Kinematical algebras in a non-relativistic expansion of the Lorentz force

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Abstract: We consider a non-relativistic expansion of the Lorentz force equation. Both the particle position and the electro-magnetic field are expanded. There are two interesting limits in the case of a constant field, called electric and magnetic, where we show that the resulting equations also follow from considering a non-linear realisation of a certain infinite-dimensional algebra.

I. INTRODUCTION

Non-Lorentzian theories refer to theories which have as their underlying symmetry algebra a different Kinematical algebra than the Poincaré one, like the Galilean algebra or the Carrol algebra. Usually, non-Lorentzian systems can be obtained as the limit of a relativistic system when some characteristic parameter goes to zero (infinity). Consider for example a relativistic free point particle and its velocity relative to the speed of light v/c. Taking this parameter to zero (infinity) one obtains the Galilean (Carrolian) free particle. The process of obtaining a non-Lorentzian algebra from a relativistic one is known as (Inönü–Wigner) Lie algebra contraction. The process is as follows, first, one introduces a dimensionless parameter λ into the original algebra $\mathfrak{g}$ and performs an invertible change of the generators $\{t_\alpha\} \to \{\lambda^{n(\alpha)} t_\alpha\}$, where the exponent $n(\alpha)$ depends on the generator, to obtain an equivalent algebra $\mathfrak{g}_\lambda$. Taking the limit as $\lambda \to \infty$ one obtains a contracted algebra $\mathfrak{g}_0$, which has the same generators as the original algebra, but different commutation relations.

Given a relativistic system, instead of considering its non-Lorentzian limits, one can perform a non-relativistic expansion in terms of the characteristic parameter, that allows to obtain, not only the non-Lorentzian limit, but also a series of corrections. However, only the first term in the expansion exhibits the symmetry of the contracted (non-Lorentzian) algebra, whereas the full expansion exhibits the relativistic symmetry. In [6] it was shown how to study the symmetry algebra of the truncated expansions at any level. The idea is to construct, from the contracted algebra, $\mathfrak{g}_0 := \mathfrak{g}^{(0)}$ with generators $\{t_\alpha^{(0)}\}$ an infinite sequence of expansions $\mathfrak{g}^{(N)}$ with generators $\{t_\alpha^{(n)}\}_{0\leq n\leq N}$, leading to an infinite dimensional Lie algebra $\mathfrak{g}^{(\infty)}$. This infinite dimensional algebra is like a non-relativistic expansion of the contracted algebra. Since $\mathfrak{g}$ acts on the space-time manifold M, we will construct an infinite dimensional space $M^{(\infty)}$ using non-linear realisations, on which this expanded algebra $\mathfrak{g}^{(\infty)}$ will act. Introducing *collective coordinates* on this generalized space, one can recover the space M and the symmetry algebra $\mathfrak{g}$. The aim of this work is to show how to use this general construction to obtain the expansion of the Lagrangian of a point-particle subject to an external constant electromagnetic field.

In [5] it was shown that the Poincaré algebra admits a non-central extension, the **Maxwell** algebra. The most general Lagrangian which realise this symmetry algebra is:

$$\mathcal{L}d\tau = -mc\sqrt{-\dot{x}^a\dot{x}_a} - \frac{1}{2}f_{ab}\Omega^{ab}$$

where $\Omega^{ab} := d\theta^{ab} + \frac{1}{2}\left(dx^a x^b + dx^b x^a\right)$ is the Maurer-Cartan form and $f_{ab}(\tau)$ and θ^{ab} are new dynamical variables. This Lagrangian describes a particle subject to an external, constant electromagnetic field. In [9] it was shown that starting from the Galilean algebra $\mathfrak{G}$, one could obtain a non-relativistic expansion of the relativistic free particle Lagrangian $\mathcal{L}d\tau = \sqrt{-\dot{x}^2}$, by considering an infinite dimensional algebra $\mathfrak{G}_\infty$. Analogously, we will show that starting from two non-relativistic limits of the **Maxwell** algebra, the electric $\mathfrak{E}$ and magnetic $\mathfrak{M}$ Maxwell algebras, we will obtain a non-relativistic expansion of the Maxwell Lagrangian, through the construction of infinite dimensional algebras $\mathfrak{E}_\infty$ and $\mathfrak{M}_\infty$. These infinite dimensional algebras admit quotients that describe the symmetries of the expansion up to a finite order in $1/c$. These algebras, can also be obtained as particular quotients of the Galilean free algebras [7].

