Exploring neuronal synchrony through Kuramoto model

Author: Montserrat Romagosa Torrallardona.
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Advisor: Jordi Soriano Fradera

Abstract: The main objective of this study is to analyze the behavior of a neuronal population described by the Kuramoto's model, in which a group of neurons is viewed as an oscillator that interacts with other groups, totaling 100 groups. Two different spatial distributions of the oscillators were studied: homogeneous distribution on a two-dimensional space and random distribution along parallel tracks. We observed that the parameters that influence the most the system's dynamics are the coupling strength between oscillators and their typical distance for connectivity. Both parameters may shape oscillators that either activate in small groups or fully synchronize.

I. INTRODUCTION

Network science and the study of complex systems has gained attention in the recent years in different scientific disciplines, as it offers awareness of the emergent properties and behaviors of interconnected elements. One particularly interesting area of research is the intersection of Kuramoto models and neuroscience, where the application of mathematics and computational simulations provides a unique opportunity to understand the dynamics of neuronal networks. The Kuramoto model [1], originally devised to comprehend the synchronization phenomena in coupled oscillators, offers a powerful and flexible tool for investigating the collective behavior of neurons in the brain. By representing a neuron population as an oscillatory unit influenced by its neighbors, the model captures the synchronization and desynchronization processes, reflecting the complex interconnections within the brain [2].

In this project, our primary focus lies in examining the relationship between the strength of neuronal couplings and the interaction radius between oscillators. By systematically varying the coupling strength and the range of interaction, we aim to explore how these factors impact the overall network dynamics and synchronization. Additionally, we delve into the consequences of imposing spatial constraints on the network connectivity, where certain tracks or pathways within the two–dimensional space are deliberately empty of neurons [3]. This approach allows us to investigate the influence of restricted information flow within the neuronal network and observe how it affects synchronization and information processing.

The implications of this research extend beyond theoretical models and mathematical simulations. Understanding the relationship between coupling strength and interaction radius, as well as the consequences of constrained connectivity, holds huge potential for advancing our knowledge of brain function and disorders. By understanding the mechanisms governing neural dynamics, we can gain knowledge in cognitive processes and neurological disorders.

By investigating the complex interplay between the Kuramoto model, neural couplings, interaction radius and restricted connectivity, this final degree project seeks to contribute to the understanding of brain dynamics and its applications in various scientific fields. Through rigorous analysis, simulation and interpretation of the results, we aim to understand fundamental principles that govern the complexity and adaptability of the brain.

II. METHODS

A. Kuramoto Model

The Kuramoto model describes a neuronal population as an oscillator. The equation that describes the behavior of an oscillator and its coupling with others is written as [1]:

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin\left(\theta_j - \theta_i\right), \tag{1}$$

where $\dot{\theta}_i$ is the phase of the ith oscillator and ω_i is the natural frequency of the ith oscillator. The adjacency matrix A_{ij} for the interaction (or connectivity) between oscillators is given by

$$A_{ij} = \begin{cases} 1 \text{ if } (i,j) \text{ are connected,} \\ 0 \text{ otherwise.} \end{cases}$$

In this case, $A_{ij} = 1$ when the oscillator i is physically able to connect with the jth and viceversa. As we will see later, this corresponds to the case in which both oscillators are close enough in a two–dimensional space, i.e., they are within a given *radius of interaction* R_{int}.

Conceptually, this R_{int} is the Euclidean distance between any two oscillators i and j below which they are connected, i.e. $A_{ij} = 1$ if $d_{ij} < R_{\text{int}}$, and 0 otherwise, with $d = ((x_i - x_j)^2 + (y_i - y_j)^2)^{1/2}$ the Euclidean distance and (x_k, y_k) the spatial coordinates of the oscillator $k = i, j$. We note that, in general, we could have the

*Electronic address: mromagto7@alumnes.ub.edu

oscillators placed on a square surface with arbitrary side length (characteristic spatial size) L. Thus, for the analysis to be size–independent, we need to scale the radius by the spatial size, *i.e.*, consider R_{int}/L. For $R_{\text{int}}/L << 1$ oscillators connect with their immediate neighborhood. For $R_{\text{int}}/L \simeq 1$ all oscillators are connected to one another.

