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Research Paper

High-order approximations to call option prices in the Heston model

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ABSTRACT

In the present paper, a decomposition formula for the call price due to Alòs is transformed into a Taylor-type formula containing an infinite series with stochastic terms. The new decomposition may be considered as an alternative to the decomposition of the call price found in a recent paper by Alòs, Gatheral and Rodoičić. We use the new decomposition to obtain various approximations to the call price in the Heston model with sharper estimates of the error term than in previously known approximations. One of the formulas obtained in the present paper has five significant terms and an error estimate of the form $O(\nu^3(|\rho| + \nu))$, where ν and ρ are the volatility-of-volatility and the correlation in the Heston model, respectively. Another approximation formula contains seven more terms and the error estimate is of the form $O(\nu^4(1 + |\rho|\nu))$. For the uncorrelated Heston model ($\rho = 0$), we obtain a formula with four significant terms and an error estimate $O(\nu^6)$. Numerical experi-

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ments show that the new approximations to the call price perform especially well in the high-volatility mode.

Keywords: computational finance; Heston model; option pricing; price approximations; stochastic volatility models; vanilla options.

1 INTRODUCTION

Stochastic volatility models were introduced to account for the various flaws of the constant volatility assumption, on which the celebrated Black–Scholes model is based. One of the most popular stochastic volatility models is the Heston model, developed in Heston (1993). For more information on stochastic volatility models, see, for example, Gatheral (2006).

In the present paper, we derive sharp approximation formulas for the call option price in the Heston model. These high-order approximations improve on previously known ones. In Alòs (2006) and Alòs (2012), special decompositions of the call option price in the Heston model were found using Malliavin calculus and Itô calculus, respectively. The main difference between those two decompositions is that the former uses the average of future variances, while the latter is based on the conditional expectation of such an average. Note that the stochastic process consisting of the conditional expectation of future variances is an adapted process, whereas the process of the genuine average is an anticipating process.

In Alòs (2012), an approximation formula with a general error term was obtained for the call option price in the Heston model. This error term was quantified in Alòs *et al* (2015), where it was shown that the error term has the form $O(\nu^2(|\rho| + \nu)^2)$. In the previous expression, ν is the volatility-of-volatility (vol-vol) parameter and ρ is the correlation coefficient in the Heston model. However, in the abovementioned approximation formula, some terms of order ν^2 were ignored, whereas other terms of the same order were kept. This may be considered a drawback of the approximation formula obtained in Alòs *et al* (2015).

Among other earlier works, we would like to mention Merino and Vives (2015), where the expansion obtained in Alòs (2012) was extended to general stochastic volatility models of diffusion type. Moreover, in Merino *et al* (2018), a general decomposition formula for a smooth functional of the log-price process was obtained for a general stochastic volatility model, along with a decomposition formula for call options in models with finite activity jumps in the spot. Merino *et al* (2018, Theorem 3.1) is used recursively in the present paper to approximate the exact call option price decomposition obtained in Alòs (2012) by an infinite series of stochastic terms. The first two terms in the new expansion are the same as in Alòs (2012) and Alòs *et al* (2015). Moreover, our result is consistent with that obtained in Alòs *et al* (2019) but is presented and obtained differently.

Using the new general approximation formula in the case of the Heston model, we add two more significant terms to the abovementioned expansion to reach an error estimate of the form $O(\nu^3(|\rho| + \nu))$ (see Theorem 4.1), and seven more significant terms to obtain an error estimate of the form $O(\nu^4(1 + |\rho|\nu))$ (see Theorem 4.2). In the particular case of zero correlation, we derive an approximation formula with four terms, obtaining an error estimate $O(\nu^6)$ (see Theorem 4.3).

We will now briefly describe the structure of the paper. In Section 2, we provide preliminary information and discuss the notation used throughout. In Section 3, we establish a general decomposition formula and show how to use it recursively to obtain higher-order approximation formulas for the call option price. In Section 4, we obtain two new approximation formulas for the call option price in the Heston model (see Theorems 4.1 and 4.2). The error estimates in those formulas are of order $O(\nu^3(|\rho| + \nu))$ and $O(\nu^4(1 + |\rho|\nu))$, respectively. In addition, we derive an approximation formula of order $O(\nu^6)$ for the uncorrelated case (see Theorem 4.3). In Section 5, we provide and discuss some numerical results. Our conclusions can be found in Section 6.

2 PRELIMINARIES AND NOTATION

Let $T > 0$ be the time horizon, and let W and $\tilde{W}$ be two independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Denote by $\mathcal{F}^W$ and $\mathcal{F}^{\tilde{W}}$ the completed natural filtrations generated by W and $\tilde{W}$, respectively. Set $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^{\tilde{W}}$, $t \in [0, T]$.

Consider a stochastic volatility model in which the asset price process $S = \{S_t, t \in [0, T]\}$ satisfies the stochastic differential equation

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma_t S_t(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\tilde{W}_t), \tag{2.1}$$

where $r \geqslant 0$ is the interest rate and $\rho \in (-1, 1)$. The volatility process σ is a square-integrable process adapted to the filtration generated by W, and it is assumed that the paths of the process σ are positive P almost surely. It is also assumed that P is a risk-free measure, that is, the discounted asset price process $t \mapsto \mathrm{e}^{-rt}S_t$, $t \in [0, T]$, is a martingale. The initial condition for the process S will be denoted by $s_0 > 0$.

We will mostly work with the log-price process $X_t = \log S_t$, $t \in [0, T]$. This satisfies

$$\mathrm{d}X_t = (r - \tfrac{1}{2}\sigma_t^2)\,\mathrm{d}t + \sigma_t(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\tilde{W}_t), \tag{2.2}$$

and the initial condition is given by $x_0 = \log s_0$.

