$-simplex in $X_n$ and then we identify common faces. In particular, for simplicial sets derived from ordered simplicial complexes (see Example 1.4.29), their barycentric subdivision will be the simplicial set associated to their (old) barycentric subdivision.

On the other hand, given $X \in \text{Ob}(\text{s}\mathcal{S}\text{ets})$ the $n$-simplices of $\text{Ex}(X)$ will be the morphisms from $\text{Sd}\Delta[n]$ to $X$.

# Chapter 2

# Model categories and homotopy theory

Algebraic topologists use category theory for generalizing topological notions. The aim of this chapter is to show how to generalize the concept of homotopy to general categories. This idea was first developed by Quillen in [Qui67], but then widely used and interpreted by a lot of mathematicians. In our case, we will use [DS95] as main reference because it summarizes very well all we need.

## 2.1. Model categories

For our goal, we first need to set some structure in our categories in order to manipulate homotopy in a suitable way.

**Definition 2.1.1.** A **model category** is a category $\mathcal{C}$ with three distinguished classes of morphisms:

- **weak equivalences** ($\xrightarrow{\sim}$),
- **fibrations** ($\twoheadrightarrow$), and
- **cofibrations** ($\hookrightarrow$),

each of which is closed under composition and contains all identity morphisms. A morphism which is both a fibration and a weak equivalence is called an **acyclic** (or **trivial**) **fibration**; analogously, a morphism which is both a cofibration and a weak equivalence is called an **acyclic** (or **trivial**) **cofibration**.

We require the following axioms:

**MC1** Finite limits and colimits exist in $\mathcal{C}$.

**MC2** Two-out-of-three property: Let $f$ and $g$ be morphisms in $\mathcal{C}$ such that $g \circ f$ is defined. If two of the three arrows $f$, $g$, $g \circ f$ are weak equivalences, then so is the third.

**MC3** If $f$ is a retract of $g$ and $g$ is a fibration, a cofibration or a weak equivalence, then so is $f$.

**MC4** Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y, \end{array}$$

where either $i$ is a cofibration and $p$ is an acyclic fibration, or $i$ is an acyclic cofibration and $p$ is a fibration, then there exists a **lift**, that is, a morphism $h\colon B \to X$ such that the resulting diagram commutes: $h \circ i = f$, $p \circ h = g$.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \nearrow^{h} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y. \end{array}$$

**MC5** Any morphism $f$ can be factored in two ways:

(I) $f = p \circ i$, where $i$ is a cofibration and $p$ is an acyclic fibration.

(II) $f = p \circ i$, where $i$ is a acyclic cofibration and $p$ is a fibration.

**Remark 2.1.2.** By **MC1** and Proposition 1.2.19, a model category $\mathcal{C}$ has both an initial object $\varnothing$ and a terminal object $*$. An object $A \in \mathrm{Ob}(\mathcal{C})$ is said to be **cofibrant** if the only morphism $\varnothing \to A$ is a cofibration and **fibrant** if the only morphism $A \to *$ is a fibration.

**Example 2.1.3.** Let $\mathcal{C}$ be a category such that finite limits and colimits exist in it (so that **MC1** holds). It is easy to check that we can always establish the **trivial model structure** as follows:

- The weak equivalences are the isomorphisms.
- The fibrations are all the morphisms.
- The cofibrations are all the morphisms.

With this model structure, every element is both fibrant and cofibrant.

**Example 2.1.4.** Let $\mathcal{C}$ be a model category. Then, the opposite category $\mathcal{C}^{\mathrm{op}}$ admits a structure of a model category given by defining a morphism $f \in \mathcal{C}^{\mathrm{op}}(Y, X) = \mathcal{C}(X, Y)$ to be

1. a weak equivalence if it is a weak equivalence in $\mathcal{C}$,
2. a cofibration if it is a fibration in $\mathcal{C}$, and
3. a fibration if it is a cofibration in $\mathcal{C}$.

**Remark 2.1.5.** This example reflects the fact that the axioms for a model category are self-dual. Then if a statement about model categories is true for all model categories, the dual statement (obtained by reversing the arrows and switching "fibration" and "cofibration") is also true.

## 2.2. Lifting properties

In **MC4**, we defined the notion of lift, which will be crucial in our definition of homotopy category. Hence, let us develop a bit this idea and prove some results which will be useful later.

**Definition 2.2.1.** A morphism $i\colon A \to B$ is said to have the **left lifting property (LLP)** with respect to another morphism $p\colon X \to Y$ or $p$ is said to have the **right lifting property (RLP)** with respect to $i$ if there exists a lift for every diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y. \end{array}$$

**Proposition 2.2.2.** *Let $\mathcal{C}$ be a model category.*

1. *The cofibrations in $\mathcal{C}$ are the morphisms which have the LLP with respect to acyclic fibrations.*
2. *The acyclic cofibrations in $\mathcal{C}$ are the morphisms which have the LLP with respect to fibrations.*
3. *The fibrations in $\mathcal{C}$ are the morphisms which have the RLP with respect to acyclic cofibrations.*
4. *The acyclic fibrations in $\mathcal{C}$ are the morphisms which have the RLP with respect to cofibrations.*

*Proof.* Let us prove the first assertion. By **MC4**, we know that cofibrations in $\mathcal{C}$ have the LLP property with respect to acyclic fibrations. Now assume that $f\colon A \to B$ has the LLP property with respect to acyclic fibrations and let us see that it is a cofibration. By **MC5**(i), we can factor $f$ as $A \hookrightarrow B' \overset{\sim}{\twoheadrightarrow} B$, where $i\colon A \hookrightarrow B'$ is a cofibration and $p\colon B' \overset{\sim}{\twoheadrightarrow} B$ is an acyclic fibration. Then, by assumption, the following commutative diagram has a lift:

$$\begin{array}{ccc} A & \xrightarrow{i} & B' \\ \downarrow{\scriptstyle f} & & \sim\downarrow{\scriptstyle p} \\ B & \xrightarrow{\mathrm{Id}_B} & B. \end{array}$$

That is, there is $g\colon B \to B'$ such that $p \circ g = \mathrm{Id}_B$ and $g \circ f = i$. This shows that $f$ is a retract of $i$ because we have the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{Id}_A} & A & \xrightarrow{\mathrm{Id}_A} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & B' & \xrightarrow{p} & B \end{array}$$

with $\mathrm{Id}_A \circ \mathrm{Id}_A = \mathrm{Id}_A$ and $p \circ g = \mathrm{Id}_B$. In consequence, by **MC3**, we have that $f$ is a cofibration.

The proof of the second assertion is analogous. Furthermore, the third assertion is the dual (Remark 2.1.5) property of the first one, and the forth assertion is the dual property of the second one. □

**Remark 2.2.3.** As a corollary of the previous proposition, when trying to set up a model category structure on some given category, if we have chosen the fibrations and the weak equivalences, then the cofibrations are already chosen too: they will be the morphisms satisfying the LLP with respect to the acyclic fibrations. Dually (Remark 2.1.5), if we have chosen the cofibrations and the weak equivalences, then the fibrations are already chosen too: they will be the morphisms satisfying the RLP with respect to the acyclic cofibrations.

In a similar way, fibrations and cofibrations determine weak equivalences in a model category. Indeed, trivial cofibrations are the morphisms having the LLP with respect to fibrations, trivial fibrations are the morphisms having the RLP with respect to cofibrations, and then the weak equivalences will be the morphisms that can be factored by **MC5** as the composition of a trivial cofibration and a trivial fibration.

**Proposition 2.2.4.** *Let $\mathcal{C}$ be a category.*

1. *If the morphism $g\colon W \to X$ is a base change of the morphism $f\colon Y \to Z$ and if $f$ has the RLP with respect to a morphism $i\colon C \to D$, then $g$ has the RLP with respect to $i$.*
2. *If the morphism $j\colon C \to D$ is a cobase change of the morphism $i\colon A \to B$ and $i$ has the LLP with respect to a morphism $f\colon X \to Y$, then $j$ also has the LLP with respect to $f$.*

*Proof.* Let us prove the first assertion. For any lifting problem for $g$, we have the following commutative diagram (where the right square is a pullback diagram):

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "i"] & W \arrow[r, "t"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ D \arrow[r, "u"] & X \arrow[r, "v"] & Z. \end{tikzcd}$$

We need to prove that there exists a lift for the left square, that is, a morphism $j\colon D \to W$ such that $g \circ j = u$ and $j \circ i = s$. Since $f$ has the RLP with respect to $i$, we have that the diagram

$$\begin{tikzcd} C \arrow[r, "t \circ s"] \arrow[d, "i"] & Y \arrow[d, "f"] \\ D \arrow[r, "v \circ u"] & Z \end{tikzcd}$$

has a lift, that is, there exists $w\colon D \to Y$ such that $f \circ w = v \circ u$ and $w \circ i = t \circ s$. In particular, we have the diagram

$$\begin{tikzcd} D \arrow[r, "w"] \arrow[d, "u"] & Y \arrow[d, "f"] \\ X \arrow[r, "v"] & Z, \end{tikzcd}$$

which justifies by the universal property of the pullback that there is a morphism $j\colon D \to W$ such that $t \circ j = w$ and $g \circ j = u$. To prove that $j$ is our desired lift, it is enough to check that $j \circ i = s$, but this holds because of the universal property of the pullback, since both $s, j \circ i\colon C \to W$ satisfy that $t \circ s = w \circ i = t \circ j \circ i$, $g \circ s = u \circ i = g \circ j \circ i$:

$$\begin{tikzcd} C \arrow[drr, bend left, "t \circ s = w \circ i"] \arrow[dr, shift left, "s"] \arrow[dr, shift right, "j \circ i"'] \arrow[ddr, bend right, "u \circ i = g \circ s"'] & & \\ & W \arrow[r, "t"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ & X \arrow[r, "v"] & Z. \end{tikzcd}$$

The second assertion is is the dual (Remark 2.1.5) property of the first one. □

**Corollary 2.2.5.** *Let $\mathcal{C}$ be a model category.*

1. *The class of cofibrations in $\mathcal{C}$ is stable under cobase change.*
2. *The class of acyclic cofibrations in $\mathcal{C}$ is stable under cobase change.*
3. *The class of fibrations in $\mathcal{C}$ is stable under base change.*
4. *The class of acylic fibrations in $\mathcal{C}$ is stable under base change.*

*Proof.* Let us prove the first assertion. Let $i\colon A \hookrightarrow B$ be a cofibration. Then by Proposition 2.2.2, it has the LLP with respect to acyclic fibrations, so by Proposition 2.2.4 any cobase change will have the LLP with respect to acyclic fibrations, which means by Proposition 2.2.2 that it will be a cofibration.

The proof of the second assertion is analogous. Furthermore, the third assertion is the dual (Remark 2.1.5) property of the first one, and the forth assertion is the dual property of the second one. □

**Proposition 2.2.6** (The retract argument)**.** *Let $\mathcal{C}$ be a category.*

1. *If the morphism $g\colon X \to Y$ can be factored as $g = p \circ i$, where $g$ has the LLP with respect to $p\colon Z \to Y$, then $g$ is a retract of $i\colon X \to Z$.*
2. *If the morphism $g\colon X \to Y$ can be factored as $g = p \circ i$, where $g$ has the RLP with respect to $i\colon X \to Z$, then $g$ is a retract of $p\colon Z \to Y$.*

*Proof.* Let us prove the first assertion. We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\mathrm{Id}_Y} & Y. \end{array}$$

Since $g$ has the LLP with respect to $p$ by hypothesis, there exists $q\colon Y \to Z$ such that $q \circ g = i$ and $p \circ q = \mathrm{Id}_Y$. In consequence, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{Id}_X} & X & \xrightarrow{\mathrm{Id}_X} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{q} & Z & \xrightarrow{p} & Y, \end{array}$$

with $\mathrm{Id}_X \circ \mathrm{Id}_X = \mathrm{Id}_X$ and $p \circ q = \mathrm{Id}_Y$, which proves that $g$ is a retract of $i$.

The proof of the second assertion is analogous. □

**Lemma 2.2.7.** *Let $\mathcal{C}$ be a category, $f \in \mathcal{C}(X,Y)$. If $S$ is a set such that for every $s \in S$ we have a morphism $g_s \in \mathcal{C}(A_s, B_s)$ that has the LLP with respect to $f$, then the morphism induced by the universal property of the coproduct $\amalg_{s\in S} g_s\colon \amalg_{s\in S} A_s \to \amalg_{s\in S} B_s$ also has the LLP with respect to $f$.*

*Proof.* Let us consider the following lifting problem:

$$\begin{array}{ccc} \amalg_{s\in S} A(s) & \xrightarrow{a} & X \\ \downarrow{\scriptstyle \amalg_{s\in S} g_s} & & \downarrow{\scriptstyle f} \\ \amalg_{s\in S} B(s) & \xrightarrow{b} & Y \end{array}.$$

Then, for each $s \in S$, we have the following commutative diagram

$$\begin{array}{ccc} A_s & \xrightarrow{a \circ in_s} & X \\ \downarrow{\scriptstyle g_s} & & \downarrow{\scriptstyle f} \\ B_s & \xrightarrow{b \circ in_s} & Y \end{array},$$

and since $g_s$ has the LLP with respect to $f$ there exists $h_s\colon B_s \to X$ such that $f \circ h_s = b \circ in_s$ and $h_s \circ f_s = a \circ in_s$. Now it is easy to see that the morphism induced by the universal property of the coproduct $h\colon \amalg_{s\in S} B_s \to X$ is the desired lift since for every $s \in S$

$$f \circ h \circ in_s = f \circ h_s = b \circ in_s,$$

and hence by the universal property of the coproduct $f \circ h = b$. □

**Lemma 2.2.8.** *Let $\mathcal{C}$ be a category, $f \in \mathcal{C}(X,Y)$. If $A_0 \xrightarrow{f_{0,1}} A_1 \xrightarrow{f_{1,2}} A_2 \xrightarrow{f_{2,3}} \dots$ is a diagram such that every $f_{n,n+1}\colon A_n \to A_{n+1}$ has the LLP with respect to $f$, then $in_0\colon A_0 \to \mathrm{colim}_n A_n$ also has the LLP with respect to $f$.*

*Proof.* Consider the lifting problem

$$\begin{array}{ccc} A_0 & \xrightarrow{g_0} & X \\ \big\downarrow{\scriptstyle in_0} & & \big\downarrow{\scriptstyle f} \\ \mathrm{colim}_n A_n & \xrightarrow{h} & Y, \end{array}$$

which yields the following lifting problem:

$$\begin{array}{ccc} A_0 & \xrightarrow{g_0} & X \\ \big\downarrow{\scriptstyle f_{0,1}} & & \big\downarrow{\scriptstyle f} \\ A_1 & \xrightarrow{h \circ in_1} & Y. \end{array}$$

Since $f_{0,1}$ has the LLP with respect to $f$ by hypothesis, there exists $g_1\colon A_1 \to X$ such that $f \circ g_1 = h \circ in_1$ and $g_1 \circ f_{0,1} = g_0$.

Assume now that we have $g_n\colon A_n \to X$ such that $f \circ g_n = h \circ in_n$ and $g_n \circ f_{n-1,n} = g_{n-1}$. Then we have the commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow{g_n} & X \\ \big\downarrow{\scriptstyle f_{n,n+1}} & & \big\downarrow{\scriptstyle f} \\ A_{n+1} & \xrightarrow{h \circ in_{n+1}} & Y. \end{array}$$

Since $f_{n,n+1}$ has the LLP with respect to $f$ by hypothesis, there exists $g_{n+1}\colon A_{n+1} \to X$ such that $f \circ g_{n+1} = h \circ in_{n+1}$ and $g_{n+1} \circ f_{n,n+1} = g_n$.

Summing up, we have inductively constructed a sequence of morphisms $g_n\colon A_n \to X$ that verify $g_{n+1} \circ f_{n,n+1} = g_n$. In consequence, by the universal property of the sequencial colimit, there exists a morphism $g\colon \mathrm{colim}_n A_n \to X$ such that $g \circ in_n = g_n$ for every $n \geq 0$. In particular $g \circ in_0 = g_0$ and $f \circ g = h$ since for every $n \geq 0$ $f \circ g \circ in_n = f \circ g_n = h \circ in_n$ and by the universal property of the sequential colimit. □

## 2.3. Homotopy relation

As mentioned, the notions of cylinder and space of paths are crucial for the homotopy of topological spaces. Now we are in conditions of defining categorically these concepts. Since they will be dual ideas (Remark 2.1.5), it is enough to prove our results only for one of them. In order to differ from [DS95], we will define right homotopy first.

### 2.3.1. Right homotopy

**Definition 2.3.1.** Let $\mathcal{C}$ be a model category. A **path object** for $X \in \mathrm{Ob}(\mathcal{C})$ is an object $X^I$ of $\mathcal{C}$ together with a factorization of the diagonal morphism $(\mathrm{Id}_X, \mathrm{Id}_X)$:

$$X \xrightarrow{(\mathrm{Id}_X,\mathrm{Id}_X)} X \times X, \qquad X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X.$$

A path object $X^I$ is called a **good path object** if $p\colon X^I \to X \times X$ is a fibration, and a **very good path object** if in addition the morphism $X \xrightarrow{\sim} X^I$ is a (necessarily acyclic) cofibration.

If $X^I$ is a path object for $X$, we will denote the two structure morphisms $X^I \to X$ by $p_0 = pr_0 \circ p$ and $p_1 = pr_1 \circ p$.

**Remark 2.3.2.** By **MC5**, for every object $X \in \mathrm{Ob}(\mathcal{C})$, there exists at least one very good path object.

**Remark 2.3.3.** If $X^I$ is a path object for $X$, then $p_0, p_1\colon X^I \to X$ are weak equivalences: the identity morphism $\mathrm{Id}_X\colon X \to X$ factors as $X \xrightarrow{\sim} X^I \xrightarrow{p_k} X$, $k = 0, 1$ and both $\mathrm{Id}_X$ and $X \xrightarrow{\sim} X^I$ are weak equivalences, so by **MC2**, $p_k$ also has to be for $k = 0, 1$.

**Lemma 2.3.4.** *Let $\mathcal{C}$ be a model category. If $X, Y \in \mathrm{Ob}(\mathcal{C})$ are fibrant objets, we have that $pr_X\colon X \times Y \to X$ and $pr_Y\colon X \times Y \to Y$ are fibrations.*

*Proof.* By Remark 1.2.16, we have the following pullback diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{pr_X} & X \\ \Big\downarrow{\scriptstyle pr_Y} & & \Big\downarrow \\ Y & \twoheadrightarrow & *. \end{array}$$

Since the lower horizontal arrow is a fibration because $Y$ is a fibrant object and fibrations are stable under base change by Corollary 2.2.5, $pr_X$ is a fibration. Symmetrically, $pr_Y$ is also a fibration. □

**Lemma 2.3.5.** *Let $\mathcal{C}$ be a model category, $X \in \mathrm{Ob}(\mathcal{C})$. If $X$ is fibrant and $X^I$ is a good path object for $X$, then the morphisms $p_0, p_1\colon X^I \to X$ are acyclic fibrations.*

*Proof.* By Lemma 2.3.4, $pr_0$ and $pr_1$ are fibrations. In consequence $p_k\colon X^I \overset{p}{\twoheadrightarrow} X \times X \overset{pr_k}{\twoheadrightarrow} X$ is a fibration because it is a composition of fibrations (the first morphism is a fibration because $X^I$ is a good path object for $X$). Moreover, since by Remark 2.3.3 $p_k$, $k = 0, 1$ are weak equivalences, we get that they are acyclic fibrations. □

**Definition 2.3.6.** Let $\mathcal{C}$ be a model category, $A, X \in \mathrm{Ob}(\mathcal{C})$. Two morphisms $f, g\colon A \to X$ in $\mathcal{C}$ are said to be **right homotopic** (written $f \overset{r}{\sim} g$) if there exists a path object $X^I$ for $X$ such that the product morphism $(f, g)\colon A \to X \times X$ lifts to a morphism $H\colon A \to X^I$ (**right homotopy**). The right homotopy is said to be **good** if $X^I$ is a good path object for$X$ and **very good** if $X^I$ is a very good path object for $X$:

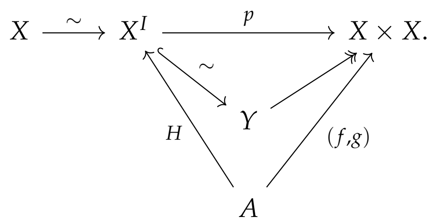
$$\begin{array}{ccccc} X & & \xrightarrow{(\mathrm{Id}_X, \mathrm{Id}_X)} & & X \times X \\ & \searrow^{\sim} & & \nearrow^{p} & \Big\uparrow{\scriptstyle (f,g)} \\ & & X^I & & \\ & & & \nwarrow^{H} & \\ & & & & A. \end{array}$$

**Remark 2.3.7.** If $f \overset{r}{\sim} g$ via the right homotopy $H$, $f$ is a weak equivalence if and only if $g$ is a weak equivalence. This follows from **MC2** and the fact that $p_0$ and $p_1$ are weak equivalences by Remark 2.3.3: if $f = p_0 \circ H$ is a weak equivalence, since $p_0$ is also a weak equivalence, then $H$ has to be a weak equivalence, and hence $g = p_1 \circ H$ will be a weak equivalence because $p_1$ also is, and viceversa.

