Birkhoff's theorem for Einstein gravity and beyond

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Birkhoff's theorem establishes that any spherically symmetric vacuum solution of Einstein's equations is necessarily static and uniquely given by the Schwarzschild metric. In this TFG we prove this theorem for Einstein gravity and several of its extensions, which include the presence of electric charge as well as higher-curvature terms in the action. More precisely, we explicitly prove Birkhoff-like theorems for: Einstein, Einstein-Maxwell, Einstein-Gauss-Bonnet and Einstein-Maxwell-Gauss-Bonnet theories in general dimensions. In all cases, starting from a general spherically symmetric ansatz, we are able to show that the equations of motion impose the staticity condition, and fully constrain the metric to take the form of the unique static and spherically symmetric spacetime of each theory. We also comment on the validity of the theorem for general Lovelock theories and its current status for more general higher-curvature theories of gravity.

I. INTRODUCTION

General Relativity describes gravity as the curvature of spacetime. The dynamics of the gravitational field are governed by Einstein's field equations. These equations can be derived from the Einstein-Hilbert action:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{|g|}R\,, \tag{1}$$

where $\mathrm{d}^D x\sqrt{|g|}$ represents the volume element, R is the Ricci scalar and we have omitted the cosmological constant.

While the Einstein-Hilbert action provides a description of gravity compatible with all observations so far, there are good reasons to believe that it is not the end of the story, and that it should be modified by higher curvature terms. These motivations include —see *e*.g., [1]:

1. Effective field theory perspective: The Einstein-Hilbert action ought to be the first in an infinite sum of higher-curvature operators. Higher-curvature operators are expected to become relevant at sufficiently high energy scales.

2. String Theory predictions: String Theory —the leading candidate for a consistent theory of quantum gravity— predicts the appearance of higher-curvature corrections in the form of an expansion in α'-weighted terms (α' being the inverse string tension).

3. AdS/CFT duality: Higher-curvature theories give rise to dual Conformal Field Theories (CFTs) that differ from those dual to Einstein gravity, opening up new avenues for exploration and understanding of quantum systems.

4. Probing universal aspects of gravity: Higher-curvature models provide an opportunity to investigate the generality of Einsteinian features by examining if and how they persist beyond the realm of Einstein gravity. This includes, for instance, the properties of black holes, gravitational waves, cosmological evolution, etc.

A notable property of Einstein gravity is Birkhoff's theorem, which states that any spherically symmetric vacuum solution of the Einstein field equations is necessarily static and asymptotically flat, and is described by Schwarzschild's metric [2]. In this TFG, we aim to investigate whether Birkhoff's theorem remains valid beyond Einstein gravity. In section II C, we consider the effect of including electric charge, whereas section III A examines the addition of higher-curvature terms and section III B contemplates both of these effects. Finally, sections III C and IV reflect upon the theorem's validity in more general theories.

II. BIRKHOFF'S THEOREM FOR EINSTEIN GRAVITY

Before proving Birkhoff's theorem for Einstein gravity, let us start with an electromagnetic analogy.

A. Gauss' law analogy

Maxwell's first law of electromagnetism (or Gauss' law) states that the electric flux through any closed surface is proportional to the electric charge enclosed within the surface. Mathematically, it can be expressed in natural units as:

$$\oint_{\mathcal{S}} \vec{E}\cdot \mathrm{d}\vec{S} = 4\pi Q\,,$$

with $\vec{E}$ being the electric field, $\mathrm{d}\vec{S}$ the surface element vector orthogonal to the area and Q the electric charge. This means that the electric field outside the charge distribution does not depend on its shape or size, but merely on the net amount of charge inside the volume delimited by the surface $\mathcal{S}$.

This is similar to Birkhoff's theorem, which states that the gravitational field outside a spherically symmetric mass distribution is described by the Schwarzschild metric, regardless of the distribution's internal structure or dynamics. As we will see in a moment, all spherically symmetric spacetimes fulfilling Einstein's equations are static. Analogously, the Coulomb field is the only spherically symmetric solution of Maxwell's equations in the vacuum.

B. Proof of Birkhoff's theorem for Einstein gravity

Let us now move to gravity. The Einstein-Hilbert action in D dimensions is given by expression (1). If we vary the action with respect to the metric, $\delta S/\delta g^{\mu\nu} = 0$, we find the equations of motion for Einstein gravity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ ,$$

where we included a possible matter stress-tensor. In the vacuum, $T_{\mu\nu} = 0$ and the equations reduce to $R_{\mu\nu} = 0$.