$\lambda = \frac{K}{N}$ is the coupling strength, with K the individual coupling strength (set equal to all oscillators) and N the number of oscillators. This factor λ is the maximum modification an oscillator can produce in the phase of another one in a time step, given that A_{ij} is equal to 0 or 1 and sinus takes values from -1 to 1. The smaller K or λ are, the harder it is to get the oscillators coupled.

The phase of the oscillator is modified by all the other ones inside of its radius of interaction with an strength K. Since K takes the same value for the entire system, the N oscillators interact with the ones they are accessible with the same strength.

To study how coupled the oscillators are, we introduce the following order parameter, taking the module of the equation:

$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right|. \tag{2}$$

For $r \simeq 1$ the system is synchronous, while for $r \simeq 0$ it is asynchronous. Fig. 1 illustrates the behavior of r in a typical simulation with 100 oscillators.

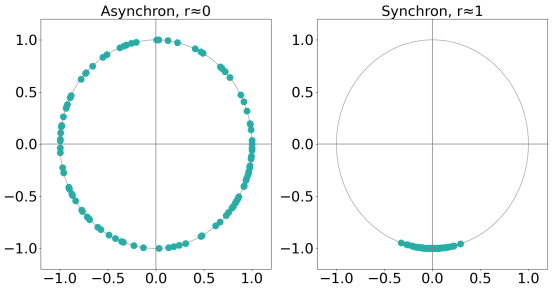

FIG. 1: Oscillator's phase in the complex plane. **Left**, weakly coupled oscillators and asynchronous behavior. **Right**, Strongly coupled oscillators and synchronous behavior.

B. Network analysis

Before starting the simulations, it is necessary to set up the network of oscillators and analyse its properties which, subsequently, will help to understand the simulation's results. The generation of the network was made in Python. Matlab resources, particularly the Brain Connectivity Toolbox, were also used to study the network's characteristics as well as the creation of a `.gexf` document for the posterior visualization of the network with the program Gephi [4].

Firstly, each node or oscillator has to be assigned to a position. This process is done randomly, creating a (x, y) coordinate and writing it in a 100×2 matrix. Two distributions of the nodes were studied: in the first one, there are no restrictions to the nodes positions, while in the second one the location of the nodes is confined within five tracks. Thus, it was necessary, in both the homogeneous and 'tracks' distributions, to prove that the randomly set locations were physically allowed and that they were not already occupied.

Once each oscillator was assigned to a position, the connections between them were established. As explained in the previous section, this information was saved in the adjacency matrix A. This matrix has a size of $N \times N$, where N is the number of oscillators and equal to 100 in our case. The process is carried out as follows:

```
for i in range(N):
        for j in range(N):
            if(((pos[i,0]-pos[j,0])**2/rx**2+
            (pos[i,1]-pos[j,1])**2/ry**2)<=1):
                A[i,j]=1
            if i==j:
                A[i,j]=0
```

Here, `pos` is a $N \times 2$ matrix and `rx` and `ry` are horizontal and vertical vectors of the radius of interaction. `rx` and `ry` are equal or different depending on the system treated. In each row of the `pos` matrix the position (x,y) of each oscillator is saved.

When represented in Gephi, each entry '1' in the adjacency matrix represents a connection and, therefore, each connected oscillator is drawn as a line between two dots, being each dot an oscillator.

To provide insight on the properties of the network, we considered the concept of *modularity* Q as the most important one for our study. A module in this context is a unit whose elements are highly connected within the module and weakly connected to the elements in other modules [5]. Therefore, modularity is the tendency of a network to show their nodes organized in modules or communities [6]. It takes values between $Q = 0$ and $Q = 1$. The higher the modularity, the larger the number of modules or communities the network has, with $Q = 0$ indicating that the whole network is the only community and $Q = 1$ indicating that each oscillator is a community. The concept of modularity will be important to understand how easy it is for the oscillators to get coupled. Modularity was calculated with the Matlab function `community_louvain()`. This function also returns the adjacency matrix reordered by communities.

