Polygonal cycles in higher Chow groups of Jacobians

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To the memory of Fabio Bardelli

Abstract

The aim of this paper is to construct non-trivial cycles in the first higher Chow group of the Jacobian of a curve having special torsion points. The basic tool is to compute the analogue of the Griffiths' infinitesimal invariant of the natural normal function defined by the cycle as the curve moves in the corresponding moduli space. We prove also a Torelli like theorem. The case of genus 2 is considered in the last section.

Key words: Algebraic cycles, Jacobians, regulator, adjunction.

Introduction

The aim of this paper is to construct non-trivial cycles in the first higher Chow group of the Jacobian of a curve having special torsion points. Let C be a curve of genus $g \geq 2$ such that there exist two points $p, q \in C$ with $n[p-q] = 0$ ($n > 1$) in its Jacobian variety $J(C)$. In other words, there exists

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a map $h : C \longrightarrow \mathbb{P}^1$ of degree n totally ramified at these points. Generalizing a construction by Collino one can produce a polygonal cycle $P(C,p,q)$ for the first higher Chow group $CH^{g-1}(J(C),1) \cong H^{g-1}(J(C),\mathcal{K}_g)$. We prove that the image of $P(C,p,q)$ in the higher Jacobian by the regulator map is non-trivial. The basic tool is to compute the analogue of the Griffiths' infinitesimal invariant of the natural normal function defined by the cycle as C moves in the corresponding moduli space. We prove also that the infinitesimal invariant contains enough information to recover the map h. The hyperelliptic case (n=2) has been considered by Collino and has inspired this note.

The genus 2 case deserves special attention since one has as many polygonal cycles as torsion points. As in the paper by Collino we prove the infinite generation of the group $H^1(J(C),\mathcal{K}_2)/Pic(J(C))\otimes\mathbb{C}^*$, by using an argument of Bardelli and Nori. Moreover we are able to compute the infinitesimal invariant in a very explicitely way.

The paper is organized as follows: section 1 is devoted to the definitions of the polygonal cycle and of the regulator map in the primitive higher Jacobian. In section 2 we study the infinitesimal deformations of the triples (C,p,q). In section 3 we introduce some properties of the adjoint of differential forms. This is the main tool to express the infinitesimal invariant of the corresponding normal function in section 4. In section 5 we prove non-triviality and Torelli-like Theorems. Finally, in section 6 we consider the genus 2 case concerning the independence of the cycles.

1 The cycle $P(C,p,q)$

1.1. We consider curves of genus $g \geq 2$ with the property that the surface $C - C = \{[x-y] \in J(C), x, y \in C\}$ intersects the subgroup of n-torsion points of $J(C)$. Equivalently, curves such that there exist different points $p, q \in C$ with $n[p-q] = 0$. We choose a parameter in $\mathbb{P}^1$ and we denote by h a rational function such that $div(h) = n \cdot p - n \cdot q$. The isomorphism classes of the objects (C,h,p,q) define an algebraic scheme. By the theory of the Hurwitz schemes its dimension is $2g-1$. We are interested in the image $\mathcal{X}_{g,n}$ of this scheme in the moduli space of 2-pointed curves, obtained by the obvious map, which sends (C,h,p,q) to (C,p,q). Clearly, the dimension of $\mathcal{X}_{g,n}$ is still $2g-1$.

1.2. Let C,h,p,q as above. We denote by $i_x : C \longrightarrow J(C)$ the embedding defined by $i_x(y) = [y-x]$. The curves $A_1 := Image(i_p)$, $A_i := A_1 + (i-1)[p-q]$, $i = 1, \ldots, n$ are isomorphic to C. Let $h_i : A_i \longrightarrow \mathbb{P}^1$ be the rational functions determined by h under the isomorphisms

$$\begin{aligned}\phi_i : C &\longrightarrow A_i \\ x &\longmapsto [x-p]+(i-1)[p-q].\end{aligned}$$

Observe that

$$0 \in A_1 \cap A_2, [p-q] \in A_2 \cap A_3, \ldots, (n-1)[p-q] \in A_n \cap A_1.$$

Hence we have a *polygon* contained in the Jacobian variety with vertexs over the multiples of the torsion point $[p-q]$. Since zeroes and poles of the functions h_i are exactly the vertices, the equality $div(h_1)+\cdots+div(h_n)=0$ holds. Then

$$P(C,p,q) := \sum_{i=1}^{n} (A_i, h_i)$$

is a well-defined cycle for the Quillen group $H^{g-1}(J(C), \mathcal{K}_g)$.