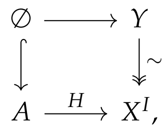
**Example 2.3.8.** Let $\mathcal{C}$ be a category such that finite limits and colimits exist in it. By Example 2.1.3, we can define a trivial model structure on $\mathcal{C}$ where every element is fibrant and cofibrant. With this model structure, two morphisms are right homotopic if and only if they are equal.

**Lemma 2.3.9.** *If $f \overset{r}{\sim} g \colon A \to X$, then there exists a good right homotopy from $f$ to $g$. If in addition $A$ is cofibrant, then there exists a very good right homotopy from $f$ to $g$.*

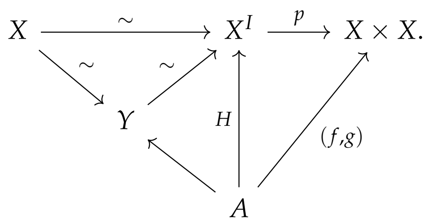
*Proof.* Let us assume that $f \overset{r}{\sim} g$. Applying **MC5**(ii) to $p\colon X^I \to X \times X$, we get that it can be factored as $X^I \overset{\sim}{\hookrightarrow} Y \twoheadrightarrow X \times X$ for some object $Y$, so $Y$ will be a good path object for $X$ and $A \overset{H}{\to} X^I \overset{\sim}{\to} Y$ will be the desired good right homotopy:




Assume now that $f \overset{r}{\sim} g$ and $A$ is cofibrant. We have just proved that we can take a good right homotopy $H\colon A \to X^I$ from $f$ to $g$. By **MC5**(ii) and **MC2**, we may factor $X \overset{\sim}{\to} X^I$ as $X \overset{\sim}{\hookrightarrow} Y \overset{\sim}{\twoheadrightarrow} X^I$ for some object $Y$. Then we have that $Y$ is a very good path object for $X$ and moreover we have the following diagram




where the left vertical arrow is a cofibration because $A$ is cofibrant and the right vertical arrow is an acyclic fibration. By **MC4** we get the desired very good left homotopy $A \to Y$:




□

**Lemma 2.3.10.** *Let $\mathcal{C}$ be a model category. If $X$ is a fibrant object, then $\overset{r}{\sim}$ is an equivalence relation on $\mathcal{C}(A, X)$ for every $A \in \mathrm{Ob}(\mathcal{C})$.*

*Proof.* Reflexivity is clear: $f \overset{r}{\sim} f\colon A \to X$ taking $X$ as path object for $X$ (factoring $(\mathrm{Id}_X, \mathrm{Id}_X)$ as $X \overset{\mathrm{Id}_X}{\to} X \overset{(\mathrm{Id}_X,\mathrm{Id}_X)}{\to} X \times X$) and $H = f\colon A \to X$ as right homotopy:

$$(\mathrm{Id}_X, \mathrm{Id}_X) \circ H = (\mathrm{Id}_X, \mathrm{Id}_X) \circ f = (f, f).$$

Next, let us prove symmetry. Assume $f \overset{r}{\sim} g$ for $f,\ g \in \mathcal{C}(A, X)$, then there exists a path object $X^I$ for $X$ and a homotopy $H\colon A \to X^I$ such that $p \circ H = (f, g)$. Notice that if $X^I$ is a path object for $X$ with the diagram $X \overset{\sim}{\to} X^I \overset{p}{\to} X \times X$ factoring $(\mathrm{Id}_X, \mathrm{Id}_X)$, then $X^I$ is also a path object for $X$ with the diagram

$X \overset{\sim}{\rightarrow} X^I \overset{(pr_1, pr_0)\circ p}{\longrightarrow} X \times X$ factoring $(\mathrm{Id}_X, \mathrm{Id}_X)$, and moreover $(g, f) = (pr_1, pr_0) \circ (f, g)$. In consequence, the morphism $H$ verifies $((pr_1, pr_0) \circ p) \circ H = (pr_1, pr_0) \circ (p \circ H) = (pr_1, pr_0)(f, g) = (g, f)$ and then $g \overset{r}{\sim} f$.

Finally, let us prove transitivity. Assume that $f \overset{r}{\sim} g$ and $g \overset{r}{\sim} h$, and let us see that $f \overset{r}{\sim} h$. By Lemma 2.3.9, we can consider good right homotopies $H \colon A \to X^I$ from $f$ to $g$ and $H' \colon A \to (X^I)'$ from $g$ to $h$. We have the following commutative diagrams:

$$
\begin{array}{ccccccc}
 & & & & & & X \\
 & & & & & \nearrow^{pr_0} & \\
X & \xrightarrow[\sim]{u} & X^I & \overset{p}{\twoheadrightarrow} & X \times X & & \\
 & & {\scriptstyle H}\uparrow & \nearrow_{(f,g)} & & \searrow^{pr_1} & \\
 & & A & & & & X,
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & & & & & X \\
 & & & & & \nearrow^{pr_0} & \\
X & \xrightarrow[\sim]{u'} & (X^I)' & \overset{p'}{\twoheadrightarrow} & X \times X & & \\
 & & {\scriptstyle H'}\uparrow & \nearrow_{(g,h)} & & \searrow^{pr_1} & \\
 & & A & & & & X.
\end{array}
$$

Now let $(X^I)''$ be the pullback of the diagram $(X^I)' \overset{p_1'}{\rightarrow} X \overset{p_0}{\leftarrow} X^I$:

$$
\begin{array}{ccc}
(X^I)'' & \overset{v}{\longrightarrow} & X^I \\
\downarrow {\scriptstyle v'} & & \sim \downarrow\!\!\downarrow {\scriptstyle p_0} \\
(X^I)' & \underset{\sim}{\overset{p_1'}{\twoheadrightarrow}} & X
\end{array} .
$$

Since the morphisms $p_0$ and $p_1'$ are acyclic fibrations by Lemma 2.3.5 and acyclic fibrations are stable under base change by Corollary 2.2.5, $v$ and $v'$ are also acyclic fibrations. Moreover, given $u \colon X \overset{\sim}{\rightarrow} X^I$ and $u' \colon X \overset{\sim}{\rightarrow} (X^I)'$, there is $u'' \colon X \overset{\sim}{\rightarrow} (X^I)''$ (which will be a weak equivalence because the $u, u'$ are and so do $v$ and $v'$, so we can apply **MC2**) such that $v \circ u'' = u$ and $v' \circ u'' = u'$. It is clear then that $X \overset{u''}{\rightarrow} (X^I)'' \overset{(p_0 \circ v, p_1' \circ v')}{\longrightarrow} X \times X$ factors $(\mathrm{Id}_X, \mathrm{Id}_X)$:

$$pr_0 \circ (p_0 \circ v, p_1' \circ v') \circ u'' = p_0 \circ v \circ u'' = p_0 \circ u = \mathrm{Id}_X,$$
$$pr_1 \circ (p_0 \circ v, p_1' \circ v') \circ u'' = p_1' \circ v' \circ u'' = p_1' \circ u' = \mathrm{Id}_X.$$

In consequence, $(X^I)''$ is a path object for $X$. Furthermore, given the morphisms $H \colon A \to X^I$ and $H' \colon A \to (X^I)'$, there is $H'' \colon A \to (X^I)''$ such that $v \circ H'' = H$ and $v' \circ H'' = H'$. This is the desired right homotopy since

$$(p_0 \circ v, p_1' \circ v') \circ H'' = (p_0 \circ v \circ H'', p_1' \circ v' \circ H'') = (p_0 \circ H, p_1' \circ H') = (f, h).$$

As a result, $f \overset{r}{\sim} h$. □

**Definition 2.3.11.** Let $\mathcal{C}$ be a model category and $A, X \in \mathrm{Ob}(\mathcal{C})$ objects. We will denote the set of equivalence classes of $\mathcal{C}(A, X)$ under the equivalence relation generated by right homotopy by $\pi^r(A, X)$.

**Remark 2.3.12.** The word "generated" in the above definition of $\pi^r(A, X)$ is important due to the fact that we will sometimes consider this set even if $X$ is not fibrant and the right homotopy is not necessarily an equivalence relation on $\mathcal{C}(A, X)$.

**Lemma 2.3.13.** *Let $\mathcal{C}$ be a model category. If $f, g\colon X \to X'$ are right homotopic morphisms and $h\colon A \to X$ is another morphism, then, $f \circ h \overset{r}{\sim} g \circ h$.*

*Proof.* Since $f \overset{r}{\sim} g$, there exists a homotopy $H\colon X \to X'^I$. Then, $H \circ h\colon A \to X'^I$ is our desired right homotopy: $p \circ (H \circ h) = (p \circ H) \circ h = (f, g) \circ h = (f \circ h, g \circ h)$. □

**Lemma 2.3.14.** *Let $\mathcal{C}$ be a model category. If $X$ is fibrant and $i\colon A \overset{\sim}{\hookrightarrow} B$ is an acyclic cofibration, then composition with $i$ induces a bijection:*

$$\begin{aligned} i^*\colon \pi^r(B, X) &\to \pi^r(A, X). \\ [f] &\mapsto i^*([f]) = [f \circ i] \end{aligned}$$

*Proof.* First of all, $i^*$ is well defined because if $f \overset{r}{\sim} g$, then $f \circ i \overset{r}{\sim} g \circ i$ by Lemma 2.3.13.

Next, let us check that $i^*$ is onto. Let $[f] \in \pi^r(A, X)$ and let us see that there is a class $[g] \in \pi^r(B, X)$ such that $i^*([g]) = [g \circ i] = [f]$. We have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & & \big\downarrow \\ B & \longrightarrow & *, \end{array}$$

where the left vertical arrow is an acyclic cofibration and the right vertical arrow is fibration because $X$ fibrant. Then, by **MC4**, there is a lift $g\colon B \to X$ such that in particular $g \circ i = f$. It is then clear that $i^*([g]) = [g \circ i] = [f]$.

Finally, let us verify that $i^*$ is one-to-one. Let $f, g\colon B \to X$ be two morphisms having the same image $i^*([f]) = [f \circ i] = [g \circ i] = i^*([g])$, that is, such that $f \circ i \overset{r}{\sim} g \circ i$, and let us see that $[f] = [g]$, that is, that $f \overset{r}{\sim} g$. Since $f \circ i \overset{r}{\sim} g \circ i\colon A \to X$, by Lemma 2.3.9 we can choose a good right homotopy $H\colon A \to X^I$ from $f \circ i$ to $g \circ i$. We have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{H} & X^I \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle h} \\ B & \xrightarrow{(f,g)} & X \times X, \end{array}$$

where the left vertical arrow is a an acyclic cofibration and the right vertical arrow is a fibration because $H$ is a good right homotopy. Then, by **MC4**, there is a lift $H'\colon B \to X^I$ such that in particular $h \circ H' = (f, g)$. Hence, $H'$ is the desired right homotopy from $f$ to $g$. □

**Lemma 2.3.15.** *Let $\mathcal{C}$ be a model category. Suppose that $A$ is cofibrant, that $f, g\colon A \to X$ are right homotopic morphisms and that $h\colon X \to X'$ is a morphism. Then $h \circ f \overset{r}{\sim} h \circ g$.*

*Proof.* By Lemma 2.3.9, we can choose a very good right homotopy $H\colon A \to X^I$ between $f$ and $g$. Moreover, we can choose a good path object for $X'$ (see Remark 2.3.2) factoring the diagoral morphism $(\mathrm{Id}_{X'}, \mathrm{Id}_{X'})\colon X' \overset{\sim}{\to} X'^I \overset{q}{\twoheadrightarrow} X' \times X'$. We therefore have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{h} & X' & \xrightarrow{\sim} & X'^I \\ \big\downarrow{\scriptstyle \sim} & & & & \big\downarrow{\scriptstyle q} \\ X^I & \xrightarrow{p} & X \times X & \xrightarrow{h \times h} & X' \times X', \end{array}$$

where the left vertical arrow is an acyclic cofibration and the right vertical arrow is a fibration, so by **MC4** there exists a lift $k\colon X^I \to X'^I$. We then get that $k \circ H$ is the desired homotopy, since

$$\begin{aligned} q \circ (k \circ H) &= (q \circ k) \circ H = ((h \times h) \circ p) \circ H \\ &= (h \times h) \circ (p \circ H) = (h \times h) \circ (f, g) = (h \circ f, h \circ g). \end{aligned}$$ □

**Lemma 2.3.16.** *Let $\mathcal{C}$ be a model category. If $A$ is cofibrant, then the composition in $\mathcal{C}$ induces a map:*

$$\begin{aligned} \pi^r(A, X) \times \pi^r(X, X') &\to \pi^r(A, X'). \\ ([h], [f]) &\mapsto [f \circ h] \end{aligned}$$

*Proof.* We only need to see that this assignation does not depend on the chosen representative. Notice that since we are not supposing $X$ nor $X'$ fibrant, we don't always have two representatives of the same class related by a right homotopy, but by a sequence of them (see Remark 2.3.12).

If $h, k\colon A \to X$ represent the same element on $\pi^r(A, X)$, we have a sequence

$$h \overset{r}{\sim} h_1 \overset{r}{\sim} \dots \overset{r}{\sim} h_n \overset{r}{\sim} k.$$

For $g\colon X \to X'$, by Lemma 2.3.15,

$$g \circ h \overset{r}{\sim} g \circ h_1 \overset{r}{\sim} \dots \overset{r}{\sim} g \circ h_n \overset{r}{\sim} g \circ k,$$

so both $g \circ h$ and $g \circ k$ represent the same element on $\pi^r(A, X')$.

Similarly,if $f, g\colon X \to X'$ represent the same element on $\pi^r(X, X')$, we have a sequence

$$f \overset{r}{\sim} f_1 \overset{r}{\sim} \dots \overset{r}{\sim} f_n \overset{r}{\sim} g.$$

For $h\colon A \to X$, by Lemma 2.3.13,

$$f \circ h \overset{r}{\sim} f_1 \circ h \overset{r}{\sim} \dots \overset{r}{\sim} f_n \circ h \overset{r}{\sim} g \circ h,$$

so both $f \circ h$ and $g \circ h$ represent the same element on $\pi^r(A, X')$. □

### 2.3.2. Left homotopy

As we already mentioned, this will be the dual (Remark 2.1.5) notion of right homotopy and therefore all the results are already proved, so we will just state them.

**Definition 2.3.17.** Let $\mathcal{C}$ be a model category. A **cylinder object** for $A \in \mathrm{Ob}(\mathcal{C})$ is an object $A \wedge I$ of $\mathcal{C}$ together with a factorization of the folding morphism $\mathrm{Id}_A + \mathrm{Id}_A$:

$$\begin{array}{ccc} A \amalg A & \xrightarrow{\mathrm{Id}_A + \mathrm{Id}_A} & A. \\ & {\scriptstyle i} \searrow \qquad \nearrow {\scriptstyle \sim} & \\ & A \wedge I & \end{array}$$

A cylinder object $A \wedge I$ is called a **good cylinder object** if $i\colon A \amalg A \to A \wedge I$ is a cofibration, and a **very good cylinder object** if in addition the morphism $A \wedge I \overset{\sim}{\to} A$ is a (necessarily acyclic) fibration.

If $A \wedge I$ is a cylinder object for $A$, we will denote the two structure morphisms $A \to A \wedge I$ by $i_0 = i \circ in_0$ and $i_1 = i \circ in_1$.

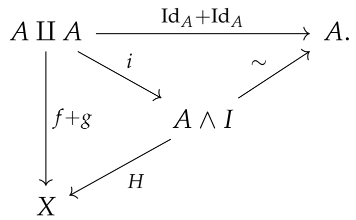
**Remark 2.3.18.** By **MC5**, for every object $A \in \mathrm{Ob}(\mathcal{C})$, there exists at least one very good cylinder.

**Remark 2.3.19.** If $A \wedge I$ is a cylinder object for $A$, then $i_0, i_1 \colon A \to A \wedge I$ are weak equivalences.

**Lemma 2.3.20.** *Let $\mathcal{C}$ be a model category. If $A, B$ are cofibrant objets, we have that $in_A \colon A \to A \amalg B$ and $in_B \colon B \to A \amalg B$ are cofibrations.*

**Lemma 2.3.21.** *Let $\mathcal{C}$ be a model category. If $A$ is cofibrant and $A \wedge I$ is a good cylinder object for $A$, then the morphisms $i_0, i_1 \colon A \to A \wedge I$ are acyclic cofibrations.*

**Definition 2.3.22.** Let $\mathcal{C}$ be a model category, $A, X \in \mathrm{Ob}(\mathcal{C})$. Two morphisms $f, g \colon A \to X$ in $\mathcal{C}$ are said to be **left homotopic** (written $f \overset{l}{\sim} g$) if there exists a cylinder object $A \wedge I$ for $A$ such that the sum morphism $f + g \colon A \amalg A \to X$ extends to a morphism $H \colon A \wedge I \to X$. That is, there exists a morphism $H \colon A \wedge I \to X$ (**left homotopy**) with $H \circ i = f + g$. The left homotopy is said to be **good** if $A \wedge I$ is a good cylinder object for $A$ and **very good** if $A \wedge I$ is a very good cylinder object for $A$:




**Remark 2.3.23.** If $f \overset{l}{\sim} g$ via the left homotopy $H$, $f$ is a weak equivalence if and only if $g$ is a weak equivalence.

**Example 2.3.24.** Let $\mathcal{C}$ be a category such that finite limits and colimits exist in it. By Example 2.1.3, we can define a trivial model structure on $\mathcal{C}$ where every element is fibrant and cofibrant. With this model structure, two morphisms are left homotopic if and only if they are equal.

**Lemma 2.3.25.** *If $f \overset{l}{\sim} g \colon A \to X$, then there exists a good left homotopy from $f$ to $g$. If in addition $X$ is fibrant, then there exists a very good left homotopy from $f$ to $g$.*

**Lemma 2.3.26.** *Let $\mathcal{C}$ be a model category. If $A$ is a cofibrant object, then $\overset{l}{\sim}$ is an equivalence relation on $\mathcal{C}(A, X)$ for every object $X \in \mathrm{Ob}(\mathcal{C})$.*

**Definition 2.3.27.** Let $\mathcal{C}$ be a model category and $A, X \in \mathrm{Ob}(\mathcal{C})$ objects. We will denote the set of equivalence classes of $\mathcal{C}(A, X)$ under the equivalence relation generated by left homotopy by $\pi^l(A, X)$.

