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SMALL HANKEL OPERATORS ON GENERALIZED WEIGHTED FOCK SPACES

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Abstract. In this work we characterize the boundedness, compactness and membership in the Schatten class of small Hankel operators on generalized weighted Fock spaces $F^{p,\ell}_\alpha(\omega)$ associated to an $\mathcal{A}^\ell_p$ weight ω, for $1<p<\infty$, $\ell\geq 1$, and $\alpha>0$.

1. Introduction

The goal of this work is to characterize the boundedness, compactness and membership in the Schatten class of small Hankel operators on generalized weighted Fock spaces.

Let ω be a *weight*, that is, a positive locally integrable function on $\mathbb{C}$. For $1\leq \ell,p<\infty$ and $\alpha\geq 0$, we define the space $L^{p,\ell}_\alpha(\omega):=L^p(\mathbb{C},e^{-\frac{\alpha p}{2}|z|^{2\ell}}\omega\,dA)$ and the *generalized weighted Fock space* $F^{p,\ell}_\alpha(\omega):=H(\mathbb{C})\cap L^{p,\ell}_\alpha(\omega)$, where $H(\mathbb{C})$ denotes the space of entire functions and $dA(z)=\frac{1}{\pi}\,dxdy$. For the weight $\omega_{\rho,p}(z)=(1+|z|)^{\rho p}$, $\rho\in\mathbb{R}$, the spaces $L^{p,\ell}_\alpha(\omega_{\rho,p})$ and $F^{p,\ell}_\alpha(\omega_{\rho,p})$ are simply denoted by $L^{p,\ell}_{\alpha,\rho}$ and $F^{p,\ell}_{\alpha,\rho}$. As usual, $L^{\infty,\ell}_{\alpha,\rho}$ consists of all measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{L^{\infty,\ell}_{\alpha,\rho}}:=\operatorname*{ess\,sup}_{z\in\mathbb{C}}|f(z)|(1+|z|)^\rho e^{-\frac{\alpha}{2}|z|^{2\ell}}<\infty.$$

Moreover, $F^{\infty,\ell}_{\alpha,\rho}:=L^{\infty,\ell}_{\alpha,\rho}\cap H(\mathbb{C})$, and $\mathfrak{f}^{\infty,\ell}_{\alpha,\rho}$ is the closure of the space of holomorphic polynomials in $F^{\infty,\ell}_{\alpha,\rho}$. The spaces $F^{p,\ell}_{\alpha,\rho}$ and $F^{p,\ell}_\alpha:=F^{p,\ell}_{\alpha,0}$ are called *generalized Fock-Sobolev spaces* and *generalized Fock spaces*, respectively. It is worth to mention that the generalized Fock-Sobolev spaces appear naturally when considering the derivatives of functions in generalized Fock spaces. Namely, $f\in F^{p,\ell}_\alpha$ if and only if $f^{(k)}\in F^{p,\ell}_\alpha((1+|z|)^{kp(1-2\ell)})$ (see [5, Theorem 1.4]). This is true even in some weighted setting (see [7, Theorem 1.1]).

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As we will show later, these Fock-Sobolev spaces also appear when we study the membership to the Schatten class for small Hankel operators on generalized weighted Fock spaces (see Theorem 1.2 below).

It is clear that $L^{2,\ell}_\alpha$ is a Hilbert space with the inner product

$$\langle f,\, g\rangle_\alpha := \int_{\mathbb{C}} f(z)\,\overline{g(z)}\,e^{-\alpha|z|^{2\ell}}\,dA(z),$$

and it is well known that $F^{2,\ell}_\alpha$ is a closed subspace of $L^{2,\ell}_\alpha$. The *Bergman projection* for $F^{2,\ell}_\alpha$ is the orthogonal projection P^ℓ_α from $L^{2,\ell}_\alpha$ onto $F^{2,\ell}_\alpha$, which is given by

$$(P^\ell_\alpha\psi)(z) = \int_{\mathbb{C}} K^\ell_\alpha(z,w)\,\psi(w)\,e^{-\alpha|w|^{2\ell}}\,dA(w) \qquad (z\in\mathbb{C},\,\psi\in L^{2,\ell}_\alpha),$$

where K^ℓ_α is the *Bergman kernel* for $F^{2,\ell}_\alpha$. The boundedness properties of this projection on the spaces $L^{p,\ell}_\alpha$ have been thoroughly studied in [3, 4].

As it is well known, the weights for which the Bergman projection is bounded in the classical weighted Bergman spaces on the unit ball of $\mathbb{C}^n$ are characterized by a Muckenhoupt type condition. They are the so called Békollé-Bonami weights (see [2] and [1]), which have become a key tool in the study of weighted norm inequalities for the Bergman projection in different settings of complex analysis (see, for instance, [12], [10], [11], and the references therein).

In the Fock setting, the weights ω for which P^ℓ_α is bounded from $L^{p,\ell}_\alpha(\omega)$ onto $F^{p,\ell}_\alpha(\omega)$ are those in the class $\mathcal{A}^\ell_p$. This result was proved for $\ell=1$ in [8], and extended to $\ell>1$ in [6]. The class $\mathcal{A}^\ell_p$ is defined as follows.

For $1<p<\infty$, $\mathcal{A}^\ell_p$ is the set of all weights ω such that

$$\sup_{z\in\mathbb{C}} \frac{\omega(D^\ell_z)\,\big(\omega'(D^\ell_z)\big)^{p/p'}}{|D^\ell_z|^p} < \infty, \tag{1.1}$$

where $D^\ell_z = \{w\in\mathbb{C} : |w-z| < (1+|z|)^{1-\ell}\}$, p' is the conjugate exponent of p, $\omega' = \omega^{-p'/p}$, $|D^\ell_z| := \int_{D^\ell_z} dA = (1+|z|)^{2(1-\ell)}$, $\omega(D^\ell_z) := \int_{D^\ell_z} d\omega$, and $d\omega := \omega\,dA$. It is worth to mention that if we replace D^ℓ_z by $D^\ell_{z,\varrho} = \{w\in\mathbb{C} : |w-z| < \varrho(1+|z|)^{1-\ell}\}$, for some ρ, in (1.1), we obtain the same class of weights.

One advantage of considering the case $\ell>1$ is that it covers a wider range of weights, for instance, exponential polynomial weights *i.e.* $\omega(z) = e^{q(|z|)}$, where q is a real polynomial. Indeed, it is proved in [6] that for such weights the boundedness of P^ℓ_α on $L^{p,\ell}_\alpha(\omega)$ is equivalent to the condition $\deg q \le \ell$.

Our first result gives a complete description of the boundedness and compactness of the small Hankel operators on our weighted Fock spaces $F^{p,\ell}_\alpha(\omega)$, $\omega\in\mathcal{A}^\ell_p$. We consider the small Hankel operators defined on the space

$$E := \{f\in H(\mathbb{C}) : |f(z)| = O(e^{\tau|z|^\ell}), \text{ for some } \tau>0\}$$

of entire functions of order ℓ and finite type, which is a dense subspace of $F^{p,\ell}_\alpha(\omega)$ (see [6, Proposition 5.6] and Proposition 2.5 below).

