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On the h-cobordism theorem and applications

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Summary

H-cobordism is a notion developed by John Milnor, for the case of smooth manifolds, which involves a combination of homotopy theory and the theory of cobordisms. Stephen Smale used this concept to prove the h-cobordism theorem, which applies to compact smooth manifolds of dimension greater than five, with boundary. Later on, Milnor proved another version of the theorem in which he replaced the h-cobordism condition by a purely topological one, though he still used differential topology to prove the theorem. We will prove this second version of the theorem and, afterwards, will compare the two of them and will see that they are equivalent indeed. Finally, we will see that one of the corollaries of h-cobordism theorem proves a version of the so called generalized Poincaré conjecture in dimensions greater than four.

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Introduction

Henri Poincaré did not invent Topology but did lay the foundations. He realized that all the paths of mathematics that he studied in depth led to this subject. In 1895, when he was forty years old, he published an article called "Analysis Situs", which was the basis for what was to become a new branch of Mathematics called Topology. In this first article he studied manifolds from different points of view, in particular, as polyhedra. He established simplicial homology groups, from which he defined the Betti numbers of a manifold and proved that two closed manifolds with the same Betti numbers will not necessarily have isomorphic homology groups and hence they are not homeomorphic in general. He gave a first version of his duality theorem, in terms of Betti numbers. Also, he introduced a new topological invariant, the fundamental group.

Four years later, Poul Heegaard published an article describing an example of a non orientable manifold which, of course, invalidated the first version of Poincaré's duality theorem, for the latter only used Betti numbers. Then, Poincaré wrote a complement to his article were he introduced torsion coefficients in order to amend his error. Finally, at the end of the complement he made the following claim

> "*Pour ne pas trop allonger ce travail, je me bornerai à énoncer le théorème suivant dont la démostration demanderait quelques développements: Tout polyèdre qui a tous ses nombres de Betti égaux à 1 et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire, homéomorphe à l'hypersphère*"

which can be translated in modern language as follows:

> *Every closed 3-dimensional manifold with the homology of S^3 is simply connected, that is, homeomorphic to S^3.*

Even though he already had the tool to disprove such a statement, that is, the fundamental group, it was not an easy task to find a counterexample among the infinite number of 3-dimensional closed manifolds. Finally, four years later he found it. He constructed a manifold whose homology groups equal those of S^3 and described on it a pair of trajectories not reducible to a point. Such a space is called Poincaré homology sphere. In Appendix A we give a modern rigorous description of it.

Then, it still remained one question to answer:

> *If a closed 3-dimensional manifold has trivial fundamental group, must it be homeomorphic to S^3?*

Poincaré conjecture asserts that the answer to previous question is true. The problem remained unsolved for the whole twentieth century. In parallel to the efforts of many mathematicians for solving the conjecture, Topology experienced a huge growth. Singular homology, defined for any topological space, was introduced, among other important results. Actually, Eilenberg and Steenrod went further by establishing a list of axioms that any homology theory must fulfill to be considered as such.

The question was generalized to any dimension and category (Top=Topological spaces, PL=Piecewise linear manifolds, Diff=Smooth manifolds) as follows.

Must every homotopy sphere (a closed n-manifold which is homotopy equivalent to the n-sphere) in the chosen category (Top, PL, Diff) be isomorphic to the standard n-sphere in the chosen category (i.e. homeomorphic, PL-isomorphic or diffeomorphic)?

Surprisingly, though, it was first proved true in dimension five of more (Top) and dimensions five and six (Diff). It was Stephen Smale who, in 1961, appreciated that such a fact was not surprising at all. Certainly, when adding more dimensions you lose geometric intuition but, on the other hand, you gain capacity of altering functions in order to cancel certain critical points.

In 1982, Michael Freedman solved the case of dimension four, only for topological manifolds.

In spite of all the advances achieved in Topology, finally it was a tool of Analysis, the Ricci flow, that Grigori Perelman used to prove the original version of Poincaré conjecture in 2002.

Note that in dimension 2, one has the classification theorem for compact connected topological surfaces, which assures that such manifolds are homeomorphic to one of the following surfaces: S^2, gT^2 or gP^2, that is, a 2-sphere, a connected sum of g tori or a connected sum of g projective planes. If we have a closed surface which is homotopy equivalent to a 2-sphere, then it must be a topological 2-sphere, otherwise it would be either gT^2 or gP^2, which have non trivial fundamental group, contradicting the hypothesis. This proves generalized Poincaré conjecture in dimension two, modulo proving the classification theorem. When it comes to dimension one, already in Poincaré's time it was known that every compact connected curve is homeomorphic to S^1.

The present work focuses on the study and proof of the so called h-cobordism theorem, which can be stated as follows.

Let W be a compact smooth manifold of dimension greater than five, whose boundary consists of the disjoint union of two closed submanifolds V and V'. If W, V and V' are simply connected and the integral homology of W relative to any of its boundary components is trivial, then W is diffeomorphic to $V \times [0,1]$.

The name of the theorem is due to the fact that, in Smale's original version, one assumes $(W; V, V')$ is an h-cobordism instead of assuming that $H_*(W, V) = 0$. In Chapter 4 we define h-cobordism notion properly and prove that the two versions of the theorem are equivalent indeed.

A remarkable corollary of the h-cobordism theorem proves the following version of generalized Poincaré conjecture:

Every closed smooth manifold which is homotopy equivalent to a sphere of dimension $n \geq 5$ is homeomorphic to S^n. Furthermore, if $n = 5$ or 6, then it is diffeomorphic to S^n.

CHAPTER 1

Preliminary concepts

Categories and Functors

In any field of mathematics one always works with certain objects together with morphisms between them. Below we make this notion precise.

Definition. A *category* $\mathfrak{C}$ consists of three ingredients: a class of *objects*, *obj* $\mathfrak{C}$; sets of *morphisms* $Hom(A, B)$, one for every ordered pair $A, B \in obj\ \mathfrak{C}$; and *composition* $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$, denoted by $(f, g) \to g \circ f$, for every $A, B, C \in obj\ \mathfrak{C}$, satisfying the following axioms:

(i) the family of $Hom(A, B)$'s is pairwise disjoint,
(ii) composition is associative when defined,
(iii) for every $A \in obj\ \mathfrak{C}$, there exists an *identity* $id_A \in Hom(A, A)$ such that if $f \in Hom(B, A)$, with $B \in obj\ \mathfrak{C}$, then $id_A \circ f = f$, and if $g \in Hom(A, C)$, with $C \in obj\mathfrak{C}$, then $g \circ id_A = g$.

Sometimes certain problems are easier when dealt with in another category, which is precisely what happens in some topological problems when treated from an algebraic point of view. Below we define the mathematical tool that allows us moving from one category to another.

Definition. A *covariant functor*[1] is an object map from a category $\mathfrak{A}$ to a category $\mathfrak{C}$, such that

(i) $A \in obj\ \mathfrak{A}$ implies $\mathcal{F}(A) \in obj\ \mathfrak{C}$.
(ii) if $f : A \to A'$ is a morphism in $\mathfrak{A}$, then $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(A')$ is a morphism in $\mathfrak{C}$,
(iii) if f, g are morphisms in $\mathfrak{A}$ for which $g \circ f$ is defined, then

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f),$$

(iv) $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$ for every $A \in obj\ \mathfrak{A}$.

Singular Homology

This section is devoted to state some basic definitions, fix the notation and give the essential notions concerning singular (co)homology which are used for proving homotopy invariance, excision, Thom isomorphism theorem, Poincaré duality, universal coefficients theorem and the existence of long exact sequences, which are tools well known to the informed reader, that, if we were to treat them rigorously, they would require another master thesis project.

Definition. The *standard n-simplex* is the following convex set of $\mathbb{R}^{n+1}$.

$$\Delta^n := \left\{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : t_i \geq 0; \sum_{i=0}^{n} t_i = 1\right\}$$

[1]With the same notation, a *contravariant functor* is also an object map $\mathcal{F} : \mathfrak{A} \to \mathfrak{C}$, except for, in this case, $\mathcal{F}(f) : \mathcal{F}(A') \to \mathcal{F}(A)$ and $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

DEFINITION. Let X be a topological space. A *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$. Define the linear imbedding $\phi_i : \Delta^{n-1} \to \Delta^n$ by

$$\phi_i(t_0, ..., t_{i-1}, t_{i+1}, ..., t_n) = (t_0, ..., t_{i-1}, 0, t_{i+1}, ..., t_n)$$

The i-th *face* of σ is the singular (n-1)-simplex

$$\sigma \circ \phi_i : \Delta^{n-1} \to X$$

DEFINITION. Let Λ be a commutative ring with unit and let X be a topological space. The *singular chain group* $S_n(X;\Lambda)$, $n \geq 0$, is the free Λ-module having one generator $[\sigma]$ for each singular n-simplex σ in X. If $n < 0$, then $S_n(X;\Lambda)$ is defined to be zero. The n-th *boundary operator* $\partial_n : S_n(X;\Lambda) \to S_{n-1}(X;\Lambda)$ is defined by

$$\partial_n[\sigma] = \sum_{i=0}^{n} (-1)^i [\sigma \circ \phi_i]$$

It can be verified that, if $n \geq 2$, then $\partial_{n-1} \circ \partial_n = 0$. The *singular chain complex* of X is the chain complex $S_*(X;\Lambda) := \{S_n(X;\Lambda), \partial\}$. Thus, one can define the following quotient module

$$H_n(X;\Lambda) := \frac{ker(\partial_n : S_n(X;\Lambda) \to S_{n-1}(X;\Lambda))}{im(\partial_{n+1} : S_{n+1}(X;\Lambda) \to S_n(X;\Lambda))}$$

which is called the n-th *singular homology group*. Define

$$\begin{aligned} Z_n(X;\Lambda) &:= ker(\partial_n : S_n(X;\Lambda) \to S_{n-1}(X;\Lambda)) \\ B_n(X;\Lambda) &:= im(\partial_{n+1} : S_{n+1}(X;\Lambda) \to S_n(X;\Lambda)) \end{aligned}$$

which are called *n-cycles* and *n-boundaries*, respectively.

DEFINITION. The *singular cochain group*, $S^n(X;\Lambda)$, is the dual module $Hom_\Lambda(S_n(X;\Lambda), \Lambda)$. The value of a cochain c in a chain γ is denoted by $\langle c, \gamma \rangle \in \Lambda$. Define the n-th *coboundary operator* as

$$\begin{aligned} \delta_n : S^n(X;\Lambda) &\to S^{n+1}(X;\Lambda) \\ c &\mapsto c \circ \partial_{n+1} \end{aligned} \tag{1}$$

So $\delta_{n+1}(\delta_n(c)) = \delta_{n+1}(c \circ \partial_{n-1}) = c \circ \partial_{n-1} \circ \partial_n = 0$. Thus $\delta_{n-1}(S^{n-1}(X;\Lambda)) \subseteq ker(\delta_n)$ and hence one can define the quotient module

$$H^n(X;\Lambda) = \frac{Z^n(X;\Lambda)}{B^n(X;\Lambda)} := \frac{ker(\delta_n : S^n(X;\Lambda) \to S^{n+1}(X;\Lambda))}{im(\delta_{n-1} : S^{n-1}(X;\Lambda) \to S^n(X;\Lambda))}$$

which is called the n-th *singular cohomology group* of X, where $Z^n(X;\Lambda)$ and $B^n(X;\Lambda)$ are the *n-cocycles* and the *n-coboundaries*.

DEFINITION. Let $\sigma : \Delta^{m+n} \to X$ be a singular (m+n)-simplex. The *front* m-face of σ is the composition $\sigma \circ \alpha_m : \Delta^m \to X$ where

$$\alpha_m(t_0, ..., t_m) = (t_0, ..., t_m, 0, ..., 0)$$

The *back* n-face of σ is the composition $\sigma \circ \beta_n : \Delta^n \to X$ where

$$\beta_n(t_m, t_{m+1}, ..., t_{m+n}) = (0, ..., 0, t_m, t_{m+1}, ..., t_{m+n})$$

Now, given cochains $c \in S^m(X;\Lambda)$ and $c' \in S^n(X;\Lambda)$, define the product cc' by the following identity

$$\langle cc', [\sigma] \rangle = (-1)^{mn} \langle c, [\sigma \circ \alpha_m] \rangle \cdot \langle c', [\sigma \circ \beta_n] \rangle \in \Lambda$$

where [ ] refers to a homology class. Such a product operation is bilinear and associative, the identity element being the constant cocycle $1 \in S^0(X;\Lambda)$. It can be verified, using (1) that

$$\delta(cc') = (\delta c)c' + (-1)^m c(\delta c')$$

inducing, then, a well-defined product in cohomology

$$H^m(X;\Lambda) \otimes H^n(X;\Lambda) \to H^{m+n}(X;\Lambda)$$

called *cup product* and written $[a] \smile [b]$, with

$$[a] \otimes [b] \in H^m(X;\Lambda) \otimes H^n(X;\Lambda)$$

REMARK 1.1. The cup product is commutative up to sign, that is,

$$[a] \smile [b] = (-1)^{mn}[b] \smile [a]$$

Thus, $H^*(X;\Lambda)$, endowed with $\smile$, is commutative as a graded ring.

DEFINITION. A *topological pair* is a pair (X, A), where X is a topological space and $A \subseteq X$ is a subspace. A *pair map* $(X, A) \to (X', A')$ is a continuous map $f : X \to X'$ such that $f(A) \subseteq A'$. If A is empty, we shall not distinguish between the pair $(X, \emptyset)$ and the space X.

DEFINITION. The *relative singular chain complex* of (X, A) is defined as

$$S_*(X, A;\Lambda) := \frac{S_*(X;\Lambda)}{S_*(A;\Lambda)}$$

with the corresponding boundary operator ∂, induced by $S_*(X;\Lambda)$, which satisfies $\partial(S_n(A;\Lambda)) \subseteq S_{n-1}(A;\Lambda)$. Therefore, one can define the n-th *relative singular homology group* $H_n(X, A;\Lambda)$.

Note that cycles in $S_*(X, A;\Lambda)$ can be identified with elements $\alpha \in S_*(X;\Lambda)$ such that $\partial\alpha$ is not necessarily zero, but an element in $S_*(A;\Lambda)$. Moreover, two cycles in $S_*(X, A;\Lambda)$, represented by $\alpha, \alpha' \in S_*(X;\Lambda)$, are homologous, that is, they represent the same homology class in $H_*(X, A;\Lambda)$, if and only if $\exists \beta \in S_{*+1}(X;\Lambda)$ such that $\alpha - \alpha' - \partial\beta \in S_*(A;\Lambda)$.

REMARK 1.2. The relation between $H_*(X, A;\Lambda)$, $H_*(X;\Lambda)$ and $H_*(A;\Lambda)$ is given by the long exact sequence in relative homology

$$... \to H_k(A;\Lambda) \to H_k(X;\Lambda) \to H_k(X, A;\Lambda) \to H_{k-1}(A;\Lambda) \to ...$$

and since, in general, $H_k(X;\Lambda) \to H_k(X, A;\Lambda)$ is not surjective, then

$$H_k(X, A;\Lambda) \ncong \frac{H_k(X;\Lambda)}{H_k(A;\Lambda)}$$

REMARK 1.3. Universal coefficients theorem provides the following remarkable fact. If Λ is a principal ideal domain (e.g. $\mathbb{Z}$ or any field), given two chain complexes C_* and C'_*, either of a topological space or a topological pair, with coefficients in Λ, then

$$H_*(C_*) \cong H_*(C'_*) \Rightarrow H^*(C^*) \cong H^*(C'^*)$$

From now on, we will assume $\Lambda = \mathbb{Z}$, unless another ring is specified, and we will omit reference to Λ, writing $H_*(X)$ and $H_*(X, A)$ instead of $H_*(X;\Lambda)$ and $H_*(X, A;\Lambda)$.

Homotopy and Fundamental Group

In the previous section we considered H_* and H^*, which are functors from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms. Now we present another functor, called fundamental group and denoted π_1, which goes from the category of *pointed topological spaces*(i.e. nonempty topological spaces with a *base point*) and continuous maps preserving base points onto the the category of groups and homomorphisms. In general, the fundamental group functor yields non abelian groups called first homotopy groups.

We start by defining homotopy in the most general case. Then we will define the concept of retraction and will use homotopy to stablish the notion of deformation retraction both in the category of topological spaces and in the homotopy category of topological spaces (*homotopy category*).

Finally, having already defined homology groups, we will state some important results relating both theories.

Homotopy.

DEFINITION. Let (X, A) and (Y, B) be topological pairs and let $(X, A) \times I := [0, 1]$ denote the pair $(X \times I, A \times I)$. Let $X' \subset X$ and suppose $f, g : (X, A) \to (Y, B)$ are pair maps agreeing on X'. Then f is *homotopic to g relative to X'*, denoted $f \simeq g\ rel\ X'$, if there exists a map

$$H : (X, A) \times I \to (Y, B)$$

such that, if $x \in X$, then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$; if $x \in X'$ and $t \in I$, then $H(x, t) = f(x)$. The map H is called a *homotopy relative to X'* from f to g and is denoted by $H : f \simeq g\ rel\ X'$. In case $X' = \emptyset$ we just say that H is a homotopy from f to g and that $f \simeq g$, that is, f is homotopic to g.

The following results can be easily proved [**1**, p. 24 thms. 5 & 6].

THEOREM 1.4. *Homotopy relative to X' defines an equivalence relation in the set of pair maps from (X, A) to (Y, B).*

THEOREM 1.5. *Let $f_0, f_1 : (X, A) \to (Y, B)$ be homotopic relative to X' and let $g_0, g_1 : (Y, B) \to (Z, C)$ be homotopic relative to Y', where $f_1(X') \subset Y'$. Then, the compositions $g_0 f_0, g_1 f_1 : (X, A) \to (Z, C)$ are homotopic relative to X'.*

The first one means there exist equivalence classes of pair maps from (X, A) to (Y, B), which are called *homotopy classes relative to X'*. The second one guarantees, in particular, the existence of a category whose objects are topological pairs and whose morphisms are homotopy classes relative to $\emptyset$. Such a category is called *homotopy category of pairs* and one of its subcategories is the homotopy category, cited in the introduction of this section.

DEFINITION. A pair map $f : (X, A) \to (Y, B)$ is called a *homotopy equivalence* if the class it represents, namely $[f]$, is an equivalence in the homotopy category of pairs, that is, if there exists a pair map $g : (Y, B) \to (X, A)$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Pairs (X, A) and (Y, B) are said to have the *same homotopy type*, or to be *homotopy equivalent*, if they are equivalent in the homotopy category of pairs.

DEFINITION. A *contraction* is a homotopy $F : X \times I \to X$ such that $F : id_X \simeq c$, where c is a constant map of X to itself, that is, the image of c is a point $x_0 \in X$. If a contraction $F : X \times I \to X$ exists, then X is said to be *contractible*.

Note that, homotopically, contractible spaces are the simplest topological spaces inasmuch as a space is contractible if and only if it has the homotopy type of a one-point space. Certainly, let $c : X \to P := x_0$ be the constant map of previous definition and let $\iota : P \subset X$. Then $c \circ \iota = id_P$. Now, $id_X \simeq c = \iota \circ c$ by hypothesis and hence X is homotopy equivalent to a one-point space; conversely, let $f : X \to x_0$ be a homotopy equivalence with homotopy inverse $g : x_0 \to X$. Then $g \circ f$ is a constant map such that $id_X \simeq g \circ f$ and hence X is contractible.

Fundamental Group. Let $\gamma, \delta : I \to X$ be paths with $\gamma(1) = \delta(0)$, then one can define another path in X by the binary operation

$$\gamma\delta(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

called *path concatenation*. If we set $A = B = \emptyset$, $X = I$, $X' = \dot{I} := \{0,\ 1\}$ in Theorem 1.4 and let $\sigma, \tau : I \to Y$ be loops based at $y_0 \in Y$, then homotopy relative to $\dot{I}$ defines an equivalence relation in the *loop space* of Y *based at* y_0 $(\Omega(Y, y_0))$, the equivalence classes being called *homotopy classes of loops at base point* y_0. It can be easily checked [**2**, pp. 42-43] that the quotient space of $\Omega(Y, y_0)$ by the equivalence relation given by homotopy relative to $\dot{I}$ has a group structure with the binary operation described above. It is denoted $\pi_1(Y, y_0)$.

THEOREM 1.6. *If Y is path connected and y_0, $y_1 \in Y$, then*

$$\pi_1(Y, y_1) \cong \pi_1(Y, y_0)$$

Proof. It suffices to build an explicit map between $\Omega(Y, y_1)$ and $\Omega(Y, y_0)$, pass to homotopy classes and check the resulting map is a well-defined homomorphism with inverse homomorphism. Indeed, let $\gamma : I \to Y$ be a path from y_0 to y_1 and define $\varphi : \Omega(Y, y_1) \to \Omega(Y, y_0)$ by $\varphi(\sigma) = (\gamma\sigma)\gamma^{-1}$. Figure 1 illustrates the situation. Passing to homotopy classes one

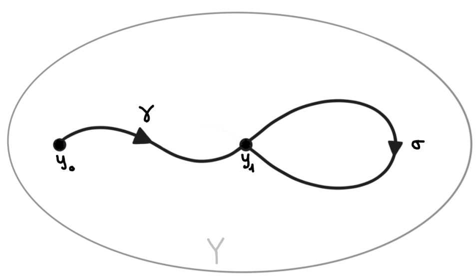


FIGURE 1

obtains the homomorphism

$$\begin{aligned} \pi_1(\varphi) : \pi_1(Y, y_1) &\to \pi_1(Y, y_0) \\ [\sigma] &\mapsto [\gamma\sigma\gamma^{-1}]. \end{aligned}$$

Certainly, $\gamma\sigma\tau\gamma^{-1} \simeq \gamma\sigma\gamma^{-1}\gamma\tau\gamma^{-1}$ *rel* $\dot{I}$ for any $\sigma, \tau \in \pi_1(Y, y_1)$. Note that, since path concatenation is associative up to homotopy, parenthesis are no longer needed. Finally, the inverse homomorphism is defined by $[\omega] \mapsto [\gamma^{-1}\omega\gamma]$. ■

Strictly speaking, though, we cannot canonically identify $\pi_1(Y, y_0)$ with $\pi_1(Y, y_1)$, because the isomorphism between them is not unique, but it depends on the path class of γ. There is a technical lemma [**2**, p. 47 lemma 3.8] which establishes the relation between such an isomorphism and the induced maps in homotopy of two homotopic maps

$H : f \simeq g : X \to Y$, through the following commutative diagram, where $\Psi_{[\lambda]}$ is the iso-

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "\pi_1(g)"] \arrow[dr, "\pi_1(f)"'] & \pi_1(Y,g(x_0)) \arrow[d, "\Psi_{[\lambda]}"] \\
& \pi_1(Y,f(x_0))
\end{tikzcd}$$

FIGURE 2

morphism which depends on the class of the path joining the points $f(x_0)$ and $g(x_0)$, with $\lambda(t) := H(x_0,t)$, $0 \le t \le 1$. The isomorphism is precisely given by $[\tau] \to [\lambda\tau\lambda^{-1}]$.