**Remark 2.3.28.** Observe the importance of the word "generated" in the above definition of $\pi^l(A, X)$.

**Lemma 2.3.29.** *Let $\mathcal{C}$ be a model category. If $f, g \colon A' \to A$ are left homotopic morphisms and $h \colon A \to X$ is another morphism, then $h \circ f \overset{l}{\sim} h \circ g$.*

**Lemma 2.3.30.** *Let $\mathcal{C}$ be a model category. If $A$ is cofibrant and $p \colon X \to Y$ is an acyclic fibration, then composition with $p$ induces a bijection:*

$$p_* \colon \pi^l(A, X) \to \pi^l(A, Y).$$
$$[f] \mapsto p_*([f]) = [p \circ f]$$

**Lemma 2.3.31.** *Let $\mathcal{C}$ be a model category. Suppose that $X$ is fibrant, that $f, g\colon A \to X$ are left homotopic morphisms and that $h\colon A' \to A$ is a morphism. Then $f \circ h \overset{l}{\sim} g \circ h$.*

**Lemma 2.3.32.** *Let $\mathcal{C}$ be a model category. If $X$ is fibrant, then the composition in $\mathcal{C}$ induces a map:*

$$\begin{aligned} \pi^l(A', A) \times \pi^l(A, X) &\to \pi^l(A', X). \\ ([h], [f]) &\mapsto [f \circ h] \end{aligned}$$

## 2.3.3. Relationship between left and right homotopy

**Lemma 2.3.33.** *Let $\mathcal{C}$ be a model category, and $f, g\colon A \to X$ morphisms.*

1. *If $A$ is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*
2. *If $X$ is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

*Proof.* Let us prove the first assertion. By Lemma 2.3.25, there exists a good cylinder object $A \wedge I$ for $A$ factoring $\mathrm{Id}_A + \mathrm{Id}_A\colon A \amalg A \overset{i_0+i_1}{\hookrightarrow} A \wedge I \underset{\sim}{\overset{j}{\to}} A$ and a good homotopy $H\colon A \wedge I \to X$ from $f$ to $g$. By Lemma 2.3.21, $i_0$ is an acyclic cofibration. Moreover, by Remark 2.3.2, we can choose a good path object $X^I$ for $X$ factoring the diagonal morphism $(\mathrm{Id}_X, \mathrm{Id}_X)\colon X \overset{q}{\to} X^I \overset{(p_0,p_1)}{\twoheadrightarrow} X \times X$. Then, since

$$\begin{aligned} (p_0, p_1) \circ q \circ f &= (\mathrm{Id}_X, \mathrm{Id}_X) \circ f = (f, f) = (f \circ \mathrm{Id}_A, H \circ i_0) \\ &= (f \circ j \circ i_0, H \circ i_0) = (f \circ j, H) \circ i_0, \end{aligned}$$

we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{q\circ f} & X^I \\ {\scriptstyle i_0}\big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle (p_0,p_1)} \\ A \wedge I & \xrightarrow{(f\circ j, H)} & X \times X. \end{array}$$

Therefore, by **MC4**, there exists a lift $K\colon A \wedge I \to X^I$ for the diagram. The composite $K \circ i_1\colon A \to X^I$ is the desired right homotopy from $f$ to $g$:

$$\begin{aligned} (p_0, p_1) \circ (K \circ i_1) &= ((p_0, p_1) \circ K) \circ i_1 = (f \circ j, H) \circ i_1 \\ &= (f \circ j \circ i_1, H \circ i_1) = (f \circ \mathrm{Id}_A, H \circ i_1) = (f, g). \end{aligned}$$

The second assertion is the dual (Remark 2.1.5) property of the first one. □

**Corollary 2.3.34.** *Let $\mathcal{C}$ be a model category and $f, g\colon A \to X$ morphisms. If $A$ is cofibrant and $X$ is fibrant, then the left and right homotopy relations on $\mathcal{C}(A, X)$ agree.*

**Definition 2.3.35.** Let $\mathcal{C}$ be a model category, $A, X \in \mathrm{Ob}(\mathcal{C})$ objects and $f, g\colon A \to X$ morphisms. If $A$ is cofibrant and $X$ is fibrant, we will denote the identical right homotopy and left homotopy equivalence relations on $\mathcal{C}(A, X)$ by the symbol "$\sim$" and say that two morphisms related by this relation are **homotopic**. The set of equivalence classes with respect to this relation is denoted by $\pi(A, X)$.

**Definition 2.3.36.** Let $\mathcal{C}$ be a model category, $A, X \in \mathrm{Ob}(\mathcal{C})$ both fibrant and cofibrant objects and $f\colon A \to X$ a morphism. We will say that $f$ is a **homotopy equivalence** if it has a **homotopy inverse**, that is, if there exists a morphism $g\colon X \to A$ such that $g \circ f \sim \mathrm{Id}_A$, $f \circ g \sim \mathrm{Id}_X$.

**Lemma 2.3.37.** *Let $\mathcal{C}$ be a model category, $f\colon A \to X$ a morphism in $\mathcal{C}$ between objects which are both fibrant and cofibrant. Then $f$ is a weak equivalence if and only if $f$ is a homotopy equivalence.*

*Proof.* Suppose that $f$ is a weak equivalence. By **MC5**(ii), we can factor $f$ as $f\colon A \underset{\sim}{\overset{q}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$, where, by **MC2**, $p$ is also a weak equivalence. We then have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle q}\downarrow{\scriptstyle \sim} & & \downarrow \\ C & \longrightarrow & *, \end{array}$$

where $q$ is an acyclic cofibration and the right vertical arrow is a fibration because $A$ is fibrant. Then, by **MC4**, there exists a lift for the diagram, that is, a morphism $r\colon C \to A$ such that in particular $r \circ q = \mathrm{Id}_A$.

Since $C$ is fibrant and $q$ is an acyclic cofibration, by Lemma 2.3.32, $q$ induces a bijection

$$\begin{aligned} q^*\colon \pi^r(C, C) &\to \pi^r(A, C). \\ [g] &\mapsto [g \circ q] \end{aligned}$$

Since $q^*([q \circ r]) = [q \circ r \circ q] = [q \circ \mathrm{Id}_A] = [q] = [\mathrm{Id}_C \circ q] = p^*([\mathrm{Id}_C])$ and $q^*$ is one-to-one, we get that $[q \circ r] = [\mathrm{Id}_C]$, that is, that $q \circ r \sim \mathrm{Id}_C$. We can then conclude that $r$ is a homotopy inverse for $q$.

A dual (Remark 2.1.5) argument will give us the existence of an arrow $s\colon X \to C$ such that $s \circ p \sim \mathrm{Id}_C$.

Finally, we can terminate saying that $r \circ s$ is a homotopy inverse for $f = p \circ q$:

$$f \circ r \circ s = p \circ q \circ r \circ s \underbrace{\sim}_{2.3.32, 2.3.30} p \circ \mathrm{Id}_C \circ s = p \circ s = \mathrm{Id}_X,$$
$$r \circ s \circ f = r \circ s \circ p \circ q \underbrace{\sim}_{2.3.32, 2.3.30} r \circ \mathrm{Id}_C \circ q = r \circ q = \mathrm{Id}_A.$$

Conversely, assume that $f$ is a homotopy equivalence. By **MC5**(ii), we can factor $f$ as the composition $f\colon A \underset{\sim}{\overset{q}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$. Notice that $C$ is both cofibrant ($\varnothing \to A \overset{q}{\hookrightarrow} C$ is a cofibration because it is the composition of cofibrations) and fibrant ($C \overset{p}{\twoheadrightarrow} X \to *$ is a fibration because it is the composition of fibrations). Since $q$ is a weak equivalence, to prove that $f$ also is a weak equivalence, it is enough to show that $p$ is a weak equivalence. Let $g\colon X \to A$ be a homotopy inverse for $f$ and $H\colon X \wedge I \to X$ a good left homotopy between $f \circ g$ and $\mathrm{Id}_X$ (exists by Lemma 2.3.25). We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{q \circ g} & C \\ {\scriptstyle i_0}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle p} \\ X \wedge I & \xrightarrow{H} & X, \end{array}$$

where $i_0$ is an acyclic cofibration by Lemma 2.3.21 and $p$ is a fibration, so by **MC4** there is a lift $H'\colon X \wedge I \to C$ commuting the diagram; in particular, $H'$ is a left homotopy between $H' \circ i_0 = q \circ g$ and $H' \circ i_1$, so $q \circ g \sim H' \circ i_1$. Let $s = H' \circ i_1$, we have that $p \circ s = p \circ H' \circ i_1 = H \circ i_1 = \mathrm{Id}_X$. Since $q$ is a weak equivalence, by previous implication we have that it has a homotopy inverse $r$. Since in particular $q \circ r \sim \mathrm{Id}_C$, by Lemma 2.3.29 we have that $p = p \circ \mathrm{Id}_C \sim p \circ q \circ r = f \circ r$. We therefore have (using Lemma 2.3.32)

$$s \circ p \sim q \circ g \circ p \sim q \circ g \circ f \circ r \sim q \circ \mathrm{Id}_A \circ r = q \circ r \sim \mathrm{Id}_C.$$

Since $\mathrm{Id}_C$ is a weak equivalence, by Remark 2.3.23, $s \circ p$ also is. Finally, the following commutative diagram shows that $p$ is a retract (1.2.3) of $s \circ p$ and hence a weak equivalence by **MC3**:

$$\begin{array}{ccccc} C & \xrightarrow{\mathrm{Id}_C} & C & \xrightarrow{\mathrm{Id}_C} & C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle s \circ p} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{s} & C & \xrightarrow{p} & X. \end{array}$$

□

## 2.4. The homotopy category of a model category

As expected, fibrant and cofibrant objects are determinant in our context, and sometimes we will want to consider only this kind of objects.

**Definition 2.4.1.** Let $\mathcal{C}$ be a model category. We will pay atention to the following categories associated to it:

- $\mathcal{C}_c$: the full subcategory of $\mathcal{C}$ generated by the cofibrant objects in $\mathcal{C}$.
- $\mathcal{C}_f$: the full subcategory of $\mathcal{C}$ generated by the fibrant objects in $\mathcal{C}$.
- $\mathcal{C}_{cf}$: the full subcategory of $\mathcal{C}$ generated by the objects of $\mathcal{C}$ which are both fibrant and cofibrant.
- $\pi\mathcal{C}_c$: the category consisting of the cofibrant objects in $\mathcal{C}$ and whose morphisms are right homotopy classes of morphisms.
- $\pi\mathcal{C}_f$: the category consisting of the fibrant objects in $\mathcal{C}$ and whose morphisms are left homotopy classes of morphisms.
- $\pi\mathcal{C}_{cf}$: the category consisting of the objects in $\mathcal{C}$ which are both fibrant and cofibrant and whose morphisms are homotopy classes of morphisms.

**Remark 2.4.2.** Notice that $\pi\mathcal{C}_c$, $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_{cf}$ are well defined because of Lemmas 2.3.32 and 2.3.30.

Moreover, it would be very nice to be able to relate every object in our category with some fibrant or cofibrant object in a factorial way. Here is when the concepts of **fibrant replacement** and **cofibrant replacement** come in handy.

**Lemma 2.4.3.** *Let $\mathcal{C}$ be a model category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. Then we can apply* **MC5***(ii) to the only morphism $X \to *$ and obtain an acyclic cofibration $i_X \colon X \overset{\sim}{\hookrightarrow} RX$ with $RX$ fibrant. Furthermore, given a morphism $f \in \mathcal{C}(X, Y)$ there exists a morphism $\overline{f} \in \mathcal{C}(RX, RY)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle i_Y} \\ RX & \xrightarrow{\overline{f}} & RY. \end{array}$$

*The morphism $\overline{f}$ depends up to right homotopy or up to left homotopy only on $f$, and it is a weak equivalence if and only if $f$ is. If $X$ is cofibrant, then $\overline{f}$ depends up to right homotopy or up to left homotopy only on the right homotopy class of $f$.*

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{i_Y \circ f} & RY \\ {\scriptstyle i_X}\Big\downarrow{\scriptstyle \sim} & & \Big\downarrow \\ RX & \longrightarrow & *, \end{array}$$

where $i_X$ is an acyclic cofibration and the right vertical arrow is a fibration because $RY$ is fibrant. Therefore, by **MC4**, there is a lift $\overline{f}\colon RX \to RY$ such that $\overline{f} \circ i_X = i_Y \circ f$. This equation also tells us that $f$ is a weak equivalence if and only if $\overline{f}$ is by **MC2** and the fact that $i_X$ and $i_Y$ are weak equivalences.

Moreover, since $RY$ is fibrant and $i_X\colon X \to RX$ is an acyclic cofibration, by Lemma 2.3.32 there is a bijection:

$$\begin{aligned} i_X^*\colon \pi^r(RX, RY) &\to \pi^r(X, RY). \\ [g] &\mapsto i_X^*([g]) = [g \circ i_X] \end{aligned}$$

In particular, if there exist $\overline{f}, \overline{\overline{f}}$ satisfying $\overline{f} \circ i_X = i_Y \circ f = \overline{\overline{f}} \circ i_X$, then $\overline{f} \overset{r}{\sim} \overline{\overline{f}}$, which implies $\overline{f} \overset{l}{\sim} \overline{\overline{f}}$ by Lemma 2.3.33 since $RY$ is fibrant. This proves the unicity up to right or left homotopy.

Moreover, if $X$ is cofibrant and $f \overset{r}{\sim} g\colon X \to Y$, then by Lemma 2.3.15 $\overline{f} \circ i_X = i_Y \circ f \overset{r}{\sim} i_Y \circ g = \overline{g} \circ i_X$. Then, again by Lemma 2.3.32, $\overline{f} \overset{r}{\sim} \overline{g}$, which implies $\overline{f} \overset{l}{\sim} \overline{g}$ by Lemma 2.3.33 since $RY$ is fibrant. This proves the dependence up to homotopy on the right homotopy class of $f$ if $X$ is cofibrant. □

**Corollary 2.4.4.** *We can define a functor $R\colon \mathcal{C} \to \pi\mathcal{C}_f$ which sends $X \in \mathrm{Ob}(\mathcal{C})$ to $RX$ and $f \in \mathcal{C}(X, Y)$ to $[\overline{f}] \in \pi^l(RX, RY)$.*

*Proof.* The following diagram obviously commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}_X} & X \\ \Big\downarrow{\scriptstyle i_X} & & \Big\downarrow{\scriptstyle i_X} \\ RX & \xrightarrow{\mathrm{Id}_{RX}} & RX. \end{array}$$

Therefore, by the unicity of $\overline{\mathrm{Id}_X}$ up to left homotopy, we get that $\overline{\mathrm{Id}_X} \overset{l}{\sim} \mathrm{Id}_{RX}$, or equivalently, that $[\overline{\mathrm{Id}_X}] = [\mathrm{Id}_{RX}] \in \pi^l(RX, RY)$.

Analogously, the following diagram obviously commutes:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \Big\downarrow{\scriptstyle i_X} & & \Big\downarrow{\scriptstyle i_Y} & & \Big\downarrow{\scriptstyle i_Z} \\ RX & \xrightarrow{\overline{f}} & RY & \xrightarrow{\overline{g}} & Z. \end{array}$$

Therefore, by the unicity of $\overline{g \circ f}$ up to left homotopy, we get that $\overline{g \circ f} \overset{l}{\sim} \overline{g} \circ \overline{f}$, or equivalently, that $[\overline{g \circ f}] = [\overline{g} \circ \overline{f}] = [\overline{g}] \circ [\overline{f}]$. We have thus proved the functoriality of $R$. □

**Lemma 2.4.5.** *The restriction of the functor $R\colon \mathcal{C} \to \pi\mathcal{C}_f$ to the full subcategory of cofibrant objects $\mathcal{C}_c$ induces a functor $R'\colon \pi\mathcal{C}_c \to \pi\mathcal{C}_{cf}$.*

*Proof.* First, notice that if $X$ is cofibrant, then $RX$ is both fibrant and cofibrant. Indeed, it is fibrant by definition and the only morphism $\emptyset \to RX$ must be factorized as $\emptyset \hookrightarrow X \overset{i_X}{\hookrightarrow} RX$, which is a composition of cofibrations and hence a cofibration. Moreover, we have already seen in Lemma 2.3.33 that, for cofibrant objects, the homotopy class of $\overline{f}$ depends only on the right homotopy class of $f$. □

We also have the dual (Remark 2.1.5) results, which we will only state.

**Lemma 2.4.6.** *Let $\mathcal{C}$ be a model category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. Then we can apply* **MC5** *to the morphism $\emptyset \to X$ and obtain an acyclic fibration $p_X\colon QX \overset{\sim}{\twoheadrightarrow} X$ with $QX$ cofibrant. Moreover, given a morphism $f \in \mathcal{C}(X,Y)$ there exists a morphism $\widetilde{f} \in \mathcal{C}(QX, QY)$ such that the following diagram commutes:*

$$\begin{array}{ccc} QX & \xrightarrow{\widetilde{f}} & QY \\ {\scriptstyle p_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

*The morphism $\widetilde{f}$ depends up to left homotopy or up to right homotopy only on $f$, and it is a weak equivalence if and only if $f$ is. If $Y$ is fibrant, then $\widetilde{f}$ depends up to left homotopy or up to right homotopy only on the left homotopy class of $f$.*

**Corollary 2.4.7.** *We can define a functor $Q\colon \mathcal{C} \to \pi\mathcal{C}_c$ which sends $X \in \mathrm{Ob}(\mathcal{C})$ to $QX$ and $f \in \mathcal{C}(X,Y)$ to $[\widetilde{f}] \in \pi^r(QX, QY)$.*

**Lemma 2.4.8.** *The restriction of the functor $Q\colon \mathcal{C} \to \pi\mathcal{C}_c$ to the full subcategory of fibrant objects $\mathcal{C}_f$ induces a functor $Q'\colon \pi\mathcal{C}_f \to \pi\mathcal{C}_{cf}$.*

Now we have all the ingredients for defining the homotopy category related to our model category.

**Definition 2.4.9.** The **homotopy category** $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and with morphisms

$$\mathrm{Ho}(\mathcal{C})(X,Y) = \pi\mathcal{C}_{cf}(R'QX, R'QY) = \pi(RQX, RQY)$$

for $X, Y \in \mathrm{Ob}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$.

**Remark 2.4.10.** It is clear that the following assignation is a functor:

$$\begin{aligned} \gamma\colon \mathcal{C} &\to \mathrm{Ho}(\mathcal{C}). \\ X \in \mathrm{Ob}(\mathcal{C}) &\mapsto X \in \mathrm{Ob}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C}) \\ f \in \mathcal{C}(X,Y) &\mapsto \gamma(f) = [\overline{\widetilde{f}}] \in \pi(RQX, RQY) = \mathrm{Ho}(\mathcal{C})(X,Y) \end{aligned}$$

**Remark 2.4.11.** When considering the homotopy category of a model category, we are basically converting the weak equivalences in isomorphisms, and moreover we are doing so in the minimal way. This has to do with the notion of localization of a category; see [DS95, section 6].

For example, if we consider $\mathcal{C}$ a category such that finite limits and colimits exist in it and the trivial model structure seen on Example 2.1.3, we know by Example 2.3.8 that two morphisms are right homotopic (or equivalently, left homotopic by Corollary 2.3.34) if and only if they are the same. Furthermore, notice that we can set $RX = X$ and $QX = X$ for every $X \in \mathrm{Ob}(X)$. In conclusion, $\mathrm{Ho}(\mathcal{C})$ has as objects the same objects as $\mathcal{C}$ and for $X, Y \in \mathrm{Ob}(\mathrm{Ho}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{Ho}(\mathcal{C})(X,Y) = \pi(RQX, RQY) = \pi(X,Y) = \{\{f\}\colon f \in \mathcal{C}(X,Y)\}.$$

In conclusion, in this case $\mathcal{C} \cong \mathrm{Ho}(\mathcal{C})$. This makes sense because since isomorphisms are already invertible, we do nothing by inverting them. But of course, all this theory of model categories was developed in order to consider as weak equivalences more morphisms than just the isomorphisms.

## 2.5. Quillen equivalences

Until now, we know that in some categories we can set a model structure that leads us to the homotopy categories of our initial categories. Now we will see how the relations between our initial categories determine the relations between their homotopy categories.

First of all, we will see two dual (2.1.5) lemmas that will be useful for us in the future.

**Lemma 2.5.1** (K. Brown). *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences then F preserves all weak equivalences between cofibrant objects.*

**Lemma 2.5.2.** *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If F carries acyclic fibrations between fibrant objects to weak equivalences then F preserves all weak equivalences between fibrant objects.*

*Proof.* Let $f: A \to B$ be a weak equivalence between fibrant objects, and let us see that its image $F(f): F(A) \to F(B)$ is also a weak equivalence. First, by **MC5**(ii) we can factor $(\mathrm{Id}_A, f): A \to A \times B$ as a composite $A \underset{\sim}{\overset{q}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} A \times B$. By Lemma 2.3.4, $pr_A: A \times B \to A$ and $pr_B: A \times B \to B$ are fibrations, so $pr_A \circ p: C \to A$ and $pr_B \circ p: C \to B$ are also fibrations. Moreover, since $f = pr_B \circ p \circ q$ and $\mathrm{Id}_A = in_A \circ p \circ q$ are weak equivalences and $q$ is also a weak equivalence, by **MC2** $pr_A \circ p$ and $pr_B \circ p$ are also weak equivalences. In addition, we have that $C \to *$ equals the composite $C \overset{pr_A \circ p}{\twoheadrightarrow} A \twoheadrightarrow *$, so it will be a fibration and hence $C$ will be a fibrant object. Summing up, $pr_A \circ p: C \to A$ and $pr_B \circ p: C \to B$ are acyclic fibrations between fibrant objects, so by hypothesis, $F(pr_A \circ p)$, $F(pr_B \circ p)$ are weak equivalences, and the same happens to $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$. Then, since

$$F(\mathrm{Id}_A) = F(pr_A \circ p \circ q) = F(pr_A \circ p) \circ F(q),$$

by **MC2**, $F(q)$ is a weak equivalence and then by **MC2** again,

$$F(f) = F(pr_B \circ p \circ q) = F(pr_B \circ p) \circ F(q)$$

is also a weak equivalence, as we wanted to prove. □

Now, let us see that if we are able to relate two model categories in a certain way, then its homotopy categories will be isomorphic. This will be very important for us, since it will give us freedom to study the same homotopy category from the point of view of different model categories.

