Testing for multiple level shifts with an integrated or stationary noise component*

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Abstract

The paper analyzes the detection and estimation of multiple level shifts regardless of the order of integration of the time series. We show that it is possible to extend the methodology of Bai and Perron (1998) to the I(1) and NI(1) nonstationary cases so that a unified framework to test for the presence of multiple level shifts in a robust way is designed. The finite sample performance of the proposed statistics is carried out, establishing a comparison with other existing approaches in the literature. The paper illustrates the implementation of the statistics focusing on the real exchange rate with time series that either cover a long time period or provide a worldwide analysis. Robust detection of multiple level shifts is of great importance to define the statistical approach that is used to test the purchasing power parity hypothesis.

Keywords: multiple structural breaks, union of rejections, robust break date estimation, long-run variance estimation, purchasing power parity

JEL codes: C12, C22

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1 Introduction

Time series modeling requires the characterization of shocks that affect their evolution so that proper estimation techniques and statistical inference are applied. The literature has distinguished between recurrent and occasional shocks, and the characterization of these shocks in terms of their persistence is of great importance for time series modeling. In this regard, the order of integration analysis of stochastic processes requires the use of test statistics that take into account features that can bias the persistence of recurrent shocks. One of these features is the existence of structural breaks. Perron (1989, 1990) note that the inference drawn from the Dickey-Fuller (DF) test statistic can be seriously plagued if the presence of structural breaks is not accounted for. This situation is due to the dependence of the unit root statistics limiting distribution on the type, number and position of the structural breaks. As pointed out in Perron (1994, 2006), one can view structural changes as infrequent events that have permanent effects on the time series level. Notwithstanding, some popular procedures that are applied to detect the presence of structural breaks rely on particular assumptions about the persistence of recurrent shocks – see Perron (2006) and Casini and Perron (2019) for an overview. As can be seen, there exists an intrinsic relationship between the modeling of structural changes and the degree of persistence of recurrent shocks.

There are different proposals in the literature to assess the presence of structural breaks affecting time series regardless of its order of integration. Perron and Yabu (2009) consider one structural break for trending time series, with three different types of effects – change in the level (Model I), in the slope (Model II) or both (Model III). Saygindoy and Vogelsang (2011) cover these three models, but also tackle the case of non-trending variables. Kejriwal and Perron (2010) generalize the proposal in Perron and Yabu (2009) to multiple structural breaks, but just focusing on Models II and III. Finally, Harvey, Leybourne and Taylor (2010) is the only proposal in the literature that deals with the case of just multiple level shifts.

In this paper, we focus on testing and estimation of multiple level shifts for non-trending time series, regardless of the order of integration of the stochastic process. We follow Bai and Perron (1998) who design testing procedures for models for which recurrent errors (shocks) have transitory effects – i.e., the error term of the model is an integrated of order zero, I(0), stationary stochastic process – but extending their methodology to the case in which recurrent shocks have permanent effects – i.e., the error term of the model is an integrated of order one, I(1), non-stationary stochastic process; for completeness, local-to-unit root stochastic processes are also considered. The first area of contribution centres on the statistic suggested in Bai and Perron (1998) that tests the null hypothesis of no structural break against the alternative hypothesis of a fixed number of structural breaks. In addition, the paper focuses on the double max statistics that allow assessing the presence of parameter instabilities considering up to a maximum number of structural breaks. In all cases, it is possible to define union statistics to test the presence of multiple level shifts regardless of the order of integration of the time series. To the best of our knowledge, this type of robust statistics have not been previously designed in the literature.

The second area of contribution concerns the use of sequential statistics to detect the presence of multiple structural breaks when dealing with stochastic processes with either a unit root or a local-to-unit root. The limiting distribution is shown to depend on the number and positions of the structural breaks, which introduces two essential issues. First, the computation of critical values for each step of the sequential testing depends on the previously estimated break dates, something that prevents offering a reasonable set of tables with the asymptotic critical values. To overcome this limitation, response surfaces are estimated to approximate asymptotic critical values so that the implementation of the new statistic in empirical applications is straightforward. Second, Bai and Perron (1998) proved that the limiting distribution of the sequential statistic applied to I(0) stochastic processes can be written as a function of independent random variables. This characteristic eases the computation of asymptotic critical values, since what is relevant is the number of structural breaks under the null hypothesis, not their position. This paper shows that it is possible to modify our initial proposal so that the limiting distribution of the modified statistics shares the same feature as the original sequential statistic in

Bai and Perron (1998). Finally, the joint use of the modified statistics for I(1) stochastic processes and the original statistic in Bai and Perron (1998) for I(0) stochastic processes allows us to propose union statistics that serve at testing the presence of multiple level shifts regardless of the order of integration of the time series.

The estimation of the long-run variance deserves special attention if empirical size and/or non-monotonic power problems of the statistics are to be prevented – see Perron (2006) and Casini and Perron (2019) for further details. In this regard, Kejriwal and Perron (2010) propose a hybrid estimation method of the long-run variance to reach a compromise between the size and power trade-off of Bai-Perron (BP) type statistics. This paper generalizes this approach and defines the so-called max-hybrid estimation method that is also used for the computation of the original Bai and Perron (1998) statistics. This estimation procedure is aimed at controlling the empirical size of the statistics when the incorrect I(0) order of integration is assumed.

An extensive simulation experiment is conducted to assess the performance of the proposed statistical inference under different scenarios. The simulation results evidence that the robust analysis that relies on the sequential BP statistics outperforms other existing proposals in the literature. In addition, the simulations allow us to study the performance of two well-known strategies that can be implemented to estimate the structural break date locations.

The paper is organized as follows. Section 2 presents the model and assumptions. Section 3 summarizes the proposals in Harvey, Leybourne and Taylor (2010) and Bai and Perron (1998), along with the new set of statistics that is designed in this paper. Section 4 deals with the estimation of the long-run variance. Section 5 investigates the finite sample performance of the proposed statistics. An empirical illustration that is based on real exchange rates using both historical and worldwide data sets is conducted in Section 6. Finally, Section 7 concludes with some remarks. The on-line supplementary material provides the proofs (Appendix A), tables of critical values and simulation results (Appendix B) and the details of the computations that are carried out in the empirical illustration (Appendix C).

2 The model

Let $\{y_t\}_{t=1}^T$ be a stochastic process with the data-generating process (DGP) given by:

$$\begin{aligned} y_t &= \mu + \sum_{i=1}^{m} \gamma_i DU_{i,t} + u_t \qquad (1) \\ u_t &= \rho u_{t-1} + \varepsilon_t, \qquad (2) \end{aligned}$$

with $u_0 = O_p(1)$, $DU_{i,t} = 1$ for $t > \lfloor \lambda_i^0 T \rfloor$, 0 otherwise, where $\lfloor \cdot \rfloor$ denotes the integer value, $\lambda_i^0 \in \Lambda(\epsilon)$, $i = 1, \dots, m$, are the break fraction parameters, $\lambda_{B,m}^0 = (\lambda_1^0, \dots, \lambda_m^0)$, $\Lambda(\epsilon) = [\epsilon, 1-\epsilon] \in (0,1)$ defines the admissible values of the break fractions, and ϵ is the amount of trimming – popular choices are $\epsilon \in \{0.05,\ 0.15,\ 0.2\}$. Throughout the paper, the "0" superscript indicates the true value of the corresponding parameter. We assume that ε_t is a stochastic process that satisfies the following linear process assumptions.