Using this lemma we can prove a result which will be used to justify that two path connected topological manifolds having the same homotopy type have isomorphic fundamental groups.

THEOREM 1.7. *If $f : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\pi_1(f) : \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is an isomorphism for every $x_0 \in X$.*

Proof. Let $g : Y \to X$ be the homotopy inverse of f. Then, in accordance with Figure 2, the lower triangle of the diagram below commutes.

$$\begin{tikzcd}
& \pi_1(Y,f(x_0)) \arrow[dr, "\pi_1(g)"] & \\
\pi_1(X,x_0) \arrow[ur, "\pi_1(f)"] \arrow[rr, "\pi_1(g\circ f)"] \arrow[dr, "id"'] & & \pi_1(X,g(f(x_0))) \arrow[dl, "\Psi"] \\
& \pi_1(X,x_0) &
\end{tikzcd}$$

Now, on the one hand, Ψ is an isomorphism and hence so is $\pi_1(g \circ f)$. On the other hand, top triangle commutes due to property (iii) of functors. Thus $\pi_1(f)$ must be injective. A similar diagram which uses the morphism induced by $f \circ g$ instead, shows that $\pi_1(f)$ is surjective. ■

COROLLARY 1.8. *If X is a contractible space, then $\pi_1(X,x_0) = \{e\}$.*

Proof. As we have already showed, if X is contractible then it is homotopy equivalent to a one-point space. But the fundamental group of the latter is trivial in that it only possesses the constant loop. Now Theorem 1.7 provides desired conclusion. ■

DEFINITION. A space X is *simply connected* if it is path connected and $\pi_1(X,x_0) = \{e\}$ for every $x_0 \in X$.

EXAMPLE 1.9. A contractible space is simply connected because it is path connected and its fundamental group is trivial.

Contractible spaces are not the only simply connected spaces. However, it is not as easy as in the contractible case to prove them so. There is a powerful tool for computing fundamental groups, the Van Kampen's theorem. For the spaces we are going to deal with, it will suffice to use the following reduced version of that result [**3**, p. 63].

THEOREM 1.10. *Let X be a topological space which is the union $X = X_1 \cup X_2$ of open subsets X_1 and X_2 such that X_1, X_2 and $X_0 := X_1 \cap X_2$ are all path connected and nonvoid. Consider the consistent diagram*

$$\begin{tikzcd} & \pi_1(X,p) & \\ \pi_1(X_1,p) \arrow[ur, "\omega_1"] & & \pi_1(X_2,p) \arrow[ul, "\omega_2"'] \\ & \pi_1(X_0,p) \arrow[ul, "\theta_1"] \arrow[ur, "\theta_2"'] \arrow[uu, "\omega_0"'] & \end{tikzcd}$$

with $\omega_0 = \omega_1\theta_1 = \omega_2\theta_2$. Then the image groups $\omega_i(\pi_1(X_i,p))$, $i = 0,\ 1,\ 2$ generate $\pi_1(X,p)$.

COROLLARY 1.11. *If θ_2 is an isomorphism, then so is ω_1.* ■

COROLLARY 1.12. *If $\pi_1(X_1,p)$ and $\pi_1(X_2,p)$ are trivial, then so is $\pi_1(X,p)$.* ■

EXAMPLE 1.13. An immediate application of Corollary 1.12, by setting $X = S^n$, with $n \geq 2$, implies that such spheres are simply connected spaces.

REMARK 1.14. Clearly, corollary 1.12 does not apply to S^1. In order to compute the fundamental group of S^1 one uses the fact that it is possible to identify a representative of each homotopy class with a loop that winds around the circle a certain amount of times, either in one sense or the other. It can be proved, then, that $\pi_1(S^1,p)$ is isomorphic to the additive group $\mathbb{Z}$ [**2**, p. 52 thm. 3.16].

Retraction and Deformation. First we define the notion of retraction for the category of topological spaces and continuous maps, and for the homotopy category. We will assume A to be a subspace of a topological space X and will consider the inclusion map $\iota : A \to X$.

DEFINITION. A *retraction* of X is a continuous map $r : X \to A$ which is a left inverse of ι, that is, $r \circ \iota = id_A$. A is said to be a *retract* of X.

DEFINITION. A *weak retraction* of X is a continuous map $r_w : X \to A$ which is a left homotopy inverse of ι, that is, $r_w \circ \iota \simeq id_A$. A is said to be a *weak retract* of X.

Now we are going to define a notion arising from considering right homotopy inverses of the inclusion map.

DEFINITION. A space X is *deformable* into A if ι has a right homotopy inverse $f : X \to A$, that is, $\iota \circ f \simeq id_X$.

REMARK 1.15. Note that there are no right inverses of the inclusion map, for if f were such an inverse, then $\iota \circ f = id_X$, which is a contradiction.

Finally, we can combine retracts and deformations into the following concept.

DEFINITION. Let $r : X \to A$ be a retraction. Then A is said to be a *deformation retract* of X when r composed with ι is a deformation, that is, there exists a deformation $D : X \times I \to X$ such that $\iota \circ r \simeq id_X$.

Such a concept can be weakened as follows.

Definition. A is said to be a *weak deformation retract* of X when

- $r \circ \iota \simeq id_A$
- $\iota \circ r \simeq id_X$

That is, when A is homotopy equivalent to X.

Hurewicz-Poincaré Theorems. Poincaré already pointed out the intimate relation between homotopy and homology [**4**] but it was Hurewicz who established the links between homotopy and homology groups. It is not difficult, but lengthy, to build a homomorphism between $\pi_1(X, x_0)$ and $H_1(X)$ [**2**, pp. 80-81 thm. 4.27] and to prove that such a homomorphism is a surjection whose kernel is the commutator $[\pi_1(X, x_0), \pi_1(X, x_0)]$ [**2**, pp. 82-84 thm. 4.29]. That is, $H_1(X)$ is isomorphic to

$$\frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]},$$

which is called the abelianization of $\pi_1(X, x_0)$, denoted $\pi_1(X, x_0)_{ab}$.

Hurewicz went further on this subject and introduced the *higher homotopy groups* $\pi_n(X, X_0)$ which, roughly speaking, are the morphisms of S^n into X. Actually, if $f : X \to Y$ is a homotopy equivalence, then Theorem 1.7 can be repeated verbatim, by applying a similar technical lemma [**2**, p. 341 lemma 11.25 & corollary 11.26], so as to conclude that $\pi_n(X, x_0) \cong \pi_n(Y, f(x_0))$. Therefore, apart from homology groups, homotopy groups in general are also invariants of homotopy type. Such groups gave rise to the so called homotopy theory, a very fruitful theory, out of the scope of present work.

Smooth Manifolds

In this section we give a precise definition of a smooth manifold of dimension n. Recall that a topological manifold $M = M^n$ is a Hausdorff topological space with a countable basis of open sets, such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. A *coordinate neighborhood* consists of an open set U of M and a *coordinate map* φ which is a homeomorphism of U onto an open subset of $\mathbb{R}^n$, so that we can assign to every $p \in U$ the n coordinates $x_1(p), ..., x_n(p)$ of its image $\varphi(p)$ in $\mathbb{R}^n$. We call *i-th coordinate function* each function $x_i(p)$ on U. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two coordinate neighborhoods, then we have the following homeomorphism

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

which is called the *transition map*. The idea behind smooth manifolds is to select a certain collection of coordinate neighborhoods, covering M, such that all transition maps and their inverses are C^∞, namely, diffeomorphisms. Below we define such a collection properly.

Smooth Structure. An *atlas* on M consists of a collection of coordinate neighborhoods

$$\mathcal{A} = \{U_\alpha, \varphi_\alpha\}_{\alpha \in A \subset \mathbb{N}}$$

such that

(i) $M = \bigcup_{\alpha \in A} U_\alpha$,
(ii) for any $\alpha, \beta \in A$, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism.

We say that a coordinate neighborhood (U, φ) is *compatible* with $\mathcal{A}$ if $\mathcal{A} \cup \{(U, \varphi)\}$ is an atlas, that is, all transition maps between coordinate maps of $\mathcal{A}$ and the new one are diffeomorphisms. An atlas $\mathcal{A}$ is *maximal* when every coordinate neighborhood compatible with $\mathcal{A}$ is itself in $\mathcal{A}$. A *smooth structure* on M is a choice of a maximal atlas on it. A *smooth manifold* is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure.

REMARK 1.16. Neither existence nor uniqueness of smooth structures is guaranteed for every topological manifold. It is, then, mandatory to separate the concept of smooth manifold from the underlying topological space [**5**, p. 104].

Partitions of Unity. A *partition of unity* on a smooth manifold M is a collection $\{f_i : i \in I\}$ of smooth functions on M such that:

(a) The collection $\{supp(f_i) : i \in I\}$ is locally finite, that is, $\forall p \in M$ there exists a neighborhood U about p such that $U \cap \{supp(f_i) : i \in I\} \neq \emptyset$ for a finite amount of indices i
(b) $\sum_{i \in I} f_i(p) = 1$, with $f_i(p) \geq 0$, for all $p \in M$.

A partition of unity $\{f_i : i \in I\}$ is *subordinate* to the cover $\{U_\alpha : \alpha \in A\}$ if, for each i, there exists an α such that $supp(f_i) \subset U_\alpha$.

The Tangent Space to a Manifold at a Point. The concept of tangent vectors on manifolds is based upon the following interpretation of the space of tangent vectors attached to a point of the euclidean space $\mathbb{E}^n$.

MOTIVATING EXAMPLE. Consider the set of smooth functions defined on neighborhoods of a point q in $\mathbb{E}^n$ (hereafter identified with $\mathbb{R}^n$). The fact that two functions agree on some neighborhood of q introduces an equivalence relation on such functions. The equivalence classes are called *germs* and we denote the set of germs at q by $\mathcal{F}_q$. If f is a smooth function on a neighborhood of q, then $\mathbf{f}$ will denote its germ. Addition and multiplication of germs, together with scalar multiplication by real numbers, induce on $\mathcal{F}_q$ the structure of an algebra over $\mathbb{R}$. A germ $\mathbf{f}$ has a well-defined value $\mathbf{f}(q)$ at q, namely, the value at q of any representative of $\mathbf{f}$. Now, the directional derivative of a function f, in the direction of a vector $X_q = (a_1, ..., a_n)$ belonging to the n-dimensional vector space $T_q\mathbb{R}^n$ attached to q, can be thought of as the following linear operator on $\mathcal{F}_q$.

$$\begin{aligned} X_q^* : \mathcal{F}_q &\to \mathbb{R} \\ \mathbf{f} &\mapsto \sum_{i=1}^n a_i \left.\frac{\partial f}{\partial r_i}\right|_q = \left(\sum_{i=1}^n a_i \left.\frac{\partial}{\partial r_i}\right|_q \right)(f) \end{aligned}$$

where

$$\begin{aligned} r_i : \mathbb{R}^n &\to \mathbb{R} \\ a &\mapsto a_i \end{aligned}$$

is the i-th canonical coordinate function. Thus, X_q^* is a linear derivation in that it satisfies the following properties.

$$X_q^*(\mathbf{f} + \lambda \mathbf{g}) = X_q^*(\mathbf{f}) + \lambda X_q^*(\mathbf{g}) \quad \text{(linearity)} \tag{2}$$

$$X_q^*(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}(q) X_q^*(\mathbf{g}) + \mathbf{g}(q) X_q^*(\mathbf{f}) \quad \text{(Leibniz rule)} \tag{3}$$

whenever $\mathbf{f}$ and $\mathbf{g}$ are smooth near q and $\lambda \in \mathbb{R}$. We will denote by $\mathcal{D}_q$ the mappings from $\mathcal{F}_q$ to $\mathbb{R}$ fulfilling those properties. The elements in $\mathcal{D}_q$ are called *derivations* on $\mathcal{F}_q$ into $\mathbb{R}$. It can be checked that $\mathcal{D}_q$ is a real vector space. Actually, the correspondence $X_q \to X_q^*$ defines a linear 1:1 map from $T_q\mathbb{R}^n$ onto $\mathcal{D}_q$, namely, there is an isomorphism of the vector space $T_q\mathbb{R}^n$ onto the vector space $\mathcal{D}_q$ [**5**, p. 34]. Note that, by identifying the elements X_q^*

with X_q through the previous isomorphism, one can take $\{\partial/\partial r_i, ..., \partial/\partial r_n\}$ as the canonical basis for $T_q\mathbb{R}^n$.

Because taking derivatives depends only on local properties of functions, then previous example can be used to extend the notion of tangent vectors to smooth manifolds. By choosing an arbitrary coordinate neighborhood (U, φ) of $p \in U \subset M$, one can check, the same way as for the case of the euclidean space, that the set formed by smooth functions on U, identifying the ones that agree on the latter, is an algebra called the algebra of germs of smooth functions at p, denoted also $\mathcal{F}_p$. It can be verified that the map $\varphi^* : \mathcal{F}_{\varphi(p)} \to \mathcal{F}_p$, given by $\varphi^*(\mathbf{f}) = \mathbf{f} \circ \varphi$ is an isomorphism of $\mathcal{F}_{\varphi(p)}$ onto $\mathcal{F}_p$.

Definition. A *tangent vector* to M at $p \in M$, denoted X_p, is a linear derivation of the algebra $\mathcal{F}_p$. T_pM denotes the set of tangent vectors to M at p and it is called the *tangent space to M at p.*

The following result establishes a vector space homomorphism between the tangent space to a manifold at a point and the tangent space to another manifold at the image of that point through a mapping. As a corollary we will be able to stablish a 1:1 relation between the tangent vectors to a manifold at a point and the tangent vectors to $\mathbb{R}^n$ at the point given by the image of the corresponding coordinate map [**5**, p. 108].

Theorem 1.17. *Let $\Psi : M \to N$ be a smooth map, that is, $\varphi \circ \Psi \circ \tau^{-1}$ is C^∞, for any two coordinate maps φ and τ of M and N, respectively. Then, for $p \in M$ the map $\Psi^* : \mathcal{F}_{\Psi(p)} \to \mathcal{F}_p$ defined by $\Psi^*(f) = f \circ \Psi$ is a homomorphism of algebras and induces a dual vector space homomorphism $\Psi_* : T_pM \to T_{\Psi(p)}N$, defined by $\Psi_*(X_p)(f) = X_p(\Psi^*(f)) = X_p(f \circ \Psi)$, which gives $\Psi_*(X_p)$ as a map of $\mathcal{F}_{\Psi(p)}$ to $\mathbb{R}$. When $\Psi : M \to M$ is the identity, both Ψ^* and Ψ_* are the identity isomorphism. If $\Upsilon = \Gamma \circ \Psi$ is a composition of smooth maps between manifolds, then $\Upsilon^* = \Psi^* \circ \Gamma^*$ and $\Upsilon_* = \Gamma_* \circ \Psi_*$.*

Remark 1.18. The homomorphism Ψ_* is called the *differential* of Ψ. Among other notations for the latter are $d\Psi$, $D\Psi$ and Ψ'.

Corollary 1.19. *If $\Psi : M \to N$ is a diffeomorphism of M onto an open set $U \subset N$ and $p \in M$, then $\Psi_* : T_pM \to T_{\Psi(p)}N$ is an isomorphism onto.*

Definition. Let $(U, \varphi = (x_1, ..., x_n))$ be any coordinate neighborhood on M. Then, for each $i \in \{1, ..., n\}$, we define a tangent vector $\frac{\partial}{\partial x_i}\big|_p \in T_pM$ by

$$\frac{\partial}{\partial x_i}\bigg|_p := \varphi_*^{-1}\left(\frac{\partial}{\partial r_i}\bigg|_{\varphi(p)}\right)$$

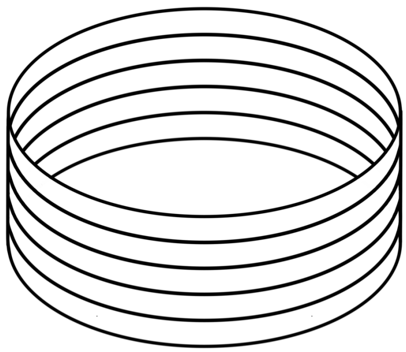
where $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is the isomorphism induced by $\varphi : U \subset M \to \mathbb{R}^n$ at $p \in M$ and $\frac{\partial}{\partial r_i}\big|_{\varphi(p)}$ is the i-th element of the canonical basis $\left\{\frac{\partial}{\partial r_i}\big|_{\varphi(p)} : i = 1, ..., n\right\}$ for $T_{\varphi(p)}\mathbb{R}^n$. Then, $\left\{\frac{\partial}{\partial x_i}\big|_p : i = 1, ..., n\right\}$ will be a basis for T_pM and hence $\dim T_pM = \dim M$.

Remark 1.20. In practice we will treat tangent vectors as operating on functions rather than on their germs. If f is a smooth function defined on a neighborhood of p and $X_p \in T_pM$, we define $X_p(f) = X_p(\mathbf{f})$. Thus, $X_p(f) = X_p(g)$ whenever f and g agree on a neighborhood of p. Moreover, f and g clearly satisfy properties (2) and (3) with $f + \lambda g$ and $f \cdot g$ defined on the intersection of the domains of definition of f and g.

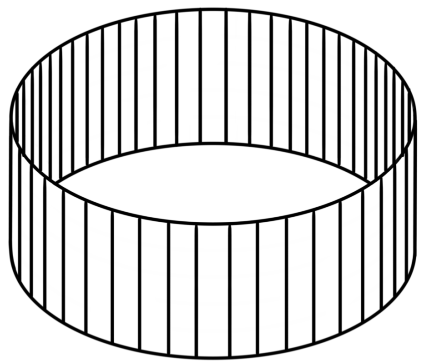
Vector Bundles

DEFINITION. A *bundle* is a triple $\zeta = (E, p, B)$, where E and B are sets called *total space* and *base space*, respectively, and $p : E \to B$ is a *projection*. For each $b \in B$, $p^{-1}(b)$ is called the *fibre* over b.

EXAMPLE 1.21. A *cylinder* is a product, for it is both an interval of circles and a circle of intervals. Thus, it is a bundle whose total space can be expressed as a product. Then it will be either a circle bundle (S^1-bundle) over an interval or an interval bundle (I-bundle) over a circle, as shown in Figure 3.



(A) $[0,1] \times S^1 \to [0,1]$



(B) $S^1 \times [0,1] \to S^1$

FIGURE 3. Cylinder.

The following example shows a bundle whose total space is not a product.

EXAMPLE 1.22. A *Möbius strip* (Figure 4) is an I-bundle over a circle, $\iota_M = (E_M, p, S^1)$, but it is not an interval of circles and hence E_M cannot be expressed like a product.

DEFINITION. A *trivial bundle* is a bundle whose total space can be expressed like a product. Example 1.21 is a trivial bundle.

DEFINITION. A *section* of a bundle (E, p, B) is a map $s : B \to E$ such that $p \circ s = id_B$, that is, $s(b) \in p^{-1}(b)$ for every $b \in B$.

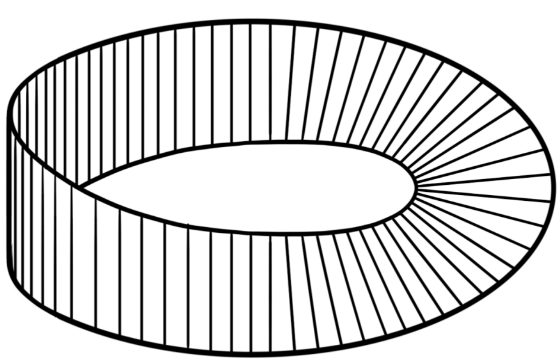


FIGURE 4. Möbius strip

In 1935, Hassler Whitney treated the case of a sphere bundle in general, that is, a bundle whose fibers are arbitrary spheres S^n. He studied the cohomology classes of the latter, which he called *characteristic classes*. Later on, the concept of characteristic cohomology class was generalized to other kinds of bundles. We will focus on smooth vector bundles.

Definition. Let B and E be topological spaces. A *real vector bundle of rank n* over B is a bundle $\zeta = (E, p, B)$ along with a real vector space structure of dimension n on each fibre, which satisfies the so called *local trivialization* condition: for each $b \in B$ there is an open neighborhood $U \subseteq B$, with $b \in U$, and a homeomorphism $h : p^{-1}(U) \to U \times \mathbb{R}^n$ so that the restriction $p^{-1}(b) \to \{b\} \times \mathbb{R}^n$ is an isomorphism of vector spaces. If we can choose $U = B$, then ζ is a trivial bundle.

Below we give two important examples of smooth real vector bundles, that is, a real vector bundle whose base and total spaces are smooth manifolds and the local trivialization map is a diffeomorphism.

The Tangent Bundle. Let M be a smooth manifold of dimension n and define

$$TM = \bigsqcup_{p \in M} T_pM$$

where T_pM is the tangent space to M at p. We are going to show that, in a natural way, TM is a 2n-dimensional smooth manifold.

Let $\mathcal{F} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A \subset \mathbb{N}}$ be a smooth structure on M and consider the projection $\pi : TM \to M$, defined by $\pi(v) = p$. Let $x_1^\alpha, ..., x_n^\alpha$ be the coordinate functions of the corresponding coordinate map and define, for every α and for all $v \in \pi^{-1}(U_\alpha)$, the map

$$\begin{aligned} \hat{\varphi}_\alpha : \pi^{-1}(U_\alpha) \subset TM &\to \mathbb{R}^n \times \mathbb{R}^n \\ v &\mapsto (x_1^\alpha(\pi(v)), ..., x_n^\alpha(\pi(v)), dx_1^\alpha(v), ..., dx_n^\alpha(v)) \end{aligned}$$

Let β be a countable basis for a topology on M and define

$$\beta' := \{U \in \beta : \exists \alpha \in A \text{ with } U \subset U_\alpha\} \subset \beta$$

Since β' is also countable, then we can define the map

$$\begin{aligned} \psi : \beta' &\to A \\ U &\mapsto \alpha \quad \text{if } U \subset U_\alpha \end{aligned}$$

Thus, $\mathcal{F}' := \{(U, \varphi_{\psi(U)}|_U) : U \in \beta'\}$ constitutes a countable smooth structure on M. The collection $\{\hat{\varphi}^{-1}(W) : W \text{ is open in } \mathbb{R}^{2n},\ (U, \varphi) \in \mathcal{F}'\}$ forms, then, a countable basis for a topology on TM and hence the latter is a second-countable topological space. Also, note that $\hat{\varphi}_\alpha$ is a 1:1 map onto an open subset of $\mathbb{R}^{2n}$, which means that TM is locally homeomorphic to $\mathbb{R}^{2n}$. Furthermore, because M is Hausdorff, then TM is also Hausdorff. Certainly, given $v_1, v_2 \in TM$ such that $v_1 \neq v_2$: if $p_1 := \pi(v_1) \neq \pi(v_2) =: p_2$, then p_1 and p_2 have disjoint neighborhoods and so v_1 and v_2 will also have disjoint neighborhoods; in case $\pi(v_1) = \pi(v_2) = p$ then v_1 and v_2 lie in T_pM, which is isomorphic to $\mathbb{R}^n$ and hence Hausdorff.