Let us consider a metric of the form:

$$\mathrm{d}s^2 = -e^{2A(r,t)}\mathrm{d}t^2 + e^{2B(r,t)}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{D-2} \ , \quad (2)$$

which is a completely general spherically symmetric ansatz. The $\mu\nu = rt, rr, tt, \theta_i\theta_i$ components of the Ricci tensor are, respectively:

$$\begin{aligned} \frac{D-2}{r}\dot{B} = 0 \ , \\ e^{2(B-A)}(\ddot{B} + \dot{B}^2 - \dot{A}\dot{B}) - A'^2 + \frac{D-2}{r}B' + A'B' - A'' = 0 \ , \\ -\ddot{B} - \dot{B}^2 + \dot{A}\dot{B} + \frac{1}{r}e^{2(A-B)}[rA'^2 + A'(D-2-rB') + rA''] = 0 \ , \\ e^{-2B}\prod_{j=1}^{i-1}\sin^2(\theta_j)[(D-3)(e^{2B}-1) - rA' + rB'] = 0 \ , \end{aligned}$$

where we use the notation $T' \equiv \frac{\mathrm{d}T}{\mathrm{d}r}$ and $\dot{T} \equiv \frac{\mathrm{d}T}{\mathrm{d}t}$. From the first equation, we immediately obtain:

$$\boxed{\dot{B} = 0 \Rightarrow B(r,t) = B(r)}$$

This tells us that the metric function $B(r,t)$ has no dependence whatsoever on the time variable. Now, from the last equation we find:

$$A'(r,t) = \frac{D-3}{r}(e^{2B(r)} - 1) + B'(r) \ .$$

So $A(r,t) = \bar{A}(r) + f(t)$, and we can set $f(t) = 0$ through a redefinition of the t coordinate: $e^{2A(r,t)}\mathrm{d}t^2 = e^{2\bar{A}(r)}e^{2f(t)}\mathrm{d}t^2 = e^{2\bar{A}(r)}\mathrm{d}\bar{t}^2$. So, without loss of generality, we can write:

$$\boxed{A(r,t) = A(r)}$$

This proves the first part of the theorem, namely, the fact that spherical symmetry implies staticity for the vacuum Einstein equations.

Combining the second and third equations, we obtain:

$$R_{tt} + e^{2(A-B)}R_{rr} = e^{2(A-B)}\frac{D-2}{r}(A' + B') \ .$$

Since $R_{rr} = 0 = R_{tt} \Rightarrow A'(r) + B'(r) = 0 \Rightarrow A(r) = -B(r)$, where we have also set the integration constant to zero, redefining time. Therefore, we have:

$$A(r) = -B(r) \ .$$

Applying this, we can rewrite the equations for the angular components as:

$$(D-3)(e^{2B} - 1) + 2rB' = 0 \Rightarrow -(D-3)\frac{\mathrm{d}r}{r} = \frac{2}{e^{2B}-1}\mathrm{d}B \, .$$

Multiplying and dividing the left-hand side by e^{-2B}, we are left with:

$$\begin{aligned} -(D-3)\frac{\mathrm{d}r}{r} &= \frac{2e^{-2B}}{1-e^{-2B}}\mathrm{d}B \ , \\ -(D-3)\log(r) + \log(C) &= \log(1 - e^{-2B}) \ , \\ 1 - e^{-2B} &= Cr^{-(D-3)} \ . \end{aligned}$$

Therefore, the metric components take the form:

$$\boxed{e^{-2B(r)} = e^{2A(r)} = 1 - \frac{16\pi G M}{(D-2)\Omega_{D-2}r^{D-3}}} \quad (3)$$

where $\Omega_{D-2} \equiv 2\pi^{\frac{D-1}{2}}/\Gamma\left[\frac{D-1}{2}\right]$ is the $(D-2)$-dimensional hypersphere volume and the integration constant C has been identified with the mass of the solution using the Arnowitt-Deser-Misner (ADM) formalism [3]. The resulting metric corresponds to the Schwarzschild-Tangherlini spacetime, a generalization of the Schwarzschild metric to higher dimensions. In the $D = 4$ case, this reduces to the somewhat more familiar expression:

$$e^{-2B(r)} = e^{2A(r)} = 1 - \frac{2GM}{r} \ .$$

C. Einstein-Maxwell

In this section, our aim is to prove that the Einstein-Maxwell system satisfies an analogous version of Birkhoff's theorem. This implies that the exterior geometry of a spherically symmetric spacetime must be described by the Reissner-Nordström metric, which generalizes Schwarzschild's solution in the presence of electric charge. We will determine the specific form of this metric for arbitrary $D \geq 4$.