Assumption LP. Let $\{\varepsilon_t\}$ be a linear stochastic process such that $\varepsilon_t = C(L)\eta_t$ with $\theta(L) = \sum_{j=0}^{\infty} C_j L^j$, $C(1)^2 > 0$ and $\sum_{j=0}^{\infty} j|C_j| < \infty$, where $\{\eta_t\}_{t=1}^T$ is an *iid* sequence of with mean zero, variance σ_η^2 and finite fourth moment. The long-run variance (LRV) of ε_t is given by $\omega_\varepsilon^2 = \lim_{T\to\infty} T^{-1} E\left(\sum_{t=1}^T \varepsilon_t\right)^2 = \sigma_\eta^2 C(1)^2$.

The paper deals with two main situations, depending on the order of integration of y_t. First, we specify the scenario in which the errors of the model in (1) follow a unit root or nearly-unit root stochastic process – denoted as I(1) and NI(1) cases, respectively – with the definition of $\rho = \rho_T := 1 - c/T$, $0 \leq c < \infty$, in (2). When $c = 0$ we have a stochastic process with a unit root – i.e., an I(1) stochastic process – whereas when $c > 0$ we have a stochastic process with a local-to-unit root – i.e., a NI(1) stochastic process. Second, we also consider the framework in which the errors of the model in (1) have a transitory effect on y_t, a situation that is covered imposing $|\rho| < 1$ in (2). Hereafter, this second situation is denoted as the I(0) case.

The proposal embeds three popular setups that can be found in the literature to specify the magnitude of the level shifts. First and following Bai and Perron (1998), we deal with structural breaks of fixed magnitude γ_i, $i = 1, \dots, m$, in (1). Second and following Harvey, Leybourne and Taylor (2010), we define magnitudes of the level shifts that depend on the sample size as:

$$\gamma_i = \sigma_d \gamma_i^* T^{d-1/2}, \tag{3}$$

where $d \in \{0,1\}$ denotes the order of integration of u_t (and y_t), $\sigma_0^2 = \lim_{T\to\infty} E[T^{-1} (\sum_{t=1}^{T} u_t)^2] = \sigma_\eta^2 C(1)^2 / (1-\rho)^2$ when $|\rho| < 1$, and $\sigma_1^2 = \omega_\varepsilon^2$ when $\rho = \rho_T := 1 - c/T$, $0 \leq c < \infty$, $|\gamma_i^*| < \infty$, $i = 1, \ldots, m$, and $T^{d-1/2}$ is the Pitman's drift. Therefore, the paper considers different scenarios depending on the order of integration of u_t and the structural break magnitudes, which can be summarized in the following assumptions.

Assumption I(1)-NI(1). Let Assumption LP hold, with $\rho = \rho_T := 1 - c/T$, $0 \leq c < \infty$, in (2) and $\gamma_i = \sigma_1 \gamma_i^* T^{1/2}$.

Assumption I(0). Let Assumption LP hold, with $|\rho| < 1$ in (2), and $\gamma_i = \sigma_0 \gamma_i^*$ (fixed structural break magnitudes) or $\gamma_i = \sigma_0 \gamma_i^* T^{-1/2}$ (shrinking structural break magnitudes).

Under Assumption I(1)-NI(1) the magnitude of the level shifts increases at a $T^{1/2}$ rate that, first, prevents structural breaks to have negligible effects in the limit and, second, implies that a consistent estimation of the break dates can be obtained – see Harvey, Leybourne and Taylor (2010). This setup has also been used in Leybourne and Newbold (2000), Kim, Leybourne and Newbold (2000) and Harvey, Leybourne and Newbold (2001), among others, and imposes that the effect of the structural breaks is of the same order of magnitude (in probability) as the stochastic trend, so that neither component totally dominates the large sample behaviour of y_t – see Leybourne and Newbold (2000). Similarly, Carrion-i-Silvestre, Kim and Perron (2009) consider the case where the magnitude of the level shifts increases at a $T^{1/2+\eta}$ rate, with $\eta > 0$. Intuitively, this parametrization of the break magnitudes can be helpful to model the effect of big structural breaks – i.e., structural breaks which effect is not dominated by the stochastic trend – and provide a better characterization of the nature of the shocks affecting the time series – recurrent shocks with permanent effects (stochastic trend) and occasional shocks (structural breaks) with permanent effects. Finally, it is worth noting that structural breaks of fixed magnitude are asymptotically negligible if $y_t \sim I(1)$ – see Perron (1990).

Under Assumption I(0) the magnitude of the level shifts given in (3) is local-to-zero and asymptotically negligible – see Harvey, Leybourne and Taylor (2012) – which defines the so-called shrinking breaks case. This specification is used to model the effect of small magnitudes of structural breaks, but also remains an adequate approximation for moderate shifts – see Bai and Perron (1998). Unfortunately, in this case, it is not possible to obtain a consistent estimation of the break fractions because the magnitude of the shifts converges to zero at a fast rate ($T^{-1/2}$). It is worth noting that Bai (1997), Bai and Perron (1998), Bai, Lumsdaine and Stock (1998), Busetti and Harvey (2001), Kurozumi and Arai (2006), Perron (2006) and Oka and Perron (2018) also deal with similar shrinking breaks configurations, although considering that the break magnitudes decrease towards zero at a rate slower than $T^{-1/2}$ – i.e., the magnitude of the level shifts decreases at a $T^{-1/2+\eta}$ rate, with $0 < \eta < 1/2$ – which allows obtaining consistent break fraction estimates. In this scenario, fixed break magnitudes can be approximated choosing η arbitrarily close to $1/2$ – see Bai and Perron (1998), Proposition 4.

3 Robust structural break test statistics

3.1 The Harvey, Leybourne and Taylor statistics

To assess the presence of multiple level shifts, Harvey, Leybourne and Taylor (2010) suggest the (sequential) use of the following two generalized fluctuation statistics:

$$S_d = \max_{t \in T\Lambda(\epsilon)} S_{d,t,\lfloor wT \rfloor} = \sigma_d^{-1} T^{1/2-d} \max_{t \in T\Lambda(\epsilon)} \left| \frac{\sum_{j=1}^{\lfloor \frac{w}{2}T \rfloor} y_{t+j} - \sum_{j=1}^{\lfloor \frac{w}{2}T \rfloor} y_{t-j+1}}{\lfloor \frac{w}{2}T \rfloor} \right|; \quad d \in \{0,1\}, \tag{4}$$

where S_0 denotes the statistic that is computed assuming that $y_t \sim I(0)$ and S_1 is the statistic that assumes that $y_t \sim I(1)$. The S_d statistics differ both on the scaling that is required ($T^{1/2-d}$) and on the LRV estimator that is used (σ_d^2), $d \in \{0,1\}$. These fluctuation statistics – in what follows, HLT statistics – are based on the difference between the mean of the $\lfloor \frac{w}{2}T \rfloor$ observations $y_{t+1}, \ldots, y_{t+\lfloor \frac{w}{2}T \rfloor}$ and the mean of the $\lfloor \frac{w}{2}T \rfloor$ observations $y_t, y_{t-1}, \ldots, y_{t-\lfloor \frac{w}{2}T \rfloor+1}$, where w is the bandwidth of the window of observations that are used – Harvey, Leybourne and Taylor (2010) essayed different values of w and recommend $w = 0.10$ as a value that produces a good compromise between the empirical size and power of the statistics. The consideration of both statistics when there is no prior knowledge about the order of integration of time series leads to define a union statistic, which is based on the union of rejection of the null hypotheses that test the S_d, $d \in \{0, 1\}$, statistics. To be specific, the union of rejection decision rule rejects the null hypothesis of no structural break if $S_1 > \kappa_\xi cv_\xi^1$ or $S_0 > \kappa_\xi cv_\xi^0$, where cv_ξ^1 and cv_ξ^0 denote the ξ significance level asymptotic critical values of S_1 and S_0, respectively. The factor κ_ξ is a positive scaling constant that warrants the union rejection decision rule to be asymptotically conservative under the null hypothesis in the presence of a wrong assumption of the order of integration that comes from one of the statistics. Note that the union rejection decision rule is equivalent to define the following union (U) statistic:

$$U = \max \left\{ S_1, \left(\frac{cv_\xi^1}{cv_\xi^0} \right) S_0 \right\}, \tag{5}$$

for which the null hypothesis of no structural break is rejected if $U > \kappa_\xi cv_\xi^1$.

