# Stochastic inflation in the Constant Roll regime

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We investigate the inhomogeneities generated during the inflationary epoch from the point of view of the stochastic formalism, which attempts to transform a problem of quantum fluctuations into a statistical one. The formalism, that we derive in the text, is based on the use of the Arnowitt-Deser-Misner (ADM) equations, which are convenient to describe inhomogeneities in the context of inflation, as well as gradient expansion, which works at zeroth order in spatial gradients but at all orders in the amplitudes of the fluctuations, and is therefore intended to capture non-perturbative effects. Finally, the perturbations are split into long- and short-wavelength modes, where the latter act as a stochastic noise for the former when crossing a certain scale.

We demonstrate that the use of certain approximations in the derivation of this formalism, which are intended to make the system of stochastic differential equations (SDEs) Markovian and described with white noises, causes the method to become restricted to the reproduction of Linear Perturbation Theory (LPT). This framework, nonetheless, is still useful since it can be used as a test for the validity of the linear approximation, signalling the coming into play of non-perturbative effects. Specifically, we solve the system of SDEs numerically for the Constant Roll (CR) inflationary scenario, and show that this regime is in accordance with LPT.

## CONTENTS

## I. INTRODUCTION

Inflation, a quasi-de Sitter expansion of the very early Universe driven by a scalar field, has been very successful in the context of cosmology due to its ability to successfully resolve the causality (or horizon) problem of the standard $\Lambda$CDM model, in addition to explaining the flatness of the Universe and predicting the distribution of its small inhomogeneities or perturbations. The latter are thought to originate as quantum fluctuations which are extended during the inflationary period to super-horizon scales, and only later on, once inflation has ended, do they reenter the horizon and re-collapse, forming galaxies and other structures that populate the current Universe.

One of the principal tests of these inhomogeneities is the spectrum of the Cosmic Microwave Background (CMB), formed by perturbations that reenter the horizon at the time of recombination, which has been successfully predicted in the context of the Slow Roll (SR) regime of inflation [1, 2]. Nonetheless, there are other regimes, such as Ultra Slow Roll (USR) and Constant Roll (CR), which might be relevant in other stages of inflation and are able to produce large inhomogeneities at scales smaller than the CMB anisotropies. These density perturbations, if large enough, could then collapse into a Primordial Black Hole (PBH) when reentering the horizon.

The motivation behind the study of PBHs comes from the fact that they are a Dark Matter candidate, as well as possibly being the seeds of the supermassive Black Holes that we encounter in galactic nuclei [3–5]. In addition, they could have generated the gravitational wave events detected by LIGO [6]. The abundance of these PBHs depends on the power spectrum of primordial fluctuations [7], and therefore their study in the context of the USR and CR regimes has a renewed interest. The case of CR is significant because it describes an interacting scalar field, as opposed to USR, in which the field is free. In addition, the USR regime has already been investigated in the context of the stochastic formalism we will now introduce [8, 9], whereas CR has not.

The aim of this work, then, is to study the CR regime and determine whether or not the perturbations generated in this scenario follow Linear Perturbation Theory (LPT). We will employ the stochastic formalism, which provides an ideal framework for this objective. In it, as will be demonstrated in detail, the long-wavelength modes of the perturbations are affected by random kicks, caused by the short-wavelength modes when crossing a certain scale; thus converting a quantum computation into a statistical one. To treat this formalism analytically, however, one has to approximate the long-wavelength modes at linear order, which implies that it is restricted to the reproduction of LPT. On the other hand, the calculations are based on gradient expansion, which includes all orders of amplitude perturbations. This apparent contradiction implies that, if the results of the stochastic formalism do not match LPT, it is due to the rise of non-perturbative effects. One can therefore employ the stochastic formalism as a consistency check for LPT, i.e. to determine if the linear theory is a plausible description of a given inflationary regime. There is some confusion on this last point in the literature, as it has been claimed that stochastic inflation in regimes other than SR can describe non-perturbative effects such as quantum diffusion [10, 11], and the formalism has even been used to calculate the contribution of such effects to the amplitude of the perturbations [12, 13]. This is not correct since it does not acknowledge the consequences of the mentioned approximation, as it has also been argued in the literature [14, 15]. In fact, the SR and USR regimes have been analysed numerically, as we mentioned above, obtaining a result which matches LPT [8, 9]. In this work, on the other hand, we will study the CR regime and we will demonstrate that the linear theory is also valid in this case.

## II. INFLATION

We begin by introducing the fundamental equations that describe the different inflationary regimes. In order for the expansion to be homogeneous and isotropic, it has to be driven by a scalar field (or multiple). The simplest model, then, is that of a single field with a standard kinetic term, described by an action of the form [16]:

$$S = \frac{1}{2}\int \sqrt{-g}\left[M_{Pl}^2 R - \nabla_\mu\phi\nabla^\mu\phi - 2V(\phi)\right], \tag{1}$$

where $M_{Pl} = 1/\sqrt{8\pi G}$ corresponds to the Planck mass, and we will be using natural units $\hbar = c = 1$ throughout the text. The shape of the potential $V(\phi)$, in turn, will determine the behaviour of the field and thus the type of inflationary scenario. The homogeneous and isotropic solution is the well-known FLRW metric:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{2}$$

Comparing the energy-momentum tensor associated to this action:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\nabla_\alpha\phi\nabla^\alpha\phi + 2V(\phi)\right], \tag{3}$$

with that of a perfect fluid in metric (2) one finds for the energy density and pressure of the field:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{4}$$

and using the energy conservation equation $\dot{\rho} = -3H(\rho + p)$ we obtain the equation of motion for the field:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0, \tag{5}$$

where $V_{,\phi}(\phi) \equiv \frac{\partial V}{\partial \phi}$ and $H \equiv \dot{a}/a$ is the Hubble rate. This equation is nothing but the general Klein-Gordon (KG) equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\partial_\nu \phi\right) - V_{,\phi}(\phi) = 0, \tag{6}$$

written in the FLRW metric. From Friedmann's equation we find:

$$H^2 = \frac{1}{3M_{Pl}^2}\rho = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{7}$$

and differentiating this last equation with respect to time and using (5) we obtain:

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_{Pl}^2}. \tag{8}$$

We can then define the SR parameters as:

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2 M_{Pl}^2}; \quad \epsilon_{i+1} \equiv \frac{\dot{\epsilon}_i}{H\epsilon_i} \quad \text{for } i \geq 1. \tag{9}$$

Now, since during inflation we want the scale factor to undergo an approximately exponential increase, $a \sim e^{Ht}$, the Hubble rate has to be approximately constant and thus $|\dot{H}| \ll H^2$, implying $\epsilon_1 \ll 1$. On the other hand, $\epsilon_1 \sim 1$ will mark the end of the inflationary period.

### A. Constant Roll

The model we are interested in is the so-called CR inflation [17, 18], in which it is assumed that the rate of roll, that is, the ratio between the acceleration and friction terms, takes the form:

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -(3+\kappa), \tag{10}$$

with $\kappa$ being an arbitrary constant. The other scenarios commonly described in the literature, SR and USR, occur for $\kappa \simeq -3$ and $\kappa = 0$, respectively. In order to solve the equations of motion we consider $H = H(\phi)$, so that $\dot{H} = \frac{dH}{d\phi}\dot{\phi}$. Substituting in (8) one finds:

$$\dot{\phi} = -2M_{Pl}^2 \frac{dH}{d\phi}, \tag{11}$$

differentiating this last expression and substituting it in (10) one finally obtains the differential equation:

$$\frac{d^2 H}{d\phi^2} = \frac{3+\kappa}{2M_{Pl}^2} H, \tag{12}$$

which leads to the general solution:

$$H(\phi) = C_1 \exp\left(\sqrt{\frac{3+\kappa}{2}}\frac{\phi}{M_{Pl}}\right) + C_2 \exp\left(-\sqrt{\frac{3+\kappa}{2}}\frac{\phi}{M_{Pl}}\right). \tag{13}$$

Some realisations of this solution, for cases with $\epsilon_2 > 0$ and $-3 < \epsilon_2 < 0$, have been studied in the literature in the context of PBH formation [19–21]. We will focus our study, however, on the case $\epsilon_2 < -6$. This particular model is obtained when $C_1 = C_2$ so that, with $\kappa > -3$:

$$H = M \cosh\left(\sqrt{\frac{3+\kappa}{2}}\frac{\phi}{M_{Pl}}\right), \tag{14}$$

with $M$ being an integration constant. From this expression, we can solve for $\phi(t)$ using (11), and for $V(\phi)$ using (7). This finally leads to:

$$V(\phi) = 3M^2 M_{Pl}^2 \left[1 + \frac{\kappa}{6}\left\{1 - \cosh\left(\sqrt{2(3+\kappa)}\frac{\phi}{M_{Pl}}\right)\right\}\right], \tag{15}$$

$$\phi(t) = M_{Pl}\sqrt{\frac{2}{3+\kappa}} \ln\left[\coth\left(\frac{3+\kappa}{2}Mt\right)\right], \tag{16}$$

$$H(t) = M \coth\left[(3+\kappa)Mt\right], \tag{17}$$

$$a = a_0 \sinh^{1/(3+\kappa)}\left[(3+\kappa)Mt\right]. \tag{18}$$

With the help of the result (17) we can also calculate the SR parameters for this model:

$$\epsilon_1 = (3+\kappa)\,\mathrm{sech}^2\left[(3+\kappa)Mt\right], \tag{19}$$

$$\epsilon_2 = -2(3+\kappa)\tanh^2\left[(3+\kappa)Mt\right], \tag{20}$$

and we can easily see that they will approach $\epsilon_1 \to 0$ and $\epsilon_2 \to -2(3+\kappa)$ very rapidly. Thus, the condition $\epsilon_2 < -6$ imposes that $\kappa > 0$. On the other hand, since $\epsilon_1 \ll 1$ is always satisfied, we will need the addition of some mechanism that takes care of ending inflation. However, we will not go into this issue.