The following notation will be used throughout the paper:

- $E_t := E(\cdot \mid \mathcal{F}_t)$.
- The Black–Scholes function will be denoted by (BS). It is given by

$$(\mathrm{BS})(t, x, y) = \mathrm{e}^x \Phi(d_+) - K\mathrm{e}^{-r\tau}\Phi(d_-),$$

where $\tau = T - t$ is the time to maturity, y is the constant volatility, K is the strike price, r is the interest rate, and Φ denotes the cumulative distribution function of the standard normal law. The symbols d_+ and d_- stand for the following functions:

$$d_\pm = \frac{x - \ln K + (r \pm (y^2/2))\tau}{y\sqrt{\tau}}.$$

- The $(\widetilde{\mathrm{BS}})$ function is defined such that $(\widetilde{\mathrm{BS}})(t, x, y^2) = (\mathrm{BS})(t, x, y)$. Note that, therefore,

$$(\widetilde{\mathrm{BS}})(t, x, y) = \mathrm{e}^x \Phi(\tilde{d}_+) - K\mathrm{e}^{-r\tau}\Phi(\tilde{d}_-),$$

with

$$\tilde{d}_\pm = \frac{x - \ln K + (r \pm (y/2))\tau}{\sqrt{y\tau}}.$$

- Throughout the paper, for simplicity, the following notation will be used:

$$(\mathrm{BS})_t := (\mathrm{BS})(t, X_t, v_t)$$

and

$$(\widetilde{\mathrm{BS}})_t := (\widetilde{\mathrm{BS}})(t, X_t, v_t^2).$$

Note that, by definition, $(\mathrm{BS})_t = (\widetilde{\mathrm{BS}})_t$.

- The call option price is given by

$$V_t = \mathrm{e}^{-r\tau} E_t[(\mathrm{e}^{X_T} - K)^+].$$

- It is known that the function (BS) satisfies the Black–Scholes equation $\mathcal{L}_y(\mathrm{BS})(t, x, y) = 0$ for any t, x, y, where

$$\mathcal{L}_y := \partial_t + \tfrac{1}{2}y^2\partial_x^2 + (r - \tfrac{1}{2}y^2)\partial_x - r. \tag{2.3}$$

Analogously, $\tilde{\mathcal{L}}_y(\widetilde{\mathrm{BS}})(t, x, y) = 0$ for any t, x, y, where

$$\tilde{\mathcal{L}}_y := \partial_t + \tfrac{1}{2}y\partial_x^2 + (r - \tfrac{1}{2}y)\partial_x - r. \tag{2.4}$$

- The following differential operators will be used in this paper: $\Lambda := \partial_x$, $\Gamma := (\partial_x^2 - \partial_x)$ and $\Gamma^2 = \Gamma \circ \Gamma$.
- We will say that a function $A(t, x, y)$ belongs to the space $\mathcal{C}^{1,2,2}((0, T) \times (0, \infty) \times (0, \infty))$ if A is one time differentiable with respect to t on $(0, T)$ and two times differentiable with respect to x and y on $(0, \infty)$. We also assume that the derivatives are continuous.
- Given two continuous semimartingales X and Y, we have
$$L[X, Y]_t := E_t\left[\int_t^T \sigma_u \,\mathrm{d}[X, Y]_u\right]$$
and
$$D[X, Y]_t := E_t\left[\int_t^T \mathrm{d}[X, Y]_u\right],$$
where the process $u \mapsto [X, Y]_u$, $u \in [0, T]$, is the quadratic covariation of the processes X and Y.

3 GENERAL EXPANSION FORMULAS

We know that under model (2.1), introduced above, in the uncorrelated case ($\rho = 0$), the following formula holds:

$$V_t = E_t[(\mathrm{BS})(t, X_t, \bar{\sigma}_t)]. \tag{3.1}$$

Here, the symbol $\bar{\sigma}^2(t)$ stands for the average future variance defined by

$$\bar{\sigma}_t^2 := \frac{1}{T-t}\int_t^T \sigma_s^2 \,\mathrm{d}s.$$

The equality in (3.1) is called the Hull–White formula (see, for example, Fouque *et al* 2000, p. 51). For correlated models – that is, models where $\rho \neq 0$ – there is a generalization of the Hull–White formula (see, for example, Fouque *et al* 2000, Formula (2.31)). However, the latter formula is significantly more complicated than the formula in (3.1).

Another way of generalizing the Hull–White formula was suggested in Alòs (2006), the idea being to obtain an expansion of the random variable V_t with the leading term equal to $E_t[(\mathrm{BS})(t, X_t, \bar{\sigma}_t)]$ and to obtain extra terms using Malliavin calculus techniques. In Alòs (2012), a similar formula was found: here, the leading term contains the adapted projection of the future variance, that is, the quantity

$$v_t^2 := E_t(\bar{\sigma}_t^2) = \frac{1}{T-t}\int_t^T E_t[\sigma_s^2]\,\mathrm{d}s,$$

instead of the future variance $\bar{\sigma}^2$. The previous remark illustrates the important concept of switching from an anticipative process $t \mapsto \bar{\sigma}_t$ to a nonanticipative (adapted) process $t \mapsto v_t$. In Merino and Vives (2015), the latter call price expansion was generalized to any stochastic volatility model.

Let us define

$$M_t = \int_0^T E_t[\sigma_s^2] \, \mathrm{d}s. \tag{3.2}$$

It is not hard to see that the following equality holds:

$$\mathrm{d}v_t^2 = \frac{1}{T-t}[\mathrm{d}M_t + (v_t^2 - \sigma_t^2) \, \mathrm{d}t].$$

The next assertion contains the abovementioned decomposition formula due to Alòs.

THEOREM 3.1 (BS expansion formula) *For every $t \in [0, T]$, define*

$$(\mathrm{I}) := \frac{\rho}{2} E_t\biggl[\int_t^T \mathrm{e}^{-r(u-t)} \Lambda\Gamma(\mathrm{BS})(u, X_u, v_u)\sigma_u \, \mathrm{d}[W, M]_u\biggr]$$

and

$$(\mathrm{II}) := \tfrac{1}{8} E_t\biggl[\int_t^T \mathrm{e}^{-r(u-t)} \Gamma^2(\mathrm{BS})(u, X_u, v_u) \, \mathrm{d}[M, M]_u\biggr].$$

Then, the call option price V_t can be written as

$$V_t = (\mathrm{BS})(t, X_t, v_t) + (\mathrm{I}) + (\mathrm{II}). \tag{3.3}$$

We will need the following statement, which was established in Merino *et al* (2018).