The sequential implementation of the HLT statistics allows the estimation of both the number and position of the structural breaks. To ease presentation on the practical implementation of this strategy, let us focus on the S_1 statistic. The first stage consists of testing the null hypothesis of no structural break against the alternative hypothesis of one level shift using all observations $t = 1, \ldots, T$. If evidence against the null hypothesis is found, the first break date is estimated as $\tilde{T}_1 = \arg\max_{t \in \Lambda_T} S_{1,t,\lfloor wT \rfloor}$, $\Lambda_T = T\Lambda(\epsilon)$. This break date estimate is used to define the exclusion area given by $\Lambda_{1,T} = [\tilde{T}_1 - \lfloor wT \rfloor + 1, \tilde{T}_1 + \lfloor wT \rfloor + 1]$, in which no further structural breaks are searched. Then, in the second stage the sequential testing procedure looks for an additional break in the range of observations defined by $t = 1, 2, 3, \ldots, \tilde{T}_1 - \lfloor wT \rfloor, \tilde{T}_1 + \lfloor wT \rfloor + 2, \ldots, T$ – i.e., the eligible break dates that are inside the set $t \in \Lambda_T - \Lambda_{1,T}$. Evidence against the null hypothesis of an additional structural break is found when $\max_{t \in \Lambda_T - \Lambda_{1,T}} S_{1,t,\lfloor wT \rfloor} > cv_\xi^1$ and the second estimated break date is obtained as $\tilde{T}_2 = \arg\max_{t \in \Lambda_T - \Lambda_{1,T}} S_{1,t,\lfloor wT \rfloor}$. The procedure continues until we find that $\max_{t \in \Lambda_T - \Lambda_{1,T} - \Lambda_{2,T} - \cdots - \Lambda_{m,T}} S_{1,t,\lfloor wT \rfloor} \leq cv_\xi^1$, in which case the null hypothesis of no (additional) structural break is not rejected. The estimated number of structural breaks from the use of the S_1 statistic is denoted by $\tilde{m}_1$.

The same approach can be applied using the S_0 statistic and obtain $\tilde{m}_0$ structural breaks. In general, $\tilde{m}_1$ has not to be equal to $\tilde{m}_0$. If $\tilde{m}_1 \geq \tilde{m}_0$, the $\tilde{m}_0$ breaks are simply a subset of the $\tilde{m}_1$ breaks. Similarly, if $\tilde{m}_0 \geq \tilde{m}_1$, the $\tilde{m}_1$ breaks are simply a subset of the $\tilde{m}_0$ breaks. If $\tilde{m}_1 = \tilde{m}_0$, both sets of break locations are identical. Note that the number of breaks that is estimated with the S_U statistic will simply be $\tilde{m}_U = \max(\tilde{m}_1, \tilde{m}_0)$. Some remarks are in order. First, the estimation of both the number and location of the structural breaks is based on the so-called *one-at-a-time* (OAAT) strategy rather than on the simultaneous estimation of multiple structural breaks. Second, this procedure relies on the argument that maximizes the sequence of $S_{d,t,\lfloor wT \rfloor}$ statistics. In what follows, we refer to this approach as the HLT detection strategy.

3.2 The Bai and Perron statistics

3.2.1 A test of no break versus some fixed number of breaks

This section proposes the use of a sup Wald-type statistic to test the null hypothesis of no structural break against the alternative hypothesis that there are m structural breaks. The specification of different values of $m \in \{1, 2, \ldots, m_{\max}\}$, with $m_{\max}$ a given maximum number of structural breaks, under the alternative hypothesis allows us to gain some insights about whether there is some evidence of structural breaks. Following Bai and Perron (1998), we consider sup Wald-type statistics of the form:

$$F_d(m|0) = m^{-1} T^{-2d} \hat{\sigma}_d^{-2} \max_{T_{B,m} \in T\Lambda(\epsilon)^m} \left[SSR(T) - SSR(T_{B,m}) \right]; \quad d \in \{0, 1\}, \qquad (6)$$

where $\hat{\sigma}_d^2$ denotes a consistent estimator of the LRV of u_t (when $d = 0$) or ε_t (when $d = 1$) to be discussed below, $SSR(T) = \sum_{t=1}^{T} (y_t - \bar{y})^2$ is the sum of squared residuals (SSR) under the null hypothesis of no structural break and $SSR(T_{B,m}) = \sum_{i=1}^{m+1} S(T_{i-1}, T_i)$, $S(T_{i-1}, T_i) = \sum_{t=T_{i-1}+1}^{T_i} (y_t - \bar{y}_{(T_{i-1}, T_i)})^2$, $\bar{y}_{(T_{i-1}, T_i)} = (T_i - T_{i-1})^{-1} \sum_{t=T_{i-1}+1}^{T_i} y_t$, denotes the SSR under the alternative hypothesis that is computed for all possible combinations of $T_{B,m} = (T_1, T_2, \ldots, T_m)$, $m \in \{1, 2, \ldots, m_{\max}\}$, break locations using the dynamic optimization algorithm in Bai and Perron (1998), with the convention that $T_0 = 0$ and $T_{m+1} = T$. The $F_0(m|0)$ statistic was proposed in Bai and Perron (1998) for the I(0) case. In this paper, we extend this sup Wald-type statistic to the I(1) case – denoted by $F_1(m|0)$. Bai and Perron (1998) also propose double maximum statistics that consider the possibility that the number of structural breaks is unknown up to some upper bound $m_{\max}$. The first double maximum statistic is the maximum of the equally weighted sequence of statistics that can be computed for all values of $m \in \{1, 2, \ldots, m_{\max}\}$:

$$UDmax_d = \max_{1 \leq m \leq m_{\max}} F_d(m|0), \tag{7}$$

and the second one is the maximum of the weighted sequence of statistics:

$$WDmax_d = \max_{1 \leq m \leq m_{\max}} a_m F_d(m|0), \tag{8}$$

where the weights are used to warrant that the marginal p-values are equal across values of m. The weights are defined as $a_1 = 1$ and, for $m > 1$, as $a_m = cv(\xi, 1)/cv(\xi, m)$ where $cv(\xi, m)$ is the asymptotic critical value of $F_d(m|0)$ for a significance level ξ. Bai and Perron (1998) derive the limiting distribution of $F_0(m|0)$, $UDmax_0$ and $WDmax_0$ statistics, whereas the corresponding limiting distribution of $F_1(m|0)$, $UDmax_1$ and $WDmax_1$ is given in the following theorem.