Moreover, for any two coordinate neighborhoods $(U_\alpha, \varphi_\alpha := (x_1^\alpha, ..., x_n^\alpha))$, $(U_\beta, \varphi_\beta := (x_1^\beta, ..., x_n^\beta))$ in $\mathcal{F}$, with $U_\alpha \cap U_\beta \neq \emptyset$, the composition $\hat{\varphi}_\beta \circ \hat{\varphi}_\alpha^{-1}$ is smooth. Indeed, if

$(\vec{a},\vec{b}) \in \mathbb{R}^n \times \mathbb{R}^n$, then

$$\begin{aligned}\hat{\varphi}_\beta \circ \hat{\varphi}_\alpha^{-1}(\vec{a},\vec{b}) =& \hat{\varphi}_\beta\left(\sum_{i=1}^n b_i \frac{\partial}{\partial x_i^\alpha}\bigg|_{\varphi_\alpha^{-1}(\vec{a})}\right)\\ =& \left(\varphi_\beta \circ \varphi_\alpha^{-1}(\vec{a}), dx_1^\beta\left(\sum_{i=1}^n b_i \frac{\partial}{\partial x_i^\alpha}\bigg|_{\varphi_\alpha^{-1}(\vec{a})}\right), ..., dx_n^\beta\left(\sum_{i=1}^n b_i \frac{\partial}{\partial x_i^\alpha}\bigg|_{\varphi_\alpha^{-1}(\vec{a})}\right)\right)\end{aligned}$$

is smooth in that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth by definition and dx_i^β is a linear transformation.

Therefore, a maximal collection $\hat{\mathcal{F}}$ containing $\{(\pi^{-1}(U_\alpha), \hat{\varphi}_\alpha) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}\}$ is a smooth structure on TM and so $(TM, \hat{\mathcal{F}})$ is a smooth manifold of dimension 2n.

Remark 1.23. The points of TM are normally written as pairs (p, v), where $p \in M$ and $v \in T_pM$.

Proposition 1.24. *$\tau_M := (TM, \pi, M)$ constitutes a smooth real vector bundle called the tangent bundle. Indeed, $\forall p \in M$,*

(i) there is a real vector space structure on the fibre $\pi^{-1}(p)$, defined by

$$t_1(p, v_1) + t_2(p, v_2) = (p, t_1v_1 + t_2v_2)$$

where $t_1, t_2 \in \mathbb{R}$ and $v_1, v_2 \in T_pM$

(ii) by construction, if (U, φ) is a coordinate neighborhood, with $p \in U$, then $\hat{\varphi} : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a diffeomorphism and the restriction $\pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is an isomorphism of vector spaces. ■

Definition. A *tangent vector field* on M, hereafter a *vector field* on M, is a section of τ_M. Thus, it assigns, to each $p \in M$, a tangent vector $X_p \in T_pM$. $\mathfrak{X}(M)$ will denote the set of all smooth vector fields on a smooth manifold M. It is, in the obvious way, a vector space over $\mathbb{R}$, but also a $C^\infty(M)$-module, for if $X \in \mathfrak{X}(M)$ is a smooth vector field on $U \subseteq M$ and $p \in U$, then, for any $f \in C^\infty(U)$, $X(f)$ is a function on U, whose value at p is $X_p(f)$, namely, the directional derivative of f in the direction of the tangent vector X_p.

The Normal Bundle. Let $M \subset \mathbb{R}^n$ be a smooth manifold and let $E \subset M \times \mathbb{R}^n$ be the set of all pairs (p, v) such that v is orthogonal to the tangent space T_pM. It can be proved that $\nu_M := (NM, \pi, M)$, with $\pi(p, v) = p$, is a smooth real vector bundle of rank n, the *normal bundle*.

Riemannian Geometry

From now on, $M = M^n$ will denote a smooth n-dimensional manifold.

Definition. A *Riemannian metric* on M is a correspondence which associates to each point $p \in M$ an inner product $\langle \ , \ \rangle$ on the tangent space T_pM, which varies smoothly, that is, given any coordinate neighborhood $(U, (x_1, ..., x_n))$ about p, $\langle \frac{\partial}{\partial x_i}\big|_p, \frac{\partial}{\partial x_j}\big|_p \rangle$ is a smooth function on U for all i, j. A smooth manifold endowed with a Riemannian metric is called a *Riemannian manifold*.

Using a partition of unity subordinate to any covering of a manifold M by coordinate neighborhoods, one easily proves M possesses a Riemannian metric [**6**, p. 43 prop. 2.10].

Definition. Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is an *isometry* if, for any coordinate neighborhood $(U, (x_1, ..., x_n))$ about $p \in U \subset M$,

$$\left\langle \frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p \right\rangle = \left\langle f_*\Big(\frac{\partial}{\partial x_i}\Big)\Big|_{f(p)}, f_*\Big(\frac{\partial}{\partial x_j}\Big)\Big|_{f(p)} \right\rangle$$

We say that two Riemannian manifolds are equivalent when there is an isometry between them.

Covariant Differentiation. We will use the differentiation of vector fields along parametrized curves in $\mathbb{R}^3$ as a model for defining differentiation of vector fields along a parametrized curve in a Riemannian manifold.

Definition. A *connection* ∇ on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoted by $\nabla : (X, Y) \to \nabla_X Y$, such that:

(1) $\nabla_X(Y + Y') = \nabla_X Y + \nabla_X Y'$
(2) $\nabla_{X+X'} Y = \nabla_X Y + \nabla_{X'} Y$
(3) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$
(4) $\nabla_{fX} Y = f\nabla_X Y$

where $f \in C^\infty(M)$ and $X, X', Y, Y' \in \mathfrak{X}(M)$.

Thus, given a coordinate neighborhood $(U, (x_1, ..., x_n))$ in M and a couple of vector fields $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $Y = \sum_j b_j \frac{\partial}{\partial x_j}$,

$$\begin{aligned}
\nabla_X Y &\overset{(1)}{=} \left(\sum_j \nabla_X \Big(b_j \frac{\partial}{\partial x_j} \Big) \right) \\
&\overset{(3)}{=} \sum_j X(b_j) \frac{\partial}{\partial x_j} + \sum_j \left(b_j \nabla_X \frac{\partial}{\partial x_j} \right) \\
&\overset{(2)}{=} \sum_j X(b_j) \frac{\partial}{\partial x_j} + \sum_j \left(b_j \sum_i \nabla_{a_i \frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \\
&\overset{(4)}{=} \sum_j X(b_j) \frac{\partial}{\partial x_j} + \sum_j \left(b_j \sum_i a_i \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \\
&= \sum_j X(b_j) \frac{\partial}{\partial x_j} + \sum_{i,j} \left(a_i b_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)
\end{aligned}$$

Now writing $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$ as a linear combination of the basis $\left\{\frac{\partial}{\partial x_k}\right\}$ at each point,

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where Γ_{ij}^k are then smooth functions on U called *Christoffel symbols* of ∇ in $(U, (x_1, ..., x_n))$.

Definition. A *vector field X along a parametrized curve* c in M is a function which assigns to each $t \in \mathbb{R}$ a tangent vector $X_{c(t)} \in T_{c(t)}M$. The differential $c_* : T_t\mathbb{R} \to T_{c(t)}M$ is a vector field along c which is called *velocity* vector field.

One easily checks [**7**, p. 46 lemma 8.1] that, given a connection ∇ on M, there is one and only one correspondence which associates, to a vector field V along a parametrized curve $c : \mathbb{R} \to M$, another vector field $\frac{DV}{dt}$ along c such that:

(a) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$, where W is a vector field along c
(b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, where f is a smooth function on $\mathbb{R}$
(c) if V is induced by a vector field $Y \in \mathfrak{X}(M)$, then $\frac{DV}{dt} = \nabla_{dc/dt}Y$

Definition. Such a vector field is called the *covariant derivative* of V.

If $x_1(t), ..., x_n(t)$ denote the local coordinates of a point $c(t)$ in some coordinate neighborhood of M and $\sum_j v_j \frac{\partial}{\partial x_j}$ is the expression of V, then, using properties (a), (b) and (c), one obtains the following expression for the covariant derivative of V along $c : \mathbb{R} \to M$

$$\frac{DV}{dt} = \sum_k \Big(\frac{dv_k}{dt} + \sum_{i,j} \Gamma^k_{i,j} v_j \frac{dx_i}{dt}\Big)\frac{\partial}{\partial x_k} \tag{4}$$

The notion of parallelism now follows naturally.

Definition. A vector field V along a parametrized curve $c : \mathbb{R} \to M$ is called *parallel* if, in the associated connection, $\frac{DV}{dt} = 0$ for all $t \in \mathbb{R}$

Remark 1.25. V is a parallel vector field along c if and only if

$$0 = \frac{dv_k}{dt} + \sum_{i,j} \Gamma^k_{i,j} v_j \frac{dx_i}{dt} \quad k = 1, ..., n$$

which is a system of ordinary differential equations. Its solutions $v_k(t)$ will be uniquely determined by the initial conditions $v_k(0)$. Thus, if V exists, it is unique. Moreover, because the system is linear, any solution is defined for all $t \in \mathbb{R}$, which proves the existence of V with the desired properties [**8**, p. 162]. Thus, if $X_p \in T_pM$, where $p = c(t_0)$, $t_0 \in \mathbb{R}$, there is one and only one parallel vector field V along c which extends $V_{c(t_0)}$.

Definition. The tangent vector $V_{c(t)}$ is said to be obtained from $V_{c(t_0)} = X_p$ by *parallel translation* along c. A connection on a Riemannian manifold M is *compatible* with the metric if parallel translation preserves inner products.

Geodesics. Let M be a Riemannian manifold together with a connection ∇. A curve $\gamma : I \subseteq \mathbb{R} \to M$ is a *geodesic* if $\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = \frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ for all $t \in I$. Then, the vector field $\frac{d\gamma}{dt}$ is parallel along γ and hence $\|\frac{d\gamma}{dt}\|$ is constant, say $\|\frac{d\gamma}{dt}\| = c$. Thus, the arc-length of the geodesic between any two points is

$$L(\gamma; a, b) := \int_a^b \Big\|\frac{d\gamma_q}{dt}\Big\| dt = c(b-a)$$

In terms of a local coordinate neighborhood $(U, (x_1, ..., x_n))$, a parametrized curve $\gamma : I \to M$ determines n smooth functions $x_1(t), ..., x_n(t)$. Thus, according to (4), γ is a geodesic if and only if

$$0 = \frac{D}{dt}\Big(\frac{d\gamma}{dt}\Big) = \sum_k \Big(\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt}\Big)\frac{\partial}{\partial x_k}$$

The existence of geodesics depends, then, on the solution of the following system of second order differential equations

$$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \quad k = 1, ..., n$$

It is very useful to turn this system of n differential equations into a system of $2n$ differential equations, by considering the tangent bundle $\tau_U = (TU, \pi, U)$. To this end, because any parametrized curve $t \to \gamma(t)$ determines a curve $t \to (\gamma(t), \frac{d\gamma}{dt}(t))$ in TU, then, if γ is a geodesic, the corresponding $2n$ coordinates $x_1(t), ..., x_n(t), \frac{dx_1}{dt}(t), ..., \frac{dx_n}{dt}(t)$ in TU satisfy the system

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j} \Gamma^k_{i,j} y_i y_j \quad k = 1, ..., n \end{cases} \tag{5}$$

Expression (5) yields the definition of a vector field on TU, with integral curves of the form $t \to (x_1(t), ..., x_n(t), y_1(t), ..., y_n(t))$, assigning, to each point $(x_1, ..., x_n, y_1, ..., y_n) \in TU$, the vector $(y_1, ..., y_n, -\sum_{i,j} \Gamma^1_{i,j} y_i y_j, ..., -\sum_{i,j} \Gamma^n_{i,j} y_i y_j)$. Then, by conclusion of Remark 1.25, such a vector field extends uniquely to TM. It is called the *geodesic field* on TM.

On the other hand, since a smooth n-manifold is locally diffeomorphic to $\mathbb{R}^n$, the fundamental theorem on existence and uniqueness of ordinary differential equations [**9**, p. 55], which is a local result, extends naturally to smooth manifolds. Therefore, we have reduced the problem of integrating a geodesic to that of finding the flow of the geodesic field on TU. In particular, we have the following result [**5**, p. 334 thm. 5.8]

THEOREM 1.26. *For every point $p \in M$ there exists a neighborhood V about p, a number $\epsilon > 0$ and a smooth map $\gamma : (-2, 2) \times \mathcal{V} \to M$, with $\mathcal{V} := \{(q, X_q) \in TV : \|X_q\| < \epsilon\}$, such that the curve $t \to \gamma(t, q, X_q)$ is the unique geodesic which, at instant $t = 0$, passes through q with velocity vector X_q.*

DEFINITION. Let $\mathcal{V}$ be chosen as in Theorem 1.26 and let $\gamma_q : [0, 1] \to M$ be the unique geodesic which, at $t = 0$, passes through q with velocity vector X_q. We define the *exponential map* to be the following smooth map

$$\begin{aligned} Exp : \mathcal{V} &\to M \\ (q, X_q) &\mapsto \gamma_q(1) \end{aligned}$$

REMARK 1.27. *$Exp(q, X_q)$* is the point on γ_q, whose distance from q along the geodesic is

$$L(\gamma_q; 0, 1) = \|X_q\|$$

Note that, in general, $Exp(q, X_q)$ is not defined for "large" vectors X_q.

DEFINITION. The manifold M is *geodesically complete* if $Exp(q, X_q)$ is defined for all $q \in M$ and all vectors $X_q \in T_qM$. That is, every geodesic $\gamma_0 : [a, b] \to M$ can be extended to a geodesic $\gamma : \mathbb{R} \to M$.

DEFINITION. A geodesic $\gamma : [a, b] \to M$ is called *minimal* if its length is less than or equal to the length of any other piecewise smooth path joining its endpoints.

Below we state a result concerning geodesically complete manifolds [**7**, p. 62 thm. 10.9].

THEOREM 1.28. *If M is geodesically complete, then any two points can be joined by a minimal geodesic*

Geometry and Topology of the Normal Bundle. In order to study characteristic classes of ν_M we need the following geometrical result.

TUBULAR NEIGHBORHOOD THEOREM 1.29. *Let $M = M^n$ be a smooth manifold which is smoothly and topologically embedded in a Riemannian manifold $A = A^{n+k}$. Then there exists an open neighborhood of M in A, which is diffeomorphic to the total space of the normal bundle $\nu_M = (NM, \pi, M)$ under a diffeomorphism that maps each point $x \in M$ to the zero normal vector at x.*

Proof.[2] Let $NM(\epsilon) = \{(x, v) \in NM : \|v\| < \epsilon\} \subset NM$. The exponential map is defined, in this case, as $Exp : NM(\epsilon) \to A$, assigning to each $(x, v) \in NM$, with $\|v\|$ small enough, the value $\gamma(1)$ of the geodesic $\gamma : [0, 1] \to A$ that has length $\|v\|$, such that $\gamma(0) = x$ and $\left.\frac{d\gamma}{dt}\right|_{t=0} = v$. The existence and uniqueness theorem for differential equations guarantees Exp is defined and smooth throughout some neighborhood of the zero section. Thus, applying the inverse function theorem at any point $(x, 0)$ on the latter, we see that some open neighborhood of $(x, 0)$ in $NM(\epsilon)$ is mapped diffeomorphically onto an open subset of A. On the other hand, since M is compact, for every integer $i > 0$ there exist points $(x_i, v_i) \neq (x'_i, v'_i)$ and a convergent subsequence $\{x_{i_j}\}$ such that

$$\lim(Exp(x_{i_j})) = Exp(x, 0) = x \neq \lim(Exp(x'_{i_j})) = Exp(x', 0) = x'$$

and so Exp is 1:1 onto a small enough neighborhood of $(x, 0)$.

Therefore, the whole open set $NM(\epsilon)$ is in fact mapped diffeomorphically onto an open neighborhood $N_\epsilon \subset A$ (tubular) by the exponential map. Finally, the correspondence $(x, v) \mapsto \left(x, v/\sqrt{1 - \|v\|^2/\epsilon(x)^2}\right)$ guarantees the desired diffeomorphism between $NM(\epsilon)$ and NM. ■

COROLLARY 1.30. *If M is closed in A, then the ring $H^*(NM, NM_0)$ associated with the normal bundle of M in A is isomorphic to the ring $H^*(A, A - M)$.*

Proof.[3] Let N_ϵ be the tubular neighborhood of Theorem 1.29. On the one hand, we have the following embedding

$$Exp : (NM(\epsilon), NM(\epsilon)_0) \to (N_\epsilon, N_\epsilon - M) \subset (A, A - M)$$

On the other hand, since $N_\epsilon \cup (A - M) = A$ and $N_\epsilon \cap (A - M) = N_\epsilon - M$, there is an excision isomorphism

$$H^*(A, A - M) \to H^*(N_\epsilon, N_\epsilon - M)$$

Thus, we have the following induced isomorphism on cohomology

$$Exp^* : H^*(A, A - M) \to H^*(NM(\epsilon), NM(\epsilon)_0).$$

Therefore, composing with the excision isomorphism $H^*(NM(\epsilon), NM(\epsilon)_0) \cong H^*(NM, NM_0)$, we obtain the desired isomorphism. ■

Now we have all we need for proving the following corollary of Thom isomorphism theorem.

LEMMA 1.31. *Let $M = M^r, M' = M'^s$ be two closed, connected and smooth submanifolds of a closed and connected manifold $V = V^{r+s}$. Then $H_0(M) \cong H_s(V, V - M)$*

Proof. Consider the normal bundle $\nu_M = (NM, p, M)$ of M in V, of rank s. On the one hand, by Corollary 1.30, $H^*(NM, NM_0) \cong H^*(V, V - M)$. On the other hand, by Thom isomorphism theorem, if $u \in H^s(NM, NM_0)$, the correspondence $c \mapsto c \smile u$ induces the

[2] We assume M to be compact. It will suffice for our purpose.

[3] NM_0 stands for the complement of zero section, that is, $NM_0 := NM - (M \times 0)$.

isomorphism $H^j(NM) \cong H^{j+s}(NM, NM_0), \forall j \in \mathbb{Z}_{\geq 0}$. Thus, $H^0(NM) \cong H^s(V, V-M)$. Now, let $\sigma : M \to NM$ denote the zero section of ν_M, inducing a canonical isomorphism

$$\sigma^* : H^*(NM) \to H^*(M)$$

Then $H^0(M) \cong H^s(V, V-M)$ and therefore $H_0(M) \cong H_s(V, V-M)$. ■

CHAPTER 2

A Rapid Course in Morse Functions

DEFINITION. $(W; V_0, V_1)$ is a *smooth manifold triad*, henceforth simply a *triad*, if W is a compact smooth manifold and ∂W is the disjoint union of two open and closed submanifolds V_0 and V_1.

DEFINITION. Let W be a smooth manifold and let $f : W \to \mathbb{R}$ be a smooth function. $p \in W$ is a *critical point* if, in some coordinate neighborhood,

$$\frac{\partial f}{\partial x_1}\bigg|_p = \cdots = \frac{\partial f}{\partial x_n}\bigg|_p = 0$$

If, in addition, $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p\right) \neq 0$, such a point is *non-degenerate*.

DEFINITION. A *Morse function* on a triad $(W; V_0, V_1)$ is a smooth function $f : W \to [a, b]$ such that

(i) $f^{-1}(a) = V_0, \;\; f^{-1}(b) = V_1$
(ii) All the critical points of f are interior and non-degenerate.

Using the fundamental theorem of calculus and basic Analysis, one can prove the following important result [**7**, p. 6].

MORSE LEMMA 2.1. *If p is a non-degenerate critical point of f, then, in some coordinate neighborhood $(U, (x_1, ..., x_n))$, with $p \in U$,*

$$f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

λ is called the index of p.

COROLLARY 2.2. *Because critical points of a Morse function on a triad $(W; V_0, V_1)$ are non-degenerate, they are all isolated. Moreover, since W is compact, there are only finitely many of them.*

DEFINITION. Among all possible Morse functions on a given triad $(W; V_0, V_1)$, the *Morse number* $\mu(W; V_0, V_1)$ is defined to be the amount of critical points of the one with the fewest.

Existence of Morse Functions

The aim of this section is to prove that every triad possesses a Morse function. Before, though, we need to define a topology on the set $F(M, \mathbb{R})$ of real functions on compact topological manifolds and prove a corollary of an important result of Analysis.

The $\mathbf{C^2}$ Topology on $\boldsymbol{F(M, \mathbb{R})}$. We will make use of the following result for defining such a topology.

LEMMA 2.3. *Let $h : U \to U'$ be a diffeomorphism of an open subset of $\mathbb{R}^n$ onto another, which sends the compact $K \subset U$ onto $K' \subset U'$. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that, for a given smooth function $f : U' \to \mathbb{R}$,*

$$|f \circ h| < \epsilon, \;\; \left|\frac{\partial (f \circ h)}{\partial x_i}\right| < \epsilon, \;\; \left|\frac{\partial^2 (f \circ h)}{\partial x_i \partial x_j}\right| < \epsilon \quad i, j = 1, ..., n$$

at all points of K if

$$|f| < \delta, \ \left|\frac{\partial f}{\partial x_i}\right| < \delta, \ \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| < \delta, \quad i, j = 1, ..., n$$

at all points of K'.

Proof. Applying the chain rule, one gets

$$\frac{\partial(f \circ h)}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial f}{\partial h_k}\frac{\partial h_k}{\partial x_i}$$

$$\frac{\partial^2(f \circ h)}{\partial x_i \partial x_j} = \sum_{k=1}^{n}\left(\frac{\partial(f \circ h)}{\partial h_k}\frac{\partial^2 h_k}{\partial x_i \partial x_j}\right) + \sum_{k=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial^2(f \circ h)}{\partial h_k \partial h_l}\frac{\partial h_k}{\partial x_i}\frac{\partial h_l}{\partial x_j}\right)$$

Thus, each of $f \circ h$, $\frac{\partial(f\circ h)}{\partial x_i}$, $\frac{\partial^2(f\circ h)}{\partial x_i \partial x_j}$ vanishes when the derivatives, from order 0 to order 2, of f vanish. Now the result follows from the fact that the derivatives of h are bounded on the compact set K. ■

Definition. Let $\{(U_\alpha, h_\alpha)\}$ be a finite collection of coordinate neighborhoods such that $\{U_\alpha\}$ is a cover of M. Let $\{C_\alpha\}$ be a compact refinement of $\{U_\alpha\}$, that is, $\{C_\alpha\}$ is a compact cover of M such that, for every C_α, there is a U_α with $C_\alpha \subset U_\alpha$. For every positive constant $\epsilon > 0$ and every $f \in F(M, \mathbb{R})$, define the subset $N(f, \epsilon)$ of $F(M, \mathbb{R})$, consisting of all maps $g : M \to \mathbb{R}$ such that, for all α,

$$|f_\alpha(p) - g_\alpha(p)| < \epsilon, \ \left|\frac{\partial f_\alpha}{\partial x_i}(p) - \frac{\partial g_\alpha}{\partial x_i}(p)\right| < \epsilon, \ \left|\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}(p)\right| < \epsilon$$

at all $p \in h_\alpha(C_\alpha)$ and $i, j = 1, ..., n$, where $f_\alpha := f \circ h_\alpha^{-1}$ and $g_\alpha := g \circ h_\alpha^{-1}$. Now, Lemma 2.3 guarantees that, given any set $N(0, \epsilon)$, we can find a set $N'(0, \epsilon')$, where $N'(f, \epsilon')$ is a subset of $F(M, \mathbb{R})$ defined as above, for another choice of coordinate covering and compact refinement. Therefore, there is a well-defined topology on $F(M, \mathbb{R})$, the C^2 *topology*, which results from taking the sets $N(f, \epsilon)$ as a basis of neighborhoods of any function f in the additive group $F(M, \mathbb{R})$.