The Einstein-Maxwell action reads:

$$S = S_G + S_{EM} = \int \mathrm{d}^D x \sqrt{|g|} \left(\frac{R}{16\pi G} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) ,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the Faraday tensor, and A^μ is the gauge potential. To obtain the field equations, we vary the action with respect to the metric and the potential, which leaves us with:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2G \left(F_{\mu\lambda} F^\lambda_\nu - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right) ,$$

$$\nabla_\alpha F^{\alpha\beta} = 0 .$$

We take the same ansatz for the metric as in (2). Assuming that the gauge potential only has the time component different from zero and that this is spherically symmetric: $A_t = g(r,t)$, $A_i = 0$. With this, the only non-vanishing components of the Faraday tensor are $F_{rt} = -F_{tr} = g'(r,t)$.

From Maxwell's equation, we find the following differential equation for $g(r,t)$:

$$\left(\frac{D-2}{r} + A' + B' \right) g'(r,t) + g''(r,t) = 0 .$$

The general solution to this equation reads:

$$g(r,t) = -\frac{C_1(t)}{(D-3) r^{D-3}} + C_2(t) .$$

We can set the second integration constant to zero, as it only depends on time and we can redefine time without loss of generality. We can also relate the first constant to the electric charge of the black hole. Hence, we find:

$$\boxed{g(r,t) = \frac{Q}{r^{D-3}}}$$

Replacing now in Einstein's equations, we get for the $\mu\nu = rt, tt, rr$ equations, respectively:

$$\frac{D-2}{r} \dot{B} = 0 ,$$

$$\frac{D-2}{2r^2} e^{2A} \left[(D-3)(e^{2B} - 1) + 2rB'(r) \right] = 2G \frac{(D-3)^2}{2} \frac{Q^2}{r^{2(D-2)}} ,$$

$$\frac{D-2}{2r^2} \left[(D-3)(1 - e^{2B}) + 2rA' \right] = -2G \frac{(D-3)^2}{2} e^{-2A} \frac{Q^2}{r^{2(D-2)}} .$$

As in the previous case, the first equation gives us the staticity condition for $B(r,t)$, while the other two give us the solution to this differential equation system. All in all, we find $A(r,t) = -B(r) = A(r)$ with:

$$\boxed{e^{2A(r)} = 1 - \frac{16\pi G M}{(D-2)\Omega_{D-2} r^{D-3}} + \frac{2G(D-3)Q^2}{(D-2) r^{2(D-3)}}} \quad (4)$$

Hence, we find that any spherically symmetric solution of the Einstein-Maxwell system is in fact static and, moreover, given by the D-dimensional Reissner-Nordström metric. Just like for Schwarzschild, we can check that the four-dimensional version of this metric reduces to the more familiar form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)} + r^2 \mathrm{d}\Omega^2 .$$

III. BIRKHOFF'S THEOREM FOR LOVELOCK GRAVITY

The Lovelock Lagrangian describes the most general theory of gravity that contains second-order field equations in D dimensions [6]. Consider a theory built from general contractions of the Riemann tensor and the metric, $\mathcal{L} = \mathcal{L}(R_{abcd}, g^{ab})$. Its equations of motion read [12]:

$$P_a^{cde} R_{bcde} - \frac{1}{2} g_{ab} \mathcal{L} - 2\nabla^c \nabla^d P_{acdb} = 0 ,$$

where $P^{abcd} \equiv \partial \mathcal{L} / \partial R_{abcd}$. These equations typically involve fourth-order derivatives with respect to the metric. However, observe that whenever $\nabla^a P_{acdb} = 0$, the equations become second order. This is precisely the defining condition of Lovelock theories.

The general Lovelock Lagrangian consists of an arbitrary linear combination of the form:

$$\mathcal{L} = \sum_{n=0}^{D/2} \alpha_n \mathcal{X}_{2n} , \quad (5)$$

where $\mathcal{X}_{2n}$ are dimensionally-continued Euler densities of the form:

$$\mathcal{X}_{2n} = \frac{1}{2^n} \delta^{\mu_1\nu_1 \dots \mu_n\nu_n}_{\alpha_1\beta_1 \dots \alpha_n\beta_n} \prod_{r=1}^{n} \mathcal{R}^{\alpha_r\beta_r}_{\mu_r\nu_r} .$$

Here, $\delta^{\mu_1\nu_1 \dots \mu_n\nu_n}_{\alpha_1\beta_1 \dots \alpha_n\beta_n}$ is the totally antisymmetric Kronecker delta. The density $\mathcal{X}_{2n}$ vanishes for $n > D/2$. For $n = D/2$, it becomes topological and, as such, does not contribute to the equations of motion. For $n < D/2$, the density becomes dynamical and does contribute to the equations of motion. The fist two densities correspond, respectively, to the cosmological constant and the Einstein-Hilbert term; $\mathcal{X}_0 = \Lambda$, $\mathcal{X}_2 = R$.

