Facultat de Matemàtiques
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GRAU DE MATEMÀTIQUES

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A Categorical Approach to Quantum Computation with Anyons

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Barcelona, 24 de gener de 2023

Abstract

Topological quantum computation is an approach to quantum computation that employs exotic quasi-particles, called anyons, to store and manipulate quantum information. Anyons can be mathematically described by a unitary modular tensor category. In this work, we give an overview of the relevant category theory necessary to introduce an anyon model, as well as the structures used in computing processes. Having developed all the necessary theoretical background, we introduce the Fibonacci anyon model, the simplest model that can give rise to universal quantum computation. Moreover, we describe how quantum computation could be carried out within such model.

Resum

La computació quàntica topològica és una aproximació a la computació quàntica que utilitza quasi-partícules exòtiques, anomenades *anyons*, per emmagatzemar i manipular informació quàntica. Matemàticament, els *anyons* es poden descriure mitjançant una categoria unitària tensorial modular. En aquest treball, proporcionem una visió general dels conceptes rellevants de teoria de categories necessaris per introduir un model d'*anyons*, així com de les estructures utilitzades en els processos computacionals. Un cop desenvolupat el marc teòric necessari, introduïm el model d'*anyons* de Fibonacci, el model més simple que pot donar lloc a la computació quàntica universal. A més, descrivim com es pot realitzar computació quàntica utilitzant aquest model.

2020 Mathematics Subject Classification. 18A05, 18M05, 18M15, 18M20, 81P68.

Acknowledgments

I would like to thank my supervisor Dra. Joana Cirici for her guidance, advise and suggestions. Her support and encouragement have been invaluable for the satisfactory elaboration of this work. I would also like to thank my friends and family for their support during all these years.

Contents

Chapter 1

Introduction

Abelian and non-Abelian Statistics

Identical particles, also called indistinguishable particles, are a fundamental concept of quantum theory. For instance, consider a system of many identical particles in a three-dimensional space and an operation that exchanges two of the particles twice. It is clear that this process is equivalent to taking one of the particles around the other, while the later remains still. We could go further and claim that the double exchange is also equivalent to a process in which none of the particles move at all. Indeed, the path followed by the encircling particle is always continuously deformable to a path that does not encircle the other particle, which is in turn fully contractible to a point. Therefore, we could conclude that the wave function describing the system should be left unchanged after the swapping process. This is only possible if it acquires a -1 or a $+1$ factor phase. These two possibilities create a natural distinction between indistinguishable particles in a three-dimensional space. If the phase factor acquired by the wave function is -1, then the particles are called *fermions* and are described by Fermi-Dirac statistics. Conversely, if the phase factor equals $+1$, the particles are *bosons* described by Bose-Einstein statistics.

If we apply the same logic to a system of identical particles in a one-dimensional space, ambiguity arises as the swapping process of two particles requires one of them to pass through the other. For instance, if after an exchange the wave function changes sign, we could assume that the particles are non-interacting fermions, but it could also be true that they are interacting bosons and that the change of sign is due to the interaction.

In between these two cases, the description of systems of identical particles in a two-dimensional space becomes quite exotic. This is due to the fact that exchanging two particles twice does not necessarily guarantee that the system will return back to the initial state. In fact, in two-dimensional spaces, a path followed by an encircling particle cannot be deformed to a point without cutting through the other particle. Therefore, the swapping process introduces an exchange phase $e^{i\theta}$, where θ can in principle have any value[1]. If $\theta = 0, \pi$, the particles are again bosons or fermions. However, if that is not the case, particles are *anyons* and are described by either *Abelian* or *non-Abelian* statistics.

A general way to describe anyons is to consider their *worldlines* in a $(2+1)$-dimensional spacetime. These can be braided, which allows us to characterize particle statistics by representations of the *braid group*. Abelian statistics make use of one-dimensional representations just as the exchange phases introduced in the preceding discussion. In these representations, the order of the braiding operations is not important, and thus they are Abelian. In non-Abelian statistics, higher-dimensional representations of the braid group are used. They are given by $k \times k$, with $k > 1$, unitary matrices that do not commute in general, forming a non-Abelian group instead.

Two-Dimensional Systems and Topological Phases of Matter

As we live in a three-dimensional world, one could ask whether anyons exist in nature or if they can be built somehow. The remarkable answer is yes since it is possible to constrain many systems to exhibit

[1] However θ must be a rational multiple of 2π due to stability considerations.

effective two-dimensional behavior. A typical example is that of a two-dimensional gas of electrons, where electrons are trapped in a thin layer between two semiconductors, freezing their motion in the direction perpendicular to the layer. Other examples are two-dimensional optical lattices of cold atoms and isolated sheets of atoms such as graphene.

Although two-dimensional systems enable anyons to exist, their emergence requires further consideration, making necessary to introduce *topological phases of matter* (TPMs). Consider a system at low temperatures and energies and long wavelengths. Such system is in a topological phase if all observable properties are invariant under smooth deformations (diffeomorphisms) of the spacetime manifold in which the system lives. Equivalently, TPMs are states of matter modeled by unitary *topological quantum field theories* (TQFTs) at low energies. A characteristic of TPMs is that, in general, their ground state(s) is separated from the lowest excited state by an energy gap. Anyons arise as finite energy particle-like excitations produced by localized disturbances on the ground state. That is why in literature they are sometimes called quasiparticles.

Without going into details, TPMs can be divided into two distinct classes [21]:

- *Symmetry-Protected Topological* (SPT) phases, which are topological only if some protecting symmetry is respected. Otherwise, they become trivial. They include systems of non-interacting fermions such as all integer quantum Hall states or topological insulators and superconductors (see [34]), but also some bosonic systems such as spin chains or lattice models [9, 10, 14].

  Anyons emerge in SPT phases when they are allowed to have defects. These may bind localized zero energy modes which can behave as anyons. For example, zero modes are described in topological superconductors by Majorana fermions, i.e. fermions that are their own antiparticle, that behave as non-Abelian anyons [16, 23, 38]. Majorana zero modes are probably one of the most experimentally accessible anyons and thus have been the subject of extensive research. Despite this, they are not universal for quantum computation[2].

- Phases with *intrinsic topological order* that do not require any symmetries to be present. They include, for instance, the strongly interacting fractional quantum Hall (FQH) states [28]. In such states, very cold electron gases are subject to high magnetic fields. This makes electrons localize in the so-called Landau levels and the system behaves as a degenerate insulator. Different magnetic fields produce gaped ground states that can be described by a non-integer filling fraction ν, which corresponds to the number of electrons per flux quantum. *Ising* anyons [6, 26], one of the simplest non-Abelian anyons that exist, arise in the $\nu = 5/2$ case. They are equivalent to Majorana modes and, hence, are not universal for quantum computation either. Another interesting case is the very fragile filling fraction $\nu = 12/5$, which has been proposed to support *Fibonacci* anyons [35, 47]. Despite being more complex, they are expected to be universal. We will present this anyon model in more detail in future chapters.

  Other examples of phases with intrinsic topological order are spin liquids, produced when strong Coulombic interactions localize electrons in a lattice configuration. This freezes their kinetic degrees of freedom in what is called a Mott-insulating state. However, their spins are still able to interact [4]. An example of a spin liquid is Kitaev's honeycomb model [19]. This model supports Ising anyons but is not likely to appear in nature. Other examples of spin liquids include the celebrated Toric Code [20], which supports Abelian anyons and is one of the simplest realizations of a topological quantum memory [8, 40].

Anyonic Quantum Computation

Quantum states are notoriously fragile. In order to store and evolve them coherently, it is important to protect them from any outside noise. This is exactly one of the fundamental challenges of quantum computation: to robustly store quantum states for long times and successfully manipulate them making use of specific quantum gates. *Topological quantum computation* (TQC) solves the fragility of quantum states by means of *topological invariants* of quantum systems. In this way, information is encoded non-locally in a fault-resilient manner.

[2] Roughly speaking, a universal quantum computer should be able to simulate any program on another quantum computer.

The presence of non-Abelian anyons in topologically ordered phases gives rise to degenerate ground states that can be used to encode and process quantum information. This computational space, known as *fusion space*, is a decoherence-free subspace in which anyonic computation is performed as follows:

1. First some number of anyon pairs are created from the vacuum. These are used to encode the input.
2. Computation is performed by moving the anyons adiabatically around each other, i.e. by braiding their worldlines.
3. Finally, anyons are *fused* together to perform the read-out. This means that they are brought pairwise physically together until they behave as a single composite anyon. This process may be repeated polynomially many times to obtain the computing result as a probability distribution of anyon types.

Moreover, as first appreciated by Kitaev [12, 20], anyons are linked to quantum error correction. Indeed, systems of Abelian anyons can be used as quantum memories protected from decoherence.

Mathematical Model

From a mathematical point of view, there are two equivalent ways to model anyons. One possible approach is to model the system in low energies by a unitary (2+1)-TQFT. This is done by focusing on the ground state manifold of the anyon system. However, it is also possible to only consider its braiding and fusion structures. Then, the system is described by a *unitary modular tensor category* (UMTC). The two notions are essentially the same [3] and, in fact, a *modular tensor category* (MTC) uniquely determines a (2 + 1)-TQFT [43].

In this work, we will focus on the second approach to introduce the *Fibonacci anyon model*. First, though, we will explore all the categorical background needed for defining the concept of UMTC. Models with up to four distinct anyonic particles have been systematically classified in [36]. Before diving into categorical definitions, let us sketch the basic properties of anyons and the necessary elements to define them. In this way, we will motivate the definitions of upcoming chapters.

- Anyons correspond to *objects* of a category.
- In any anyonic model, there are a finite number of anyon types or charges given by isomorphism classes of *simple objects*.
- Anyons are the basic entities of the theory and any object in the category is constructed from them. In categorical terms, we say that the category must be *semisimple*.
- Compound systems of anyons are expressed by a *monoidal structure*. In this way, we represent a compound system of anyons as the *monoidal product* of their respective charges. The trivial charge is given by the *unit object* $\mathbb{1}$ of the category. This is physically important because the monoidal product describes the fusion of anyons.
- Given a charge A, we can define its conjugate charge A^* as the unique charge that can fuse with A to yield the trivial charge. It is given by the dual of A. In categorical terms, the category must be *rigid*.
- Exchanging anyons corresponds to braiding their worldlines, which in categorical terms is equivalent to applying an appropriate *braiding isomorphism* to the system of anyons. In other words, the category must be *braided*.
- Representing graphically the movements of anyons with strands in a $(2+1)$-dimensional spacetime is not enough. That is because they are extended objects. Therefore, a more faithful graphical representation is given by *ribbons*, which can be also twisted. The corresponding mathematical formalism is that of a *braided ribbon category*.

- The fusion of anyons follows some *fusion rules* that decompose the monoidal product as direct sums of anyon types. Together with the preceding formalism, these rules can be mapped into the context of *Hilbert spaces*. Again, this is allowed by the semisimple structure of the category, which is compatible with the other structures.
- Finally, a UMTC can be defined as a special case of a *semisimple ribbon category*. These categories do not allow an infinite number of possible charges for an anyon of a given theory. Moreover, they contain information about the fusion rules within their defining data.

Work Structure

The work is structured as follows. We begin by introducing in Chapter 2 some basic concepts of category theory. We define *category* in section 2.1, introduce *functors and natural transformations* in section 2.2, and define *abelian category* in section 2.3. Then, in Chapter 3, we present all the concepts leading up to the definition of UMTC. We define *monoidal categories* in section 3.1, *rigid categories* in section 3.2, *braided categories* in section 3.3, *ribbon categories* in section 3.4 and *semisimple categories* in section 3.5. We finally define MTCs and UMTCs in section 3.6. In Chapter 4, we introduce the *Fibonacci anyon model*. We first introduce *fusion rules* and *fusion spaces* in section 4.1 and then introduce the *F* and *R matrices* in section 4.2. Then, the anyon model is described in section 4.3 and quantum computation with Fibonacci anyons is explained in section 4.4. Finally, in Chapter 5 we give an overview of anyonic quantum computation.

Chapter 2

Category Theory

2.1 Categories

A category is a mathematical system of related objects. Roughly speaking, it is built from objects and morphisms between them. We begin by giving the definition of a category and illustrate it with several examples following the explanations of [22, 24, 42].

Definition 2.1 (Category). *A **category** $\mathcal{C}$ consists of:*

- *a collection $Ob(\mathcal{C})$ of **objects**;*
- *for each $X, Y \in Ob(\mathcal{C})$, a set $Hom(X, Y)$ of **morphisms** from X to Y represented by arrows $X \to Y$;*
- *for each $X, Y, Z \in Ob(\mathcal{C})$, a map*

$$\circ : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

  *called **composition**. The image of a pair (g, f) under this map is denoted $g \circ f$ or just gf;*
- *for each $X \in Ob(\mathcal{C})$, a morphism $id_X \in Hom(X, X)$ called the **identity** of X,*

satisfying the following conditions:

- ***associativity**: for each $f \in Hom(X, Y)$, $g \in Hom(Y, Z)$ and $h \in Hom(Z, W)$, it holds $(h \circ g) \circ f = h \circ (g \circ f)$;*
- ***identity laws**: for each $f \in Hom(X, Y)$, the identity morphisms id_X and id_Y satisfy $f \circ id_X = f = id_Y \circ f$.*

Within the literature, different notations are also used. It is common to write $X \in \mathcal{C}$ instead of $X \in Ob(\mathcal{C})$ and $\mathcal{C}(X, Y)$ instead of $Hom(X, Y)$. We will however stick to the notation used in definition 2.1.

Given a morphism $f \in Hom(X, Y)$, the object X is called the **source** and the object Y the **target** of f. Every morphism in every category has a definite source and a definite target. For $X \in Ob(\mathcal{C})$, the set $Hom(X, X)$ is the set of **endomorphisms** and is denoted by $End(X)$.

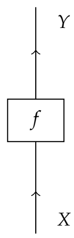
Definition 2.2 (Monomorphism). *A morphism $f : X \to Y$ in $\mathcal{C}$ is called a **monomorphism** if its left-cancellative, i.e. if $f \circ g = f \circ h \Rightarrow g = h$ for any morphisms $g, h : Z \to X$ in $\mathcal{C}$.*

Definition 2.3 (Epimorphism). *A morphism $f : X \to Y$ in $\mathcal{C}$ is called an **epimorphism** if its right-cancellative, i.e. if $g \circ f = h \circ f \Rightarrow g = h$ for any morphisms $g, h : Y \to Z$ in $\mathcal{C}$.*

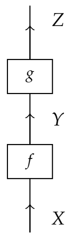
Definition 2.4 (Isomorphism). *A morphism $f : X \to Y$ in $\mathcal{C}$ is called an **isomorphism** if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. The morphism g is called the inverse of f and is often denoted by f^{-1}. Two objects $X, Y \in Ob(\mathcal{C})$ are **isomorphic** if there exists an isomorphism $X \to Y$, then we write $X \cong Y$.*

As first suggested by Penrose in [32] for a special case, most concepts in category theory can be represented using *string diagrams*. The idea is that, given a category $\mathcal{C}$, an object $X \in Ob(\mathcal{C})$ is represented by a sting

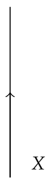
and morphisms $f : X \to Y$ in $\mathcal{C}$ by boxes with an input string for the source and an output string for the target



Note that these diagrams should be read from bottom to top as indicated by the arrows, although some authors use other conventions. The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ is obtained by placing the diagram of g on top of that of f



and when a morphism is the identity id_X the box is omitted



There are countless examples of categories. We will consider just some familiar ones:

Example 2.5. [Sets]. As a first example, we consider the category of sets **Set** described as follows. Its objects are all possible sets, which implies that the collection of objects itself is not in general a set. Instead, the collection of objects forms a class. Morphisms in this category are maps between sets and composition is given by the ordinary composition of maps. The identity morphisms are what we would expect: given $X \in Ob(\mathbf{Set})$, the identity morphism $id_X : X \to X$ is $id_X(a) = a$ for all $a \in X$. Isomorphisms in this category are bijections.