The following types of functors will be the most handy for us.

**Definition 2.5.3.** Let $\mathcal{C}$, $\mathcal{D}$ be two model categories.

- We call a functor $F: \mathcal{C} \to \mathcal{D}$ a **left Quillen functor** if it is a left adjoint and preserves cofibrations and trivial cofibrations.
- We call a functor $G: \mathcal{D} \to \mathcal{C}$ a **right Quillen functor** if it is a right adjoint and preserves fibrations and trivial fibrations.
- We will say that the adjoint pair $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ is a **Quillen adjunction** if $F$ preserves cofibrations and $G$ preserves fibrations.
- We will say that a Quillen adjunction $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ is a **Quillen equivalence** if for each cofibrant $A \in \mathrm{Ob}(\mathcal{C})$ and fibrant $X \in \mathrm{Ob}(\mathcal{D})$, a morphism $f: A \to G(X)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^\flat: F(A) \to X$ is a weak equivalence in $\mathcal{D}$.

**Proposition 2.5.4.** *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjoint pair between model categories. Then the following are equivalent:*

1. *$F$ and $G$ form a Quillen adjunction.*
2. *$F$ is a left Quillen functor.*
3. *$G$ is a right Quillen functor.*

*Proof.* Let us see, for example, that the first assertion is equivalent to the second one, and the equivalence between the first assertion and the third one will be dual (Remark 2.1.5).

First, let us see that if $G$ preserves fibrations, then $F$ preserves acyclic cofibrations. Let then $f\colon A \to B$ be an acyclic cofibration in $\mathcal{C}$ and let us see that $F(f)$ is an acyclic cofibration in $\mathcal{D}$, or equivalently by Proposition 2.2.2, that it has the LLP with respect to fibrations. Consider then the following lifting problem, with $g$ a fibration:

$$\begin{array}{ccc} F(A) & \xrightarrow{u} & X \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow{v} & Y. \end{array}$$

By adjointness, we have the following lifting problem, with $G(g)$ a fibration (or equivalently by Proposition 2.2.2, with the RLP with respect to acyclic cofibrations) because $G$ preserves fibrations by hypothesis:

$$\begin{array}{ccc} A & \xrightarrow{u^\sharp} & G(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow{v^\sharp} & G(Y). \end{array}$$

Hence, there is $w\colon B \to G(X)$ such that $G(g) \circ w = v^\sharp$, $w \circ f = u^\sharp$. In consequence, by adjointness, we have $w^\flat\colon G(B) \to X$ such that $g \circ w^\flat = v$, $w^\flat \circ F(f) = v$, that is, our initial diagram has a lift, as we wanted to prove.

Conversely, let us see that if $F$ preserves acyclic cofibrations, then $G$ preserves fibrations. Let $g\colon X \to Y$ be a fibration in $\mathcal{D}$ and let us see that $G(g)$ is a fibration in $\mathcal{C}$, or equivalently by Proposition 2.2.2, that it has the RLP with respect to acyclic cofibrations. Consider then the following lifting problem, with $f$ an acyclic cofibration:

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow{v} & G(Y). \end{array}$$

By adjointness, we have the following lifting problem, with $F(f)$ an acyclic cofibration (or equivalently by Proposition 2.2.2, with the LLP with respect to fibrations) because $F$ preserves acyclic cofibrations by hypothesis:

$$\begin{array}{ccc} F(A) & \xrightarrow{u^\flat} & X \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow{v^\flat} & Y. \end{array}$$

Hence, there is $w\colon F(B) \to X$ such that $g \circ w = v^\flat$, $w \circ F(f) = u^\flat$. In consequence, by adjointness, we have $w^\sharp\colon B \to G(X)$ such that $G(g) \circ w^\sharp = v$, $w^\sharp \circ f = u$, that is, our initial diagram has a lift, as we wanted to prove. □

**Proposition 2.5.5.** *Let $\mathcal{C}$, $\mathcal{D}$ be model categories and $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ a Quillen adjunction. Then, the following are equivalent:*

1. *F and G form a Quillen equivalence.*
2. $i^{\sharp}_{F(X)} \underbrace{=}_{1.3.4} G(i_{F(X)}) \circ \mathrm{Id}^{\sharp}_{F(X)}$ *is a weak equivalence for every cofibrant object* $X \in \mathrm{Ob}(\mathcal{C})$ *and* $p^{\flat}_{G(X)} \underbrace{=}_{1.3.4} \mathrm{Id}^{\flat}_{G(X)} \circ F(p_{G(X)})$ *is a weak equivalence for every fibrant object* $X \in \mathrm{Ob}(\mathcal{D})$.

*Proof.* Let us begin by proving that the first assertion implies the second one. Let then $F$ and $G$ form a Quillen equivalence. If $X \in \mathrm{Ob}(\mathcal{C})$ is a cofibrant object, $i_{F(X)}\colon F(X) \overset{\sim}{\hookrightarrow} RF(X)$ is a weak equivalence with $X$ cofibrant and $RF(X)$ fibrant, so $i^{\sharp}_{F(X)}$ will also be a weak equivalence. Similarly, if $X \in \mathrm{Ob}(\mathcal{D})$ is a fibrant object, $p_{G(X)}\colon QG(X) \to G(X)$ is a weak equivalence with $QG(X)$ cofibrant and $X$ fibrant, so $p^{\flat}_{G(X)}$ will also be a weak equivalence.

Conversely, assume that the second assertion holds and let $A \in \mathrm{Ob}(\mathcal{C})$ be cofibrant and $X \in \mathrm{Ob}(\mathcal{D})$ fibrant. First, given $f\colon A \to G(X)$ a weak equivalence, let us see that $f^{\flat} \underbrace{=}_{1.3.4} \mathrm{Id}^{\flat}_{G(X)} \circ F(f)$ is a weak equivalence. First, notice that since $F$ preserves acyclic cofibrations, in particular takes acyclic cofibrations between cofibrant objects to weak equivalences, so by Lemma 2.5.1, $F$ preserves all weak equivalences between cofibrant objects. Moreover, since $f$ is a weak equivalence, $\widetilde{f}\colon QA \to QG(X)$ will also be a weak equivalence by Lemma 2.4.6, and since both $QA$ and $QF(X)$ are cofibrant objects, $G(\widetilde{f})$ will also be a weak equivalence. By assumption, we also have that $\mathrm{Id}^{\flat}_{G(X)} \circ F(p_{G(X)})$ is a weak equivalence, so

$$\mathrm{Id}^{\flat}_{G(X)} \circ F(p_{G(X)}) \circ F(\widetilde{f}) = \mathrm{Id}^{\flat}_{G(X)} \circ F(f) \circ F(p_A)$$

will also be a weak equivalence. Since $p_A\colon QA \to A$ is a weak equivalence between cofibrant objects, $F(p_A)$ will also be a weak equivalence and hence by **MC2**, $f^{\flat} = \mathrm{Id}^{\flat}_{G(X)} \circ F(f)$ is a weak equivalence. Dually, given $f\colon F(A) \to X$ a weak equivalence, we get that $f^{\sharp} = G(f) \circ \mathrm{Id}^{\sharp}_{F(A)}$ is a weak equivalence, which finishes our proof. □

**Corollary 2.5.6.** *Let $\mathcal{C}$, $\mathcal{D}$ be model categories and $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ a Quillen adjunction. Then, the following are equivalent:*

1. *F and G form a Quillen equivalence.*
2. $\mathrm{Id}^{\sharp}_{F(X)}$ *is a weak equivalence for every cofibrant object* $X \in \mathrm{Ob}(\mathcal{C})$ *and* $\mathrm{Id}^{\flat}_{G(X)}$ *is a weak equivalence for every fibrant object* $X \in \mathrm{Ob}(\mathcal{D})$.

*Proof.* Since $F$ and $G$ form a Quillen adjunction, $F$ preserves acyclic cofibrations. Since $i_{F(X)}$ is an acyclic cofibration for every $X \in \mathrm{Ob}(\mathcal{C})$ (in particular for $X$ cofibrant), $G(i_{F(X)})$ will be an acyclic cofibration, hence a weak equivalence. Dually, $F(p_{G(X)})$ will also be a weak equivalence. Therefore, by **MC2**, the second assertion of previous proposition is equivalent to the second assertion of this corollary. □

Next theorem shows the importance of the functors that we have defined.

**Theorem 2.5.7.** *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ a Quillen adjunction. Then we have an adjoint pair $\mathbf{L}F\colon \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D})\colon \mathbf{R}G$ between their homotopy categories. Moreover, if F and G form a Quillen equivalence, then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories.*

*Proof.* See [DS95, section 9, Theorem 9.7]. The proof uses some special kind of functors called derived functors. □

# Chapter 3

# Modeling the homotopy theory of spaces

Now that we have everything settled, let us see some examples of model categories. Of course, the category of topological spaces admits a model structure with weak equivalences the weak homotopy equivalences. Moreover, there are other different categories which are Quillen-equivalent to this one. In particular, we will overview the case of simplicial sets and later we will focus on the category of small categories and the category of posets.

## 3.1. Topological spaces and simplicial sets

The development of homotopy theory for model categories was mainly motivated by the existence of this kind of structure in the category of topological spaces and the possibility to mimic it for simplicial sets, for later seeing that both structures are closely related. This was originally studied by the American mathematician Daniel Quillen in [Qui67].

### 3.1.1. The Quillen model structure for topological spaces

As we said before, this structure was firstly described by Quillen, and then completed and reviewed by a lot of mathematicians, such as the American Philip S. Hirschhorn in [Hir19].

Let us first recall that the category of topological spaces is formed by the class of topological spaces together with continuous maps between them. Now, let us briefly describe the classes of morphisms involved in our structure.

#### Weak homotopy equivalences

Weak homotopy type is a widely used term by topologists. In this case, we will follow [Hat02] and [Cro78].

**Definition 3.1.1.** Let $X$ be a topological space, we define $\pi_0(X)$ as the set of the path-connected components of $X$: $\pi_0(X) := X/\sim$, with $\sim$ the equivalence relation "being joint by a path".

**Definition 3.1.2.** Let $X$ be a topological space, $x_0 \in X$ and $n \in \mathbb{Z}_{\geq 1}$. We consider the set of continuous maps from the $n$-cube $I^n = [0,1]^n$ into $X$ such that the image of the boundary $\partial I^n$ (points with some of the coordinates equal to 1 or to 0) is $x_0$:

$$F_n(X,x_0) = \left\{ f\colon I^n = [0,1] \times \overset{(n)}{\cdots} \times [0,1] \to X\colon \begin{array}{l} f \text{ continuous} \\ f(\partial I^n) = x_0 \end{array} \right\}.$$

On this set, we define the following inner operation:

$$\begin{aligned} f * g\colon I^n &\to X. \\ (t_1,\dots,t_n) \mapsto (f*g)(t_1,\dots,t_n) &= \begin{cases} f(2t_1,\dots,t_n), & 0 \le t_1 \le \frac{1}{2} \\ g(2t_1-1,\dots,t_n), & \frac{1}{2} \le t_1 \le 1 \end{cases} \end{aligned}$$

Now, we consider the quotient set $\pi_n(X,x_0) = F_n(X,x_0)/\sim$, with $\sim$ the equivalence relation "being homotopic relative to $\partial I^n$". On it, we define the following inner operation:

$$[f] \circ [g] := [f * g].$$

With it, $\pi_n(X,x_0)$ is a group denominated the **$n$-th homotopy group** of $X$. It is abelian for $n \ge 2$.

**Remark 3.1.3.** Since $I^n/\partial I^n \cong \mathbb{S}^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1}\colon x_1^2 + \dots + x_{n+1}^2 = 1\}$, we can see the elements of $F_n(X,x_0)$ as continuous maps from $\mathbb{S}^n$ to $X$ such that the image of $(0,\overset{(n)}{\dots},0,1)$ is $x_0$.

**Remark 3.1.4.** We can extend this definition for $n = 0$ if we take $I^0$ a point and $\partial I^0$ the empty set.

**Definition 3.1.5.** Let $X$, $Y$ be topological spaces and $f\colon X \to Y$ a continuous map. Then, for every $n \ge 1$ and every $x_0 \in X$, the following map is a group homomorphism that we will call **homomorphism induced by** $f$. For $n = 0$ it is just a map.

$$\begin{aligned} \pi_n(f)\colon \pi_n(X,x_0) &\to \pi_n(Y,f(x_0)). \\ [g] &\mapsto \pi_n(f)([g]) := [f \circ g] \end{aligned}$$

**Definition 3.1.6.** Let $X$ and $Y$ be topological spaces. We say that a continuous map $f\colon X \to Y$ is a **weak homotopy equivalence** if for every choice of basepoint $x_0 \in X$, $f$ induces isomorphisms between all the homotopy groups of $X$ and $Y$ and $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection. In this case, we say that $X$ and $Y$ have the same **weak homotopy type**.

#### Relative cell complexes

**Definition 3.1.7.** If $X$ is a subspace of $Y$ such that there is a pushout square

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ \mathbb{D}^n & \longrightarrow & Y \end{array}$$

for some $n \ge 0$, $\mathbb{D}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n\colon x_1^2 + \dots + x_n^2 \le 1\}$ and $i$ the inclusion, then we will say that $Y$ is obtained from $X$ by **attaching a cell**.

**Definition 3.1.8.** A **(finite) relative cell complex** is an inclusion of a subspace $f\colon X \to Y$ such that $Y$ can be constructed from $X$ by a (finite) process of repeatedly attaching cells.

#### Serre fibrations

**Definition 3.1.9.** A continuous map $f\colon X \to Y$ between two topological spaces is called a **Serre fibration** if it has the RLP with respect to the inclusion $(\mathrm{Id}_{\mathbb{D}^n}, 0)\colon \mathbb{D}^n \to \mathbb{D}^n \times [0,1]$ of the standard topological $n$-disc into its cylinder for every $n \geq 0$:

$$\begin{array}{ccc} \mathbb{D}^n & \longrightarrow & X \\ {\scriptstyle (\mathrm{Id}_{\mathbb{D}^n},0)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathbb{D}^n \times [0,1] & \longrightarrow & Y. \end{array}$$

Now we are in conditions of establishing Quillen's model structure.

**Definition 3.1.10.** We will say that a morphism $f\colon X \to Y$ in $\mathcal{T}\mathrm{op}$ is:

- A **weak equivalence** if it is a weak homotopy equivalence.
- A **cofibration** if it is a relative cell complex or a retract of a relative cell complex.
- A **fibration** if it is a Serre fibration.

**Theorem 3.1.11.** *There is a model category structure on the category of topological spaces in which the weak equivalences, cofibrations and fibrations are as above. All the topological spaces are fibrant and the cofibrant objects are the retracts of CW-complexes.*

*Proof.* See [Hir19, Theorem 2.5]. □

**Remark 3.1.12.** Of course, this is not the only non-trivial model structure on the category of topological spaces. Strøm has shown in [Str72] that there is a model category structure on the category of topological spaces in which the weak equivalences are the homotopy equivalences, the fibrations are the Hurewicz fibrations and the cofibrations are the closed inclusions with the homotopy extension property.

### 3.1.2. The Kan-Quillen model structure for simplicial sets

Quillen also established a model structure on the category of simplicial sets in [Qui67], and then again, a lot of mathematicians, such as the Americans Paul Goerss and Rick Jardine or Edward B. Curtis, summarized and enriched this theory, showing the similarities with the model category of topological spaces; see [GJ09] or [Cur71]. A great reference for getting familiar with the basic notions of this topic is the survey by Greg Friedman [Fri12].

#### Kan fibrations

**Definition 3.1.13.** A morphism $f\colon X \to Y$ of simplicial sets is called a **Kan fibration** if it has the RLP with respect to all the inclusions $i_{n,k}\colon \Delta[n,k] \to \Delta[n]$, $0 \leq k \leq n$, $n \geq 0$:

$$\begin{array}{ccc} \Delta[n,k] & \longrightarrow & X \\ \downarrow{\scriptstyle i_{n,k}} & \nearrow & \downarrow{\scriptstyle f} \\ \Delta[n] & \longrightarrow & Y. \end{array}$$

A simplicial set $X$ such that the only morphism $X \to \Delta[0]$ is a Kan fibration is called a **Kan complex**.

**Definition 3.1.14.** A morphism $f\colon X \to Y$ of simplicial sets is called an **acyclic Kan fibration** if it has the RLP with respect to all the inclusions $i_n\colon \partial\Delta[n] \to \Delta[n]$, $n \geq 0$:

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \Big\downarrow{\scriptstyle i_n} & \nearrow & \Big\downarrow{\scriptstyle f} \\ \Delta[n] & \longrightarrow & Y. \end{array}$$

With the notion of Kan fibration in mind, we can define Kan-Quillen's model structure.

**Definition 3.1.15.** We will say that a morphism $f\colon X \to Y$ in s$\mathcal{S}$ets is:

- A **fibration** if it is a Kan fibration, and a trivial fibration if it is an acyclic Kan fibration.
- A **cofibration** if it has the LLP with respect to acyclic Kan fibrations, and a trivial cofibration if it has the LLP with respect to Kan fibrations.
- A **weak equivalence** if it can be factored as the composition of a trivial cofibration followed by a trivial fibration.

**Remark 3.1.16.** The cofibrations are exactly the monomorphisms, that is, the simplicial morphisms that are injective set maps in each level; see [Qui67, Chapter II, section 3, Proposition 2].

**Theorem 3.1.17.** *There is a model category structure on the category of simplicial sets in which the weak equivalences, cofibrations and fibrations are as above. Every simplicial set will be a cofibrant object and the fibrant objects will be the Kan complexes.*

*Proof.* See [Qui67, Chapter II, section 3, Theorem 3]. □

**Remark 3.1.18.** Weak equivalences in this model structre verify that if we have a sequence of weak equivalences $A_i \to A_{i+1}$ for $i \geq 0$, then the canonical morphism $A_0 \to \mathrm{colim}_n A_n$ is also a weak equivalence. They also verify that the coproduct of weak equivalences is a weak equivalence.

Moreover, the defined structures are closely related.

**Theorem 3.1.19.** *The geometric realization and singular functors form a Quillen equivalence between Quillen's model structures in $\mathcal{T}$op and s$\mathcal{S}$ets:*

$$|\cdot|\colon \mathcal{T}\mathrm{op} \rightleftarrows \mathrm{s}\mathcal{S}\mathrm{ets}\colon \mathrm{Sing}.$$

*Proof.* See [Hov99, Theorem 3.6.7] or [GJ09, Theorem 11.4]. □

As a consequence, we get two points of view for studying the same homotopy category: one topological and another one purely combinatorial. Next sections will give us even more points of view to carry out the same study.

## 3.2. The Thomason model structure for small categories

In this section, we will set up a model structure on the category of small categories which will be Quillen-equivalent to the one defined on simplicial sets.

The first candidate for the Quillen equivalence would be the adjunction

$$\tau\colon s\mathcal{S}\text{ets} \rightleftarrows \mathcal{C}\text{at}\colon \mathrm{N},$$

but this fails to give us what we want since $\mathrm{Id}^{\sharp}_{\tau(X)}$ is not a weak equivalence for every (cofibrant) object $X$ in s$\mathcal{S}$ets (see Corollary 2.5.6). For example, for $X = \partial\Delta[n]$ with $n > 2$, we have that

$$\mathrm{Id}^{\sharp}_{\tau\partial\Delta[n]}\colon \partial\Delta[n] \to \mathrm{N}\tau\partial\Delta[n] \cong \Delta[n]$$

cannot be a weak equivalence since the geometric realizations of the domain and the codomain do not have the same homotopy groups, so the geometric realization of our morphism cannot induce isomorphisms between them and hence it cannot be a weak equivalence of topological spaces.