## III. ADM FORMALISM

The inflationary picture we have presented up to know is completely homogeneous, and therefore it cannot reproduce our Universe, marked by the presence of small inhomogeneities or perturbations generated, in the context of inflation, by the quantum fluctuations of the scalar field. In order to introduce them it is convenient to work in the so-called Arnowitt-Deser-Misner (ADM) formalism [22], in which we break spacetime into spacelike hypersurfaces of constant time, $\Sigma_t$. In this formulation the metric takes the form:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{21}$$

where we have introduced [23]:

- The lapse function $\alpha$, which measures the rate of flow of proper time with respect to $t$ as one moves normally to $\Sigma_t$.
- The shift vector $\beta^i$, which measures the shift tangential to $\Sigma_t$ when moving along the time direction.
- The metric induced on the hypersurface $\Sigma_t$, $\gamma_{ij}$. It will be useful to decompose it as $\gamma_{ij} = a^2(t)e^{2\zeta}\tilde{\gamma}_{ij}$, with $\det\tilde{\gamma}_{ij} = 1$, and $\zeta$ being the curvature perturbation.

Note that the FLRW metric (2) is recovered when we set $\alpha = 1$, $\beta^i = 0$ and $\gamma_{ij} = a^2(t)\delta_{ij}$. It is also useful to introduce the extrinsic curvature of $\Sigma_t$, which takes the form:

$$K_{ij} \equiv -\nabla_i n_j = -\frac{1}{2\alpha}(\dot{\gamma}_{ij} - D_i\beta_j - D_j\beta_i), \tag{22}$$

where $n_i = \left(-\alpha, \vec{0}\right)$ is the unit vector normal to $\Sigma_t$, and $\nabla_i$, $D_i$ denote the covariant derivative with respect to $g_{\mu\nu}$ and $\gamma_{ij}$ respectively. The extrinsic curvature can also be decomposed in a convenient way:

$$K_{ij} = \frac{1}{3}\gamma_{ij}K + a^2(t)e^{2\zeta}\tilde{A}_{ij}; \quad \tilde{\gamma}^{ij}\tilde{A}_{ij} = 0, \tag{23}$$

with the first and second terms being the trace and traceless part of the tensor, respectively; and $K \equiv \gamma^{ij} K_{ij}$. Notice also that, in the homogeneous limit, with $\gamma^{ij} = \frac{1}{a^2(t)}\delta^{ij}$, the extrinsic curvature becomes, using (22), $K_{ij} = -\dot{a}a\delta_{ij}$, and therefore $K = -3H^b$. This result allows us to define a general inhomogeneous Hubble rate as:

$$H \equiv -\frac{K}{3}, \tag{24}$$

which will be used later on.

In the ADM formalism, $\alpha$ and $\beta^i$ serve as Lagrange multipliers, imposing the Hamiltonian and momentum constraints [23–25]:

$$R^{(3)} - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = \frac{2}{M_{Pl}^2}E, \tag{25}$$

$$D^j\tilde{A}_{ij} - \frac{2}{3}D_iK = \frac{1}{M_{Pl}^2}J_i, \tag{26}$$

where $R^{(3)}$ corresponds to the Ricci scalar of the induced spatial metric, $E \equiv T_{\mu\nu}n^\mu n^\nu$ and $J_i \equiv T_{\mu j}n^\mu \gamma_i^j$, with $T_{\mu\nu}$ being the energy-momentum tensor defined in (3). The variables $\gamma_{ij}$ and $K_{ij}$, on the other hand, are dynamical, and governed by the following equations [9, 25]:

$$(\partial_t - \beta^k\partial_k)\zeta + H = -\frac{1}{3}(\alpha K - \partial_k\beta^k), \tag{27}$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k, \tag{28}$$

$$(\partial_t - \beta^k\partial_k)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - D_kD^k\alpha + \frac{1}{2M_{Pl}^2}\alpha(E + S_k^k), \tag{29}$$

plus one last equation which we will not use and can be found in [25]. We have also defined $S_{ij} = T_{ij}$, $S_k^k = \gamma^{kl}S_{lk}$.

## IV. LINEAR PERTURBATION THEORY

Before deriving the stochastic formalism, we need to recover some results from LPT which will be useful in our calculations and, as will be seen later, provide a test for the results of the stochastic method. LPT is based on the assumption that all deviations from the ideal homogeneous description of spacetime can be expanded to a linear order correction, so that we can decompose the metric into a background FLRW solution, given by (2), plus a small perturbation:

$$g_{\mu\nu} \simeq g_{\mu\nu}^b + \delta g_{\mu\nu}; \quad \delta g_{\mu\nu} \ll g_{\mu\nu}^b. \tag{30}$$

Note also that from now on the superscript 'b' will refer to a background variable, i.e. its homogeneous solution. In the same way, we could decompose $\alpha \simeq 1 + A$, $\beta^i \simeq aB^i$, $\phi \simeq \phi^b + \delta\phi$, etc. knowing that the background values are $\alpha = 1$ and $\beta^i = 0$ as we explained above. These perturbations can be decomposed, based on how they transform under rotations in the background space, into three types: scalar, vector and tensor perturbations. At linear order, nonetheless, they decouple from each other and therefore can be analysed independently [26]. Our object of interest will thus be the scalar perturbations, since they couple to the inflaton field perturbation.

There is an issue that arises when attempting to calculate these perturbations, however. In principle, to find, e.g., $\delta\phi$, we would need to determine the difference between its actual value, $\phi$, and its value in the homogeneous background, $\phi^b$. These values have to be compared at the same point, but since they exist in two different geometries, we first need to establish a correspondence that connects the specific point in the two distinct spacetimes. This mapping is known as the gauge choice [26]. This implies, nevertheless, that the value that a given perturbation, say, $\delta\phi$, takes will depend on the choice of gauge that we have adopted. To resolve this ambiguity, then, we need to define some gauge-invariant quantities, and the one which we are particularly interested in is the Mukhanov-Sasaki (MS) variable:

$$Q \equiv \delta\phi + \frac{\dot{\phi}^b}{H^b}\left(D + \frac{1}{3}\nabla^2 E\right), \tag{31}$$

where $D$ and $E$ arise from the decomposition of the induced metric:

$$\gamma_{ij} \simeq a^2(t)[(1 + 2D)\delta_{ij} - 2E_{ij}]; \quad E_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E. \tag{32}$$

It can be shown that, linearising the ADM equations (25)-(29), as well as the KG equation (6), one obtains an equation of motion for the MS variable in terms of SR parameters [25]:

$$\ddot{Q} + 3H^b\dot{Q} + \left[-\frac{\nabla^2}{a^2} + (H^b)^2\left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right] Q = 0. \tag{33}$$

In order to solve this equation it is convenient to write it in terms of conformal time $\tau$, defined as $dt = ad\tau$, which in terms of SR parameters takes the form:

$$\tau = \int \frac{dt}{a} = \int \frac{da}{a^2H} = -\frac{1}{aH} + \int \frac{da}{a^2H}\epsilon_1, \tag{34}$$

where we have integrated by parts in the last equality. Equation (33) then reads, in Fourier space [9]:

$$Q''_{\mathbf{k}} + 2\mathcal{H}^bQ'_{\mathbf{k}} + \left[k^2 + (\mathcal{H}^b)^2(2-\epsilon_1) + \frac{z''}{z}\right] Q_{\mathbf{k}} = 0, \tag{35}$$

where prime denotes derivative with respect to $\tau$, $\mathcal{H} = a'/a$, and $z = (\phi^b)'/\mathcal{H}^b$, so that in terms of SR parameters:

$$\frac{z''}{z} = a^2(H^b)^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 - \frac{1}{2}\epsilon_1\epsilon_2 + \frac{1}{4}\epsilon_2^2 + \frac{1}{2}\epsilon_2\epsilon_3\right). \tag{36}$$