Example 2.6. [Groups]. The category of groups, **Grp**, whose objects are groups and whose morphisms are group homomorphisms. Isomorphisms in this category are isomorphisms between groups.

Other important examples are:

Example 2.7. [Categories as mathematical structures]. Not all categories have sets as objects and morphisms that preserve the structure. In fact, a category can be directly described by its objects, morphisms, composition and identities. For example there is a category $\emptyset$ with no objects or morphisms and a category **1** with just one object and the identity morphism.

Groups can be considered in this way: a group is a category that has only one object and in which all the morphisms are isomorphisms. To make this more clear, consider a category $\mathcal{G}$ with just one object, which can be denoted by $Ob(\mathcal{G}) = \{\bullet\}$. Then, $\mathcal{G}$ consists of a set $Hom(\bullet, \bullet)$, a binary associative operation

$$\circ : Hom(\bullet, \bullet) \times Hom(\bullet, \bullet) \to Hom(\bullet, \bullet)$$

and a two-sided identity element $id_\bullet \in Hom(\bullet, \bullet)$. Finally, saying that every element in $Hom(\bullet, \bullet)$ is an isomorphism is the same as saying that every element in $Hom(\bullet, \bullet)$ has an inverse. Therefore, $\mathcal{G}$ has the structure of a group.

Example 2.8. [Vector spaces]. The category $\mathbf{Vect}_{\mathbb{K}}$ whose objects are vector spaces over the field $\mathbb{K}$ and whose morphisms are linear maps between them.

The next example, *Hilbert spaces*, plays a major role in quantum mechanics:

Example 2.9. [Hilbert spaces]. A **Hilbert space** is a complex vector space H equipped with an inner product $\langle f, g \rangle$ for $f, g \in H$ such that the norm, defined as $|f| = \sqrt{\langle f, f \rangle}$, turns H into a complete metric space, i.e. every Cauchy series in H is convergent. The category **Hilb** of Hilbert spaces consists of

- a collection of objects $X \in Ob(\mathbf{Hilb})$, where X is a finite-dimensional Hilbert space and
- morphisms $f \in Hom(X, Y)$ that are linear operators.

We can also encounter categories in which objects represent *space* at a given time and morphisms represent portions of spacetime. The simplest example is the category of $n\mathbf{Cob}$ [13]:

Example 2.10. [Cobordisms]. In the category $n\mathbf{Cob}$

- objects $X \in Ob(n\mathbf{Cob})$ are $(n-1)$-dimensional manifolds and
- morphisms $f \in Hom(X, Y)$ are n-dimensional *cobordisms*. A **cobordism** $f : X \to Y$ is an n-dimensional manifold whose boundary is the disjoint union of the $(n-1)$-dimensional manifolds X and Y. Even though f is not a function from X to Y, it is written as $f : X \to Y$ since we may think of f as the process of time passing from the moment X to the moment Y.

In figure 2.1 we depict a 2-dimensional manifold f going from a 1-dimensional manifold X (a single circle) to a 1-dimensional manifold Y (a pair of circles). Figure 2.2 depicts the composition of f with an other 2-dimensional manifold g.

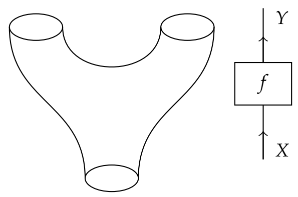


Figure 2.1: Example of cobordism for $n = 2$.

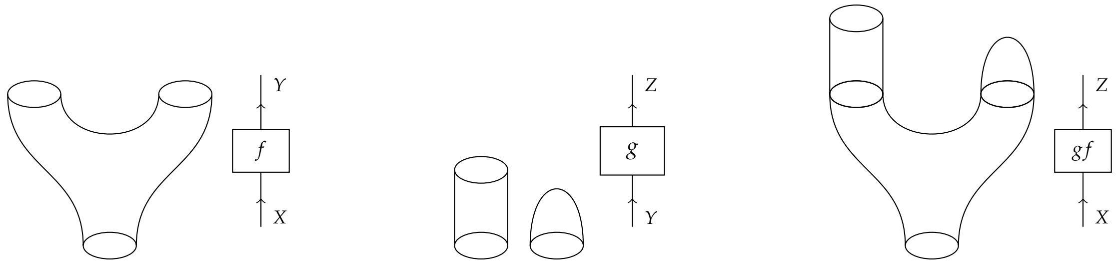


Figure 2.2: Example of composition of cobordisms for $n = 2$.

Other typical examples are *tangles* [39]:

Example 2.11. [Tangles]. Let $S_{a,b}$ be the disjoint union of a circles and b intervals $[0, 1]$. A **tangle** is a smooth embedding $T : S_{a,b} \to \mathbb{R}^2 \times [0, 1]$ such that the boundary maps to the boundary and the interior to the interior. The intersection of a tangle with the boundary of the cube is required to be transverse, i.e. to lie in X $\times(\{0\} \cup \{1\})$, where X is the x-axis.

In the language of category theory, in the category $\mathcal{T}$ of tangles

- objects $X \in Ob(\mathcal{T})$ are non-negative integers and
- morphisms $T_{X,Y} \in Hom(X, Y)$ are tangles with X inputs (points of the image of $T_{X,Y}$ with $z = 0$) and Y outputs (points of the image of $T_{X,Y}$ with $z = 1$).

The composition of two tangles $T_{X,Y}$ and $T_{Y,Z}$ is the tangle formed by putting $T_{X,Y}$ on top of $T_{Y,Z}$

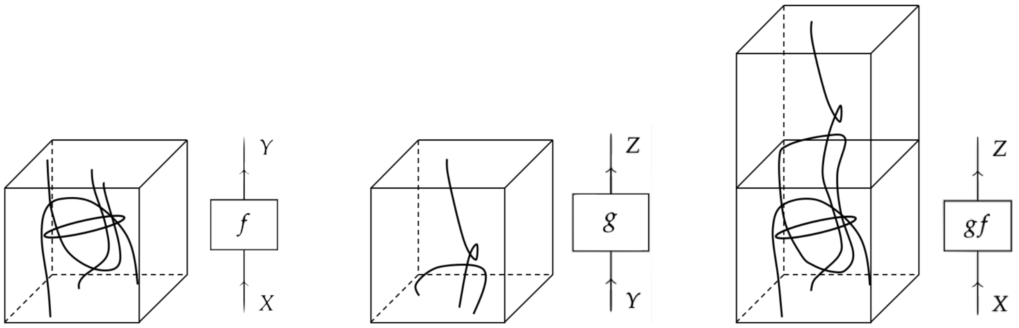


Figure 2.3: Example of composition of tangles.

2.2 Functors and Natural Transformations

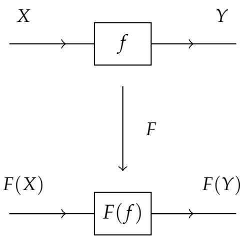
A category relates its objects through morphisms that preserve the structure of the category. We could ask ourselves whether there is a sensible notion of map between categories. The answer is given by *functors*.

Definition 2.12 (Functor)**.** *A **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a map that assigns to each object $X \in Ob(\mathcal{C})$ an object $F(X) \in Ob(\mathcal{D})$ and to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that*

$$F(g \circ f) = F(g) \circ F(f) \quad \text{and} \quad F(id_X) = id_{F(X)}$$

for all composable morphisms g, f in $\mathcal{C}$ and all $X \in Ob(\mathcal{C})$.

Graphically, functors are depicted in the following way



Remark 2.13. We are now familiar with the idea that structures and morphisms between them can form a category but, in fact, we could apply this exact idea to categories and functors and obtain a new category **CAT**, whose objects are categories and whose morphisms are functors. Behind this statement, there is the idea that functors can be composed: given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ we can consider a new functor $G \circ F : \mathcal{C} \to \mathcal{E}$ as expected. Another idea is that for every category $\mathcal{C}$ there is an identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

One of the easiest examples of functors are the so-called *forgetful functors* that, informally speaking, are functors that forget some of the structure of the input category:

Example 2.14. Consider the category of groups, **Grp**, and the category of sets, **Set**. There is a functor $F : \mathbf{Grp} \to \mathbf{Set}$ defined as follows: if $G \in Ob(\mathbf{Grp})$ is a group then $F(G) \in Ob(\mathbf{Set})$ is the set of elements of G, and if $f : G \to H$ is a group homomorphism then $F(f) : F(G) \to F(H)$ is simply the map f itself. In a way, we could say that F forgets the group structure and that group homomorphisms are homomorphisms.

Functors are also fundamental in algebraic topology:

Example 2.15. The category $\mathbf{Top}_*$ is the category of **pointed spaces**, topological spaces equipped with a base point, together with the continuous basepoint-preserving maps. There is a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$$

that assigns each pointed space (X, x) with its *fundamental group* $\pi_1(X, x)$. That also means that each continuous basepoint-preserving map

$$f : (X, x) \to (Y, y)$$

is assigned to a homomorphism

$$f_* \equiv \pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$$

We can also think of a functor $F : \mathcal{C} \to \mathcal{D}$ as modeling the category $\mathcal{C}$ by creating an image or representation of it in the category $\mathcal{D}$. The next example develops this idea:

Example 2.16. A functor can be used to implement symmetries on a system. For example, let $\mathcal{G}$ be the group of symmetries and **Hilb** the category of finite-dimensional Hilbert spaces. Then the functor $F : \mathcal{G} \to \mathbf{Hilb}$ is the representation of $\mathcal{G}$ in **Hilb**. The single object $\bullet \in Ob(\mathcal{G})$ (example 2.7) is mapped to a finite-dimensional Hilbert space $F(\bullet) = \mathbb{C}^d \in Ob(\mathbf{Hilb})$. The morphisms of $\mathcal{G}$ are mapped to linear operators between Hilbert spaces.

Finally, in an analogous way we defined isomorphisms for morphisms, we can define isomorphisms for functors.

Definition 2.17 (Functor isomorphism)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an **isomorphism** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. Such functor is called the inverse of F. Two categories are isomorphic if there exists an isomorphism between them.*

The maps between functors are called *natural transformations* and they only apply when functors have the same domain and codomain. With the interpretation that functors model categories, natural transformations can be used to translate between models.

Definition 2.18 (Natural transformation)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : F \to G$ is a family of morphisms*

$$(F(X) \xrightarrow{\alpha_X} G(X))_{X \in Ob(\mathcal{C})}$$

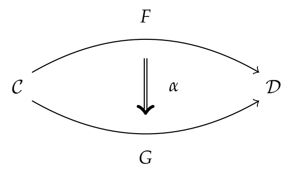
in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \alpha_X} \downarrow & & \downarrow {\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

*The maps α_X are called the **components** of α.*

From the definition it is clear that from each morphism $f : X \to Y$ in $\mathcal{C}$ it is possible to construct exactly one morphism $F(X) \to G(Y)$ in $\mathcal{D}$. When $f = id_X$ this morphism is α_X. Otherwise is $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$.

Natural transformations can be depicted as



Remark 2.19. Natural transformations can be composed as one would expect. Given natural transformations $\alpha_X : F(X) \to G(X)$ and $\beta_X : G(X) \to H(X)$, there is a composite natural transformation defined by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$. There is also an identity natural transformation on any functor F defined by $(id_F)_X = id_{F(X)}$. Therefore we can define a functor category for any two categories, whose objects are functors and whose morphisms are natural transformations.

Again, we can define isomorphisms for natural transformations:

Definition 2.20 (Natural isomorphism)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural isomorphism** is a natural transformation $\alpha : F \to G$ such that $\alpha_X : F(X) \to G(X)$ is an isomorphism for every $X \in Ob(\mathcal{C})$.*

2.3 Abelian Categories

We now introduce the concept of *abelian category* as it will be useful for future definitions (see section 3.5). The properties of abelian categories will serve us as a first step towards an interpretation of our formalism within the context of Hilbert spaces. But before, we need to introduce some new concepts.

Following [30], an object of a category $\mathcal{C}$ is called a **zero object**, and is denoted **0**, if its both **initial** and **terminal**, i.e. for any $X \in Ob(\mathcal{C})$ there is a unique morphism $\mathbf{0} \to X$ and a unique morphism $X \to \mathbf{0}$. The zero object is unique up to an isomorphism. However, a category need not have initial or terminal objects and even if it does there might not be a zero object.

Example 2.21. In the category **Set** the only initial object is $\emptyset$, final objects are sets of cardinality one and there is no zero object.

Example 2.22. In the category **Grp** the only zero object is the trivial group $\{1\}$ which consists only of its neutral element.

In a category $\mathcal{C}$ with zero object $\mathbf{0}$ we can define a unique morphism for any $X, Y \in Ob(\mathcal{C})$ via the zero object

$$0 : X \to \mathbf{0} \to Y$$

called the **zero morphism**, which does not depend on the choice of $\mathbf{0}$. The composition of the zero morphism with any arbitrary morphism in $\mathcal{C}$ is a zero morphism.

Definition 2.23 (Additive category)**.** *A category $\mathcal{C}$ is called an **additive category** if the following conditions are satisfied:*

- *every set $Hom(X, Y)$ is equipped with a structure of an additive abelian group such that the composition of morphisms is **biadditive** with respect to this structure, i.e.*

$$g \circ (f + f') = (g \circ f) + (g \circ f')$$
$$(g + g') \circ f = (g \circ f) + (g' \circ f)$$

  for composable morphisms f, f', g, g' in $\mathcal{C}$;

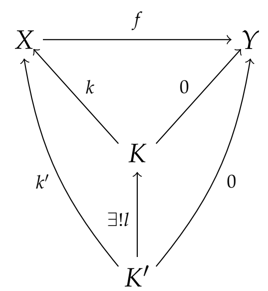
- *there exists a zero object;*

- *for any objects $X_1, X_2 \in Ob(\mathcal{C})$, there exists an object $Y \in Ob(\mathcal{C})$ and morphisms $p_1 : Y \to X_1$, $p_2 : Y \to X_2$, $i_1 : X_1 \to Y$, $i_2 : X_2 \to Y$ in $\mathcal{C}$ such that*

$$p_1 \circ i_1 = id_{X_1}, \quad p_2 \circ i_2 = id_{X_2} \quad \textit{and} \quad i_1 \circ p_1 + i_2 \circ p_2 = id_Y$$

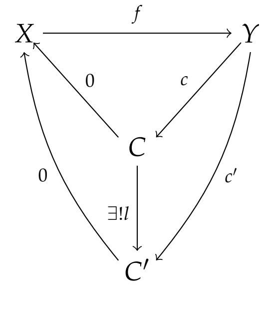
A category that satisfies the first condition is called an **Ab-category** [24]. In the third condition, the object Y is unique up to a unique isomorphism and it is often denoted by $X_1 \oplus X_2$, which is called the **direct sum** of X_1 and X_2.

Let $\mathcal{C}$ be an additive category and $f : X \to Y$ a morphism in $\mathcal{C}$. The **kernel** of f, denoted $Ker(f)$, is an object $K \in Ob(\mathcal{C})$ together with a morphism $k : K \to X$ such that $f \circ k = 0$. If $k' : K' \to X$ is such that $f \circ k' = 0$ then there exists a unique morphism $l : K' \to K$ such that $k \circ l = k'$. Expressed in a commutative diagram, the situation can be depicted as:



If $Ker(f)$ exists, then it is unique up to a unique isomorphism.

Dually, the **cokernel** of a morphism $f : X \to Y$ in $\mathcal{C}$, denoted $Coker(f)$, is an object $C \in Ob(\mathcal{C})$ together with a morphism $c : Y \to C$ such that $c \circ f = 0$. Again, if $c' : Y \to C'$ is such that $c' \circ f = 0$ then there exists a unique morphism $l : C \to C'$ such that $l \circ c = c'$. Expressed in a commutative diagram, the situation can be depicted as:



If $Coker(f)$ exists, then it is unique up to a unique isomorphism.