Theorem 1.1. *Let $1<p<\infty$, $\alpha>0$, and $\omega\in\mathcal{A}^\ell_p$. For $\beta\in(0,\frac{3}{2}\alpha)$ and $b\in F^{\infty,\ell}_\beta$, let $\mathfrak{h}^\ell_{b,\alpha}$ be the small Hankel operator defined by $\mathfrak{h}^\ell_{b,\alpha}f := \overline{P^\ell_\alpha(\overline{f}\,b)}$, $f\in E$. Then $\mathfrak{h}^\ell_{b,\alpha}$ extends to a bounded (compact) operator from $F^{p,\ell}_\alpha(\omega)$ to $\overline{F^{p,\ell}_\alpha(\omega)}$ if and only if $b\in F^{\infty,\ell}_{\frac{\alpha}{2}}$ (respectively, $b\in\mathfrak{f}^{\infty,\ell}_{\frac{\alpha}{2}}$). Moreover, $\|\mathfrak{h}^\ell_{b,\alpha}\|_{F^{p,\ell}_\alpha(\omega)} \simeq \|b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}$.*

The hypothesis $\beta\in(0,\frac{3}{2}\alpha)$ assures that the small Hankel operator $\mathfrak{h}^\ell_{b,\alpha}$ is well defined, as we will see at the beginning of the proof of the above theorem.

Finally, we characterize the membership to the Schatten class of our small Hankel operators.

Theorem 1.2. *Let $1<p<\infty$, $\alpha>0$, $\omega\in\mathcal{A}^\ell_p$, and $b\in\overline{F^{\infty,\ell}_\beta}$, for some $\beta\in(0,\frac{3}{2}\alpha)$. Then $\mathfrak{h}^\ell_{b,\alpha}$ belongs to the Schatten class $S_p(F^{2,\ell}_\alpha(\omega),\overline{F^{2,\ell}_\alpha(\omega)})$ if and only if $b\in F^{p,\ell}_{\frac{\alpha}{2},\frac{2(\ell-1)}{p}}$. Moreover, $\|\mathfrak{h}^\ell_{b,\alpha}\|_{S_p(F^{2,\ell}_\alpha(\omega),\overline{F^{2,\ell}_\alpha(\omega)})}\simeq\|b\|_{F^{p,\ell}_{\frac{\alpha}{2},\frac{2(\ell-1)}{p}}}$.*

Our theorems extent the characterization of boundedness, compactness and membership to the Schatten class of the small Hankel operators on the generalized Fock spaces to the setting of the generalized weighted Fock spaces $F^{p,\ell}_\alpha(\omega)$, where ω is an $\mathcal{A}^\ell_p$ weight (see [9, 16, 5] for the known unweighted cases). The main tools to prove these results are a weak decomposition for the Bergman kernel associated to the projection P^ℓ_α (see (2.8) below) with precise weighted estimates of its terms (see (2.13) below) and the main properties of the $\mathcal{A}^\ell_p$ weights. The fact that both the weak decomposition (2.8) and the upper bound in (2.13) do not depend on the weight ω explains somehow that the characterizations obtained in Theorems 1.1 and 1.2 are independent of ω.

The paper is organized as follows. In Section 2 we collect some necessary results on Fock spaces associated to $\mathcal{A}^\ell_p$ weights that we need to prove our results. Finally, Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2.

Notations

Along the paper, unless otherwise stated, α, ℓ, and p are real numbers such that $\alpha>0$, $\ell\ge1$, and $p>1$. As usual, the notation $\Phi\lesssim\Psi$ ($\Psi\gtrsim\Phi$) means that there exists a constant $C>0$, which does not depend on the involved variables, such that $\Phi\le C\,\Psi$. We write $\Phi\simeq\Psi$ if $\Phi\lesssim\Psi$ and $\Psi\lesssim\Phi$.

2. Fock spaces associated to $\mathcal{A}^\ell_p$ weights

In this section we collect some results on $\mathcal{A}^\ell_p$-weighted Fock spaces that we will use in the proofs of our results.

Lemma 2.1. *For any $\omega\in\mathcal{A}^\ell_p$, the dual $(L^{p,\ell}_\alpha(\omega))^*$ of $L^{p,\ell}_\alpha(\omega)$, with respect to the pairing $\langle\cdot,\cdot\rangle_\alpha$, is $L^{p',\ell}_\alpha(\omega')$. Namely, the mapping*

$$g\in L^{p',\ell}_\alpha(\omega')\longmapsto\langle\cdot,g\rangle_\alpha\in(L^{p,\ell}_\alpha(\omega))^*$$

is an isometric antilinear isomorphism.

Proof. It is an immediate consequence of the classical $L^p(\mathbb{C})-L^{p'}(\mathbb{C})$ duality and the fact that, for any weight ω and $1<q<\infty$, the operator $\Phi:L^{q,\ell}_\alpha(\omega)\to L^q(\mathbb{C})$, defined by

$$(\Phi g)(z)=g(z)e^{-\frac{\alpha}{2}|z|^{2\ell}}\omega(z)^{1/q}\qquad(g\in L^{q,\ell}_\alpha(\omega),\,z\in\mathbb{C}).$$

is an isometric linear isomorphism. □

Lemma 2.2 ([5, Lemma 2.15], [6, Theorem 1.1 and Proposition 5.7])**.**

a) *P^ℓ_α is a bounded projection from $L^{p,\ell}_\alpha(\omega)$ onto $F^{p,\ell}_\alpha(\omega)$, for any $1\le p<\infty$ and $\omega\in\mathcal{A}^\ell_p$.*

b) $P^\ell_\alpha f = f$, *for every* $f \in F^{1,\ell}_\beta$ *and* $0 < \beta < 2\alpha$.

c) *If* $1 < p < \infty$ *and* $0 < \gamma < \alpha < \beta$, *we have the embedding* $L^{\infty,\ell}_\alpha \hookrightarrow L^{p,\ell}_\alpha(\omega) \hookrightarrow L^{1,\ell}_\beta$, *for any* $\omega \in \mathcal{A}^\ell_p$.

As it is usual, the duality $L^{p,\ell}_\alpha(\omega)$-$L^{p',\ell}_\alpha(\omega')$ (see Lemma 2.1) together with the boundedness of P^ℓ_α on $L^{p,\ell}_\alpha(\omega)$, for $\omega \in \mathcal{A}^\ell_p$, gives the duality $F^{p,\ell}_\alpha(\omega)$-$F^{p',\ell}_\alpha(\omega')$:

Lemma 2.3. *Let* $\omega \in \mathcal{A}^\ell_p$. *Then the dual* $(F^{p,\ell}_\alpha(\omega))^*$ *of* $F^{p,\ell}_\alpha(\omega)$, *with respect to the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$, *is* $F^{p',\ell}_\alpha(\omega')$, i.e. *the mapping*

$$g \in F^{p',\ell}_\alpha(\omega') \longmapsto \langle\,\cdot\,, g\rangle_\alpha \in (F^{p,\ell}_\alpha(\omega))^*$$

is a topological antilinear isomorphism.