An analytical solution for (35) exists if:

$$\nu^2 \equiv \frac{1}{4} + \tau^2\frac{z''}{z}, \tag{37}$$

is a constant. From (34) and (36) we immediately see it is indeed constant up to order $\mathcal{O}(\epsilon_1)$. The solution then reads [9]:

$$Q_{\mathbf{k}} = \frac{e^{\frac{i}{2}\pi\left(\nu+\frac{1}{2}\right)}}{a}\frac{\sqrt{\pi}}{2}\sqrt{-\tau}H_\nu^{(1)}(-k\tau), \tag{38}$$

where $H_\nu^{(1)}$ denotes the Hankel function of first class, and the Bunch-Davies vacuum [27] has been imposed as initial condition. Expanding $H_\nu^{(1)}$ for $(-k\tau) \ll 1$ and $\nu > 1$ (we will later see this is always the case in our study), one finally obtains:

$$Q_{\mathbf{k}} \simeq -i\frac{e^{\frac{i}{2}\pi\left(\nu+\frac{1}{2}\right)}2^{\nu-1}}{a\sqrt{\pi}}\sqrt{-\tau}(-k\tau)^{-\nu}\Gamma[\nu], \tag{39}$$

as well as:

$$\frac{Q'_{\mathbf{k}}}{\mathcal{H}^bQ_{\mathbf{k}}} \simeq \frac{1-2\nu-2\mathcal{H}^b\tau}{2\mathcal{H}^b\tau}. \tag{40}$$

Finally, we need to be able to write $\nu$ in terms of the CR parameter we defined in Section II, $\kappa$. It can be shown that the relationship takes the form [9]:

$$\nu = \frac{3}{2}\sqrt{1 + \frac{4}{9}(3\kappa+\kappa^2)} + \mathcal{O}(\epsilon_1). \tag{41}$$

## V. GRADIENT EXPANSION

The stochastic formalism we want to derive is not based on LPT but rather on gradient expansion [28], an expansion of the ADM equations which is non-perturbative in terms of the amplitudes of the inhomogeneities. This approximation is valid when the characteristic scale of the density perturbations, $L$, is taken to be much bigger than the Hubble radius of a given local patch of the Universe, $L \gg H_l^{-1}$ (from now on the subscript '$l$' refers to a local variable or coordinate). The expansion parameter is thus defined as $\sigma \equiv H_l^{-1}/L \ll 1$, in such a way that, at leading order in $\sigma$, every local patch with size $\sigma H_l^{-1}$ (the so-called coarse-grained scale) can be approximately described as a FLRW Universe. Higher order terms in $\sigma$ will, in turn, describe the local inhomogeneities of these patches. Any function $X$ which is approximately homogeneous in local coordinates can be written as $X = X(t, \sigma x^i)$, therefore:

$$\partial_i X(t, \sigma x^i) = \sigma \frac{\partial}{\partial(\sigma x^i)} X(t, \sigma x^i) = \sigma \frac{\partial}{\partial(\sigma x^i)} X(t, \sigma x^i)\Big|_{\sigma x^i=0} + \mathcal{O}(\sigma^2), \tag{42}$$

and we can assume $\partial_i X \sim X \times \mathcal{O}(\sigma)$, since $\frac{\partial}{\partial(\sigma x^i)} X(t, \sigma x^i)\Big|_{\sigma x^i=0}$ can be of the same order as $X(t, \sigma x^i)$. In other words, we are assuming that a local patch can always be found such that any spatial gradient is of order $\mathcal{O}(\sigma)$.

As in the case of LPT, we need to define a global background metric, in the form of (2), as well as a local metric, which can be written as:

$$ds_l^2 = -{}_{(0)}\alpha^2 dt_l^2 + a^2(t_l) e^{2{}_{(0)}\zeta}\, {}_{(0)}\tilde{\gamma}_{ij}(dx_l^i + {}_{(0)}\beta^i dt_l)(dx_l^j + {}_{(0)}\beta^j dt_l), \tag{43}$$

where the subscript '(0)' indicates leading order in gradient expansion. It is important to note that this leading order can be different for each variable [25]:

- ${}_{(0)}\alpha$, ${}_{(0)}\zeta$ and ${}_{(0)}\phi$ are $\sim \mathcal{O}(\sigma^0)$.
- ${}_{(0)}\beta^i \sim \mathcal{O}(\sigma^{-1})$. This will not be problematic as it will always appear with a spatial derivative in the equations, so that ${}_{(0)}\partial_i\beta^i \sim \mathcal{O}(\sigma^0)$.
- ${}_{(0)}\tilde{\gamma}_{ij} = \delta_{ij} \sim \mathcal{O}(\sigma^0)$ and ${}_{(0)}(\tilde{\gamma}_{ij} - \delta_{ij}) \sim \mathcal{O}(\sigma)$.

In order for (43) to describe a homogeneous and isotropic Universe, the following conditions must be satisfied [9, 25]:

- ${}_{(0)}\alpha = {}_{(0)}\alpha(t_l)$.
- ${}_{(0)}\beta^i = b(t_l) x_l^i$.
- ${}_{(0)}\zeta = {}_{(0)}\zeta(t_l)$.
- $\tilde{\gamma}_{ij} \simeq \delta_{ij} - 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) C$, with $C$ being a scalar function.

Note that ${}_{(0)}\alpha(t_l)$, $b(t_l)$, ${}_{(0)}\zeta(t_l)$ and $C$ will depend on the choice of gauge.

## VI. STOCHASTIC FORMALISM

The stochastic approach to inflation aims to study the evolution of inhomogeneities in a non-perturbative way by combining both LPT and $\mathcal{O}(\sigma^0)$ gradient expansion. For a given variable of interest, and a certain coarse-grained scale, the goal is to split said variable into an Infrared (IR) part, with characteristic wavelength $\lambda > (\sigma H_l)^{-1}$, and an Ultraviolet (UV) one, with $\lambda < (\sigma H_l)^{-1}$. Since the UV part evolves well inside the Hubble horizon, we assume that it is perturbatively small and therefore can be described by LPT. The IR part, on the other hand, is composed of long wavelengths and hence can be studied using gradient expansion. As we will see later, the UV mode will act as a random excitation for the IR part when it exits the $(\sigma H_l)^{-1}$ scale, and will thus act as a stochastic variable.

Throughout this section we will be using, for convenience, the uniform-$N$ gauge [25, 29]. The number of e-folds $N$ is defined as follows:

$$N \equiv -\frac{1}{3}\int K dt_l, \tag{44}$$

where $K = -3H_l$ as was seen in (24). Using (22) the previous expression can be rewritten in terms of the ADM coordinates and variables as:

$$N = \int \left( H^b + \dot{\zeta} - \frac{1}{3} D_i \beta^i \right) dt. \tag{45}$$

The uniform-$N$ gauge is then defined so that $N = \int H^b dt$, that is, the number of e-folds in any local patch coincides with that of the background, and therefore $\zeta_{\delta N} = 0$, $\beta^i_{\delta N} = 0$. We are specifying that a given variable is calculated in this gauge by using the subscript '$\delta N$'. Throughout the rest of this section, however, we will abstain from using this notation for simplicity. The most natural choice of time coordinate in this gauge is, logically, $N$, and hence we will employ it in the rest of our calculations. On top of that, the use of other time variables leads to different stochastic processes, and it can be shown that the only coordinate choice that allows our formalism to reproduce Quantum Field Theory (QFT) calculations is indeed $N$ [30].