The organization of this work is as follows: In Section II we will obtain a non-relativistic expansion of the Lorentz equation in powers of $1/c^2$ when the constant electric (magnetic) field is dominant.

In Section III we will study the same problem, that of a non-relativistic expansion of the Lorentz equation, through its algebra of symmetries, the **Maxwell** algebra.

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II. NON-RELATIVISTIC EXPANSION OF THE LORENTZ EQUATION

The Lorentz force describes the evolution of a particle under an electromagnetic field. In covariant form, it reads:

$$\frac{dp^a}{d\tau} = qF^{ab}\frac{dx_b}{d\tau} \tag{1}$$

Where p^a denotes the relativistic 4-momentum, F^{ab} is the electromagnetic tensor, x^b is the 4-position of the particle, q the charge of the particle and τ the proper time. Throughout this text we will be using the Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

This equation is relativistic, which means that it is invariant under the Poincaré Lie group $SO(1, 3) \ltimes \mathbb{R}^4$. We wish to consider a non-relativistic expansion of this equation in powers of the speed of light c.

To do this note that the Lorentz equation may be rewritten as

$$mc\frac{d}{d\tau}\left(\frac{\dot{x}^a}{\sqrt{-x^2}}\right) = qF^{ab}\dot{x}_b, \tag{2}$$

where $\dot{}$ denotes derivative with respect to τ. Setting $q = 1$ and separating time and space indices,

$$m\frac{d}{d\tau}\left(\frac{1}{\sqrt{1 - \dot{\vec{x}}^2/(c\dot{t})^2}}\right) = \frac{1}{c^2}\tilde{F}^{ti}\dot{x}_i, \tag{3}$$

$$m\frac{d}{d\tau}\left(\frac{\dot{x}^i}{\dot{t}\sqrt{1 - \dot{\vec{x}}^2/(c\dot{t})^2}}\right) = \tilde{F}^{ti}\dot{t} + F^{ij}\dot{x}_j, \tag{4}$$

where we have introduced a rescaling of F^{0i} such that $\tilde{F}^{ti} = cF^{0i}$, so that $\tilde{F}^{ti}$ now has units of electric field. To obtain the non-relativistic expansion, we propose an expansion of

$$t = t_{(0)} + \frac{1}{c^2}t_{(1)} + \ldots, \quad x^i = x^i_{(0)} + \frac{1}{c^2}x^i_{(1)} + \ldots \tag{5}$$

Together with an expansion of the fields

$$\tilde{F}^{ti} = \tilde{F}^{ti}_{(0)} + \frac{1}{c^2}F^{ti}_{(1)} + \ldots, \quad F^{ij} = F^{ij}_{(0)} + \frac{1}{c^2}F^{ij}_{(1)} + \ldots \tag{6}$$

Substituting in (3-4), Taylor expanding the γ-factor and collecting term by term in powers of c^2, one ends up with the following non-relativistic expansion of the Lorentz force.

$$m\frac{d}{d\tau}\left[\frac{\dot{\vec{x}}^2_{(0)}}{2\dot{t}^2_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{x}_{(0)i} \tag{7}$$

$$m\frac{d}{d\tau}\left[\frac{\dot{x}^i_{(0)}}{\dot{t}_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{t}_{(0)} + F^{ij}_{(0)}\dot{x}_{(0)j} \tag{8}$$

$$m\frac{d}{d\tau}\left[\frac{3}{8}\frac{(\dot{\vec{x}}^2_{(0)})^2}{\dot{t}^4_{(0)}} + \frac{\dot{\vec{x}}_{(0)}\cdot\dot{\vec{x}}_{(1)}}{\dot{t}^2_{(0)}} - \frac{\dot{\vec{x}}^2_{(0)}\dot{t}_{(1)}}{\dot{t}^3_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{x}_{(1)i} + \tilde{F}^{ti}_{(1)}\dot{x}_{(0)i} \tag{9}$$