The American mathematician Robert Wayne Thomason found in [Tho80] the solution for this problem adding the Sd and Ex functors to our adjunction, being our new candidate for Quillen equivalence

$$\tau\mathrm{Sd}^2\colon s\mathcal{S}\text{ets} \rightleftarrows \mathcal{C}\text{at}\colon \mathrm{Ex}^2\mathrm{N}.$$

Let us then define how our classes of morphisms will be.

**Definition 3.2.1.** A morphism $f$ in $\mathcal{C}$at is:

- A **weak equivalence** if $Ex^2\mathrm{N}f$ is a weak equivalence in s$\mathcal{S}$ets.
- A **fibration** if $Ex^2\mathrm{N}f$ is a fibration in s$\mathcal{S}$ets.
- A **cofibration** if it has the LLP with respect to trivial fibrations.

Next, let us analyze how the functor $\tau\mathrm{Sd}^2$ will behave in some particular cases.

**Remark 3.2.2.** Let $X_K$ be a simplicial set arising from an ordered simplicial complex $K$ (see Example 1.4.29). Then we know that its barycentric subdivision $\mathrm{Sd}X_K$ is the simplicial set associated to the (old) barycentric subdivision of $K$. In particular, the 0-simplices of $\mathrm{Sd}X_k$ are the non-degenerate simplices of $X_K$ (that is, the simplices of $K$), and there is a 1-simplex $e \in (X_K)_1$ such that $d_0(e) = v$ and $d_1(e) = w$ if and only if $v \subseteq w$ as simplices of $K$. In conclusion, $\tau\mathrm{Sd}X_K = i(\mathcal{X}(K))$ and in addition we have that $\mathrm{N}\tau\mathrm{Sd}X_K = \mathrm{N}(i(\mathcal{X}(K)))$ is the simplicial set associated to $\mathcal{K}(\mathcal{X}(K))$, the (old) barycentric subdivision of $K$, so $\mathrm{N}\tau\mathrm{Sd}X_K = \mathrm{Sd}X_K$.

Specifically, for

$$K = \begin{cases} 2^{[n]} \setminus \{\varnothing\} \\ 2^{[n]} \setminus \{\varnothing, \{0,\dots,n\}\} \\ 2^{[n]} \setminus \{\varnothing, \{0,\dots,n\}, \{0,\dots,k-1,k+1,\dots,n\}\}, \end{cases} \qquad X_K = \begin{cases} \Delta[n] \\ \partial\Delta[n] \\ \Delta[n,k], \end{cases}$$

we have that $\tau\mathrm{Sd}^2\Delta[n]$, $\tau\mathrm{Sd}^2\partial\Delta[n]$ and $\tau\mathrm{Sd}^2\Delta[n,k]$ are the small categories associated to the posets of faces of the barycentric subdivisions of their respective $K$'s.

Furthermore, for $i_{n,k}\colon \Delta[n,k] \to \Delta[n]$ the inclusion morphism seen on Example 1.4.29,

$$\tau\mathrm{Sd}^2(i_{n,k})\colon \tau\mathrm{Sd}^2\Delta[n,k] \to \tau\mathrm{Sd}^2\Delta[n]$$

is the obvious inclusion.

The morphisms mentioned in this remark will be very useful for us.

**Definition 3.2.3.** We define the **categorical horns** as:

$$\tau\mathrm{Sd}^2\Delta[n,k] \xrightarrow{\tau\mathrm{Sd}^2(i_{n,k})} \tau\mathrm{Sd}^2\Delta[n].$$

Let us see a characterization of fibrations in $\mathcal{C}$at in terms of the categorial horns.

**Proposition 3.2.4.** *Let $p \in \mathcal{C}\mathrm{at}(C, D)$. Then $p$ is a fibration if and only if for every categorical horn $\tau\mathrm{Sd}^2(i_{n,k})$, the following diagram has a lift:*

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \longrightarrow & C \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \longrightarrow & D. \end{array}$$

*Proof.* By definition, $p$ is a fibration if and only if $\mathrm{Ex}^2\mathrm{N}p$ is a fibration in s$\mathcal{S}$ets, which is equivalent to all the diagrams of the form

$$\begin{array}{ccc} \Delta[n,k] & \longrightarrow & \mathrm{Ex}^2\mathrm{N}C \\ \downarrow{\scriptstyle i_{n,k}} & & \downarrow{\scriptstyle \mathrm{Ex}^2\mathrm{N}p} \\ \Delta[n] & \longrightarrow & \mathrm{Ex}^2\mathrm{N}D \end{array}$$

having a lift, which by adjointness is equivalent to all diagrams of the form

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \longrightarrow & C \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \longrightarrow & D \end{array}$$

having a lift. □

Acyclic fibrations also admit an analogous characterization.

**Proposition 3.2.5.** *Let $p \in \mathcal{C}\mathrm{at}(C, D)$. Then $p$ is a trivial fibration if and only if every diagram of the following form has a lift:*

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \longrightarrow & X \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_n)} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \longrightarrow & Y. \end{array}$$

*Proof.* By definition, $p$ is a trivial fibration if and only if $\mathrm{Ex}^2\mathrm{N}p$ is a trivial fibration in s$\mathcal{S}$ets, which is equivalent to all the diagrams of the form

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & \mathrm{Ex}^2\mathrm{N}X \\ \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle \mathrm{Ex}^2\mathrm{N}p} \\ \Delta[n] & \longrightarrow & \mathrm{Ex}^2\mathrm{N}Y \end{array}$$

having a lift, which by adjointness is equivalent to all diagrams of the form

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \longrightarrow & X \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_n)} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \longrightarrow & Y \end{array}$$

having a lift. □

Next, let us see an equivalent, but more simple, characterization of weak equivalences.

**Proposition 3.2.6.** *Let $f \in \mathcal{C}\mathrm{at}(C, D)$ be a morphism. Then,* $\mathrm{Ex}^2\mathrm{N}(f)$ *is a weak equivalence if and only if* $\mathrm{N}(f)$ *is a weak equivalence.*

*Proof.* Dan Kan proved in [Kan57] that there is a natural transformation $e\colon \mathrm{Id}_{\mathrm{s}\mathcal{S}\mathrm{ets}} \Rightarrow \mathrm{Ex}$ such that $e_X$ is a weak equivalence for every $X \in \mathrm{Ob}(\mathrm{s}\mathcal{S}\mathrm{ets})$. In consequence, a morphism in s$\mathcal{S}$ets is a weak equivalence if and only if its extension is a weak equivalence. Indeed, let $f \in \mathrm{s}\mathcal{S}\mathrm{ets}(X, Y)$ and consider the following commutative diagram, derived from the naturality of $e$:

$$\begin{array}{ccc} X & \xrightarrow[\sim]{e_X} & \mathrm{Ex}X \\ \big\downarrow f & & \big\downarrow \mathrm{Ex}(f) \\ Y & \xrightarrow[\sim]{e_Y} & \mathrm{Ex}Y. \end{array}$$

By **MC2**, we have that $f$ is a weak equivalence if and only if $e_Y \circ f = \mathrm{Ex}(f) \circ e_X$ is a weak equivalence, and this is equivalent again by **MC2** to $\mathrm{Ex}(f)$ being a weak equivalence.

In consequence, $\mathrm{N}(f)$ is a weak equivalence if and only if $\mathrm{ExN}(f)$ is a weak equivalence, fact that happens if and only if $\mathrm{Ex}^2\mathrm{N}(f)$ is a weak equivalence. □

**Corollary 3.2.7.** *Categorical horns* $\tau\mathrm{Sd}^2(i_{n,k})\colon \tau\mathrm{Sd}^2\Delta[n,k] \to \tau\mathrm{Sd}^2\Delta[n]$ *are weak equivalences.*

*Proof.* We have that their nerves

$$\mathrm{N}\tau\mathrm{Sd}^2(i_{n,k})\colon \mathrm{N}\tau\mathrm{Sd}^2\Delta[n,k] = \mathrm{Sd}^2\Delta[n,k] \to \mathrm{N}\tau\mathrm{Sd}^2\Delta[n] = \mathrm{Sd}^2\Delta[n]$$

(see Remark 3.2.2) are weak equivalences since their geometric realization is a continuous function between two contractible spaces and hence a weak homotopy equivalence. □

Now, for understanding our next steps, we need to introduce a certain type of morphisms in $\mathcal{C}$at, as well as some previous notions.

**Definition 3.2.8.** Let $\mathcal{B}$ be a category and $\mathcal{A}$ a subcategory of $\mathcal{B}$. We say that $\mathcal{A}$ is a **sieve** of $\mathcal{B}$ if for every morphism $f \in \mathcal{B}(b, b')$ such that $b' \in \mathrm{Ob}(\mathcal{A})$, we have that $b \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathcal{A}(b, b')$. Dually, $\mathcal{A}$ is a **cosieve** of $\mathcal{B}$ if for every morphism $f \in \mathcal{B}(b, b')$ such that $b \in \mathrm{Ob}(\mathcal{A})$, we have that $b' \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathcal{A}(b, b')$. In particular, sieves and cosieves are full subcategories.

**Remark 3.2.9.** Let $\mathcal{B}$ be a category and $\mathcal{A}$ a sieve of $\mathcal{B}$. Then, $\mathcal{V}$, the full subcategory of $\mathcal{B}$ of objects not in $\mathrm{Ob}(\mathcal{A})$ is a cosieve of $\mathcal{B}$. Indeed, if $f \in \mathcal{B}(b, b')$ is such that $b \in \mathrm{Ob}(\mathcal{V})$, then $b' \in \mathrm{Ob}(\mathcal{V})$ (otherwise, $b' \in \mathrm{Ob}(\mathcal{A})$ and then by definition of cosieve $b \in \mathrm{Ob}(\mathcal{A})$ and we would get to a contradiction) and since $\mathcal{V}$ is a full subcategory, $f \in \mathcal{V}(b, b')$. Dually, if $\mathcal{A}$ is a cosieve of $\mathcal{B}$, $\mathcal{V}$ is a sieve of $\mathcal{B}$.

**Example 3.2.10.** For $\mathcal{I} = \{0 \to 1\}$, the subcategory 0 which has 0 as only object and $\mathrm{Id}_0$ as only morphism is a sieve of $\mathcal{I}$. Indeed, the only morphism in $\mathcal{I}$ which has 0 as codomain is $\mathrm{Id}_0$, which is a morphism in 0 and whose domain is also an object of 0. Dually, the subcategory 1 which has 1 as only object and $\mathrm{Id}_1$ as only morphism is a cosieve of $\mathcal{I}$.

Let us see the following characterization of sieves and cosieves.

**Lemma 3.2.11.** *Let $\mathcal{B}$ be a category and $\mathcal{A}$ a subcategory of $\mathcal{B}$. Then, $\mathcal{A}$ is a sieve of $\mathcal{B}$ if and only if there exists a functor $\chi\colon \mathcal{B} \to \mathcal{I}$ such that $\chi^{-1}(0) = \mathcal{A}$. Dually, $\mathcal{A}$ is a cosieve of $\mathcal{B}$ if and only if there exists a functor $\chi\colon \mathcal{B} \to \mathcal{I}$ such that $\chi^{-1}(1) = \mathcal{A}$.*

*Proof.* Assume that $\mathcal{A}$ is a sieve of $\mathcal{B}$ and let us define:

$$\chi\colon \mathcal{B} \to \mathcal{I}.$$
$$b \in \mathrm{Ob}(\mathcal{B}) \mapsto \begin{cases} 0 \text{ if } b \in \mathrm{Ob}(\mathcal{A}) \\ 1 \text{ if } b \notin \mathrm{Ob}(\mathcal{A}) \end{cases}$$
$$f \in \mathcal{B}(b,b') \mapsto \begin{cases} \mathrm{Id}_0 \text{ if } b' \in \mathrm{Ob}(\mathcal{A})(\Rightarrow b \in \mathrm{Ob}(\mathcal{A})) \\ \mathrm{Id}_1 \text{ if } b \notin \mathrm{Ob}(\mathcal{A})(\Rightarrow b' \notin \mathrm{Ob}(\mathcal{A})) \\ 0 \to 1 \text{ if } b \in \mathrm{Ob}(\mathcal{A}),\ b' \notin \mathrm{Ob}(\mathcal{A}) \end{cases}$$

This is obviously a functor and moreover $\chi^{-1}(0) = \mathcal{A}$.

Conversely, suppose that we have a functor $\chi\colon \mathcal{B} \to \mathcal{I}$ such that $\chi^{-1}(0) = \mathcal{A}$ and let us see that $\mathcal{A}$ is a sieve of $\mathcal{B}$. Let $f \in \mathcal{B}(b,b')$ be a morphism such that $b' \in \mathrm{Ob}(\mathcal{A}) = \chi^{-1}(0)$. Then, $\chi(f) \in \mathcal{I}(\chi(b),0)$, and since 0 is a cosieve of $\mathcal{I}$ (see Example 3.2.10), $\chi(b) = 0$ and $\chi(f) = \mathrm{Id}_0$, so $b \in \chi^{-1}(0) = \mathrm{Ob}(\mathcal{A})$ and $f \in \mathcal{A}(b,b')$. □

**Definition 3.2.12.** An inclusion $i \in \mathcal{C}\mathrm{at}(A,B)$ is a **Dwyer map** if $i$ embeds $A$ as a sieve in $B$ and factors as composite of inclusions $f \in \mathcal{C}\mathrm{at}(A,W)$, $j \in \mathcal{C}\mathrm{at}(W,B)$ such that:

- $f$ admits a **deformation retraction**, that is, there exists $r \in \mathcal{C}\mathrm{at}(W,A)$ such that $r \circ f = \mathrm{Id}_A$ together with a natural transformation $t\colon f \circ r \Rightarrow \mathrm{Id}_W$ such that for every $a \in \mathrm{Ob}(A)$ we have $t_{f(a)} = \mathrm{Id}_{f(a)}$.
- $j$ embeds $W$ as a cosieve of $B$.

**Remark 3.2.13.** Originally, Thomason's definition of Dwyer morphism asked $r$ to be right adjoint, but this restriction caused problems on proving that this model structure is left proper. The French mathematician Denis-Charles Cisinski noticed this in [Cis99] and gave this new definition of "pseudo-Dwyer morphism", which nowadays is much more useful, so we will omit the term "pseudo".

Next proposition shows us an example of Dwyer maps.

**Proposition 3.2.14.** *Let $L \subseteq K$ be an inclusion of simplicial complexes and $i\colon X_L \to X_K$ the inclusion between their associated simplicial sets. Then, $\tau\mathrm{Sd}^2 i\colon \tau\mathrm{Sd}^2 X_L \to \tau\mathrm{Sd}^2 X_K$ is a Dwyer map.*

*Proof.* First, by Remark 3.2.2, $\tau\mathrm{Sd}^2 X_K = i(\mathcal{X}(\mathcal{K}(\mathcal{X}(K))))$ is the small category associated to the poset of faces of the (old) barycentric subdivision of $K$, $\mathcal{K}(\mathcal{X}(K))$ and, in the same way, $\tau\mathrm{Sd}^2 X_L$ is the small category associated to the poset of faces of the old barycentric subdivision of $L$. Moreover, $\tau\mathrm{Sd}^2 i$ will be the functor induced by the inclusion between these posets.

Let us first see that $\tau\mathrm{Sd}^2 X_L$ is a sieve in $\tau\mathrm{Sd}^2 X_K$. A morphism in $\tau\mathrm{Sd}^2 X_K(\sigma,\sigma')$ will mean that $\sigma \subseteq \sigma'$. If $\sigma' \in \mathrm{Ob}(\tau\mathrm{Sd}^2 X_L)$, by definition of simplicial complex, $\sigma \in \mathrm{Ob}(\tau\mathrm{Sd}^2 L)$ and $\sigma \subseteq \sigma'$ in $\tau\mathrm{Sd}^2 X_L$ too.

Now, let $W = \{\sigma \in \mathcal{K}(\mathcal{X}(K))\colon \sigma \cap \mathcal{K}(\mathcal{X}(L)) \neq \emptyset\}$ be the subposet of $\mathcal{X}(\mathcal{K}(\mathcal{X}(K)))$ containing all the simplices of $\mathcal{K}(\mathcal{X}(K))$ that meet $\mathcal{K}(\mathcal{X}(L))$. Let us see that $i(W)$ is a cosieve in $\tau\mathrm{Sd}^2 X_K$ which obviously contains $\tau\mathrm{Sd}^2 X_L$. A morphism in $\tau\mathrm{Sd}^2 X_K(\sigma,\sigma')$ will mean that $\sigma \subseteq \sigma'$. If $\sigma \in \mathrm{Ob}(i(W))$, then $\sigma \cap \mathcal{K}(\mathcal{X}(L)) \neq \emptyset$, so $\sigma' \cap \mathcal{K}(\mathcal{X}(L)) \neq \emptyset$ and $\sigma \subseteq \sigma'$ in $W$ too.

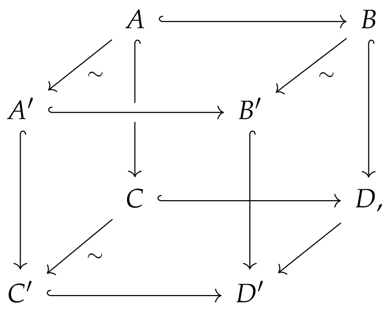
Finally, let us see that the inclusion $f\colon \tau\mathrm{Sd}^2 X_L \to i(W)$ admits a deformation retraction. Let us define

$$r\colon i(W) \to \tau\mathrm{Sd}^2 X_L,$$
$$\sigma \mapsto \sigma \cap \mathcal{K}(\mathcal{X}(L))$$
$$(\sigma \to \sigma') \mapsto (\sigma \cap \mathcal{K}(\mathcal{X}(L)) \to \sigma' \cap \mathcal{K}(\mathcal{X}(L)))$$

which obviously verifies $r \circ f = \mathrm{Id}_{\tau\mathrm{Sd}^2 X_L}$. Moreover, we can define $t\colon f \circ r \Rightarrow \mathrm{Id}_{i(W)}$ a natural transformation assigning to every $\sigma \in \mathrm{Ob}(i(W))$ the inclusion morphism $t_\sigma\colon i(r\sigma) = \sigma \cap \mathcal{K}(\mathcal{X}(L)) \to \sigma$. The naturality is clear, and in addition it is obvious that for $\sigma \in \mathrm{Ob}(\tau\mathrm{Sd}^2 X_L)$, $t_\sigma$ is the identity, which finishes our proof. □

Next, let us see a lemma that will be useful later.

**Lemma 3.2.15** (Gluing/Pasting lemma)**.** *Given a commutative cube in* s$\mathcal{S}$ets



*where the front and back faces are pushout diagrams with all arrows cofibrations, if* $A \xrightarrow{\sim} A'$, $B \xrightarrow{\sim} B'$ *and* $C \xrightarrow{\sim} C'$ *are weak equivalences, then so is* $D \to D'$.

*Proof.* First, it is worth to mention that this lemma holds for model categories in general if the objects involved in the diagram are cofibrant; see [And78, Lemma 2.5].

We will see a proof of it that uses some of the results we have already seen. If $\mathcal{D}$ is a small category and $\mathcal{C}$ is a *cofibrantly generated* model category (such as s$\mathcal{S}$ets), then the category $\mathcal{C}^{\mathcal{D}}$ admits a **projective model structure**, where given two objects $X, Y\colon \mathcal{D} \to \mathcal{C}$, a morphism $t\colon X \Rightarrow Y$ between them is a weak equivalence (or respectively, a fibration) if and only if for every $d \in \mathrm{Ob}(\mathcal{C})$, $t_d\colon X(d) \to Y(d)$ is a weak equivalence (or respectively, a fibration) in $\mathcal{C}$. Cofibrations will be those morphisms having the LLP with respect to acyclic fibrations; see [nLa22a]. In particular, a morphism $s\colon X \Rightarrow Y$ between two functors $X, Y\colon \mathcal{D} \to \mathcal{C}$ such that $s_d\colon X(d) \to Y(d)$ is a cofibration for every $d \in \mathrm{Ob}(\mathcal{D})$ will be a cofibration. Moreover, our model categories are Quillen-equivalent: $\mathrm{colim}\colon \mathcal{C}^{\mathcal{D}} \rightleftarrows \mathcal{C}\colon \mathrm{Const}$; see [nLa22b]. In particular, the colimit functor is a left Quillen functor, so it preserves acyclic cofibrations. Specifically, it carries trivial cofibrations between cofibrant objects to weak equivalences, so by Lemma 2.5.1, it preserves all weak equivalences between cofibrant objects.