### A. Calculation example

To illustrate the functionality of this formalism we will study in detail the Hamiltonian constraint (25). The first step is to expand the equation at $\mathcal{O}(\sigma^0)$ in gradient expansion. Since ${}_{(0)}\tilde{\gamma}_{ij} = \delta_{ij}$, ${}_{(0)}R^{(3)} = 0$. From (27) we see that $K = -\frac{3H^b}{{}_{(0)}\alpha}$. From (28) we find:

$$\tilde{A}_{ij} = -\frac{H^b}{2\,{}_{(0)}\alpha}\frac{\partial \tilde{\gamma}_{ij}}{\partial N}, \tag{46}$$

which can be neglected as $\frac{\partial \tilde{\gamma}_{ij}}{\partial N} \sim \mathcal{O}(\sigma)$. On the other hand:

$$\begin{aligned} E = T_{\mu\nu}n^\mu n^\nu = \frac{1}{{}_{(0)}\alpha^2}T_{00} =& \frac{1}{2}\left[(H^b)^2\left(\frac{\partial\,{}_{(0)}\phi}{\partial N}\right)^2 g^{00} + ({}_{(0)}\partial_i\phi)({}_{(0)}\partial^i\phi)\right] + V({}_{(0)}\phi) \\ \simeq& \frac{1}{2}\left(\frac{H^b}{{}_{(0)}\alpha}\right)^2\left(\frac{\partial\,{}_{(0)}\phi}{\partial N}\right)^2 + V({}_{(0)}\phi), \end{aligned} \tag{47}$$

where in the last step we have taken into account that ${}_{(0)}\partial_i\phi \sim \mathcal{O}(\sigma)$. Equation (25) now becomes:

$$6\left(\frac{H^b}{{}_{(0)}\alpha}\right)^2 - \frac{2}{M_{Pl}^2}\left[\frac{1}{2}\left(\frac{H^b}{{}_{(0)}\alpha}\right)^2\left(\frac{\partial\,{}_{(0)}\phi}{\partial N}\right)^2 + V({}_{(0)}\phi)\right] = 0. \tag{48}$$

In the following calculations, we will abstain from using the subscript '(0)' so as to not overload the notation, but the reader should keep in mind that we are always taking the variables at leading order in gradient expansion. The next step is to split our variables into a UV and IR part:

$$\begin{aligned} \alpha &= \alpha^{IR} + \alpha^{UV}, \\ \phi &= \phi^{IR} + \phi^{UV}. \end{aligned} \tag{49}$$

The aim now is to expand equation (48) at leading order in the UV variables, which we are assuming to be perturbatively small, obtaining:

$$\begin{aligned} &6\left(\frac{H^b}{\alpha^{IR}}\right)^2 - \frac{2}{M_{Pl}^2}\left[\frac{1}{2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2 + V(\phi^{IR})\right] \\ =& 12\frac{(H^b)^2}{(\alpha^{IR})^3}\alpha^{UV} + \frac{2}{M_{Pl}^2}\left[\left(\frac{H^b}{\alpha^{IR}}\right)^2\frac{\partial\phi^{IR}}{\partial N}\frac{\partial\phi^{UV}}{\partial N} - \frac{(H^b)^2}{(\alpha^{IR})^3}\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2\alpha^{UV} + V_{,\phi}(\phi^{IR})\phi^{UV}\right]. \end{aligned} \tag{50}$$

We can use Fourier analysis to give a more rigorous definition of the IR and UV modes. For a function $X$, we can define:

$$\begin{aligned} X^{IR}(t,\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta(\sigma a_l(N)H_l(N) - k)\mathcal{X}^{IR}_{\mathbf{k}}(t,\mathbf{x}), \\ X^{UV}(t,\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta(k - \sigma a_l(N)H_l(N))\mathcal{X}^{UV}_{\mathbf{k}}(t,\mathbf{x}), \end{aligned} \tag{51}$$

where we are particularly interested in $\mathcal{X}^{UV}_{\mathbf{k}}(t,\mathbf{x})$, as it is the perturbative term. It is defined as:

$$\mathcal{X}^{UV}_{\mathbf{k}}(t,\mathbf{x}) = e^{-i\mathbf{k}\cdot\mathbf{x}}X_{\mathbf{k}}(N)a_{\mathbf{k}} + e^{i\mathbf{k}\cdot\mathbf{x}}X^*_{\mathbf{k}}(N)a^\dagger_{\mathbf{k}}\,, \tag{52}$$

with $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ being the usual QFT creation and annihilation operators, satisfying the commutation relation:

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{53}$$

while the other commutators are zero. $X_{\mathbf{k}}(N)$, on the other hand, is the solution for the evolution of the perturbation of $X$ in the local background (43), and at sub-horizon scales [9, 25]. Note also that, in the uniform-$N$ gauge,

$H_l = H^b/\alpha^{IR}$ and $a_l = a^b \equiv a$. With this in mind, substituting the definition (51) for $\alpha^{UV}$ and $\phi^{UV}$ into (50) one finds:

$$\begin{aligned} &6\left(\frac{H^b}{\alpha^{IR}}\right)^2 - \frac{2}{M_{Pl}^2}\left[\frac{1}{2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2 + V(\phi^{IR})\right] \\ &= -\frac{2}{M_{Pl}^2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\frac{\partial\phi^{IR}}{\partial N}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\varphi_{\mathbf{k}}^{UV} \\ &+\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\left\{12\frac{(H^b)^2}{(\alpha^{IR})^3}\boldsymbol{\alpha}_{\mathbf{k}}^{UV} + \frac{2}{M_{Pl}^2}\left[\left(\frac{H^b}{\alpha^{IR}}\right)^2\frac{\partial\phi^{IR}}{\partial N}\frac{\partial\varphi_{\mathbf{k}}^{UV}}{\partial N}\right.\right. \\ &\left.\left. -\frac{(H^b)^2}{(\alpha^{IR})^3}\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2\boldsymbol{\alpha}_{\mathbf{k}}^{UV} + V_{,\phi}(\phi^{IR})\varphi_{\mathbf{k}}^{UV}\right]\right\}, \end{aligned} \tag{54}$$

with $\boldsymbol{\alpha}_{\mathbf{k}}^{UV}$ and $\varphi_{\mathbf{k}}^{UV}$ defined as in (52). We can identify two terms in this last equation:

1. The term multiplying the Heaviside theta. This is nothing but the Hamiltonian constraint at sub-horizon scales. We can assume this constraint will be satisfied once we impose the Bunch-Davies vacuum as initial condition. In other words, we are choosing a vacuum in which the Hamiltonian constraint for the operators $\boldsymbol{\alpha}_{\mathbf{k}}^{UV}$ and $\varphi_{\mathbf{k}}^{UV}$ is satisfied. Therefore, we can set this term to zero [9].

2. The integral with a Dirac delta, which will act as a stochastic white noise, as will be proven later on.

We can define:

$$\xi_1 \equiv -\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\varphi_{\mathbf{k}}^{UV}, \tag{55}$$

so that (54) becomes:

$$6\left(\frac{H^b}{\alpha^{IR}}\right)^2 - \frac{2}{M_{Pl}^2}\left[\frac{1}{2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2 + V(\phi^{IR})\right] = \frac{2}{M_{Pl}^2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\frac{\partial\phi^{IR}}{\partial N}\xi_1, \tag{56}$$

and solving for $\left(\frac{H^b}{\alpha^{IR}}\right)^2$ we find:

$$\left(\frac{H^b}{\alpha^{IR}}\right)^2 = \frac{V(\phi^{IR})}{3M_{Pl}^2 - \frac{1}{2}\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2 - \frac{\partial\phi^{IR}}{\partial N}\xi_1}. \tag{57}$$

### B. System

The system of (stochastic, as we will see later) differential equations that describe the inflationary perturbations can be found by following the same method we have demonstrated in Section VI A with the rest of the ADM equations. For the equation of the trace of the extrinsic curvature (29) one obtains:

$$\begin{aligned} &-3\frac{H^b}{\alpha^{IR}}\frac{\partial}{\partial N}\left(\frac{H^b}{\alpha^{IR}}\right) - 3\left(\frac{H^b}{\alpha^{IR}}\right)^2 - \frac{1}{M_{Pl}^2}\left[\left(\frac{H^b}{\alpha^{IR}}\right)^2\left(\frac{\partial\phi^{IR}}{\partial N}\right)^2 - V(\phi^{IR})\right] \\ &= -3\frac{(H^b)^2}{(\alpha^{IR})^3}\xi_3 + \frac{2}{M_{Pl}^2}\left(\frac{H^b}{\alpha^{IR}}\right)^2\frac{\partial\phi^{IR}}{\partial N}\xi_1, \end{aligned} \tag{58}$$

where we have defined:

$$\xi_3 \equiv -\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\boldsymbol{\alpha}_{\mathbf{k}}^{UV}. \tag{59}$$

On the other hand, the equation of motion for the field (6) leads:

$$\begin{aligned}&\frac{\partial^2\phi^{IR}}{\partial N^2}+\left[3+\frac{\alpha^{IR}}{H^b}\frac{\partial}{\partial N}\left(\frac{H^b}{\alpha^{IR}}\right)\right]\frac{\partial\phi^{IR}}{\partial N}+\left(\frac{\alpha^{IR}}{H^b}\right)^2V_{,\phi}(\phi^{IR})\\&=-\frac{\partial\xi_1}{\partial N}-\xi_2-\left[3+\frac{\alpha^{IR}}{H^b}\frac{\partial}{\partial N}\left(\frac{H^b}{\alpha^{IR}}\right)\right]\xi_1+\frac{\partial\phi^{IR}}{\partial N}\frac{\xi_3}{\alpha^{IR}},\end{aligned}\tag{60}$$

with:

$$\xi_2\equiv-\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\frac{\partial\varphi^{UV}_{\mathbf{k}}}{\partial N}.\tag{61}$$