Theorem 1 *Let $\{y_t\}_{t=1}^T$ be a stochastic process with the DGP given by (1) and (2) with $\rho = 1 - c/T$, $0 \leq c < \infty$. Under the null hypothesis that there are no structural breaks, the $F_1(m|0)$, $UDmax_1$ and $WDmax_1$ statistics given in (6) to (8) converge as $T \to \infty$*

to:

$$
\begin{aligned}
F_1(m|0) &\Rightarrow \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^m} m^{-1}\left[\sum_{i=1}^{m+1}(\lambda_i-\lambda_{i-1})\left(\int_{\lambda_{i-1}}^{\lambda_i} W_c(s)\,ds\right)^2-\left(\int_0^1 W_c(s)\,ds\right)^2\right] \\
&\equiv \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^m} K_c(\lambda_{B,m}) \\
UDmax_1 &\Rightarrow \max_{1\leq m\leq m_{\max}} \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^m} K_c(\lambda_{B,m}) \\
WDmax_1 &\Rightarrow \max_{1\leq m\leq m_{\max}} \frac{cv(\xi,1)}{cv(\xi,m)} \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^m} K_c(\lambda_{B,m}),
\end{aligned}
$$

where $\Rightarrow$ denotes weak convergence to the associated measure of probability and $W_c(s)$ is a standard Ornstein-Uhlenbeck (OU) process.

The proof is given in Appendix A. Table B.1 collects asymptotic critical values for $F_1(m|0)$, $UDmax_1$ and $WDmax_1$ statistics under I(1) errors with $c = 0$ for different values of the trimming parameter – for completeness, it also reports the critical values of $F_0(m|0)$, $UDmax_0$ and $WDmax_0$. The critical values have been obtained using Monte Carlo simulations with 300 steps to approximate the Brownian motions of the limiting distribution with $c = 0$ and 10,000 replications. In addition and following Harvey, Leybourne and Taylor (2010), each pair of $F_d(m|0)$, $UDmax_d$ and $WDmax_d$, $d \in \{0,1\}$, statistics can be combined to design union statistics as defined in (18). This leads us to propose three additional statistics, namely, $F_U(m|0)$, $UDmax_U$ and $WDmax_U$. Table B.1 presents the κ_ξ^j constants, $j \in \{F(m|0), UDmax, WDmax\}$, that are required for the implementation of the statistical rejection rule for these union statistics.

Bai and Perron (1998) suggest the application of the double max statistics in the first place and, in the case in which some evidence of structural breaks is found, compute the sequential statistics described in the next section to estimate the number of structural breaks that are affecting the time series.

3.2.2 The sequential test statistics

The second set of statistics designed in this paper builds upon the sequential approach advocated by Bai and Perron (1998) that allows testing the null hypothesis of m structural breaks against the alternative hypothesis of $m + 1$ structural breaks for I(0) time series.

The procedure starts testing the null hypothesis of no structural break ($m = 0$) against the alternative hypothesis of one structural break ($m = 1$). If the null hypothesis is rejected, we can proceed in a second stage to test the null hypothesis of $m = 1$ against the alternative hypothesis of $m = 2$, and so on. The testing process ends when the corresponding null hypothesis is not rejected. The sequential statistic in Bai and Perron (1998) is based on the computation of the SSR under both the null and alternative hypotheses. Let us consider the SSR computed using the vector of break points $T^0_{B,m} = (T^0_1, T^0_2, \ldots, T^0_m)$:

$$SSR\left(T^0_{B,m}\right) = \sum_{i=1}^{m+1} S(T^0_{i-1}, T^0_i), \tag{9}$$

where $S(T^0_{i-1}, T^0_i) = \sum_{t=T^0_{i-1}+1}^{T^0_i} (y_t - \bar{y}_{(T^0_{i-1},T^0_i)})^2$, $\bar{y}_{(T^0_{i-1},T^0_i)} = (T^0_i - T^0_{i-1})^{-1} \sum_{t=T^0_{i-1}+1}^{T^0_i} y_t$, with the convention that $T^0_{B,0} = T$ for the model with no structural break ($m = 0$). Similarly, we can compute the SSR for which an additional break is considered inside the i-th segment:

$$SSR\left(T^0_{B,m}, \tau\right) = \sum_{j=1}^{i-1} S(T^0_{j-1}, T^0_j) + S(T^0_{i-1}, \tau) + S(\tau, T^0_i) + \sum_{j=i+1}^{m+1} S(T^0_{j-1}, T^0_j). \tag{10}$$

Following Bai and Perron (1998), the sup Wald statistic to test the null hypothesis of m structural breaks against the alternative hypothesis of $m + 1$ structural breaks is:

$$F_d\left(m+1|m\right) = \max_{1\leq i\leq m+1} \max_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[T^{-2d}\hat{\sigma}_d^{-2}\left[S(T^0_{i-1}, T^0_i) - (S(T^0_{i-1}, \tau) + S(\tau, T^0_i))\right]\right], \tag{11}$$

with $d \in \{0, 1\}$, $\Lambda_i(\epsilon) = \{(\lambda_{i-1}, \lambda_\tau, \lambda_i); |\lambda_\tau - \lambda_j| \geq \epsilon,\ j \in \{i-1, i\}\}$, $\lambda_\tau = \tau/T$, and where $\hat{\sigma}^2_d$ denotes a consistent estimator of the LRV of u_t (when $d = 0$) or ε_t (when $d = 1$) to be discussed below. The original Bai and Perron (1998) sequential statistic is obtained setting $d = 0$ in (11), whereas in this paper we extend its use to I(1) stochastic processes. The limiting distribution of $F_1\left(m+1|m\right)$ given in (11) when $\rho = 1 - c/T$, $0 \leq c < \infty$, in (2) is provided in the following theorem.

Theorem 2 *Let* $\{y_t\}_{t=1}^T$ *be a stochastic process with the DGP given by (1) and (2) with* $\rho = 1 - c/T$, $0 \leq c < \infty$. *Under the null hypothesis that there are* m *structural breaks* *with* $T^0_{B,m}/T \to \lambda^0_{B,m}$ *as* $T \to \infty$*, the* $F_1(m+1|m)$ *statistic given in (11) converges as* $T \to \infty$ *to:*

$$\begin{aligned} F_1(m+1|m) &\Rightarrow \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)}(\lambda_i^0-\lambda_{i-1}^0)^2\left[\int_0^1 W_c^*(a)^2\,da \right. \\ &\quad \left. -\int_0^l W_{c,1}^*(a)^2\,da - \int_l^1 W_{c,2}^*(a)^2\,da\right] \qquad (12) \\ &= \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)}\left[\Delta\lambda_i^0\frac{\left(\int_0^l W_c(a)\,da - l\int_0^1 W_c(a)\,da\right)^2}{l(1-l)}\right] \qquad (13) \\ &\equiv \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)} H_c\left(\lambda_{i-1}^0,\lambda_\tau,\lambda_i^0\right), \end{aligned}$$

where $W_c^*(a)$*,* $W_{c,1}^*(a)$ *and* $W_{c,2}^*(a)$ *are three demeaned OU processes, with* $l=(\lambda_\tau-\lambda_{i-1}^0)/(\lambda_i^0-\lambda_{i-1}^0)$*, and* $W_c(a)$ *is a standard OU process.*

The proof is given in Appendix A. As can be seen, the limiting distribution depends both on the number (m) and position $(\lambda^0_{B,m})$ of the structural breaks that are specified under the null hypothesis. For subsequent derivations, Theorem 2 provides two equivalent representations of the limiting distribution of $F_1(m+1|m)$. Table B.2 reports asymptotic critical values for $F_1(m+1|m)$ under I(1) errors $(c=0)$ for selected combinations of $\lambda^0_{B,m}$, $m\in\{0,1,2\}$ and $\epsilon=0.15$. These critical values are computed using Monte Carlo simulations with 1,000 steps to approximate the Brownian motions of the limiting distributions with $c=0$ and 10,000 replications.