Remark 2.24. A kernel is necessarily a monomorphism and a cokernel an epimorphism by the unique factorization requirement in their definitions.

Definition 2.25 (Abelian category)**.** *An* ***abelian category*** *is an additive category where*

- *every morphism admits a kernel and a cokernel, and*
- *every monomorphism is a kernel and every epimorphism is a cokernel.*

The second condition is powerful in the sense that implies that any morphism in an abelian category that is both a monomorphism and an epimorphism is also an isomorphism. Moreover, it can be shown that, in an abelian category, every morphism is a composition of an epimorphism followed by a monomorphism [24].

Example 2.26. [Abelian groups]. The category **Ab** of abelian groups has abelian groups as objects and group homomorphisms as morphisms. It is an example of an abelian category since:

- Given two abelian groups $A, B \in Ob(\mathbf{Ab})$, there is an obvious group structure on $Hom(A, B)$ as for $f, g \in Hom(A, B)$ the sum of the two homomorphisms is defined as

  $$(f + g)(a) = f(a) + g(a)$$

  for $a \in A$. Moreover, given $f, f' \in Hom(A, B)$ and $g, g' \in Hom(B, C)$ we have

  $$g \circ (f + f') = (g \circ f) + (g \circ f') \quad \text{and} \quad (g + g') \circ f = (g \circ f) + (g' \circ f)$$

  which means that the group structures are compatible with composition.

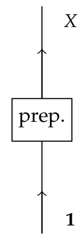
- It has as zero object the trivial group $\{0\}$, which consists only of the neutral element.
- It is well-known that we can define the direct sum of two abelian groups $A, B \in Ob(\mathbf{Ab})$ as another abelian group $A \oplus B$ consisting of the ordered pairs (a, b) with $a \in A$ and $b \in B$.
- In the category **Ab** kernels and cokernels always exist. The notion of kernel in the category sense coincides with the notion of kernel in the algebraic sense, i.e let $f : A \to B$ be a group homomorphism, then $Ker(f) = \{a \in A : f(a) = 0\}$ together with the inclusion morphism $i : Ker(f) \to A$. The cokernel is the quotient group $C = B/f(A)$ together with the natural projection $p : B \to C$.
- Every monomorphism is a kernel: let $f : A \to B$ be a monomorphism between abelian groups. Consider the projection $\pi : B \to B/Im(f)$. Then (A, f) is a kernel of π. Suppose there are $A' \in Ob(\mathbf{Ab})$ and $f' : A' \to B$ with $\pi \circ f' = 0$, so $Im(f') \subset Ker(\pi) = Im(f)$. The map f is injective and has and inverse morphism $f^{-1} : Im(f) \to A$. Then, the morphism $\gamma = f^{-1} \circ f'$ is the unique morphism with $f \circ \gamma = f'$.
- Every epimorphism is a cokernel: let $f : A \to B$ be an epimorphism between abelian groups. Consider the inclusion $i : Ker(f) \to A$. Then (B, f) is a cokernel of i. Suppose there are $B' \in Ob(\mathbf{Ab})$ and $f' : A \to B'$ with $f' \circ i = 0$. The map f is surjective and because $Ker(f) \subset Ker(f')$ we can define the morphism $\gamma : B \to B'$ such that $\gamma \circ f = f'$, which is the unique morphism with this property.

Chapter 3

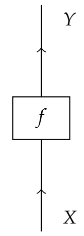
Modular Tensor Categories

We have mainly focused on understanding categories, morphisms between them and morphisms between morphisms of categories. In the framework of category theory, objects can be seen as physical systems and morphisms as processes that turn a state of one physical system into a state of another one. According to this analogy, composition can be seen as performing one process after another. We can use this idea to build a whole experimental setup.

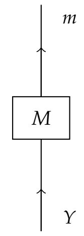
For example, consider a process in which a system X is prepared from the vacuum or a thermal state $\mathbf{1}$



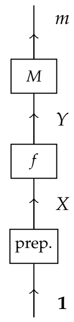
then the system X evolves to another system Y following the evolution f



and finally a measurement M is performed with output m



Composing these three processes, we get the following setup



A problem arises when we try to build a similar example to describe experiments that consist of multiple subsystems or in which several processes take part in parallel, since we do not know yet a way of doing so. These setups can be described in physics using *tensor products* and *monoidal categories*.

3.1 Monoidal Categories

Before introducing the concept of product of categories, let us consider the following examples:

Example 3.1. [Tensor product of vector spaces]. Given two vector spaces V and W over some field $\mathbb{K}$, we define their **tensor product** as the vector space denoted by

$$V \otimes W$$

that has associated a bilinear map

$$V \times W \to V \otimes W$$

out of the Cartesian product of the underlying sets, that maps a pair (v, w), with $v \in V$ and $w \in W$, to an element $v \otimes w \in V \otimes W$. Such element is called the **tensor product** of v and w. If V and W are finite dimensional, then so is $V \otimes W$ with $dim(V \otimes W) = dimV \cdot dimW$.

Example 3.2. [Tensor product of Hilbert spaces]. Let H and K be Hilbert spaces with scalar products $(x|x')$ and $(y|y')$ respectively. The **tensor product** of H with K is a Hilbert space $H \otimes K$ together with a bilinear map

$$\phi : H \times K \to H \otimes K$$

such that:

- the set of vectors $\phi(x, y)$ with $x \in H$ and $y \in K$ forms a total subset of $H \otimes K$, i.e. its closed linear span is equal to $H \otimes K$;
- $(\phi(x_1, y_1)|\phi(x_2, y_2)) = (x_1|x_2)(y_1|y_2)$ for $x_1, x_2 \in H$ and $y_1, y_2 \in K$

It is customary to write $x \otimes y$ instead of $\phi(x, y)$.

In the context of category theory, a similar product can be defined for a category $\mathcal{C}$, taking two objects $X, Y \in Ob(\mathcal{C})$ and returning an object $X \otimes Y \in Ob(\mathcal{C})$. This leads us to the topic of *monoidal categories*. We will be following the explanations from [3, 11, 17, 42, 43].

Definition 3.3 (Product of two categories)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. The **product of the two categories** is the category $\mathcal{C} \times \mathcal{C}'$ where*

- *objects are pairs (X, X') with $X \in Ob(\mathcal{C})$ and $X' \in Ob(\mathcal{C}')$;*
- *morphisms are pairs $(f, f') : (X, X') \to (Y, Y')$ where $f : X \to Y$ is a morphism in $\mathcal{C}$ and $f' : X' \to Y'$ is a morphism in $\mathcal{C}'$.*

Composition is defined componentwise $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$ and identity morphisms are $id_{(X,X')} = (id_X, id_{X'})$.

Now we are able to give a formal definition of a *monoidal category*:

Definition 3.4 (Monoidal category)**.** *A* ***monoidal category*** *is a category $\mathcal{C}$ equipped with*

- *a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called* ***monoidal product****;*
- *an object $\mathbb{1} \in Ob(\mathcal{C})$ called the* ***unit object****;*
- *a family of natural isomorphisms*

  $$\alpha = \{\alpha_{X,Y,Z} = (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}_{X,Y,Z \in Ob(\mathcal{C})}$$

  called the ***associativity constraint****;*
- *a family of natural isomorphisms*

  $$l = \{l_X : \mathbb{1} \otimes X \to X\}_{X \in Ob(\mathcal{C})}$$

  called the ***left unit constraint****;*
- *a family of natural isomorphisms*

  $$r = \{r_X : X \otimes \mathbb{1} \to X\}_{X \in Ob(\mathcal{C})}$$

  called the ***right unit constraint****;*

such that

- *for all objects $X, Y, Z, W \in Ob(\mathcal{C})$ the* ***pentagon equation*** *commutes:*

$$\begin{array}{ccc} & (X \otimes Y) \otimes (Z \otimes W) & \\ {\scriptstyle \alpha_{X \otimes Y, Z, W}} \nearrow & & \searrow {\scriptstyle \alpha_{X, Y, Z \otimes W}} \\ ((X \otimes Y) \otimes Z) \otimes W & & X \otimes (Y \otimes (Z \otimes W)) \\ {\scriptstyle \alpha_{X,Y,Z} \otimes id_W} \downarrow & & \uparrow {\scriptstyle id_X \otimes \alpha_{Y,Z,W}} \\ (X \otimes (Y \otimes Z)) \otimes W & \xrightarrow[\alpha_{X, Y \otimes Z, W}]{} & X \otimes ((Y \otimes Z) \otimes W) \end{array}$$

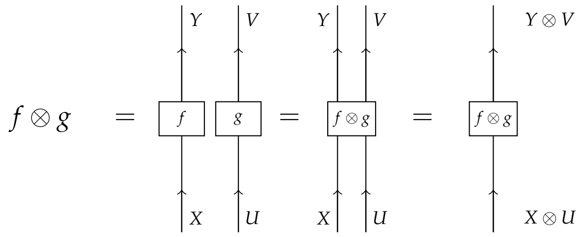
- *for all objects $X, Y \in Ob(\mathcal{C})$ the* ***triangle equation*** *commutes:*

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\ {\scriptstyle r_X \otimes id_Y} \searrow & & \swarrow {\scriptstyle id_X \otimes l_Y} \\ & X \otimes Y & \end{array}$$

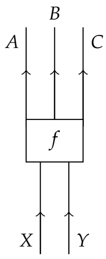
Given a monoidal category $\mathcal{C}$, that $\otimes$ is a functor means that we have an object $X \otimes Y \in Ob(\mathcal{C})$ associated to any $X, Y \in Ob(\mathcal{C})$. Moreover, $id_X \otimes id_Y = id_{X \otimes Y}$ for all $X, Y \in Ob(\mathcal{C})$ and

$$(g \otimes g') \circ (f \otimes f') = (g \circ f) \otimes (g' \circ f')$$

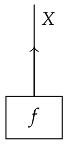
for all pairs of composable morphisms g, f and g', f' in $\mathcal{C}$. When the functor $\otimes$ acts on morphisms, they can be thought of as being done in parallel. We can easily illustrate this idea using string diagrams. Suppose we have two morphisms $f : X \to Y$ and $g : U \to V$ in $\mathcal{C}$. Their monoidal product is represented by



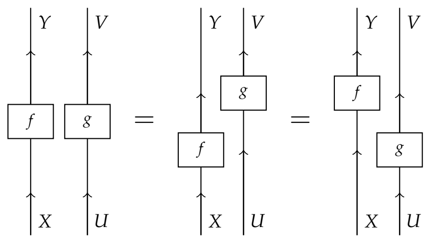
We can also use boxes with several arrows attached. For example the morphism $f : X \otimes Y \to A \otimes B \otimes C$ with $X, Y, A, B, C \in Ob(\mathcal{C})$ may be depicted as



We draw the unit object as a blank space. For example $f : \mathbb{1} \to X$ is



By calculating the composition and the product of morphisms we can build up elaborate pictures that can be deformed without changing the morphism they describe. For example boxes lying on the same horizontal level can be pushed up or down. To illustrate this, we consider morphisms $f : X \to Y$ and $g : U \to V$ in a category $\mathcal{C}$. Then



which graphically expresses

$$f \otimes g = (id_Y \otimes g) \circ (f \otimes id_U) = (f \otimes id_V) \circ (id_X \otimes g)$$

Remark 3.5. If we take a closer look at the definition 3.4,

- the associativity constraint $\alpha = \{\alpha_{X,Y,X}\}_{X,Y,Z \in Ob(\mathcal{C})}$ is a natural isomorphism from the functor $\otimes \circ (\otimes \times 1_{\mathcal{C}})$ to the functor $\otimes \circ (1_{\mathcal{C}} \times \otimes)$;
- the left unit constraint $l = \{l_X\}_{X \in Ob(\mathcal{C})}$ is a natural isomorphism from the functor $\mathbb{1} \times ? : \mathcal{C} \to \mathcal{C}$ to the functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$;
- and the right unit constraint $r = \{r_X\}_{X \in Ob(\mathcal{C})}$ is a natural isomorphism from the functor $? \otimes \mathbb{1} : \mathcal{C} \to \mathcal{C}$ to the functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

Here, the functors $\mathbb{1}\otimes ?$ and $? \otimes \mathbb{1}$ are given by

$$(\mathbb{1}\otimes ?)(X) = \mathbb{1} \otimes X, \qquad (? \otimes \mathbb{1})(X) = X \otimes \mathbb{1},$$
$$(\mathbb{1}\otimes ?)(f) = id_{\mathbb{1}} \otimes f, \qquad (? \otimes \mathbb{1})(f) = f \otimes id_{\mathbb{1}},$$

for $X \in Ob(\mathcal{C})$ and f morphism in $\mathcal{C}$.

The monoidal product of anyons A and B is used to represent the fusion of the two. We will regard an $A \in Ob(\mathcal{C})$ as a label for a set of anyons. Note that this set may contain just a single anyon or more than one. In the first case, the object will have to satisfy some other properties that will be introduced in section 3.5. Nonetheless, to make explanations much simpler, from now on we will assume that an object A is the charge of a single anyon. The unit object $\mathbb{1} \in Ob(\mathcal{C})$ will indicate the trivial charge.

The existence of the associativity constraint ensures that $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are isomorphic. This is used to change the pattern of fusion of anyons, meaning that

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

changes the order of fusion from $A \otimes B$ to $B \otimes C$ occurring first. The left and right unit constraints simply tell us that combining an anyon with a certain charge with the trivial charge changes nothing about the compound charge. When considering four objects $X, Y, Z, W \in Ob(\mathcal{C})$, the pentagon equation ensures that there are five isomorphic ways to order their monoidal product. The triangle equation ensures that all isomorphisms with the same source and target constructed from the natural isomorphisms α, l and r are equal.

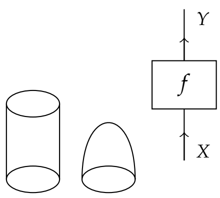
A monoidal category $\mathcal{C}$ is said to be **strict** if for any objects X, Y, Z we have $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$. Then the associativity and unit constraints are the identity morphisms.

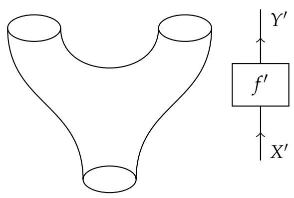
Example 3.6. Several of the previous examples can be made into a monoidal category:

- The category **Set** of sets is a monoidal category with monoidal product of two sets, $X \otimes Y$, given by the cartesian product of sets $X \times Y$. In this case, the unit object is any one-element set. The associativity constraint is given by the natural isomorphism $(X \times Y) \times Z \to X \times (Y \times Z)$ and the unit constraints are obvious. The monoidal product of two morphisms $f : X \to Y$ and $f' : X' \to Y'$ is just

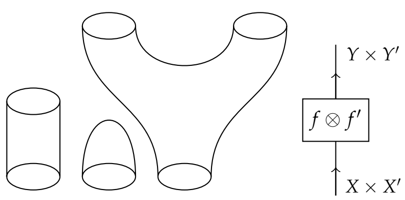
$$\begin{aligned} f \times f' : X \times X' &\to Y \times Y' \\ (x, x') &\mapsto (f(x), f'(x')) \end{aligned}$$

- In n**Cob** the monoidal product is the disjoint union. For example, the monoidal product of these two morphisms

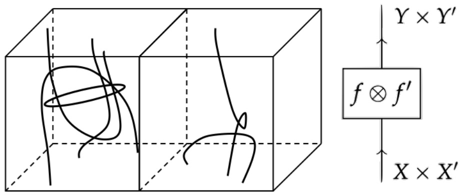




is



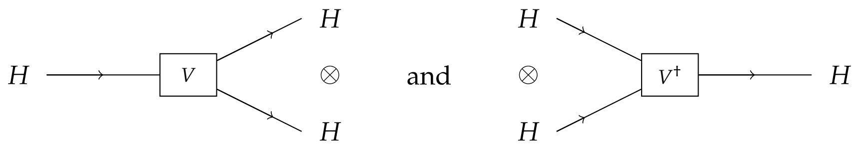
- It is similar for the category of tangles $\mathcal{T}$, where the monoidal product is also given by the disjoint union



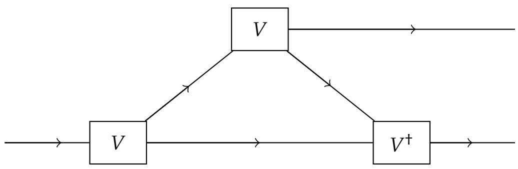
Example 3.7. Let $\mathcal{C}$ be a category. The category $\mathbf{End}(\mathcal{C})$ of all functors from $\mathcal{C}$ to itself is a monoidal category, where the monoidal product is given by the composition of functors. The associativity isomorphism is the identity, the unit object is the identity functor and the unit constraints are obvious.