C. Numerical simulations

A script in Python was written to solve the differential equation of Eq. (1) and obtain the evolution of the order parameter over the time, as follows:

```
for t in range(1, tmax):
    for i in range(N):
        sum = 0
        for j in range(N):
            s = np.sin(theta[j, t-1] -
                - theta[i, t-1])*A[i,j]
            suma = suma + s
        theta[i, t] = theta[i, t-1] +
            + dt * (omega[i] + K / N * suma)
```

where `t` is time, `tmax` the maximum number of time steps, `N` the number of oscillators, `sum` the accumulation of `s`, which is the interior part of the sum in Eq. (1). This brings to the solution of the differential equation. The vector `omega[i]` indicates ω_i in Eq. (1), and is generated randomly for each oscillator and simulation.

Modifying different parameters of the system enabled us to study the change in the behavior of the oscillators with the variation of K or the interaction radius.

For each simulation study, we considered a coupling strength K value and an interaction radius. Then, 10 numerical realizations were carried out in the same conditions, changing w and the position of oscillators. In each realization, once the system had an stable order parameter over the time, an average among all realizations was calculated, providing a final value for the particular set of K and interaction radius.

III. RESULTS AND DISCUSSION

A. Random geometric graph

By using Python functions that generate pseudorandom numbers, each one of the 100 nodes or oscillators were assigned a random position in a 50×50 grid, that represents our Euclidean space. For a given distribution in the space, it was interesting to study the coupling of the system as a function of the normalized interaction radius R_{int}/L and the coupling strength K.

In Fig. 2 one can observe how different the system of oscillators is as R_{int}/L changes, a construction that is known as *random geometric graph*. Indeed, by varying R_{int}/L between 0.1 and 1 (a tenth part of the dimension $L = 50$ of our space to the total dimension of the space) we evolve from a highly fragmented, locally–coupled oscillators to a fully integrated system. Clearly, the normalized radius restricts how far the oscillators are able to connect, allowing to interact to one another inside the circumference described by the radius. We note that the latter case with $R_{\text{int}}/L = 1$ corresponds to a situation in which the spatial location of the oscillators is irrelevant, strictly shaping a *random graph*.

Fig. 3 explores the impact in the dynamics of the system when both R_{int}/L and K are changed. Here, the coupling strength varies from 5 to 20 and, thus, the λ factor in Eq. (1) takes values from 0.05 to 0.2. From the figure one can easily see that the higher interaction

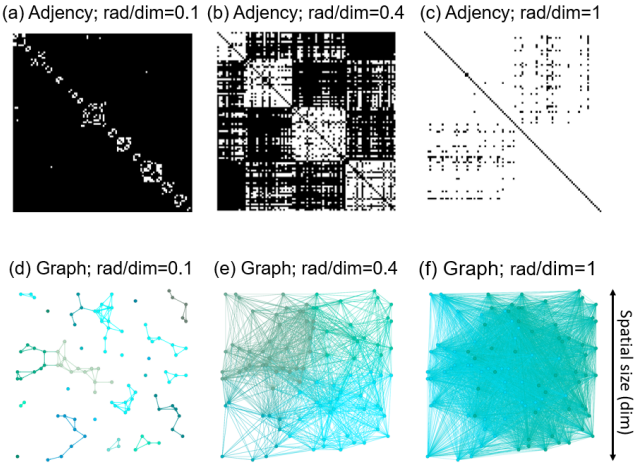

FIG. 2: Visualization of the oscillators' connectivity for the same oscillators system with different normalized interaction radius R_{int}/L. From (a) to (c) the adjacency matrix A is represented. Each white pixel represents a connection between two nodes. From (d) to (f) each plot shows the corresponding location and connectivity of the oscillators. Each color in the graph represents a different community. The number of square white areas along the diagonal of the adjacency matrices coincides with the number of different colors in the graph and equals the number of modules of the network.