$$m\frac{d}{d\tau}\left[\frac{1}{2}\frac{\dot{\vec{x}}^2_{(0)}\dot{x}^i_{(0)}}{\dot{t}^3_{(0)}} - \frac{\dot{t}_{(1)}\dot{x}^i_{(0)}}{\dot{t}^2_{(0)}} + \frac{\dot{x}^i_{(1)}}{\dot{t}_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{t}_{(1)} + F^{ij}_{(0)}\dot{x}_{(1)j} + \tilde{F}^{ti}_{(1)}\dot{t}_{(0)} + F^{ij}_{(1)}\dot{x}_{(0)j} \tag{10}$$

$$\ldots$$

They encode the non relativistic Lorentz force, together with energy conservation.

Due to Lévy-Leblond [2], we know that there is not an unique well-defined non-relativistic limit of electromagnetism. Instead, different non-relativistic regimes appear depending on the relative strength of the electric and magnetic field. We will study the two main limits, namely:

1. The magnetic limit, where $|\mathbf{E}| / |\mathbf{B}| \ll c$.
2. The electric limit, where $|\mathbf{E}| / |\mathbf{B}| \gg c$.

Translating these conditions into our field expansion, the magnetic limit is obtained by having $F^{ij}_{(0)} \neq 0$ and keeping $\tilde{F}^{ti}$ fixed since we already have $F^{ij} \gg \frac{1}{c}\tilde{F}^{ti}$. The electric limit is obtained by setting $F^{ij}_{(0)}$ so that the expansion now reads $F^{ij} = \frac{1}{c^2}F^{ij}_{(1)} + \ldots$, and it is clear that $F^{ij} \ll \frac{1}{c}\tilde{F}^{ti}$.

The Lorentz equations in the magnetic limit are simply (7)-(10). Introducing an expansion of the boost parameter $v^i = v^i_{(0)} + \frac{1}{c^2}v^i_{(1)} + \ldots$, and space-time translations $\epsilon = \epsilon_{(0)} + \frac{1}{c^2}\epsilon_{(1)} + \ldots$, $\epsilon^i = \epsilon^i_{(0)} + \frac{1}{c^2}\epsilon_{(1)} + \ldots$ it can be seen that the Lorentz equations in the magnetic limit are invariant under:

$$\delta t_{(n)} = \epsilon_{(n)} + \sum_{m=0}^{n-1}\delta_{ij}v^i_{(m)}x^j_{(n-m-1)} \tag{11}$$

$$\delta x^i_{(n)} = \epsilon^i_{(n)} + \sum_{m=0}^{n} v^i_{(m)}t_{(n-m)} \tag{12}$$

$$\delta\tilde{F}^{ti}_{(n)} = \sum_{m=0}^{n} v_{(m)k}F^{ki}_{(n-m)}, \quad \delta F^{ij}_{(n)} = \sum_{m=0}^{n-1} -2\tilde{F}^{t[i}_{(m)}v^{j]}_{(n-m-1)} \tag{13}$$

for all n.

In the electric limit we have the following equations,

$$m\frac{d}{d\tau}\left[\frac{\dot{\vec{x}}^2_{(0)}}{2\dot{t}^2_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{x}_{(0)i} \tag{14}$$

$$m\frac{d}{d\tau}\left[\frac{\dot{x}^i_{(0)}}{\dot{t}_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{t}_{(0)} \tag{15}$$

$$m\frac{d}{d\tau}\left[\frac{3}{8}\frac{(\dot{\vec{x}}^2_{(0)})^2}{\dot{t}^4_{(0)}} + \frac{\dot{\vec{x}}_{(0)}\cdot\dot{\vec{x}}_{(1)}}{\dot{t}^2_{(0)}} - \frac{\dot{\vec{x}}^2_{(0)}\dot{t}_{(1)}}{\dot{t}^3_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{x}_{(1)i} + \tilde{F}^{ti}_{(1)}\dot{x}_{(0)i} \tag{16}$$