Having introduced monoidal categories, now we are able to build up systems in which processes can take part in parallel. The next example is an illustration of this:

Example 3.8. The category **Hilb** of finite-dimensional Hilbert spaces can be made as a monoidal category by using the usual tensor product of Hilbert spaces as the monoidal product. This allows us to build up elaborate experimental setups by composing string diagrams either in series (composition) or in parallel (tensor product). For example, suppose we have the following processes



We could consider the system



Notice that in the example lines do not cross each other. We will see what happens when they do further on.

3.2 Rigid Categories

We will now discuss duality in monoidal categories since this is used in anyon theory to express the conjugation of charges. Remember that the fusion of two conjugate charges yields the trivial charge.

Definition 3.9 (Left duality)**.** *Let $\mathcal{C}$ be a monoidal category with unit object $\mathbb{1}$. It is said to be a monoidal category with **left duality** if for every object $X \in Ob(\mathcal{C})$ there exists an object $X^* \in Ob(\mathcal{C})$ called **left dual** and morphisms*

$$ev_X : X^* \otimes X \to \mathbb{1},$$
$$coev_X : \mathbb{1} \to X \otimes X^*,$$

*called the **left evaluation** and the **left coevaluation**, such that the following diagrams commute*

$$\begin{array}{ccccc}
\mathbb{1} \otimes X & \xrightarrow{coev_X \otimes id_X} & (X \otimes X^*) \otimes X & \xrightarrow{\alpha_{X,X^*,X}} & X \otimes (X^* \otimes X) \\
\Big\downarrow{\scriptstyle l_X} & & & & \Big\downarrow{\scriptstyle id_X \otimes ev_X} \\
X & & \xleftarrow[\quad r_X \quad]{} & & X \otimes \mathbb{1}
\end{array}$$

$$\begin{array}{ccccc}
X^* \otimes \mathbb{1} & \xrightarrow{id_{X^*} \otimes coev_X} & X^* \otimes (X \otimes X^*) & \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} & (X^* \otimes X) \otimes X^* \\
\Big\downarrow{\scriptstyle r_{X^*}} & & & & \Big\downarrow{\scriptstyle ev_X \otimes id_{X^*}} \\
X^* & & \xleftarrow[\quad l_{X^*} \quad]{} & & \mathbb{1} \otimes X^*
\end{array}$$

Similarly,

Definition 3.10 (Right duality)**.** *Let $\mathcal{C}$ be a monoidal category with unit object $\mathbb{1}$. It is said to be a monoidal category with **right duality** if for every object $X \in Ob(\mathcal{C})$ there exists an object ${}^*X \in Ob(\mathcal{C})$ called **right dual** and morphisms*

$$ev'_X : X \otimes {}^*X \to \mathbb{1},$$
$$coev'_X : \mathbb{1} \to {}^*X \otimes X,$$

*called the **right evaluation** and the **right coevaluation**, such that the following diagrams commute*

$$\begin{array}{ccccc}
X \otimes \mathbb{1} & \xrightarrow{id_X \otimes coev'_X} & X \otimes ({}^*X \otimes X) & \xrightarrow{\alpha^{-1}_{X,{}^*X,X}} & (X \otimes {}^*X) \otimes X \\
\Big\downarrow{\scriptstyle r_X} & & & & \Big\downarrow{\scriptstyle ev'_X \otimes id_X} \\
X & & \xleftarrow[\quad l_X \quad]{} & & \mathbb{1} \otimes X
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{1} \otimes {}^*X & \xrightarrow{coev'_X \otimes id_{{}^*X}} & ({}^*X \otimes X) \otimes {}^*X & \xrightarrow{\alpha_{{}^*X,X,{}^*X}} & {}^*X \otimes (X \otimes {}^*X) \\
\Big\downarrow{\scriptstyle l_{{}^*X}} & & & & \Big\downarrow{\scriptstyle id_{{}^*X} \otimes ev'_X} \\
{}^*X & & \xleftarrow[\quad r_{{}^*X} \quad]{} & & {}^*X \otimes \mathbb{1}
\end{array}$$

It can be shown that the left or right dual of an object is unique up to a unique isomorphism [11]. So we usually speak of *the* right or left dual if it exists.

If X, Y are objects in $\mathcal{C}$ with left duals X^*, Y^*, and $f : X \to Y$ is a morphism, we define the **left dual** $f^* : Y^* \to X^*$ of f by

$$f^* = (ev_Y \otimes id_{X^*}) \circ (id_{Y^*} \otimes f \otimes id_{X^*}) \circ (id_{Y^*} \otimes coev_X) : Y^* \to X^*$$

Similarly, if X, Y are objects in $\mathcal{C}$ with right duals ${}^*X, {}^*Y$, and $f : X \to Y$ is a morphism, we define the **right dual** ${}^*f : {}^*Y \to {}^*X$ of f by

$${}^*f = (id_{{}^*X} \otimes ev'_Y) \circ (id_{{}^*X} \otimes f \otimes id_{{}^*Y}) \circ (coev'_X \otimes id_{{}^*Y}) : {}^*Y \to {}^*X$$

Notice that in the two definitions the natural transformations l and r have been omitted. Finally, we can give the general definition of a rigid category:

Definition 3.11 (Rigid category)**.** *A **left (right) rigid category** is a monoidal category admitting left (right) duality. A **rigid category** is a monoidal category that is both left rigid and right rigid.*

Remark 3.12. If X^* is the left dual of an object X, then X is the right dual of X^* with $ev'_{X^*} = ev_X$ and $coev'_{X^*} = coev_X$, and vice versa. Therefore ${}^*(X^*) \cong X \cong ({}^*X)^*$ for any object X admitting left and right duals. Also $\mathbb{1}^* = {}^*\mathbb{1} = \mathbb{1}$ in any monoidal category.

Example 3.13. The category $\mathbf{Vect}_{\mathbb{K}}$ of finite-dimensional $\mathbb{K}$-vector spaces is rigid. The right and left duals to a finite-dimensional vector space V are its dual space V^*, consisting of linear functions $f : V \to \mathbb{K}$. Letting $\{e_i\}_{i=1}^n$ be the basis vectors of V and $f_j : V \to \mathbb{K}$ be the functional in V^* such that $f_j(e_i) = \delta_{ij}$, the evaluation map is given by

$$\begin{aligned} ev_V : V^* \otimes V &\to \mathbb{K} \\ f_j \otimes e_j &\mapsto f_j(e_j) \end{aligned}$$

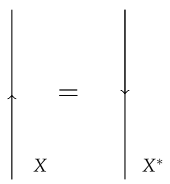
and the coevaluation map by

$$\begin{aligned} coev_V : \mathbb{K} &\to V \otimes V^* \\ 1 &\mapsto \sum_i e_i \otimes f_i \end{aligned}$$

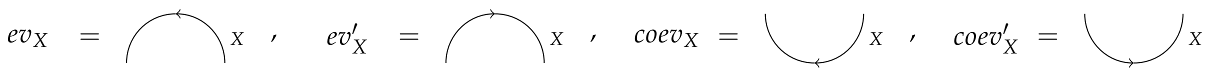
Although here we have introduced the concepts of left and right duality, when talking about charge conjugation in the theory of anyons we will only use left duality. That is because, in a category describing an anyon model, the left and right duals of objects and morphisms coincide. This will be further explained in section 3.4, when introducing a *ribbon structure*. Physically speaking, we will interpret the duality morphisms in the following way:

- the left evaluation is the annihilation of an antiparticle-particle pair - anyons with respective dual charges - and
- the left coevaluation is the creation of a particle-antiparticle pair.

In terms of string diagrams, an upward arrow labeled by an object X is equal to a downward arrow labeled by its dual X^*



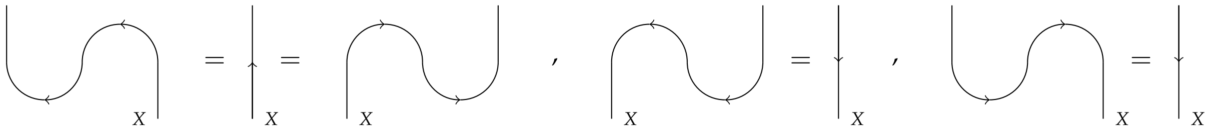
The duality morphisms can be pictured as follows



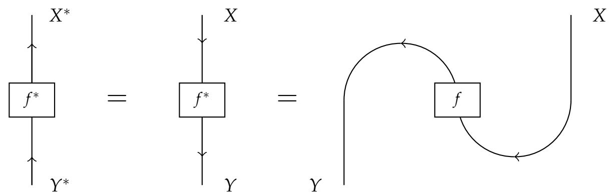
The two leftmost pictures are called *caps* and the two rightmost pictures are called *cups*. The condition that the diagrams in definitions 3.9 and 3.10 commute can be rewritten as the dual identities

$$\begin{aligned} (id_X \otimes ev_X) \circ (coev_X \otimes id_X) = id_X, &\quad (ev_X \otimes id_{X^*}) \circ (id_{X^*} \otimes coev_X) = id_{X^*} \\ (ev'_X \otimes id_X) \circ (id_X \otimes coev'_X) = id_X, &\quad (id_{{}^*X} \otimes ev'_X) \circ (coev'_X \otimes id_{{}^*X}) = id_{{}^*X} \end{aligned}$$

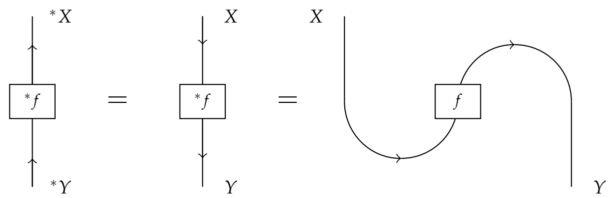
Notice that here the natural transformations α, l and r have been omitted again. These identities can be represented as follows



The left dual morphism f^* is given by



And the right dual morphism ${}^*f$ by



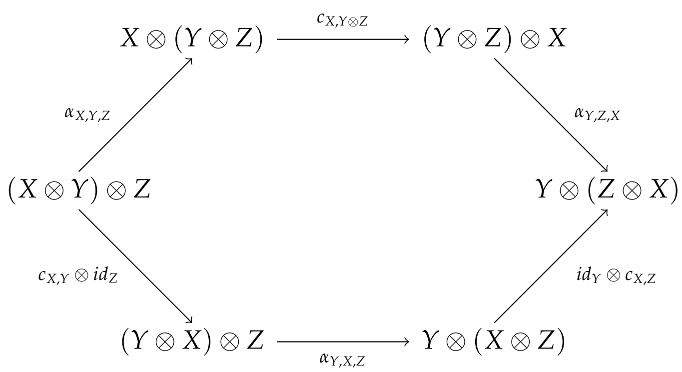
3.3 Braided Categories

When anyons are present in a two-dimensional surface, they may move around one another tracing a *braid* in time. This swapping process can be described by a *braiding structure* that must behave coherently with the monoidal structure. Therefore, we will need what is called a *braided monoidal category*.

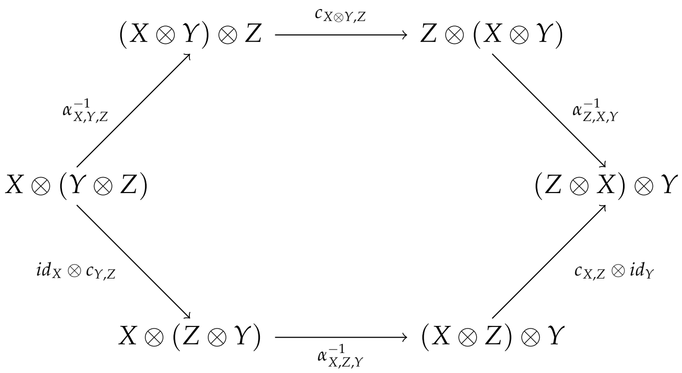
Definition 3.14 (Braiding)**.** *A **braiding** (or commutative constraint) on a monoidal category $\mathcal{C}$ is a natural family of isomorphisms*

$$c = \{c_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in Ob(\mathcal{C})}$$

such that the hexagonal diagrams



and



commute for any $X, Y, Z \in Ob(\mathcal{C})$.

Note that if c is a braiding so it is its inverse. When a monoidal category $\mathcal{C}$ is strict, the hexagonal diagrams become

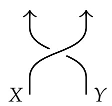
$$c_{X,Y\otimes Z} = (id_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes id_Z)$$
$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes id_Y) \circ (id_X \otimes c_{Y,Z})$$

This implies that braiding X with $Y \otimes Z$ all at once is the same as braiding first X with Y and then X with Z. Similarly for $X \otimes Y$ and Z. As a consequence, a braiding with the unit object $\mathbb{1} \in Ob(\mathcal{C})$ is given by

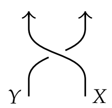
$$c_{X,\mathbb{1}} = c_{\mathbb{1},X} = id_X$$

for any $X \in Ob(\mathcal{C})$.

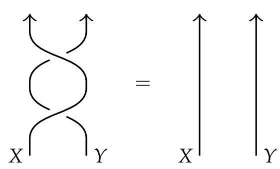
Braiding isomorphisms can be represented by *Reidemeister diagrams*, which generalize Penrose diagrams by allowing the strings to cross. The braiding isomorphism $c_{X,Y}$ is drawn as

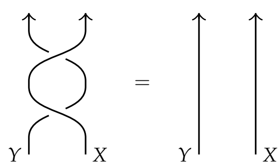


and its inverse $c^{-1}_{X,Y}$ by



Composing the braiding isomorphism and its inverse yields the identity morphism





Now, we are able to give the definition of *braided monoidal category*:

Definition 3.15 (Braided monoidal category)**.** *A **braided monoidal category** is a monoidal category endowed with a braiding.*

Example 3.16. In the category **Hilb** of finite-dimensional Hilbert spaces with its standard tensor product a braiding is given by

$$\begin{aligned} c_{X,Y} : X \otimes Y &\to Y \otimes X \\ x \otimes y &\mapsto y \otimes x \end{aligned}$$

for all $X, Y \in Ob(\mathbf{Hilb})$ and $x \in X$, $y \in Y$.

Definition 3.17 (Symmetric braiding)**.** *Let $\mathcal{C}$ be a monoidal category. A braiding c on $\mathcal{C}$ is symmetric if*

$$c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y} : X \otimes Y \to X \otimes Y$$

for all $X, Y \in Ob(\mathcal{C})$.

For shortness, these braidings are called symmetries. A **symmetric category** is a monoidal category equipped with a symmetry.