The use of brute force algorithms to obtain critical values for all possible combinations of m structural break locations generates a computational burden of order T^m, which makes the implementation of the proposal almost unfeasible when $m>2$. To address this issue, we follow Bai and Perron (1998) and develop a dynamic optimization algorithm that relies on the minimization of the SSR over all possible segments defined between $t=1$ and $t=T$, which reduces the computation cost to a problem of order T^2. It is worth mentioning that the dynamic optimization algorithm that is implemented in this paper for the computation of $F_1(m+1|m)$ is different and faster than the one in Bai and Perron (1998), but serves the same purpose of reducing the number of calculations. Reporting a complete set of tables with critical values for all possible combinations of m structural break locations makes no sense, especially for $m > 2$. In this regard, a Matlab code is available from the authors upon request to compute the critical values with $c = 0$ for whatever combination of m and ϵ values that is desired. In addition, Table B.3 provides estimated response surfaces to approximate asymptotic critical values for the I(1) case ($c = 0$) for a given combination of m structural breaks with $\epsilon \in \{0.15, 0.2\}$.

Bai and Perron (1998) derive the limiting distribution of $F_0(m+1|m)$ assuming that $y_t \sim I(0)$, a limiting distribution that involves the maximum of $m+1$ independent variables. The presence of λ_{i-1}^0 and λ_i^0 in (12) and (13) prevents reaching a similar neat result in the sense that, implicitly, the limiting distribution of $F_1(m+1|m)$ depends on the length of the $m+1$ regimes. Besides, it complicates the computation of critical values, although the use of the estimated response surfaces can solve this issue for the trimming parameters that have been considered. Fortunately, we can think of designing modified statistics that get rid of the regime length in the limit. The first modified statistic is computed using the SSR in (9) and (10), but with the SSR of each segment weighted (WSSR) by the inverse of the square of the regime length:

$$WSSR\left(T_{B,m}^0\right) = \sum_{i=1}^{m+1} \left(T_i^0 - T_{i-1}^0\right)^{-2} S(T_{i-1}^0, T_i^0), \tag{14}$$

and

$$\begin{aligned} WSSR\left(T_{B,m}^0, \tau\right) &= \sum_{j=1}^{i-1} \left(T_j^0 - T_{j-1}^0\right)^{-2} S(T_{j-1}^0, T_j^0) + \left(T_i^0 - T_{i-1}^0\right)^{-2} \left(S(T_{i-1}^0, \tau) + S(\tau, T_i^0)\right) \\ &\quad + \sum_{j=i+1}^{m+1} \left(T_j^0 - T_{j-1}^0\right)^{-2} S(T_{j-1}^0, T_j^0), \end{aligned} \tag{15}$$

so that the first version of the modified $F_1(m+1|m)$ statistic that we propose is:

$$\begin{aligned} F_1^a(m+1|m) &= \hat{\sigma}_1^{-2} \left[WSSR\left(T_{B,m}^0\right) - \min_{1 \le i \le m+1} \min_{\tau/T \in \Lambda_i(\epsilon)} WSSR\left(T_{B,m}^0, \tau\right) \right] \\ &= \max_{1 \le i \le m+1} \max_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[\left(T_i^0 - T_{i-1}^0\right)^{-2} \hat{\sigma}_1^{-2} \left[S(T_{i-1}^0, T_i^0) - \left(S(T_{i-1}^0, \tau) + S(\tau, T_i^0)\right) \right] \right]. \end{aligned} \tag{16}$$

The use of the $(T_i^0 - T_{i-1}^0)^{-2}$ rescaling term in the WSSR is suggested by the presence of $(\lambda_i^0 - \lambda_{i-1}^0)^2$ in the limit distribution given by (12). This transformation is inspired by Busetti and Harvey (2001) and Harvey (2005), who propose unit root and stationarity statistics allowing for structural breaks. The second modified statistic is designed from the expression of the limit distribution of $F_1(m+1|m)$ given in (13), which suggests that the WSSR of each regime should be rescaled by $T^{-1}(T_i^0 - T_{i-1}^0)^{-1}$ and defines:

$$\begin{aligned} F_1^b(m+1|m) &= \max_{1\le i\le m+1} \max_{\lambda_\tau \in \Lambda_i(\epsilon)} \Big[T^{-1}\left(T_i^0 - T_{i-1}^0\right)^{-1} \hat{\sigma}_1^{-2}\left[S(T_{i-1}^0, T_i^0)\right. \\ &\quad \left.\left. -(S(T_{i-1}^0,\tau) + S(\tau, T_i^0))\right]\right]. \end{aligned} \tag{17}$$

The limit distribution of the modified statistics is given in the following theorem.

Theorem 3 *Let $\{y_t\}_{t=1}^T$ be a stochastic process with the DGP given by (1) and (2) with $\rho = 1 - c/T$, $0 \le c < \infty$. Under the null hypothesis that there are m structural breaks with $T_{B,m}^0/T \to \lambda_{B,m}^0$ as $T \to \infty$, the $F_1^a(m+1|m)$ and $F_1^b(m+1|m)$ statistics given in (16) and (17), respectively, converge as $T \to \infty$ to:*

$$\begin{aligned} F_1^a(m+1|m) &\Rightarrow \sup_{1\le i\le m+1} \sup_{\lambda_\tau\in\Lambda(\epsilon)} \left[\int_0^1 W_c^*(a)^2\,da - \int_0^l W_{c,1}^*(a)^2\,da - \int_l^1 W_{c,2}^*(a)^2\,da\right] \\ &\equiv \sup_{1\le i\le m+1} J_c^a(x) \\ F_1^b(m+1|m) &\Rightarrow \sup_{1\le i\le m+1} \sup_{\lambda_\tau\in\Lambda(\epsilon)} \frac{\left(\int_0^l W_c(a)\,da - l\int_0^1 W_c(a)\,da\right)^2}{l(1-l)} \equiv \sup_{1\le i\le m+1} J_c^b(x), \end{aligned}$$

where $W_c^*(a)$, $W_{c,1}^*(a)$ and $W_{c,2}^*(a)$ are three demeaned OU processes, with $l = (\lambda_\tau - \lambda_{i-1}^0)/(\lambda_i^0 - \lambda_{i-1}^0)$, and $W_c(a)$ is a standard OU process.

The proof is given in Appendix A. An important feature shown by the limiting distribution of $F_1^j(m+1|m)$, $j \in \{a,b\}$, is that it involves the maximum of $m+1$ independent variables, which simplifies the computation of critical values. From a pure minimalism point of view, this feature also equalizes both types of Bai-Perron sequential statistics in the limit, since $F_0(m+1|m)$ and $F_1^j(m+1|m)$, $j \in \{a,b\}$, share this common feature of having an asymptotic distribution that involves $m+1$ independent variables.

Tables B.4 and B.5 report approximate asymptotic critical values for $F_0(m+1|m)$ and $F_1^j(m+1|m)$, $j \in \{a,b\}$, under I(1) errors with $c=0$ and considering up to $m=9$ structural breaks for different values of ϵ. As above, the critical values are computed using Monte Carlo simulations with 1,000 steps to approximate the Brownian motions of the limiting distribution and 10,000 replications.[1]