1.3. Here we recall some standard facts about higher Jacobians and regulators. We follow closely the notations used in [2].

Let us denote by $JK(J(C))$ the subgroup of the Deligne-Beilinson cohomology group $H_{\mathcal{D}}(J(C), \mathbb{Z}(g))$ given by:

$$\begin{aligned}JK(J(C)) =& H^{2g-2}(J(C),\mathbb{C})/(F^g H^{2g-2}(J(C),\mathbb{C}) + H^{2g-2}(J(C),\mathbb{Z}(g))) \\ &\cong F^1 H^2(J(C),\mathbb{C})^*/H_2(J(C),\mathbb{Z}(1)).\end{aligned}$$

Following Collino, we will refer to $JK(J(C))$ as the higher Jacobian of $J(C)$.

One defines a regulator map over the subgroup

$$BL(J(C)) \subset H^{g-1}(J(C), \mathcal{K}_g)$$

of cycles homologous to zero:

$$reg : BL(J(C)) \longrightarrow JK(J(C)).$$

The cycle $P(C,p,q)$ is homologous to zero. Indeed, denote by γ_i the preimage $h^{-1}(\lambda)$ of a path λ from 0 to ∞ in $\mathbb{P}^1$. By definition the cycle $P(C,p,q)$ is homologically trivial if the chain $\sum_{i=1}^{n} \gamma_i$ is a boundary in $J(C)$. This is so, since the integral of any holomorphic one-form along γ_i is zero, since it is equal to the integral of the trace of the form (which is equal to zero) along the path λ.

1.4. The definition of the cycle $P(C,p,q)$ depends on the choice of a parameter in $\mathbb{P}^1$. Different choices of the parameter change the cycle by an element in the image of the cup-product map

$$Chow^{g-1}(J(C)) \otimes \mathbb{C}^* \longrightarrow BL(J(C)).$$

So, it is convenient to consider the primitive higher Jacobian

$$PJK(J(C)) := PF^1H^2(J(C),\mathbb{C})^*/H_2(J(C),\mathbb{Z}(1)),$$

where $PF^1H^2(J(C),\mathbb{C})^*$ is the space orthogonal to the class of C. It is a consequence of [5], Proposition 3.1, that for C general (with our property) the Hodge-group of algebraic one-cycles has dimension 1.

We denote also by reg the regulator with image in the primitive higher Jacobian.

2 Infinitesimal deformations of (C, p, q)

We fix C and p, q as in the last section. Let h be, as above, a map $h : C \longrightarrow \mathbb{P}^1$ such that $h(p) = 0$ and $h(q) = \infty$.

2.1. Recall the Hurwitz formula $\omega_C \cong h^*(\omega_{\mathbb{P}^1}) \otimes \mathcal{O}_C(R)$, where R is the ramification divisor of h. Under our hypothesis, we can put the equality of effective divisors $R = R_0+(n-1)(p+q)$. Since $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-0-\infty)$, we get that the divisor R_0 belongs to the linear series $|\omega_C(p+q)|$. Moreover, this divisor corresponds to a meromorphic differential form $dh/h \in H^0(C, \omega_C(p + q))$. Observe that dh/h is $h^*(dz/z)$, where z is a standard parameter in $\mathbb{P}^1$. It vanishes exactly at the points in R_0 and it has poles in p and q with residues n and $-n$ respectively.

2.2. The vector space $H^1(C, T_C(-p - q))$ parametrizes the first-order deformations of (C, p, q). For an element $\eta \in H^1(C, T_C(-p - q))$, we denote by

$$\mathcal{C}_\eta \xrightarrow{\pi} Spec\,\mathbb{C}[\epsilon]/(\epsilon^2)$$

an infinitesimal deformation representing η, and P, Q sections of π with values p, q respectively at the point 0. We identify P, Q with its images in $\mathcal{C}_\eta$. It is a standard fact that the image of η by the forgetful map

$$H^1(C, T_C(-p - q)) \longrightarrow H^1(C, T_C) \cong Ext^1(C, \omega_C)$$

is represented by the extension:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \Omega^1_{\mathcal{C}_\eta\,|C} \longrightarrow \omega_C \longrightarrow 0.$$