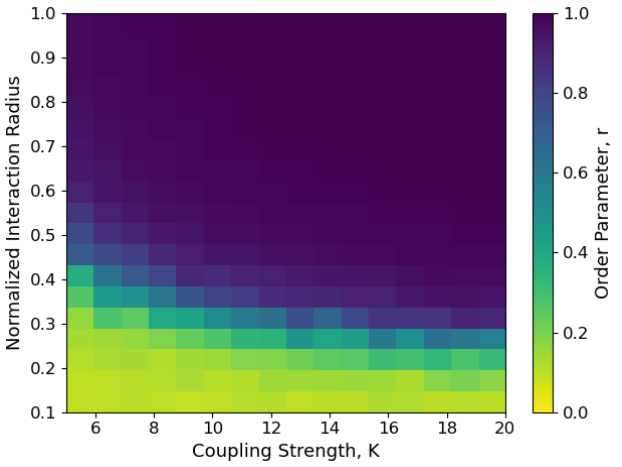

FIG. 3: Colormap of the order parameter, which goes from 0 to 1 (as represented in the colorbar), as a function of the normalized interaction radius and the coupling strength.

radius and coupling strength are, the more strongly the system gets synchronized. This can give an idea of how easy it is to get coupled with the other nodes due to the increase of R_{int}/L or K.

B. Random geometric graph limited to tracks

Once the system in a random spatial embedding was understood, it was interesting to add a restriction to the

oscillators' position. By applying this constraint, modularity is strengthened and isotropy of connectivity broken. Thus, and inspired by the experimental work of Montalà et al. [3] in which biological neurons were constrained to parallel tracks, here we reduced the space available to the oscillators so that they can be placed only in five parallel tracks. In this way, it is more difficult for the oscillators to connect with their horizontal neighbours from the consecutive tracks.

Furthermore, now, the interaction allowed is not in a circle around the node, it is in an ellipse of major axis equal to the magnitude given to the radius in the vertical direction and the minor axis equal to the radius in the horizontal direction, being the vertical radius grater than the horizontal one. Therefore, for the system it is easier to connect vertically than horizontally. The overall results of this construction are shown in Fig. 4, which represents the oscillators' network in two different situations: one with a very short horizontal radius combined with a vertical radius equal to the system's dimension [Fig. 4(a)-(c)], and one where the horizontal and vertical radius are equal to the system's size [Fig. 4(b)-(d)]. Clearly, for the first case, the identified communities in the adjacency matrix match the groups of oscillators within each track.

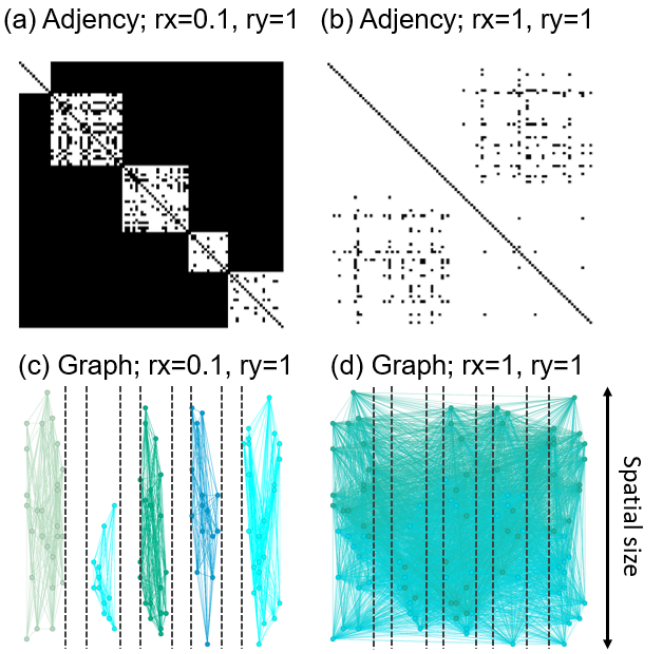

FIG. 4: Visualization of the oscillators' connectivity with different interaction radius. In (a) and (b) the adjacency matrix is represented. In (c) and (d) each the disposition and connectivity of the network is represented. Each color in the graph represents a different community. The number of central squares in the adjacency matrices coincides with the number of different colors in the graph and equals the number of modules of the network.