In our case, we have $\mathcal{D} = \{c \leftarrow a \to b\}$ and the colimit will be the pushout. Our diagrams $C \leftarrow A \to B$ and $C' \leftarrow A' \to B'$ will be cofibrant objects since the only natural transformation from the initial diagram $\varnothing \leftarrow \varnothing \to \varnothing$ to them will have only cofibrations, so it will be a cofibration in $\mathcal{C}^{\mathcal{D}}$. Moreover, the natural transformation $s$ formed by the three weak equivalences in the formulation of the lemma will of course be a weak equivalence, so it is a weak equivalence between cofibrant objects and hence $\mathrm{colim}(s)$ (the induced morphism between the pushouts) will be a weak equivalence, as we wanted to prove. □

Now consider a pushout diagram in $\mathcal{C}$at:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & B \amalg_A C. \end{array}$$

Applying the nerve functor N we get the following commutative diagram in s$\mathcal{S}$ets:

$$\begin{array}{ccc} \mathrm{N}A & \xrightarrow{\mathrm{N}(f)} & \mathrm{N}C \\ \big\downarrow{\scriptstyle \mathrm{N}(i)} & & \big\downarrow{\scriptstyle \mathrm{N}(j)} \\ \mathrm{N}B & \xrightarrow{\mathrm{N}(g)} & \mathrm{N}(B \amalg_A C). \end{array}$$

Then, by the universal property of the pushout $NB \amalg_{NA} NC$ there is a morphism

$$\varphi\colon \mathrm{N}B \amalg_{\mathrm{N}A} \mathrm{N}C \to \mathrm{N}(B \amalg_A C).$$

Let us see that this morphism is very important for us.

**Proposition 3.2.16.** *In a pushout diagram as above in* $\mathcal{C}$at*, if $i$ is a Dwyer map, then $j$ is a Dwyer map and $\varphi$ is a weak equivalence.*

*Proof.* Since $i$ is a Dwyer map, we have that it embeds $A$ as a sieve in $B$ and it factors as a composite of $i_1 \in \mathcal{C}\mathrm{at}(A, W)$, $i_2 \in \mathcal{C}\mathrm{at}(W, B)$ such that $i_1$ admits a deformation retraction $r \in \mathcal{C}\mathrm{at}(W, A)$ (with associated natural transformation $t\colon i_1 \circ r \Rightarrow \mathrm{Id}_W$) and $i_2$ embeds $W$ as a cosieve of $B$.

First, we will see that $j$ is a Dwyer map. Let us consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle j_1} \\ W & \xrightarrow{h} & W \amalg_A C = W' \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_2} \\ B & \xrightarrow{g} & B \amalg_A C. \end{array}$$

We claim that $j$ is a Dwyer map with the factorization $j = j_2 \circ j_1$. First, let us see that $C$ is a sieve in $D$. Since $A$ is a sieve in $B$, by Lemma 3.2.11 there exists a functor $\chi\colon B \to \mathcal{I}$ such that $\chi^{-1}(0) = A$. We also have the functor $\mathrm{Const}(0)\colon C \to \mathcal{I}$, which makes the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \mathrm{Const}(0)} \\ B & \xrightarrow{\chi} & \mathcal{I}, \end{array}$$

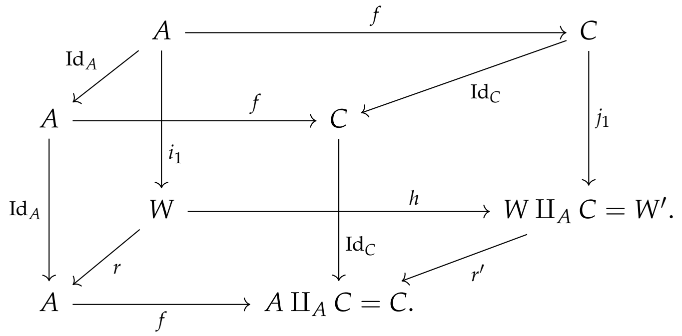
so by the universal property of the pushout, there exists a functor $\chi'\colon B \amalg_A C \to \mathcal{I}$ such that in particular $\chi'^{-1}(0) = C$, which by Lemma 3.2.11 shows what we wanted.

In a similar way, let us see that $W'$ is a cosieve in $D$. Since $W$ is a cosieve in $B$, by Lemma 3.2.11 there exists a functor $\mu\colon B \to \mathcal{I}$ such that $\mu^{-1}(1) = W$. We also have the functor $\mathrm{Const}(1)\colon C \to \mathcal{I}$, which makes the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \mathrm{Const}(1)} \\ B & \xrightarrow{\mu} & \mathcal{I}, \end{array}$$

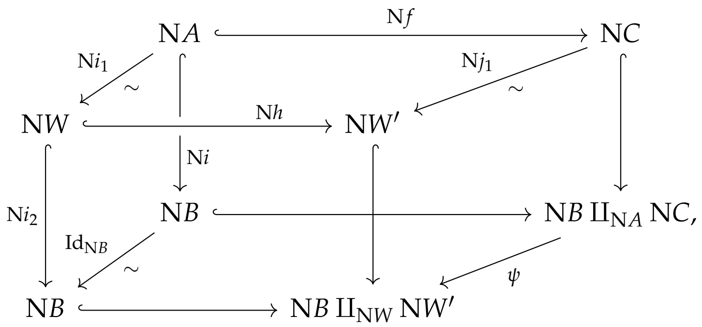
so by the universal property of the pushout, there exists a functor $\mu'\colon B \amalg_A C \to \mathcal{I}$ such that in particular $\mu'^{-1}(1) = W'$, which by Lemma 3.2.11 shows what we wanted.

Moreover, we have a morphism $r' = r \amalg_{\mathrm{Id}_A} \mathrm{Id}_C \colon W' \to A \amalg_A C = C$ such that $r' \circ j_1 = \mathrm{Id}_C \circ \mathrm{Id}_C = \mathrm{Id}_C$:




Similarly, we have a natural transformation $t' = t \amalg_{\mathrm{Id}} \mathrm{Id} \colon j_1 \circ r' \Rightarrow \mathrm{Id}'_W$ such that for $x \in \mathrm{Ob}(C)$, $t'_{j_1(x)} = \mathrm{Id}_{j_1(x)}$, which finishes the first part of our proof.

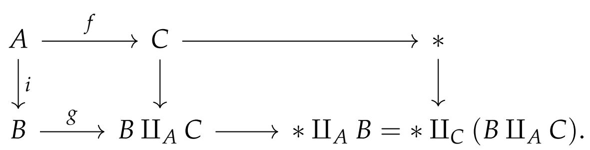
Next, we will see that $\varphi$ is a weak equivalence. Notice that that by Remark 1.1.13, having a natural transformation $t \colon i_1 \circ r \Rightarrow \mathrm{Id}_W$ is equivalent to having a functor $H \colon W \times \{0 \to 1\} \to W$ such that $H(w, 0) = r(i_1(w))$ and $H(w, 1) = w$ for every object $w \in \mathrm{Ob}(W)$ and $H(f, \mathrm{Id}_0) = r(i_1(f))$ and $H(f, \mathrm{Id}_1) = G(f)$ for every morphism $f$ in $W$. The geometric realization of the nerve of this functor $|\mathrm{N}H| \colon |\mathrm{N}W| \times [0,1] \to |\mathrm{N}W|$ is a continuous map such that $|\mathrm{N}H|(w,0) = |\mathrm{N}r|(|\mathrm{N}i_1|(w))$ and $|\mathrm{N}H|(w,1) = w$ for every $w \in |\mathrm{N}W|$, that is, a homotopy between $|\mathrm{N}r| \circ |\mathrm{N}i_1|$ and $\mathrm{Id}_{|\mathrm{N}W|}$. In addition, we have that $|\mathrm{N}i_1| \circ |\mathrm{N}r| = |\mathrm{N}(i_1 \circ r)| = |\mathrm{N}(\mathrm{Id}_A)| = \mathrm{Id}_{|\mathrm{N}A|}$. In consequence, $|\mathrm{N}i_1| \colon |\mathrm{N}A| \to |\mathrm{N}W|$ is a homotopy equivalence (with homotopy inverse $|\mathrm{N}r|$) between both fibrant and cofibrant objects (CW-complexes), and hence by Lemma 2.3.37 a weak equivalence. Therefore, $\mathrm{N}i_1 \colon \mathrm{N}A \to \mathrm{N}W$ is a weak equivalence. Similarly, $\mathrm{N}j_1 \colon \mathrm{N}C \to \mathrm{N}W'$ is also a weak equivalence. We then have the following commutative cube




where we can apply the Gluing lemma 3.2.15 to claim that the canonical morphism $\psi$ is a weak equivalence.

Now let $V$ be the full subcategory of objects of $B$ not in $A$, which is a cosieve in $B$ since $A$ is a sieve by Remark 3.2.9. Notice that $W \cap V$ is a cosieve in $W$. Indeed, if $f \in W(w, w') \subseteq B(w, w')$ is such that $w \in \mathrm{Ob}(W \cap V) \subseteq \mathrm{Ob}(V)$, since $V$ is a cosieve in $B$ we have that $w' \in \mathrm{Ob}(V)$ and $f \in V(w, w')$ so $w' \in \mathrm{Ob}(W \cap V)$ and $f \in W \cap V(w, w')$. By symmetry, $W \cap V$ is a cosieve in $V$. For the same reasons, if we put $V'$ the full subcategory of objects of $B \amalg_A C$ not in $C$, we will have a cosieve in $B \amalg_A C$ and $W' \cap V'$ will be a cosieve in both $V'$ and $W'$.

Take a look at the following diagram (see Remark 1.2.10):



But observe that $* \amalg_A B$ is the category $V$ with one more object $*$ and a morphism $* \to v$ if and only if $v \in \mathrm{Ob}(V \cap W)$. Indeed, since $A$ is a sieve of $B$, morphisms with codomain in $A$ are collapsed; since $V$ is a cosieve of $B$, morphisms with domain in $V$ go to themselves; hence, only morphisms with domain in $A \subseteq W$ and codomain in $V$ will be of our interest, but since $W$ is a cosieve in $B$, the codomain will also have to be in $W$, so there will only be morphisms $* \to v$ if $v \in \mathrm{Ob}(V \cap W)$. Conversely, given $v \in \mathrm{Ob}(V \cap W)$ we have $t_v \colon i_1(r(v)) = r(v) \to v$ with $r(v) \in A$, so in the pushout we will have a morphism $* \to v$. In an analogous way, $* \amalg_C (B \amalg_A C)$ is the category $V'$ with one more object $*$ and a morphism $* \to v$ if and only if $v \in \mathrm{Ob}(V' \cap W')$. This shows $V \cong V'$, $V \cap W \cong V' \cap W'$.

Notice also that $B = B \amalg_{W \cap V} V$. Indeed, the inclusions between this categories commute:

$$\begin{array}{ccc} W \cap V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow{\scriptstyle v} \\ W & \xrightarrow{\;w\;} & B. \end{array}$$

Moreover, let $C \in \mathrm{Ob}(\mathcal{C}\mathrm{at})$ and $f \colon V \to C$, $g \colon W \to C$ functors such that the following diagram commutes (or equivalently, such that $f(x) = g(x)$ for every $x \in \mathrm{Ob}(W \cap V)$ and $f(z) = g(z)$ for every $z \in W \cap V(x, x')$):

$$\begin{array}{ccc} W \cap V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow{\;g\;} & C. \end{array}$$

Then, there is a well defined functor

$$\begin{aligned} h \colon B &\to C \\ b \in \mathrm{Ob}(B) &\mapsto \begin{cases} g(b) \text{ if } b \in A \subseteq W \\ f(b) \text{ if } b \in V \end{cases} \\ h \in B(b, b') &\mapsto \begin{cases} g(h) \text{ if } b' \in A \subseteq W \\ f(h) \text{ if } b \in V \end{cases} \end{aligned}$$

such that $h \circ v = f$, $h \circ w = g$.

In a similar way, $\mathrm{N}B = \mathrm{N}W \amalg_{\mathrm{N}(W \cap V)} \mathrm{N}V$. Indeed, the nerves of the inclusions commute:

$$\begin{array}{ccc} \mathrm{N}(W \cap V) & \longrightarrow & \mathrm{N}V \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{N}(v)} \\ \mathrm{N}W & \xrightarrow{\;\mathrm{N}(w)\;} & \mathrm{N}B. \end{array}$$

Moreover, let $S \in \mathrm{Ob}(\mathrm{s}\mathcal{S}\mathrm{ets})$ and $f \colon \mathrm{N}V \to S$, $g \colon \mathrm{N}W \to S$ morphisms such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{N}(W \cap V) & \longrightarrow & \mathrm{N}V \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathrm{N}W & \xrightarrow{\;g\;} & S. \end{array}$$

Then, there is a well defined morphism $h \colon \mathrm{N}B \to S$ with

$$\begin{aligned} h_n \colon (\mathrm{N}B)_n &\to S_n \\ (b_0 \to \ldots \to b_n) &\mapsto \begin{cases} g(b_0 \to \ldots \to b_n) \text{ if } b_n \in A \subseteq W \\ f(b_0 \to \ldots \to b_n) \text{ if } b_0 \in V \end{cases} \end{aligned}$$

which obviously commutes with face and degeneracy maps and such that $h \circ \mathrm{N}(v) = f$, $h \circ \mathrm{N}(w) = g$. In a resembling way, $\mathrm{N}(B \amalg_A C) = \mathrm{N}W' \amalg_{\mathrm{N}(W' \cap V')} \mathrm{N}V'$.

Putting all this together, we get that

$$\begin{aligned} \mathrm{N}B \amalg_{\mathrm{N}W} \mathrm{N}W' &= \mathrm{N}V \amalg_{\mathrm{N}(V\cap W)} \mathrm{N}W \amalg_{\mathrm{N}W} \mathrm{N}W' = \mathrm{N}V \amalg_{\mathrm{N}(V\cap W)} \mathrm{N}W' \\ &= \mathrm{N}V' \amalg_{\mathrm{N}(V'\cap W')} \mathrm{N}W' = \mathrm{N}(B \amalg_A C), \end{aligned}$$

and we can conclude that the canonical morphism $\varphi = \psi$ is a weak equivalence. □

**Corollary 3.2.17.** *In a pushout diagram as above in $\mathcal{C}$at with $i$ a Dwyer map, if $i$ is a weak equivalence then $j$ is also a weak equivalence.*

*Proof.* Pay attention to the diagram derived of the universal property of the pushout:

$$\begin{array}{ccccc} \mathrm{N}A & \xrightarrow{\mathrm{N}(f)} & \mathrm{N}C & & \\ \downarrow{\scriptstyle \mathrm{N}(i)} & & \downarrow{\scriptstyle \alpha} & \searrow{\scriptstyle \mathrm{N}(j)} & \\ \mathrm{N}B & \longrightarrow & \mathrm{N}B \amalg_{\mathrm{N}A} \mathrm{N}C & & \\ & \searrow{\scriptstyle \mathrm{N}(g)} & & \searrow{\scriptstyle \varphi} & \\ & & & & \mathrm{N}(B \amalg_A C). \end{array}$$

Since $i$ is a Dwyer map, in particular it is an inclusion, so $\mathrm{N}(i)$ will be a monomorphism and hence a cofibration in s$\mathcal{S}$ets (see Remark 3.1.16). Moreover, since $i$ is a weak equivalence, by Proposition 3.2.4, $\mathrm{N}(i)$ will also be a weak equivalence. Since acyclic cofibrations are stable under cobase change by Corollary 2.2.5, $\alpha$ will be an acyclic cofibration, and in particular a weak equivalence. Moreover, by Proposition 3.2.16 $\varphi$ will also be a weak equivalence, so in consequence $\mathrm{N}(j)$ will be a weak equivalence, which means again by Proposition 3.2.4 that $j$ is a weak equivalence in $\mathcal{C}$at. □

Let us now see some properties of the cofibrations in $\mathcal{C}$at.

**Lemma 3.2.18.** *Let $f \in \mathrm{s}\mathcal{S}\mathrm{ets}(X, Y)$ be a cofibration. Then $\tau\mathrm{Sd}^2 f \in \mathcal{C}\mathrm{at}(\tau\mathrm{Sd}^2 X, \tau\mathrm{Sd}^2 Y)$ is a cofibration.*

*Proof.* We need to see that $\tau\mathrm{Sd}^2 f$ has the LLP with respect to trivial fibrations. Let us then consider the following lifting problem:

$$\begin{array}{ccc} \tau\mathrm{Sd}^2 X & \longrightarrow & C \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2 f} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2 Y & \longrightarrow & D, \end{array}$$

where $p$ is a trivial fibration in $\mathcal{C}$at (that is, $\mathrm{Ex}^2\mathrm{N}p$ is a trivial fibration in s$\mathcal{S}$ets). By adjointness, we have the following diagram:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Ex}^2\mathrm{N}C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Ex}^2\mathrm{N}p} \\ Y & \longrightarrow & \mathrm{Ex}^2\mathrm{N}D, \end{array}$$

which, since $f$ is a cofibration and $\mathrm{Ex}^2\mathrm{N}p$ is an acyclic fibration, has a lift $h\colon Y \to \mathrm{Ex}^2\mathrm{N}C$, which by adjointess gives us our desired lift $h^\flat\colon \tau\mathrm{Sd}^2 Y \to C$. □

**Corollary 3.2.19.** *The categorical horns are trivial cofibrations.*

*Proof.* They are cofibrations by Proposition 3.2.18 and weak equivalences by Corollary 3.2.7. □

**Lemma 3.2.20.** *The following hold:*

1. *If $A \to B$ is a cofibration in $\mathcal{C}$at and $A \to C$ is any morphism, the induced morphism into the pushout $C \to B \amalg_A C$ is a cofibration.*
2. *If $A_0 \to A_1 \to A_2 \to \dots$ is a sequence of cofibrations in $\mathcal{C}$at, then the morphism $A_0 \to \mathrm{colim}_n(A_n)$ is a cofibration. If in addition each morphism is a weak equivalence, so does $A_0 \to \mathrm{colim}_n(A_n)$.*
3. *If $A_i \to B_i$ is a cofibration in $\mathcal{C}$at, then their coproduct $\amalg_i A_i \to \amalg_i B_i$ is a cofibration. If moreover each $A_i \to B_i$ is a weak equivalence, then $\amalg_i A_i \to \amalg_i B_i$ is also a weak equivalence.*

*Proof.* The first assertion holds by Proposition 2.2.4: if $A \to B$ is a cofibration, it has the LLP with respecto to trival fibrations, and in consequence its cobase change will have the same property, so it will also be a cofibration.

The first part of the second assertion holds by Lemma 2.2.8: if $A_n \to A_{n+1}$ is a cofibration, it has the LLP with respect to trivial fibrations, and in consequence $A_0 \to \mathrm{colim}_n A_n$ will have the same property, so it will also be a cofibration. Moreover, if each $A_n \to A_{n+1}$ is a weak equivalence, by Propositon 3.2.6 we have a sequence of weak equivalences $\mathrm{N}A_n \to \mathrm{N}A_{n+1}$, so by Remark 3.1.18 the canonical morphism $\mathrm{N}A_0 \to \mathrm{colim}_n \mathrm{N}A_n$ is also a weak equivalence. But $\mathrm{colim}_n \mathrm{N}A_n = \mathrm{N}(\mathrm{colim}_n A_n)$ because of how sequential colimits work for simplicial sets and categories and because of the definition of the nerve functor. Consequently, by Proposition 3.2.6 the canonical morphism $A_0 \to \mathrm{colim}_n A_n$ is also a weak equivalence.

The first part of the third assertion holds by Lemma 2.2.7: if $A_i \to B_i$ is a cofibration, it has the LLP with respect to trivial fibrations, and in consequence $\amalg_i A_i \to \amalg_i B_i$ will have the same property, so it will also be a cofibration. Moreover, if each $A_i \to B_i$ is a weak equivalence, by Proposition 3.2.6 $\mathrm{N}A_i \to \mathrm{N}B_i$ will also be weak equivalences, so by Remark 3.1.18 its coproduct will also be a weak equivalence. But by construction, the coproduct of the nerves will be the nerve of the coproducts and then our coproduct will be a weak equivalence. □

Now we have all the ingredients in order to prove our aimed theorems.