$$m\frac{d}{d\tau}\left[\frac{1}{2}\frac{\dot{\vec{x}}^2_{(0)}\dot{x}^i_{(0)}}{\dot{t}^3_{(0)}} - \frac{\dot{t}_{(1)}\dot{x}^i_{(0)}}{\dot{t}^2_{(0)}} + \frac{\dot{x}^i_{(1)}}{\dot{t}_{(0)}}\right] = \tilde{F}^{ti}_{(0)}\dot{t}_{(1)} + \tilde{F}^{ti}_{(1)}\dot{t}_{(0)} + F^{ij}_{(1)}\dot{x}_{(0)j} \tag{17}$$

$$\dots$$

which are instead invariant under

$$\delta t_{(n)} = \epsilon_{(n)} + \sum_{m=0}^{n-1}\delta_{ij}v^i_{(m)}x^j_{(n-m-1)} \tag{18}$$

$$\delta x^i_{(n)} = \epsilon^i_{(n)} + \sum_{m=0}^{n} v^i_{(m)}t_{(n-m)} \tag{19}$$

$$\delta\tilde{F}^{ti}_{(n)} = \sum_{m=0}^{n-1} v_{(m)k}F^{ki}_{(n-m-1)}, \quad \delta F^{ij}_{(n)} = \sum_{m=0}^{n} -2\tilde{F}^{t[i}_{(m)}v^{j]}_{(n-m)} \tag{20}$$

for all n.

III. LIE ALGEBRA REFORMULATION

The **Maxwell** algebra is a non-central extension of the Poincaré algebra, with commutation relations:

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{bd}M_{ac} - \eta_{ac}M_{bd} + \eta_{ad}M_{bc} \tag{21}$$

$$[M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b \tag{22}$$

$$[P_a, P_b] = Z_{ab} \tag{23}$$

Where P_a denote space-time translation generators, M_{ab} Lorentz generators and Z_{ab} are the new generators. As seen in [7], it admits two non-relativistic limits, the Electric and Magnetic Maxwell algebras. Defining boost generators $G_i := M_{0i}$, rotation generators $J_{ij} := M_{ij}$ and time translation generators $H := P_0$, the Electric Maxwell algebra satisfies (omitting rotations):

$$[G_i, P_j] = 0, \qquad [G_i, Z_j] = 0, \tag{24}$$

$$[H, G_i] = P_i, \qquad [P_i, P_j] = 0, \tag{25}$$

$$[H, P_i] = Z_i, \qquad [G_k, Z_{ij}] = 2\delta_{k[i}Z_{j]}, \tag{26}$$

$$[G_i, G_j] = 0, \tag{27}$$

Whereas the Magnetic Maxwell algebra differs in these three commutators:

$$[G_i, Z_j] = -Z_{ij}, \quad [P_i, P_j] = Z_{ij}, \quad [G_k, Z_{ij}] = 0. \tag{28}$$

In [9] it was shown that a particular expansion of the Poincaré algebra, the $\mathfrak{G}_\infty$ algebra, led to a non-relativistic series expansion of the free particle Lagrangian $\mathcal{L}d\tau = \sqrt{-\dot{x}^2}$.

We will show that a similar process can be done for the Maxwell Lagrangian, leading to an electric and magnetic non-relativistic expansion, which coincide with (7)-(10) and (14)-(17) respectively.

A. Electric limit

To obtain $\mathfrak{E}_\infty$, one starts with **Maxwell** and construct the expanded algebra $\mathfrak{E}_\infty$ via the method of infinite Lie algebra expansion, with generators

$$H^{(m)} = P_0 \otimes c^{-2m+1}, \qquad P^{(m)}_i = P_i \otimes c^{-2m}$$
$$G^{(m)}_i = M_{0i} \otimes c^{-2m-1} \qquad J^{(m)}_{ij} = M_{ij} \otimes c^{-2m}$$
$$Z^{(m)}_i = Z_{0i} \otimes c^{-2m+1} \qquad Z^{(m)}_{ij} = Z_{ij} \otimes c^{-2m+2}$$