The corresponding dynamics is shown in Fig. 5. Here,

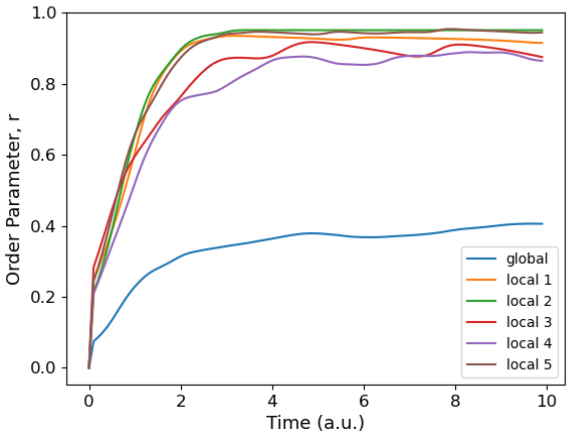

FIG. 5: Evolution of the order parameter over the time. This parameter has been represented for each track of the system (local 1, 2, 3, 4, 5) and globally.

we represent the *global* order parameter over the time (the one that corresponds to the whole system), and the *local* one that corresponds to each track.

Interestingly, for the local situation, the interaction radius in the vertical direction is equal to the matrix dimension, so the nodes can connect with all other nodes in their track; while the horizontal interaction radius is equal to the tenth part of the system's dimension, a fact that makes almost impossible to connect with nodes that do not belong to the same track. Thus, as Fig. 5 shows, the local order parameter is high, while the global is relatively low.

In general, this exploration indicates that it is interesting to study how a system is able to interact globally or locally, and that both trends can be very different depending on connectivity. The importance of connectivity, and its impact on dynamics, can be summarized in the color map of Fig. 6, which shows the ratio between average local and global order parameters, Ratio_{LG}, as a function of the horizontal and vertical connectivity. This ratio is calculated as

$$\text{Ratio}_{\text{LG}} = \frac{\left(\sum_{j=1}^{5} r_j^{loc}\right)/5}{r^{glob}}. \tag{3}$$

Here, in the lightest colors, the global and local order parameters take similar values, while in the darker ones, the local order parameter is larger.

Clearly, for weak horizontal connectivity, the system is unable to connect globally, and the map of Fig. 6 takes blue colors, which means that only the tracks achieve a coupled state. The nodes in each track are not able to connect with other tracks, therefore, the global coupling is not possible. On the contrary, for a fully connected system, the global and local order parameter take the same value because all tracks and the global network can oscillate synchronously.

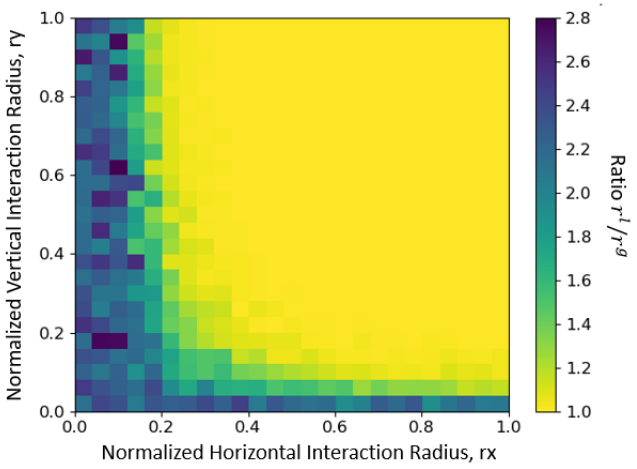

FIG. 6: Colormap of the ratio between the mean local order parameter of the five tracks and the global order parameter, as a function of the normalized horizontal and vertical interaction radius.