Lastly, we have to study the momentum constraint (26). This equation is special as in the uniform-$N$ gauge it can be written as:

$$D^j\left(-\frac{H^b}{2\alpha}\frac{\partial\tilde{\gamma}_{ij}}{\partial N}\right)-\frac{2}{3}D_iK=-\frac{1}{\alpha M_{Pl}^2}\frac{\partial\phi}{\partial N}\partial_i\phi,\tag{62}$$

and we immediately see that it only contains $\mathcal{O}(\sigma)$ in gradient expansion terms. Using the decomposition for $\tilde{\gamma}_{ij}$ explained in Section V, and following the steps from Section VI A one finds:

$${}_{(0)}\partial_i\left[\frac{\partial}{\partial N}\left(\frac{1}{3}{}_{(0)}\nabla^2C^{IR}\right)\right]-\frac{{}_{(0)}\partial_i\alpha^{IR}}{{}_{(0)}\alpha^{IR}}+\frac{\partial_{(0)}\phi^{IR}}{\partial N}\frac{{}_{(0)}\partial_i\phi^{IR}}{2M_{Pl}^2}=-{}_{(0)}\partial_i\xi_4,\tag{63}$$

defining:

$$\xi_4\equiv-\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a\frac{H^b}{\alpha^{IR}}\right)\left(-\frac{k^2}{3}\mathcal{C}^{UV}_{\mathbf{k}}\right),\tag{64}$$

with $\mathcal{C}^{UV}_{\mathbf{k}}$ defined as in (52), just like the other perturbative variables. Notice that in (63) the ${}_{(0)}\partial_i$ terms are $\mathcal{O}(\sigma)$ in gradient expansion, while ${}_{(0)}\alpha^{IR}$, ${}_{(0)}\phi^{IR}$ and ${}_{(0)}\nabla^2C$ are $\mathcal{O}(\sigma^0)$. The fact that ${}_{(0)}\nabla^2C\sim\mathcal{O}(\sigma^0)$ is not in contradiction with the statement ${}_{(0)}\left(\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2\right)C\sim\mathcal{O}(\sigma)$ from Section V, as is proven in Appendix A.

If the full $\mathcal{O}(\sigma^0)$ information is to be extracted, then, all ${}_{(0)}\partial_i$ terms have to be integrated from the momentum constraint. Nonetheless, we do not know how to perform this integration in a non-linear way, and so we have to approximate it at linear order [25].

With all this in mind, in addition to the definition of an auxiliary variable $\pi^{IR}$, we obtain the following system of equations by combining (57), (58), (60) and (63), as well as neglecting $\xi_i^2$ terms:

$$\pi^{IR}=\frac{\partial\phi^{IR}}{\partial N}+\xi_1,\tag{65}$$

$$\frac{\partial\phi^{IR}}{\partial N}+\left[3-\frac{(\pi^{IR})^2}{2M_{Pl}^2}\right]\pi^{IR}+\left[3M_{Pl}^2-\frac{(\pi^{IR})^2}{2}\right]\frac{V_{,\phi}(\phi^{IR})}{V(\phi^{IR})}=-\xi_2,\tag{66}$$

$$\frac{\partial}{\partial N}\left(\frac{1}{3}\nabla^2C^{IR}\right)-\left[H^b\sqrt{\frac{3M_{Pl}^2-\frac{(\pi^{IR})^2}{2}}{V(\phi^{IR})}}-1\right]+\frac{1}{2M_{Pl}^2}\frac{\partial\phi^{IR}}{\partial N}(\phi^{IR}-\phi^b)=-\xi_4.\tag{67}$$

### C. White noises

In order to characterise the $\xi_i$ variables as stochastic white noises, we need to compute their two-point correlation function at equal space point. In the case of $\xi_1$, for example:

$$\begin{aligned}&\langle\xi_1(N_1)\xi_1(N_2)\rangle=\\&\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_1}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_2}\int\frac{d^3\mathbf{k_1}d^3\mathbf{k_2}}{(2\pi)^3}\delta\left(k_1-\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_1}\right)\delta\left(k_2-\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_2}\right)\langle\varphi^{UV}_{\mathbf{k_1}}(N_1)\varphi^{UV}_{\mathbf{k_2}}(N_2)\rangle.\end{aligned}\tag{68}$$

Using the definition (52) and the commutation relations one obtains:

$$\langle\varphi^{UV}_{\mathbf{k_1}}(N_1)\varphi^{UV}_{\mathbf{k_2}}(N_2)\rangle = \delta\phi_{\mathbf{k_1}}(N_1)\delta\phi^*_{\mathbf{k_2}}(N_2)\delta^{(3)}(\mathbf{k_1}-\mathbf{k_2}), \tag{69}$$

where we have considered the fact that $\delta\phi_{\mathbf{k}}(N)$ is indeed the solution for the evolution of the perturbation of $\phi$ in the local background, as we defined around (52); and thus, after integrating in $\mathbf{k_2}$, and taking advantage of the spherical symmetry:

$$\begin{aligned}
&\langle\xi_1(N_1)\xi_1(N_2)\rangle = \\
&\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_1}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_2}\int\frac{k_1^2dk_1}{2\pi^2}\delta\left(k_1-\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_1}\right)\delta\left(\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_1}-\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_2}\right)\delta\phi_{\mathbf{k_1}}(N_1)\delta\phi^*_{\mathbf{k_2}}(N_2)\\
&=\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_1}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_2}\int\frac{k_1^2dk_1}{2\pi^2}\delta\left(k_1-\sigma a\frac{H^b}{\alpha^{IR}}\Big|_{N_1}\right)\frac{\delta(N_1-N_2)}{\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\Big|_{N_1}}|\delta\phi_{\mathbf{k_1}}(N_1)|^2,
\end{aligned} \tag{70}$$

where in the last step we have used the properties of the Dirac delta function. After integrating we finally obtain the result:

$$\langle\xi_1(N_1)\xi_1(N_2)\rangle = \frac{1}{2\pi^2}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)^2\left|\delta\phi(N_1)_{k=\sigma a\frac{H^b}{\alpha^{IR}}}\right|^2\delta(N_1-N_2). \tag{71}$$

Following the same procedure we also find:

$$\langle\xi_1(N_1)\xi_2(N_2)\rangle = \frac{1}{2\pi^2}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)^2\left(\delta\phi(N_1)_{k=\sigma a\frac{H^b}{\alpha^{IR}}}\frac{\partial\delta\phi^*(N_1)}{\partial N}_{k=\sigma a\frac{H^b}{\alpha^{IR}}}\right)\delta(N_1-N_2), \tag{72}$$

$$\langle\xi_2(N_1)\xi_2(N_2)\rangle = \frac{1}{2\pi^2}\frac{\partial}{\partial N}\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)\left(\sigma a\frac{H^b}{\alpha^{IR}}\right)^2\left|\frac{\partial\delta\phi(N_1)}{\partial N}_{k=\sigma a\frac{H^b}{\alpha^{IR}}}\right|^2\delta(N_1-N_2), \tag{73}$$

as well as analogous expressions for $\xi_4$. Let us now focus on (71). We are evaluating $\delta\phi_{\mathbf{k}}$ at the coarse-grained scale, well outside the local Hubble radius. At this scale, any UV perturbation which started at profound sub-horizon scales will have evolved into a squeezed state, or, in other words, it will behave as a classical random variable [31]. Therefore, we can interpret the term $\left|\delta\phi(N_1)_{k=\sigma a\frac{H^b}{\alpha^{IR}}}\right|^2$ as the power spectrum of one such variable. In addition, the fact that $\langle\xi_1(N_1)\xi_1(N_2)\rangle \propto \delta(N_1-N_2)$ allows us to characterise $\xi_1$ as a white noise. This is a consequence of splitting the modes in (51) with the use of a Heaviside theta function, and employing other functions would lead to coloured noises, which are much more difficult to treat [30].