Which satisfy the following commutation relations

$$\left[H^{(m)}, G^{(n)}_i\right] = P^{(m+n)}_i, \qquad \left[G^{(m)}_i, P^{(n)}_j\right] = \delta_{ij}H^{(m+n+1)} \tag{29}$$

$$\left[G^{(m)}_i, G^{(n)}_j\right] = J^{(m+n+1)}_{ij}, \qquad \left[G^{(m)}_k, Z^{(n)}_{ij}\right] = 2\delta_{k[i}Z^{(m+n)}_{j]}, \tag{30}$$

$$\left[G^{(m)}_i, Z^{(n)}_j\right] = Z^{(m+n)}_{ij}, \qquad \left[H^{(m)}, P^{(n)}_i\right] = Z^{(m+n)}_i, \tag{31}$$

$$\left[P^{(m)}_i, P^{(n)}_j\right] = Z^{(m+n+1)}_{ij}. \tag{32}$$

We have omitted transformations with respect to generalized rotations $J^{(m)}_{ij}$, since everything transforms as tensors. Note that quotienting by the ideal consisting of all generators from levels $m \geq 1$ we recover the Electric Maxwell algebra. As shown in the introduction, we shall now define the generalised space $M^{(\infty)}$ on which this infinite-dimensional algebra acts by quotienting by generalized "Lorentz" generators: $\mathcal{L}_\infty := \{G^{(m)}_i, J^{(m)}_{ij}\}_{m\geq 0}$, i.e. the formal coset $\exp\mathfrak{E}_\infty/\exp\mathcal{L}_\infty$. We introduce in this space local coordinates $x^i_{(m)}$, $t_{(m)}$, $\theta^i_{(m)}$, $\phi^{ij}_{(m)}$, dual to: $P^{(m)}_i, H^{(m)}, Z^{(m)}_i, Z^{(m)}_{ij}$ respectively. The infinitesimal action of an element of the form

$$\sum_n \epsilon_{(n)}H^{(n)} + \epsilon^i_{(n)} + v^i_{(n)}G^{(n)}_i + \varepsilon^i_{(n)}Z^{(n)}_i + \varepsilon^{ij}_{(n)}Z^{ij}_{(n)} \tag{33}$$

is, on the local coordinates:

$$\delta t_{(m)} = \epsilon_{(m)} + \sum_n v^i_{(m-n-1)} x_i^{(n)} \tag{34}$$

$$\delta x^i_{(m)} = \epsilon^i_{(m)} + \sum_n v^i_{(m-n)} t_{(n)} \tag{35}$$

$$\begin{aligned}\delta\theta^i_{(m)} = &\varepsilon^i_{(m)} + \sum_n \frac{1}{2}\epsilon^i_{(m-n)} t_{(n)} \\ &- \frac{1}{2}\epsilon_{(m-n)} x^i_{(n)} - 2v^k_{(m-n-1)}\phi^i_{k(n)}\end{aligned} \tag{36}$$

$$\begin{aligned}\delta\phi^{ij}_{(m)} = &\varepsilon^{ij}_{(m)} + \frac{1}{2}\sum_n \epsilon^i_{(m-n)} x^j_{(n)} - v^i_{(m-n)}\theta^j_{(n)} \\ &- \sum_r v^{[i}_{(m-n-r)} t_{(n)} x^{j]}_{(r)}\end{aligned} \tag{37}$$

Restricting to the level zero generators, we recover the transformation laws of "electric" Galilean electromagnetism [2]. Consider the following collective coordinates

$$\begin{aligned} X^i &= \sum_{m=0}^{\infty} c^{-2m} x^i_{(m)} & T &= \sum_{m=0}^{\infty} c^{-2m} t_{(m)} \\ \Theta^i &= \sum_{m=0}^{\infty} c^{-2m+1}\theta^i_{(m)} & \Phi^{ij} &= \sum_{m=0}^{\infty} c^{-2m+2}\phi^{ij}_{(m)} \\ F_{0i} &= \sum_{m=0}^{\infty} c^{-2m-1} f^{(m)}_{0i} & F_{ij} &= \sum_{m=0}^{\infty} c^{-2m-2} f^{(m)}_{ij} \end{aligned}$$

And the Maxwell action of the collective coordinates:

$$S = \int -mc\sqrt{-\dot{X}^\mu \dot{X}_\mu} - \frac{1}{2}F_{ab}\overline{\Omega}^{ab}.$$