Another observation interesting to mention is that, with small vertical radius, when the normalized horizontal radius takes values larger than 0.2, it is easy for the system to couple, while for a vertical radius equal to the dimension of the system, it is impossible to globally couple if the normalized horizontal radius does not exceed 0.2. This is because the nodes are forbidden to take some concrete positions. The horizontal radius must exceed the forbidden zone to allow the interaction between populated tracks and, therefore, get a synchronized system.

As an interesting discussion, we could qualitatively link our results with neurological disorders, such as schizophrenia, epilepsy, Alzheimer's disease and Parkinson's disease. What these diseases have in common is that the brain experiences alterations in neuronal synchrony [7].

In the case of schizophrenia [8], neurons fail at synchronizing at large distances and, thus, in Fig. 3, this disease would qualitatively correspond to the greener areas. Alzheimer's have a same behavior. The opposite happens in epilepsy and Parkinson's: there is an excessive and extended neuronal synchronization as in Fig. 4(d), as well as in the yellowish region of Fig. 6, a clearly damaging state for proper brain functions.

IV. CONCLUSIONS

The Kuramoto model for neuronal populations was successfully simulated. Firstly, we studied the relation between the interaction radius, the coupling strength and the order parameter in a free disposition of the oscillators in the space. The higher the interaction radius and coupling strength are, the more linked oscillators are. The interaction radius is also related to the number of modules the network has, therefore, as the number of communities or modules decrease, the easier it is for the system to achieve the synchrony.

On the other hand, when the space is limited to tracks, we showed that the forced modularity leads to the coupling of the local systems while it is not possible to couple globally. In other words, the nodes in each track can oscillate synchronously but the five tracks have different phases of oscillation. The non populated zones in this second disposition of the nodes in the space makes it more difficult to connect with other oscillators in the horizontal axis (as compared to the vertical one). Therefore, the horizontal radius has to be grater than the vertical to achieve the global synchronization of the system.

Conceptually, in a real brain system, an interaction radius either too high or too low leads to a rupture of the correct degree of synchronization, a trait of different neurological disorders.

Acknowledgments

I would like to express my sincere gratitude to my advisor Dr. Jordi Soriano for his excellent guidance and for introducing me to such an interesting field. I also express my gratitude to my parents and brother for supporting me during this time and my friends for always being there when their advice and support was needed.

[1] Y. Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*. In: Araki, H. (eds) *International Symposium on Mathematical Problems in Theoretical Physics*. Lecture Notes in Physics, **39**, Springer, Berlin, 1975.

[2] G. Ódor, J. Kelling, *Critical synchronization dynamics of the Kuramoto model on connectome and small world graphs*. Scientific Reports **9**, 19621 (2019).

[3] M. Montalà-Flaquer et al., *Rich dynamics and functional organization on topographically designed neuronal networks in vitro*. iScience **25**, 105680 (2022).

[4] M. Bastian, S. Heymann, M. Jacomy, *Gephi: an open source software for exploring and manipulating networks*. Proc. Int. AAAI Conf. Web. Soc. Media **3**, 361-362 (2010)

[5] C. Y. Baldwin, K. B. Clark, *Design Rules: The power of modularity*. MIT Press, Boston, MA, 2000.

[6] E. A. Variano, J. H. McCoy, H. Lipson, *Networks, Dynamics, and Modularity*. Phys. Rev. Lett. **92**, 188101 (2004).

[7] P. J. Uhlhaas, W. Singer *Neural Synchrony in Brain Review Disorders: Relevance for Cognitive Dysfunctions and Pathophysiology*. Neuron **52**, 155-168 (2006).

[8] P. J. Uhlhaas, W. Singer, *Abnormal neural oscillations and synchrony in schizophrenia*. Nature Reviews Neuroscience **11**, 100-113 (2010).