A. Einstein-Gauss-Bonnet

Gauss-Bonnet gravity is a modified theory of gravity that includes, in addition to the Ricci scalar, the quadratic Lovelock piece, given by:

$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 .$$

Therefore, the Gauss-Bonnet action reads:

$$S_{GB} = \frac{1}{16\pi G}\int d^D x\sqrt{|g|}\Big[R + \alpha\mathcal{X}_4\Big] ,$$

where we set the cosmological constant to zero. Also, α is called the Gauss-Bonnet parameter. Varying the action with respect to the metric $\delta S/\delta g^{\mu\nu} = 0$, we obtain the following vacuum field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \alpha\Big[\frac{1}{2}g_{\mu\nu}(R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2) - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}_{\nu} + 4R_{\gamma\delta}R^{\gamma\delta}_{\mu\nu} - 2R_{\mu\gamma\delta\lambda}R^{\gamma\delta\lambda}_{\nu}\Big] .$$

Taking the same ansatz for the metric as in (2), the equations for the $\mu\nu = rt, rr, tt$ components read, respectively:

$$\frac{\dot{B}}{r} = \alpha\frac{2(D-3)(D-4)}{r^3}(e^{-2B}-1)\dot{B} ,$$

$$\frac{1}{2r^2}\big[(D-3)(1-e^{2B}) + 2rA'\big] = \alpha\frac{(D-4)(D-3)}{2r^4} \times (1-e^{-2B})\big[(D-5)(e^{2B}-1) - 4rA'\big] ,$$

$$\frac{1}{2r^2}\big[(D-3)(e^{2B}-1) + 2rB'\big] = \frac{-\alpha(D-4)(D-3)}{2r^4} \times (1-e^{-2B})\big[(D-5)(e^{2B}-1) + 4rB'\big] .$$

From the first equation, we get $B = B(r)$. Rewriting and adding the second and third equations, we get $A' + B' = 0 \Rightarrow A' = -B'$ and therefore $A(r,t) = -B(r) + f(t)$, where we can set $f(t) = 0$ redefining time. Thus, we have found that the spherically symmetric solution of the Lovelock theory with a Gauss-Bonnet term is static.

Now, we can substitute the result $A(r) = -B(r)$ into either the rr or tt equations (they are equivalent), and solve the differential equation for $A(r)$ or $B(r)$. We get:

$$\boxed{e^{2A(r)} = e^{-2B(r)} = 1 + \frac{r^2}{2\alpha(D-3)(D-4)} \cdot\left[1 \pm \sqrt{1 + \frac{64\pi GM\alpha(D-3)(D-4)}{(D-2)\Omega_{D-2}r^{D-1}}}\right]}$$

where we identified the integration constant with the ADM mass.

The above metric corresponds to the Gauss-Bonnet gravity generalization of the Schwarzschild-Tangherlini metric in general dimensions [7]. We can observe there are actually two possible solutions to the system, corresponding to the $\pm$ branches. We can prove only the $-$ branch makes physical sense by studying its behaviour for small α and checking its correspondence with the Schwarzschild solution. So, expanding our solution in a series for $\alpha \to 0$, we find:

$$e^{2A(r)} = +1 - \frac{16\pi GM}{(D-2)\Omega_{D-2}r^{D-3}} + \frac{256\pi^2\alpha(D-3)(D-4)G^2M^2}{\Omega^2_{D-2}r^{2(D-2)}} + \mathcal{O}(\alpha^2) , \tag{6}$$

where the first terms match the ones obtained in (3).

B. Einstein-Maxwell-Gauss-Bonnet

We now aim to unify the electromagnetic action with the gravitational action, incorporating the Gauss-Bonnet quadratic term. Unlike what happened in section II C, we require a spacetime dimension of $D \geq 5$, since the Gauss-Bonnet term assumes a topological nature in $D = 4$ and, consequently, does not contribute to the field equations. Thereupon, the combined action is:

$$S = \int d^D x\sqrt{|g|}\Big[\frac{R}{16\pi G} + \alpha\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big] .$$

Varying with respect to the metric, we arrive to the following field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\cdot 4\pi G\left(T^{EM}_{\mu\nu} + \alpha T^{GB}_{\mu\nu}\right) ,$$

$$T^{EM}_{\mu\nu} = F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} ,$$

$$T^{GB}_{\mu\nu} = g_{\mu\nu}(R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2) - 4RR_{\mu\nu} + 8R_{\mu\gamma}R^{\gamma}_{\nu} + 8R_{\gamma\delta}R^{\gamma\delta}_{\mu\nu} - 4R_{\mu\gamma\delta\lambda}R^{\gamma\delta\lambda}_{\nu} .$$

Again, we take the starting metric as (2). We have separately calculated both sides of the field equations in the previous sections, so the combined case will simply contain the sum of the Gauss-Bonnet and Maxwell contributions, with the first weighed by the α parameter.