Lemma 2.1. *The extension*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \Omega^1_{\mathcal{C}_\eta}(\log(P + Q))_{\,|C} \longrightarrow \omega_C(p + q) \longrightarrow 0.$$

represents the class

$$\eta \in Ext^1(\omega_C(p + q), \mathcal{O}_C) \cong H^1(C, T_C(-p - q)).$$

Proof. Consider a generator of the tangent space of $Spec\,\mathbb{C}[\epsilon]/(\epsilon^2)$ at the origin and lift it to a vector field on $\mathcal{C}_\eta$. By contracting with this vector field the following diagram holds:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega^1_{\mathcal{C}_\eta\,|C} & \longrightarrow & \omega_C \\
\downarrow & & \downarrow \\
\Omega^1_{\mathcal{C}_\eta}(\log(P+Q))_{|C} & \longrightarrow & \omega_C(p+q) \\
\downarrow & & \downarrow \\
\mathcal{O}_{p+q} & = & \mathcal{O}_{p+q} \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

Then, adding the kernels in the first two rows we reach to:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \Omega^1_{\mathcal{C}_\eta\,|C} & \longrightarrow & \omega_C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \Omega^1_{\mathcal{C}_\eta}(\log(P+Q))_{|C} & \longrightarrow & \omega_C(p+q) & \longrightarrow & 0,
\end{array}
$$

proving the lemma. □

2.3. We would like to identify the subspace $\mathbb{T}_h$ of the elements η corresponding to infinitesimal deformations $\mathcal{C}_\eta$ which preserve the condition $n[p-q]=0$. This is equivalent to say that h extends to $H:\mathcal{C}_\eta \longrightarrow \mathbb{P}^1$ such that $H^*(0)=n\cdot P$ and $H^*(\infty)=n\cdot Q$. Observe that in this case dh/h extends to the meromorphic form dH/H on $\mathcal{C}_\eta$. Hence dh/h belongs to the kernel of the coboundary operator in the long sequence of cohomology associated with the sequence of the lemma:

$$\partial_\eta : H^0(C,\omega_C(p+q)) \longrightarrow H^1(C,\mathcal{O}_C).$$

This means that $\mathbb{T}_h$ is contained in the orthogonal subspace of $< dh/h >$ with respect to the cup-product map

$$H^1(C,T_C(-p-q)) \otimes H^0(C,\omega_C(p+q)) \longrightarrow H^1(C,\mathcal{O}_C).$$

2.4. Since $\mathbb{T}_h$ has the expected dimension $2g-1=\dim \mathcal{X}_{\mathrm{g,n}}$, then it is exactly the orthogonal of dh/h and it is isomorphic to the tangent space $T_{\mathcal{X}_{g,n}}(C,p,q)$ at the general element.

2.5. By using the dual of the exact sequence

$$0 \longrightarrow \mathbb{T}_h \longrightarrow H^1(C, T_C(-p-q)) \stackrel{\cup dh/h}{\longrightarrow} H^1(C, \mathcal{O}_C) \longrightarrow 0$$

we can prove the following lemma, which summarizes the previous results:

Lemma 2.2. *The space $\mathbb{T}_h$ of the infinitesimal deformations of (C, p, q) preserving the condition $n[p-q] = 0$ is the orthogonal of dh/h by the cup product. Its dual is the cokernel of the map*

$$H^0(C, \omega_C(-p-q)) \longrightarrow H^0(C, \omega_C^{\otimes 2})$$

given by the multiplication with dh/h.

2.6. Remark. Assume that $p + q$ is not a g^1_2 linear series. In this case it is easy to check that $\mathbb{T}_h$ maps isomorphically into a subspace of $H^1(C, T_C)$, whose dual is the cokernel of the map given by the product with dh/h:

$$H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C^{\otimes 2}(p+q)).$$

3 The adjoint image

In this section we recover some basic properties on adjunction of sections of a line bundle on a curve. The proof of (3.1) can be found in [3], or in section 3 of [6].

3.1. Let C be a curve equipped with an invertible sheaf L. Let $\eta \in Ext^1(L, \mathcal{O}_C) \cong H^1(C, L^*)$ be an extension class represented by the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow L \longrightarrow 0.$$

Denote by $K(\eta)$ the kernel of the coboundary map

$$\partial_\eta : H^0(C, L) \longrightarrow H^1(C, \mathcal{O}_\mathcal{C}),$$

sending x to $\eta \cup x$.