The Bergman kernel K^ℓ_α is given in terms of the classical two parametric Mittag-Leffler functions

$$E_{a,b}(\lambda) := \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(ak+b)} \qquad (\lambda \in \mathbb{C},\ a, b > 0)$$

by the formula

$$K^\ell_\alpha(w,z) = K^\ell_{\alpha,z}(w) = \ell\alpha^{1/\ell} E_{1/\ell,1/\ell}(\alpha^{1/\ell} w\bar{z}) \qquad (w, z \in \mathbb{C}). \tag{2.1}$$

Precise pointwise estimates of K^ℓ_α can be given in terms of the functions $\psi^\ell_{\alpha,\beta} := e^{-\frac{\alpha}{2}\phi^\ell_\beta}$, for $0 < \beta < \frac{\pi}{2\ell}$, where

$$\phi^\ell_\beta(z,w) := \begin{cases} |z|^{2\ell} + |w|^{2\ell} - 2\,\mathrm{Re}((w\bar{z})^\ell), & \text{if } z \in \mathbb{C}\setminus\{0\} \text{ and } w \in S_{z,\beta}, \\ |z|^{2\ell} + |w|^{2\ell} - 2|z|^\ell |w|^\ell \cos(\ell\beta), & \text{otherwise.} \end{cases}$$

Here

$$S_{z,\beta} := \{w \in \mathbb{C}\setminus\{0\} :\, |\arg(z\overline{w})| \le \beta\} \cup \{0\} \qquad (z \in \mathbb{C}\setminus\{0\}),$$

where $\arg\lambda$ denotes the principal branch of the argument of $\lambda \in \mathbb{C}\setminus\{0\}$, *i.e* $-\pi < \arg\lambda \le \pi$. We will need the following technical estimate.

Proposition 2.4 ([6, Proposition 4.2])**.** *Let* $s \in \mathbb{R}$ *and* $\omega \in \mathcal{A}^\ell_{p,\varrho}$. *Then*

$$\int_{\mathbb{C}} (1+|w|)^s\, \psi^\ell_{\alpha,\beta}(z,w)\, d\omega(w) \simeq (1+|z|)^s\, \omega(D^\ell_{z,\varrho}). \tag{2.2}$$

We recall the precise estimates of the Bergman kernel that we will use:

$$|\mathbf{K}^\ell_\alpha(w,z)| \lesssim (1+|w|)^{\ell-1}(1+|z|)^{\ell-1}\, \psi^\ell_{\alpha,\beta}(w,z) \qquad (z, w \in \mathbb{C}), \tag{2.3}$$

for every $0 < \beta < \frac{\pi}{2\ell}$, where

$$\mathbf{K}^\ell_\alpha(w,z) := e^{-\frac{\alpha}{2}|w|^{2\ell}} K^\ell_\alpha(w,z)\, e^{-\frac{\alpha}{2}|z|^{2\ell}} \tag{2.4}$$

is the socalled twisted Bergman kernel (see [6, (6.22)]). In particular, we have the following rough estimate:

$$|K^\ell_\alpha(w,z)| \lesssim (1+|z|)^{\ell-1}(1+|w|)^{\ell-1} e^{\alpha|z|^\ell|w|^\ell} \qquad (z, w \in \mathbb{C}). \tag{2.5}$$

Proposition 2.5. *Let* $\omega \in \mathcal{A}^\ell_p$, *for some* $1 < p < \infty$ *and* $\ell \ge 1$. *Then* E *is dense in* $F^{p,\ell}_\alpha(\omega)$, *for every* $\alpha > 0$.

Proof. Let $f \in F^{p,\ell}_\alpha(\omega)$. Then $f_R = f\chi_{D(0,R)} \in L^{p,\ell}_\alpha(\omega)$, for any $R > 0$, and

$$\|f - f_R\|_{L^{p,\ell}_\alpha(\omega)} \to 0, \quad \text{as } R \to \infty. \tag{2.6}$$

Now consider the entire function $h_R = P^\ell_\alpha f_R$. Taking into account the estimate (2.5), it is clear that

$$|h_R(z)| \lesssim (1 + |z|)^{\ell-1} e^{\alpha R^\ell |z|^\ell} \lesssim e^{\tau\,|z|^\ell} \qquad (z \in \mathbb{C}),$$

for every $\tau > \alpha R^\ell$, and, in particular, $h_R \in E$. Moreover, since $\omega \in \mathcal{A}^\ell_p$, [6, Theorem 1.1] and Lemma 2.2 show that P^ℓ_α is a bounded projection from $L^{p,\ell}_\alpha(\omega)$ onto $F^{p,\ell}_\alpha(\omega)$, and so

$$\|f - h_R\|_{F^{p,\ell}_\alpha(\omega)} = \|P^\ell_\alpha(f - f_R)\|_{F^{p,\ell}_\alpha(\omega)} \lesssim \|f - f_R\|_{L^{p,\ell}_\alpha(\omega)} \quad (R > 0). \tag{2.7}$$

Finally, (2.6) and (2.7) give that $h_R \to f$ in $F^{p,\ell}_\alpha(\omega)$, as $R \to \infty$, and that ends the proof. □

Besides the pointwise estimates of the Bergman kernel, a key tool to prove our results is a decomposition formula for K^ℓ_α obtained in [5, Theorem 1.3]:

$$K^\ell_\alpha(w, z) = G_0(w, z)^2 + G_1(w, z), \tag{2.8}$$

where

$$G_0(w, z) = \sqrt{2}\left(\frac{\alpha}{2}\right)^{1/(2\ell)} E_{\frac{1}{\ell},\frac{\ell+1}{2\ell}}\left(\left(\frac{\alpha}{2}\right)^{1/\ell} w\overline{z}\right) \quad \text{and} \quad G_1(w, z) = R_\ell(\alpha^{1/\ell} w\overline{z}).$$

The functions $G_j's$ satisfy the pointwise estimates satisfy

$$|\mathbf{G}_j(w, z)| \lesssim (1 + |w|)^{\frac{\ell-1}{2}} (1 + |z|)^{\frac{\ell-1}{2}}\, \psi^\ell_{\alpha,\beta}(w, \tfrac{z}{2^{1/\ell}}) \qquad (z, w \in \mathbb{C}), \tag{2.9}$$

for every $0 < \beta < \frac{\pi}{2\ell}$, where as above the $\mathbf{G}_j$'s are the twisted functions

$$\mathbf{G}_j(w, z) := e^{-\frac{\alpha}{2}|w|^{2\ell}} G_j(w, z)\, e^{-\frac{\alpha}{2}\left|\frac{z}{2^{1/\ell}}\right|^{2\ell}} = e^{-\frac{\alpha}{2}|w|^{2\ell}} G_j(w, z)\, e^{-\frac{\alpha}{8}|z|^{2\ell}} \tag{2.10}$$

for $j = 0, 1$.