Example 3.18. The monoidal category of sets (**Set**) carries a symmetry given by the flips

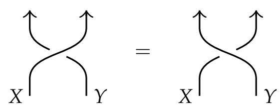
$$\begin{aligned} X \times Y &\to Y \times X \\ (x, y) &\mapsto (y, x) \end{aligned}$$

for all sets X, Y and $x \in X$, $y \in Y$.

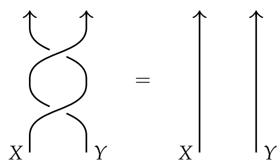
Definition 3.19 (Reverse braiding)**.** *Let $\mathcal{C}$ be a braided monoidal category with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$. We define the* ***reverse braiding*** *on $\mathcal{C}$ by*

$$c'_{X,Y} = c^{-1}_{Y,X}$$

Given a braiding c, symmetry implies that $c_{X,Y} = c^{-1}_{Y,X}$. This can be represented using string diagrams as



and implies that



A useful result, proven in [3], is:

Proposition 3.20. *Let $\mathcal{C}$ be a rigid braided monoidal category and $X, Y, Z \in Ob(\mathcal{C})$. Then, there are canonical isomorphisms*

$$\begin{aligned} Hom(X \otimes Y, Z) &\cong Hom(X, Z \otimes Y^*) \\ Hom(X, Y \otimes Z) &\cong Hom(Y^* \otimes X, Z) \end{aligned}$$

Finally, the next example explains the name braiding:

Example 3.21. [The category of braids]. Recall that in example 2.11 we introduced the notion of tangles. Braids form a special case of tangles: a **braid** on n strands is a tangle obtained from n disjoint intervals $[0,1]$ (i.e. no circles allowed) such that, for each $t \in [0,1]$, exactly one point of each interval belongs to $\mathbb{R}^2 \times \{t\}$.

Isotopy classes of n braids form a group B_n, called the *braid group*, under the operation of composition. For $n \geq 2$,the **braid group** B_n can be defined as the group generated by $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j = 1, ..., n-1$ with $|i-j| \geq 2$, and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, ..., n-2$. More information about the braid group can be found in [18].

When composing or tensoring two braids, as with tangles, we get another braid. This proves that braids form a monoidal category $\mathcal{B}$. It can be proved [17] that, in fact, it is a braided category with braidings $c_{n,m} : n \otimes m \to m \otimes n$ for any couple (n, m) of non-negative integers, which can be visualized as n strands passing over m strands.

3.4 Ribbon Categories

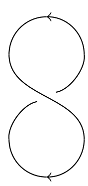
Until now we have introduced various morphisms in terms of monoidal products. Yet, another important ingredient to fully characterize an anyon model are *twists*. This is because an anyon can revolve around some center by 2π and the change induced on the system is not an identity. Let us consider what this means considering the following process:

- a particle-antiparticle pair is created with charges A and A^* respectively;
- the two anyons are swapped, with the anyon of charge A going behind the anyon of charge A^*;
- finally they annihilate.

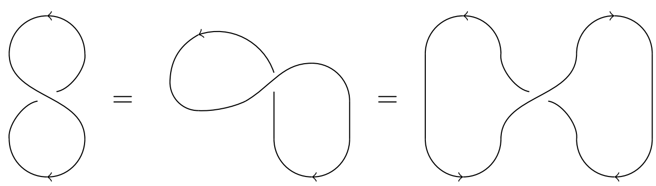
Based on the language we already have of *rigid braided monoidal categories*, such process can be expressed by the following composition

$$ev_A \circ c^{-1}_{A^*,A} \circ coev_A$$

or graphically,



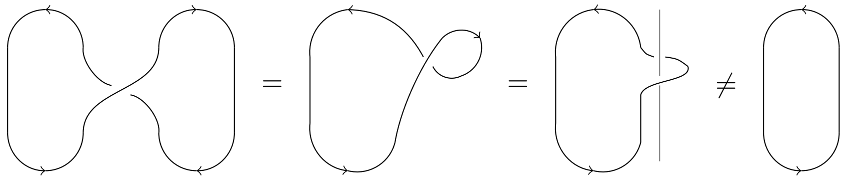
The essential idea here is that the amplitude of the process is non-trivial, i.e. not *equal* to the amplitude of a circle, since there is an exchange occurring. It is easy to see that



and, therefore, the amplitude of the first process is *equal* to the amplitude of the third, in which

- an antiparticle-particle and a particle-antiparticle pairs are created;
- the particle from the left pair and the one from the right pair are exchanged;
- finally both of the remaining pairs are annihilated.

Furthermore, this last process can be deformed such as follows



In third picture an antiparticle-particle pair is created and the particle gets rotated about 2π around some center before the pair is finally annihilated. Now, it is clear that this amplitude is different from the trivial amplitude depicted in the fourth picture.

Therefore, from this example we conclude that it is not completely faithful to illustrate the world-lines of anyons with strands. Instead we will use *ribbons* (see example 3.24) so that they can be twisted. In categorical terms, we will ask the category to be a *ribbon category*. Let us develop these ideas more formally.

Definition 3.22 (Twist). *Let $\mathcal{C}$ be a braided rigid monoidal category. A* ***twist*** *(or balancing transformation) on $\mathcal{C}$ is a natural family of isomorphisms*

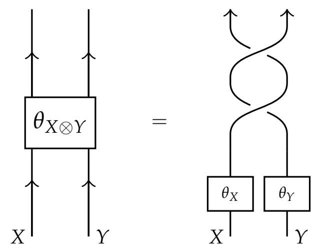
$$\theta_X : X \to X$$

such that

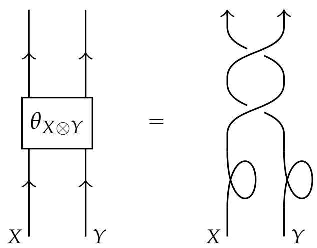
$$\theta_{X\otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y),$$

for all $X, Y \in Ob(\mathcal{C})$.

Graphically,



Another way to represent this identity which displays the action of the twist slightly better is



Using the naturality of the braiding, we may rewrite the previous condition as follows

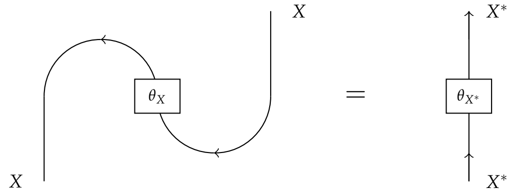
$$\theta_{X\otimes Y} = c_{Y,X} \circ (\theta_Y \otimes \theta_X) \circ c_{X,Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$$

Note that $\theta_{\mathbb{1}} = id_{\mathbb{1}}$. This follows from the invertibility of $\theta_{\mathbb{1}}$ and

$$(\theta_{\mathbb{1}})^2 = (\theta_{\mathbb{1}} \otimes id_{\mathbb{1}}) \circ (id_{\mathbb{1}} \otimes \theta_{\mathbb{1}}) = \theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}} = \theta_{\mathbb{1}}$$

Finally, the naturality of θ means that for any morphism $f : X \to Y$ in $\mathcal{C}$ we have $\theta_Y \circ f = f \circ \theta_X$.

A twist is called a **ribbon structure** if $(\theta_X)^* = \theta_{X^*}$. Graphically,



Definition 3.23 (Ribbon monoidal category)**.** *A* ***ribbon monoidal category*** *is a braided rigid monoidal category equipped with a ribbon structure.*

In a ribbon category left rigidity also implies right rigidity and vice versa. In fact, given the left duality morphisms ev_X and $coev_X$, we can write

$$ev'_X = ev_X \circ c_{X,X^*} \circ (\theta_X \otimes id_{X^*}) \quad \text{and} \quad coev'_X = (id_{X^*} \otimes \theta_X) \circ c_{X,X^*} \circ coev_X$$

which are the right duality morphisms. This makes the category also right rigid. With this right duality, the left and right duals of objects and morphisms coincide.

A ribbon category is called **strict** if its underlying monoidal category is strict. Given a monoidal category $\mathcal{C}$, we say that the duality in $\mathcal{C}$ is compatible with the braiding c and the twist θ if for any object $X \in Ob(\mathcal{C})$ we have

$$(\theta_X \otimes id_{X^*}) \circ coev_X = (id_X \otimes \theta_{X^*}) \circ coev_X$$

Equivalently, we can define a ribbon category as a monoidal category $\mathcal{C}$ equipped with a braiding c, a twist θ and a compatible duality $(*, ev_X, coev_X)$.

Example 3.24. [Ribbons]. The theory of ribbon categories is related to the theory of *links* in a 3-dimensional Euclidean space considering *ribbon graphs* embedded in $\mathbb{R}^2 \times [0,1]$. A **ribbon graph** consists of small rectangles (**coupons**) and long bands instead of vertices and edges. The bands are attached to the bases of the coupons and, possibly, to certain intervals in the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$.

Given a category $\mathcal{C}$, bands can be *colored* with objects of $\mathcal{C}$ whilst coupons can be *colored* with morphisms of $\mathcal{C}$. In this way, we can construct a monoidal category $\mathbf{Rib}_{\mathcal{C}}$, where

- objects are finite sequences $\{(X_1, \epsilon_1), ..., (X_m, \epsilon_m)\}$ where $X_1, ..., X_m \in Ob(\mathcal{C})$ and $\epsilon_1, ..., \epsilon_m \in \{+1, -1\}$ represent the orientations of the bands, being $\epsilon_i = +1$ if the band i is directed out of the coupon and $\epsilon_i = -1$ in the opposite case;
- morphisms $\eta \to \eta'$ are isotopy classes of a colored ribbon graph such that η (resp. η') is the sequence of colors and orientations of the bands which hit either the bottom or the top boundary interval.

The composition of morphisms is obtained by gluing ribbon graphs on top of each other and the monoidal product is given by the disjoint union of ribbon graphs. The category of $\mathbf{Rib}_{\mathcal{C}}$ admits a natural braiding, twist and duality and thus becomes a ribbon category. To read more about ribbon graphs and the category $\mathbf{Rib}_{\mathcal{C}}$ see [43].

A ribbon graph with no coupons is called a **ribbon tangle**. Ribbon tangles form a subcategory of $\mathbf{Rib}_{\mathcal{C}}$ with the same objects but less morphisms. In turn, they are a generalization of the tangles of example 2.11.

One of the most important features of ribbon categories are *traces* and *dimensions* of objects, which distinguish them from arbitrary monoidal categories.

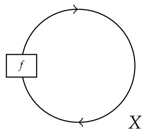
Definition 3.25 (Trace)**.** *Let $\mathcal{C}$ be a ribbon category with unit object $\mathbb{1}$. For any object $X \in Ob(\mathcal{C})$ and any endomorphism f of $\mathcal{C}$, we define the* ***trace*** *of f as the element*

$$tr(f) = ev_X \circ c_{X,X^*} \circ ((\theta_X \circ f) \otimes id_{X^*}) \circ coev_X$$

of $End(\mathbb{1})$, i.e. as the composition

$$\mathbb{1} \xrightarrow{coev_X} X \otimes X^* \xrightarrow{(\theta_X \circ f) \otimes id_{X^*}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{ev_X} \mathbb{1}$$

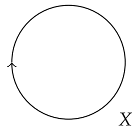
A graphical representation of $tr(f)$ is given by the string diagram

$tr(f) \quad =$ 

Definition 3.26 (Dimension)**.** *Let $\mathcal{C}$ be a ribbon category with unit object $\mathbb{1}$, For any object $X \in Ob(\mathcal{C})$ we define the **dimension** of X as*

$$dim(X) = tr(id_X) = ev_X \circ c_{X,X^*} \circ (\theta_X \otimes id_{X^*}) \circ coev_X \in End(\mathbb{1})$$

Graphically,

$dim(X) \quad =$ 

3.5 Semisimple Categories

As we have already explained, when considering an anyon model as a certain monoidal category $\mathcal{C}$, the monoidal product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ serves as a mathematical tool to describe the fusion of two anyon charges $A, B \in Ob(\mathcal{C})$ into a compound charge $A \otimes B \in Ob(\mathcal{C})$. This object may be isomorphic to another one, say $C \in Ob(\mathcal{C})$, and then we may write $A \otimes B \cong C$. For this reason it might look reasonable to define isomorphism classes of objects of $\mathcal{C}$ such that objects in a particular class can be regarded as being the same.

Definition 3.27 (Simple object)**.** *Let $\mathcal{C}$ be an abelian category. An object $X \in Ob(\mathcal{C})$ is called **simple** if any injection $Y \hookrightarrow X$ is either the zero morphism 0 or an isomorphism for any $Y \in Ob(\mathcal{C})$.*

This characterization ensures that a simple object X does not contain any non-trivial *subobjects* that are isomorphic to X. Given two objects $X, Y \in Ob(\mathcal{C})$, we can define an equivalence relation between them if there exists an isomorphism $f : Y \to X$ in $\mathcal{C}$. Each equivalence class can then be represented by a simple object. Let $[\mathcal{C}]$ be the set of isomorphism classes of non-zero simple objects. The simple objects in $[\mathcal{C}]$ may be regarded as the primitive objects from which any object in $Ob(\mathcal{C})$ can be constructed. In the anyon context, different primitive types of anyons are given by isomorphism classes of simple objects. This makes anyon charges irreducible in the sense that they cannot be decomposed into more elementary entities. Sometimes we say that charges specify a *superselection sector* of the theory. This just means that charge is a property that cannot be changed by any local physical process. That is, if one anyon is at all times well isolated from other anyons, its charge will never change. This property is in fact the essential reason why anyons are suitable for fault-tolerant quantum computation.

Definition 3.28 (Semisimple category)**.** *An abelian category $\mathcal{C}$ is called **semisimple** if every object $X \in Ob(\mathcal{C})$ is a direct sum of simple objects*

$$X \cong \bigoplus_{i \in [\mathcal{C}]} N_i X_i$$

where X_i are simple objects and $N_i \in \mathbb{Z}_+$ with only a finite number of them being non-zero.

Note that we use $i \in [\mathcal{C}]$ meaning X_i. This is enough to give the last definitions, in which there are now two distinct monoidal products: one from the abelian structure denoted by $\oplus$ and one from the ribbon structure written as $\otimes$. We will consider $\mathcal{C}$ to be a *semisimple ribbon category*. Let $[\mathcal{C}]$ be the set of equivalence classes of non-zero simple objects in $\mathcal{C}$ and let $\{X_i\}_{i\in[\mathcal{C}]}$ be representatives of those classes. We will additionally require the monoidal product $\otimes$ to be *bilinear* acting on the spaces of morphism and we will assume that $\mathbb{1} \in Ob(\mathcal{C})$ is a simple object denoted by $X_0 = \mathbb{1}$. Moreover, for each simple object $i \in [\mathcal{C}]$, $End(X_i) \cong \mathbb{K}$, where $\mathbb{K}$ is an arbitrary field. Although much of the following theory can be developed for an arbitrary field $\mathbb{K}$, it is natural to assume $\mathbb{K} = \mathbb{C}$ for our purposes, which are the description of quantum mechanical systems.

Now, having such a semisimple structure on $\mathcal{C}$ has many consequences. For example, for $i \in [\mathcal{C}]$, $(X_i)^*$ is also simple, hence $(X_i)^* \cong X_{i^*}$ for some $i^* \in [\mathcal{C}]$. The map $* : [\mathcal{C}] \to [\mathcal{C}]$ is an involution and $0^* \cong 0$. Furthermore, these properties will allow us to define the *fusion rules* of anyons and the *fusion spaces* where all anyon processes take place.