Given $\alpha_1, \alpha_2 \in K(\eta) \subset H^0(C, L)$, there exist, by construction, liftings $\tilde{\alpha}_1, \tilde{\alpha}_2 \in H^0(C, \mathcal{E})$. We denote by $\omega_{\eta, \tilde{\alpha}_1, \tilde{\alpha}_2} \in H^0(C, L)$ the image of the wedge product $\tilde{\alpha}_1 \wedge \tilde{\alpha}_2$ under the map:

$$\wedge^2 H^0(C, \mathcal{E}) \longrightarrow H^0(C, \wedge^2 \mathcal{E}) \cong H^0(C, L).$$

It is easy to check that different choices of the liftings $\tilde{\alpha}_1, \tilde{\alpha}_2$ change the section $\omega_{\eta,\tilde{\alpha}_1,\tilde{\alpha}_2}$ by an element in the subspace generated by the initial forms α_1 and α_2. Therefore we get a well-defined element

$$[\omega_{\eta,\alpha_1,\alpha_2}] := [\omega_{\eta,\tilde{\alpha_1},\tilde{\alpha}_2}] \in H^0(C,L))/ < \alpha_1, \alpha_2 >,$$

called the adjoint image of α_1 and α_2.

3.2. One has the following non-vanishing result for the adjoint image:

Proposition 3.1. *Let D be the fixed divisor of the sub-linear system of $|\,L\,|$ generated by two linearly independent sections $\alpha_1, \alpha_2 \in K(\eta)$. Then $[\omega_{\eta,\alpha_1,\alpha_2}] = 0$ if and only if η belongs to the kernel of the map*

$$H^1(C, L^*) \longrightarrow H^1(C, L^*(D)).$$

Hence, $D = 0$ implies

$$\omega_{\eta,\tilde{\alpha}_1,\tilde{\alpha}_2} \notin < \alpha_1, \alpha_2 > .$$

3.3. Typically we will apply the above construction in the following case: L will be the sheaf $\omega_C(p+q)$ and $\eta \in H^1(C, T_C(-p-q))$ will be a first order deformation of (C, p, q). We keep the notation h for the degree n map $C \longrightarrow \mathbb{P}^1$. We will assume that η belongs to the subspace of the deformations preserving the condition $n[p-q] = 0$. In particular dh/h lives in $K(\eta)$.

Let $\alpha \in K(\eta)$ be a differential form linearly independent of dh/h. Then, the construction described in 3.1, with $L = \omega_C(p+q)$, yields a meromorphic differential form $\omega_{\eta,\tilde{\alpha},dH/H} \in H^0(C, \omega_C(p+q))$, where H is a rational function on C_η extending h.

3.4. Now we give a different approach to adjunction that points out that it can be described as a Massey product.

As in the beginning of the section we fix a curve C, an invertible sheaf L and elements $\alpha_1, \alpha_2 \in H^0(C, L)$, $\xi \in H^1(C, L^*)$ such that $\alpha_1 \cup \xi = \alpha_2 \cup \xi = 0$. Assume that $< \alpha_1, \alpha_2 > \subset H^0(C, L)$ is base-point-free. Let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow L \longrightarrow 0$$

be an extension representing $\xi \in H^1(C, L^*) \cong Ext^1(L, \mathcal{O}_C)$. Then, choosing liftings of α_1, α_2 to global sections of $\mathcal{E}$, is equivalent to giving a diagram as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^* & \longrightarrow & < \alpha_1, \alpha_2 > \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

By definition, the dual of $L^* \longrightarrow \mathcal{O}_C$ corresponds to the adjoint image defined in (3.1). By taking cohomology in the diagram one finds that the adjunction image is simply a section of L mapping to ξ by the coboundary map attached to the first row in the diagram

$$H^0(C, L) \longrightarrow H^1(C, L^*).$$

Observe that this proves the Proposition 3.1 for the case $D = 0$. The general case can be proved in the same way by changing slightly the exact sequence above.

We use now Dolbeault resolution of the exact sequence

$$0 \longrightarrow L^* \longrightarrow < \alpha_1, \alpha_2 > \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0$$

to find an explicit formula for the adjoint image as follows. A representative of ξ is locally of the form $s\,\overline{dz}$, where $s \in \mathcal{C}^\infty(L^*)$. The conditions $\alpha_i \cup \xi$ translates into

$$\alpha_i \cdot s\,\overline{dz} \;=\; \overline{\partial}\rho_i$$

where ρ_i are $\mathcal{C}^\infty$ functions on C. By the definition of the coboundary map, the section represented by

$$\rho_1\,\alpha_2 - \rho_2\,\alpha_1$$

maps to ξ. Therefore, also represents the adjoint image (defined up to an element in $< \alpha_1, \alpha_2 >$). Observe that this expression is just the definition of the Massey product in Dolbeault cohomology.