Nevertheless, the system described by the noises $\xi_i$ is non-Markovian, as $\delta\phi_{\mathbf{k}}$ has to be computed every time step over a local background which is dependent on all the previous time steps, i.e. it is modified by the noises themselves [9]. Let us illustrate this point by switching over to the spatially-flat gauge, in which the MS variable is $Q_{\mathbf{k}} = \delta\phi_{\mathbf{k},f}$ (the subscript '$f$' indicates the use of the spatially-flat gauge). Rewriting (33) in Fourier space:

$$H_l^2\frac{\partial^2\delta\phi_{\mathbf{k},f}}{\partial N^2}+3H_l^2\frac{\partial\delta\phi_{\mathbf{k},f}}{\partial N}+\left[\frac{k^2}{a^2}+H_l^2\left(-\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\epsilon_2-\frac{1}{4}\epsilon_2^2-\frac{1}{2}\epsilon_2\epsilon_3\right)\right]\delta\phi_{\mathbf{k},f}=0. \tag{74}$$

Let us emphasise that we are computing $\delta\phi_{\mathbf{k},f}$ in the local patch since, as we mentioned, it is a UV variable and therefore evolves at sub-horizon scales, and we will later evaluate it at the coarse-grained scale, when it has become a squeezed state. Rewriting (74) with our usual variables we obtain:

$$\frac{\partial^2\delta\phi_{\mathbf{k},f}}{\partial N^2}+3\frac{\partial\delta\phi_{\mathbf{k},f}}{\partial N}+\left[(\alpha^{IR})^2\frac{k^2}{(aH^b)^2}+\left(-\frac{3}{2}\epsilon_2+\frac{1}{2}\epsilon_1\epsilon_2-\frac{1}{4}\epsilon_2^2-\frac{1}{2}\epsilon_2\epsilon_3\right)\right]\delta\phi_{\mathbf{k},f}=0. \tag{75}$$

But $\alpha^{IR}$ is given by (57), i.e. it depends on both $\phi^{IR}$, which is a stochastic variable according to (66); and $\xi_1$. In other words, to find the power spectrum of $\delta\phi_{\mathbf{k},f}$, and therefore the variance of $\xi_1$ given by (71), we must already know the values of $\xi_1$ and $\xi_2$.

It is possible to solve this non-Markovian system numerically, as in [32]. Nonetheless, we will perform a further approximation in order to make the system Markovian and hence easier to solve. We will assume that the IR quantities are approximately equal to the background ones at first order in the UV variables, that is:

$$X^{IR}Y^{UV} \simeq X^bY^{UV}+\mathcal{O}\left((Y^{UV})^2\right), \tag{76}$$

or, equivalently, $X^{IR} - X^b = \mathcal{O}(Y^{UV})$. This means that we are considering the IR variables to be of linear order in UV variables, which implies that, under this approximation, stochastic inflation is only able to reproduce the LPT results, as long as the linear theory holds [9]. This seems to be in contradiction with gradient expansion, which, as we have stressed above, captures all orders in the amplitudes of the perturbations. Nonetheless, this apparent incoherence means that, if non-perturbative effects are present, we will be able to detect them as our stochastic formalism will not be in accordance with LPT as the previous approximation will not hold.

Under this assumption, then, the $\alpha^{IR}$ term in (75) becomes:

$$\frac{k^2}{(aH^b)^2}(\alpha^{IR})^2\delta\phi_{\mathbf{k},f} \simeq \frac{k^2}{(aH^b)^2}\delta\phi_{\mathbf{k},f} + \mathcal{O}\left((\delta\phi_{\mathbf{k},f})^2\right), \tag{77}$$

where we have used $\alpha^b = 1$. Now we have eliminated the dependence on $\xi_1$ and $\xi_2$, recovering equation (35), written in the global (and analytical) background, and making the system Markovian with additive noises [25]. Since we will evaluate the perturbation at the coarse-grained scale, where $(-k\tau) \simeq \sigma \ll 1$, we can make use of solution (38).

Finally, we want to recover the perturbation calculated in the uniform-$N$ gauge, $\delta\phi_{\mathbf{k},\delta N}$. It can be shown that the gauge transformation between $\delta\phi_{\mathbf{k},f}$ and $\delta\phi_{\mathbf{k},\delta N}$ is of order $\mathcal{O}(\epsilon_1)$ [29, 32]. Since $\epsilon_1$ is negligible in CR, as we have shown in Section II A, we can safely make the approximation $\delta\phi_{\mathbf{k},\delta N} = \delta\phi_{\mathbf{k},f} = Q_{\mathbf{k}}$.

### D. Comparison against Linear Perturbation Theory

In order to be able to test our stochastic formalism against LPT results, we need to define a gauge-invariant observable. The most obvious candidate is the MS variable, of which a linear counterpart can be defined as [9, 25]:

$$Q^{IR} = \phi^{IR} - \phi^b - \frac{\partial\phi^{IR}}{\partial N}\frac{1}{3}\nabla^2 C^{IR}. \tag{78}$$

Since, as explained previously, the approximation (76) causes our formalism to be limited to the reproduction of LPT, and consequently we expect, as long as LPT holds, $\langle Q^{lin}Q^{lin}\rangle \simeq \langle Q^{IR}Q^{IR}\rangle$ (where $Q^{lin}$ denotes the analytical MS variable (31)). On the other hand, as we have stressed above, if our stochastic formalism diverges from the linear result it will signal a breakdown of LPT and the coming into play of non-perturbative effects.

Since $Q^{IR}$ is a stochastic variable, its two point correlator will simply be its statistical variance:

$$\langle Q^{IR}(N)Q^{IR}(N)\rangle = \mathrm{Var}[Q^{IR}(N)], \tag{79}$$

whereas for the linear MS variable [9]:

$$\langle Q^{lin}(N)Q^{lin}(N)\rangle = \int_{\sigma a(N_0)H^b(N_0)}^{\sigma a(N)H^b(N)} \frac{dk}{k}\mathcal{P}_Q(k,N), \tag{80}$$

where the integration limits correspond to the modes inside the coarse-grained scale during our stochastic simulation, and the power spectrum is defined as:

$$\mathcal{P}_Q(k,N) \equiv \frac{k^3}{2\pi^2}|Q_{\mathbf{k}}(N)|^2. \tag{81}$$

## VII. ALGORITHM

Before presenting our results we want to briefly explain the numerical algorithm employed to solve each formalism in the CR regime. To calculate the two point correlator of the linear MS variable we simply use the solution (39) to integrate (80) numerically. As we explained in the previous section, this has to be compared to the non-linear MS variable (78), which implies solving the system (65)-(67) numerically. One further simplification can be made: since, as explained in Section VI C, $Q_{\mathbf{k}} = \delta\phi_{\mathbf{k},\delta N}$ at zeroth order in $\epsilon_1$, we can neglect the last term in (78) as it will only provide $\mathcal{O}(\epsilon_1)$ information, and, as a consequence, equation (67) can be ignored. In addition, according to (65):

$$(\pi^{IR})^2 \sim (\dot{\phi}^{IR}/H^b)^2 \sim \epsilon_1, \tag{82}$$

where in the last step we have used (9). Thus we can neglect the $(\pi^{IR})^2$ terms in (66).

To solve the system of stochastic equations, then, we will use an order 1.5 strong Stochastic Runge-Kutta method [33, 34]. We can write a generic stochastic differential equation (SDE) in the form:

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \tag{83}$$

where $W(t)$ is a Wiener process related to a white noise $\xi(t)$ as $dW(t) = \xi(t)dt$, and imposing the additivity of the noises implies $b(t, X(t)) = b(t)$. The method is recursive, with the solution for every time step of size $h_n$ being:

$$X_{n+1} = X_n + \sum_{i=1}^{3} \alpha_i a\left(t_n + c_i^{(0)} h_n, H_i^{(0)}\right) h_n + \sum_{i=1}^{3} \left(\beta_i^{(1)} I_{(1)} + \beta_i^{(2)} \frac{I_{(1,0)}}{h_n}\right) b\left(t_n + c_i^{(1)} h_n\right), \tag{84}$$

with stages:

$$H_i^{(0)} = X_n + \sum_{j=1}^{3} A_{ij}^{(0)} a\left(t_n + c_j^{(0)} h_n, H_j^{(0)}\right) h_n + \sum_{j=1}^{3} B_{ij}^{(0)} b\left(t_n + c_j^{(1)} h_n\right) \frac{I_{(1,0)}}{h_n}, \tag{85}$$

where $I_{(1)}$ and $I_{(1,0)}$ will be specified later, and the rest of the constants can be written in a compact way using a Butcher tableau, shown in Table I.

| $c^{(0)}$ | $A^{(0)}$ | $B^{(0)}$ | $c^{(1)}$ |
|---|---|---|---|
| | $\alpha^T$ | $\beta^{(1),T}$ | $\beta^{(2),T}$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | 1 |
| 1 | 1 | | | 0 | | | 0 |
| 1/2 | 1/4 | 1/4 | | 1 | 1/2 | | 0 |
| | 1/6 | 1/6 | 2/3 | 1 | 0 | 0 | 1 -1 0 |

TABLE I. Butcher tableau (left) and its specific entries (right).

$I_{(1)}$ and $I_{(1,0)}$ are some Ito stochastic integrals, which can be implemented numerically by defining two independent random variables, $U_1$ and $U_2$, that follow a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$, so that:

$$I_{(1)} = U_1 \sqrt{h_n}\ ;\ \ I_{(1,0)} = \frac{1}{2} h_n^{3/2} \left(U_1 + \frac{U_2}{\sqrt{3}}\right). \tag{86}$$

In our case, since we have a system of SDEs rather than a single one, we simply have to apply this algorithm to every equation simultaneously for every time step, with the Ito integrals (86) being the same for every equation as the noises are completely correlated.