**Theorem 3.2.21.** *There is a model category structure on the category of small categories in which the weak equivalences, cofibrations and fibrations are as in Definition 3.2.1.*

*Proof.* The axiom **MC1** follows from Proposition 1.4.13.

Let us now check the closure under composition and the contention of the identity of the different classes of morphisms. First of all, for weak equivalences and fibrations it holds because it does for simplicial sets. Now, for cofibrations, let us see that the identity morphisms are cofibrations. If we have the following lifting problem ($p \circ f = g \circ \mathrm{Id}_A$), with $p$ an acyclic fibration

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & Y, \end{array}$$

it is clear that there is a morphism $f\colon A \to X$ such that $p \circ f = g$ and $f \circ \mathrm{Id}_A = f$, so the identities are cofibrations. Let us now check the closure under composition. Assume that $i\colon A \to B$ and $j\colon B \to C$

are cofibrations, and let us see that $j \circ i$ is a cofibration, that is, that it has the left lifting property with respect to trivial fibrations. Consider then the following lifting problem, with $p$ an acyclic fibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle j\circ i} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & Y \end{array}.$$

Then, we also have the following lifting problem, with $p$ an acyclic fibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g\circ j} & Y. \end{array}$$

Since $i$ is a cofibration, there is $u\colon B \to X$ such that $p \circ u = g \circ j$ and $u \circ i = f$. This yields another lifting problem, with $p$ an acyclic fibration:

$$\begin{array}{ccc} B & \xrightarrow{u} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & Y. \end{array}$$

Since $j$ is a cofibration, there is $w\colon C \to X$ such that $p \circ w = g$ and $w \circ j = u$. In conclusion, there is $w\colon C \to X$ such that $p \circ w = g$ and $w \circ j \circ i = u \circ i = f$, which is what we wanted to see. Hence, the composition of cofibrations is also a cofibration.

The second axiom **MC2** holds because it does for simplicial sets: if we have $f\colon A \to B$, $g\colon B \to C$ such that two out the three $f$, $g$, $g \circ f$ are weak equivalences, then two out of the three $Ex^2Nf$, $Ex^2Ng$, $Ex^2N(f \circ g) = Ex^2Nf \circ Ex^2Ng$ are weak equivalences, so by **MC2** for s$\mathcal{S}$ets, the third one will also be a weak equivalence and hence so will be the third of our initial morphisms.

The third axiom **MC3** holds for weak equivalences and fibrations because it does for simplicial sets, and for cofibrations because of Lemma 2.3.20.

Let us now prove that this structure verifies the fifth axiom **MC5** because we will use it for proving **MC4** next. Let then $f\colon A \to B$ be any morphism in $\mathcal{C}$at.

Let us first see that we can factor $f = p \circ i$ with $p$ a fibration and $i$ a trivial cofibration.

First, consider the set $I$ of all diagrams involving the categorical horns

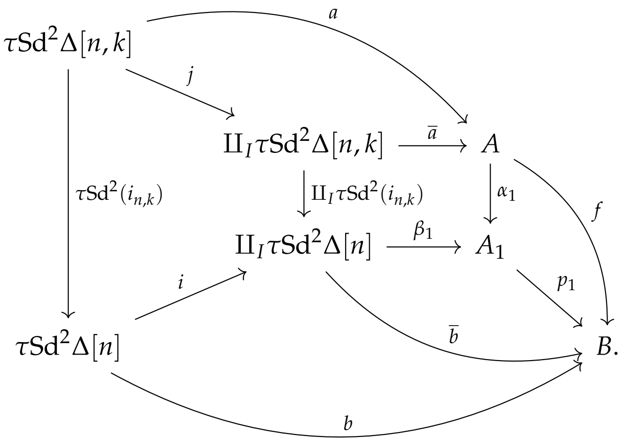
$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{a} & A \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle f} \\ \tau\mathrm{Sd}^2\Delta[k] & \xrightarrow{b} & B \end{array}$$

and take the coproducts $\amalg_I \tau\mathrm{Sd}^2\Delta[n,k]$, $\amalg_I \tau\mathrm{Sd}^2\Delta[n]$. The morphism

$$\amalg_I \tau\mathrm{Sd}^2(i_{n,k})\colon\ \amalg_I \tau\mathrm{Sd}^2\Delta[n,k] \to \amalg_I \tau\mathrm{Sd}^2\Delta[n]$$

is a trivial cofibration because categorical horns are trivial cofibrations by Corollary 3.2.19 and the coproduct of trivial cofibrations is a trivial cofibration by Lemma 3.2.20. Moreover, it is a Dwyer map by Proposition 3.2.14, since the coproduct of the inclusions will also be a inclusion of simplicial sets coming from ordered simplicial complexes.

Now let $\overline{a}\colon\ \amalg_I \tau\mathrm{Sd}^2\Delta[n,k] \to A$ be the morphism induced by the universal property of the coproduct and take the pushout $A_1 = A \amalg_{\amalg_I \tau\mathrm{Sd}^2\Delta[n,k]} \amalg_I \tau\mathrm{Sd}^2\Delta[n]$. Then the canonical morphism $\alpha_1\colon A \to A_1$ is a cofibration by Lemma 3.2.20, a weak equivalence by Corollary 3.2.17 and a Dwyer map by Proposition 3.2.16. Now let $\overline{b}\colon \amalg_I \tau\mathrm{Sd}^2\Delta[n] \to B$ be the morphism induced by the universal property of the coproduct. We are in the following situation:




It is clear that $f = p_1 \circ \alpha_1$. Moreover, by construction, any of the initial morphisms $b\colon \tau\mathrm{Sd}^2\Delta[n] \to B$ lifts to $\beta_1 \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_1$ extending $\alpha_1 \circ a\colon \tau\mathrm{Sd}^2\partial\Delta[n] \to A_1$:

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{\alpha_1\circ a} & A_1 \\ {\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})}\downarrow & \nearrow_{\beta_1\circ i} & \downarrow{\scriptstyle p_1} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{b} & B. \end{array}$$

Notice also that $\alpha_1$ has the LLP with respect to fibrations. Indeed, consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{c} & X \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle w} \\ A_1 & \xrightarrow{d} & Y, \end{array}$$

where $w\colon X \to Y$ is a fibration (that is, has the RLP with respect to categorical horns by Proposition 3.2.4). For every commutative diagram in $I$, consider the composite commutative diagram

$$\begin{array}{ccccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{a} & A & \xrightarrow{c} & X \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle w} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{\beta_1\circ i} & A_1 & \xrightarrow{d} & Y. \end{array}$$

Since $w$ is a fibration, there is a lift $l\colon \tau\mathrm{Sd}^2\Delta[n] \to X$ such that $w \circ l = d \circ \beta_1 \circ i$ and $l \circ \tau\mathrm{Sd}^2(i_{n,k}) = c \circ a$. Let $L\colon \amalg_I \tau\mathrm{Sd}^2\Delta[n] \to X$ be the morphism induced by these lifts in the coproduct, so that $L \circ i = l$. By the universal property of the pushout, there is $k\colon A_1 \to X$ such that $k \circ \alpha_1 = c$ and $k \circ \beta_1 = L$. This is the lift we were looking for, since $k \circ \alpha_1 = c$ is clear and moreover $w \circ k = d$ by the universal property of the pushout and the fact that

$$w \circ k \circ \alpha_1 = w \circ c = d \circ \alpha_1,$$
$$w \circ k \circ \beta_1 = w \circ L = d \circ \beta_1,$$

with this last equality derived from the universal property of the coproduct and the fact that

$$w \circ L \circ i = w \circ l = d \circ \beta_1 \circ i.$$

Now we can apply all this method to the morphism $p_1\colon A_1 \to B$, obtaining a factorization $p_1 = p_2 \circ \alpha_2$, so $f = p_2 \circ \alpha_2 \circ \alpha_1$, and such that $\alpha_2$ has the LLP with respect to fibrations and by construction, any of the initial morphisms $b\colon \tau\mathrm{Sd}^2\Delta[n] \to B$ lifts to a morphism $\beta_2 \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_2$ extending $\alpha_2 \circ a\colon \tau\mathrm{Sd}^2\Delta[n,k] \to A_1$:

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{\alpha_2\circ a} & A_2 \\ {\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})}\downarrow & \nearrow{\scriptstyle \beta_2\circ i} & \downarrow{\scriptstyle p_2} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{b} & B. \end{array}$$

Repeating the operation countably many times we obtain a sequence of cofibrations and weak equivalences

$$A \overset{\alpha_1}{\to} A_1 \overset{\alpha_2}{\to} A_2 \overset{\alpha_3}{\to} \dots$$

and a family of morphisms $p_i\colon A_i \to B$ such that $p_{i+1} \circ \alpha_{i+1} = p_i$. Then, for $A_\infty = \mathrm{colim}_n A_n$ we have a morphism $p\colon A_\infty \to B$ which is a fibration by Proposition 3.2.4. Indeed, given a commutative diagram

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{\widetilde{a}} & A_\infty \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{b} & B, \end{array}$$

since $\tau\mathrm{Sd}^2\Delta[n,k]$ is a finite category, by the construction of the sequential colimit we get that $\widetilde{a}$ factors through some $A_m$, that is, $\widetilde{a} = in_m \circ a$ for some $a\colon \tau\mathrm{Sd}^2\Delta[n,k] \to A_m$, which yields the commutative diagram

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\Delta[n,k] & \xrightarrow{a} & A_m \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_{n,k})} & & \downarrow{\scriptstyle p\circ in_m = p_m} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{b} & B, \end{array}$$

which as observed has a lift $\beta_m \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_m$. In consequence, we have a lift for our initial problem $in_m \circ \beta_m \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_\infty$.

Moreover, $i\colon A \to A_\infty$ is a cofibration and a weak equivalence by Lemma 3.2.20, and in addition has the LLP with respect to fibrations by Lemma 2.2.7. Hence, we have finally reached our factorization (notice that $f = p \circ i$ by construction).

Now let us see that we can factor $f = p \circ i$ with $p$ a trivial fibration and $i$ a cofibration.

First, consider the set $I$ of all diagrams of the form

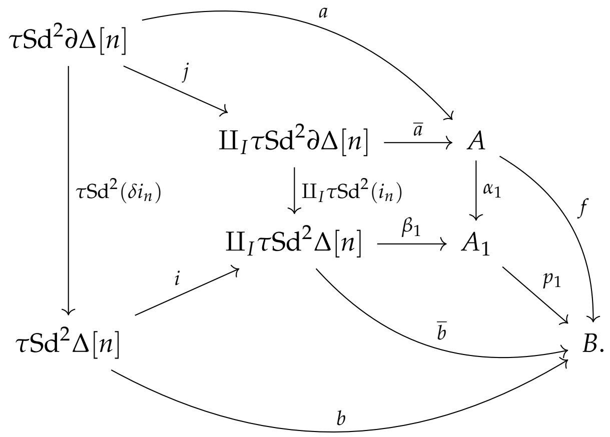
$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \xrightarrow{a} & A \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_n)} & & \downarrow{\scriptstyle f} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{b} & B \end{array}$$

and take the coproducts $\amalg_I \tau\mathrm{Sd}^2\partial\Delta[n]$, $\amalg_I \tau\mathrm{Sd}^2\Delta[n]$. The morphism

$$\amalg_I \tau\mathrm{Sd}^2(i_n)\colon\ \amalg_I \tau\mathrm{Sd}^2\partial\Delta[n] \to \amalg_I \tau\mathrm{Sd}^2\Delta[n]$$

is a cofibration by Lemma 3.2.18 since the coproduct of the inclusions will also be a inclusion of simplicial sets coming from ordered simplicial complexes.

Now let $\bar{a}\colon\ \amalg_I \tau\mathrm{Sd}^2\partial\Delta[n] \to A$ be the morphism induced by the universal property of the coproduct and take the pushout $A_1 = A \amalg_{\amalg_I \tau\mathrm{Sd}^2\partial\Delta[n]} \amalg_I \tau\mathrm{Sd}^2\Delta[n]$. Then the canonical morphism $\alpha_1\colon A \to A_1$ is a cofibration by Lemma 3.2.20. Now let $\overline{b}\colon \amalg_I \tau\mathrm{Sd}^2\Delta[n] \to B$ be the morphism induced by the universal property of the coproduct. We are in the following situation:




It is clear that $f = p_1 \circ \alpha_1$. Moreover, by construction, any of the initial morphisms $b\colon \tau\mathrm{Sd}^2\Delta[n] \to B$ lifts to $\beta_1 \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_1$ extending $\alpha_1 \circ a\colon \tau\mathrm{Sd}^2\partial\Delta[n] \to A_1$:

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \xrightarrow{\alpha_1\circ a} & A_1 \\ {\scriptstyle \tau\mathrm{Sd}^2(i_n)}\downarrow & \overset{\beta_1\circ i}{\nearrow} & \downarrow{\scriptstyle p_1} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{\ b\ } & B. \end{array}$$

Now we can apply all this method to the morphism $p_1\colon A_1 \to B$, obtaining a factorization $p_1 = p_2 \circ \alpha_2$, so $f = p_2 \circ \alpha_2 \circ \alpha_1$, and such that by construction, any of the initial morphisms $b\colon \tau\mathrm{Sd}^2\Delta[n] \to B$ lifts to a morphism $\beta_2 \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_2$ extending $\alpha_2 \circ a\colon \tau\mathrm{Sd}^2\Delta[n,k] \to A_1$:

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \xrightarrow{\alpha_2\circ a} & A_2 \\ {\scriptstyle \tau\mathrm{Sd}^2(i_n)}\downarrow & \overset{\beta_2\circ i}{\nearrow} & \downarrow{\scriptstyle p_2} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{\ b\ } & B. \end{array}$$

Repeating the operation countably many times we obtain a sequence of cofibrations

$$A \overset{\alpha_1}{\to} A_1 \overset{\alpha_2}{\to} A_2 \overset{\alpha_3}{\to} \dots$$

and a family of morphisms $p_i\colon A_i \to B$ such that $p_{i+1} \circ \alpha_{i+1} = p_i$. Then, for $A_\infty = \mathrm{colim}_n A_n$ we have a morphism $p\colon A_\infty \to B$ which is a trivial fibration by Proposition 3.2.5. Indeed, given a commutative diagram

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \xrightarrow{\ \widetilde{a}\ } & A_\infty \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_n)} & & \downarrow{\scriptstyle p} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{\ b\ } & B, \end{array}$$

since $\tau\mathrm{Sd}^2\Delta[n]$ is a finite category, we get that $\widetilde{a}$ factors through some $A_m$, that is, $\widetilde{a} = in_m \circ a$ for some $a\colon \tau\mathrm{Sd}^2\partial\Delta[n] \to A_m$, which yields the commutative diagram

$$\begin{array}{ccc} \tau\mathrm{Sd}^2\partial\Delta[n] & \xrightarrow{\ a\ } & A_m \\ \downarrow{\scriptstyle \tau\mathrm{Sd}^2(i_n)} & & \downarrow{\scriptstyle p\circ in_m = p_m} \\ \tau\mathrm{Sd}^2\Delta[n] & \xrightarrow{\ b\ } & B, \end{array}$$

which as observed has a lift $\beta_m \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_m$. In consequence, we have a lift for our initial problem $in_m \circ \beta_m \circ i\colon \tau\mathrm{Sd}^2\Delta[n] \to A_\infty$.

Moreover, $i\colon A \to A_\infty$ is a cofibration by Lemma 3.2.20. Hence, we have finally reached our factorization (notice that $f = p \circ i$ by construction).

There is only left to check the forth axiom **MC4**. Half of it holds by definition of cofibration, so let us prove the other half. Let us consider the following lifting problem

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow{b} & Y, \end{array}$$

where $k$ is a trivial cofibration and $q$ a fibration. By the proof of **MC5**(i) above, we know that we can factor $k$ as $k = p \circ i$, with $i$ a trivial cofibration and $p$ a fibration (which will also be trivial by **MC2** and the fact that $i$ and $k$ are weak equivalences). Now consider the following lifting problem:

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle q} \\ A_\infty & \xrightarrow{b\circ p} & Y. \end{array}$$

As observed earlier, $i$ has the LLP with respect to fibrations, so there is a morphism $h\colon A_\infty \to X$ such that $q \circ h = b \circ p$ and $h \circ i = a$. Next, consider the following lifting problem:

$$\begin{array}{ccc} A & \xrightarrow{i} & A_\infty \\ \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{\mathrm{Id}_B} & B. \end{array}$$

Since $k$ is a cofibration and $p$ is an acyclic fibration, by definition of cofibration there exists $s\colon B \to A_\infty$ such that $p \circ s = \mathrm{Id}_B$ and $s \circ k = i$. In consequence, we have $h \circ s\colon B \to X$ verifying

$$q \circ h \circ s = b \circ p \circ s = b \circ \mathrm{Id}_B = b,$$
$$h \circ s \circ k = h \circ i = a,$$

so our initial diagram has a lift.

We have finished the checking of all the axioms of model categories, and with it, our proof. □

From this theorem we can extract a corollary which will be useful in a bit.

**Corollary 3.2.22.** *Let $f\colon A \to B$ be a cofibration in $\mathcal{C}$at, and $f = p \circ i$ the factorization described above, with $i\colon A \to A_\infty$ a cofibration and $p\colon A_\infty \to B$ a trivial fibration. Then $f$ is a retract of $i$.*

*Proof.* This holds by the first part of Proposition 2.2.6 since $f$ has the LLP with respect to the trivial fibration $p$ by definition of cofibration. □

**Theorem 3.2.23.** *The category of small categories and the category of simplicial sets with the described model structures, are Quillen equivalent under the adjunction*

$$\tau\mathrm{Sd}^2\colon \mathrm{s}\mathcal{S}\mathrm{ets} \rightleftarrows \mathcal{C}\mathrm{at}\colon \mathrm{Ex}^2\mathrm{N}.$$

*Proof.* First, we need to see that we are dealing with a Quillen adjunction. Indeed, $\tau\mathrm{Sd}^2$ preserves cofibrations by Lemma 3.2.18 and $\mathrm{Ex}^2\mathrm{N}$ preserves fibrations by definition of fibration in $\mathcal{C}$at.

The fact that it is indeed a Quillen equivalence was proved by the German topologist Rudolf Fritsch and the American mathematician Dana May Latch in [FL81]. □

## 3.3. The Raptis-Thomason model structure for posets

Previous theorems tell us that now we have the option of studying the homotopy theory of spaces from the point of view of small categories. But we can refine this result with just a couple of steps in order to get even another model for the homotopy category of spaces: the category of partially ordered sets. Thomason already mentioned some of these results in his paper [Tho80], and the Greek mathematician George Raptis analyzed them more deeply in [Rap10].

**Lemma 3.3.1.** *The following hold:*

1. *Consider a sequence* $i(P_0) \to i(P_1) \to \dots$ *in* $\mathcal{C}\mathrm{at}$ *with* $P_j$ *a poset for every* $j \geq 0$. *Then, the sequencial colimit* $\mathrm{colim}_n i(P_n)$ *is equal to* $i(P)$ *for some poset* $P$.
2. *Let* $P$ *be a poset. Then any subcategory* $\mathcal{C}$ *of* $i(P)$ *is equal to* $i(Q)$ *for some* $Q$.
3. *Let* $i(P_A)$, $i(P_B)$, $i(P_C) \in \mathrm{Ob}(\mathcal{C}\mathrm{at})$ *with* $P_A, P_B, P_C \in \mathrm{Ob}(\mathcal{P}\mathrm{os})$ *and* $f\colon i(P_A) \to i(P_B)$ *a Dwyer map. Then, for any morphism* $i(P_A) \to i(P_C)$, *the pushout* $i(P_B) \amalg_{i(P_A)} i(P_C)$ *equals* $i(P)$ *for some poset* $P$.

*Proof.* The first assertion follows from the fact that the inclusions of posets have at most one morphism between two objects and then their sequential colimit by construction (see Proposition 1.4.35) has at most one morphism between two objects, so it will be an inclusion of some poset (see Remark 1.4.39).

The second assertion follows from the fact that the inclusions of posets have at most one morphism between two objects and then any subcategory will have the same property, so it will also be the inclusion of some poset (see Remark 1.4.39).