Another interesting result is the following variant of *Schur's lemma*:

Lemma 3.29. *Let $\mathcal{C}$ be a semisimple abelian category and X_i and X_j be simple objects such that $i \neq j$. Then $Hom(X_i, X_j) \cong \{0\}$.*

Proof. Let $f : X_i \to X_j$ be arbitrary and consider $Ker(f) : K \to X_i$, which is necessary a monomorphism since the category is abelian. As X_i is simple by assumption, $Ker(f)$ is either 0 or an isomorphism. If the kernel is 0, then f is an injection and hence an isomorphism since X_j is simple. However, this cannot be as $i \neq j$. Therefore, $Ker(f)$ is an isomorphism and we have $K \cong X_i$ and $f = 0$, from which we conclude that $Hom(X_i, X_j) \cong \{0\}$ since f is arbitrary. □

3.6 Modular Tensor Categories

Semisimple ribbon categories with the additional assumptions made in section 3.5 can already be used to describe the kinematics of a system of anyons. Nevertheless, we will be working with a particular class of these categories, namely *unitary modular tensor categories* (UMTC). Such categories only allow a finite set of isomorphism classes of simple objects. Although there is no reason to believe that this is true in nature, applications in quantum computation can be done with a finite number of anyon types.

Let $\mathcal{C}$ be a semisimple ribbon category with the additional assumptions made in section 3.5:

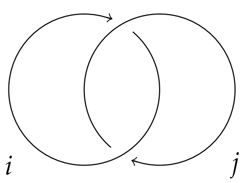
Definition 3.30 (Modular tensor category)**.** *The category $\mathcal{C}$ is a **modular tensor category** (MTC) if satisfies the following properties:*

- *$\mathcal{C}$ has only a finite number of isomorphism classes of simple objects;*
- *and the so called **S-matrix** with entries*

$$s_{ij} \equiv tr(c_{X_j,X_i} \circ c_{X_i,X_j}) \in \mathbb{K}, \quad i, j \in [\mathcal{C}]$$

is invertible.

Graphically, we can depict the entries of the S-matrix as



The components of the S-matrix form the so-called *Hopf link*. They can be thought of as the following process:

- two pairs of particle-antiparticle are created,
- the particle of one pair is wound around the antiparticle from the other pair and, finally,

- the two pairs annihilate.

It is easy to see that

$$s_{ij} = s_{ji} = s_{i^*j^*} = s_{j^*i^*}, \quad s_{i0} = d_i = dim(X_i)$$

The number d_i is called the **quantum dimension** of X_i. In the context of anyons, it is a positive real number ≥ 1 and an important quantum number since it is related to the dimension of the degeneracy of the ground state of the system. Many authors (for example in [43]) impose weaker conditions, not necessarily requiring semisimplicity in our sense. However, for our discussion, definition 3.30 is absolutely sufficient.

MTCs encode the symmetries of the physical system in which they are described. Since our ultimate goal is to define a valid quantum theory of anyons, they must also be equipped with the notion of *hermitian* and *unitary* morphisms in order to properly describe the observables and the time evolution of the theory.

Definition 3.31 (Hermitian)**.** *A ribbon category $\mathcal{C}$ is **hermitian** if it has a conjugation*

$$\dagger : Hom(X, Y) \to Hom(Y, X)$$

such that

$$(f^\dagger)^\dagger = f, \quad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \quad \textit{and} \quad (f \circ g)^\dagger = g^\dagger \circ f^\dagger$$

If $\mathbb{K} \subset \mathbb{C}$ then $\dagger$ must also act as the usual conjugation. Furthermore, $\dagger$ must be compatible with the other structures present, i.e.

$$\begin{aligned}
(c_{X,Y})^\dagger &= c^{-1}_{X,Y}, \\
(\theta_X)^\dagger &= (\theta_X)^{-1}, \\
(coev_X)^\dagger &= ev_X \circ c_{X,X^*} \circ (\theta_X \otimes id_{X^*}), \\
(ev_X)^\dagger &= (id_{X^*} \otimes \theta^{-1}_X) \circ c^{-1}_{X^*,X} \circ coev_X
\end{aligned}$$

Definition 3.32 (Unitary)**.** *A ribbon category $\mathcal{C}$ is called **unitary** if every morphism $f : X \to Y$ in $\mathcal{C}$ has an inverse that satisfies $f^{-1} = f^\dagger$.*

For hermitian ribbon categories $\mathcal{C}$ the categorical dimensions d_i are always real numbers. If they are also unitary, the morphism spaces $End(X)$ for $X \in Ob(\mathcal{C})$ are Hilbert spaces. A **unitary modular tensor category** (UMTC) is a MTC that is unitary.

Chapter 4

Fibonacci Anyons

4.1 Fusion Rules and Fusion Spaces

For now on, we fix $\mathcal{C}$ to be a UMTC. As already mentioned, the charge of an anyon is represented by an isomorphism class of a simple object in $\mathcal{C}$. Consider the fusion of two anyons with charges S_i and S_j into an anyon of charge S_k in N_{ij}^k ways. We will write this as

$$S_i \otimes S_j \cong N_{ij}^k S_k$$

The lower indices of N_{ij}^k indicate which charges fuse together in order to yield the charge identified by the upper index.

Remark 4.1. Note that such an expression always makes sense since the category is assumed to be semisimple, thus each object is isomorphic to a direct sum of simple objects.

More generally, the fusion process could produce different charges, which takes us to the next definition:

Definition 4.2 (Fusion rule)**.** *Let X_i and X_j be simple objects in $\mathcal{C}$ and $[\mathcal{C}]$ be the set of equivalence classes of non-zero simple objects in $\mathcal{C}$. The **fusion rule** of X_i and X_j is given by*

$$X_i \otimes X_j \cong \bigoplus_{k \in [\mathcal{C}]} N_{ij}^k X_k$$

*where the coefficients $N_{ij}^k = dim(Hom((X_i \otimes X_j), X_k))$ are called **fusion coefficients**.*

The fusion coefficients satisfy

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*}, \quad N_{ij}^0 = \delta_{ij^*}$$

and hence, we can lower and raise indices. Returning to the context of anyon theory, the first equality also implies that the fusion rule is symmetric with respect to the charges S_i and S_j since the possible charges of the compound system do not depend on whether S_i is on the left or right. Read backwards, the fusion rule specifies the possible ways for the charge S_k to split into parts with charges S_i and S_j. For most physical models, $N_{ij}^k = 0, 1$. If $N_{ij}^k = 0$, then the charge S_k cannot be obtained from the fusion of S_i and S_j. If for all $i, j \in [\mathcal{C}]$ there is only one N_{ij}^k that is different from zero, then the fusion outcome of each pair of charges is unique and the model is Abelian. On the other hand, if for some pair of charges S_i and S_j there are two or more fusion coefficients that satisfy $N_{ij}^k \neq 0$, then the model is non-Abelian. This implies that the fusion of S_i and S_j can result in several different charges. Finally, if $N_{ij}^k > 1$, then the charge S_k can be obtained in N_{ij}^k distinguishable ways.

Given a UMTC $\mathcal{C}$, we can easily describe its *Grothendieck ring* $K(\mathcal{C})$, from which we can build a finite dimensional commutative associative algebra $K = K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{K}$ [3]. This algebra is frequently called the

fusion algebra or **Verlinde algebra** and has for basis the set $x_i = [X_i]$ for $i \in [\mathcal{C}]$ and for unit $x_0 = \mathbb{1}$. Here $[X_i]$ denotes the isomorphism class of X_i. The multiplication in K is given by the fusion rules, i.e

$$x_i x_j = [X_i \otimes X_j] = \sum_k N_{ij}^k x_k$$

This algebra can be diagonalised by a renormalization of the S-matrix, the *modular* S-matrix, given by

$$S = \frac{1}{D} s \quad with \quad D = \sqrt{\sum_i d_i^2}$$

Indeed, we have from [3] the following:

Proposition 4.3. *For a fixed $i \in [\mathcal{C}]$, let N_i be the matrix of multiplication by x_i in the basis $\{x_j\}$, i.e. $(N_i)_{ab} = N_{ib}^a$ and also let D_i be the diagonal matrix $(D_i)_{ab} = \delta_{ab} S_{ia}/S_{0a}$. Then,*

$$S N_i S^{-1} = D_i$$

Note that N_i gives the action of multiplying some fixed x_i on the basis $\{x_j\}$. This proposition is usually formulated by saying that the *S-matrix diagonalises the fusion rules*. A more complete discussion of this fact along with its proof is given in the source of this proposition and in [33].

Rearranging the expression in the previous proposition as $SN_i = D_i S$, this immediately implies the following well-known result:

Theorem 4.4 (Verlinde Formula)**.**

$$N_{ij}^k = \sum_r \frac{S_{ir} S_{jr} S_{k^* r}}{S_{0r}}$$

Thus, the S-matrix is not only related to the braidings in the category used to define it, but also to the fusion coefficients.

Additionally, we can relate the quantum dimension of simple objects with its fusion coefficients. In fact, it can be proved [19, 33] that given simple objects X_i, X_j and X_k of $\mathcal{C}$

$$d_i d_j = \sum_k N_{ij}^k d_k$$

From this expression is easy to see that the quantum dimension of Abelian anyons is always 1 independently of the charge since $d_i = d_{i^*}$. However, for a non-Abelian anyon of charge S_i we have $d_i > 1$.

Following [31], all the notions introduced so far can be expressed in the language of Hilbert spaces. Such a translation is already built-in the UMTC. Indeed, we will use the fact that hom-sets of $\mathcal{C}$ are vector spaces over $\mathbb{C}$ and that, for all simple objects X_i, $End(X_i) \cong \mathbb{C}$ to build the so-called *fusion and splitting spaces*.

As a consequence of semisimplicity, the hom-set

$$Hom(X_i \otimes X_j, X_k)$$

is a complex vector space whose dimension is fixed by the fusion rule. This fact allows us to relate the concept of UMTC with that of a finite dimensional Hilbert space.

Definition 4.5 (Fusion space)**.** *Let X_i, X_j and X_k be simple objects in $\mathcal{C}$. A **fusion space** is a complex vector space*

$$V_{ij}^k \cong Hom(X_i \otimes X_j, X_k)$$

of dimension $N_{ij}^k = dim(Hom((X_i \otimes X_j), X_k))$. This vector space forms a Hilbert space.

In the anyon context, vectors in the fusion space $V_{ij}^k \cong Hom(S_i \otimes S_j, S_k)$ correspond to different ways in which charges S_i and S_j can fuse into S_k. We call these states **fusion states**. A basis for V_{ij}^k may be given by

$$\{|ij;k,\mu\rangle\,, \quad \mu = 1,2,...,N_{ij}^k\}$$

Dually, we can also define the **splitting space** $V_k^{ij} \cong Hom(X_k, X_i \otimes X_j)$. If a state $|\psi\rangle \in V_{ij}^k$, then $|\psi^\dagger\rangle = \langle\psi| \in V_k^{ij}$. Although from now on we will mainly focus on fusion spaces and develop concepts from them, most of the following results will still hold for splitting spaces with the appropriate changes.

In the light of lemma 3.29, spaces with different values of k are mutually orthogonal, so that the basis elements satisfy

$$\langle ij;k',\mu'|ij;k,\mu\rangle = \delta_{kk'}\delta_{\mu\mu'}$$

An anyon model is non-Abelian if

$$dim\left(\bigoplus_k V_{ij}^k\right) = \sum_k N_{ij}^k \geq 2$$

for at least some pair of charges S_i and S_j. Otherwise, the model is Abelian. The states in such a fusion space are perfectly degenerate and, as it is a collective non-local property of the anyons, no local perturbation can lift the degeneracy. It is hence a decoherence-free subspace and an ideal place to non-locally encode quantum information. This is precisely why non-Abelian anyons are suitable for TQC. However, it is important to bear in mind that the fusion space of a pair of non-Abelian anyons cannot be used directly to encode a qubit. This is because two states $|ij;k\rangle$ and $|ij;k'\rangle$ belong to different superselection sectors, given by k and k' respectively, and hence cannot be superposed. Instead, one needs more than two anyons in the system such that they can be fused in various different ways that give the same total charge. We will develop this idea further on when introducing the Fibonacci anyon model in Chapter 4.

There are some natural isomorphisms among fusion spaces. For instance, $V_{ij}^k \cong V_{ji}^k$. These vector spaces should be regarded as different if $i \neq j$ as they are associated to different labelings, but are isomorphic since fusion is symmetric. We may also rise and lower indices such as with fusion coefficients, which allows us to obtain the isomorphism $V_{i0}^i \cong V_{ii^*}^0$. This tells us that S_i^* is the unique charge that can fuse with S_i to yield $\mathbb{1}$ and that this fusion can occur in only one way. In addition, we obtain $V_{i0}^i \cong V_0^{ii^*}$, meaning that pairs of anyons created from the vacuum have always conjugate charges.

4.2 The F-matrix and the R-matrix

To properly describe an anyon model, one needs to know which are the fusion rules and braiding properties of such anyons. These are determined by the F and R matrices and are constrained by some consistency conditions, which arise because one can make a series of *F-moves* and *R-moves* to obtain an isomorphism that relates two fusion spaces. Such isomorphisms can be regarded as unitary matrices that relate the two bases of the spaces.

One consistency condition is that fusion has to be associative. This is physically imposed because the total charge of a system of anyons is an intrinsic property of the anyons themselves and, therefore, ought not depend on the fusion order. Consider three anyon charges S_i, S_j and S_k that fuse into a charge S_l. There are two natural ways to construct their fusion space from the direct sum, corresponding to the parenthesis structures $(S_i \otimes S_j) \otimes S_k$ and $S_i \otimes (S_j \otimes S_k)$,

$$V_{ijk}^l \cong \bigoplus_m V_{ij}^m \otimes V_{mk}^l \cong \bigoplus_n V_{in}^l \otimes V_{jk}^n$$

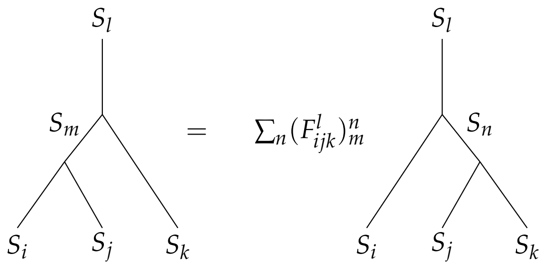
The basis vectors for these two fusion spaces can be denoted by

$$|(ij)k;mk;l\rangle \quad \text{and} \quad |i(jk);in;l\rangle$$

and must be related by a unitary transformation as

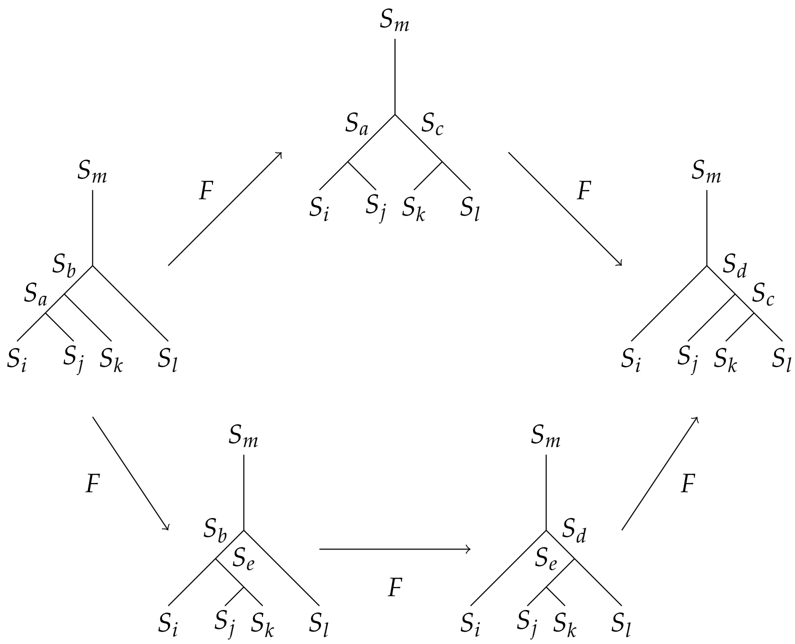
$$|(ij)k;mk;l\rangle = \sum_n (F^l_{ijk})^n_m \,|i(jk);in;l\rangle$$

where $(F^l_{ijk})^n_m$ are the matrix elements of F^l_{ijk}, which is the **F-matrix**. Here n is summed over all the anyons that S_j and S_k can fuse to, i.e. for which $N^n_{jk} \neq 0$, but complications from fusion coefficients $N^k_{ij} > 1$ have been ignored. Graphically, each fusion order is represented by a *tree diagram* and the basis vectors are its admissible labelings. The change from one fusion tree to another is what we call an F-move.