The joint use of $F_0(m+1|m)$ and $F_1^j(m+1|m)$, $j \in \{a,b\}$, statistics lead us to define union statistics similar to the one in Harvey, Leybourne and Taylor (2010), with an expression that mimics (5):

$$F_U^j(m+1|m) = \max\left\{F_1^j(m+1|m), \left(\frac{cv_\xi^{F_1^j(m+1|m)}}{cv_\xi^{F_0(m+1|m)}}\right) F_0(m+1|m)\right\}; \quad j \in \{a,b\}, \tag{18}$$

where $cv_\xi^{F_1^j(m+1|m)}$ and $cv_\xi^{F_0(m+1|m)}$ denote the ξ significance level asymptotic critical values of $F_1^j(m+1|m)$, $j \in \{a,b\}$, and $F_0(m+1|m)$, respectively, that are collected in Tables B.4 and B.5. The statistical rejection rule decision for the $F_U^j(m+1|m)$, $j \in \{a,b\}$, statistics is given by:

$$\text{Reject } H_0 \text{ if } F_U^j(m+1|m) > \kappa_\xi^{F^j(m+1|m)} cv_\xi^{F_1^j(m+1|m)}, \tag{19}$$

where the positive constant $\kappa_\xi^{F^j(m+1|m)}$ is computed as in Harvey, Leybourne and Taylor (2010). Specifically, $\kappa_\xi^{F^j(m+1|m)}$ is obtained by simulating the limit distribution of $\max\{F_1^j(m+1|m), (cv_\xi^{F_1^j(m+1|m)}/cv_\xi^{F_0(m+1|m)})F_0(m+1|m)\}$, computing the critical value of this distribution at a ξ-level of significance – denoted by $cv_\xi^{F_U^j(m+1|m)}$ – and, finally, calculating $\kappa_\xi^{F^j(m+1|m)} = cv_\xi^{F_U^j(m+1|m)}/cv_\xi^{F_1^j(m+1|m)}$, $j \in \{a,b\}$. Despite the constant used by HLT statistics, here $\kappa_\xi^{F^j(m+1|m)}$, $j \in \{a,b\}$, depend on the number of structural breaks that are considered under the null hypothesis of each step of the sequential testing strategy. Tables B.4 and B.5 provide the values of $\kappa_\xi^{F^j(m+1|m)}$, $j \in \{a,b\}$.

[1] The critical values for the $F_0(m+1|m)$ statistic are reported for completeness, although they are comparable to the ones in Bai and Perron (1998, 2003b).

Unknown structural breaks So far, the computation of the sequential statistics has assumed known structural breaks under the null hypothesis of m structural breaks – i.e., $T^0_{B,m}$ is known a priori. This situation is rarely found in practice so that it is desirable to design a procedure to estimate the break dates. Bai and Perron (1998) show that, under the assumption that $y_t \sim I(0)$, it is possible to obtain consistent estimates of the break fractions through the minimization of the SSR of the model given by (1). In the limit, the use of the estimated break dates that are obtained from the minimization of the SSR of the model given by (1) under the null hypothesis of m structural breaks is as good as knowing $T^0_{B,m}$ – see proof of Proposition 7 in Bai and Perron (1998). For the $y_t \sim I(1)$ case, Harvey, Leybourne and Taylor (2010) suggest the specification of the model given by (1) in first differences:

$$\Delta y_t = \sum_{i=1}^{m} \gamma_i D(T_i)_t + v_t \qquad t = 2, \ldots, T, \tag{20}$$

with $D(T_i)_t = 1$ for $t = T_i + 1$ and 0 otherwise, $i = 1, \ldots, m$, and estimate the break dates through the minimization of the SSR of the model in (20), SSR_{DIF}, so that:

$$\hat{T}_{B,m} = \underset{T_{B,m} \in T\Lambda(\epsilon)^m}{\arg\min}\ SSR_{DIF}(T_{B,m}), \tag{21}$$

which provides consistent estimates of the break fraction parameters iff $\gamma_i = \gamma_i^* T^{1/2}$ – see Harvey, Leybourne and Taylor (2010). Note that the use of $\hat{T}_{B,m}$ also provides consistent estimates of the break fraction parameters when $y_t \sim I(0)$. Since the approach that is adopted in this paper aims at proposing structural break test statistics that are robust to the order of integration of the time series, we suggest to use $\hat{T}_{B,m}$ on each step of the sequential testing procedure. The estimated break dates can be treated as if they were the true ones and proceed to compute the different statistics defined above, which limiting distribution is given in the previous theorems. This is the approach that is used in this paper to compute the sequential $F_1(m+1|m)$ and $F_1^j(m+1|m)$, $j \in \{a, b\}$, statistics. Consequently, the break dates that are specified under the null hypothesis when $m > 0$ are obtained from the joint minimization of $SSR_{DIF}(T_{B,m})$ over all possible combinations of

m break dates. It is worth noting that the implementation of this procedure has required the design of a new dynamic programming algorithm to jointly estimate the break dates associated to the m impulse dummies in (20) for each stage of the sequential testing strategy. Finally, one might also think of implementing the sequential BP statistics using the HLT-detection strategy discussed above, where the structural breaks are selected OAAT on the basis of the maximization of the sequence of Wald statistics. This approach is also essayed below.

4 Estimation of the long-run variance

The computation of all statistics discussed above requires a consistent estimation of the long-run variance, an issue that turns out to be of special relevance. The implementation of the Bai and Perron (1998) statistics could be carried out using the long-run variance that is estimated under the null hypothesis of the corresponding statistic, although this might imply non-monotonic empirical power problems – see Perron (2006). The estimation of the long-run variance could be done under the alternative hypothesis, although this might cause size distortions in case of persistent errors. As can be seen, these considerations create a trade-off between the empirical size and power that can be important in empirical applications.

Harvey, Leybourne and Taylor (2010) recommend a parametric estimation method of the long-run variance that relies on the use of the maximum number of potential structural breaks that admits the specified trimming parameter – i.e., $m_{\max} = 1 + \lfloor ((1-\epsilon) - \epsilon)/\epsilon \rfloor$. Consequently, the long-run variance is estimated under the alternative of $m_{\max}$ structural breaks for both the S_1 and S_0 statistics. Allowing for the maximum number of structural breaks when computing the long-run variance avoids obtaining a biased estimation due to unaccounted structural breaks, a situation that might be found if we were using long-run variance estimates for each stage of the sequential testing. This is relevant in the case where there are more structural breaks than the ones considered under the null hypothesis of each step of the (sequential) testing procedure. However, this might come at the price of losing power, if more structural breaks than exist are specified in the estimation of (20). Conditional on $m_{\max}$, the break locations are estimated assuming that $y_t \sim I(1)$, which implies computing the argument that minimizes the SSR of the model in (20) following the OAAT strategy in Harvey, Leybourne and Taylor (2010).[2] This determines the set of estimates $\hat{T}_{B,m_{\max}} = (\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_{m_{\max}})$ that, for coherence, is maintained regardless of d.[3] Then, the long-run variance is estimated following a two-step procedure with some particularities that depend on d. For the S_1 statistic, it first requires the Ordinary Least Squares (OLS) estimation of the model:

$$\Delta y_t = \sum_{i=1}^{m_{\max}} \gamma_i D(\hat{T}_i)_t + v_t, \tag{22}$$

$t = 2, \ldots, T$ and, second, use the estimated residuals of (22) to estimate the augmented Dickey-Fuller (ADF) type regression equation:

$$\Delta \hat{v}_t = \pi \hat{v}_{t-1} + \sum_{j=1}^{k-1} \psi_j \Delta \hat{v}_{t-j} + e_t, \tag{23}$$

$t = k+2, \ldots, T$, with $\hat{\sigma}_e^2 = (T - 2k - 1)^{-1} \sum_{t=k+2}^{T} \hat{e}_t^2$ and k selected so that, as $T \to \infty$, $1/k + k^3/T \to 0$ – Harvey, Leybourne and Taylor (2010) suggest using the Bayesian information criterion (BIC) to choose k, although other criteria such as the modified information criteria in Ng and Perron (2001) and Perron and Qu (2007) might be applied. The long-run variance is obtained as $\hat{\sigma}_1^2 = \hat{\sigma}_e^2/\hat{\pi}^2$, where the subscript indicates that it has been assumed that $y_t \sim I(1)$. A similar strategy is implemented to estimate the long-run variance for the S_0 statistic. Conditional on $\hat{T}_{B,m_{\max}}$, the following regression

[2] A shortcoming of the OAAT strategy is that the location of an additional break date depends on the previously allocated break dates, so that it might be the case that the maximum number of breaks cannot be reached if there is not enough space left for the inclusion of an additional structural break. This limitation is not found when the breaks location is carried out using a joint estimation procedure, as we suggest here for the implementation of the BP statistics.