On the Measure of Critical Values of $\mathbf{C^1}$ Maps. [10]

Definition. In $\mathbb{R}^n$, a set is said to be *of measure zero* if there is a finite or countably infinite covering by balls for which the sum of the volumes is arbitrarily small.

Below we state a theorem which is a consequence of the result proved in Sard's article, cited above.

Sard's Theorem 2.4. *Let $M = M^m$ and $N = N^n$, with $m \leq n$, be two smooth manifolds and let $\mu : M \to N$ be C^1. Then the image $\mu(E)$ of the set E of critical points of μ is a set of measure zero in N.*

Corollary 2.5. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2, then, for almost all*[1] *linear mappings $L : \mathbb{R}^n \to \mathbb{R}$, the function $f + L$ has only nondegenerate critical points.*

Proof. Consider the product manifold $U \times Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ and the submanifold formed by all elements (x, L) such that $d(f(x) + L(x)) = 0$. Since L is a linear transformation, its differential is itself and so we can write such a submanifold as $M = \{(x, L) : L = -df(x))\}$.

[1]By *almost all* we mean except for a set of measure zero in $Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}$.

Thus, the correspondence $x \mapsto (x, -df(x))$ is a diffeomorphism of U onto M and hence the map

$$\pi : U \subset \mathbb{R}^n \to Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$$
$$x \mapsto -df(x)$$

is C^1. Now applying Theorem 2.4, the image of the set of critical points of π is of measure zero in $\mathbb{R}^n$. On the other hand, by definition, each $(x, L) \in M$ corresponds to a critical point of the function $f + L$, which is precisely degenerate when π is critical, that is, when the matrix $\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)$ is singular. Therefore, $f + L$ has a degenerate critical point if and only if L is of measure zero, that is, $f + L$ has only nondegenerate critical points for almost all L. ■

Existence Theorem of Morse Functions. In order to prove the existence of Morse functions for any triad we need the following lemma.

LEMMA 2.6. *Let K be a compact subset of an open U in $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a C^2 function whose critical points are all nondegenerate in K. Then there is a number $\epsilon > 0$ such that, if $g : U \to \mathbb{R}$ is C^2 and, at all K,*

$$(1)\ \Big|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\Big| < \epsilon, \quad (2)\ \Big|\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}\Big| < \epsilon$$

where $i, j = 1, ..., n$, then g likewise has only nondegenerate critical points in K.

Proof. Let $|df| = \big[\big(\frac{\partial f}{\partial x_1}\big)^2 + \cdots + \big(\frac{\partial f}{\partial x_n}\big)^2\big]^{1/2}$. By hypothesis, for all $p \in K$, $\det\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)\big|_p \neq 0$ and hence $|df| + \big|\det\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)\big| > 0 \quad \forall p \in K$. Let $\mu := \min\big(|df| + \big|\det\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)\big|\big)$ and choose $\epsilon > 0$ such that (1) and (2) imply $\big||df| - |dg|\big| < \mu/2$ and $\Big|\big|\det\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)\big| - \big|\big(\frac{\partial^2 g}{\partial x_i \partial x_j}\big)\big|\Big| < \mu/2$, respectively. Then $|dg| + \big|\det\big(\frac{\partial^2 g}{\partial x_i \partial x_j}\big)\big| > |df| + \big|\det\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)\big| - \mu \geq 0$ at all points in K. Therefore g cannot have degenerate critical points in K, otherwise, if $p \in K$ were a degenerate critical point, then

$$|dg(p)| + \left|\Big(\frac{\partial^2 g}{\partial x_i \partial x_j}\Big)\Big|_p\right| = 0$$

which is a contradiction. ■

THEOREM 2.7. *Every triad $(W; V_0, V_1)$ possesses a Morse function.*

Proof. Let $U_1, ..., U_k$ be a finite cover of W by coordinate neighborhoods such that, $\forall i = 1, ..., k$, $U_i \cap V_0 \cap V_1 = \emptyset$ and, if $U_i \cap \partial W \neq \emptyset$, the corresponding coordinate map $h_i : U_i \to \mathbb{R}^n_+$ sends U_i onto the intersection of the open unit ball with $\mathbb{R}^n_+$. Define $f_i : W \to [0, 1]$ as follows

$$f_i(p) = \begin{cases} 0 & \text{if} \quad p \notin U_i \\ 1/2 & \text{if} \quad U_i \cap \partial W = \emptyset \\ x_n & \text{if} \quad U_i \cap V_0 \neq \emptyset \\ 1 - x_n & \text{if} \quad U_i \cap V_1 \neq \emptyset \end{cases}$$

where $x = h_i(p)$. Then, choosing a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{U_i\}$, one can define the following smooth function on W onto $[0, 1]$

$$f(p) = \varphi_1(p) f_1(p) + \cdots + \varphi_k(p) f_k(p) \quad \text{with} \quad f^{-1}(0) = V_0,\ f^{-1}(1) = V_1.$$

Let $p \in \partial W$. Then, on the one hand,

$$\sum_{j=1}^{k} f_j \frac{\partial \varphi_j}{\partial x_n} = f_j \frac{\partial}{\partial x_n}\Big(\sum_{j=1}^{k} \varphi_j\Big) \quad (f_j \text{ has the same value for all } j) \tag{6}$$

$$= 0 \qquad \qquad \left(\sum_{j=1}^{k} \varphi_j = 1 \quad \text{by definition}\right)$$

On the other hand, for some i, $\varphi_i(p) > 0$ and $p \in U_i$ since $\{\varphi_i\}$ is a partition of unity subordinate to $\{U_i\}$. Moreover, $\frac{\partial f_i}{\partial x_n}(p)$ equals either 1 or -1 and it can be checked that the rest of derivatives $\frac{\partial f_j}{\partial x_n}(p)$ all have the same sign as $\frac{\partial f_i}{\partial x_n}(p)$. Thus,

$$\frac{\partial f}{\partial x_n}(p) = \sum_{j=1}^{k} f_j \frac{\partial \varphi_j}{\partial x_n}(p) + \sum_{j=1}^{k} \varphi_j \frac{\partial f_j}{\partial x_n} \overset{(6)}{=} \sum_{j=1}^{k} \varphi_j \frac{\partial f_j}{\partial x_n} \neq 0$$

and hence we have built a function $f : W \to [0,1]$ such that:

(i) $f^{-1}(0) = V_0, \quad f^{-1}(1) = V_1$

(ii) f has no critical points in some neighborhood of ∂W.

Now we need to modify f in such a way that the resulting function does not have degenerate critical points in $W - \partial W$ and it preserves properties (i) and (ii). Firstly, we build a neighborhood of f formed by functions which satisfy such properties. Let U be an open neighborhood of ∂W in which f has no critical points. Because W is a normal topological space, there exists an open V of ∂W such that $\overline{V} \subset U$. We may also assume that each U_i of the finite cover $\{U_i\}$, defined above, lies either in U or in $W - \overline{V}$. Now take a compact refinement $C_1, ..., C_k$ of $\{U_i\}$ and let C_0 be the union of those which lie in U. Then, by Lemma 2.6, there exists a small enough neighborhood N of f, where no function can have a degenerate critical point in C_0. Also, on the compact set $W - V$, f is strictly bounded between 0 and 1. Thus, there is a neighborhood N' of f such that, for every $g \in N'$, $0 < g(q) < 1$ for all $q \in W - V$. Notice that any function in $N_0 := N \cap N'$ satisfies properties (i) and (ii). However, such functions could have degenerate critical points in $W - C_0$. From this point on we proceed as follows.

Consider a smooth function $\lambda : W \to [0,1]$ whose value equals 1 in a neighborhood of C_1 and 0 in a neighborhood of $W - U_1$. Then, Corollary 2.5 guarantees, for almost all choices of a linear map $L : \mathbb{R}^n \to \mathbb{R}$, that the function

$$f_1(p) := f(p) + \lambda(p) L(h_1(p)),$$

has no degenerate critical points in $C_1 \subset U_1$. Manipulating the previous equation one obtains

$$f_1 h_1^{-1}(x) - f h_1^{-1}(x) = (\lambda h_1^{-1}(x)) \sum_{i=1}^{n} \ell_i x_i \tag{7}$$

By choosing the ℓ_i sufficiently small one can guarantee that the difference (7) and its first and second derivatives are less than some $\epsilon > 0$ throughout $h_1(K)$. Now it is crucial to observe that f_1 differs from f only on a compact set $K = supp(\lambda) \subset U_1$. Thus, if ϵ is small enough, it follows from Lemma 2.3 that f_1 belongs to N_0. So we have obtained a function f_1 in N_0, and hence fulfilling properties (i) and (ii), which has no degenerate critical points in $C_0 \cup C_1$. Now, applying Lemma 2.6 again one can choose a neighborhood N_1 of f_1, with $N_1 \subset N_0$, such that no function in N_1 has a degenerate critical point in $C_0 \cup C_1$.

Repeating previous process k times we can choose a function $f_k \in N_k \subset N_{k-1} \subset \cdots \subset N_0$, and hence fulfilling desired properties, which has no degenerate critical points in $C_0 \cup C_1 \cup \cdots \cup C_k = W$. Moreover, since $f_k\big|_V = f\big|_V$, then f_k is a Morse function on (W, V_0, V_1). ■

Gradient-like Vector Fields

Below we prove there exists a special kind of vector field on W for any Morse function on a triad $(W; V_0, V_1)$.

Definition. Let f be a Morse function for the triad $(W; V_0, V_1)$. A vector field ξ is *gradient-like* for f if:

(1) $\xi(f) > 0$ throughout the complement of the set of critical points of f

(2) for every critical point p of f there exist coordinates $(\vec{x}, \vec{y}) = (x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n)$ in a neighborhood U of p such that $f = f(p) - \|\vec{x}\|^2 + \|\vec{y}\|^2$ and

$$\xi_q = (-x_1, ..., -x_\lambda, x_{\lambda+1}, ..., x_n) \quad \forall q \in U$$

Lemma 2.8. *Every Morse function possesses a gradient-like vector field.*

Proof. We can assume, for the sake of simplicity, that f is a Morse function for the triad $(W; V_0, V_1)$ with only one critical point p. By Lemma 2.1, we may choose coordinates $(\vec{x}, \vec{y}) = (x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_n)$ in a neighborhood U_0 of p such that $f = f(p) - \|\vec{x}\|^2 + \|\vec{y}\|^2$ at all points in U_0. By hypothesis, $p' \in W - U_0$ cannot be a critical point of f. Thus, applying implicit function theorem, we can find coordinates in a neighborhood U' of p' such that $f = constant + x_1'$ in U' and hence, if U is a neighborhood of p such that $\overline{U} \subset U_0$, then we can find a finite cover $\{(U_i, (x_1^i, ..., x_n^i))\}_{i=1,...,k}$ of the compact $W - U_0$, by coordinate neighborhoods, such that

- $f = constant + x_1^i$
- $U \cap U_i = \emptyset$

Now, on the one hand, there is the vector field on U_0, whose coordinates are

$$(-x_1, ..., -x_\lambda, x_{\lambda+1}, ..., x_n).$$

On the other hand, on U_i, there is the vector field $\frac{\partial}{\partial x_i}$ whose coordinates are $(1, 0, ..., 0)$. Therefore, we can piece together these vector fields using a partition of unity subordinate to the cover $U_0, ..., U_k$ of W, obtaining a vector field ξ fulfilling conditions (1) and (2). ■

With the help of such a vector field we can now prove a result that justifies the conclusion of h-cobordism theorem.

Theorem 2.9. *If $\mu(W; V_0, V_1) = 0$, then W is diffeomorphic to $V_0 \times [0, 1]$.*

Proof. Let $f : W \to [0, 1]$ be a Morse function without critical points. By Lemma 2.8 there exists a gradient-like vector field ξ for f. Then, $\xi_p(f) > 0$ for all $p \in W$. Thus, multiplying the function $\xi(f)$, at each point, by the positive real number $\frac{1}{\xi_p(f)}$, we can assume that $\xi(f) = 1$ identically on W. On the other hand, applying the fundamental existence and uniqueness theorem for ordinary differential equations, we can define an integral curve $\varphi : [a, b] \to W$ for ξ, that is, $\frac{d\varphi}{dt}(t) = \xi_{\varphi(t)}$. Then $\frac{d}{dt}(f \circ \varphi)(t)$ is identically equal to 1 and hence $f(\varphi(t)) = t + constant$ or, for the curve $\varphi(s) := \varphi(s - constant)$, $f(\varphi(s)) = s$.

Now, because each curve through $y \in W$ can be extended uniquely over a maximal interval, which must be $[0, 1]$, for W is compact, there exists a unique maximal integral curve

$$\Psi_y : [0, 1] \to W,$$

which passes through y satisfying $f(\Psi_y(s)) = s$ and is smooth as a function of both variables [**7**, p. 10]. The desired diffeomorphism is now given by

$$h : V_0 \times [0,1] \to W$$
$$(y_0, s) \mapsto \Psi_{y_0}(s)$$

with $h^{-1}(y) = (\Psi_y(0), f(y))$. ■

As a corollary we get the following result.

COLLAR NEIGHBORHOOD THEOREM 2.10. *Let W be a compact smooth manifold with boundary. There exists a neighborhood of ∂W (collar neighborhood) diffeomorphic to $\partial W \times [0,1)$.*

Proof. In the proof of Theorem 2.7 we showed there exists a smooth function $f : W \to \mathbb{R}_+$ such that $f^{-1}(0) = \partial W$ and $df \neq 0$ on a neighborhood U of ∂W. Then f is a Morse function on $f^{-1}[0, \epsilon/2]$, where $\epsilon > 0$ is a lower bound of f on the compact $W - U$. Now Theorem 2.9 guarantees a diffeomorphism between $f^{-1}[0, \epsilon/2)$ and $\partial W \times [0,1)$. ■

Triads with Morse Number Equal to One

Next we will study the particular case in which $\mu(W; V_0, V_1) = 1$.

NOTATION. Let OD^n_r denote the open ball of radius r with center 0 in $\mathbb{R}^n$ and set $OD^n := OD^n_1$.

DEFINITION. Let $(W; V_0, V_1)$ be a triad with Morse function $f : W \to [a,b]$ and a gradient-like vector field ξ for f, such that $p \in W$, with value $c = f(p)$, is the only critical point. Due to (2), there exists a neighborhood U of p and a coordinate diffeomorphism $g : OD^n_{2\epsilon} \to U$ such that $f \circ g(\vec{x}, \vec{y}) = c - \|\vec{x}\|^2 + \|\vec{y}\|^2$, where $0 < 4\epsilon^2 < \min(|c-a|, |c-b|)$, and such that ξ has coordinates $(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$ throughout U. Then, level $V_{-\epsilon} := f^{-1}(c - \epsilon^2)$ lies between V_0 and $f^{-1}(c)$, while level $V_\epsilon := f^{-1}(c + \epsilon^2)$ lies between $f^{-1}(c)$ and V_1. Figure 1 illustrates the situation.

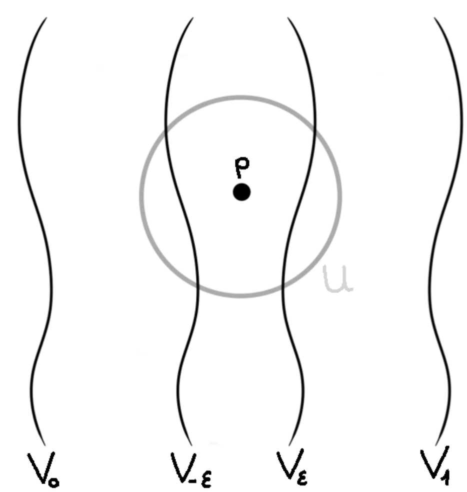


FIGURE 1

We define the *characteristic embedding* $\varphi_L : S^{\lambda-1} \times OD^{n-\lambda} \to V_0$ as follows. Consider the embedding

$$(u, \theta v) \mapsto g(\epsilon u \cosh\theta, \epsilon v \sinh\theta) : S^{\lambda-1} \times OD^{n-\lambda} \to V_{-\epsilon}$$

with $u \in S^{\lambda-1}$, $v \in S^{n-\lambda-1}$ and $0 \leq \theta < 1$. Now, starting at a point $g(\epsilon u \cosh\theta, \epsilon v \sinh\theta) \in V_{-\epsilon}$, we follow an integral curve of ξ up until a well defined point $\varphi_L(u, \theta v) \in V_0$. Define the *left-hand sphere* S_L of p in V_0 to be the intersection of V_0 with all integral curves of ξ coming from the critical point p, that is, the image $\varphi_L(S^{\lambda-1} \times 0)$. The *left-hand disk* D_L is a smoothly embedded disc with boundary S_L, defined to be the union of the segments of these integral curves from S_L to p. Similarly one can define a characteristic embedding $\varphi_R : OD^\lambda \times S^{n-\lambda-1} \to V_1$. The *right-hand sphere* S_R of p in V_1 is defined to be $\varphi_R(0 \times S^{n-\lambda-1})$ which is the boundary of the *right-hand disk*, defined as the union of segments of integral curves of ξ from p to S_R. Figure 2 illustrates the concepts defined above, for $n = 2$ and index $\lambda = 1$.

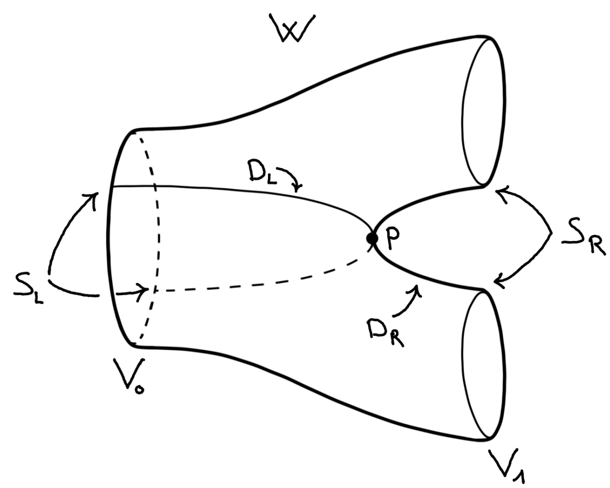


FIGURE 2

Now we can state the following important result.

THEOREM 2.11. *Let $\mu(W; V, V') = 1$. Then $V \cup D_L$ is a deformation retract of W.*

REMARK 2.12. Theorem 2.11 is due to Milnor [**11**, p. 32]. Actually, he proves that the previous result can be generalized to the case of more than one critical point, that is, if $\mu(W; V, V') = k$ then $V \cup D_L^1 \cup ... \cup D_L^k$ is a deformation retract of W, where D_L^i denotes the left-hand disk of the critical point p_i, $i = 1, ..., k$. In Chapter 3 we will make use of this crucial tool for cancelling critical points of Morse functions. In Smale's terminology [**12**], $V \cup D_L^1 \cup ... \cup D_L^k$ is called a *handlebody*, the topological *handles* being D_L^i, one for each critical point.

Self-indexing Morse Functions

In this section we will prove the following result.

FINAL REARRANGEMENT THEOREM 2.13. *Given a Morse function on a triad $(W^n; V_0, V_1)$, there exists another Morse function f with the same critical points, each with the same index, such that*

(1) $f(V_0) = -\frac{1}{2}$; $\quad f(V_1) = n + \frac{1}{2}$

(2) $f(p) = index(p)$ *at each critical point p of f.*

DEFINITION. Such a Morse function will be called *self-indexing*.

EXAMPLE 2.14. Consider the triad of Figure 3 with Morse function h being the height function (projection into the z-axis). Then, clearly, $index(p_1) = 1$ and $index(p_2) = 0$, while $h(p_1) < h(p_2)$. According to final rearrangement theorem 2.13, we can find a new Morse function f which sends $f(p_2) = 0$ and $f(p_1) = 1$.

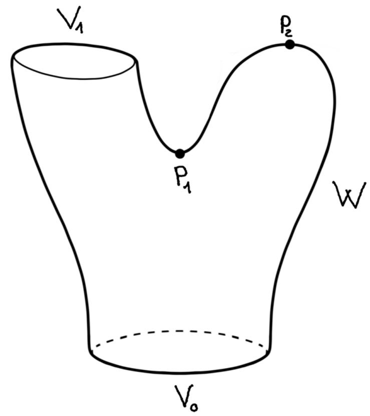


FIGURE 3

The proof of Theorem 2.13 will follow straightforwardly from the two arguments we develop below.

Firstly, we show that any triad can be factored in triads whose Morse number equals 1. To this end, it is immediate to see that, given a non-critical value $0 < c < 1$ of a Morse function $f : (W^n; V_0, V_1) \to ([0,1], 0, 1)$, both $f^{-1}[0,c]$ and $f^{-1}[c,1]$ are smooth manifolds with boundary. Indeed, if $p \in f^{-1}(c)$, then, in some coordinate neighborhood $(U, (x_1, ..., x_n))$ about p, f looks like the projection map $(x_1, ..., x_n) \mapsto x_n : \mathbb{R}^n \to \mathbb{R}$, due to implicit function theorem. On the other hand, let $p_1, ..., p_k$ be the critical points of f. Then it can be proved that the latter can be approximated by a Morse function g with the same critical points such that $g(p_i) \neq g(p_j)$ for $i \neq j$ [**11**, p. 17]. Therefore, one obtains the desired result, which is illustrated in Figure 4, for two critical points and $n = 2$.

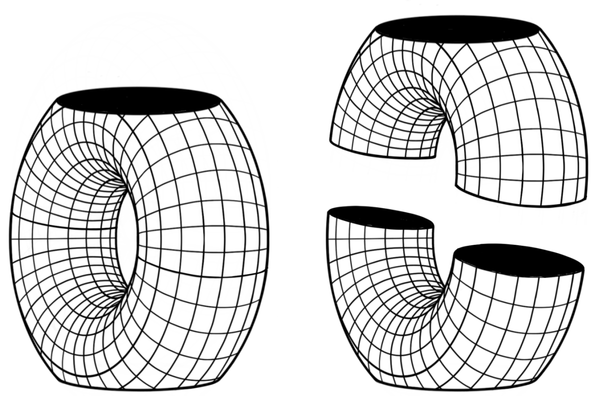


FIGURE 4

REMARK 2.15. Strictly speaking, if $\{(W_i; V_{i-1}, V_i)\}$ are to be the n triads in which the triad $(W; V_0, V_n)$ is factored, in order to prove that W is diffeomorphic to $\bigcup_i W_i$, where

$\bigcup_i W_i$ is the space formed from W_i by identifying points of corresponding boundaries, we need to guarantee the existence and uniqueness of a smooth structure for $\bigcup_i W_i$ compatible with the smooth structures on W_i [**11**, p. 25].