Lemma 2.6. *Let $\omega \in \mathcal{A}^\ell_p$. Then we have that*

$$\|K^\ell_\alpha(\,\cdot\,, z)\|^p_{F^{p,\ell}_\alpha(\omega)} \lesssim e^{\frac{\alpha p}{2}|z|^{2\ell}} (1 + |z|)^{2p(\ell-1)} \omega(D^\ell_z) \qquad (z \in \mathbb{C}). \tag{2.11}$$

$$\|G_j(\cdot\,\overline{z})\|^p_{F^{p,\ell}_\alpha(\omega)} \lesssim e^{\frac{\alpha p}{8}|z|^{2\ell}} (1 + |z|)^{p(\ell-1)} \omega(D^\ell_{\frac{z}{2^{1/\ell}}}) \quad (z \in \mathbb{C},\ j = 0, 1). \tag{2.12}$$

$$\|G_0(\cdot\,\overline{z})\|_{F^{p,\ell}_\alpha(\omega)} \|G_0(\cdot\,\overline{z})\|_{F^{p',\ell}_\alpha(\omega')} + \|G_1(\cdot\,\overline{z})\|_{F^{p,\ell}_\alpha(\omega)} \|1\|_{F^{p',\ell}_\alpha(\omega')} \lesssim e^{\frac{\alpha}{4}|z|^{2\ell}} \tag{2.13}$$

Proof. Estimate (2.11) follows from (2.4), (2.3) and (2.2). Estimate (2.12) follows from (2.10), (2.9) and (2.2):

$$\begin{aligned}
\|G_j(\cdot\,\overline{z})\|^p_{F^{p,\ell}_\alpha(\omega)} &= e^{\frac{\alpha p}{8}|z|^{2\ell}} \int_{\mathbb{C}} |\mathbf{G}_j(w, z)|^p\, d\omega(w) \\
&\lesssim e^{\frac{\alpha p}{8}|z|^{2\ell}} (1 + |z|)^{\frac{p(\ell-1)}{2}} \int_{\mathbb{C}} (1 + |w|)^{\frac{p(\ell-1)}{2}}\, \psi^\ell_{\alpha,\beta}(w, \tfrac{z}{2^{1/\ell}})\, d\omega(w) \\
&\lesssim e^{\frac{\alpha p}{8}|z|^{2\ell}} (1 + |z|)^{p(\ell-1)} \omega(D^\ell_{\frac{z}{2^{1/\ell}}}).
\end{aligned}$$

Finally, we are going to prove (2.13). By (2.12) we have that

$$\|G_0(\cdot\,\overline{z})\|_{F^{p,\ell}_\alpha(\omega)}\|G_0(\cdot\,\overline{z})\|_{F^{p',\ell}_\alpha(\omega')}\lesssim e^{\frac{\alpha}{4}|z|^{2\ell}}(1+|z|)^{2(\ell-1)}\,\omega(D^\ell_{\frac{z}{2^{1/\ell}}})^{1/p}\,\omega'(D^\ell_{\frac{z}{2^{1/\ell}}})^{1/p'}$$
$$\lesssim e^{\frac{\alpha}{4}|z|^{2\ell}}(1+|z|)^{2(\ell-1)}\,|D^\ell_{\frac{z}{2^{1/\ell}}}|\simeq e^{\frac{\alpha}{4}|z|^{2\ell}}.$$

Moreover, (2.12) also gives that

$$\|G_1(\cdot\,\overline{z})\|_{F^{p,\ell}_\alpha(\omega)}\lesssim e^{\frac{\alpha}{8}|z|^{2\ell}}(1+|z|)^{\ell-1}\omega(D^\ell_{\frac{z}{2^{1/\ell}}})^{1/p}.$$

Since we know that there is a constant $C>1$ such that

$$\omega(D^\ell_z)\le C^{\phi^\ell_\beta(z,w)^{1/2}}\,\omega(D^\ell_w)\qquad (z,w\in\mathbb{C})$$

(see [6, Theorem 1.3]), we deduce that

$$\|G_1(\cdot\,\overline{z})\|_{F^{p,\ell}_\alpha(\omega)}\lesssim e^{\frac{\alpha}{8}|z|^{2\ell}}(1+|z|)^{\ell-1}C^{\frac{1}{p}\phi^\ell_\beta\left(\frac{z}{2^{1/\ell}},0\right)^{1/2}}\,\omega(D^\ell_0)^{1/p}$$
$$\lesssim e^{\frac{\alpha}{8}|z|^{2\ell}}(1+|z|)^{\ell-1}e^{c|z|^\ell}\lesssim e^{\frac{\alpha}{4}|z|^{2\ell}}.$$

Hence, since $\|1\|_{F^{p',\ell}_\alpha(\omega')}<\infty$, we conclude that estimate (2.13) holds. □

3. Proof of Theorem 1.1

We start by proving the characterization of the boundedness stated in Theorem 1.1. By duality (see Lemma 2.3), we have to show that the norm of the bilinear form

$$\Lambda_b(f,g):=\langle g,\,\overline{\mathfrak{h}^\ell_{b,\alpha}f}\,\rangle_\alpha\qquad (f,g\in E)$$

on $F^{p,\ell}_\alpha(\omega)\times F^{p',\ell}_\alpha(\omega')$ satisfies that

$$\|\Lambda_b\|\simeq\|b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}. \tag{3.1}$$

Observe that, for any $f,g\in E$, we have

$$\Lambda_b(f,g)=\langle g,\,P^\ell_\alpha(\overline{f}b)\rangle_\alpha\overset{(*)}{=}\langle P^\ell_\alpha g,\,\overline{f}b\rangle_\alpha=\langle g,\,\overline{f}b\rangle_\alpha=\langle fg,\,b\rangle_\alpha, \tag{3.2}$$

where equality $(*)$ follows from Fubini's theorem and the fact that

$$\Psi_{f,g,b}(z,w):=K^\ell_\alpha(w,z)f(w)\,\overline{b}(w)\,e^{-\alpha|w|^{2\ell}}g(z)\,e^{-\alpha|z|^{2\ell}}$$

is in $L^1(\mathbb{C}\times\mathbb{C})$. This is a consequence of the rough pointwise estimate (2.5). Indeed, if $\lambda>0$ we have that

$$|\Psi_{f,g,b}(w,z)|\lesssim\|b\|_{F^{\infty,\ell}_\beta}(1+|w|)^{\ell-1}(1+|z|)^{\ell-1}e^{\tau|w|^\ell}e^{\tau|z|^\ell}e^{\alpha|z|^\ell|w|^\ell-(\alpha-\frac{\beta}{2})|w|^{2\ell}-\alpha|z|^{2\ell}}$$
$$\lesssim\|b\|_{F^{\infty,\ell}_\beta}\big((1+|w|)(1+|z|)\big)^{\ell-1}e^{\tau(|w|^\ell+|z|^\ell)}e^{\frac{\alpha}{2}(\lambda-2+\frac{\beta}{\alpha})|w|^{2\ell}+\alpha(\frac{1}{2\lambda}-1)|z|^{2\ell}},$$

for some $\tau>0$. Therefore, by choosing $\frac12<\lambda<2-\frac{\beta}{\alpha}$ (which is possible since $\beta<\frac{3\alpha}{2}$), we see that $\Psi_{f,g,\varphi}\in L^1(\mathbb{C}\times\mathbb{C})$.