For the third assertion, let us set the same notation as in Proposition 3.2.16. We have that $i(P_C)$ is a sieve of $i(P_B) \amalg_{i(P_A)} i(P_C)$ and that $W'$ and $V'$ are cosieves of $i(P_B) \amalg_{i(P_A)} i(P_C)$. Moreover, $i(P_C)$ is the inclusion of a poset by hypothesis, and $W' \cong W$, $V' \cong V$ will also be inclusions of posets since $W$ and $V$ are subcategories of $i(P_B)$ and then we can apply the second part of this lemma. Let us now take two different objects $x, x'$ in $i(P_B) \amalg_{i(P_A)} i(P_C)$. If there is not any morphism between them, we would be done. Assume now, for example, that there is $w \in i(P_B) \amalg_{i(P_A)} i(P_C)(x, x')$. If $x' \in \mathrm{Ob}(i(P_c))$, since $i(P_C)$ is a sieve of $i(P_B) \amalg_{i(P_A)} i(P_C)$, $x \in \mathrm{Ob}(i(P_C))$ and then there is only one morphism in $i(P_C)$ between our objects. Since $i(P_C)$ is a full subcategory of $i(P_B) \amalg_{i(P_A)} i(P_C)$, there is only one morphism in $i(P_B) \amalg_{i(P_A)} i(P_C)$ between our objects. In a similar way, if $x \in \mathrm{Ob}(V')$, since $V'$ is a cosieve of $i(P_B) \amalg_{i(P_A)} i(P_C)$, $x' \in \mathrm{Ob}(V')$ and then there is only one morphism in $V'$ between our objects. Since $V'$ is a full subcategory of $i(P_B) \amalg_{i(P_A)} i(P_C)$, there is only one morphism in $i(P_B) \amalg_{i(P_A)} i(P_C)$ between our objects. Finally, if $x \in i(P_C) \subseteq W'$ and $x' \in V'$, since $W'$ is a cosieve of $i(P_B) \amalg_{i(P_A)} i(P_C)$, $x' \in W'$ and then there is only one morphism in $W'$ between our objects. Since $W'$ is a full subcategory of $i(P_B) \amalg_{i(P_A)} i(P_C)$, there is only one morphism in $i(P_B) \amalg_{i(P_A)} i(P_C)$ between our objects. □

**Proposition 3.3.2.** *Every cofibrant object in* $\mathcal{C}\mathrm{at}$ *is equal to* $i(P)$ *for some poset* $P$.

*Proof.* Let $C \in \mathrm{Ob}(\mathcal{C}\mathrm{at})$ be a cofibrant category. Then, the empty morphism from the empty category $\varnothing \to C$ is a cofibration. By Corollary 3.2.22, $\varnothing \to C$ is a retract of $i\colon \varnothing \to \varnothing_\infty$. In particular, $C$ is a subcategory of $\varnothing_\infty$. Then, by the second assertion of Lemma 3.3.1, it is enough to see that $\varnothing_\infty$ is the inclusion of some poset, and by the first assertion of Lemma 3.3.1, for seeing this is enough to see that every $\varnothing_n$ is the inclusion of some poset. Let us then prove this by induction. First of all, we have that $\varnothing_0 = \varnothing$ is the inclusion of the empty poset. Now suppose that $\varnothing_{n-1}$ is the inclusion of some poset. Then, by the third assertion of Lemma 3.3.1,

$$\varnothing_n = \varnothing_{n-1} \amalg_{\amalg \tau \mathrm{Sd}^2 \Delta[n]} (\amalg \tau \mathrm{Sd}^2 \partial \Delta[n])$$

is the inclusion of some poset since $\amalg\tau\mathrm{Sd}^2\partial\Delta[n]$ and $\amalg\tau\mathrm{Sd}^2\Delta[n]$ are inclusions of posets by Remark 3.2.2 and $\amalg\tau\mathrm{Sd}^2(i_n)$ is a Dwyer map by Proposition 3.2.14. □

**Remark 3.3.3.** We just learned that cofibrant objects in $\mathcal{C}$at are inclusions of posets. In consequence, if a category has more than one morphism between two objects, it cannot be cofibrant. Nevertheless, the reciprocal result does not hold in general. The German mathematicians Roman Bruckner and Christoph Pegel published the article [BP16] where they show that every finite semilattice, every chain, every countable tree, every finite zigzag and every poset with five or less elements is cofibrant in all of those structures.

In a similar way, cofibrations in Thomason's model structure do not carry a very intuitive meaning. In his paper [Rap10], Raptis shows that there exist different choices of cofibrations (and hence, of fibrations) that yields a model structure in $\mathcal{C}$at and $\mathcal{P}$os with the same weak equivalences as Thomason's structure.

Previous proposition has a corollary that will be crucial for us.

**Corollary 3.3.4.** *For every simplicial set $X$, $\tau\mathrm{Sd}^2X$ is the inclusion of some poset.*

*Proof.* Let $X$ be a simplicial set. Since every object in s$\mathcal{S}$ets is cofibrant, $\emptyset \to X$ is a cofibration in s$\mathcal{S}$ets and then by Proposition 3.2.18, $\emptyset = \tau\mathrm{Sd}^2\emptyset \to \tau\mathrm{Sd}^2X$ will be a cofibration in $\mathcal{C}$at. Hence, $\tau\mathrm{Sd}^2X$ is a cofibrant object and in consequence, by Proposition 3.3.2, $\tau\mathrm{Sd}^2X$ is the inclusion of some poset. □

This corollary suggests that $\mathcal{P}$os can inherit the model structure from $\mathcal{C}$at, as we will see in the following theorem.

**Definition 3.3.5.** We say that an order-preserving map $f\colon P \to Q$ between posets is:

- A **weak equivalence** if $i(f)$ is a weak equivalence in $\mathcal{C}$at.
- A **cofibration** if $i(f)$ is a cofibration in $\mathcal{C}$at.
- A **fibration** if $i(f)$ is a fibration in $\mathcal{C}$at.

**Theorem 3.3.6.** *There is a model category structure on the category of posets with the classes of morphisms as above.*

*Proof.* The first axiom **MC1** follows from Proposition 1.4.13.

The closure under composition, the contention of the identities and the second and third axioms **MC2** and **MC3** are verified because they hold for small categories.

Let us now check that the forth axiom **MC4** holds. Consider a lifting problem in $\mathcal{P}$os

$$\begin{array}{ccc} P & \xrightarrow{a} & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Q & \xrightarrow{b} & S, \end{array}$$

with $f$ a cofibration and $g$ an acyclic fibration (or with $f$ an acyclic cofibration and $g$ a fibration). Then, applying $i$ we get another lifting problem

$$\begin{array}{ccc} i(P) & \xrightarrow{i(a)} & i(R) \\ \downarrow{\scriptstyle i(f)} & & \downarrow{\scriptstyle i(g)} \\ i(Q) & \xrightarrow{i(b)} & i(S), \end{array}$$

with $i(f)$ a cofibration and $i(g)$ an acyclic fibration (or with $i(f)$ an acyclic cofibration and $i(g)$ a fibration). Then, there exists $h\colon i(Q) \to i(R)$ such that $i(g) \circ h = i(b)$ and $h \circ i(f) = i(a)$

$$\begin{array}{ccc} i(P) & \xrightarrow{i(a)} & i(R) \\ {\scriptstyle i(f)}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle i(g)} \\ i(Q) & \xrightarrow{i(b)} & i(S), \end{array}$$

and applying pos we get $\mathrm{pos}(h)\colon Q \to R$ such that $g \circ \mathrm{pos}(h) = b$ and $\mathrm{pos}(h) \circ f = a$

$$\begin{array}{ccc} \mathrm{pos}(i(P)) = P & \xrightarrow{\mathrm{pos}(i(a))=a} & \mathrm{pos}(i(R)) = R \\ {\scriptstyle \mathrm{pos}(i(f))=f}\downarrow & \overset{\mathrm{pos}(h)}{\nearrow} & \downarrow{\scriptstyle \mathrm{pos}(i(g))=g} \\ \mathrm{pos}(i(Q)) = Q & \xrightarrow{\mathrm{pos}(i(b))=b} & \mathrm{pos}(i(S)) = S, \end{array}$$

that is, a lift for our initial problem.

Let us now check that the fifth axiom **MC5** holds. Let $f\colon A \to B$ be a morphism in $\mathcal{P}$os. Then $i(f)$ is a morphism in $\mathcal{C}$at, which we can factor as $i(f) = \widetilde{p} \circ \widetilde{j}$ with $\widetilde{j}$ an acyclic cofibration and $\widetilde{p}$ a fibration (or $\widetilde{j}$ a cofibration and $\widetilde{p}$ an acyclic fibration). Moreover, by Lemma 3.3.1 and the construction of these factorizations, we have that $A_\infty$ is an inclusion of some poset and since $i\colon \mathcal{P}\mathrm{os} \to \mathcal{C}\mathrm{at}$ is a full functor, $\widetilde{p} = i(p)$ for some $p$ and $\widetilde{j} = i(j)$ for some $j$. In consequence,

$$f = \mathrm{pos}(i(f)) = \mathrm{pos}(i(p)) \circ \mathrm{pos}(i(j)) = p \circ j$$

with $j$ an ayclic cofibration and $p$ a fibration (or $j$ a cofibration and $p$ an acyclic fibration).

All the axioms of model categories are satisfied, and hence our proof is finished. □

**Theorem 3.3.7.** *The adjunction* $\mathrm{pos}\colon \mathcal{C}\mathrm{at} \rightleftarrows \mathcal{P}\mathrm{os}\colon i$ *forms a Quillen equivalence.*

*Proof.* It is a Quillen adjunction because $i$ is right adjoint and preserves fibrations and trivial fibrations. It is a Quillen equivalence by Corollary 2.5.6 since $\mathrm{Id}^{\sharp}_{\mathrm{pos}(X)} = \mathrm{Id}_X$ is a weak equivalence for every object $X \in \mathrm{Ob}(\mathcal{C}\mathrm{at})$ (in particular for cofibrants) and $\mathrm{Id}^{\flat}_{i(X)} = \mathrm{Id}_X$ is a weak equivalence for every object $X \in \mathrm{Ob}(\mathcal{P}\mathrm{os})$ (in particular for fibrants) by Proposition 1.4.40. □

So we finally are able to model the homotopy theory of spaces via posets! Nevertheless, with all the steps done, we have lost a bit the intuition about the relation between posets and topological spaces. For example, we do not have the notion of Dwyer morphisms in topological spaces, but we will see a nice characterization of them using the isomorphism between posets and Alexandroff $T_0$ spaces. Recall that an open map is a continuous map between topological spaces that takes open sets into open sets, and a closed map is a continuous map between topological spaces that takes closed sets into closed sets.

**Lemma 3.3.8.** *Let* $f\colon P \to Q$ *be a morphism in* $\mathcal{P}$os*. Then* $i(f)$ *embeds* $i(P)$ *as a cosieve of* $i(Q)$ *if and only if* $\mathcal{T}(f)$ *is an open inclusion of spaces. Dually,* $i(f)$ *embeds* $i(P)$ *as a sieve of* $i(Q)$ *if and only if* $\mathcal{T}(f)$ *is a closed inclusion of spaces.*

*Proof.* By construction of the $\mathcal{T}$ functor (see Definition 1.4.14), $f$ is a inclusion of posets if and only if $\mathcal{T}(f)$ is also an inclusion of topological spaces. Moreover, the fact that $i(f)$ embeds $i(P)$ as a cosieve of $i(Q)$ is equivalent to the condition that if $q_1 \leq q_2$ in $Q$ and $q_1 = f(p_1) \in f(P)$, then $q_2 = f(p_2) \in f(P)$ and $p_1 \leq p_2$ in $P$.

On the other hand, we have that $\mathcal{T}(f)$ is an open map if and only if the image of every basic open set $U_p$ is an open set. But this is equivalent to see that for every $x = f(x') \in f(U_p)$, $U_x \subseteq f(U_p)$; that is, that for every $y \geq x$, $y = f(y')$ for $y' \geq p$.

But this is equivalent to the cosieve condition. Indeed, if $y \geq x = f(x') \in f(U_p) \subseteq f(P)$, then $y = f(y')$ and $y' \geq x' \geq p$. Conversely, if $q_1 \leq q_2$ in $Q$ and $q_1 = f(p_1) \in f(U_{p_1})$, then $q_2 = f(p_2)$ and $p_2 \geq p_1$. □

This lemma tells us how the notion of sieve is translated to Alexandroff $T_0$ spaces. Now, let us see how to rephrase the whole definition of Dwyer morphism in topological terms. Recall that the **Sierpiński topological space** $S$ is the set $\{0,1\}$ with the topology $\{\emptyset, \{1\}, \{0,1\}\}$, and notice that it coincides with $\mathcal{T}(0 \leq 1)$.

**Proposition 3.3.9.** *A morphism $f\colon P \to Q$ is such that $i(f)$ is a Dwyer morphism if and only if $\mathcal{T}(f)$ is a closed inclusion and there exists an open neigborhood $U$ of $\mathcal{T}(P)$ in $\mathcal{T}(Q)$ and a* **Sierpiński homotopy** *$H\colon U \times S \to \mathcal{T}(Q)$ with $H(u,1) = u$ for every $u \in U$, $H(u,0) \in \mathcal{T}(P)$ and $H(p,t) = p$ for every $(p,t) \in \mathcal{T}(P) \times S$.*

*Proof.* $f\colon P \to Q$ is such that $i(f)$ is a Dwyer morphism if and only if, by Lemma 3.3.1 and the fact that $i\colon \mathcal{P}\mathrm{os} \to \mathcal{C}\mathrm{at}$ is a full functor, the inclusion $f\colon P \subseteq Q$ can be decomposed as $f_1\colon P \subseteq W$, $f_2\colon W \subseteq Q$ in such a way that:

- If we have $u \leq v$ in $Q$ and $v \in P$, then $u \in P$ and $u \leq v$ in $P$.
- If we have $u \leq v$ in $Q$ and $u \in W$, then $v \in W$ and $u \leq v$ in $W$.
- There is an order-preserving map $r\colon W \to P$ such that $r(p) = p$ for every $p \in P$ and there is a natural transformation $t\colon i(f_1 \circ r) \Rightarrow \mathrm{Id}_{i(W)}$ such that $t_p = \mathrm{Id}_p$ for every $p \in P$.

By previous lemma, the first condition is equivalent to $\mathcal{T}(f)$ being a closed inclusion.

Also by previous lemma, the second condition is equivalent to having an open inclusion of spaces $\mathcal{T}(W) \subseteq \mathcal{T}(Q)$ with $\mathcal{T}(P) \subseteq \mathcal{T}(W)$. In particular, $\mathcal{T}(W)$ will be an open set of $\mathcal{T}(Q)$ containing $\mathcal{T}(P)$. Conversely, if we have an open neighbourhood $U$ of $\mathcal{T}(P)$ in $\mathcal{T}(Q)$, we have that $\mathcal{X}(U)$ is a cosieve of $Q$ which obviously contains $P$. Indeed, if $u \leq v$ in $Q$ and $u \in \mathcal{X}(U)$, then $v \in U_u \subseteq U$ and hence $v \in \mathcal{X}(U)$ and $u \leq v$ in $\mathcal{X}(U)$.

Finally, the natural transformation $t\colon i(f_1 \circ r) \Rightarrow \mathrm{Id}_{i(W)}$ is equivalent by Remark 1.1.13 to a functor $H\colon i(W) \times \{0 \to 1\} \to i(W)$ such that $H(u,0) = f_1(r(u))$ and $H(u,1) = u$ for every $u \in \mathrm{Ob}(i(W))$ and $H(g, \mathrm{Id}_0) = f_1(r(g))$ and $H(g, \mathrm{Id}_1) = g$ for every morphism $g$ in $i(W)$, with the additional condition of $H(p,t) = p$ for every $(p,t) \in i(P) \times \{0 \to 1\}$. Since $i\colon \mathcal{P}\mathrm{os} \to \mathcal{C}\mathrm{at}$ is a full functor, this is equivalent to having an order preserving map $H\colon W \times \{0 \leq 1\} \to W$ such that $H(u,0) = f_1(r(u)) \in P$ and $H(u,1) = u$ for every $u \in W$ and $H(p,t) = p$ for every $(p,t) \in P \times \{0 \leq 1\}$. This implies the existence of the desired Sierpiński homotopy $\mathcal{T}(H)$. Conversely, if we have one such map $H$, then its associated order-preserving map $\mathcal{P}(H)\colon \mathcal{P}(U) \times \{0 \leq 1\} \to Q$ is such that $\mathcal{P}(H)(u,1) = u \in \mathcal{P}(U)$ and $\mathcal{P}(H)(u,0) \in P \subseteq \mathcal{P}(U)$ for every $u \in \mathcal{P}(U)$ and $\mathcal{P}(H)(p,t) = p$ for all $(p,t) \in P \times \{0 \leq 1\}$. Therefore, first of all, we can restrict $\mathcal{P}(H)\colon \mathcal{P}(U) \times \{0 \leq 1\} \to \mathcal{P}(U)$ and moreover, this yields a functor $i(\mathcal{P}(H))\colon i(\mathcal{P}(U)) \times \{0 \to 1\} \to i(\mathcal{P}(U))$, which again by Remark 1.1.13 brings a natural transformation between the restriction of the functor to the subcategory 0 and the restriction of the functor to the subcategory 1. Since $H(u,1) = u$ for every $u \in \mathcal{P}(U)$, the restriction of the functor to the subcategory 1 is the identity functor. Moreover, since $H(u,0) \in P$ for every $u \in \mathcal{P}(U)$, we can consider $r\colon \mathcal{P}(U) \to P$ the order-preserving map associated to the restriction of the functor $i(\mathcal{P}(H))$ to the subcategory 0 and restricting the codomain to $P$, and hence our functor $i(\mathcal{P}(H))$ yields a natural transformation $t$ between $f_1 \circ r$ and the identity, and moreover, this natural transformation is such that $t_p = i(\mathcal{P}(H))(\mathrm{Id}_p, 0 \to 1) = \mathrm{Id}_p$ for every $p \in P$ and moreover our morphism $r$ verifies that $r(f_1(p)) = r(p) = H(p,0) = p$ for every $p \in P$.

This completes our proof. □

So, after all this long way, we are back in our initial setting of topological spaces, cylinders and homotopies, showing once more the importance of all this theory and giving even more reasons to try to understand it.

# Conclusion

This work has allowed the student to widely increase her mathematical knowledge and now gives her a lot of options for further work.

First of all, the background on category theory acquired enables the student to face a lot of topics, not only topological, but even also more algebraic.

More concretely, the student introduced herself in the world of homotopy theory, one of the most popular matters in Algebraic Topology nowadays. Even though she focused on a model structure for small categories, she learned the basic properties of the categories that are used by most topologists, as well as the functors relating them. All notions learned will help her to understand more articles and talks on the field. For example, on the one hand, she could keep studying the Thomason model structure for small categories and posets and deal with topics such as the characterization of fibrant and cofibrant objects, which we already mentioned is non-trivial. On the other hand, she should be ready to tackle the basics on $\infty$-categories and operads, concepts handled by her supervisor for example in [CG19] or [CG20]. For instance, she could try to generalize the relations between simplicial sets, small categories and posets to dendroidal sets (d$\mathcal{S}$ets), operads ($\mathcal{O}$per) and broad posets (Br$\mathcal{P}$os) in order to set model structures there too.

Also, the topic dealt with here has to do with the Bachelor Final Project of the student [SB21]. As Raptis mentions in his article [Rap10], the equivalence between $A$-spaces and posets can be restricted to finite spaces and finite posets and is related to a notion of simple homotopy type, developed by the Argentinian mathematicians Jonathan Barmak and Gabriel Minian in [BM08], which stands strictly between homotopy type and weak homotopy type and which is equivalent to a notion of simple homotopy type for finite simplicial complexes. This fact makes us wonder if there is a model structure on finite posets such that in the homotopy category posets with the same simple homotopy type are isomorphic; and if so, if it can be extended to $A$-spaces in general and related to simplicial sets and small categories via Quillen equivalences. Furthermore, as a result of a collaboration scholarship with the Mathematics and Computer Science Department at Universitat de Barcelona, the student is analyzing if there is a relation between the simple homotopy type of finite simplicial complexes and their associated Stanley-Reisner rings; see [HH11]. In case there is, we could also even study the possibility of having the category of rings as a model for the simple homotopy type of finite spaces.

In conclusion, this work resulted to be very fruitful for the student, who now knows a bit about homotopy theory and is very eager to use this knowledge and expand it!

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