4 The infinitesimal invariant

Now we will assume that the curve C belongs to a family of curves $\pi : \mathcal{C} \longrightarrow B$ such that the cycle $P(C, p, q)$ moves with the curve in the corresponding family of Jacobian varieties $\pi : \mathcal{JC} \longrightarrow B$, that is to say: there exists a map over B, $H : \mathcal{C} \longrightarrow \mathbb{P}^1 \times B$ and two sections $P, Q : B \longrightarrow \mathcal{C}$ of π, such that for all $b \in B$, the map $h_b : \mathcal{C}_b \longrightarrow \mathbb{P}^1 \times \{b\}$ induced by H has degree n and $h_b^*(0) = nP_b$, $h_b^*(\infty) = nQ_b$.

In this case the definition of normal function and the construction of the attached Griffiths' infinitesimal invariant can be extended to our situation, as we recall briefly below (we refer to [2] and [7] for the details).

4.1. Associated to π we can consider the family of primitive higher Jacobians $\mathcal{PJK}(\mathcal{JC}) \longrightarrow B$. We will consider a relative cycle in $\mathcal{JC}$

$$\mathcal{P} = \sum_{i=1}^{n} (\mathcal{A}_i, H_i)$$

such that $P_b \in BL(\mathcal{J}C_b)$, $\forall b \in B$. Then one has a holomorphic section

$$\nu : B \longrightarrow \mathcal{PJK}(\mathcal{JC})$$
$$b \longmapsto reg_b(P_b)$$

where reg_b stands for the regulator map in $\mathcal{J}C_b$.

4.2. Similarly to the classical case one can define the invariant $\delta\nu$ as an element in the middle cohomology of the complex

$$F^g\mathcal{H}^{2g-2} \longrightarrow F^{g-1}\mathcal{H}^{2g-2} \otimes \Omega^1_B \longrightarrow F^{g-2}\mathcal{H}^{2g-2} \otimes \Omega^2_B$$

where $F^i\mathcal{H}^j$ are the Hodge subsheaves of $\mathcal{H}^j := \mathcal{R}^j\pi_*\mathbb{C} \otimes \mathcal{O}_B$. Briefly we recall that the infinitesimal invariant is defined by applying the Gauss-Manin connection to a lifting of the normal function to the Hodge bundle. Therefore the non vanishing of this invariant implies that the normal function is not torsion.

4.3. From now on we will assume that the dimension of B is $2g-1$ and that the corresponding moduli map $\Phi : B \longrightarrow \mathcal{X}_{g,n}$ is generically finite. Therefore, for a general $b \in B$, $d\Phi$ provides an isomorphism between $T_B(b)$ and $\mathbb{T}_{h_b} =< dh_b/h_b >^{\perp} \subset H^1(C_b, T_{C_b}(-p-q))$. We point out that locally the normal function can be thought as a normal function on an open set of the moduli space $\mathcal{X}_{g,n}$ of elements (C,p,q). Moreover, assuming that $h^0(C, \mathcal{O}_C(p+q)) = 1$, the normal function can be defined on an open set of the image of $\mathcal{X}_{g,n}$ in the moduli space of curves. In this case the description of the tangent space is given by the Remark 2.6.

4.4. Next, we consider the restriction of the complex in 4.2 to a generic $b \in B$ and we project onto the graded primitive part. With our identifications one obtains:

$$P^{g,g-2} \longrightarrow \mathbb{T}^*_{h_b} \otimes P^{g-1,g-1} \longrightarrow \wedge^2\mathbb{T}^*_{h_b} \otimes P^{g-2,g}.$$

By dualizing, we see $\delta\nu([C_b])$ as a linear map on the middle cohomology of the complex

$$\wedge^2\mathbb{T}_{h_b} \otimes P^{2,0} \longrightarrow \mathbb{T}_{h_b} \otimes P^{1,1} \longrightarrow P^{0,2}.$$

4.5. In section 5 of [2] Collino relates the infinitesimal invariant of a normal function with forms obtained by adjunction. Taking care of minor changes one checks that the same computation works in our case showing the following statement:

Theorem 4.1. *With the above notations, fix $C = C_b$, for a generic b. For every $\eta \in \mathbb{T}_h$, $\alpha \in K(\eta)$ and μ (0,1) form such that $\int_C \alpha \wedge \mu = 0$, it holds:*

$$\delta\nu[C](\eta \otimes (\alpha \wedge \mu)) = n \int_C \mu \wedge \omega_{\eta,\alpha},$$

where $\omega_{\eta,\alpha,}$ is the holomorphic part of the form $\omega_{\eta,\tilde{\alpha},dH/H}$ obtained by adjunction on C.