THEOREM 3.2 (General expansion formula) *Let $\{B_t, t \in [0, T]\}$ be a continuous semimartingale with respect to the filtration $\mathcal{F}^W$, and let $A(t, x, y)$ be a continuous function on the space $[0, T] \times [0, \infty) \times [0, \infty)$ such that $A \in C^{1,2,2}((0, T) \times (0, \infty) \times (0, \infty))$ (see the definition in Section 2). Let us also assume that $\tilde{\mathcal{L}}_y A = 0$ and v_t^2 and M_t are as above. Then, for every $t \in [0, T]$, the following formula holds:*

$$\begin{aligned} &E_t[\mathrm{e}^{-r(T-t)} A(T, X_T, v_T^2) B_T] \\ &\quad = A(t, X_t, v_t^2) B_t \\ &\qquad + E_t\biggl[\int_t^T \mathrm{e}^{-r(u-t)} \partial_y A(u, X_u, v_u^2) B_u \frac{1}{T-u}(v_u^2 - \sigma_u^2) \, \mathrm{d}u\biggr] \\ &\qquad + E_t\biggl[\int_t^T \mathrm{e}^{-r(u-t)} A(u, X_u, v_u^2) \, \mathrm{d}B_u\biggr] \end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{1}{2} E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} (\partial_x^2 - \partial_x) A(u, X_u, v_u^2) B_u (\sigma_u^2 - v_u^2)\, \mathrm{d}u\bigg] \\
&+ \tfrac{1}{2} E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \partial_y^2 A(u, X_u, v_u^2) B_u \frac{1}{(T-u)^2}\, \mathrm{d}[M, M]_u\bigg] \\
&+ \rho E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \partial_{x,y}^2 A(u, X_u, v_u^2) B_u \frac{\sigma_u}{T-u}\, \mathrm{d}[W, M]_u\bigg] \\
&+ \rho E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \partial_x A(u, X_u, v_u^2) \sigma_u\, \mathrm{d}[W, B]_u\bigg] \\
&+ E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \partial_y A(u, X_u, v_u^2) \frac{1}{T-u}\, \mathrm{d}[M, B]_u\bigg],
\end{aligned}
$$

provided $A(t, X_t, v_t^2)$ and B_t satisfy enough integrability conditions to guarantee the existence of all the conditional expectations.

The following statement can be derived from Theorem 3.2.

COROLLARY 3.3 *Let function A and process B exist as in Theorem 3.2. Suppose that function A satisfies*

$$
\partial_y A(t, x, y) = \frac{T-t}{2} (\partial_x^2 - \partial_x) A(t, x, y). \tag{3.4}
$$

Let $A_t := A(t, X_t, v_t^2)$ for all $t \in [0, T]$. Then, for every $t \in [0, T]$, the following formula holds:

$$
\begin{aligned}
\mathrm{e}^{-r(T-t)} E_t[A_T B_T] = A_t B_t &+ \frac{\rho}{2} E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \Lambda \Gamma A_u B_u \sigma_u\, \mathrm{d}[W, M]_u\bigg] \\
&+ \tfrac{1}{8} E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \Gamma^2 A_u B_u\, \mathrm{d}[M, M]_u\bigg] \\
&+ \rho E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \Lambda A_u \sigma_u\, \mathrm{d}[W, B]_u\bigg] \\
&+ \tfrac{1}{2} E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} \Gamma A_u\, \mathrm{d}[M, B]_u\bigg] \\
&+ E_t\bigg[\int_t^T \mathrm{e}^{-r(u-t)} A_u\, \mathrm{d}B_u\bigg].
\end{aligned}
$$

PROOF By substituting (3.4) in Theorem 3.2 and using the definitions of Λ and Γ, the proof straightforward. □

REMARK 3.4 Note that $(\widetilde{\mathrm{BS}})_t$ or any of its derivatives with respect to x fulfills the conditions of Corollary 3.3.

REMARK 3.5 Theorem 3.1 follows from Corollary 3.3 with $B \equiv 1$ and $A = (\widetilde{\mathrm{BS}})$. Recall that $(\widetilde{\mathrm{BS}})(t, x, y^2) = (\mathrm{BS})(t, x, y)$ and that this equality holds, deriving with respect to x.

The terms (I) and (II) in (3.3) are not easy to evaluate. Therefore, it becomes important to find simpler approximations to (I) and (II) and to estimate the error terms. Below, we will explain how to get an infinite expansion of the call price V_t, and in the next section, higher-order approximations to V_t will be obtained in the case of the Heston model.

The starting point in the construction of an infinite expansion of V_t is the formula in (3.3). In Alòs (2012), Corollary 3.3 was applied to the equality in (3.3). Only the two main terms in the expansion were kept; the remaining terms were ignored. The main idea used in the present paper is to apply Corollary 3.3 to each new term, obtaining an infinite series with stochastic terms. By selecting which terms to keep in the approximation formula and which ones to discard, the approximation error can be controlled.

The process described above leads to the following expansion of V_t:

$$
\begin{aligned}
V_t = {} & (\mathrm{BS})_t + \Lambda\Gamma(\mathrm{BS})_t\left(\frac{\rho}{2}L[W, M]_t\right) + \tfrac{1}{2}\Lambda^2\Gamma^2(\mathrm{BS})_t\left(\frac{\rho}{2}L[W, M]_t\right)^2 \\
& + \Gamma^2(\mathrm{BS})_t(\tfrac{1}{8}D[M, M]_t) + \tfrac{1}{2}\Gamma^4(\mathrm{BS})_t(\tfrac{1}{8}D[M, M]_t)^2 \\
& + \Lambda\Gamma^3(\mathrm{BS})_t\left(\frac{\rho}{2}L[W, M]_t\right)(\tfrac{1}{8}D[M, M]_t) \\
& + \cdots .
\end{aligned} \tag{3.5}
$$

REMARK 3.6 Note that the approximation formula can be obtained without specifying the volatility process: that is, the formula is valid for a general stochastic volatility model. In order to find an estimate of the error, a model must be specified. In the Heston model, Lemma 7.1 (in the online appendix) and the concrete semimartingales used guarantee the integrability conditions of Theorem 3.2.