Secondly, we prove that if $f : (W; V_0, V_1) \to ([0,1], 0, 1)$ is a Morse function with two critical points p, p' with indices $\lambda \geq \lambda'$, such that $f(p) < \frac{1}{2} < f(p')$, then we can find another Morse function g, whose critical points are still p, p', but $g(p') < \frac{1}{2} < g(p)$. It is said that the triad can be *rearranged*. The proof consists of two steps: in the first step we prove that we can alter the gradient-like vector field of f such that the new right-hand sphere of p does not intersect with the new left-hand sphere of p'; in the second step we prove this fact implies that the triad can be rearranged. We need a couple of definitions.

DEFINITION. A *product neighborhood* is an open neighborhood U of a submanifold $M^m \subset V^v$, which is diffeomorphic to $M^m \times \mathbb{R}^{v-m}$ such that M^m corresponds to $M^m \times 0$.

DEFINITION. Two diffeomorphisms $h_0, h_1 : M \to M'$ are (smoothly) *isotopic* if there exists a map (*isotopy*) $F : M \times I \to M'$ such that

(1) F is smooth
(2) each $f_t(x) := F(x,t)$ is a diffeomorphism
(3) $f_0 = h_0, \quad f_1 = h_1$

THEOREM 2.16. *Let f be a Morse function on $(W^n; V_0, V_1)$ with two critical points p, p', of indices λ, λ', such that $f(p) < \frac{1}{2} < f(p')$ and $V := f^{-1}(\frac{1}{2})$. Let ξ be a gradient-like vector field for f, with spheres S_R and S'_L in V. If $\lambda \geq \lambda'$, then it is possible to alter ξ on a small neighborhood of V, such that the corresponding new spheres $\overline{S}_R$ and $\overline{S}'_L$ do not intersect.*

Proof. The proof consists of showing that we can alter ξ on certain $f^{-1}[a, \frac{1}{2}]$ in a prescribed product neighborhood of V, such that the corresponding new right-hand sphere $\overline{S}_R$ is the image of a diffeomorphism $h : V \to V$ smoothly isotopic to the identity, causing that $\overline{S}_R \cap \overline{S}'_L = h(S_R) \cap S'_L = \emptyset$. Note that $\overline{S}'_L = S'_L$ because ξ is only being altered on $f^{-1}[a, \frac{1}{2}]$, as illustrated in Figure 5.

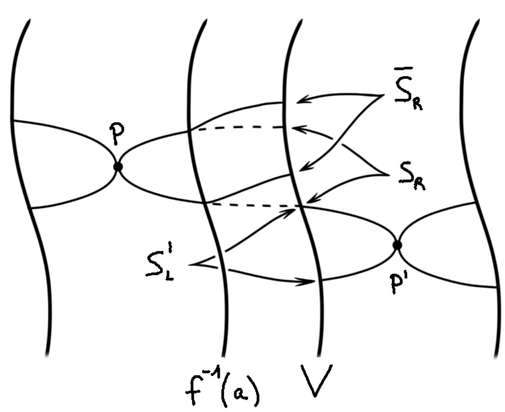


FIGURE 5

By definition, $\dim S_R = n - \lambda - 1$ and $\dim V = n - 1$. Moreover, by construction, S_R has a product neighborhood U in V. Thus, we can define a diffeomorphism $k : S_R \times \mathbb{R}^\lambda \to U \subset V$ such that $k(S_R \times 0) = S_R$. Now let $S'_{L_0} = U \cap S'_L$ and consider $g := \pi \circ k^{-1}|_{S'_{L_0}}$, where $\pi : S_R \times \mathbb{R}^\lambda \to \mathbb{R}^\lambda$. Then $k(S_R \times \vec{x}) \subset V$ intersects S'_L if and only if $\vec{x} \in g(S'_{L_0})$. Since

$\dim(S'_{L_0}) = \lambda' - 1$ and $\lambda' \leq \lambda$ by hypothesis, then, if $S'_{L_0} \neq \emptyset$, $g(S'_{L_0})$ is of measure zero in $\mathbb{R}^\lambda$ due to Sard's theorem 2.4. Thus, we may choose $\vec{u} \in \mathbb{R}^\lambda - g(S'_{L_0})$ and build a diffeomorphism $h : V \to V$, isotopic to the identity, which sends S_R to $k(S_R \times \vec{u})$ so that $h(S_R) \cap S'_L = \emptyset$. Define the following smooth vector field on $\mathbb{R}^\lambda$

$$\zeta(\vec{x}) = \begin{cases} \vec{u} & \|\vec{x}\| \leq \|\vec{u}\| \\ 0 & \vec{x} \geq 2\|\vec{u}\| \end{cases}$$

Because $supp(\zeta) = \{\vec{x} \in \mathbb{R}^\lambda : \|\vec{x}\| \leq \|\vec{u}\|\}$ is compact and $\mathbb{R}^\lambda$ has no boundary, then, for each fixed $\vec{x} \in supp(\zeta)$, the differential equation

$$\frac{d\Psi(t, \vec{x})}{dt} = \zeta_{\Psi(t,\vec{x})}, \quad \Psi(0, \vec{x}) = \vec{x}$$

has a unique solution for all $t \in [0, 1]$, namely, the integral curve $\Psi(t, \vec{x})$, which is smooth as a function of both variables(cf. *proof* 2.9). Thus, $\Psi(t, \vec{x})$ gives an isotopy from the identity to $\Psi(1, \vec{x})$, the latter being a diffeomorphism which carries 0 to $\vec{u}$. Therefore, since this isotopy leaves all points fixed outside a bounded set in $\mathbb{R}^\lambda$, we can define the following isotopy

$$\begin{aligned} \iota_t : V &\to V \\ w &\mapsto k(q, \Psi(t, \vec{x})) \quad \text{if } w = k(q, \vec{x}) \in U \\ w &\mapsto w \qquad\qquad\qquad \text{if } w \in V - U \end{aligned}$$

Then $h := \iota_1$ is the desired diffeomorphism. Now, for a large enough, $f^{-1}[a, \frac{1}{2}] \subset U$ and hence the integral curves of the normalized gradient-like vector field $\hat{\xi} := \xi/\xi(f)$ determine a diffeomorphism

$$\varphi : \left[a, \frac{1}{2}\right] \times V \to f^{-1}\left[a, \frac{1}{2}\right]$$

such that $f(\varphi(t, q)) = t$ and $\varphi(\frac{1}{2}, q) = q \in V$. Now define the diffeomorphism

$$\begin{aligned} H : \left[a, \frac{1}{2}\right] \times V &\to \left[a, \frac{1}{2}\right] \times V \\ (t, q) &\mapsto (t, h_t(q)) \end{aligned}$$

where $h_t(q)$ is an isotopy $\left[a, \frac{1}{2}\right] \times V \to V$ from the identity to h such that h_t is the identity for t near a and h_t is h for t near $\frac{1}{2}$. Then

$$\xi' := (\varphi \circ H \circ \varphi^{-1})_* \hat{\xi}$$

is a smooth vector field defined on $f^{-1}\left[a, \frac{1}{2}\right]$, which agrees with $\hat{\xi}$ near $f^{-1}(a)$ and $f^{-1}\left(\frac{1}{2}\right)$, and satisfies $\xi'(f) = 1$ identically on $f^{-1}\left[a, \frac{1}{2}\right]$. Thus, the gradient-like vector field $\overline{\xi}$ on W defined to be $\xi(f)\xi'$ on $f^{-1}\left[a, \frac{1}{2}\right]$ and ξ elsewhere is a new gradient-like vector field for f. We claim that $h(S_R)$ is the new right-hand sphere of p associated to $\overline{\xi}$, that is, $\overline{S}_R$. Certainly, for each $v \in V$, $\varphi(t, h_t(v))$ describes an integral curve of $\overline{\xi}$ from $\varphi(a, v) \in f^{-1}(a)$ to $\varphi(\frac{1}{2}, h(v)) = h(v)$ in $f^{-1}\left(\frac{1}{2}\right) = V$ and hence $\varphi(a \times S_R)$ in $f^{-1}(a)$ is carried to $h(S_R)$ in V. ■

COROLLARY 2.17. *Let $V := f^{-1}(b)$ be a non-critical level and $h : V \to V$ a diffeomorphism isotopic to the identity. If $f^{-1}[a, b]$ does not contain critical points, then it is possible to build a new gradient-like vector field $\overline{\xi}$ for f such that* *(a)* $\overline{\xi}$ *agrees with* ξ *outside* $f^{-1}(a,b)$

(b) $\overline{\varphi} = h \circ \varphi$, *where* φ *and* $\overline{\varphi}$ *are diffeomorphisms* $f^{-1}(a) \to V$ *determined by following the trajectories of* ξ *and* $\overline{\xi}$, *respectively.*

REMARK 2.18. Replacing f by $-f$ one deduces a proposition which is analogous to Corollary 2.17, except for now ξ is altered on $f^{-1}(b,c)$, with $b < c$.

REMARK 2.19. The proof of Theorem 2.16 can be generalized to the case in which f has several index λ critical points and several index λ' critical points.

Once we have proved we can alter a gradient-like vector field for f such that the corresponding new spheres $\overline{S}_R$ and $\overline{S}_L$ do not intersect, the second step consists of proving that $(W; V_0, V_1)$ can be rearranged as the following result shows.

THEOREM 2.20. *Let* $(W; V_0, V_1)$ *be a triad with Morse function* f *having two critical points* p, p'. *Let* K_p *be the compact set formed by those points in the trajectories that go or come from* p, *which is assumed to be disjoint from the compact set* $K_{p'}$, *formed by those points in the trajectories that go or come from* p'. *Then, for any* $a, a' \in (0,1)$, *there exists a new Morse function* g *such that:*

(a) p *and* p' *are still critical points of* g,

(b) ξ *is still a gradient-like vector field for* g.

(c) $g(p) = a$ *and* $g(p') = a'$,

(d) g *agrees with* f *near* $V_0 \cup V_1$, *but equals* f *plus a constant in some neighborhood of* p *and in some neighborhood of* p'.

Proof. It is clear that all trajectories through points outside $K := K_p \cup K_{p'}$ go from V_0 to V_1. Let $\pi : W - K \to V_0$ be the smooth map that assigns to each point q the unique intersection of the integral curve through q with V_0 (cf. *proof* 2.9), such that when q lies near K, then $\pi(q)$ lies near K in V_0. Thus, if $\mu : V_0 \to [0,1]$ is a smooth function zero near $K_p \cap V_0$ and one near $K_{p'} \cap V_0$, then it extends uniquely to a smooth function $\overline{\mu} : W \to [0,1]$ which is constant on each trajectory, zero near K_p and one near $K_{p'}$.

One easily checks that a new Morse function

$$\begin{aligned} g : W &\to [0,1] \\ q &\mapsto G(f(q), \overline{\mu}(q)) \end{aligned}$$

satisfies the desired properties provided that $G(x,y)$ is any smooth function $[0,1] \times [0,1] \to [0,1]$ with the properties:

(i) For all x and y, $\frac{\partial G}{\partial x}(x,y) > 0$ and $G(x,y)$ increases from 0 to 1 as x increases from 0 to 1. (This proves g satisfies (b))

(ii) $G(f(p), 0) = a$; $G(f(p'), 1) = a'$ (This proves g satisfies (c))

(iii)

$$\left.\begin{array}{ll} G(x,y) = x & \text{for } x \text{ near 0 or 1 and for all } y, \\ \frac{\partial G}{\partial x}(x,0) = 1 & \text{for } x \text{ in a neighborhood of } f(p), \\ \frac{\partial G}{\partial x}(x,1) = 1 & \text{for } x \text{ in a neighborhood of } f(p') \end{array}\right\} \quad \text{(This proves } g \text{ satisfies (d))}$$

Note that property (a) is implicitly satisfied by the definition of g, for if $(U, (x_1, ..., x_n))$ is any coordinate neighborhood, then $\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i}$, $i = 1, ..., n$. ■

REMARK 2.21. Using Remark 2.19 to allow f having two sets of critical points $p = \{p_1, ..., p_m\}$, $p' = \{p'_1, ..., p'_m\}$ with all points of p with the same critical value, say $f(p)$, and all points of p' with the same critical value, say $f(p')$, then the proof of Theorem 2.20 can be repeated literally.

All in all, we have seen that any triad can be factored in a finite number of triads with Morse number equal to one (first argument above) and so applying Remark 2.21 to sets of points p and p', whose indices λ and λ' are such that $\lambda \geq \lambda'$, one obtains the conclusion of Theorem 2.13.

CHAPTER 3

Cancellation

In this chapter we will see that, imposing certain conditions to a triad $(W^n; V_0, V_1)$, it is possible to alter the gradient-like vector field for a Morse function, in order for the latter to have no critical points at all (*cancellation*). To simplify notation, we will omit the base point when dealing with fundamental groups of manifolds because they will always be path connected. We will make use of the following notions.

Definition. Two submanifolds $M^r, M'^s \subset V^v$ have *transverse intersection* if the tangent space to V at each point $q \in M \cap M'$ is generated by the vectors tangent to M and the vectors tangent to M' at this point. Note that if $r + s < v$, then transverse intersection is impossible unless $M \cap M' = \emptyset$. Thus, in case M and M' have non empty transverse intersection, $r + s = v$ and hence it will consist of a finite number of isolated points. Let p_i be one of the points $p_1, ..., p_k$ in which M and M' intersect transversely and choose there a positively oriented frame of linearly independent vectors $X^1_{p_i}, ..., X^r_{p_i}$ spanning $T_{p_i}M$, which will also represent a basis for the fibre at p_i of the normal bundle $\nu_{M'}$. Assuming the total space NM' of $\nu_{M'}$ is oriented, the *intersection number of M and M' at p_i* is defined to be $+1$ or -1 depending on whether $X^1_{p_i}, ..., X^r_{p_i}$ represent a positively or negatively oriented basis. The *intersection number $M' \cdot M$ of M and M'* is the sum of the intersection numbers at the points p_i.

In the cancellation procedure, previous concepts will be applied to $M = S_R$ and $M' = S'_L$, which are to be the right-hand sphere of a critical point p and the left-hand sphere of a critical point p', having indices λ, $\lambda + 1$, respectively. The following result guarantees transverse intersection between such spheres.

Theorem 3.1. *Let f be a Morse function on $(W^n; V_0, V_1)$ with two critical points p, p' of indices λ, $\lambda + 1$, such that $f(p) < b < f(p')$. Let ξ be a gradient-like vector field determining, in $V := f^{-1}(b)$, a right-hand sphere S_R of p and a left-hand sphere S'_L of p'. Then it is possible to obtain a new gradient-like vector field $\overline{\xi}$ for f such that the corresponding new spheres $\overline{S}_R$ and $\overline{S}'_L$ intersect transversely.*

Proof. First, note that

$$\dim S_R + \dim S'_L = (n - \lambda - 1) + \lambda = n - 1 = \dim V$$

and so transverse intersection is possible. Using the same notation as in the proof of Theorem 2.16, the image $g(C)$ of the set $C \subset S'_{L_0}$ of all critical points of g is of measure zero in $\mathbb{R}^\lambda$ (Theorem of Sard 2.4). Then, if we choose a point $q \in C$ such that $\vec{u} := g(q) \in \mathbb{R}^\lambda - g(C)$, then g will have maximal rank λ at q and hence the manifold $k(M \times \vec{u})$ will have transverse intersection with S'_L. On the other hand, we can build an isotopy of the identity map of V to a diffeomorphism $h : V \to V$ that carries S_R to $k(S_R \times \vec{u})$ (cf. *proof* Theorem 2.16). Now, because we can alter ξ such that the new right-hand sphere $\overline{S}_R$ is $h(S_R)$ and the left-hand sphere is unchanged (Corollary 2.17), the proof is complete. ■

Due to previous result, if $(W'; V'_0, V'_1)$ is one of the triads in which our initial triad $(W; V_0, V_1)$ can be factored, possessing a Morse function f with two critical points p, p'

whose indices are λ, $\lambda+1$, then we can assume there exists a gradient-like vector field for f, such that the corresponding spheres S_R, S'_L of p, p' have transverse intersection.

Note that, when defining the intersection number of S_R and S'_L at a given point of their intersection, we are assuming implicitly that both our original manifold W and its boundary $X := \partial W$ are oriented. Certainly, from an algebraic-topological point of view, W can be given an orientation $[W] \in H_n(W, X)$ [**1**, p. 304] and hence, under the boundary homomorphism $H_n(W, X) \to H_{n-1}(X)$ of the exact sequence in homology for the pair (W, X), there is an *induced orientation generator* $[X] \in H_{n-1}(X)$ for X.

The tools we will use for cancelling critical points are the first and second cancellation theorems. For the first one, let us just assume we have a triad $(W; V_0, V_1)$ having a Morse function f with two critical points p, p' whose indices are λ, $\lambda+1$, respectively.

First Cancellation Theorem

First Cancellation Theorem 3.2. *(Weak) If S_R and S'_L intersect transversely in a single point, then W is diffeomorphic to $V_0 \times [0,1]$, namely, it is possible to alter the gradient-like vector field ξ on an arbitrary small neighborhood of the trajectory T joining p and p', producing a nowhere zero vector field ξ'' whose trajectories all proceed from V_0 to V_1, such that ξ'' is gradient-like for a Morse function g without critical points that agrees with f near $V_0 \cup V_1$.*

Proof. On the one hand, it can be proved that the hypothesis of the theorem implies the following fact [**11**, p. 55 Assertion 6].

Fact: One can choose a new gradient-like vector field ξ' for f so that there is a coordinate neighborhood $(U_T, g = (x_1, ..., x_n))$ of the trajectory T such that:

(1) p and p' correspond to the points $o := (0, ..., 0)$ and $e := (1, 0, ..., 0)$.
(2) $g_*(\xi'_q) = \eta_x = (v(x_1), -x_2, ..., -x_\lambda, -x_{\lambda+1}, x_{\lambda+2}, ..., x_n)$ where $g(q) = x$ and $v(x_1)$ is a smooth function of x_1, positive on $(0,1)$, zero at 0 and 1, and negative elsewhere. Also, $\left|\frac{\partial v}{\partial x_1}(x_1)\right| = 1$ near 0 and 1.

On the other hand, note that, given an open neighborhood U of T, it is always possible to find another neighborhood U', with $T \subset U' \subset U$, such that no trajectory leaving U' and going outside of U comes back into U'. Otherwise, there would exist a partial sequence of trajectories $T_1, ..., T_k$ which start and end at $r_k, t_k \in U'$, respectively, passing through a point s_k outside U, such that $\{r_k\}$ and $\{t_k\}$ approach T. But, because $W - U$ is compact, if $s \in W - U$ is the limit of $\{s_k\}$, then for any s' in a certain neighborhood of s, the minimum distance (in any metric) between the trajectories T and $T_{s'}$, which depends continuously on s', would be strictly greater than zero and hence the points r_k would not approach T as $k \to \infty$, which is a contradiction. Let then U' be such a neighborhood for a given neighborhood U of T such that $\overline{U} \subset U_T$. That is, $T \subset U' \subset U \subset \overline{U} \subset U_T$.

Now, let us replace η_x by a smooth vector field $\eta'_x = (v'(x_1, \rho), -x_2, ..., x_n)$, where $\rho = (x_2^2 + \cdots + x_n^2)^{1/2}$ and

(i) $v'(x_1, \rho(x)) = v(x_1)$ outside a compact neighborhood of $g(T)$ in $g(U')$.
(ii) $v'(x_1, 0)$ is everywhere negative.

(See Figure 1).

Thus we obtain a nowhere zero vector field ξ'' on W, whose integral curves satisfy, on U_T, the following differential equations.

$$\frac{dx_1}{dt} = v'(x_1, \rho);\ \frac{dx_2}{dt} = -x_2;\ \dots\ ;\ \frac{dx_{\lambda+1}}{dt} = -x_{\lambda+1};\ \frac{dx_{\lambda+2}}{dt} = x_{\lambda+2};\ \dots\ ;\ \frac{dx_n}{dt} = x_n$$

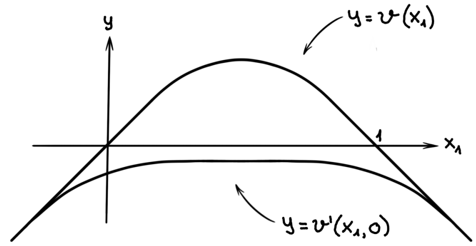


FIGURE 1

Consider the integral curve $x(t) = (x_1(t), ..., x_n(t))$ with initial value $(x_1^0, ..., x_n^0)$. Then, as t increases,

- if one of $x_{\lambda+2}^0, ..., x_n^0$ is not zero, say $x_n^0 \neq 0$, then, by solving $\int \frac{dx_n}{x_n} = \int dt$, we get
$$\left|x_n(t)\right| = \left|x_n^0 e^t\right|$$
and hence $x(t)$ leaves $g(U)$ eventually.
- if $x_{\lambda+2}^0 = \cdots = x_n^0 = 0$, then $\rho(x(t)) = [(x_2^0)^2 + \cdots + (x_{\lambda+1}^0)^2]^{1/2} e^{-t}$, so that in case $x(t)$ remained in $g(U)$, there would exist a compact $K_\delta = \{x \in g(\overline{U}) : 0 \leq \rho(x) \leq \delta\}$ with $v'(x_1, \rho(x))$ having a negative upper bound $-\alpha < 0$ on K_δ. But, since $\rho(x(t))$ decreases exponentially, eventually $\rho(x(t)) \leq \delta$ and hence $\frac{dx_1(t)}{dt} \leq -\alpha$, which is a contradiction. Therefore $x(t)$ leaves $g(U)$ eventually.

Similarly, it can be proved that $x(t)$ goes outside $g(U)$ as t increases.

We have just proved that every trajectory of the vector field ξ'' goes from V_0 to V_1. Certainly: if, at some time, an integral curve of ξ'' is in U', then, by last argument above, it eventually must go outside U and, by doing so, it follows a trajectory of ξ' up until V_1 because it cannot come back in U'. Similarly, it can be proved that the trajectory comes from V_0; in case the integral curve of ξ'' never goes inside U', then it is an integral curve of ξ', which, of course, goes from V_0 to V_1.