So we are left to prove (3.1). By Lemma 2.2 b), (2.8), and (2.13) show that

$$|b(z)|=|\langle K_\alpha(\cdot,z),b\rangle_\alpha|\le|\Lambda_b(G_0(\cdot,z),G_0(\cdot,z))|+|\Lambda_b(G_1(\cdot,z),1)|\lesssim\|\Lambda_b\|e^{\frac{\alpha}{4}|z|^{2\ell}}.$$

Therefore $b\in F^{\infty,\ell}_{\frac{\alpha}{2}}$ and $\|b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}\lesssim\|\Lambda\|$.

Now we prove the opposite estimate. Assume that $b\in F^{\infty,\ell}_{\frac{\alpha}{2}}$. Then [5, Proposition 2.8] shows that there exists $\varphi\in L^\infty$ such that $P_\alpha(\varphi)=b$ and $\|\varphi\|_{L^\infty}\simeq\|b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}$. Therefore $\Lambda_b(f,g)=\langle fg,b\rangle_\alpha=\langle fg,\varphi\rangle_\alpha$.

It follows that the bilinear form $\widetilde{\Lambda}_b : L^p_\alpha(\omega)\times L^{p'}_\alpha(\omega')\to\mathbb{C}$ defined by $\widetilde{\Lambda}_b(f,g)=\langle fg,\varphi\rangle_\alpha$ extends Λ_b. Moreover, $\|\widetilde{\Lambda}_b\|\le\|\varphi\|_{L^\infty}$, because, by Hölder's inequality,

$$|\widetilde{\Lambda}_b(f,g)|\le\|\varphi\|_{L^\infty}\|fg\|_{L^{1,\ell}_\alpha}\le\|\varphi\|_{L^\infty}\|f\|_{F^p_\alpha(\omega)}\|g\|_{F^{p'}_\alpha(\omega')}.$$

Hence $\|\varphi\|_{L^\infty}\simeq\|b\|_{F^\infty_{\frac{\alpha}{2}}}\lesssim\|\Lambda_b\|\le\|\widetilde{\Lambda}_b\|\le\|\varphi\|_{L^\infty}$. And that ends the proof of the boundedness.

Next we prove the compactness part. Assume that $b\in\mathfrak{f}^{\infty,\ell}_{\frac{\alpha}{2}}$. Then there is a sequence of polynomials $\{q_k\}_{k\in\mathbb{N}}$ such that $\|q_k-b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}\to 0$, and so $\|\mathfrak{h}^\ell_{q_k}-\mathfrak{h}_b\|_{F^{p,\ell}_\alpha(\omega)}\to 0$, because $\|\mathfrak{h}^\ell_{q_k}-\mathfrak{h}^\ell_b\|_{F^{p,\ell}_\alpha(\omega)}=\|\mathfrak{h}^\ell_{q_k-b}\|_{F^{p,\ell}_\alpha(\omega)}\lesssim\|q_k-b\|_{F^{\infty,\ell}_{\frac{\alpha}{2}}}$. Consequently, since $\{\mathfrak{h}^\ell_{q_k}\}_{k\in\mathbb{N}}$ is a sequence of finite rank operators, it follows that $\mathfrak{h}^\ell_b : F^{p,\ell}_\alpha(\omega)\to\overline{F^{p,\ell}_\alpha(\omega)}$ is compact.

Now assume that $\mathfrak{h}^\ell_b : F^{p,\ell}_\alpha(\omega)\to\overline{F^{p,\ell}_\alpha(\omega)}$ is compact. In particular, it is a bounded operator, or equivalently, the bilinear form Λ_b is bounded on $F^{p,\ell}_\alpha(\omega)\times F^{p',\ell}_\alpha(\omega')$. Moreover, $b\in F^{\infty,\ell}_{\frac{\alpha}{2}}$ and $\overline{b(z)}=\langle K(\cdot,z),b\rangle_\alpha$ (by Lemmas 2.2 and 2.2). Hence, (2.8) gives that

$$\overline{b(z)}=\langle G_0(\cdot,z)^2,b\rangle_\alpha+\langle G_1(\cdot,z),b\rangle_\alpha=\Lambda_b(G_0(\cdot,z),G_0(\cdot,z))+\Lambda_b(G_1(\cdot,z),1).$$

Since $E\times E$ is dense in $F^{p,\ell}_\alpha(\omega)\times F^{p',\ell}_\alpha(\omega')$, identity (3.2) also holds for any $f\in F^{p,\ell}_\alpha(\omega)$ and $g\in F^{p',\ell}_\alpha(\omega')$. So we may apply twice (3.2) (to $f=g=G_0(\cdot,z)$ and $f=G_1(\cdot,z)$, $g=1$) and obtain that

$$\overline{b(z)}=\langle G_0(\cdot,z),\overline{\mathfrak{h}^\ell_b(G_0(\cdot,z))}\rangle+\langle 1,\overline{\mathfrak{h}^\ell_b(G_1(\cdot,z))}\rangle. \tag{3.3}$$

If we consider the normalized functions

$$\widetilde{G}_{k,z}:=G_k(\cdot,z)/\|G_k(\cdot,z)\|_{F^{p,\ell}_\alpha(\omega)}\qquad(k=0,1)$$
$$\widehat{G}_{0,z}:=G_0(\cdot,z)/\|G_0(\cdot,z)\|_{F^{p',\ell}_\alpha(\omega')},$$

we may write (3.3) as

$$\begin{aligned}\overline{b(z)}=&\|G_0(\cdot,z)\|_{F^{p,\ell}_\alpha(\omega)}\|G_0(\cdot,z)\|_{F^{p',\ell}_\alpha(\omega')}\langle\widehat{G}_{0,z},\overline{\mathfrak{h}^\ell_b(\widetilde{G}_{0,z})}\rangle\\&+\|G_1(\cdot,z)\|_{F^{p,\ell}_\alpha(\omega)}\langle 1,\overline{\mathfrak{h}^\ell_b(\widetilde{G}_{1,z})}\rangle.\end{aligned}$$

By (2.13) we have

$$|b(z)|\lesssim e^{\frac{\alpha}{4}|z|^{2\ell}}\Big(\big\|\mathfrak{h}^\ell_b(\widetilde{G}_{0,z})\big\|_{F^{p,\ell}_\alpha(\omega)}+\big\|\mathfrak{h}^\ell_b(\widetilde{G}_{1,z})\big\|_{F^{p,\ell}_\alpha(\omega)}\Big).$$

Consequently, in order to show that $b\in\mathfrak{f}^{\infty,\ell}_{\frac{\alpha}{2}}$, it is enough to prove that

$$\|\mathfrak{h}_b(\widetilde{G}_{k,z})\|_{F^{p,\ell}_\alpha(\omega)}\to 0\quad\text{as } |z|\to\infty,\text{ for } k=0,1.$$

Since $\mathfrak{h}_b$ is compact, we only have to check that $\widetilde{G}_{k,z}\to 0$ weakly in $F^{p,\ell}_\alpha(\omega)$ as $|z|\to\infty$, for $k=0,1$. Indeed, that follows from the fact that $\|\widetilde{G}_{k,z}\|_{F^{p,\ell}_\alpha(\omega)}=1$ and $\widetilde{G}_{k,z}\to 0$ uniformly on compact sets as $|z|\to\infty$, for $k=0,1$. The uniform convergence on compacta is a direct consequence of estimates (2.12) and $|G_k(w\overline{z})|\lesssim e^{\alpha(R^\ell+1)|z|^\ell}$, for $|w|\le R$.