To illustrate the precision that can be achieved with this method, as opposed, for example, to the much simpler Euler method [35], we will use it to solve a paradigmatic SDE: one that describes the velocity $v(t)$ of a particle of mass $m$ undergoing Brownian motion in one dimension inside a fluid with friction coefficient $\alpha$:

$$m \frac{dv(t)}{dt} = -\alpha v(t) + \xi(t), \tag{87}$$

which has an analytical solution:

$$v(t) = v(0) e^{-\frac{\alpha}{m} t} + \frac{1}{m} e^{-\frac{\alpha}{m} t} \int_0^t e^{\frac{\alpha}{m} s} dW_s, \tag{88}$$

with $v(0)$ being the initial velocity of the particle. In Figure 1 the comparison between the Euler and Runge-Kutta methods with respect to the analytical solution is shown. It can be readily seen that, using only 100 steps, the Runge-Kutta algorithm is already reproducing the analytic solution with very high precision, as opposed to the Euler method.

## VIII. RESULTS

In Figures 2 and 3 we show the result obtained for the stochastic formalism in the CR regime, i.e. using the equations from Section II A as the potential, the background solution of the field, etc. We also compare this solution to the LPT result. It should be noted that we have set $M_{Pl} = 1$ in equations (15)-(18) for simplicity, and we have arbitrarily chosen $\kappa = 0.5$ and $N_0 = 1$. The code that we have developed and used can be found in Appendix B.

From these results we can readily see that we have been able to reproduce LPT theory using the stochastic formalism, and therefore it seems that non-perturbative effects are not relevant for the CR regime.

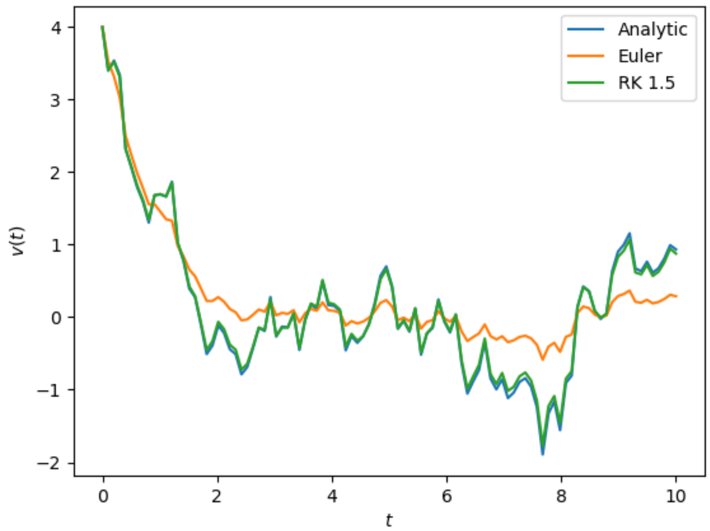


FIG. 1. Analytical solution of the SDE (87), together with the numerical solution, employing the Euler and order 1.5 Runge-Kutta methods, respectively. Both velocity and time are shown in arbitrary units. We have set $m = \alpha = 1$ and $v(0) = 4$. We are using 100 time steps.

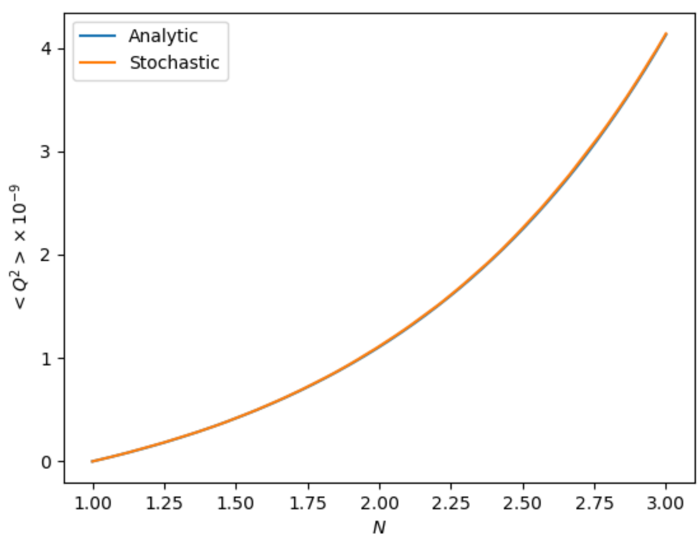


FIG. 2. Two point correlator of the MS variable as a function of the number of e-folds $N$, solved both in the analytical LPT regime (80), and in the stochastic formalism (79) using the order 1.5 strong Runge-Kutta algorithm with 1000 time steps, where we have used $n = 10^5$ different trajectories to calculate $\mathrm{Var}[Q^{IR}(N)]$.

## IX. CONCLUSIONS

We have studied the cosmological perturbations generated in the CR regime, in which the acceleration of the scalar field is proportional to the friction term, from the point of view of stochastic inflation. We have attempted to derive the method step by step, using the ADM formalism, $\mathcal{O}(\sigma^0)$ gradient expansion and the splitting of our variables of interest into IR (long-wavelength) and UV (short-wavelength) modes, with the latter being treated perturbatively as they are well inside the Hubble horizon. After these calculations one obtains a system of SDEs, with the UV modes

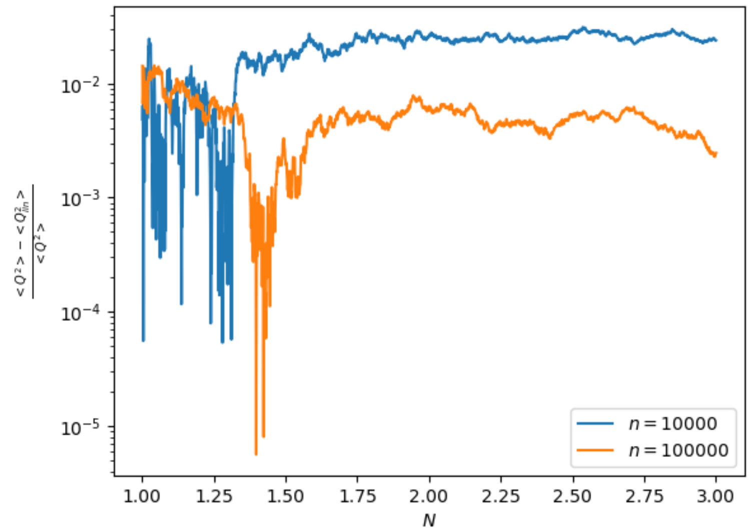


FIG. 3. Relative error, normalised to 1, between the two point correlator of the non-linear MS variable, which we have called $Q$ for simplicity, calculated using the order 1.5 strong Runge-Kutta algorithm with 1000 time steps; and the two point correlator of the linear MS variable, which we have called $Q_{lin}$. We are comparing the result when using $n = 10^4$ and $n = 10^5$ different trajectories in the calculation of $\text{Var}[Q^{IR}(N)]$.

behaving, once they cross the coarse-grained scale, as a white noise for the IR part. This system can then be solved numerically.

As we have tried to stress in the text, however, in order to characterise the variables $\xi_i$ as white noises, as well as making the system of SDEs Markovian, one has to restrict the IR variables to the linear order. This last approximation, as we have shown, implies that, if a contradiction between the stochastic results and those of LPT were to arise, it could be an indication of the presence of non-perturbative effects and the collapse of LPT. The results we have obtained by restricting ourselves to zeroth order in $\epsilon_1$, and using the very precise order 1.5 strong Runge-Kutta algorithm, however, match their linear counterpart with a high level of precision, which leads us to conclude that the linear approximation does not break down in the CR regime.

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### Appendix A: Clarification on the gradient expansion

We have claimed that ${}_{(0)}\nabla^2 C \sim \mathcal{O}(\sigma^0)$ is not in contradiction with the statement ${}_{(0)}\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) C \sim \mathcal{O}(\sigma)$. To demonstrate it we can take for example $C = \mathbf{x}\cdot\mathbf{x}g(t,\sigma\mathbf{x})$, with $g(t,\sigma\mathbf{x})$ being an arbitrary function. In this case:

$$\partial_i\partial_j C - \frac{1}{3}\nabla^2 C \sim \mathcal{O}(\sigma), \tag{A1}$$

$$\frac{1}{3}\nabla^2 C = 2g(t,0) + \mathcal{O}(\sigma), \tag{A2}$$

and hence ${}_{(0)}\nabla^2 C \sim \mathcal{O}(\sigma^0)$.