Now a comment about fusion trees is necessary. Although we do not draw them with arrows, following our convention they are drawn upwards. This is because, as interpreted as a physical process, time also goes upwards. If we read them downwards, they would become splitting trees.

F-matrices are obtained as a solution of the pentagon equation in definition 3.4, which can also be expressed diagrammatically as



The basis shown furthest to the left is $|left;a;b;m\rangle$, in which anyons with charges S_i and S_j are fused first, the resulting charge S_a is fused with S_k to yield charge S_b, and finally S_b is fused with S_l into the total charge S_m. The basis shown furthest to the right is $|right;c;d;m\rangle$, in which anyons are fused from right to left instead that from left to right. These two bases are related across the top of the pentagon by two F-moves

$$|left;a;b;m\rangle = \sum_{c,d} (F^m_{ijc})^d_a (F^m_{akl})^c_b \,|right;c;d;m\rangle$$

and across the bottom by three F-moves

$$|left;a;b;m\rangle = \sum_{c,d,e} (F^d_{jkl})^c_e (F^m_{iel})^d_b (F^b_{ijk})^e_a \,|right;c;d;m\rangle$$

Equating for $|left;a;b;m\rangle$, we obtain the matrix equation

$$(F^{m}_{ijc})^{d}_{a}(F^{m}_{akl})^{c}_{b} = \sum_{e}(F^{d}_{jkl})^{c}_{e}(F^{m}_{iel})^{d}_{b}(F^{b}_{ijk})^{e}_{a}$$

Solving this in conjunction with a given set of fusion rules yields the F-matrix. To solve such an equation one has to fix the charges for all possible states in the fusion basis and solve the resulting system of equations.

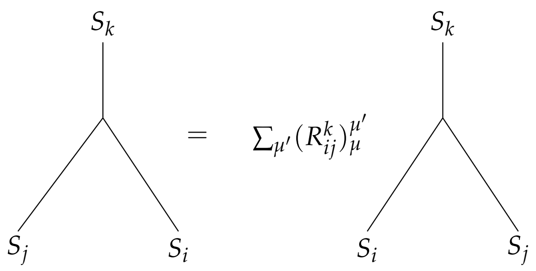
Another consistency condition is found when considering the exchange of two anyons. When two anyons are exchanged, their state in the fusion space undergoes a unitary evolution. This is categorically expressed by the braiding morphism. Suppose that two charges S_i and S_j, of total charge S_k, are swapped. Such process induces an isomorphism

$$R : V^{k}_{ji} \to V^{k}_{ij}$$

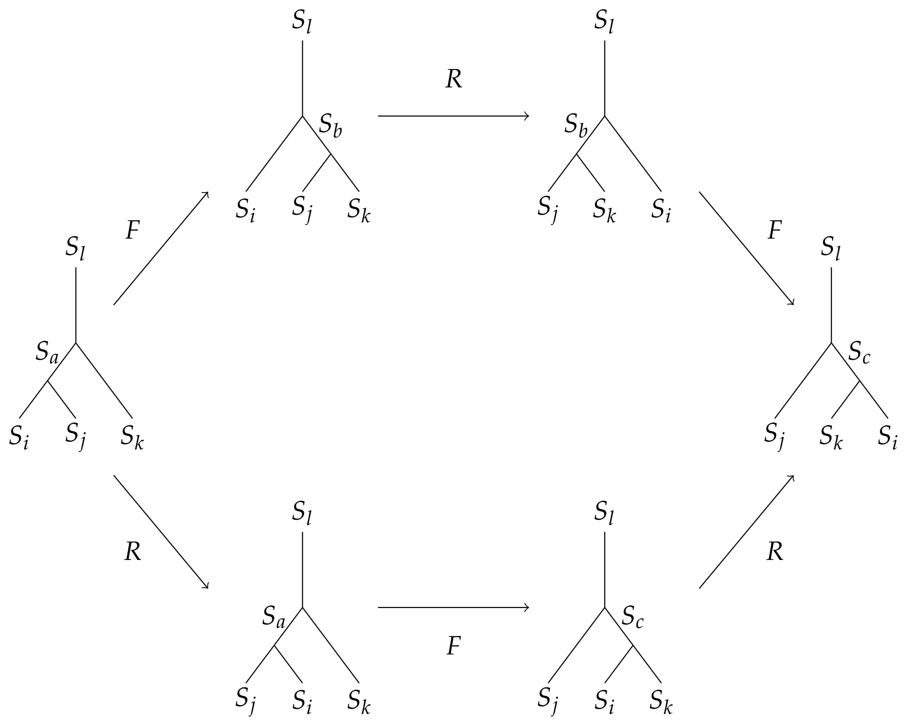
which, choosing the bases $|ji;k,\mu\rangle$ and $|ij;k,\mu'\rangle$, can be expressed as the unitary matrix

$$R : |ji;k,\mu\rangle \mapsto \sum_{\mu'}(R^{k}_{ij})^{\mu'}_{\mu}\, |ij;k,\mu'\rangle$$

Diagrammatically,



This is the **R-matrix** and it can be obtained as a solution of the hexagonal diagrams from the braiding monoidal structure in definition 3.14. For example, let us consider



The basis furthest to the left $|left;a;l\rangle$ is obtained when the charges are arranged in the order S_i, S_j and S_k and charges S_i and S_j are fused first. The basis furthest to the right $|right;c;l\rangle$ is obtained if charges are arranged S_j, S_k and S_i and charges S_k and S_i are fused first. According to the top part of the hexagonal diagram, the two bases are related by two F-moves and one R-move

$$|left;a;l\rangle = \sum_{b,c} (F^l_{jki})^c_b R^l_{ib} (F^l_{ijk})^b_a \, |right;c;l\rangle$$

and according to the bottom part by one F-move and two R-moves

$$|left;a;l\rangle = \sum_{c} R^c_{ik} (F^l_{jik})^c_a R^a_{ij} \, |right;c;l\rangle$$

Equating for $|left;a;l\rangle$, we obtain the matrix equation

$$R^c_{ik} (F^l_{jik})^c_a R^a_{ij} = \sum_{b} (F^l_{jki})^c_b R^l_{ib} (F^l_{ijk})^b_a$$

Finally, sometimes is useful to apply the F-matrix to move from a basis to another in which the R-matrix is block diagonal, apply R, and then apply F^{-1} to return to the initial basis. The composition of these three operations, which expresses the effect of braiding in the initial basis, is denoted B and sometimes called the **braid matrix**, or simply the **B-matrix**:

$$B = F^{-1} R F$$

4.3 Introduction to Fibonacci Anyons

We have seen that the F-matrix and the R-matrix impose the conditions necessary to ensure consistency of fusing and braiding of anyons. For any choice of an initial and final basis describing some fusion spaces, all sequences of F-moves and R-moves that take the initial basis to the final basis yield the same isomorphism, provided that the pentagon and the hexagon equations are satisfied. These equations together are called the *Moore-Seiberg polynomial equations* [27].

A solution to the polynomial equations defines a viable anyon model. Therefore, the procedure for constructing any anyon model is as follows:

1. choose a set of charges,
2. assume some fusion rule,
3. solve the polynomial equations for F and R.

If no solution exists, then the hypothetical fusion rule is incompatible with the principles of local quantum physics and must be rejected. If there is more than one solution not related to one another, then each distinct solution defines an anyon model with the assumed fusion rule.

To illustrate this procedure, in this section we will construct the *Fibonacci anyon model* [21, 28, 31, 33, 41]. Based on their anyon model structure, Fibonacci anyons are the simplest non-Abelian anyons that can give rise to universal quantum computation. However, such simplicity as an anyon model by no means correlates with the accessibility of the macroscopic systems that support them, quite the contrary. Elaborate schemes to realize them have been proposed in coupled domain wall arrays of Abelian FQH states [25]. The most plausible candidate is the Read-Rezayi state proposed to describe the filling fraction $\nu = 12/5$ FQH state [35]. Nevertheless, it remains unclear whether it can be ever realized in a laboratory since it is a very fragile state.

In the Fibonacci anyon model there are only two particle types: the trivial charge $\mathbb{1}$ and the non-trivial charge τ. The quantum dimension of the trivial charge is $d_0 = 1$ and the quantum dimension of the τ charge is $d_1 = \phi$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio, a characteristic of the Fibonacci series. Both charges are their own conjugate charge and the fusion rules are given by

$$\begin{aligned} &\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1} \\ &\mathbb{1} \otimes \tau \cong \tau \otimes \mathbb{1} \cong \tau \\ &\tau \otimes \tau \cong \mathbb{1} \oplus \tau \end{aligned}$$

Let us inspect the fusion rules. While the two first hold trivially, from the third one we see that the fusion of two charges τ can result into either a charge $\mathbb{1}$ or a charge τ. It is precisely from here that we see that the anyons are non-Abelian as they can fuse in two distinct ways.

Categorically speaking, the anyon model is described by the UMTC **Fib**. This category

- has two simple objects $\mathbb{1}$ and τ, where $\mathbb{1}$ is the monoidal unit object,
- $dim(\mathbb{1}) = 1$ and $dim(\tau) = \phi$,
- both objects are self-dual, i.e. $\mathbb{1}^* = \mathbb{1}$ and $\tau^* = \tau$ and,
- both $\mathbb{1}$ and τ satisfy the fusion rules given above.

Using the Verlinde Formula, we can calculate the S-matrix

$$S = \frac{1}{\sqrt{1+\phi^2}} \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}$$

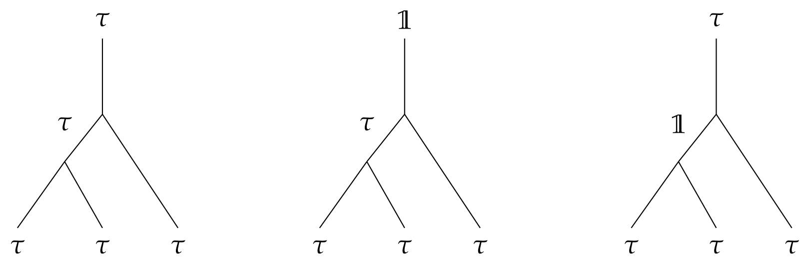
which is invertible with inverse

$$S^{-1} = \sqrt{1+\phi^2} \begin{pmatrix} \frac{1}{\phi^2+1} & \frac{\phi}{\phi^2+1} \\ \frac{\phi}{\phi^2+1} & \frac{-1}{\phi^2+1} \end{pmatrix}$$

Now, back to the model, consider the fusion of three anyons of charge τ in the order fixed by the parenthesis structure $(\tau \otimes \tau) \otimes \tau$. This process can be described by

$$\begin{aligned}(\tau \otimes \tau) \otimes \tau &\cong (\mathbb{1} \oplus \tau) \otimes \tau \\ &\cong (\mathbb{1} \otimes \tau) \oplus (\tau \otimes \tau) \\ &\cong \tau \oplus (\mathbb{1} \oplus \tau) \\ &\cong \mathbb{1} \oplus 2 \cdot \tau\end{aligned}$$

Hence, the fusion process of three τ anyons yields either a final charge $\mathbb{1}$ or a final charge τ in two different ways. These three scenarios can be depicted as



In the context of fusion spaces,

$$\begin{aligned}Hom((\tau \otimes \tau) \otimes \tau, b) &\cong Hom(\mathbb{1} \oplus 2 \cdot \tau, b) \\ &\cong Hom(\mathbb{1}, b) \oplus Hom(2 \cdot \tau, b) \\ &\cong Hom(\mathbb{1}, b) \oplus 2 \cdot Hom(\tau, b)\end{aligned}$$

for $b \in \{\mathbb{1}, \tau\}$. Now, using lemma 3.29 and the property that $End(b) \cong \mathbb{C}$, we see that

$$\begin{aligned}Hom((\tau \otimes \tau) \otimes \tau, \mathbb{1}) &\cong \mathbb{C} \oplus 2 \cdot 0 \\ Hom((\tau \otimes \tau) \otimes \tau, \tau) &\cong 0 \oplus 2 \cdot \mathbb{C}\end{aligned}$$

Therefore considering the space of states with global charge $b \in \{\mathbb{1}, \tau\}$ is the same as considering the fusion space $Hom((\tau \otimes \tau) \otimes \tau, b)$. Similarly,

$$\begin{aligned}((\tau \otimes \tau) \otimes \tau) \otimes \tau &\cong 2 \cdot \mathbb{1} \oplus 3 \cdot \tau \\ (((\tau \otimes \tau) \otimes \tau)) \otimes \tau &\cong 3 \cdot \mathbb{1} \oplus 5 \cdot \tau\end{aligned}$$

and so on. From this we conclude that the dimensionality of the fusion space in both superselection sectors grows as the Fibonacci series, hence the name of the anyon model.

Now that we have talked about the fusion spaces in **Fib**, we will solve the polynomial equations for F and R. There are many pentagon equations for the Fibonacci anyons depending on the four charges to be fused and their total global charge. Written explicitly, the pentagon equation for the Fibonacci anyon reads

$$(F^{\tau}_{\tau\tau c})^d_a (F^{\tau}_{a\tau\tau})^c_b = (F^{d}_{\tau\tau\tau})^c_e (F^{\tau}_{\tau e\tau})^d_b (F^{b}_{\tau\tau\tau})^e_a$$

There are only a few different matrices appearing, four of which are uniquely determined by the fusion rules

$$F^{\tau}_{\tau\tau\mathbb{1}} = F^{\tau}_{\mathbb{1}\tau\tau} = F^{\tau}_{\tau\mathbb{1}\tau} = F^{\mathbb{1}}_{\tau\tau\tau} = 1$$

The only non-trivial matrix is $F^{\tau}_{\tau\tau\tau}$. Writing $F^{\tau}_{\tau\tau\tau} \equiv F$ and setting $b = c = \mathbb{1}$ we have

$$F^{\mathbb{1}}_{\mathbb{1}} = F^{\mathbb{1}}_{\tau} F^{\tau}_{\mathbb{1}}$$

which, combined with the condition that F is unitary, constrains the matrix, up to arbitrary phases, to be

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} F^{\mathbb{1}}_{\mathbb{1}} & F^{\mathbb{1}}_{\tau} \\ F^{\tau}_{\mathbb{1}} & F^{\tau}_{\tau} \end{pmatrix} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

Therefore, in the three-dimensional Hilbert space of three τ anyons we have

$$\left(\begin{array}{c|cc} 1 & & \\ \hline & \phi^{-1} & \phi^{-1/2} \\ & \phi^{-1/2} & -\phi^{-1} \end{array}\right)$$

Note that this matrix can be applied to any three anyons, each of them having a charge τ. Each object, though, can consist of more than one anyon. As the other F-matrices for the Fibonacci anyons are trivial, this is indeed the only matrix needed to make arbitrary basis changes for any number of Fibonacci anyons.