4. Proof of Theorem 1.2

For $0<p<\infty$ and separable complex Hilbert spaces H_0 and H_1, we recall that the Schatten class $S_p(H_0,H_1)$ consists of all compact linear operators T from H_0 to H_1 such that

$$\|T\|^p_{S_p(H_0,H_1)} := \sum_{k=1}^{\infty} s_k(T)^p < \infty,$$

where $\{s_k(T)\}_{k\in\mathbb{N}}$ is the sequence of singular values of T Moreover, $S_\infty(H_0,H_1)$ is the space of all the bounded linear operators from H_0 to H_1. (See, for instance, [13, Chapter 7] and [5, §6.2], for more details)

Note that $(S_p(H_0,H_1), \|\cdot\|_{S_p(H_0,H_1)})$ is a Banach space for $p\ge 1$ and a quasi-Banach space for $p<1$. Moreover, since $\|T\|_{S_q(H_0,H_1)} \le \|T\|_{S_p(H_0,H_1)}$ for $p<q$ and $T\in S_p(H_0,H_1)$, we have the embedding

$$S_p(H_0,H_1) \hookrightarrow S_q(H_0,H_1), \qquad (0<p<q\le\infty).$$

The polar decomposition of T gives the existence of two orthonormal systems $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ of H_0 and H_1, respectively, such that

$$T(f) = \sum_{k=1}^{\infty} s_k(T)\langle f, u_k\rangle_{H_0} v_k.$$

Note that if $T_k(f) := s_k(T)\langle f,u_k\rangle_{H_0} v_k$, then $\|T_k\|_{S_p(H_0,H_1)} = s_k(T)$. So if $T\in S_1(H_0,H_1)$, then the rank one operators T_k satisfy

$$\sum_{k=1}^{n} T_k \to T \text{ in } S_1(H_0,H_1) \ \text{ and } \ \Big\|\sum_{k=1}^{n} T_k\Big\|_{S_1(H_0,H_1)} = \sum_{k=1}^{n} \|T_k\|_{S_1(H_0,H_1)}. \tag{4.1}$$

Moreover, recall the following complex interpolation identity [15, Theorem 2.6]:

$$(S_1(H_0,H_1), S_\infty(H_0,H_1))_{[\theta]} = S_{1/(1-\theta)}(H_0,H_1) \qquad (0<\theta<1). \tag{4.2}$$

In order to prove our results, we need the following lemma on rank one operators from $F^{2,\ell}_\alpha(\omega)$ to $\overline{F^{2,\ell}_\alpha(\omega)}$. We omit its proof since it can be easily deduced from Lemma 2.3.

Lemma 4.1. *Let $\omega\in\mathcal{A}^\ell_2$. Then $T: F^{2,\ell}_\alpha(\omega)\to\overline{F^{2,\ell}_\alpha(\omega)}$ is a bounded linear operator of rank one if and only if there are non zero functions $g\in F^{2,\ell}_\alpha(\omega')$ and $h\in F^{2,\ell}_\alpha(\omega)$ such that $Tf=\langle f,g\rangle_\alpha \overline{h}$, for any $f\in F^{2,\ell}_\alpha(\omega)$. Moreover, in this case, $\|T\|_{S_p(F^{2,\ell}_\alpha(\omega))} \simeq \|g\|_{F^{2,\ell}_\alpha(\omega')}\|h\|_{F^{2,\ell}_\alpha(\omega)}$, for any $0<p<\infty$.*

4.0.1. ***Proof of the sufficient condition.***

The sufficient condition is a direct consequence of the following result.

Proposition 4.2. *For $1\le p\le\infty$, the operator $b\mapsto\mathfrak{h}_b$ is bounded from $F^{p,\ell}_{\frac{\alpha}{2},\frac{2(\ell-1)}{p}}$ to $S_p(F^{2,\ell}_\alpha(\omega))$.*

In order to prove Proposition 4.2, we will need the following interpolation Lemma.

Lemma 4.3 ([5, Lemma 6.4])**.** *Let $1<p<\infty$. Then*

$$(L^{1,\ell}_{\alpha/2,2(\ell-1)}, L^{\infty,\ell}_{\alpha/2})_{[1/p']} = L^{p,\ell}_{\alpha/2,2(\ell-1)/p}, \qquad \textit{and} \tag{4.3}$$

$$(F^{1,\ell}_{\alpha/2,2(\ell-1)}, F^{\infty,\ell}_{\alpha/2})_{[1/p']} = F^{p,\ell}_{\alpha/2,2(\ell-1)/p}, \tag{4.4}$$

Proof of Proposition 4.2. By the interpolation identities (4.4) and (4.2) it is enough to prove the result for $p=1$ and $p=\infty$. Since the last case has been done in the previous section, we only have to deal with the case $p=1$.

Assume $b\in F^{1,\ell}_{\frac{\alpha}{2},2(\ell-1)}$. By the pointwise estimate [5, Corollary 2.9] and Lemma 2.2, $b\in F^{\infty,\ell}_{\frac{\alpha}{2}}$ and $b=P_{\frac{\alpha}{2}}b$. Therefore, for $f\in E$ we have

$$\begin{aligned}(\mathfrak{h}^\ell_b f)(z)&=\int_{\mathbb{C}} f(u)\,\overline{b(u)}\,K^\ell_\alpha(u,z)\,e^{-\alpha|u|^{2\ell}}dA(u)\\&=\int_{\mathbb{C}} f(u)\Big(\int_{\mathbb{C}}\overline{b(w)}\,K^\ell_{\frac{\alpha}{2}}(w,u)\,e^{-\frac{\alpha}{2}|w|^{2\ell}}dA(w)\Big)K^\ell_\alpha(u,z)\,e^{-\alpha|u|^{2\ell}}dA(u),\end{aligned}$$

and Fubini's theorem gives

$$(\mathfrak{h}^\ell_b f)(z)=\int_{\mathbb{C}}\overline{b(w)}\,(\mathfrak{h}^\ell_{K^\ell_{\frac{\alpha}{2}}(\cdot,w)}f)(z)\,e^{-\frac{\alpha}{2}|w|^{2\ell}}dA(w). \tag{4.5}$$

This allows us to consider the following Bochner integral

$$\int_{\mathbb{C}}\overline{b(w)}\,\mathfrak{h}^\ell_{K^\ell_{\alpha/2}(\cdot,w)}\,e^{-\frac{\alpha}{2}|w|^{2\ell}}dA(w). \tag{4.6}$$

By Bochner's integrability theorem (see, for instance, [14, p. 133]), the $S_1(F^{2,\ell}_\alpha(\omega))$-convergence of the Bochner's integral (4.6) means that the integrand $S(w):=\overline{b(w)}\,\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}$ is an $S_1(F^{2,\ell}_\alpha(\omega))$-valued strongly measurable function on $\mathbb{C}$ which satisfies

$$\int_{\mathbb{C}}\|S(w)\|_{S_1(F^{2,\ell}_\alpha(\omega))}\,e^{-\frac{\alpha}{2}|w|^{2\ell}}dA(w)<\infty. \tag{4.7}$$

We are going to show that $S(w)$ is an operator of rank at most one, for every $w\in\mathbb{C}$, and next we estimate its $S_1(F^{2,\ell}_\alpha(\omega))$-norm.