REMARK 3.7 In Alòs *et al* (2019), an exact representation of V_t is given in terms of a forest of iterated integrals (which are also called diamonds). The expansion of the call price found in the present section is equivalent to that obtained in Alòs *et al* (2019).

4 CALL PRICE APPROXIMATIONS IN THE HESTON MODEL

The log-price process X in the Heston model satisfies the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}X_t &= (r - \tfrac{1}{2}\sigma_t^2)\,\mathrm{d}t + \sigma_t(\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}\tilde{W}_t), \\ \mathrm{d}\sigma_t^2 &= \kappa(\theta - \sigma_t^2)\,\mathrm{d}t + \nu\sigma_t\,\mathrm{d}W_t. \end{aligned}$$

Here, the process σ^2 models the stochastic variance of the asset price, $\theta > 0$ is the long-run mean level of the variance, $\kappa > 0$ is the rate at which σ reverts to the mean θ, $\nu > 0$ is the vol-vol parameter, and $r \geqslant 0$ is the interest rate. The initial conditions for the volatility process σ and the log-price process X will be denoted by $\sigma_0 > 0$ and x_0, respectively.

In this section, the general results established in Section 3 are used to obtain new approximation formulas for the call price in the Heston model. In this case, the terms of the approximations can be calculated explicitly. The results in the present paper generalize and sharpen the approximation formula obtained in Alòs (2012) and Alòs *et al* (2015), providing more terms in the small vol-vol asymptotic expansion of the call price. Moreover, the error terms in our formulas are of higher order than the error term of the form $O(\nu^2(|\rho| + \nu)^2)$ appearing in Alòs (2012) and Alòs *et al* (2015). The proofs of these results will be given in the online appendix.

We start with an assertion that provides an approximation of order $O(\nu^3(|\rho| + \nu))$.

THEOREM 4.1 (Second-order approximation formula) *For every $t \in [0, T]$, the following formula holds:*

$$\begin{aligned} V_t = {} & (\mathrm{BS})(t, X_t, v_t) + \Gamma^2(\mathrm{BS})(t, X_t, v_t)(\tfrac{1}{8}D[M, M]_t) \\ & + \Lambda\Gamma(\mathrm{BS})(t, X_t, v_t)\left(\frac{\rho}{2}L[W, M]_t\right) \\ & + \tfrac{1}{2}\Lambda^2\Gamma^2(\mathrm{BS})(t, X_t, v_t)\left(\frac{\rho}{2}L[W, M]_t\right)^2 \\ & + \rho\Lambda^2\Gamma(\mathrm{BS})(t, X_t, v_t)L\left[W, \frac{\rho}{2}L[W, M]\right]_t \\ & + \varepsilon_t, \end{aligned}$$

where ε_t is the error term satisfying

$$|\varepsilon_t| \leqslant \nu^3(|\rho| + |\rho|^3 + \nu)\left(\frac{1}{r} \wedge (T - t)\right)\Pi(\kappa, \theta),$$

and $\Pi(\kappa, \theta)$ is a positive constant depending on κ and θ.

The next assertion contains an approximation formula with an error term of the form $O(\nu^4(1+|\rho|\nu))$.

THEOREM 4.2 (Third-order approximation formula) *For every $t \in [0,T]$, the following formula holds:*

$$\begin{aligned}
V_t &= (\mathrm{BS})_t \\
&\quad + \Lambda\Gamma(\mathrm{BS})_t\left(\frac{\rho}{2}L[W,M]_t\right) + \tfrac{1}{2}\Lambda^2\Gamma^2(\mathrm{BS})_t\left(\frac{\rho}{2}L[W,M]_t\right)^2 \\
&\quad + \tfrac{1}{6}\Lambda^3\Gamma^3(\mathrm{BS})_t\left(\frac{\rho}{2}L[W,M]_t\right)^3 \\
&\quad + \Lambda\Gamma^3(\mathrm{BS})_t\left(\frac{\rho}{2}L[W,M]_t\right)(\tfrac{1}{8}D[M,M]_t) \\
&\quad + \rho\Lambda^2\Gamma(\mathrm{BS})_t L\left[W,\frac{\rho}{2}L[W,M]\right]_t + \rho\Lambda\Gamma^2(\mathrm{BS})_t L[W,\tfrac{1}{8}D[M,M]]_t \\
&\quad + \tfrac{1}{2}\Lambda\Gamma^2(\mathrm{BS})_t D\left[M,\frac{\rho}{2}L[W,M]\right]_u \\
&\quad + \rho\Lambda^3\Gamma^2(\mathrm{BS})_t\frac{\rho}{2}L[W,M]_t L\left[W,\frac{\rho}{2}L[W,M]\right]_t \\
&\quad + \rho\Lambda^3\Gamma(\mathrm{BS})_t L\left[W,\rho L\left[W,\frac{\rho}{2}L[W,M]\right]\right]_t \\
&\quad + \Gamma^2(\mathrm{BS})_t(\tfrac{1}{8}D[M,M]_t) \\
&\quad + \varepsilon_t,
\end{aligned}$$

where ε_t is the error term satisfying

$$|\varepsilon_t| \leqslant \nu^4(1+\rho^2(1+\rho^2)+|\rho|\nu(1+\rho^2))\left(\frac{1}{r}\wedge(T-t)\right)\Pi(\kappa,\theta),$$

and $\Pi(\kappa,\theta)$ is a positive constant depending on κ and θ.

For the uncorrelated Heston model, we obtain a similar expansion with fewer terms and a better error estimate.