### Appendix B: Code

Below we present the code, written in Python, that we have developed and used in our study. We have employed it in the case of CR, but it is completely general and can be applied to solve the stochastic formalism for any potential, as long as the parameter $\nu$ defined in (37) is constant. One would simply have to change the different functions for $\phi^b(t)$, $V(\phi)$, $H^b(t)$, etc. in lines 9-37, the initial condition for the velocity of the field $\left.\frac{\partial\phi}{\partial N}\right|_{N_0}$ in line 51, as well as $\nu$ in line 54, to correspond to the model of interest.

```
1 import math
2 import numpy as np
3 import random
4 import matplotlib.pyplot as plt
5 
6 #Functions for Constant Roll
7 kappa=0.5 #Parameter kappa in Starobinsky's paper
8 sigma=0.1
9 def phib(t): #Inflaton
10     phib=math.sqrt(2/(3+kappa))*math.log(1/math.tanh((3+kappa)/2*t))
11     return phib
12 def v(phi): #Potential
13     v=3*(1+(kappa/6)*(1-np.cosh(math.sqrt(2*(3+kappa))*phi))) #using M=1
14     return v
15 def dv(phi): #Derivative of the potential
16     dv=-math.sqrt((3+kappa)/2)*kappa*np.sinh(math.sqrt(2*(3+kappa))*phi)
17     return dv
18 def Hb(t): #Hubble rate
19     Hb=1/math.tanh((3+kappa)*t)
20     return Hb
21 def t(N): #Cosmic time as a function of the number of e-folds
22     t=np.arcsinh(math.exp((N-1)*(3+kappa))*math.sinh(3+kappa))/(3+kappa)
23     return t
24 def eps1(t):
25     eps1=(3+kappa)/(math.cosh((3+kappa)*t)**2)
26     return eps1
27 def eps2(t):
28     eps2=-2*(3+kappa)*math.tanh((3+kappa)*t)**2
29     return eps2
30 def a(t): #Scale factor
31     a=math.sinh((3+kappa)*t)**(1/(3+kappa))
32     return a
33 def tau(t): #Conformal time
34     tau=-1/(a(t)*Hb(t))
35     return tau
36 
37 #Steps, initial conditions, etc.
38 steps=1000
39 N0=1
40 Nf=3
41 N=np.linspace(N0,Nf,steps)
42 fi=[]
43 for Ni in N:
44     ttt=t(Ni)
45     fi.append(phib(ttt))
46 for pi in fi:
47     potential.append(v(pi))
48     dpot.append(dv(pi))
49 dphib0=-1/(math.sinh((3+kappa)*t(1)))*math.sqrt(6+2*kappa)*Hb(t(1))
50 
51 o=10**(-9)
52 nu=3/2*math.sqrt(1-4/9*(-3*kappa-kappa**2)) #Nu for the Henckel functions
53 def k(t): #k at coarse-grained scale
54     k=sigma*a(t)*Hb(t)
55     return k
56 def var1(t,k,nu): #Variance of white noise 1, also POWER SPECTRUM OF Q
57     conft=tau(t)
58     Q2=k**3/(2*math.pi**2)*abs(2**(2*nu-2)*(-conft)*(-k*conft)**(-2*nu)*math.gamma(nu)**2/(a(t)**2*
       math.pi))*o
59     return Q2
```

```python
def var2(t,k,nu): #Variance of white noise 2
    dQ2=var1(t,k,nu)*(nu-3/2)**2
    return dQ2

#System
def a1(N,phi,pi):
    a1=pi
    return a1
def b1(N):
    time=t(N)
    kk=k(time)
    b1=-math.sqrt(var1(time,kk,nu))
    return b1
def a2(N,phi,pi):
    if abs(phi/math.sqrt(2*(3+kappa)))<100:
        a2=-3*(pi+dv(phi)/v(phi))
    else:
        if phi>0:
            a2=-3*(pi+math.sqrt(2*(3+kappa)))
        else:
            a2=-3*(pi-math.sqrt(2*(3+kappa)))
    return a2
def b2(N):
    time=t(N)
    kk=k(time)
    b2=-math.sqrt(var2(time,kk,nu))
    return b2

#Initial conditions
phiIR0=fi[0]
piIR0=dphib0

# RK1.5
# As in 2107.12735

#Butcher tableau
c0=[0,1,0.5]
A0=[[0,0,0],[1,0,0],[0.25,0.25,0]]
B0=[[0,0,0],[0,0,0],[1,0.5,0]]
c1=[1,0,0]
alphaT=[1/6,1/6,2/3]
beta1T=[1,0,0]
beta2T=[1,-1,0]

def rk3(N,a1,b1,a2,b2,y10,y20):
    hn=N[1]-N[0]
    y1=[]
    y2=[]
    y1.append(y10)
    y2.append(y20)
    for pas in range(1,len(N)):
        #Ito integrals, according to (65)
        u1=np.random.normal()
        u2=np.random.normal()
        I1=u1*math.sqrt(hn)
        I10=((hn**(3/2))/2)*(u1+u2/math.sqrt(3))
        #Calculate H^(0)_i according to eqn (62) in the paper
        H01=[]
        H01.append(y1[pas-1])
        H02=[]
        H02.append(y2[pas-1])
        for i in range(1,3):
            sumone1=0
            sumone2=0
            sumtwo1=0
            sumtwo2=0
            for j in range(i):
                sumone1=sumone1+A0[i][j]*a1(N[pas-1]+c0[j]*hn,H01[j],H02[j])*hn
                sumone2=sumone2+B0[i][j]*b1(N[pas-1]+c1[j]*hn)*I10/hn
                sumtwo1=sumtwo1+A0[i][j]*a2(N[pas-1]+c0[j]*hn,H01[j],H02[j])*hn
```

```
                sumtwo2=sumtwo2+B0[i][j]*b2(N[pas-1]+c1[j]*hn)*I10/hn
            HH1=y1[pas-1]+sumone1+sumone2
            H01.append(HH1)
            HH2=y2[pas-1]+sumtwo1+sumtwo2
            H02.append(HH2)
        #Calculate Y_(n+1) according to (61)
        s1=0
        s2=0
        for i in range(3):
            s1=s1+alphaT[i]*a1(N[pas-1]+c0[i]*hn,H01[i],H02[j])*hn
            s2=s2+(beta1T[i]*I1+beta2T[i]*I10/hn)*b1(N[pas-1]+c1[i]*hn)
        y11=y1[pas-1]+s1+s2
        y1.append(y11)
        s1=0
        s2=0
        for i in range(3):
            s1=s1+alphaT[i]*a2(N[pas-1]+c0[i]*hn,H01[j],H02[i])*hn
            s2=s2+(beta1T[i]*I1+beta2T[i]*I10/hn)*b2(N[pas-1]+c1[i]*hn)
        y21=y2[pas-1]+s1+s2
        y2.append(y21)
    return y1,y2


# Computing <Q^IR Q^IR>
def QIR(N,phiIR): #Input: vector N and vector phi (solved via SDEs)
    QIR=[]
    for i in range(len(N)):
        ti=t(N[i])
        Q=phiIR[i]-phib(ti)
        QIR.append(Q)
    return QIR #Output: vector Q^IR/(N)
def varQ(Q): #Input: vector of many different solutions for Q
    var=[]
    for i in range(len(Q[0])):
        QN=[]
        for j in range(len(Q)):
            QN.append(Q[j][i]) #We take all the solutions for a given N and compute the variance
        var.append(np.var(QN))
    return var #Output: vector with VarianceQ(N)

Qs=[]
stats=100000
for j in range(stats):
    phiIR,piIR=rk3(N,a1,b1,a2,b2,phiIR0,piIR0)
    qir=QIR(N,phiIR)
    Qs.append(qir)

variance=varQ(Qs)


#Computing < Qlin Qlin >
def I(f,N,ki,kf,points):
    I=0
    ks=np.linspace(ki,kf,points)
    for j in range(points-1):
        I=I+f(ks[j],N)*(ks[j+1]-ks[j])
    return I
def integrand(k,N):
    ti=t(N)
    f=var1(ti,k,nu)/k
    return f
def varQlin(N):
    var=[]
    ti=t(N[0])
    k0=k(ti)
    for i in range(len(N)):
        tf=t(N[i])
        kf=k(tf)
        var.append(I(integrand,N[i],k0,kf,1000))
    return var
```

```

variance2=varQlin(N)

vari=np.array(variance)*10**9
vari2=np.array(variance2)*10**9
h=plt.plot(N,vari2,label='Analytic')
hh=plt.plot(N,vari,label='Stochastic')
plt.xlabel('$N$')
plt.ylabel(r'$<Q^2>$')
plt.legend()
plt.savefig('VarQ_UV')
plt.show()

y=[]
for i in range(len(N)):
    yi=abs((variance[i]-variance2[i])/variance[i])
    y.append(yi)
plt.plot(N,y)
plt.yscale('log')
plt.xlabel('$N$')
plt.ylabel(r'$\frac{<Q^2>-<Q_{lin}^2>}{<Q^2>}$')
plt.savefig('Error')
plt.show()
```

Listing 1. Code employed in our study, written in Python.


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