[3] Note that proceeding in this way the implementation of the HLT statistics implies working with, potentially, two different sets of estimated break dates: (i) the $\tilde{m}_1$ (or $\tilde{m}_0$) break dates $\tilde{T}_{B,\tilde{m}_1} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_{\tilde{m}_1})$ (or $\tilde{T}_{B,\tilde{m}_0} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_{\tilde{m}_0})$) that are obtained from the maximization of the sequence of $S_{1,t,\lfloor wT \rfloor}$ (or $S_{0,t,\lfloor wT \rfloor}$) statistics and (ii) $\hat{T}_{B,m_{\max}} = (\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_{m_{\max}})$, the break dates that are obtained using the OAAT strategy on the minimization of the SSR of (20).

model is estimated by OLS:

$$y_t = \mu + \sum_{i=1}^{m_{\max}} \gamma_i DU_{i,t} + u_t \qquad t = 1, \ldots, T, \tag{24}$$

and the estimated residuals of (24) are in turn used to estimate a Perron's ADF-type regression equation:

$$\Delta \hat{u}_t = \pi \hat{u}_{t-1} + \sum_{j=1}^{k-1} \psi_j \Delta \hat{u}_{t-j} + \sum_{i=1}^{m_{\max}} \sum_{j=0}^{k-1} \psi_{i,j} D(\hat{T}_i)_{t-j} + e_t \qquad t = k+1, \ldots, T. \tag{25}$$

The parametric long-run variance estimator is then given by $\hat{\sigma}_0^2 = \hat{\sigma}_e^2 / \hat{\pi}^2$ with $\hat{\sigma}_e^2 = (T - (2 + m_{\max})k)^{-1} \sum_{t=k+1}^{T} \hat{e}_t^2$. The computation of the long-run variance for the BP test statistics that is suggested in this paper combines some features described so far. The discussion that follows distinguishes between the $y_t \sim I(0)$ case covered in Bai and Perron (1998) and the extension to the $y_t \sim I(1)$ case that is proposed in this paper.

4.1 The long-run variance for the BP statistics when $y_t \sim I(0)$

To reach a compromise between the size and power trade-off discussed above, we proceed following the spirit of the hybrid non-parametric estimation method of the long-run variance proposed in Kejriwal and Perron (2010), which involves using the OLS estimated residuals under both the null and alternative hypotheses. The hybrid method implies selecting the bandwidth (h) of the (quadratic) spectral window required by the non-parametric estimator of the long-run variance in Andrews (1991), but using the estimated residuals under the alternative hypothesis instead – henceforth, we denote the estimated bandwidth as $\hat{h}_a$. Then, the long-run variance is computed using the estimated residuals under the null hypothesis with $\hat{h}_a$ as the bandwidth parameter.

However, the approach that is essayed in this paper relies on the so-called *max-hybrid method*. The max-hybrid method modifies the first stage of the hybrid method in Kejriwal and Perron (2010) since the bandwidth parameter is selected applying Andrews (1991) AR(1)-based automatic procedure using the estimated residuals of the model given by

(24) – note that $\hat{T}_{B,m_{\max}}$ is obtained assuming that $y_t \sim I(1)$, as in Harvey, Leybourne and Taylor (2010), and using either the OAAT or the joint break dates estimation strategies. This defines the bandwidth parameter $\hat{h}_a^{\max}$. Then, the long-run variance is computed as $\hat{\sigma}_0^2 = T^{-1}\sum_{t=1}^{T}\tilde{u}_t^2 + 2T^{-1}\sum_{j=1}^{T-1} w(j/\hat{h}_a^{\max})\sum_{t=j+1}^{T}\tilde{u}_t\tilde{u}_{t-j}$, where $\tilde{u}_t$ are the estimated residuals under the null hypothesis and $w(\cdot)$ is the quadratic spectral kernel function. Provided that $\hat{\sigma}_0^2 \xrightarrow{p} \sigma_0^2$, where $\xrightarrow{p}$ denotes convergence in probability, the $F_0(m|0)$ and $F_0(m+1|m)$ statistics converge to the limiting distribution given in Propositions 6 and 7 of Bai and Perron (1998), respectively. It is also of interest to analyze the limiting distribution of the $F_0(m+1|m)$ statistic when $y_t \sim I(1)$, which is provided in the following theorem.

Theorem 4 *Let $\{y_t\}_{t=1}^{T}$ be a stochastic process with the DGP given by (1) and (2) with $\rho = 1 - c/T$, $0 \leq c < \infty$. Under the null hypothesis that there are m structural breaks with $T_{B,m}^0/T \to \lambda_{B,m}^0$ as $T \to \infty$, the $F_0(m+1|m)$ statistic given in (11) converges as $T \to \infty$ to:*

$$F_0(m+1|m) \Rightarrow \sup_{1\leq i\leq m+1}\ \sup_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[\kappa \int_0^1 W_c^*(r, \lambda_{B,m}^0)^2 dr\right]^{-1} H_c\left(\lambda_{i-1}^0, \lambda_\tau, \lambda_i^0\right).$$

The proof is given in Appendix A. An interesting feature that derives from the limiting distribution given in Theorem 4 is that it is independent of nuisance parameters and, hence, approximate critical values can be computed. Further, if we compare the limiting distributions given in Theorems 2 and 4, we will realize that they only differ on $\kappa \int_0^1 W_c^*(r, \lambda_B^0)^2 dr$, a strictly positive term.

4.2 The long-run variance for the BP statistics when $y_t \sim I(1)$

The procedure that is essayed in this paper basically mimics the one described above for the S_1 statistic, but instead of relying on an autoregressive parametric estimator of the long-run variance, we restore on a non-parametric one. To be specific, the estimated residuals from (22) – i.e., $\hat{v}_t$, which correspond to the alternative hypothesis of $\hat{T}_{B,m_{\max}}$ structural breaks – are used to estimate the long-run variance as $\hat{\sigma}_1^2 = (T-1)^{-1}\sum_{t=2}^{T}\hat{v}_t^2 +$

$2\,(T-1)^{-1}\sum_{j=2}^{T-1} w(j/\hat{h}_a^{\max})\sum_{t=j+1}^{T}\hat{v}_t\hat{v}_{t-j}$, where the bandwidth of the (quadratic) spectral window is selected with Andrews (1991) automatic procedure using $\hat{v}_t$ as well. Provided that $\hat{\sigma}_1^2 \xrightarrow{p} \sigma_1^2$, the $F_1\,(m|\,0)$, $F_1\,(m+1|m)$ and $F_1^j\,(m+1|m)$, $j \in \{a,b\}$, statistics converge to the limiting distribution given in Theorems 1 to 3, respectively.