Where $\overline{\Omega}^{ab} := d\Theta^{ab} + \frac{1}{2}\left(dX^a X^b + dX^b X^a\right)$ is the Maurer-Cartan form, $\Theta^{0i} := \Theta^i$ and $\Theta^{ij} := \Phi^{ij}$. Expanding each collective coordinate as its power series we obtain a series $S = S_{(0)} + S_{(1)} + S_{(2)} + \dots$,

$$S_{(0)} = -mc^2 \int d\tau \left[\dot{t}_{(0)}\right] \tag{38}$$

$$\begin{aligned} S_{(1)} = \int d\tau \Bigg\{ &-m\left[\dot{t}_{(1)} - \frac{\dot{x}^2_{(0)}}{2\dot{t}_{(0)}}\right] \\ &-f^{(0)}_{0i}\left(\dot{\theta}^i_{(0)} + \frac{1}{2}\left(\dot{t}_{(0)}x^i_{(0)} - \dot{x}^i_{(0)}t_{(0)}\right)\right) - \frac{1}{2}f^{(0)}_{ij}\left(\dot{\phi}^{ij}_{(0)}\right)\Bigg\} \end{aligned} \tag{39}$$

$$\begin{aligned} S_{(2)} = \frac{1}{c^2}\int d\tau \Bigg\{ &-m\left[\dot{t}_{(2)} - \frac{\dot{x}^i_{(0)}\dot{x}^j_{(1)}\delta_{ij}}{\dot{t}_{(0)}} + \frac{\dot{t}_{(1)}\dot{x}^2_{(0)}}{2\dot{t}^2_{(0)}} - \frac{\dot{x}^4_{(0)}}{8\dot{t}^3_{(0)}}\right] \\ &- f^{(0)}_{0i}\left(\dot{\theta}^i_{(1)} + \frac{1}{2}\left(\dot{t}_{(1)}x^i_{(0)} + x^i_{(1)}\dot{t}_{(0)} - \dot{x}^i_{(0)}t_{(1)} - t_{(0)}\dot{x}^i_{(1)}\right)\right) \\ &- f^{(1)}_{0i}\left(\dot{\theta}^i_{(0)} + \frac{1}{2}\left(\dot{t}_{(0)}x^i_{(0)} - \dot{x}^i_{(0)}t_{(0)}\right)\right) \\ &- \frac{1}{2}f^{(0)}_{ij}\left(\dot{\phi}^{ij}_{(1)} + \frac{1}{2}\left(\dot{x}^i_{(0)}x^j_{(0)} - \dot{x}^j_{(0)}x^i_{(0)}\right)\right) - \frac{1}{2}f^{(1)}_{ij}\dot{\phi}^{ij}_{(0)}\Bigg\} \end{aligned} \tag{40}$$

Which is invariant under the transformations laws of $\mathfrak{E}_\infty$ (34)-(37). Computing the equations of motion for the action $S_{(2)}$, we recover the non-relativistic expansion of the Lorentz equation (14)-(17). Therefore, by expanding the non-relativistic algebra $\mathfrak{E}$ we have recovered the relativistic correction we lost by performing a Lie algebra contraction.

B. Magnetic limit

The process is completely analogous to the electric case, so we will just sketch the differences. For the magnetic case $\mathfrak{M}_\infty$, we define the following generators

$$\begin{aligned} H^{(m)} &= P_0 \otimes c^{-2m+1} & P^{(m)}_i &= P_i \otimes c^{-2m} \\ G^{(m)}_i &= M_{0i}\otimes c^{-2m-1} & J^{(m)}_{ij} &= M_{ij}\otimes c^{-2m} \\ Z^{(m)}_i &= Z_{0i}\otimes c^{-2m+1} & Z^{(m)}_{ij} &= Z_{ij}\otimes c^{-2m} \end{aligned}$$

Which satisfy the same commutation relations as (29)-(32) except for

$$\left[G^{(m)}_k, Z^{(n)}_{ij}\right] = 2\delta_{k[i}Z^{(m+n+1)}_{j]}, \quad \left[P^{(m)}_i, P^{(n)}_j\right] = Z^{(m+n)}_{ij} \tag{41}$$