For any $w\in\mathbb{C}$ and $f\in E$, we have

$$\big(\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}f\big)(z)=2^{-1/\ell}\langle f,K^\ell_\alpha(\cdot,2^{-1/\ell}w)\rangle_\alpha\,K^\ell_\alpha(2^{-1/\ell}w,z). \tag{4.8}$$

Indeed, by (2.1), $K^\ell_{\alpha/2}(\,\cdot\,,w)=2^{-1/\ell}K^\ell_\alpha(\,\cdot\,,2^{-1/\ell}w)$. Therefore

$$\begin{aligned}\big(\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}f\big)(z)&=2^{-1/\ell}\langle fK^\ell_\alpha(\,\cdot\,,z),\,K^\ell_\alpha(\,\cdot\,,2^{-1/\ell}w)\rangle_\alpha\\&=2^{-1/\ell}f(2^{-1/\ell}w)K^\ell_\alpha(2^{-1/\ell}w,z)\\&=2^{-1/\ell}\langle f,K^\ell_\alpha(\cdot,2^{-1/\ell}w)\rangle_\alpha\,K^\ell_\alpha(2^{-1/\ell}w,z).\end{aligned}$$

So $\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}$ is an operator of rank one and, by Lemma 4.1 and estimate (2.11), we obtain

$$\begin{aligned}\|\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}\|_{S_1(F^2_\alpha(\omega))}&\simeq\|K^\ell_\alpha(\cdot,2^{-1/\ell}w)\|_{F^2_\alpha(\omega')}\|K^\ell_\alpha(\cdot,2^{-1/\ell}w)\|_{F^2_\alpha(\omega)}\\&\lesssim(1+|\tfrac{w}{2^{1/\ell}}|)^{4(\ell-1)}\,e^{\frac{\alpha}{4}|w|^{2\ell}}\omega(D^\ell_{\frac{w}{2^{1/\ell}}})^{\frac12}\omega'(D^\ell_{\frac{w}{2^{1/\ell}}})^{\frac12}\\&\lesssim(1+|w|)^{2(\ell-1)}\,e^{\frac{\alpha}{4}|w|^{2\ell}}.\end{aligned} \tag{4.9}$$

Observe that (4.8) shows that S is an $S_1(F^{2,\ell}_\alpha(\omega))$-valued function on $\mathbb{C}$. Moreover, it is $S_1(F^{2,\ell}_\alpha(\omega))$-strongly measurable because

$$w\in\mathbb{C}\longmapsto\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)}\in S_1(F^{2,\ell}_\alpha(\omega))$$

is continuous. That follows because $\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)} - \mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,v)}$ has rank at most 2 and so

$$\begin{aligned}\|\mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,w)} - \mathfrak{h}_{K^\ell_{\alpha/2}(\cdot,v)}\|_{S_1(F^{2,\ell}_\alpha(\omega))} &\le 2\,\|\mathfrak{h}_{\{K^\ell_{\alpha/2}(\cdot,w)-K^\ell_{\alpha/2}(\cdot,v)\}}\|_{S_\infty(F^{2,\ell}_\alpha(\omega))}\\ &\overset{(1)}{\lesssim} \|K^\ell_{\alpha/2}(\cdot,w) - K^\ell_{\alpha/2}(\cdot,v)\|_{F^{\infty,\ell}_{\alpha/2}}\\ &\overset{(2)}{\lesssim} \|K^\ell_{\alpha/2}(\cdot,w) - K^\ell_{\alpha/2}(\cdot,v)\|_{F^{1,\ell}_{\frac{\alpha}{2},2(\ell-1)}} \overset{(3)}{\longrightarrow} 0,\end{aligned}$$

as $w \to v$, where (1), (2) and (3) are consequences of Theorem 1.1, the pointwise estimate [5, Corollary 2.9] and the dominated convergence theorem, respectively.

Now (4.9) gives (4.7):

$$\int_{\mathbb{C}} \|S(w)\|_{S_1(F^{2,\ell}_\alpha(\omega))}\, e^{-\frac{\alpha}{2}|w|^{2\ell}} dA(w) \lesssim \int_{\mathbb{C}} |b(w)|\,(1+|w|)^{2(\ell-1)} e^{-\frac{\alpha}{4}|w|^{2\ell}}\, dA(w).$$

Therefore, by (4.5), $\mathfrak{h}_b \in S_1(F^{2,\ell}_\alpha(\omega))$ and $\|\mathfrak{h}_b\|_{S_1(F^{2,\ell}_\alpha(\omega))} \lesssim \|b\|_{F^{1,\ell}_{\alpha/2,2(\ell-1)}}$. □

4.0.2. ***Proof of necessary condition.***

The necessity will follow the ideas from the case $\omega \equiv 1$ and $\ell > 1$ (see [5]), which ultimately are inspired by the classical case $\ell = 1$ (see [9]). The following definition is suggested by (3.3).

Definition 4.4. For $T \in S_\infty(F^{2,\ell}_\alpha(\omega))$, let

$$\Phi_T(z) := \langle G_0(\cdot,z), \overline{T(G_0(\cdot,z))}\rangle_\alpha + \langle 1, \overline{T(G_1(\cdot,z))}\rangle_\alpha \qquad (z\in\mathbb{C}).$$

Since $\Phi_{\mathfrak{h}_b} = \overline{b}$, the necessary part in Theorem 1.2 is a direct consequence of the following result.

Proposition 4.5. *For $1 \le p \le \infty$, the linear operator $T \mapsto \Phi_T$ is bounded from $S_p(F^{2,\ell}_\alpha(\omega))$ to $L^{p,\ell}_{\alpha/2,2(\ell-1)/p}$.*

Proof. It is easy to check that Φ_T is a continuous function on $\mathbb{C}$. Indeed, if $z_j \to z$ in $\mathbb{C}$, [6, Proposition 5.6] and the dominated convergence theorem imply that $G_k(\cdot\,\overline{z}_j) \to G_k(\cdot\,\overline{z})$ in both spaces $F^{2,\ell}_\alpha(\omega')$ and $F^{2,\ell}_\alpha(\omega)$.

So, taking into account the interpolation identities (4.2) and (4.3), it is enough to prove the proposition for $p = 1$ and $p = \infty$.