THEOREM 4.3 *Suppose $\rho = 0$. Then, for every $t \in [0,T]$, the following formula holds:*

$$\begin{aligned}
V_t = (\mathrm{BS})_t &+ \Gamma^2(\mathrm{BS})_t(\tfrac{1}{8}D[M,M]_t) + \tfrac{1}{2}\Gamma^4(\mathrm{BS})_t(\tfrac{1}{8}D[M,M]_t)^2 \\
&+ \tfrac{1}{2}\Gamma^3(\mathrm{BS})_t D[M,\tfrac{1}{8}D[M,M]]_t \\
&+ \varepsilon_t,
\end{aligned}$$

where ε_t is the error term satisfying

$$|\varepsilon_t| \leqslant \nu^6 \left(\frac{1}{r} \wedge (T-t)\right) \Pi(\kappa, \theta),$$

and $\Pi(\kappa, \theta)$ is a positive constant depending on κ and θ.

REMARK 4.4 The call option price approximations obtained above for the Heston model can be easily extended to the more general Bates model following the ideas developed in Merino *et al* (2018).

5 NUMERICAL RESULTS

In this section, we compare the performance of the call option price approximation formula proposed in Alòs (2012) and Alòs *et al* (2015) with the new approximation formulas obtained in the present paper. To simplify our notation, we call the formula obtained in Alòs (2012) and Alòs *et al* (2015) the formula with an error estimate $O(\nu^2)$, while the two formulas obtained in the present paper are referred to as the formulas with error estimates $O(\nu^3)$ and $O(\nu^4)$ (see Theorems 4.1 and 4.2). We make a similar comparison in the uncorrelated case. Here, we compare the formula with an error estimate $O(\nu^4)$ established in Alòs (2012) and Alòs *et al* (2015) with the new formula with an error estimate $O(\nu^6)$ found in the present paper (see Theorem 4.3). As our benchmark price, we choose a call option price obtained using a Fourier transform-based pricing formula. This is one of the standard approaches to pricing European options under stochastic volatility models. In particular, we use a semi-closed-form solution with one numerical integration as a reference price (see Mrázek and Pospíšil 2017).[1] The comparison between approximations is made with two important aspects in mind: the practical precision of the pricing formula and the efficiency of the formula expressed in terms of the computational time needed for particular pricing tasks.

Analytical approximations of the implied volatility exist in the literature (see, for example, Forde *et al* 2012; Lorig *et al* 2017). We will compare these approximations with the implied volatilities obtained from the approximation formula with an error estimate $O(\nu^4)$ for the correlated case and the formula with an error estimate $O(\nu^6)$ for the uncorrelated case.

Our next goal is to illustrate the quality of our new approximation formulas for the call option price in the Heston model for various values of ρ and ν while keeping the other parameters fixed. Concretely, we choose the following parameters: $S_0 = 100$, $r = 0.001$, $v_0 = 0.25$, $\kappa = 1.5$ and $\theta = 0.2$. We understand the error in the

[1] With a slight modification, mentioned in Gatheral (2006), in order not to suffer from the "Heston trap" issues.

price as the relative error in a $\log_{10}$ scale. In the following figures, the blue line illustrates the approximation with an error estimate $O(\nu^2)$, the red line represents the approximation with an error estimate $O(\nu^3)$ and the yellow line corresponds to the approximation with an error estimate $O(\nu^4)$.

Figure 1 shows approximations of the call option price when the vol-vol, ν, and the absolute value of the correlation, ρ, are both small. In this case, $\nu = 5\%$ and $\rho = -0.2$. We observe that, in general, the approximation formula with an error estimate $O(\nu^3)$ performs better than the formula with an error estimate $O(\nu^2)$. In some cases, however, there are exceptions, such as the in-the-money options for $\tau = 3$. The call option price approximation with an error estimate $O(\nu^4)$ is much better, with an error around 10^{-7}–10^{-10}.

In Figure 2, we discuss the case where ν is small while $|\rho|$ is close to 1. In this case, $\nu = 5\%$ and $\rho = -0.8$. We observe that the new approximation formulas perform better than the previously known formula. The approximation error is in the range 10^{-4}–10^{-8} for the formula with an error estimate $O(\nu^3)$ and 10^{-7}–10^{-10} for the formula with an error estimate $O(\nu^4)$.

Figure 3 refers to the case of high vol-vol and low absolute correlation. In this case, $\nu = 50\%$ and $\rho = -0.2$. Here, we note that the three approximation formulas show similar performances. The approximation formula where the error estimate is $O(\nu^4)$ seems to perform a little, but not significantly, better.

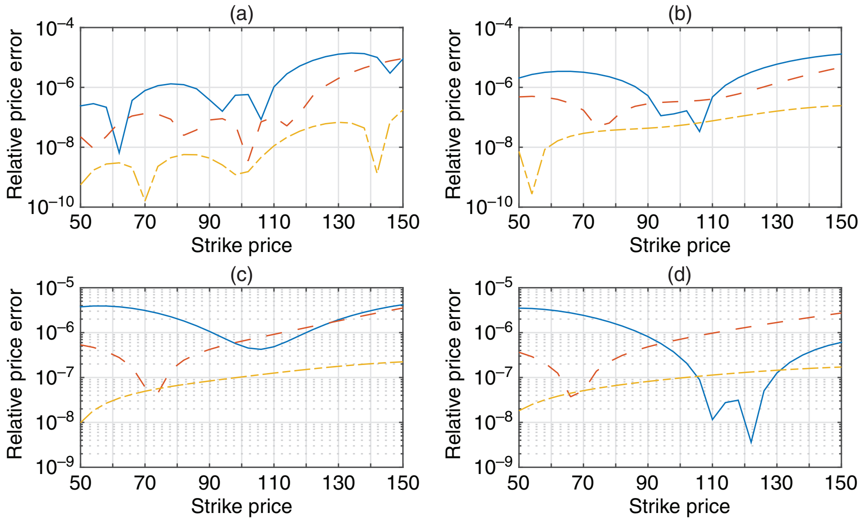
Figure 4 illustrates the performance of the formulas when both parameters are not suitable for the approximation, eg, when $\nu = 50\%$ and $\rho = -0.8$. Here, we observe that the approximations have a similar quality. The approximation formula with an error estimate $O(\nu^4)$ seems to perform better than the other formulas, while the formula with an error estimate $O(\nu^3)$ performs better only in the short term.