We point out that one has the decomposition

$$H^0(C, \omega_C(p+q)) = H^0(C, \omega_C) \oplus < dh/h >,$$

hence the form $\omega_{\eta,\tilde{\alpha},dH/H}$ decomposes into an holomorphic part and a multiple of dh/h. Notice also that different choices of the liftings change the adjunction form by a multiple of α, hence orthogonal to μ.

5 Non-triviality of the cycle

5.1. Now we can prove the regulator of our cycle is not torsion in general. Equivalently, we have to prove the non-vanishing of the infinitesimal invariant.

Theorem 5.1. *For a general (C, p, q), the cycle $P(C, p, q)$ is not torsion.*

Proof. We proceed by contradiction. Assume the infinitesimal invariant is trivial and then, by Theorema 4.1, we have

$$\int_C \mu \wedge \omega_{\eta,\alpha} = 0;$$

for every η, α and μ as in the Theorem. We fix an holomorphic form $\alpha \in H^0(C, \omega_C)$ such that the pencil $< \alpha, dh/h >\subset H^0(C, \omega_C(p+q))$ is base point free. Let us define

$$\mathbb{T}_{h,\alpha} = \{\eta \in \mathbb{T}_h \,|\, \eta \cup \alpha = 0\}.$$

Set V the 2 dimensional vector space generated by dh/h and α. The base point free condition implies that V fits in the following exact sequence

$$0 \longrightarrow T_C(-p-q) \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \omega_C(p+q) \longrightarrow 0.$$

Hence, one has

$$0 \to V \longrightarrow H^0(C, \omega_C(p+q)) \longrightarrow H^1(C, T_C(-p-q)) \xrightarrow{f} V \otimes H^1(C, \mathcal{O}) \to 0,$$

where f stands for the map $(\cup\alpha, \cup dh/h)$. Hence $\mathbb{T}_{h,\alpha} \cong H^0(C, \omega_C(p+q))/V$ has dimension $g-1$. We denote by $< \alpha >^{\perp}$ the subspace of $H^1(C, \mathcal{O}_C)$ orthogonal to α with respect to the standard pairing

$$(\beta, \mu) \mapsto \int_C \beta \wedge \mu.$$

The vanishing of the invariant says that the pairing

$$\begin{aligned} b_\alpha : \mathbb{T}_{h,\alpha} \otimes < \alpha >^\perp &\longrightarrow \mathbb{C} \\ (\eta, \mu) \quad &\longmapsto \int_C \mu \wedge \omega_{\eta,\alpha} \end{aligned}$$

is trivial for our α. Hence, for all $\eta \in \mathbb{T}_{h,\alpha}$ one gets $\omega_{\eta,\alpha} \in < \mu >^\perp$. Since the adjunction map for fixed α,

$$\mathbb{T}_{h,\alpha} \longrightarrow H^0(C, \omega_C)$$

is linear and injective by Proposition 3.1, one has that the image has dimension $g-1$ and so is equal to $< \mu >^\perp$. Then, there exists a deformation $\eta_0 \in \mathbb{T}_{h,\alpha}$ such that the adjunction form $\omega_{\eta_0,\alpha}$ is a multiple of α, this contradicts Proposition 3.1. □

5.2. Observe that one deduces from the proof the following statement: if $< \alpha, dh/h >$ is base point free, then the pairing b_α defined above is non-degenerated.

Assume now that $< \alpha, dh/h >$ is not base point free and denote by F the base locus of the corresponding linear system. Then $V =< dh/h, \alpha >$ fits now in the exact sequence:

$$0 \longrightarrow T_C(-p-q+F) \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \omega_C(p+q-F) \longrightarrow 0.$$

One deduces from this that cupping with α and dh/h induces a surjective map

$$H^1(C, T_C(-p-q+F)) \longrightarrow V \otimes H^1(C, \mathcal{O}_C).$$

Since $H^1(C, T_C(-p-q)) \longrightarrow H^1(C, T_C(-p-q+F))$ is also surjective, the kernel $\mathbb{T}_{h,\alpha}$ of the composition has dimension $g-1$ and the pairing b_α is well defined.