The case $p = \infty$ follows from Schwarz inequality, the boundedness of T and (2.13):

$$\begin{aligned}|\Phi_T(z)| &\lesssim \|T\|_{S_\infty(F^{2,\ell}_\alpha(\omega))}\left(\|G_0(\cdot,z)\|_{F^{2,\ell}_\alpha(\omega)}\|G_0(\cdot,z)\|_{F^{2,\ell}_\alpha(\omega')} + \|G_1(\cdot,z)\|_{F^{2,\ell}_\alpha(\omega)}\right)\\ &\lesssim \|T\|_{S_\infty(F^{2,\ell}_\alpha(\omega))}\, e^{\frac{\alpha}{4}|z|^{2\ell}}.\end{aligned}$$

Now we prove the case $p = 1$, that is,

$$\|\Phi_T\|_{L^{1,\ell}_{\alpha/2,2(\ell-1)}} \lesssim \|T\|_{S_1(F^{2,\ell}_\alpha(\omega))} \qquad (T \in S_1(F^{2,\ell}_\alpha(\omega))). \tag{4.10}$$

Taking into account (4.1), the case $p = \infty$, and Fatou's lemma, it is easy to show that we only have to prove (4.10) for operators of rank one. So, by Lemma 4.1, we may assume that T satisfies

$$Tf = \langle f, g\rangle_\alpha\, \overline{h} \qquad (f \in F^{2,\ell}_\alpha(\omega)),$$

for some functions $g \in F^{2,\ell}_\alpha(\omega')$ and $h \in F^{2,\ell}_\alpha(\omega)$.

In this case,

$$\Phi_T(z) = \langle G_0(\cdot,z), g\rangle_\alpha \, \langle G_0(\cdot,z), h\rangle_\alpha + \langle G_1(\cdot,z), g\rangle_\alpha \, \langle 1, h\rangle_\alpha,$$

and Schwarz inequality using that

$$\omega(D^\ell_{\frac{z}{2^{1/\ell}}})\omega'(D^\ell_{\frac{z}{2^{1/\ell}}}) \simeq (1+|z|)^{4(1-\ell)}$$

gives

$$\|\Phi_T\|_{L^{1,\ell}_{\frac{\alpha}{2},2(\ell-1)}} \lesssim I_0\, J_0 + I_1\, J_1,$$

where

$$\begin{aligned}
I_k^2 &:= \int_{\mathbb{C}} |\langle G_k(\cdot\overline{z}), g\rangle_\alpha|^2 \;(1+|z|)^{4(\ell-1)} e^{-\frac{\alpha}{4}|z|^{2\ell}} \omega'(D^\ell_{\frac{z}{2^{1/\ell}}})\, dA(z)\\
J_0^2 &:= \int_{\mathbb{C}} |\langle G_0(\cdot\overline{z}), h\rangle_\alpha|^2 \;(1+|z|)^{4(\ell-1)} e^{-\frac{\alpha}{4}|z|^{2\ell}} \omega(D^\ell_{\frac{z}{2^{1/\ell}}})\, dA(z)\\
J_1^2 &:= \int_{\mathbb{C}} |\langle 1, h\rangle_\alpha|^2 \;(1+|z|)^{8(\ell-1)} e^{-\frac{\alpha}{4}|z|^{2\ell}} \omega(D^\ell_{\frac{z}{2^{1/\ell}}})\, dA(z).
\end{aligned}$$

Next we prove that $I_k \lesssim \|g\|_{F^{2,\ell}_\alpha(\omega')}$ and $J_k \lesssim \|h\|_{F^{2,\ell}_\alpha(\omega)}$, which, by Lemma 4.1, give

$$\|\Phi_T\|_{L^1_{\frac{1}{2},2n(\ell-1)}} \lesssim \|g\|_{F^{2,\ell}_\alpha(\omega')}\, \|h\|_{F^{2,\ell}_\alpha(\omega)} \simeq \|T\|_{S_1(F^{2,\ell}_\alpha(\omega))}.$$

In order to prove the estimate $I_k \lesssim \|g\|_{F^{2,\ell}_\alpha(\omega')}$, first note that Schwarz's inequality gives

$$|\langle G_k(\cdot,z), g\rangle_\alpha|^2 \lesssim \left(\int_{\mathbb{C}} |g|^2\, |G_k(\cdot,z)|\, \omega'\, dA_{3\alpha}\right)\left(\int_{\mathbb{C}} |G_k(\cdot,z)|\, \omega\, dA_\alpha\right),$$

where $dA_\beta(w) = e^{-\frac{\beta}{2}|w|^{2\ell}} dA(w)$, for any $\beta>0$. Then, by (2.12), we obtain

$$|\langle G_k(\cdot,z), g\rangle_\alpha|^2 \lesssim (1+|z|)^{\ell-1} e^{\frac{\alpha}{8}|z|^{2\ell}} \omega(D^\ell_{\frac{z}{2^{1/\ell}}}) \int_{\mathbb{C}} |g(w)|^2 |G_k(w,z)|\, \omega'(w) dA_{3\alpha}(w)$$

Therefore

$$\begin{aligned}
I_k^2 &\lesssim \int_{\mathbb{C}} (1+|z|)^{5(\ell-1)}\, \omega(D^\ell_{\frac{z}{2^{1/\ell}}})\omega'(D^\ell_{\frac{z}{2^{1/\ell}}}) \left(\int_{\mathbb{C}} |g|^2\, |G_k(\cdot,z)|\, \omega' dA_{3\alpha}\right) dA_{\frac{\alpha}{4}}(z)\\
&\lesssim \int_{\mathbb{C}} \left(\int_{\mathbb{C}} |G_k(w,z)|\, (1+|z|)^{\ell-1}\, dA_{\frac{\alpha}{4}}(z)\right) |g(w)|^2 \omega'(w)\, dA_{3\alpha}(w)\\
&\lesssim \int_{\mathbb{C}} \|G_k(\cdot,w)\|_{F^{1,\ell}_{\frac{\alpha}{4},\ell-1}} |g(w)|^2 \omega'(w)\, dA_{3\alpha}(w)
\end{aligned}$$

and Proposition 4.3 in [5] with $\gamma=1$, $\alpha=\frac14$ and $\theta=\frac12$ gives

$$\|G_k(\cdot,w)\|_{F^{1,\ell}_{\frac{\alpha}{4},\ell-1}} \lesssim e^{\frac{\alpha}{2}|w|^{2\ell}}$$

so we have that

$$I_k^2 \lesssim \int_{\mathbb{C}} |g(w)|^2 e^{-\alpha|w|^{2\ell}} \omega'(w) dA(w) = \|g\|^2_{F^{2,\ell}_\alpha(\omega')}.$$

Similarly, we obtain $J_0 \lesssim \|h\|_{F^{2,\ell}_\alpha(\omega)}$.

Finally, we estimate J_1:

$$\begin{aligned}J_1^2 &= c_{\alpha,\ell}\,|h(0)|^2\int_{\mathbb{C}}(1+|z|)^{8(\ell-1)}e^{-\frac{\alpha}{4}|z|^{2\ell}}\omega(D^\ell_{\frac{z}{2^{1/\ell}}})\,dA(z)\\ &\lesssim |h(0)|^2\int_{\mathbb{C}}(1+|z|)^{8(\ell-1)}e^{-\frac{\alpha}{4}|z|^{2\ell}+M|z|^\ell}\,dA(z)\lesssim \|h\|^2_{F^{2,\ell}_\alpha(\omega)},\end{aligned}$$

since $\omega(D^\ell_{\frac{z}{2^{1/\ell}}})\lesssim e^{M|z|^\ell}$ and $|h(0)|^2\lesssim \|h\|^2_{F^{2,\ell}_\alpha(\omega)}$, by [6, Theorem 1.3] and [6, Lemma 5.5], respectively. □

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