Comparing Figure 4 with Figure 3, we observe that the new approximation formulas are more efficient in the former figure than in the latter. This can be explained by the fact that most of the terms in the expansion include the parameter ρ. When $|\rho|$ is small, the new approximations are closer to the known ones than when $|\rho|$ is close to 1.

We have already observed that the approximation formulas obtained in the present paper perform better than the previously known formula when $|\rho|$ is close to 1 and ν is small. However, the improvement in the performance is not significant for large ν. This can be fixed by adding more terms. As an example, we compare the benchmark prices with their approximations in the uncorrelated case. In Figures 5 and 6, the blue line is the approximation with an error estimate $O(\nu^4)$, while the red line is the approximation with an error estimate $O(\nu^6)$.

In Figure 5, we illustrate the case of low vol-vol. In this case, $\nu = 5\%$ and $\rho = 0$. The formulas with error estimates $O(\nu^4)$ and $O(\nu^6)$ have a very small error,

FIGURE 1 Comparison of the three different approximation formulas and reference prices for $\nu = 5\%$ and $\rho = -0.2$.



although the new approximation behaves much better. Figure 6 shows the approximations when ν is large. In this case, $\nu = 50\%$ and $\rho = 0$. We can see that the new approximation behaves better, especially in the long term.

One of the main advantages of the proposed option price approximations is their computational efficiency. To compare the amount of time each method spent on computations, we replicated the computational effort of performing in different calibration situations with three pricing tasks. We used a batch of 100 various call options with different strikes and times to maturity, including out-of-the-money, at-the-money and in-the-money options with short-, mid- and long-term times to maturity. Our first task was to evaluate the option prices in the batch with respect to 100 (uniformly) randomly sampled parameter sets. This task has a similar number of price evaluations to a market calibration task with a very good initial guess. Further on, we repeated the same trials for 1000 and 10 000 parameter sets to mimic the number of evaluations for a typical local-search calibration and a global-search calibration (for more information about calibration tasks, see, for example, Mikhailov and Nögel (2003) and Pospíšil and Sobotka (2016)).

Our results are listed in Table 1. The call prices were analytically calculated in all cases. We observe that for the trials of 100 and 1000 parameter sets, the amounts of time spent on computations were quite similar. For the trial of 10 000 sets, the

FIGURE 2 Comparison of the three different approximation formulas and reference prices for $\rho = -0.8$ and $\nu = 5\%$.

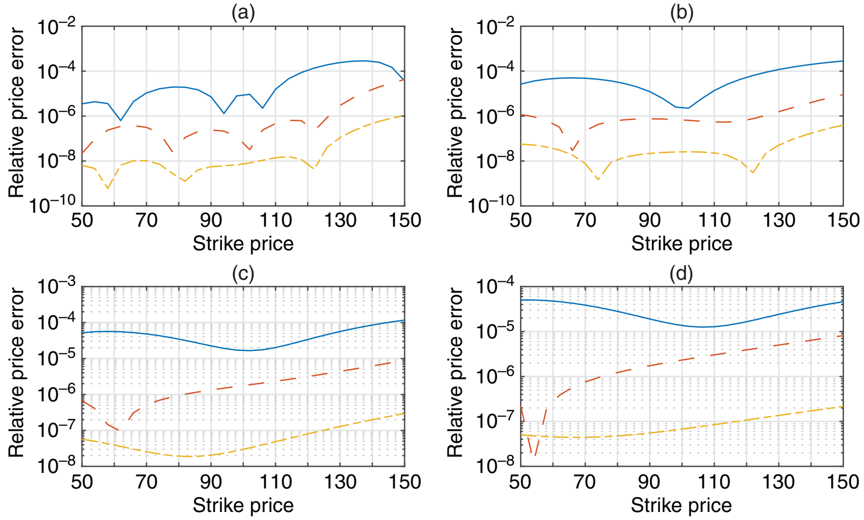


FIGURE 3 Comparison of the three different approximation formulas and reference prices for $\rho = -0.2$ and $\nu = 50\%$.

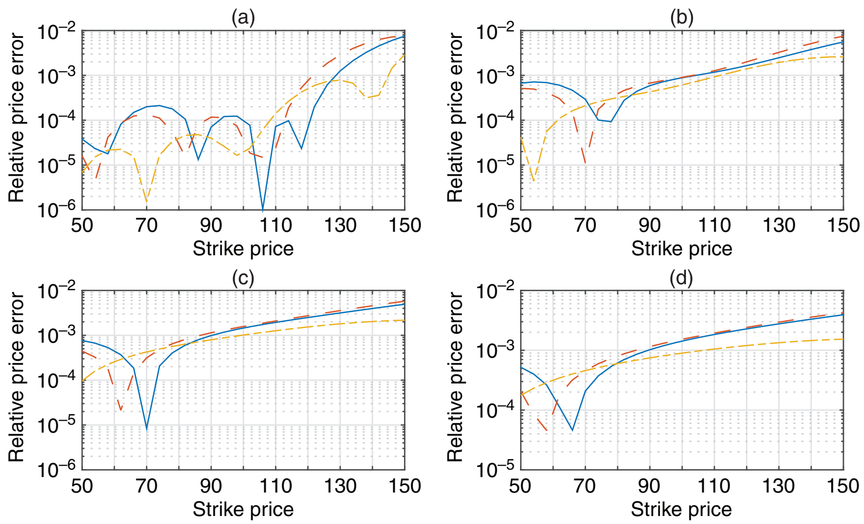
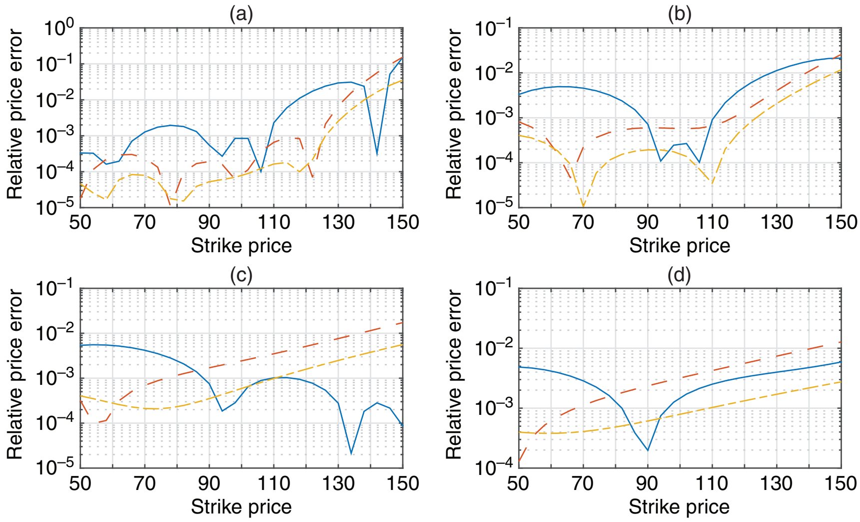