Again, quotienting by the ideal consisting of all generators from levels $m \geq 1$ we recover the Magnetic Maxwell algebra. Defining the generalised space as the formal coset $\exp\mathfrak{M}_\infty / \exp\mathcal{L}_\infty$, with local coordinates $x^i_{(m)}, t_{(m)}, \theta^i_{(m)}, \phi^{ij}_{(m)}$, associated to: $P^{(m)}_i, H^{(m)}, Z^{(m)}_i, Z^{(m)}_{ij}$ respectively. The infinitesimal action of an element of the form (33) is now

$$\delta t_{(m)} = \epsilon_{(m)} + \sum_n v^i_{(m-n-1)} x_i^{(n)} \tag{42}$$

$$\delta x^i_{(m)} = \epsilon^i_{(m)} + \sum_n v^i_{(m-n)} t_{(n)} \tag{43}$$

$$\begin{aligned}\delta\theta^i_{(m)} = &\varepsilon^i_{(m)} + \sum_n \frac{1}{2}\epsilon^i_{(m-n)} t_{(n)} - \frac{1}{2}\epsilon_{(m-n)} x^i_{(n)} \\ &- 2v^k_{(m-n)}\phi^i_{k(n)}\end{aligned} \tag{44}$$

$$\begin{aligned}\delta\phi^{ij}_{(m)} = &\varepsilon^{ij}_{(m)} + \frac{1}{2}\sum_n \epsilon^i_{(m-n-1)} x^j_{(n)} - v^i_{(m-n-1)}\theta^j_{(n)} \\ &- \sum_r v^{[i}_{(m-n-r-1)} t_{(n)} x^{j]}_{(r)}\end{aligned} \tag{45}$$

Again, restricting to the level $m = 0$, we recover the transformations laws for the Magnetic Galilean electromagnetism [2].

The collective coordinates will be the same as the electric case, except for $F_{ij} = \sum_{m=0}^{\infty} c^{-2m} f^{(m)}_{ij}$.

The magnetic expansion of the Maxwell action is

$$S_{(0)} = -mc^2 \int d\tau \left[\dot{t}_{(0)}\right] \tag{46}$$

$$\begin{aligned} S_{(1)} = &\int d\tau \left\{ -m \left[\dot{t}_{(1)} - \frac{\dot{x}_{(0)}^2}{2\dot{t}_{(0)}} \right] \right. \\ &- f_{0i}^{(0)} \left(\dot{\theta}_{(0)}^i + \frac{1}{2}\left(\dot{t}_{(0)} x_{(0)}^i - \dot{x}_{(0)}^i t_{(0)} \right) \right) \\ &\left. - \frac{1}{2} f_{ij}^{(0)} \left(\dot{\phi}_{(0)}^{ij} + \dot{x}_{(0)}^{[i} x_{(0)}^{j]} \right) \right\} \end{aligned} \tag{47}$$

$$\begin{aligned} S_{(2)} = &\frac{1}{c^2} \int d\tau \left\{ m \left[-\dot{t}_{(2)} + \frac{\dot{x}_{(0)}^i \dot{x}_{(1)}^j \delta_{ij}}{\dot{t}_{(0)}} - \frac{\dot{t}_{(1)} \dot{x}_{(0)}^2}{2\dot{t}_{(0)}^2} + \frac{\dot{x}_{(0)}^4}{8\dot{t}_{(0)}^3} \right] \right. \\ &- f_{0i}^{(1)} \left(\dot{\theta}_{(0)}^i + \frac{1}{2}\left(\dot{t}_{(0)} x_{(0)}^i - \dot{x}_{(0)}^i t_{(0)} \right) \right) \\ &- f_{0i}^{(0)} \left(\dot{\theta}_{(1)}^i + \frac{1}{2}\left(\dot{t}_{(1)} x_{(0)}^i + x_{(1)}^i \dot{t}_{(0)} - \dot{x}_{(0)}^i t_{(1)} - t_{(0)} \dot{x}_{(1)}^i \right) \right) \\ &- f_{ij}^{(0)} \left(\dot{\phi}_{(1)}^{ij} + \left(\dot{x}_{(1)}^{[i} x_{(0)}^{j]} + \dot{x}_{(0)}^{[j} x_{(1)}^{i]} \right) \right) \\ &\left. - f_{ij}^{(1)} \left(\dot{\phi}_{(0)}^{ij} + \dot{x}_{(0)}^{[i} x_{(0)}^{j]} \right) \right\} \end{aligned} \tag{48}$$

Which are invariant under the $\mathfrak{M}_\infty$ algebra. Computing the equations of motion for $S_{(2)}$ one recovers the non-relativistic expansion of the Lorentz force in the magnetic limit (7)-(10).