Now let $\tau_1(q)$ be the function that assigns to each point $q \in W$ the time at which the integral curve $\Psi(t, q)$ for ξ'' reaches V_1. Similarly let $\tau_0(q)$ assign minus the time $\Psi(t, q)$ reaches V_0. Because, by definition, $\frac{d\Psi}{dt}(t, q)$ is nowhere zero on ∂W, then $\tau_i(q)$ depends smoothly on q due to implicit function theorem. Thus, the trajectories of the smooth vector field $\tau_1(\Psi(-\tau_0(q), q))\xi''$ go from V_0 to V_1. Let us assume, for the sake of simplicity, that ξ'' already had this property from the outset. Then its integral curves determine a diffeomorphism

$$\begin{aligned} \phi : [0, 1] \times V_0 &\to W \\ (t, q_0) &\mapsto \Psi(t, q_0) \end{aligned}$$

whose inverse is the smooth map defined by $q \mapsto (\tau_0(q), \Psi(-\tau_0(q), q))$. Now, if $\lambda : [0, 1] \to [0, 1]$ is a smooth function zero for $t \in [\delta, 1 - \delta]$ and one for t near 0 and 1, then, choosing δ sufficiently small, one checks that the function

$$\begin{aligned} g \circ \phi : [0, 1] \times V_0 &\to [0, 1] \\ (u, q) &\mapsto \int_0^u \Big\{\lambda(t) \frac{\partial (f \circ \phi)}{\partial t}(t, q) + [1 - \lambda(t)] k(q)\Big\} dt, \end{aligned}$$

where

$$k(q) = \frac{\left\{1 - \int_0^1 \lambda(t) \frac{\partial(f\circ\phi)}{\partial t}(t,q)dt\right\}}{\int_0^1 [1-\lambda(t)]dt},$$

satisfies $\frac{\partial(g\circ\phi)}{\partial t} > 0$ and agrees with $f \circ \phi$ near $(0 \times V_0) \cup (1 \times V_1)$. Then, also $\frac{\partial g}{\partial t} > 0$. Therefore, g is a Morse function on W, agreeing with f near $V_0 \cup V_1$, for which ξ'' is a gradient-like vector field. This completes the proof of first cancellation theorem. ■

Second Cancellation Theorem

In general, though, $S_R \cap S'_L$ will consist of more than one point. Thus, we cannot use the first cancellation theorem, initially, for canceling p and p', but we will need a stronger result, the second cancellation theorem. Basically, the latter says ξ can be altered near the non-critical level where S_R and S'_L lie, in such a way that these intersect transversely in a single point and hence first cancellation theorem applies.

Throughout this section we will make use of the following tools, due to Munkres [**13**, p. 54], Milnor [**14**, p. 62] and Whitney [**15**].

LEMMA 3.3. *Let A_0 be a closed subset of a compact metric space A. Let $f : A \to B$ be a local homeomorphism such that $f|_{A_0}$ is 1:1. Then there is a neighborhood W of A_0 such that $f|_W$ is 1:1.*

LEMMA 3.4. *Let $f : M_1 \to M_2$ be a continuous map of smooth manifolds which is smooth on a closed subset A of M_1. Then there exists a smooth map $g : M_1 \to M_2$ such that $g \simeq f$ (g is homotopic to f) and $g|_A = f|_A$.*

LEMMA 3.5. *Let $f : M_1 \to M_2$ be a smooth map of smooth manifolds which is an embedding on the closed subset A of M_1. Assume that* $\dim M_2 \geq 2 \dim M_1 + 1$. *Then there exists an embedding $g : M_1 \to M_2$ approximating f such that $g \simeq f$ and $g|_A = f|_A$.*

COROLLARY 3.6. *Under the hypothesis of Lemma 3.5 and due to Lemma 3.4, such a g is, actually, smooth.*

COROLLARY 3.7. *If two smooth embeddings of a smooth manifold M^m into a smooth manifold N^n are homotopic, then they are smoothly isotopic provided that $n \geq 2m + 3$.*

The proof of second cancellation theorem is based on the repeated application of a delicate theorem which follows from a technical lemma we will prove below. In order to prove such a lemma we first need the following result.

LEMMA 3.8. *If V_1^n, $n \geq 5$, is a smooth manifold and M_1 is a smooth submanifold of codimension at least 3, then a loop in $V_1 - M_1$ which is contractible in V_1 is also contractible in $V_1 - M_1$.*

Proof. Let $g : (D^2, S^1) \to (V_1, V_1 - M_1)$ be the map which defines the contraction in V_1 of a loop in $V_1 - M_1$. Then

$$g|_{S^1} \simeq constant \tag{8}$$

On the other hand, because $\dim (V_1 - M_1) \geq 5$, applying Corollary 3.6, there exists a smooth embedding $h : (D^2, S^1) \to (V_1, V_1 - M_1)$ such that

$$g|_{S^1} \simeq h|_{S^1} \quad \text{in} \quad V_1 - M_1 \tag{9}$$

Then $h_{S^1} \simeq constant$ and hence $h(D^2)$ is contractible. Thus the normal bundle of $h(D^2)$ is trivial [**16**, p. 30]. So there exists an embedding H of $D^2 \times \mathbb{R}^{n-2}$ in V_1 such that $H(u, 0) = h(u)$ for $u \in D^2$.

Now, since $V_1 - M_1 \hookrightarrow V_1$ and $h(D^2) \subset V_1 - M_1$, there exists $\epsilon > 0$ such that $H(D^2 \times x) \subset V_1 - M_1$, with $\|x\| < \epsilon$. Using Sard's Theorem 2.4, we can guarantee there exists $x_0 \in \mathbb{R}^{n-2}$, with $\|x_0\| < \epsilon$, such that $H(D^2 \times x_0) \cap M_1 = \emptyset$. Therefore, in $V_1 - M_1$, $H|_{S^1 \times 0} \simeq H|_{S^1 \times x_0}$. But $H|_{S^1 \times 0} = h|_{S^1} \overset{(9)}{\simeq} g|_{S^1} \overset{(8)}{\simeq}$ *constant*, that is, the loop is also contractible in $V_1 - M_1$. ∎

Let us now prove the technical lemma.

LEMMA 3.9. *Let M^r and M'^s be smooth, closed, transversely intersecting submanifolds of a smooth manifold V^{r+s} without boundary, with $r + s \geq 5$, $s \geq 3$ and, in case $r = 1$ or 2, the map induced in homotopy $\pi_1(V - M') \to \pi_1(V)$ is 1:1 into. Suppose M and the normal bundle NM' of M' are oriented. Let us assume that the intersection numbers at $p, q \in M \cap M'$ are $+1$ and -1, respectively, and let C and C' be smoothly embedded arcs in M and M', respectively, through p and q, defining a loop L, contractible in V, which does not intersect $M \cap M' - \{p, q\}$. Let C_0 and C'_0 be open arcs in $\mathbb{R}^2$ intersecting transversely at a, b, and thus defining a disk D with two corners. Let $\varphi_1 : C_0 \cup C'_0 \to M \cup M'$ be an embedding such that $\varphi_1(C_0) = C$, $\varphi_1(C'_0) = C'$, $\varphi_1(a) = p$ and $\varphi_1(b) = q$. Then, for some neighborhood U of D, one can extend $\varphi_1|_{U \cap (C_0 \cup C'_0)}$ to an embedding $\varphi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to V$ such that $\varphi^{-1}(M) = (U \cap C_0) \times \mathbb{R}^{r-1} \times 0$ and $\varphi^{-1}(M') = (U \cap C'_0) \times 0 \times \mathbb{R}^{s-1}$.*

Proof. Figure 2 illustrates in a simple way the tedious hypothesis of the lemma.

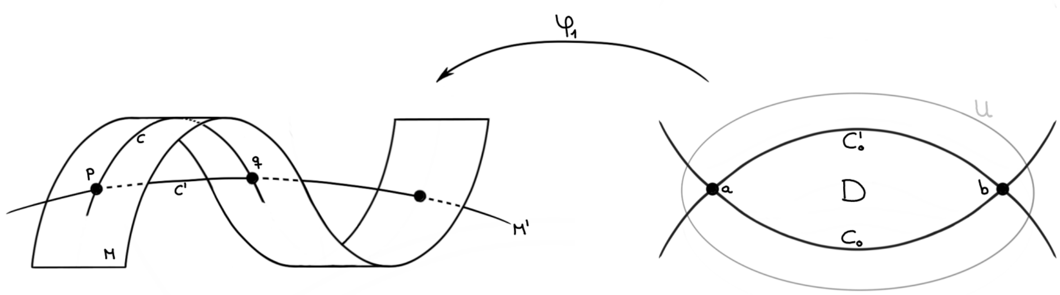


FIGURE 2

Suppose that $M \cap M' = \{p_1, ..., p_k\}$, with $p := p_1$ and $q := p_2$. Cover $M \cup M'$ with coordinate neighborhoods $(W_1, h_1), ..., (W_m, h_m)$, where $h_i : W_i \to \mathbb{R}^{r+s}$ are diffeomorphisms such that $h_i(N_i \cap C)$ and $h_i(N_i \cap C')$ are straight line segments in $\mathbb{R}^{r+s}$, $i = 1, ..., k$, with $p_i \in N_i \subset \overline{N}_i \subset W_i$. Define a Riemannian metric $\langle \ , \ \rangle$ in $W_0 := W_1 \cup ... \cup W_m$, by piecing together the metrics on W_j induced by the h_j, $j = 1, ..., m$, using a partition of unity. Then, using Theorem 1.29, one can build open tubular neighborhoods T and T' like those in Figure 3.

Now consider the metric $\langle X, Y \rangle_A := \frac{1}{2}(\langle X, Y \rangle + \langle A_* X, A_* Y \rangle)$, where $A : T \to T$ is the antipodal map on each fibre of T. Note that

$$\begin{aligned}\langle A_* X, A_* Y \rangle_A &= \frac{1}{2}(\langle A_* X, A_* Y \rangle + \langle A_* A_* X, A_* A_* Y \rangle) \\ &= \frac{1}{2}(\langle A_* X, A_* Y \rangle + \langle X, Y \rangle) \\ &= \langle X, Y \rangle_A\end{aligned}$$

and hence the map A is an isometry. We claim that, with respect to this new Riemannian metric, M is a totally geodesic submanifold of T, that is, in the associated connection, if a

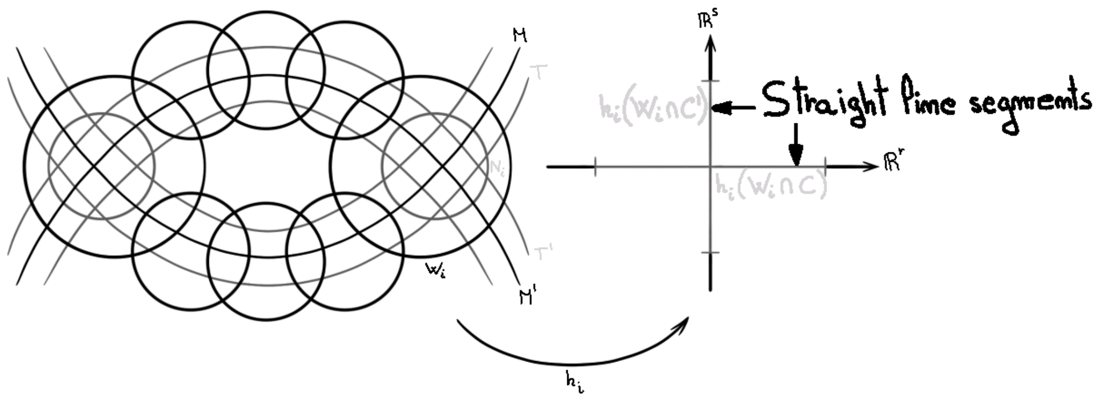


Figure 3

geodesic in T is tangent to M at any point, then it lies entirely in M. Certainly, let γ be a geodesic in T tangent to M at the point $z \in M$. On the one hand, A sends geodesics to geodesics, for it is an isometry. On the other hand, $A(z) = z$ because points in M remain fixed under the map A by construction of the tubular neighborhood around M. Thus, $A(\gamma)$ and γ are geodesics with the same tangent vector at z and hence, by uniqueness, A must be the identity on γ. Therefore, $\gamma \subset M$.

Similarly, one can define a new metric $\langle\ ,\ \rangle_{A'}$ on T', with respect to which M' is a totally geodesic submanifold of T'. It follows, by construction (Figure 3), that these new metrics agree with the old one on $T \cap T'$ and hence together define a metric on $T \cup T'$. One can now extend the restriction of this new metric to an open set O, with $M \cup M' \subset O \subset \overline{O} \subset T \cup T'$, to all of V. This way we have built a Riemannian metric on V satisfying the following properties:

(1) In a compatible connection, M and M' are totally geodesic submanifolds of V.
(2) There exist coordinate neighborhoods N_p and N_q around p and q in which the metric is the euclidean metric and $N_p \cap C$, $N_p \cap C'$, $N_q \cap C$, $N_q \cap C'$ are straight line segments (Figure 3).

Let τ and τ' be the normalized velocity vectors along C and C' such that $\tau(p)$, $\tau(q)$, $\tau'(p)$ and $\tau'(q)$ are oriented from p to q. Because C is a contractible space, the bundle defined by the set of orthogonal vectors to M over C will be trivial. Then one can build a field of unit vectors orthogonal to M along C and equal to the parallel translates of $\tau'(p)$ and $-\tau'(q)$ along $N_p \cap C$ and $N_q \cap C$, respectively, as well as the corresponding vector field in $\mathbb{R}^2$ via φ_1^{-1}. Now the exponential map guarantees the existence of a neighborhood A_0 of C_0 such that $\varphi_1|_{A_0}$ is locally an embedding into V(cf. *proof* Theorem 1.29). Thus, applying Lemma 3.3, there exists a neighborhood W of A_0 such that $\varphi_1|_W$ is an embedding into V. Similarly one can extend $\varphi_1|_{C_0'}$ to an embedding of a neighborhood of C_0' into V. Now property 2 guarantees an embedding $\varphi_2 : N \to V$ of a closed annular neighborhood N of ∂D, with $\varphi_2^{-1}(M) = N \cap C_0$ and $\varphi_2^{-1}(M') = N \cap C_0'$. Let $D_0 \subset D$ and $S := \partial D_0$, so that $U = N \cup D_0$ (Figure 4).

Note that $\varphi_2(S)$ is contractible in V, since L is and $L \simeq \varphi_2(S)$. Then, $\varphi_2(S)$ is also contractible in $V - M'$ both when $r = 2$, because $\pi_1(V - M') \to \pi_1(V)$ is 1:1 by hypothesis, and when $r \geq 3$ due to Lemma 3.8. Thus, because $s \geq 3$, $\varphi_2(S)$ is also contractible in $(V - M') - M = V - (M \cup M')$ again by Lemma 3.8. Then we can clearly extend φ_2 continuously to a map $\varphi_2' : U \to V$ that sends $\mathring{D}$ into $V - (M \cup M')$ and hence, applying

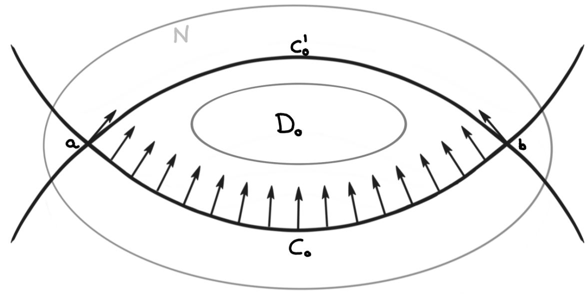


FIGURE 4

Corollary 3.6 to $\varphi_2'|_{\mathring{D}}$, we obtain a smooth embedding $\varphi_3 : U \to V$ which coincides with φ_2 on a neighborhood of $U - \mathring{D}$ and such that $\varphi_3(u) \notin M \cup M'$ for $u \notin C_0 \cup C_0'$. Now, on the one hand, because $U \subset \mathbb{R}^2$, in the whole argument above we have built a subspace $\varphi_3(U)$ of V, which is contractible and has codimension $r+s-2$. On the other hand, parallel translations along a curve in totally geodesic submanifolds, of vectors which are tangent to the latter, yield tangent vectors to these submanifolds. Therefore, we can build smooth vector fields $X^1, ..., X^{r-1}, Y^1, ..., Y^{s-1}$ along $\varphi_3(U)$ which are orthonormal and orthogonal to $\varphi_3(U)$ [**11**, p. 81]. Moreover, since inner products are preserved under parallel translations because we assumed a compatible connection was chosen, then

(i) $X^1, ..., X^{r-1}$ are tangent to M along $\varphi_3(U) \cap C$

(ii) $Y^1, ..., Y^{s-1}$ are tangent to M' along $\varphi_3(U) \cap C'$.

Finally, one can check that the following local diffeomorphism

$$(u, x_1, ..., x_{r-1}, y_1, ..., y_{s-1}) \mapsto \exp\left[\sum_{i=1}^{r-1} x_i X^i_{\varphi_3(u)} + \sum_{j=1}^{s-1} y_j Y^j_{\varphi_3(u)}\right] : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to V,$$

restricted to $\tilde{U} \times N_\epsilon \subseteq U \times N_\epsilon$ for some neighborhood N_ϵ about the origin in $\mathbb{R}^{r+s-2}$, defines an embedding $\varphi_4 : \tilde{U} \times N_\epsilon \to V$ (Lemma 3.3). The embedding

$$\varphi : \tilde{U} \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to V$$

$$(u, z) \mapsto \varphi_4\left(u, \frac{\epsilon z}{\sqrt{1 + \|z\|^2}}\right)$$

has, then, the desired properties. Indeed, if $z = (x_1, ..., x_{r-1}, 0, ..., 0)$, then $\varphi(u, z)$, with $u \in C_0$, will be tangent to M, due to (i), and hence $\varphi(C_0 \times \mathbb{R}^{r-1} \times 0) \subset M$, for M is totally geodesic. Similarly one checks that $\varphi(C_0' \times 0 \times \mathbb{R}^{s-1}) \subset M'$. But $\varphi(\tilde{U} \times 0)$ intersects M and M' in C and C', transversely. Therefore, there exists $\epsilon > 0$ small enough such that $\varphi^{-1}(M) = C_0 \times \mathbb{R}^{r-1} \times 0$ and $\varphi^{-1}(M') = C_0 \times 0 \times \mathbb{R}^{s-1}$. ■

REMARK 3.10. If M and M' are connected and $r \geq 2$, then $M - S$ and $M' - S$, where $S := M \cap M' - \{p, q\}$, are complete as metric spaces and hence they are geodesically complete manifolds [**6**, p. 147 *proof* c) $\Rightarrow$ d)]. If, in addition, V is simply connected, then we can always build a contractible loop L with desired properties by applying Theorem 1.28.

THEOREM 3.11. *With the hypothesis of Lemma 3.9, there exists an isotopy F_t of the identity $i : V \to V$, such that*

(i) F_t fixes i near $M \cap M' - \{p, q\}$
(ii) $F_1(M) \cap M' = M \cap M' - \{p, q\}$

Proof. Let $G_t : U \to U$ be an isotopy of the identity like the one represented schematically in Figure 5. Clearly $G_1(U \cap C_0) \cap C_0' = \emptyset$.

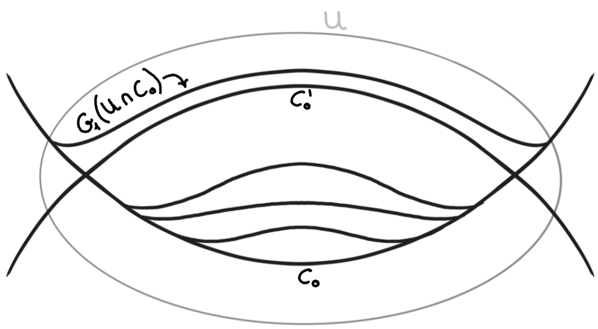


FIGURE 5

Now consider the isotopy

$$\begin{aligned} H_t : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} &\to U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \\ (u, x, y) &\mapsto (G_{t\rho(x,y)}(u), x, y) \end{aligned}$$

where $u \in U$ and

$$\begin{aligned} \rho : \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} &\to [0, 1] \\ (x, y) &\mapsto 1 \quad \text{if} \quad \|x\|^2 + \|y\|^2 \leq 1 \\ (x, y) &\mapsto 0 \quad \text{if} \quad \|x\|^2 + \|y\|^2 \geq 2 \end{aligned}$$

Define the isotopy $F_t(w) := \varphi \circ H_t \circ \varphi^{-1}(w)$, where $\varphi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to V$ is the embedding obtained by Lemma 3.9. Then

- if $t = 0$: $H_0(u, x, y) = (G_0(u), x, y) = (u, x, y)$ and hence $F_0(w) = \varphi \circ H_0 \circ \varphi^{-1}(w) = \varphi \circ \varphi^{-1}(w) = w$.
- if $0 < t < 1$: when $\|x\|^2 + \|y\|^2 \geq 2$, then $\rho(x, y) = 0$ and hence $H_t(u, x, y) = (G_0(u), x, y) = (u, x, y)$, that is, F_t is the identity outside the image of φ; when $\|x\|^2 + \|y\|^2 \leq 1$, then $\rho(x, y) = 1$ and hence, since $G_t(U \cap C_0) \cap C_0' = \{a, b\}$, $F_t(M) \cap M' = M \cap M'$, that is, F_t fixes i near $M \cap M' - \{p, q\}$.
- if $t = 1$: F_1 is still the identity when we are outside the image of φ, but when $\|x\|^2 + \|y\|^2 \leq 1$, since $G_1(U \cap C_0) \cap C_0' = \emptyset$, then $F_1(M) \cap M' = M \cap M' - \{p, q\}$.

Therefore, F_t defines the required isotopy. ■

Second cancellation theorem assures that, if a triad $(W^n; V_0, V_1)$ satisfies certain conditions, then the hypothesis of Theorem 3.11 is fulfilled and hence one can use it repeatedly, by setting $M = S_R$ and $M' = S_L'$, up until there is only one point left in the intersection of resulting new spheres. The conclusion of first cancellation theorem then applies. Nonetheless, there is yet another crucial lemma we need to prove in order to guarantee that the intersection numbers $M \cdot M'$ of new resulting spheres do not change due to the isotopies or deformations M and M' suffer when applying Theorem 3.11.

Lemma 3.12. *Let M^r and M'^s be smooth, closed, connected transversely intersecting submanifolds of a smooth, closed, connected manifold V^{r+s}. Suppose that M and the total space NM' of the normal bundle $\nu_{M'}$ in V are oriented. Then, in the sequence*

$$H_r(M) \xrightarrow{g} H_r(V) \xrightarrow{g'} H_r(V, V - M'),$$

where g and g' are induced by inclusion, we have $g' \circ g([M]) = M' \cdot M\Psi(\alpha)$, where $[M] \in H_r(M)$ is the orientation generator and α is the canonical generator of $H_0(M') \cong \mathbb{Z}$.