Let x be a point in the support of F. Denote by $\xi_x \in H^1(C, T_C)$ the Schiffer variation attached to this point. By definition of x and the general properties of the Schiffer variations we have

$$\xi'_x \cup \alpha = 0 \qquad\qquad \xi'_x \cup dh/h = 0,$$

where ξ'_x stands for a lifting of ξ_x to $H^1(C, T_C(-p-q))$. Therefore $\xi'_x \in \mathbb{T}_{h,\alpha}$. Since the deformation ξ'_x is in the kernel of the map

$$H^1(C, T_C(-p-q)) \longrightarrow H^1(C, T_C(-p-q+x)),$$

Proposition 3.1 implies that $b_\alpha(\xi'_x, \mu) = 0$ for all $\mu \in < \alpha >^\perp$. Then b_α degenerates.

Proposition 5.2. *If C is general, the rational function h can be recovered, up to constant, by the infinitesimal invariant.*

Proof. Since C is general we can assume that the divisor R_0 is simple. We deduce from the discusion above that R_0 is recovered from the infinitesimal invariant as the locus where the linear form b_α is identicaly 0. Since $\omega_C(-R_0) \cong \mathcal{O}_C(p+q)$, we get also the points p, q and then h is given by the equality $div(h) = n \cdot p - n \cdot q$. □

6 Independence of polygonal cycles for $g = 2$

6.1. In this section we restrict ourselves to the case $g = 2$. Since the difference map $C \times C \longrightarrow J(C), (x, y) \mapsto [x-y]$ is surjective, any torsion point gives rise to a polygonal cycle. Then we have countably many meromorphic functions on C, defined up to constant, each of them giving a map into $\mathbb{P}^1$ totally ramified at 0 and ∞.

6.2. We consider the subgroup $H(C)$ of the multiplicative group of the function field $K(C)^*$ generated by these functions. Then there is a natural map

$$H(C) \longrightarrow H^1(J(C), \mathcal{K}_2).$$

By using a standard monodromy argument we can see that the image of this map is infinitely generated. We do not enter into the details since this result is not new: Collino used ideas of Nori [4] and Bardelli[1] to get the same result.

6.3. As a first step to understand the image of this regulator we will compute the infinitesimal invariant analyticaly using meromorphic differential forms.

We use the notations n, h, p, q and R_0 as in section 2. Fix a point $x \in C$ different from p, q, and the points of the support of R_0. Let α, β be a basis of holomorphic forms such that α vanishes at x. Let z_x be a local coordinate in a small neighborhood $\mathcal{U}_x$ of the point x, and let ξ_x be the Schiffer variation represented in Dolbeault cohomology by the element

$$\frac{\overline{\partial}\rho_x}{z_x} \frac{\partial}{\partial z_x},$$

where ρ_x stands for a $\mathcal{C}^\infty$ function on C which is constant 1 in $\mathcal{U}_x$ and constant 0 on the complementary of a small open set containing the closure of $\mathcal{U}_x$.

We denote by $\xi_x^h \in H^1(C, T_C(-p-q))$ the unique element mapping to ξ_x and such that $\xi_x^h \cup dh/h = 0$. The condition $\xi_x^h \cup \alpha = 0$ is automatically fullfiled. Finally, we put $z_x\, A_x^0(z_x)\, dz_x$ and $B_x(z_x)\, dz_x$ for the restrictions to $\mathcal{U}_x$ of α and β respectively. Then:

Proposition 6.1. *The following formula holds*

$$\delta\nu([C])(\xi_x^h \otimes (\alpha \wedge \xi_x^h \beta)) \;=\; 2\pi i\, n\, A_x^0(0) B_x(0)\, \frac{\frac{dh}{h}(x) \cdot \frac{dh}{h}(\sigma(x))}{\frac{dh}{h}(x) + \frac{dh}{h}(\sigma(x))},$$

σ being the hyperelliptic involution on C.

Proof. We do the computation in several steps:

Step 1. Computation of ξ_x^h.

By definition is represented by an element of the form

$$\frac{\overline{\partial}\rho_x}{z_x}\frac{\partial}{\partial z_x} \;+\; c_1 \overline{\partial}(\rho_p)\frac{\partial}{\partial z_p} \;+\; c_2 \overline{\partial}(\rho_q)\frac{\partial}{\partial z_q},$$

for some constants c_1, c_2, where ρ_p, ρ_q are defined in a similar way that ρ_x and z_p, z_q are local coordinates at p and q such that h has local expressions z_p^n and z_q^{-n} respectively. We will denote by $A_p(z_p)dz_p, A_q(z_q)dz_q$ and $B_p(z_p)dz_p, B_q(z_q)dz_q$ the local expressions of α and β.