FIGURE 4 Comparison of the three different approximation formulas and reference prices for $\rho = -0.8$ and $\nu = 50\%$.

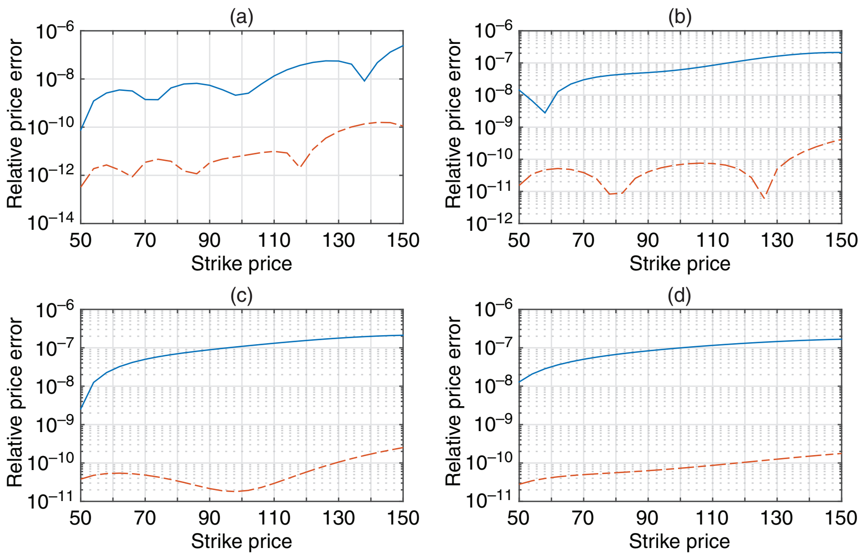


experiment based on the approximation with an error estimate $O(\nu^4)$ was a little bit slower than the other experiments.

This table shows that the approximations where the error estimate is $O(\nu^2)$ or $O(\nu^3)$ are around forty-three to forty-five times faster than the approximation based on the fast Fourier transform methodology, while the approximation with an error estimate $O(\nu^4)$ is around thirty-six times faster than the latter. Therefore, the approximations with error estimates $O(\nu^2)$ or $O(\nu^3)$ are around 1.14–1.25 times less time-consuming than the $O(\nu^4)$ approximation.

Our next goal is to compare the approximation formulas presented in this paper with other analytical approximation methods. In Forde *et al* (2012), based on saddle-point methods, the authors derive a small-maturity expansion formula for prices that are transformed into a closed-form implied volatility for the Heston model. In Lorig *et al* (2017), the authors derive an explicit implied volatility for local-stochastic volatility models, including the Heston case, using a perturbation technique for parabolic equations. We choose the following values for the Heston parameters: $S_0 = 100$, $r = 0$, $v_0 = 0.20$, $\kappa = 1.15$, $\theta = 0.04$, $\nu = 0.2$ and $\rho = -0.4$. We understand the error in the implied volatility to be the absolute error in a $\log_{10}$ scale. The blue line illustrates the approximation with an error estimate $O(\nu^4)$, the red line is the third-order approximation of the implied volatility in Lorig *et al* (2017), the

FIGURE 5 Comparison of the two different approximation formulas and reference prices for $\rho = 0$ and $\nu = 5\%$.



yellow line is the second-order approximation of the implied volatility in Lorig *et al* (2017) and the purple line corresponds to the approximation in Forde *et al* (2012). We compare our methodology with that of Forde *et al* (2012) only for maturities less than one year.

In Figure 7, we observe that our approximation is more accurate in almost all cases. As expected, the Forde *et al* (2012) approximation is competitive for short-term maturities, but the error increases with the time to maturity, and the third-order approximation of the implied volatility generally behaves better than the second-order approximation. We observe that the results of our approximation are very close to the third-order expansion of the implied volatility in Lorig *et al* (2017).

In Figure 8, we compare all the approximations when $\rho = 0$. We observe that the second- and third-order approximations coincide. In general, our approximation is better than the other methods, especially when the time to maturity is increasing.

6 CONCLUSIONS

In the present paper, we develop a method to obtain approximations of a call option price under a general stochastic volatility model. In the special case of the Heston model, we derive sharp approximation formulas with error estimates for the call

FIGURE 6 Comparison of the two different approximation formulas and reference prices for $\rho = 0$ and $\nu = 50\%$.