Proof. First of all, note that $H_r(M) \cong \mathbb{Z}$, for M is closed, connected and orientable. Also $H_0(M') \cong \mathbb{Z}$, for M' is connected. Let then α and $[M]$ be the generators of $H_0(M')$ and $H_r(M)$, respectively. Now let $U_1, ..., U_k$ be disjoint open r-cells in M containing $p_1, ..., p_k$, respectively. Obviously, by the orientability of M, we will have $H_r(U_i, U_i - p_i) \cong \mathbb{Z}$ for all $i = 1, ..., k$. Let γ_i be the corresponding generators. Then, using the inclusion induced map $H_r(U_i, U_i - p_i) \to H_r(V, V - M')$ and the Thom isomorphism $\Psi : H_0(M') \to H_r(V, V - M')$ (Lemma 1.31), we can build the following composition

$$\begin{aligned} H_r(U_i, U_i - p_i) \to H_0(M') &\xrightarrow{\cong} H_r(V, V - M') \\ \gamma_i \mapsto \epsilon_i\alpha \qquad &\mapsto \Psi(\pm\alpha) = \epsilon_i\Psi(\alpha) \end{aligned}$$

where ϵ will be $+1$ or -1 depending on whether the vectors generating $T_{p_i}M$ represent a positive or negative basis for the fibre of $\nu_{M'}$ at p_i, that is, ϵ_i is the intersection number of M and M' at p_i. Thus,

$$H_r\left(\bigsqcup_{i=1}^{k}(U_i, U_i - p_i)\right) \cong \bigoplus_{i=1}^{k} H_r(U_i, U_i - p_i) \to H_0(M') \xrightarrow{\cong} H_r(V, V - M')$$

$$\gamma_i \oplus \cdots \oplus \gamma_k \mapsto \sum_{i=1}^{k} \epsilon_i\alpha \ \mapsto \sum_{i=1}^{k} \epsilon_i\Psi(\alpha) = M' \cdot M\Psi(\alpha)$$

On the other hand, the excision map

$$\bigsqcup_{i=1}^{k}(U_i, U_i - p_i) \subset (M, M - M \cap M')$$

induces the isomorphism $\bigoplus_{i=1}^{k} H_r(U_i, U_i - p_i) \cong H_r(M, M - M \cap M')$. Therefore, the following commutative diagram, gives the desired conclusion.

$$\begin{tikzcd} H_r(M) \arrow[r, "g"] \arrow[d] & H_r(V) \arrow[r, "g'"] & H_r(V, V - M') \\ H_r(M, M - M \cap M') \arrow[rr, "\cong"'] \arrow[urr] & & \bigoplus_{i=1}^{k} H_r(U_i, U_i - p_i) \arrow[u] \end{tikzcd}$$

Remark 3.13. Previous result implies that $M' \cdot M$ does not change under deformations of M' or ambient isotopy of M. Therefore, from a differential view point, it is in fact a topological invariant. Then we can make use of Theorem 3.11 consistently, even though we lose transversely intersecting condition throughout its iterative application.

Remark 3.14. Suppose we are in a general situation where we have n-dimensional triads $(W; V, V')$, $(W'; V', V'')$, $(W \cup W'; V, V'')$ and f is a Morse function on $(W \cup W'; V, V'')$ with several critical points $q_1, ..., q_\ell \in W$, all of index λ on one level, and several critical points $q'_1, ..., q'_m \in W'$, all of index $\lambda+1$ on another level. Then it can be proved, using Lemma 3.12, that the induced map by inclusion $h : H_\lambda(S'_L(q'_j)) \to H_\lambda(W, V)$ is given by

$$h([S'_L(q'_j)]) = S_R(q_1) \cdot S'_L(q'_j)[D_L(q_1)] + ... + S_R(q_\ell) \cdot S'_L(q'_j)[D_L(q_\ell)]$$

where [ ] denotes the corresponding orientation generators and $S_R(q_i) \cdot S'_L(q'_j)$ is the intersection number of the spheres [**11**, p. 86]. As a corollary, with respect to the bases here represented by the oriented left-hand disks, the boundary map

$$\partial : H_{\lambda+1}(W \cup W', W) \cong H_{\lambda+1}(W', V') \to H_\lambda(W, V)$$

for the triple $W \cup W' \supset W \supset V$ is given by the matrix (a_{ij}) of intersection numbers $a_{ij} = S_R(q_i) \cdot S'_L(q'_j)$ in V', naturally determined by the orientations assigned to the left-hand disks.

We can finally prove second cancellation theorem.

Second Cancellation Theorem 3.15. *(Strong) Suppose W, V_0 and V_1 are simply connected, $\lambda \geq 2$ and $\lambda + 1 \leq n - 3$. If $S_R \cdot S'_L = \pm 1$, then W is diffeomorphic to $V_0 \times [0, 1]$.*

Proof. On the one hand, by hypothesis, $n - \lambda - 1 \geq 3$ and $\lambda \geq 2$. On the other hand, by definition of the spheres and using Theorem 3.11 terminology, $s = \dim S_R = n - \lambda - 1$ and $r = \dim S'_L = \lambda$. Thus, $s \geq 3$ and $r \geq 2$.

If $S_R \cap S'_L$ is not a single point, then $S_R \cdot S'_L = \pm 1$ implies that there exist a couple of points $p_1, p_2 \in S_R \cap S'_L$ with opposite intersection numbers.

Now, thanks to Remark 3.10, in order to prove that the rest of the hypothesis in Theorem 3.11 is satisfied it suffices to check that V is simply connected provided that $\lambda \geq 3$. However, in case $\lambda = 2$, we also have to check that $V - S_R$ is simply connected, for if $\pi_1(V) = \{e\}$, then the map $\pi_1(V - S_R) \to \pi_1(V) = \{e\}$ is 1:1 into if and only if $\pi_1(V - S_R) = \{e\}$.

Because W is simply connected, to prove V is simply connected it suffices to show $\pi_1(V) \cong \pi_1(W)$. Firstly, if we apply Van Kampen's Theorem 1.10 taking into account that the associated disks are simply connected due to dimensionality conditions, we obtain the following diagrams

$$\begin{tikzcd}
 & \pi_1(D_R \cup V) & \\
\pi_1(V) \arrow[ur] & & \pi_1(D_R) \cong \{e\} \arrow[ul] \\
 & \pi_1(D_R \cap V) \cong \{e\} \arrow[ul] \arrow[ur, "\cong"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(D_R \cup V \cup D'_L) & \\
\pi_1(D_R \cup V) \arrow[ur] & & \pi_1(D'_L) \cong \{e\} \arrow[ul] \\
 & \pi_1((D_R \cup V) \cap D'_L) \cong \{e\} \arrow[ul] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Hence, by Corollary 1.11, $\pi_1(V) \cong \pi_1(D_R \cup V) \cong \pi_1(D_R \cup V \cup D'_L)$.

Secondly, in accordance with Remark 2.12, $D_R \cup V \cup D'_L$ is a deformation retract of W, namely, given the inclusion map $i : D_R \cup V \cup D'_L \hookrightarrow W$, there exists a map $r : W \to D_R \cup V \cup D'_L$ such that $r \circ i = id_{D_R \cup V \cup D'_L}$ and $i \circ r \simeq id_W$. Then, in particular, the former is also a weak deformation retract, that is, $i : D_R \cup V \cup D'_L \to W$ is an homotopy equivalence and hence $\pi_1(D_R \cup V \cup D'_L) \cong \pi_1(W)$.

If $\lambda = 2$, then the left-hand sphere of p, that is S_L, is a 1-sphere. Thus, if N is a product neighborhood of S_L in V_0, $\pi_1(N) \cong \mathbb{Z}$. Moreover, since $n - \lambda - 1 \geq 3$, then $\pi_1(N - S_L) \cong \mathbb{Z}$. Note that $(V_0 - S_L) \cap N = N - S_L$ so that, applying Van Kampen's theorem to the spaces $V_0 - S_L$ and N, we obtain $\pi_1(V_0 - S_L) = \{e\}$, due to Corollary 1.11.

Finally, since trajectories of the corresponding gradient-like vector field determine a diffeomorphism of $V_0 - S_L$ onto $V - S_R$, then $\pi_1(V - S_R) = \pi_1(V_0 - S_L) = \{e\}$. ∎

COROLLARY 3.16. *The dimensional condition is equivalent to* $\lambda \geq 3$, $\lambda + 1 \leq n - 2$.

Proof. We just need to consider the triad $(W; V_1, V_0)$ with Morse function $-f$ and gradient-like vector field $-\xi$. ∎

Thus, second cancellation theorem 3.15 can be used to cancel two critical points p and p' with indices λ and $\lambda + 1$, provided that $2 \leq \lambda \leq \lambda + 1 \leq n - 2$. Note that in case $\lambda = 1$, V is not necessarily simply connected and hence we cannot apply Theorem 3.11. Therefore, we will need to treat points with indices 0 and 1 or, analogously, $n - 1$ and n (Corollary 3.16) differently in order to cancel them.

Cancellation of Critical Points

In this section we will construct a relative singular chain complex, using the boundary map ∂ defined in Remark 3.14, and we will prove that if we assume all the homology groups of such a chain complex are zero, then we can eliminate all critical points with indices ranging from 2 to $n - 2$. Afterwards, we will show how to cancel critical points with indices $0, 1$ or $n - 1, n$.

We already showed that any triad can be factored into simpler ones (cf. *proof* of Theorem 2.13). Let us denote by $w = w_0 w_1 ... w_n$ the factorization of a triad (W^n, V, V') into triads admitting a Morse function all of whose critical points are on the same level and have the same index. Let $w_0 w_1 ... w_\lambda$ denote the manifold $W_\lambda \subset W$ and set $W_{-1} = V$ so that

$$V = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_n = W \tag{10}$$

Let $C_\lambda := H_\lambda(W_\lambda, W_{\lambda-1})$ and hence ∂ is the boundary map for the exact sequence of the triple $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$. Actually, $C_* = \{C_\lambda, \partial\}$ is a chain complex, since clearly $\partial^2 = 0$ by the definition. Moreover, we claim that

$$H_\lambda(C_*) \cong H_\lambda(W, V) \tag{11}$$

for all λ. Certainly, the following commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & & \\
C_{\lambda+1} = H_{\lambda+1}(W_{\lambda+1}, W_\lambda) \arrow[r] \arrow[dr, "\partial"'] & H_\lambda(W_\lambda, W_{\lambda-2}) \arrow[r] \arrow[d] & H_\lambda(W_{\lambda+1}, W_{\lambda-2}) \arrow[r] & 0 \\
& C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1}) \arrow[ur] \arrow[d, "\partial"'] & & \\
& C_{\lambda-1} = H_{\lambda-1}(W_{\lambda-1}, W_{\lambda-2}) & &
\end{tikzcd}$$

shows that the λ-th homology group of C_* is given by $\mathcal{Z}_\lambda / \mathcal{B}_\lambda = H_\lambda(W_{\lambda+1}, W_{\lambda-2})$. But $H_\lambda(W_{\lambda+1}, W_{\lambda-2}) \cong H_\lambda(W, V)$ [**17**, p. 261].

Below we prove the main tool for cancelling critical points with indices ranging from 2 to $n-2$.

THEOREM 3.17. *Let $(W^n; V, V')$ be a triad, with $n \geq 6$, possessing a Morse function f with no critical points of indices 0, 1 or $n-1$, n. Furthermore, assume that W, V and V' are all simply connected (hence orientable) and that $H_*(W, V) = 0$. Then W is diffeomorphic to $V \times [0,1]$.*

Proof. Let $w = w_2 w_3 ... w_{n-2}$ denote the triad $(W; V, V')$, so that w admits a Morse function f whose restriction to each w_λ is a Morse function all of whose critical points are on the same level and have index λ. Now, because $0 = H_*(W, V) \overset{(11)}{\cong} H_*(C_*)$, then the sequence

$$C_{n-2} \xrightarrow{\partial} C_{n-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{\lambda+1} \xrightarrow{\partial} C_\lambda \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2$$

is exact, that is, $im(\partial : C_{\lambda+1} \to C_\lambda) = ker(\partial : C_\lambda \to C_{\lambda-1})$. Thus, given a basis $z_1^{\lambda+1}, ..., z_{k_{\lambda+1}}^{\lambda+1}$, for each λ, of $ker(\partial : C_{\lambda+1} \to C_\lambda)$, we may choose $b_1^{\lambda+1}, ..., b_{k_\lambda}^{\lambda+1} \in C_{\lambda+1}$ such that $b_i^{\lambda+1} \xrightarrow{\partial} z_i^\lambda$ for $i = 1, ..., k_\lambda$. Then $z_1^{\lambda+1}, ..., z_{k_{\lambda+1}}^{\lambda+1}, b_1^{\lambda+1}, ..., b_{k_\lambda}^{\lambda+1}$ is a basis of $C_{\lambda+1}$. On the other hand, since $2 \leq \lambda \leq \lambda + 1 \leq n-2$, it is possible to find a Morse function f' and a gradient-like vector field ξ', both agreeing with original ones in a neighborhood of $V \cup V'$, such that f' has the same critical points as f, all on the same level, and such that the orientation generators of the homology groups of the left-hand disks of w_λ and $w_{\lambda+1}$ represent the bases given above for C_λ and $C_{\lambda+1}$ [**11**, p. 92 Basis Theorem]. Let then p and q be the critical points in w_λ and $w_{\lambda+1}$, respectively, corresponding to z_1^λ and $b_1^{\lambda+1}$. According to Remark 2.21, it is possible to increase f' in a neighborhood of p and decrease f' in a neighborhood of q so as to obtain a new triad $w'_\lambda w_p w_q w'_{\lambda+1}$ equivalent to the triad $w_\lambda w_{\lambda+1}$, where w_p only has the critical point p and w_q only has the critical point q.

It can be verified that a level manifold V_0 between w_p and w_q is simply connected (cf. *proof* Theorem 3.15). Also $w_p w_q$ and its two end manifolds will be simply connected by the same argument. Since $\partial b_1^{\lambda+1} = z_1^\lambda$, then, by definition of the boundary map, $S_R(p)$ and $S_L(q)$ have intersection number ± 1 in V_0. The conclusion of second cancellation theorem 3.15 or Corollary 3.16 then applies. Repeating such a process for the rest of critical points and, in general, for the rest of levels, one clearly eliminates all critical points of f. Now Theorem 2.9 provides desired conclusion. ■

It remains showing how to cancel critical points with indices 0, 1 or equivalently, by Corollary 3.16, with indices $n-1$, n. For the remaining of the section we will consider any triad $(W^n; V, V')$ carrying a self-indexing Morse function f and an associated gradient-like vector field ξ. Also

$$W_k = f^{-1}\Big[-\frac{1}{2}, k + \frac{1}{2}\Big], \quad k = 0, 1, ..., n$$

will be the manifolds defined in (10), along with the non-critical level manifolds $V_{k^+} = f^{-1}\big(k + \frac{1}{2}\big)$ in between.

In order to prove the theorem which guarantees the elimination of such points we will need the following result.

LEMMA 3.18. *Suppose W and V are simply connected, $n \geq 5$ and there are no critical points of index 0. Let S_R^{n-2} be a right-hand sphere in V_{1^+}. Then there exists a 1-sphere embedded in V_{1^+} that has one transverse intersection with S_R^{n-2} and meets no other right-hand sphere.*

Proof. $\dim V_{1^+} = n - 1 \geq 4$. Consider then a 1-disk $D \subset V_{1^+}$ such that S_R^{n-2} is the only right-hand sphere intersecting D. Obviously, we can assume D transversely intersects S_R^{n-2}

in V_{1^+} and hence $\dim D = 1$. Let $q_0 = D \cap S_R^{n-2}$ be its midpoint. Then, if we translate the endpoints of D through the trajectories of ξ up until V, we may join them by a smooth path avoiding any 0-sphere. Certainly, this is possible because V is connected and of dimension greater than 1. Afterwards, such a path may be translated back to a smooth path α in V_{1^+} joining the endpoints of D and avoiding all right-hand spheres. Then $\gamma := D \cup \alpha$ will be a closed path in V_{1^+} which transversely intersects S_R^{n-2} at q_0 and meets no other right-hand sphere. Now construct a smooth map $f : S^1 \to V_{1^+}$ which, restricted to a closed subset A about $a \in S^1$, is an embedding such that $f^{-1}(q_0) = a \in S^1$ and $f(S^1 - a) = \gamma - q_0$. Then, since $\dim V_1 \geq 2 \cdot \dim S^1 + 1$, by Corollary 3.6, there exists a smooth embedding $g : S^1 \to V_{1^+}$ such that $g \simeq f$ and $g|_A = f|_A$. ■

Now we can prove the main tool for cancelling critical points with indices 0 and 1.

Theorem 3.19.

(i) If $H_0(W, V) = 0$, then the critical points of index 0 can be cancelled against an equal amount of critical points of index 1.

(ii) With the assumptions of Lemma 3.18, one can insert, for each critical point of index 1, one critical point of index 2 and one critical point of index 3 in such a way that the index 1 critical points can be cancelled against the auxiliary index 2 critical points.

Proof.

(i) Consider homology with coefficients in $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Then

$$H_1(W_1, W_0; \mathbb{Z}_2) \xrightarrow{\partial} H_0(W_0, V; \mathbb{Z}_2)$$

is onto, that is, for every S_R^{n-1} of an index 0 critical point there exists a S_L^0 of an index 1 critical point. But ∂ is given by the matrix of intersection numbers, modulo 2, of the right-hand (n-1)-spheres and left-hand 0-spheres in V_{0^+}. Thus, for every S_R^{n-1} there exists, at least, one S_L^0 such that $S_R^{n-1} \cdot S_L^0 \equiv 1$ $(mod.\ 2)$, namely, $S_R^{n-1} \cdot S_L^0$ is an odd number and hence, because S_L^0 is the disjoint union of two points, $S_R^{n-1} \cap S_L^0$ consists of a single point. Therefore, applying first cancellation theorem 3.2, we can cancel the corresponding index 0 critical point against one of the index 1 critical points. Applying such a procedure iteratively we can eliminate all critical points of index 0 against the same amount of critical points of index 1.

(ii) Note that the inclusion

$$D_R^{n-2} \cup D_R^{n-1} \cup V_{2^+} \cup D_L^3 \cup \ldots \cup D_L^n \subset W,$$

is a homotopy equivalence (cf. 2.12). On the other hand, applying Van Kampen's theorem iteratively, one obtains

$$\pi_1(V_{2^+}) \cong \pi_1(D_R^{n-2} \cup D_R^{n-1} \cup V_{2^+} \cup D_L^3 \cup \ldots \cup D_L^n)$$

Thus, because W is simply connected, $\pi_1(V_{2^+}) \cong \pi_1(W) = \{e\}$, that is, V_{2^+} is also simply connected. Let p be an index 1 critical point and consider the topological 1-sphere S in V_{1^+} given by Lemma 3.18. Then, after adjusting ξ, we can translate S right to a 1-sphere S_1 in V_{2^+} (cf. 2.17) along the corresponding trajectories.

Now, since W_2 is compact, there exists a collar neighborhood N of ∂W_2 (cf. 2.10), that is, N is diffeomorphic to $\partial W_2 \times [0, 1)$. Let $x_1, \ldots, x_n$ be coordinate functions in N embedding an open set $U \subset N$ into $\mathbb{R}^n$ such that $f|_U = x_n$ (cf. 1st argument in *proof* Theorem 2.13). Then we can alter f on a compact subset of U inserting a pair q, r of auxiliary critical points of index 2 and 3, with $f(q) < f(r)$ [**11**, p. 101 Lemma 8.2]. Figure 6 illustrates the situation.

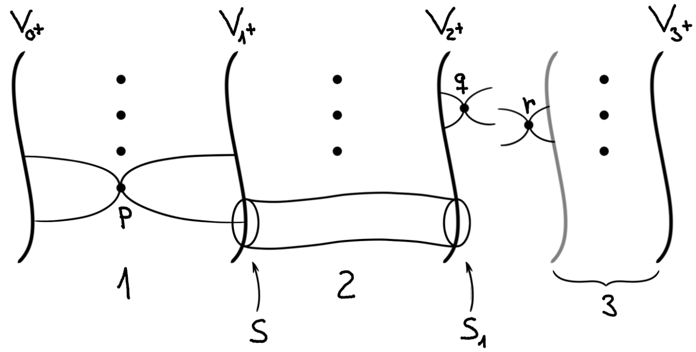


FIGURE 6

Let S_2 be the left-hand 1-sphere of q in V_{2+}. Since the latter is simply connected, applying Corollary 3.7 and Isotopy Extension Theorem [**18**], we can find an isotopy of the identity $V_{2+} \to V_{2+}$ which sends S_2 to S_1 and hence, after adjusting ξ to the right of V_{2+}, the left-hand sphere of q in V_{2+} will be S_1. Thus, the left-hand sphere of q in V_{1+} will be S which, by construction, intersects the right-hand sphere of p transversely in a single point.

Increasing the level of p and lowering the level of q without altering ξ (by 2.21) we can apply first cancellation theorem on a triad $f^{-1}\left[1+\delta, 2-\delta\right]$ in which f has only the critical points p and q, for a small enough $\delta > 0$, in order to eliminate these two points. Finally, we just move the critical value of r up until level 3 using again 2.21.

To sum up, we have traded an index 1 critical point for an index 3 critical point. The process can be repeated until there are no more index 1 critical points left. ■

CHAPTER 4

The H-cobordism Theorem

Below we state a version of the h-cobordism theorem. Its proof follows straightforwardly from the results we have proved in previous chapter.

h-Cobordism Theorem 4.1. *(Milnor) Let $(W^n; V, V')$ be a triad such that:*

(1) W, V and V' are simply connected
(2) $H_*(W, V) = 0$
(3) $n \geq 6$

Then W is diffeomorphic to $V \times [0,1]$.

Proof. Let f be a self-indexing Morse function on $(W; V, V')$. Using Theorem 3.19, we can cancel all critical points of indices 0, 1 and, after replacing f by $-f$, all critical points of original indices $n-1$, n. Now, applying Theorem 3.17, we can cancel the rest of critical points. ■

Remark 4.2. Condition (2) is equivalent to $H_*(W, V') = 0$. Certainly, $H_*(W, V) = 0$ implies $H^*(W, V') = 0$ by Poincaré duality applied to the pair (W, V) [**11**, p. 90]. But $H^*(W, V') = 0$ implies $H_*(W, V') = 0$. The converse is proved similarly.

On the Original Version of the H-cobordism Theorem

The original version of h-cobordism theorem is due to Stephen Smale [**19**], who proved it in 1961 using the notion of h-cobordism. It is an interesting fact, though, that we did not mention such a concept neither in the statement of Theorem 4.1 nor in the proof. Actually, there is no even need for defining the cobordism category. In this chapter, though, we will define such a mathematical tool in order to state Smale's version of the theorem and prove it is equivalent to that of Milnor.

The Cobordism Category.

Definition. Let M_0 and M_1 be closed n-manifolds. A *cobordism* from M_0 to M_1 is a 5-tuple $(W; V_0, V_1; h_0, h_1)$, where $(W; V_0, V_1)$ is a triad and $h_i : V_i \to M_i$, with $i = 0, 1$, is a diffeomorphism. Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ are *equivalent* if there exists a diffeomorphism $g : W \to W'$ sending V_0 to V_0' and V_1 to V_1' such that the following diagram commutes

$$\begin{tikzcd} V_i \arrow[rr, "g|_{V_i}"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\ & M_i & \end{tikzcd}$$

Then we have a category whose objects are closed manifolds and whose morphisms are equivalence classes of cobordisms.

Remark 4.3. Note that any triad $(W; V_0, V_1)$ can be identified with the cobordism $(W; V_0, V_1; id, id)$. Such a cobordism is commonly written just $(W; V_0, V_1)$.

Definition. A cobordism $(W; V, V')$ is an *h-cobordism* if $V \hookrightarrow W$ and $V' \hookrightarrow W$ are homotopy equivalences, that is, both V and V' are weak deformation retracts of W. If $(W; V, V')$ is an h-cobordism, then V is said to be *h-cobordant* to V'.

The Original Version of H-cobordism Theorem.

h-Cobordism Theorem 4.4. *(Smale) Let V and V' be two closed, oriented and simply connected manifolds of dimension greater than four, which are h-cobordant. Then V and V' are diffeomorphic by an orientation preserving diffeomorphism.*

It is not evident to see that the two versions of the theorem are equivalent. Let us check it.