We impose

$$\xi_x^h \cup dh/h \;=\; \overline{\partial}\rho$$

and we get the existence of a meromorphic function $f \in H^0(C, \mathcal{O}_C(p+q+x))$ such that

$$\rho \;=\; f + \frac{\rho_x g(z_x)}{z_x} \;+\; \frac{c_1 n \rho_p}{z_p} \;-\; \frac{c_2 n \rho_q}{z_q},$$

where $g(z_x)dz_x$ is the local expression of dh/h. Observe that this equality give us the local expressions of f at x, p and q. By imposing that the sum of the residues of the forms $f\alpha$ and $f\beta$ are zero, we get

$$\begin{aligned} -nc_1 A_p(0) + nc_2 A_q(0) = 0 \\ -nc_1 B_p(0) + nc_2 B_q(0) - g(0)B_x(0) = 0. \end{aligned}$$

Hence

$$c_1 = \frac{g(0)A_q(0)B_x(0)}{n\, A_p(0)B_q(0) - n\, A_q(0)B_p(0)}$$

$$c_2 = \frac{g(0)A_p(0)B_x(0)}{n\, A_p(0)B_q(0) - n\, A_q(0)B_p(0)}$$

Step 2. Computation of the adjoint form.

Here we use the description given in 3.4 in terms of representatives in Dolbeault cohomology. The form $\omega_{\xi_x^h,\alpha,dh/h}$ (defined in the quotient by $<\alpha, dh/h>$) is represented by

$$(A_x^0\rho_x + c_1\rho_p A_p + c_2\rho_q A_q)dh/h \,-\, (f + \frac{\rho_x g(z_x)}{z_x} + \frac{c_1 n\rho_p}{z_p} - \frac{c_2 n\rho_q}{z_q})\alpha.$$

It is a straightforward computation to simplify the expression above to find that

$$[\omega_{\xi_x^h,\alpha,dh/h}] \,=\, [-f\alpha].$$

Step 3. Final computation.

By definition we have

$$\delta\nu([C])(\xi_x^h \otimes (\alpha \wedge \xi_x^h \beta)) = n\int_C (\frac{\overline{\partial}\rho_x}{z_x}\frac{\partial}{\partial z_x} + c_1\overline{\partial}\rho_p \frac{\partial}{\partial z_p} + c_2\overline{\partial}\rho_q \frac{\partial}{\partial z_q})\beta \wedge (-f\alpha).$$

We point out that we should replace the adjoint form by its holomorphic part, but adding to $-f\alpha$ a multiple of dh/h does not modify the value of the integral.

Due to the definition of the functions ρ_x, ρ_p and ρ_q the integral can be separated into three parts, each one on a small open set around the points. Then we only need to compute the residues at each point. The result we get is

$$2\pi i\, n\, (A_x^0(0)B_x(0)g(0) + c_1^2 n A_p(0)B_p(0) - c_2^2 n A_q(0)B_q(0)).$$

Next, by replacing the values of c_1 and c_2 found in the step 1 and simplifying:

$$\delta\nu([C])(\xi_x^h \otimes (\alpha \wedge \xi_x^h \beta)) \,=\, ng(0)B_x(0)(A_x^0(0) - \frac{1}{n}\frac{B_x(0)g(0)}{\frac{B_q(0)}{A_q(0)} - \frac{B_p(0)}{A_p(0)}}).$$

Finally we compute the residues of the meromorphic differential form

$$\beta/\alpha\, dh/h$$

and we get

$$0 \,=\, n\frac{B_p(0)}{A_p(0)} - n\frac{B_q(0)}{A_q(0)} + \frac{B_x(0)}{A_x^0(0)}g(0) + \frac{B_{\sigma(x)}(0)}{A_{\sigma(x)}^0(0)}g_\sigma(0),$$

where the functions $A_{\sigma(x)}^0, B_{\sigma(x)}$ and $g_{\sigma(x)}$ come, with the obvious meaning, from the local expressions at $\sigma(x)$ of α, β and dh/h.

Combining the last two formulas and using that

$$B_x(0)/A_x^0(0) = B_{\sigma(x)}(0)/A_{\sigma(x)}^0(0)$$

it is easy to end the proof of the Proposition. □

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