Next, we will compute the R-matrix, which gives the phase factor of two anyons that are moved around one another. Written explicitly, the hexagon equation reads

$$R^{c}_{\tau\tau} (F^{\tau}_{\tau\tau\tau})^c_a R^{a}_{\tau\tau} = \sum_{b\in\{\mathbb{1},\tau\}} (F^{\tau}_{\tau\tau\tau})^c_b R^{\tau}_{\tau b} (F^{\tau}_{\tau\tau\tau})^b_a$$

Inserting the F-matrix $F^{\tau}_{\tau\tau\tau}$ and realizing that braiding a charge around the trivial charge is trivial, i.e. $R^{\mathbb{1}}_{\tau\mathbb{1}} = R^{\tau}_{\tau\mathbb{1}} = 1$, the hexagon equation becomes

$$\begin{pmatrix} (R^{\mathbb{1}}_{\tau\tau})^2\phi^{-1} & R^{\mathbb{1}}_{\tau\tau}R^{\tau}_{\tau\tau}\phi^{-1/2} \\ R^{\mathbb{1}}_{\tau\tau}R^{\tau}_{\tau\tau}\phi^{-1/2} & -(R^{\tau}_{\tau\tau})^2\phi^{-1} \end{pmatrix} = \begin{pmatrix} R^{\tau}_{\tau\tau}\phi^{-1} + \phi^{-2} & (1 - R^{\tau}_{\tau\tau})\phi^{-3/2} \\ (1 - R^{\tau}_{\tau\tau})\phi^{-3/2} & R^{\tau}_{\tau\tau}\phi^{-2} + \phi^{-1} \end{pmatrix}$$

which has the solution

$$R^{\mathbb{1}}_{\tau\tau} = e^{-4\pi i/5}, \quad R^{\tau}_{\tau\tau} = e^{+3\pi i/5}$$

Therefore the R-matrix is given by

$$R = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{+3\pi i/5} \end{pmatrix}$$

Such a diagonal form is not surprising: the global charge of a couple of anyons must remain unchanged even if the two anyons are exchanged. Again, this matrix applies if we exchange two anyons both with charge τ, even if these objects consist of more than one Fibonacci anyon.

4.4 Universal Quantum Computation with Fibonacci Anyons

In TQC, quantum information is stored in states which are intrinsically protected from decoherence and quantum gates are carried out by adiabatically braiding anyons around one another in a $(2+1)$-dimensional spacetime.

4.4.1 Simulating Qubits

In quantum computing, a *qubit* is the basic unit of information. It is the quantum analogue of the classical bit, which can have value zero or one, used to encode information in classical computing. A **qubit** is a two-level quantum system, where the two logical qubit states are usually given by

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

in the Dirac notation. These two orthonormal states, called the computational basis, span the two-dimensional Hilbert space of the qubit. Qubits can be either in one of the basis states or in a linear combination of both, known as a superposition.

The possible quantum states for a single qubit can be visualized using a **Bloch sphere**, a 2-sphere where the north and south poles correspond to the $|0\rangle$ and $|1\rangle$ states respectively. Then, every pure qubit state is given by

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$$

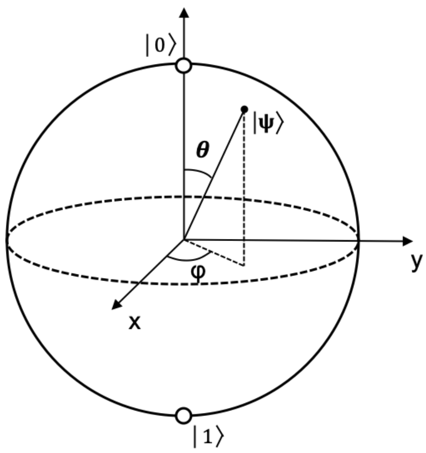


Figure 4.1: Bloch sphere

Going back to the anyon context, recall that the fusion rules for the Fibonacci anyons imply that the fusion space of two charges is two-dimensional. However, such space cannot be used to simulate a qubit since the basis states belong to different superselection sectors and thus cannot be superposed. When a third charge is added, the fusion space becomes three-dimensional and is spanned by the fusion states

$$|(\tau\otimes\tau)\otimes\tau;\mathbb{1}\otimes\tau;\tau\rangle\,, \quad |(\tau\otimes\tau)\otimes\tau;\tau\otimes\tau;\tau\rangle \quad \text{and} \quad |(\tau\otimes\tau)\otimes\tau;\tau\otimes\tau;\mathbb{1}\rangle$$

We can use this Hilbert space for quantum computation and encode qubits into triplets of anyons with global charge τ, taking the logical qubit states to be

$$|0\rangle = |(\tau\otimes\tau)\otimes\tau;\mathbb{1}\otimes\tau;\tau\rangle \quad \text{and} \quad |1\rangle = |(\tau\otimes\tau)\otimes\tau;\tau\otimes\tau;\tau\rangle$$

The remaining fusion state, with global charge $\mathbb{1}$, is then a non-computational state

$$|NC\rangle = |(\tau \otimes \tau) \otimes \tau; \tau \otimes \tau; \mathbb{1}\rangle$$

This encoding can be illustrated as

Even if we set the global charge of the triplet of anyons to be τ, it is still possible to end up measuring a charge of $\mathbb{1}$. Therefore, care must be taken to minimize transitions to non-computational states, known as *leakage errors*, when carrying out computations.

Equivalently, some authors prefer to encode their qubits within a quadrupole of anyons of individual charge τ with global charge $\mathbb{1}$. This is because, since **Fib** is rigid, we can apply proposition 3.20 and therefore

$$Hom((\tau \otimes \tau) \otimes \tau, \tau) \cong Hom((\tau \otimes \tau) \otimes \tau, \mathbb{1} \otimes \tau) \cong Hom(((\tau \otimes \tau) \otimes \tau) \otimes \tau, \mathbb{1})$$

Nonetheless, from now on we will only use the three-anyon encoding.

4.4.2 Quantum Computation

To perform actual quantum computation, one would like first to be able to apply any gate on the simulated qubits. A powerful theorem, due to Solovay and Kitaev [29], guarantees that given a set of gates generated by finite braids, which is sufficiently dense in the space of all gates, it is possible to find braids approximating arbitrary gates to any required accuracy. This theorem is valid for both one- and two-qubit gates.

Therefore, in any TQC process, quantum algorithms are carried out by quantum circuits consisting of one- and two-qubit gates, which are translated into a braid. Then, qubits are initialized by pulling pairs of conjugate charges out of the vacuum. These charges are then adiabatically dragged around one another so that their worldlines trace out a braid equivalent to the one compiled from the quantum algorithm. Finally, individual qubits are measured by fusing together the two leftmost charges.

Following [7, 15], we can illustrate a computation in which four anyons are generated from the vacuum, a single-qubit operation is carried out by braiding within the qubit, and the final state of the qubit is measured by fusing as follows

Using the F and R matrices, we can determine the elementary matrices that act on the three-dimensional fusion space of three Fibonacci anyons. For instance, a clockwise interchange of the two leftmost charges in a qubit is given by the matrix

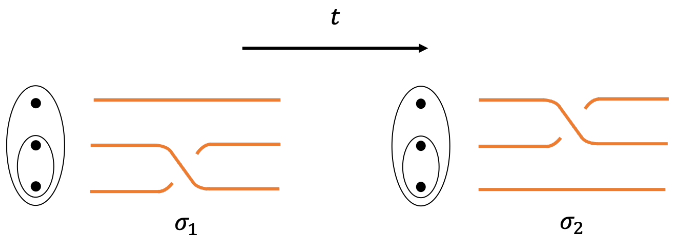
$$\sigma_1 = R = \left(\begin{array}{cc|c} e^{-4\pi i/5} & 0 & \\ 0 & e^{+3\pi i/5} & \\ \hline & & e^{+3\pi i/5} \end{array}\right)$$

To find the matrix corresponding to a clockwise interchange of the two rightmost charges in a qubit, it is convenient to first use the F-matrix to change basis to one in which the global charge of these anyons is well-defined. Then, the R-matrix is simply σ_1, and so, after changing back to the original basis,

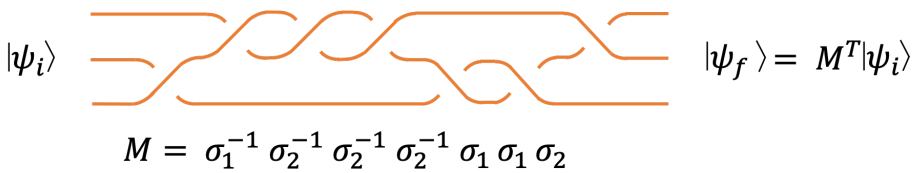
$$\sigma_2 = B = F^{-1}\sigma_1 F = \left(\begin{array}{cc|c} -\phi^{-1}e^{-\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} & \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} & \\ \hline & & e^{+3\pi i/5} \end{array}\right)$$

In both matrices, the upper 2×2 blocks act on the computational qubit space and are used to perform single-qubit rotations, while the lower right elements are phase factors acquired by the non-computational state $|NC\rangle$. The diagonal form of the 2×2 block of the first matrix illustrates that if a group of anyons has a certain global charge, then braiding them within this group does not change their global charge. Consequently, single-qubit gates performed by braiding anyons within a qubit will not lead to leakage error.

The elementary braiding operations can be represented as follows, with time flowing from left to right



Any three-anyon braid can be constructed out of these elementary operations and their inverses. For example



We can also implement two-qubit gates. Constructions are based on two essential ideas. First, a pair of anyons from one qubit (the control qubit) are woven through the anyons forming the second qubit (the target qubit). By weaving we mean that the target charges remain fixed while the control pair is moved around them as a single immutable object. We choose the control pair such that its global charge determines the logical state of the qubit. In such way, the construction automatically yields a controlled (conditional) operation. If the control qubit is in the state $|0\rangle$, the control pair has global charge $\mathbb{1}$ and weaving through the target qubit has no effect. Therefore, the identity operation is performed on the target qubit. However, if the control qubit is in the state $|1\rangle$, the control pair has global charge τ and a transition on the target qubit is induced.

Second, we only weave the control pair through two target charges at a time. Since the only non-trivial case is when the control pair has global charge τ, and so behaves as a single Fibonacci anyon, the problem of constructing two-qubits gates is reduced to that of finding a finite number of specific three-anyon braids. In addition, no leakage error is introduced out of the computational qubit space, or at least leakage is kept as small as required for a particular computation.

Using these ideas, we can build for instance a **CNOT** gate, which flips the target qubit state if and only if the control qubit state is $|1\rangle$. Consider first a braid in which one charge is woven through two static charges and it is not returned to its original position. The unitary transformation produced by this weave approximates the identity operation and it can permute the three anyons involved without changing any of their charges. Therefore, it can be used to safely *inject* an anyon, or any compound system with global charge τ, into a qubit.

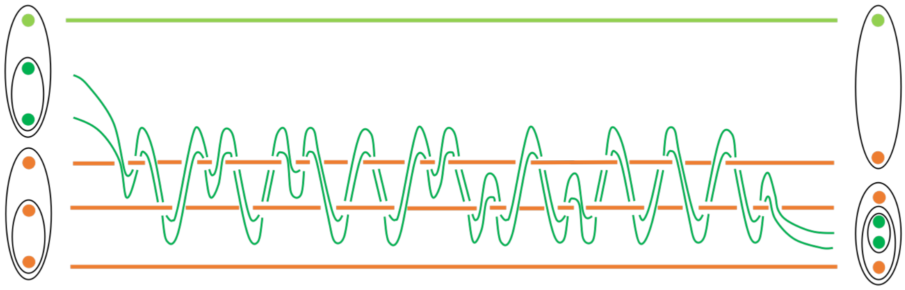


Figure 4.2: Injection weave and step one of the **CNOT** gate. If the control qubit is in the state $|1\rangle$, then the global charge of the injected anyons is τ and the result is to produce a target qubit with the same charges as the original but with its middle anyon replaced by the control pair.

Next, consider a braid which performs an approximate **NOT** gate - actually an $i \cdot$ **NOT** - on the target qubit. A **NOT** gate simply maps $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

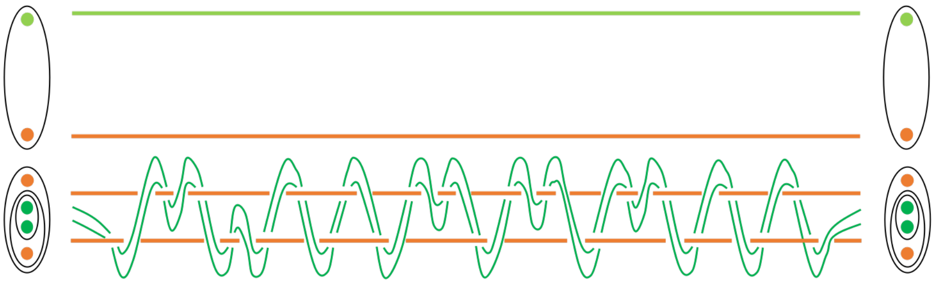


Figure 4.3: Step two of the **CNOT** gate. The control pair is woven within the injected target qubit. An approximate $i \cdot$ **NOT** gate is carried out when the control pair has global charge τ.

These two braids can be used to construct a $i \cdot$ **CNOT** gate as follows: first the control pair is injected into the target qubit. When the global charge of the control pair is $\mathbb{1}$, the state of the target qubit is unchanged after the injection. However, if the control pair has global charge τ, it replaces one of the target charges. Then, a $i \cdot$ **NOT** operation is performed on the injected target qubit and, finally, the control pair is *ejected* from the target qubit using the inverse of the injection braid, thereby returning the control qubit to its original state. It can be shown that all this together with a single-qubit rotation is equivalent to a **CNOT** gate.

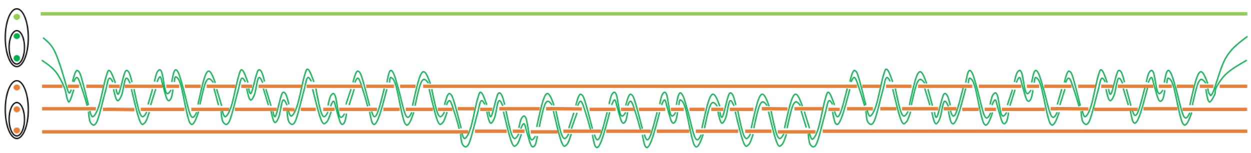


Figure 4.4: Full construction of a $i \cdot$ **CNOT** gate, which can be decomposed into the operations of injection, $i \cdot$ **NOT** weave and inverse injection.

By replacing the $i \cdot$ **NOT** braid with any other even winding weave that carries out an operation U, the construction will give a controlled-U gate.

Chapter 5

Outlook

Ever since the concept of quantum computation was proposed, making it a reality has been an aspiration for many. Quantum states are delicate and decoherence upon interactions with the environment, making the task of building a quantum computer inherently challenging. A quantum computer must be able not only to manipulate quantum information, but also to protect it from error and environmental noise. TQC offers a particular elegant way to achieve this by making use of non-Abelian anyons.

As we have seen in the previous chapters, the fundamental ingredients to perform TQC are: (i) To have access to a system of non-Abelian anyons, (ii) to be able to build quantum gates by adiabatically moving them around each other and (iii) to be able to measure their fusion outcomes. If we were to have access to Fibonacci anyons, we would be able to implement universal quantum computation with just these steps. However, reality is much more complex and the emergence of those anyons is experimentally challenging as the theoretically states that support them are extremely fragile. That is why research is currently focusing on other types of anyons such as Ising anyons or Majorana zero modes.

In this work, we have given an explicit description of the algebra of anyons in terms of UMTCs and have introduced the Fibonacci anyon model. The very rich nature of this subject makes hard to write a complete introduction concisely. Therefore, to complement our attempt, we refer the interested reader to [5, 19, 31, 37, 44] for more information on UMTCs and anyons, to [19, 20, 28] for the physical side of anyons and to [7, 15, 41] to go deeper into Fibonacci anyons.

Perhaps one could ask if TQC is the future of quantum computation. Unfortunately no one still knows the answer to this question. Maybe a more realistic route would be some midway point between a fully topological quantum computer and a system that enjoys some of the benefits of non-locally information encoding and processing. Nevertheless, what is true for sure is that TQC is still at its infancy compared to more conventional quantum computation approaches and that there is still a long way to go.

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