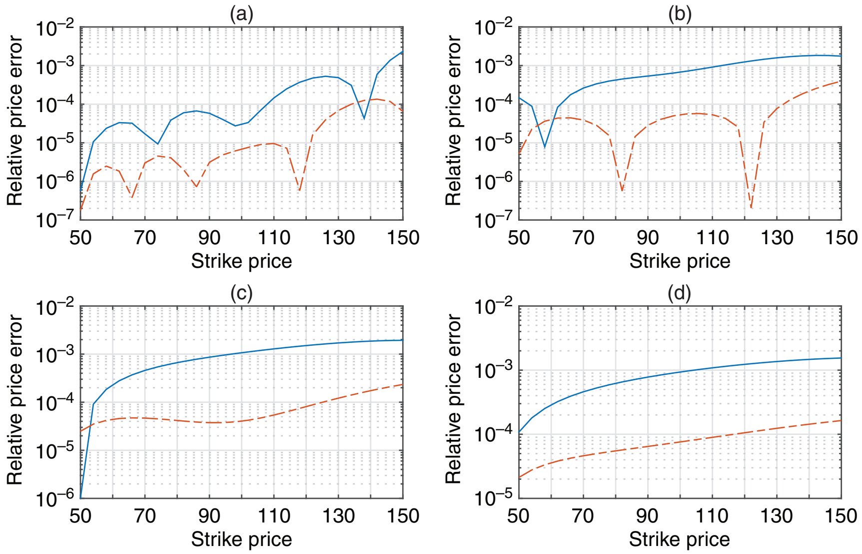


TABLE 1 Efficiency of the Heston call price approximations.

Pricing approach	Task	Time† [sec]	Speed-up factor
Heston–Lewis	#1	3.63	–
	#2	33.52	–
	#3	336.59	–
Approximation of order $O(\nu^2)$	#1	0.08	45×
	#2	0.76	44×
	#3	7.41	45×
Approximation of order $O(\nu^3)$	#1	0.08	45×
	#2	0.78	43×
	#3	7.77	43×
Approximation of order $O(\nu^4)$	#1	0.10	36×
	#2	0.91	37×
	#3	8.87	38×

† The results were obtained on a PC with Intel Core i7-7700HQ CPU @ 2.80 GHz 2.80GHz and 16 GB RAM.

FIGURE 7 Comparison with other analytical approximation methods.

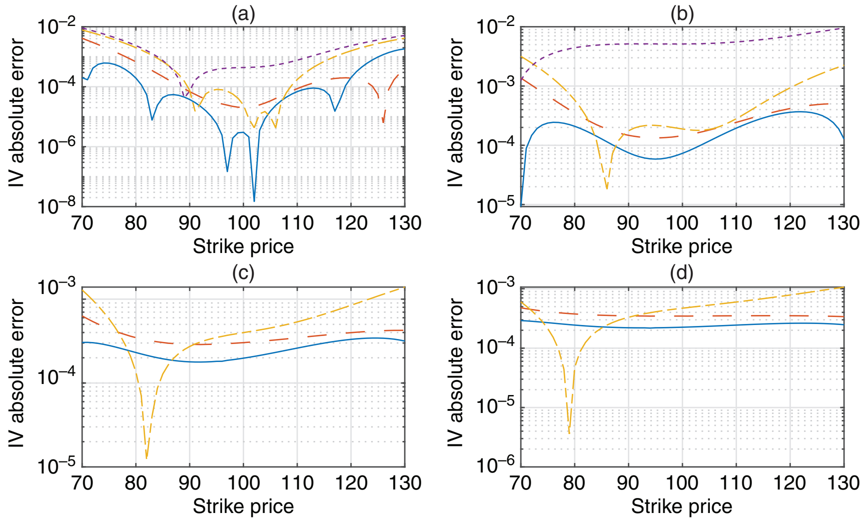
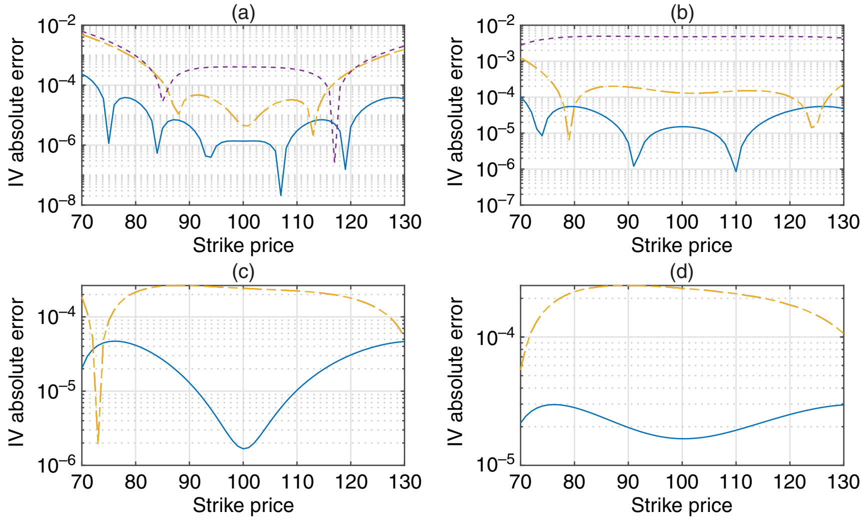


FIGURE 8 Comparison with other analytical approximation methods when $\rho = 0$.



option price. These formulas have a higher order of accuracy than previously known ones. In Section 4, an exact second-order approximation formula is established for the call option price in the Heston model that has an error estimate $O(\nu^3(|\rho| + \nu))$, where ν is the vol-vol parameter and ρ is the correlation coefficient. We also find a sharper formula with an error estimate $O(\nu^4(1 + |\rho|\nu))$ and seven additional significant terms. In Section 5, the numerical performance of the approximation formulas obtained in the present paper is illustrated. We observe that the new formulas are very efficient for low values of the vol-vol parameter ν or when the time to maturity is small. We also observe that for the uncorrelated Heston model, the number of terms that must be taken into account in computations decreases substantially. For the call option price in the uncorrelated Heston model, we find an approximation that has an error estimate of order $O(\nu^6)$. The approximations to the call option price obtained in the present paper are computationally more efficient than those proposed in Alòs (2012) and Alòs *et al* (2015). We compare the implied volatility of the approximation formula of an error estimate $O(\nu^4)$ for the correlated case, and of an error estimate $O(\nu^6)$ for the uncorrelated case, with the implied volatility expansions of Forde *et al* (2012) and Lorig *et al* (2017). In general, our approximation method is more accurate, but in the correlated case it is of similar magnitude to the third-order expansion of Lorig *et al* (2017). In the uncorrelated case, our approximation is much better, especially when the time to maturity increases.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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