Firstly, the fact that V and V' are h-cobordant means that there exists a compact smooth manifold W^n, $n > 5$, with the homotopy type of V and V'. Thus, W will also be simply connected. The condition that V and V' are oriented is implicitly assumed in Milnor's version 4.1 when using the second cancellation theorem 3.15. Indeed, since W is simply connected, then it is orientable and hence ∂W can be given an orientation induced from that of W as we have already explained in the introduction of Chapter 3. The fact that V and V' are diffeomorphic is also implicit in Milnor's version 4.1, due to Remark 4.2.

So it remains to prove that $(W; V, V')$ is an h-cobordism if and only if $H_*(W, V) = 0$. The implication to the right is immediate, for if $(W; V, V')$ is an h-cobordism then $H_*(V) \cong H_*(W)$, by homotopy invariance, and hence $H_*(W, V) = 0$. As for the converse, we need to work the argument a little bit more. The existence of homotopy classes of pair maps is justified from the general definition we gave for homotopy in section (1) of preliminary concepts. Actually, the existence of an *exact homotopy sequence of the pair* (W, V) [**2**, p. 354 thm. 11.43] implies that $\pi_n(W, V)$ is a group and that it is abelian for all $n \geq 2$. Now the fact that $\pi_1(V) = \{e\}$ and $\pi_1(W, V) = \{e\}$ together with $H_*(W, V) = 0$ imply that $\pi_i(W, V) = 0$, $i = 0, 1, 2, \ldots$, by the relative Hurewicz isomorphism theorem [**20**, p. 103 thm. 2.6]. Then, because (W, V) is a triangulable pair [**13**, p. 103 thm. 10.6], a deformation retraction $W \to V$ can be constructed [**20**, p. 98 Thm 1.7]. Similarly, V' is also a deformation retract of W.

CHAPTER 5

Applications

Characterization of the Smooth n-disk $\mathbf{D^n}$, $\mathbf{n \geq 6}$

THEOREM 5.1. *Let W^n be a smooth, closed, simply connected manifold, with $n \geq 6$, such that $V := \partial W$ is simply connected. Then W has the homology of a point if and only if W is diffeomorphic to D^n.*

Proof. Let D_0 be a smooth n-disk embedded in $\mathring{W}$. Then

$$\begin{aligned} H_*(W - \mathring{D}_0, \partial D_0) &\cong H_*(W, D_0) \quad \text{(Excision)} \\ &\cong 0 \qquad\qquad\quad (W \text{ has the homology of a point}) \end{aligned}$$

Then, applying h-cobordism theorem to the triad $(W - \mathring{D}_0; \partial D_0, V)$ we obtain

$$\begin{cases} W - \mathring{D}_0 \text{ is diffeomorphic to } \partial D_0 \times [0,1] \\ W - \mathring{D}_0 \text{ is diffeomorphic to } V \times [0,1] \end{cases}$$

Therefore, V is diffeomorphic to ∂D_0 and hence W is diffeomorphic to D_0.

The converse is trivial, for if W is diffeomorphic to D^n, then it is also homeomorphic to D^n and hence contractible. Therefore, it will have the homology of a point. ■

Generalized Poincaré Conjecture

Below we will prove a result that, combined with a result of Milnor and Kervaire, yields a corollary which is the version of generalized Poincaré conjecture we announced at the end of the introduction. Before, though, we need to define a concept and prove a lemma.

DEFINITION. A *twisted n-sphere* is the disjoint union of two n-disks D_1, D_2 with the boundaries identified under a diffeomorphism $h : \partial D_1 \to \partial D_2$, namely, $D_1 \cup_h D_2$.

LEMMA 5.2. *Any twisted sphere is homeomorphic to S^n.*

Proof. Let $g_1 : D_1 \to S^n$ be an embedding onto the south hemisphere of S^n, that is, $\{\vec{x} \in \mathbb{R}^{n+1} : \|x\| = 1, x_{n+1} \leq 0\}$. Then, the map

$$\begin{aligned} g : D_1 \cup_h D_2 &\to S^n \\ u &\mapsto g_1(u) \quad \text{if } u \in D_1 \\ tv &\mapsto \sin\frac{\pi t}{2} g_1(h^{-1}(v)) + \cos\frac{\pi t}{2}(0, ..., 0, 1) \quad \text{if } tv \in D_2;\ 0 \leq t \leq 1 \end{aligned}$$

is a well defined 1:1 continuous map onto S^n. ■

THEOREM 5.3. *Let W^n, $n \geq 6$, be a smooth, closed, simply connected manifold with the homology of S^n. Then W is homeomorphic to S^n.*

Proof. Let $D_0 \subset W$ be a smooth n-disk. Then

$$
\begin{aligned}
H_i(W - \mathring{D}_0) &\cong H^{n-i}(W - \mathring{D}_0, \partial D_0) && \text{(Poincaré duality)} \\
&\cong H^{n-i}(W, D_0) && \text{(Excision)} \\
&\cong \begin{cases} 0 & \text{if } i > 0 \\ \mathbb{Z} & \text{if } i = 0 \end{cases} && \text{(Exact sequence)}
\end{aligned}
$$

Namely, $W - \mathring{D}_0$ has the homology of a point and hence, by Theorem 5.1, $W - \mathring{D}_0$ is diffeomorphic to D^n. Therefore, $W = (W - \mathring{D}_0) \cup D_0$ is diffeomorphic to a twisted sphere which, according to Lemma 5.2, is homeomorphic to S^n. ■

Now, due to Kervaire and Milnor [**21**], given W^n with the same assumptions of Theorem 5.3 but $n = 4$, 5 or 6, W bounds a smooth, compact and contractible manifold.

Combining both results, one obtains the following corollary.

COROLLARY 5.4. *If W^n, $n \geq 5$, is a homotopy n-sphere, that is, a closed n-manifold which is homotopy equivalent to S^n, then W is homeomorphic to S^n. Furthermore, if $n = 5$ or 6, W is diffeomorphic to S^n.*

Proof. First of all, if W is a homotopy n-sphere, then it is closed and simply connected. Moreover, due to homotopy invariance, it also has the homology of S^n. Thus:

- if $n \geq 6$, W is homeomorphic to S^n (by Theorem 5.3)
- if $n = 5$ or 6, W bounds a closed, compact and contractible manifold (by Kervaire and Milnor) and hence it is actually diffeomorphic to S^n (by Theorem 5.1). ■

REMARK 5.5. Note that h-cobordism theorem guarantees the generalized Poincaré conjecture in dimensions greater than four is true for a topological manifold W provided that it is endowed with a smooth structure. Nonetheless, in spite of the fact that one uses Morse functions to conclude that the so called handlebodies are deformation retracts of W, the latter fact is purely topological and hence it suggests that the h-cobordism theorem may be proved for the category of topological spaces and continuous maps as well. Certainly, there is a much more complicated topological version of the theorem [**22**] which proves the generalized Poincaré conjecture is true in such a category.

APPENDIX A

Poincaré Homology Sphere

Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, that is, $a = x_1 + x_2 i$, $b = x_3 + x_4 i$ such that $P^*P = I$ and $detP = 1$. Then, $P = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ and hence $\overline{a}a + \overline{b}b = 1$. Thus, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, which is the unit 3-sphere in $\mathbb{R}^4$. Such a correspondence, as well as its inverse, is obviously continuous. Therefore, $SU(2)$ is homeomorphic to S^3. We will represent an element $P \in SU(2)$ as $(a, b) \in \mathbb{C}^2$ or $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ indistinctly.

The "latitudes" of S^3 will be the intersections with the hyperplane $x_1 = constant$.

$$\begin{cases} x_1 = c \\ x_2^2 + x_3^2 + x_4^2 = (1 - c^2) \quad \text{where } -1 < c < 1 \end{cases} \tag{12}$$

They are, then, 2-spheres in $\mathbb{R}^4$ formed by matrices with the same trace. In order to visualize, for example, the latitude of trace zero matrices, we must project S^3 onto the tridimensional space. Figure 1 shows such a latitude by flattening it.

Now $\{I_2\}$ and $\{-I_2\}$ conform two conjugacy classes, for if $P \in SU(2)$ then $PI_2P^{-1} = I_2$ and $P(-I_2)P^{-1} = -I_2$. As for the rest of classes, on the one hand, one observes that the characteristic polynomial of $P \in SU(2)$ is $t^2 - tr(P)t + 1$ and hence two matrices belonging to $SU(2)$ will have the same eigenvalues if and only if they have the same trace, that is, they correspond to points with the same latitude of S^3. On the other hand, by spectral theorem for normal operators, there exists a unitary matrix Q such that QPQ^* is diagonal and hence there exists a matrix $Q_1 = \overline{\epsilon}Q$, with $\overline{\epsilon}\epsilon = 1$, such that

$$\begin{pmatrix} \lambda & \\ & \overline{\lambda} \end{pmatrix} = Q_1 P Q_1^* = Q_1 P Q_1^{-1}$$

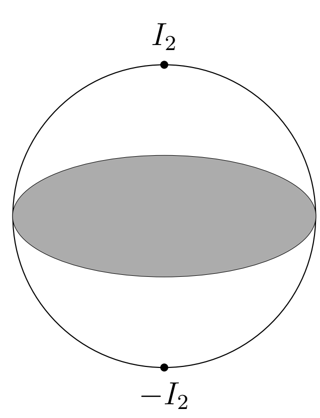


FIGURE 1. In gray, zero latitude (trace zero matrices).

Thus, every point on a given latitude will have the same eigenvalues as its conjugate and hence they both lie on the same latitude. Therefore, except for $\{I_2\}$ and $\{-I_2\}$, conjugacy classes of $SU(2)$ are the different latitudes of S^3.

Note that conjugacy operation induces a homomorphism

$$\varphi : SU(2) \to GL(3, \mathbb{R})$$

Indeed, $(PQ)A(PQ)^* = P(QAQ^*)P^*$ is the composition of conjugacy operations by P and Q which means that conjugacy by a product equals the product of conjugations. Thus,

$$I_2 A I_2^* = P^{-1}PA(P^{-1}P)^* = A$$

and hence $\varphi(I_2) = I_3$. So $\varphi(P)$ is an invertible matrix for any $P \in SU(2)$. An element A in the conjugacy class C formed by trace zero matrices will be a matrix of the form

$$A = \begin{pmatrix} y_2 i & y_3 + y_4 i \\ -y_3 + y_4 i & -y_2 i \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} y_2 + \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} y_3 + \begin{pmatrix} & i \\ i & \end{pmatrix} y_4,$$

with $y_2^2 + y_3^2 + y_4^2 = 1$. Therefore, C is a unit 2-sphere in a real vector space of dimension 3. In order to describe φ it will suffice studying how $SU(2)$ operates on the space V that contains C. Conjugation by a matrix $P \in SU(2)$ provides a linear operator on V, since

$$\begin{cases} P(X + X')P^* & = PXP^* + PX'P^* \\ P(rX)P^* & = rPXP^* \end{cases}$$

where $r \in \mathbb{R}$ and $X, X' \in V$. The matrix of this operator is $\varphi(P)$ whose columns are the coordinates, with respect to the basis $\beta_V = \left\{ \begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix} \right\}$, of conjugations of the elements of this basis by P. Thus,

$$\varphi(P) = \begin{pmatrix} a\overline{a} - b\overline{b} & i(\overline{a}b - a\overline{b}) & \overline{a}b + a\overline{b} \\ i(\overline{ab} - ab) & \frac{1}{2}(a^2 + \overline{a}^2 + b^2 + \overline{b}^2) & \frac{i}{2}(a^2 - \overline{a}^2 - b^2 + \overline{b}^2) \\ -(\overline{ab} + ab) & \frac{i}{2}(\overline{a}^2 - a^2 + \overline{b}^2 - b^2) & \frac{1}{2}(a^2 + \overline{a}^2 - b^2 - \overline{b}^2) \end{pmatrix}$$

which is a real matrix, for it is the matrix of a linear operator on a real vector space. One observes that $\varphi(P)\varphi(P)^T = I_3$ and hence such an operator is a rotation. Moreover, since $det\varphi(P) = 1$, $\varphi(P) \in SO(3, \mathbb{R})$ so that we will write $\varphi : SU(2) \to SO(3, \mathbb{R})$. Now we prove the following result concerning the homomorphism φ.

Theorem A.1. $ker\varphi = \{\pm I_2\} \quad \& \quad im\varphi = SO(3, \mathbb{R})$

Proof. If $P \in ker\varphi$, then $PXP^* = X$ for any $X \in \beta_V$. Thus, $b = 0$, $a = \overline{a}$ and hence $P = \pm I_2$.

Let us now check that $im\varphi = SO(3, \mathbb{R})$, that is, φ is surjective. Let us have a look on how matrix $\varphi(P)$ acts on an element of $SU(2)$, for example the following zero trace diagonal matrix.

$$A = \begin{pmatrix} y_2 i & y_3 + y_4 i \\ -y_3 + y_4 i & -y_2 i \end{pmatrix}$$

By conjugation, one obtains

$$PAP^* = \begin{pmatrix} y_2 i & a^2(y_3 + y_4 i) \\ -\overline{a}^2(-y_3 + y_4 i) & -y_2 i \end{pmatrix}$$

Now, using the change of variables $a^2 = \exp(2\theta i)$, one observes that $\varphi(P)$ is a 2θ rotation around the point $(1,0,0) \in \mathbb{R}^3$, that is,

$$\varphi(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Then the image, by φ, of the set of zero trace diagonal matrices is the subgroup $H \subset SO(3,\mathbb{R})$ of rotations around the point $(1,0,0)$ corresponding to the matrix

$$\begin{pmatrix} i & \\ & -i \end{pmatrix} \in C$$

Now, let $B \in C$ correspond to a unit vector $u \in \mathbb{R}^3$. Then, by transitivity, there must exist $Q \in SU(2)$ such that $\varphi(Q) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = u$, that is, $Q \begin{pmatrix} i & \\ & -i \end{pmatrix} Q^* = B$. Thus, $\varphi(Q)H\varphi(Q)^*$ will be the subgroup of rotations around any point in the 2-sphere and therefore $SO(3,\mathbb{R}) = \sum \varphi(Q)H\varphi(Q)^*$. ■

Corollary A.2. *$SO(3,\mathbb{R}) \cong SU(2)/\{\pm I_2\}$ and the cosets $\{\pm P\}$ conform the homomorphism fibers of φ. Thus, each element of $SO(3,\mathbb{R})$ corresponds to a pair of unitary matrices of $SU(2)$ with opposite signs.*

Remark A.3. Considering $SU(2)$ and $SO(3,\mathbb{R})$ as Lie groups and hence smooth manifolds, then φ is a covering map and $SU(2)$ is a covering space of $SO(3,\mathbb{R})$ with two sheets.

Below we give a definition as well as a couple of important results concerning covering maps[1].

Definition. Let $p : \tilde{X} \to X$ be a covering map. A covering transformation is a homeomorphism $\varphi : \tilde{X} \to \tilde{X}$ such that $p \circ \varphi = p$. It can be proved that the set of all covering transformations forms a group which is denoted $G(\tilde{X}|X)$.

Theorem A.4. *Let $p : \tilde{X} \to X$ be a covering map. Let $\tilde{X}$ be path connected both global and locally and let $\tilde{x}_0 \in \tilde{X}$. Let $H := p_*(\pi_1(\tilde{X}, \tilde{x_0})) \subset \pi_1(X, x_0)$. Then*

$$G(\tilde{X}|X) \cong N(H)/H.$$

Notation. $N(H)$ denotes the normalizer of H in $\pi_1(X, x_0)$, that is, the subgroup of $\pi_1(X, p(\tilde{x}_0))$ consisting of elements $[\omega] \in \pi_1(X, p(\tilde{x}_0))$ such that $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is invariant under conjugation by $[\omega]$.

Corollary A.5. *If $\tilde{X}$ is simply connected, then H is trivial, because $p_*(\pi_1(\tilde{X}, \tilde{x_0})) \cong \{e\}$, and hence $G(\tilde{X}|X) \cong \pi_1(X, x_0)$.*

Theorem A.6. *Let $\mathcal{Y}$ be a Hausdorff topological space which is connected. Let G be a finite group of homeomorphisms acting without fixed points on $\mathcal{Y}$. Then $p : \mathcal{Y} \to \mathcal{Y}/\Gamma$ is a covering map and $G(\mathcal{Y}|\mathcal{Y}/G) = G$.*

[1]Theorems A.4 and A.6 are reformulations of theorems 2 and 7 in pages 85-88 of [**1**], where theorem 2 is stated for the general case of a *fibration*, which is the bundle projection of a *fibre bundle*, that is, a type of bundle, the structure we defined in Chapter 1.

Now consider the icosahedral group $\mathcal{I} \subset SO(3,\mathbb{R})$ formed by all the isometries of a regular dodecahedron. Then, $\#\mathcal{I} = 60$. Define $2\mathcal{I} := \varphi^{-1}(\mathcal{I})$ which is then a subgroup of $SU(2)$ of order 120 in that $SU(2)$ is a covering space with two sheets. Define the Poincaré homology sphere

$$\mathcal{S} := SO(3,\mathbb{R})/\mathcal{I} \cong SU(2)/2\mathcal{I} \cong S^3/2\mathcal{I}$$

Let us show that $\mathcal{S}$ has the homology groups of S^3 but it is not simply connected. Firstly, since S^3 is path connected, so is $\mathcal{S}$ and hence $H_0(\mathcal{S}) \cong \mathbb{Z}$; secondly, $\mathcal{S}$ is a smooth manifold, for S^3 is; and thirdly, because $2\mathcal{I} \subset SO(4,\mathbb{R})$, the determinants of the corresponding rotation matrices are positive and hence that manifold is orientable. Thus, by Poincaré duality, $H_3(\mathcal{S}) \cong \mathbb{Z}$. In order to obtain $H_1(\mathcal{S})$ we just abelianize $\pi_1(\mathcal{S})$. So let's compute the fundamental group of $\mathcal{S}$. Observe that

$$p : S^3 \to S^3/2\mathcal{I} \cong \mathcal{S}$$

is a covering map between both locally and globally path connected spaces. Thus, since S^3 is simply connected, by Corollary A.5, $\pi_1(\mathcal{S}) \cong G(S^3|S^3/2\mathcal{I})$. On the other hand, by Theorem A.6, $G(S^3|S^3/2\mathcal{I}) = 2\mathcal{I}$. Then $\pi_1(\mathcal{S}) \cong 2\mathcal{I}$. As for the abelianization of $2\mathcal{I}$, it is very useful to describe the latter as a subgroup of the group of quaternions with norm one, that is,

$$Sp(1) = \{a + bi + cj + dk : i^2 = j^2 = k^2 = ijk = -1; a, b, c, d \in \mathbb{R}\} \subset \mathbb{R}^4$$

which is isomorphic to $SU(2)$. A presentation of $2\mathcal{I}$ using quaternions is the following.

$$2\mathcal{I} = \langle s, t : (st)^2 = s^3 = t^5\rangle, \text{ with } \begin{cases} s = \frac{1}{2}(1 + i + j + k) \\ t = \frac{1}{2}(\Phi + \Phi^{-1}i + j) \end{cases}$$

where $\Phi := \frac{1+\sqrt{5}}{2}$ (Golden ratio). Thus,

$$\begin{aligned}(2\mathcal{I})_{ab} =& \langle s, t : (st)^2 = s^3 = t^5, st = ts\rangle \\ =& \langle s, t : t^2 = ss^2 = t^3, st = ts\rangle \\ =& \langle s, t : t^2 = ss = t\rangle \\ =& \langle s, t : s^2 = s\rangle \\ =& \{0\}\end{aligned}$$

so $H_1(\mathcal{S}) = 0$ and again, by Poincaré duality, $H_2(\mathcal{S}) = 0$. Then $\mathcal{S}$ has the same homology groups as S^3 but $\pi_1(\mathcal{S}) = 2\mathcal{I}$ and therefore $S^3 \ncong \mathcal{S}$.

Remark A.7. The simplicial analogous of $\mathcal{S}$ is called Poincaré dodecahedral space, $\mathcal{D}$, which is obtained by identifying the opposite sides of a dodecahedron with a rotation of $\frac{2\pi}{10}$. In order to see that $\mathcal{S} \cong \mathcal{D}$, let us take the dodecaplex (120-cell), which is a simplicial complex with 120 dodecahedrons, 720 pentagons, 1200 edges and 600 vertices, and consider the triangulation $f : S^3 \xrightarrow{\cong} |\text{120-cell}|$ of S^3. Then, if D is a regular dodecahedron, it can be checked that

$$\text{120-cell}/2\mathcal{I} \cong D/\sim \iff \sim \text{ is the equivalence relation described above}$$

Therefore,

$$\mathcal{D} \cong D/\sim \cong |\text{120-cell}|/2\mathcal{I} \cong S^3/2\mathcal{I}$$

Bibliography

[1] Spanier, Edwin H. *Algebraic Topology.* McGraw-Hill Pub., 1966.

[2] Rotman, Joseph J. *An Introduction to Algebraic Topology.* Graduate Texts in Mathematics, **119**. Springer-Verlag, 1988.

[3] Crowell, Richard H. For, Ralph H. *Introduction to Knot Theory.* Graduate Texts in Mathematics, **57**. Springer-Verlag, 1977.

[4] Poincaré, H. *Géométrie; Analysis situs.* Oeuvres, TOME VI, 1854-1912.

[5] Boothby, William M. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Academic Press Inc., 1986.

[6] Carmo, M. do *Riemannian Geometry.* Birkhäuser, 1992.

[7] Milnor, John W. *Morse Theory.* Princeton University Press, 1969.

[8] Graves, Lawrence M. *The Theory of Functions of Real Variables.* Dover Books on Mathematics, 2009.

[9] S. Lang *Introduction to Differentiable Manifolds.* Interscience, New York, 1962.

[10] Sard, A. *The Measure of the Critical Values of Differentiable Maps.* Bulletin of the American Mathematical Society **48**, 883-897 (1942).

[11] Milnor, John W. *Lectures on the h-cobordism Theorem.* Princeton University Press, 1965.

[12] Smale, S. *Generalized Poincaré's Conjecture in Dimensions Greater than 4.* Annals of Mathematics, vol. **74**, pp. 391-406 (1961).

[13] Munkres, J. *Elementary Differential Topology.* Princeton University Press, 1966.

[14] Milnor, John W. *Differential Topology.* mimeographed Princeton University, 1958.

[15] Whitney, H. *Differentiable Manifolds.* Annals of Mathematics, vol. **37**, pp. 645-680 (1936).

[16] Husemoller, D. *Fibre Bundles.* Graduate Texts in Mathematics, **20**. Springer-Verlag, 1994.

[17] Milnor, John W. *Characteristic Classes.* Princeton University Press, 1974.

[18] Thom, R. *La classification des immersions.* Séminaire Bourbaki, 1957.

[19] Smale, S. *On the Structure of Manifolds.* American Journal of Mathematics, vol. **84**, pp. 387-399 (1962).

[20] Hilton, Peter J. *An Introduction to Homotopy Theory.* Cambridge University Press, 1953.

[21] Kervaire, M. Milnor, John W. *Groups of Homotopy Spheres.* Annals of Mathematics, vol. **77**, No. **3** pp. 504-537 (1963).

[22] Connell, E. H. *A topological H-cobordism theorem.* Illinois J. Math., vol. **11**, pp. 300-309 (1967).