One may ask what the role of the θ^i and ϕ^{ij} variables is on the non-relativistic expansion of the Lorentz equation. The key idea connecting these two formalisms is that if one wants to obtain a Lagrangian from which the non-relativistic expansion of Lorentz equations arise, one would need to add new variables θ^{ab} as Lagrange multipliers to assure that F^{ab} is constant on-shell. They transform under the action of the symmetry algebra as (36-37) and (44-45). These extra variables can be regarded as the dipole moment of the particle [4], since in the Maxwell Lagrangian, $\dot{\theta}^{ab}$ is proportional to the angular momentum (magnetic moment) of the particle on-shell.

IV. CONCLUSIONS

We showed how to obtain a non-relativistic expansion in powers of $1/c^2$ of the Lorentz equation for a constant electro-magnetic field from a completely algebraic point of view, applying the construction presented in [6], and following the results obtained in [8], which we also satisfactorily reproduced. From the non-relativistic limits of the Maxwell algebra, $\mathfrak{M}$ and $\mathfrak{E}$, we constructed the infinite dimensional algebras $\mathfrak{M}_\infty$ and $\mathfrak{E}_\infty$, which allowed us to obtain an order by order non-relativistic expansion of the Lorentz force, which matched the non-relativistic expansion one obtains from the Lorentz equation. Furthermore, the truncation of these algebras at level ℓ gives us the symmetry algebra of the non-relativistic expansion up to level $c^{2-2\ell}$.

These results will be explained in much further detail in a paper currently in progress.

This formalism is completely general, and can be applied to a number of situations. For instance, the post-Newtonian expansion of the two body problem for gravitational wave emission. Also mention that in [11] a one to one correspondence between stationary motion of a particle and orbits in a constant E.M. field was established, implying that the equations of motion obtained correspond to stationary motion.

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[1] Carles Batlle and Joaquim Gomis. Space-time schrödinger symmetries of a post-galilean particle. *Journal of High Energy Physics*, (8), 2020.

[2] M. Le Bellac and J.M Lévy-Leblond. Galilean electromagnetism. *Nuovo Cimento B*, (14), 2008.

[3] Eric Bergshoeff, José Figueroa-O'Farrill, and Joaquim Gomis. A non-lorentzian primer. *SciPost Physics Lecture Notes*, 69, 2023.

[4] Sotirios Bonanos and Joaquim Gomis. Infinite sequence of poincaré group extensions: structure and dynamics. *Journal of Physics A: Mathematical and Theoretical*, 43(1), 2009.

[5] Joaquim Gomis and Axel Kleinschmidt. On free lie algebras and particles in electro-magnetic fields. *Journal of High Energy Physics*, 2017(7), 2017.

[6] Joaquim Gomis and Axel Kleinschmidt. Infinite-dimensional algebras as extensions of kinematic algebras, 2022.

[7] Joaquim Gomis, Axel Kleinschmidt, and Jakob Palmkvist. Galilean free lie algebras. *Journal of High Energy Physics*, 2019(9), 2019.

[8] Joaquim Gomis, Axel Kleinschmidt, Jakob Palmkvist, and Patricio Salgado-Rebolledo. Newton-hooke/carrollian expansions of (a)dS and chern-simons gravity. *Journal of High Energy Physics*, (2), 2020.

[9] Joaquim Gomis, Axel Kleinschmidt, Jakob Palmkvist, and Patricio Salgado-Rebolledo. Symmetries of post-galilean expansions. *Phys. Rev. Lett.*, 124, 2020.

[10] Jorge G Russo and Paul K Townsend. Relativistic kinematics and stationary motions. *Journal of Physics A: Mathematical and Theoretical*, 42(44), 2009.

[11] A. H. Taub. Orbits of charged particles in constant fields. *Phys. Rev.*, 73:786–798, 1948.