Following the same steps as in previous sections, we find that the metric is once more static and characterized by:

$$\boxed{e^{A(r)} = e^{-2B(r)} = 1 + \frac{r^2}{2\alpha(D-4)(D-3)} \cdot\left[1 \pm \sqrt{1 - \frac{8GQ^2\alpha(D-3)^2(D-4)}{(D-2)r^{2(D-2)}} + \frac{64\pi GM\alpha(D-3)(D-4)}{(D-2)\Omega_{D-2}r^{D-1}}}\right]}$$

We see there are two branches to this solution, corresponding to the $\pm$ sign. As we did in section III A, we can prove only the $-$ branch makes physical sense by studying its behaviour for small α and checking its correspondence with the Reissner-Nordström solution. So, expanding our solution in a series for $\alpha \to 0$ up to zeroth order, we obtain the same result as in (4) and, just like in (6), we can determine the first terms in α.

C. General Lovelock gravity

Birkhoff-like theorems for completely general Lovelock theories of the form given by (5) can be similarly proven. In that case, the generalized version of the Schwarzschild solution is again a metric of the form (2) with $e^{-2B(r)} = e^{2A(r)} \equiv f(r)$ where the time-independent metric function $f(r)$ satisfies [9]:

$$\sum_{n=1}^{D/2}\alpha_n\frac{(D-2n)}{2}r^{D-2n-1}(1-f)^n = \frac{16\pi GM}{(D-2)\Omega_{D-2}} .$$

It is easy to verify that this reduces to the Einstein — setting $\alpha_1 \equiv 2/(D-2)$— and Gauss-Bonnet results studied in detail above in the corresponding cases.

IV. BEYOND LOVELOCK GRAVITY?

Birkhoff's theorem does not hold for general higher-curvature gravities. However, there may be additional theories that do satisfy it. A simple example is the so-called cubic Quasi-topological gravity theory [11].

While a full systematic analysis seems to be lacking in the literature, some additional progress has been made in [10]. In that paper, more families of theories satisfying the Birkhoff theorem were found. These are characterized by Lagrangian densities which, while possessing fourth-order equations on general backgrounds, the trace of such equations is second order. Additionally, the full set of field equations becomes second order for spherically symmetric spacetimes. The number of such theories can be written in terms of the amount of independent conformal invariants of order n existing in D dimensions, $N_D^{(n)}$. In any dimension D, they found that there exists a p-parameter family of (non-trivial) Lagrangian densities which generically admit Birkhoff's theorem, where

$$p = 1 + \left(\sum_{1<n<D/2} N_{2n}^{(n)} \right) + \left(\sum_{n \geq D/2} (N_D^{(n)} - 1) \right) , \quad (7)$$

for even D and

$$p = 1 + \left(\sum_{1<n<\frac{D+1}{2}} N_{2n}^{(n)} - 1 \right) + \left(\sum_{n>\frac{D+1}{2}} (N_D^{(n)} - 1) \right) , \quad (8)$$

odd D. As the authors acknowledge, their analysis is partial, and additional theories may be missing.

V. CONCLUSIONS

In this TFG, we have explored the validity of the Birkhoff theorem for and beyond Einstein gravity in general spacetime dimensions. In particular, we have explicitly proved it for the Einstein and Einstein-Maxwell systems, as well as for Einstein-Gauss-Bonnet with and without electric charge. We have also discussed the general Lovelock case, and commented on the state of the art for general theories. In all the cases explored, we have determined the geometry of the spherically symmetric spacetimes, which is given by the corresponding static black hole metrics.

The study of higher-curvature gravitational theories and their properties is an ongoing research field. As we saw, exploring the validity of Birkhoff's theorem for higher-curvature theories is a task which has not been addressed in full generality yet. Natural candidates for additional theories which may satisfy it include the so-called Generalized Quasi-topological gravities [9]. On a different front, it would be interesting to include higher-order terms in the Maxwell field as well. No results have been obtained in this direction whatsoever, so it is natural to expect that at least some subset of such theories will satisfy Birkhoff theorems. It would be interesting to pursue this research direction.

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