5 Monte Carlo simulations

The finite sample performance of the statistics is analyzed using the DGP:

$$y_t = \sum_{i=1}^{m} \gamma_i DU_{i,t} + u_t; \quad u_t = \rho u_{t-1} + \varepsilon_t, \tag{26}$$

with $u_0 = 0$, $\varepsilon_t \sim iid\ N\,(0,1)$, and $\rho = 1 - c/T$, $0 \leq c \leq 30$, for the I(1) and NI(1) cases, and $\rho \in \{0, 0.5, 0.8\}$ for the I(0) case. The simulation experiment has dealt with up to two structural breaks located at $\lambda_{B,1}^0 = 0.5$, for $m = 1$, and $\lambda_{B,2}^0 = (0.2, 0.8)$, for $m = 2$, with the sample size given by $T \in \{100, 300, 1000\}$ and considering up to $m_{\max} = 5$ structural breaks. Following Harvey, Leybourne and Taylor (2010), the magnitude of the level shifts is held constant across the I(1), NI(1) and I(0) cases that are used in this section, rather than scaling it according to the order of integration, so as to provide some coherence across different values of ρ.[4] This has implied the definition of $\gamma_i = \gamma^* \ \forall i$, with $\gamma^* \in \{1, 5\}$. The nominal size is set at the 5% level of significance and 1,000 replications are conducted using Matlab.[5] Throughout this section, the value of k in (23) and (25) is selected using the BIC with a maximum of $k_{\max} = \lfloor 4(T/100)^{1/4} \rfloor$ lags.

5.1 No break versus some fixed number of breaks test statistics

5.1.1 Empirical size

Tables B.6 and B.8 present the empirical size – i.e., $\gamma_i = 0 \ \forall i$ in (26) – of the statistics for the I(1) and NI(1) cases considering three selected values of $c \in \{0, 15, 30\}$ and where

[4] Simulation results for the increasing and shrinking break magnitudes are available upon request.

[5] The simulation experiment uses the asymptotic critical values in Bai and Perron (1998) for the BP statistics when $d = 0$, and the ones computed in this paper when $d = 1$. The computation of the HLT statistics is carried out setting $w = 0.10$ and using the asymptotic critical values in Harvey et al. (2010).

the LRV is estimated using the OAAT and the joint break dates estimation procedures, respectively, with $m_{\max} = 5$. In general, results are similar for both LRV estimation procedures. As can be seen, the empirical size of $F_1(m|0)$ is close to the nominal one when the assumed order of integration of the time series matches the true d, whereas the statistic becomes conservative otherwise. For $m = 1$, the empirical size of $F_0(m|0)$ tends to the nominal one as $c > 0$ increases and, interestingly, it does not exceed the 5% when the wrong d is assumed (i.e., when $c = 0$). For $m > 1$, $F_0(m|0)$ becomes conservative. These features define the performance of the union statistic $F_U(m|0)$, which shows rejection rates that are close to the nominal size when $c = 0$ and becomes conservative when $c > 0$. The double maximum statistics perform similarly. The $UDmax_d$ and $WDmax_d$, $d \in \{0, 1\}$, statistics have the right empirical size when the true d is assumed, and they become conservative otherwise. This behavior translates to the double maximum union statistics. $UDmax_U$ and $WDmax_U$ under-reject the null hypothesis when $c = 0$ and become very conservative when $c > 0$.

Tables B.7 and B.9 contain the empirical size results for the I(0) case with the LRV estimated using the OAAT and joint break dates estimation methods, respectively. As above, similar conclusions are obtained regardless of the LRV estimate that is used. As predicted by the theory, $F_1(m|0)$ never rejects the null hypothesis, whereas the empirical size of $F_0(m|0)$ either attains the nominal one ($m = 1$) or takes a lower value ($m > 1$ with $T = 100$ and $T = 300$), regardless of ρ. These features imply that the empirical size of $F_U(m|0)$ is smaller than 5% in all cases. The $UDmax_1$ and $WDmax_1$ statistics never reject their null hypotheses, and $UDmax_0$ and $WDmax_0$ have an empirical size close to the nominal one, regardless of ρ. Finally, $UDmax_U$ presents mild under-size distortions, whereas $WDmax_U$ is very conservative. All these elements lead us to recommend the use of the proposed union statistics in empirical analyses to conduct robust statistical inference about the presence of level shifts affecting the time series.

5.1.2 Empirical power

Tables B.6 and B.8 summarize the empirical power for the I(1) and NI(1) cases for the two LRV estimation strategies, respectively, when $m = 1$ and $c \in \{0, 15, 30\}$. In general, the picture is qualitatively similar for both LRV estimates, although the use of the joint break dates estimation method produces higher power as γ increases for given values of T and c. Let us first analyze the results that are obtained for the I(1) case. As expected, the rejection rates of $F_1\,(m|0)$ approximate to the nominal size as T increases. Interestingly, $F_1\,(m|0)$ shows some ability to detect the presence of structural breaks in finite samples as γ increases. $F_0\,(m|0)$ also behaves as expected, with rejection rates that, in the limit, are below the nominal size.

As for the NI(1) case, we observe that the rejection rates of $F_1\,(m|0)$ tend to zero as T increases, as predicted by the theory, regardless of γ and the LRV estimate that is used. It is worth noting that for small T, $F_1\,(m|0)$ tends to detect the existence of parameter instabilities for large break magnitudes. The empirical power of $F_0\,(m|0)$ increases with γ for given c and T values, although, due to the near-to-unit set-up that has been defined, it obviously reduces as T increases. The double maximum statistics show a similar pattern, with the UDmax statistics outperforming the WDmax ones. As mentioned above, the use of the LRV estimate that is based on the joint break dates estimation produces more powerful statistics. Consequently, the statistics feature non-negligible empirical power when the correct d is specified. When this is not the case, the behavior of the statistics either does not harm (the rejection rates are below the nominal size) or helps in the detection of structural instabilities. This establishes the basis for the definition of a union statistic that will allow performing statistical inference that is robust to d. The simulation results confirm the utility of these union statistics, either when considering a fixed alternative or when computing the double maximum statistics. As expected, the empirical power of the union statistics is lower than the one offered by the corresponding statistic that assumes the correct d. As above, $UDmax_U$ outperforms $WDmax_U$ in terms of empirical power.

Tables B.7 and B.9 provide the empirical power for the I(0) case. As above, the LRV obtained with the use of the joint break dates estimation strategy produces empirical power improvements, more evident as γ increases, regardless of the given values of T and ρ. The $F_1(m|0)$ statistic never rejects the null hypothesis, whereas the empirical power of $F_0(m|0)$ approaches one as γ and/or T increase, although it reduces in finite samples as ρ gets large. The empirical power of the union statistic is somewhat lower than the one shown by $F_0(m|0)$, but this is the price that needs to be paid to have a robust approach. Finally, the $UDmax_1$ statistic is very conservative, and the empirical power of $UDmax_0$ and $UDmax_U$ is marginally below the $F_0(1|0)$ and $F_U(1|0)$ ones. In all cases, UDmax outperforms WDmax.

The specification that includes $m = 2$ structural breaks leads to similar conclusions – see Tables B.13 and B.14. In all, the simulation evidence recommends the use of the union statistics in empirical applications to test the presence of level shifts without prior knowledge of the order of integration of time series. Although both LRV estimates produce similar results, the one that is based on the joint estimation of the break dates leads to empirical power improvements.

5.2 Sequential test statistics

5.2.1 Empirical size

Table B.10 reports the empirical size ($\gamma = 0$) of the BP and HLT statistics to test the null hypothesis of m structural breaks against the alternative hypothesis of $m+1$ structural breaks, $0 \leq m < m_{\max}$, on the I(1) and NI(1) frameworks. For simplicity and unless required, $F_d(m+1|m)$ is denoted by F_d, $d \in \{0,1\}$, and, similarly, $F_1^j(m+1|m)$ and $F_U^j(m+1|m)$ are designated as F_1^j and F_U^j, $j \in \{a,b\}$, respectively.

In general, the empirical size of the BP sequential statistics is close to the nominal one when the correct d is assumed. As can be seen, the empirical size of F_1 and F_1^j, $j \in \{a,b\}$, tends to the nominal one as T increases, whereas F_0 shows mild under-size distortions for the NI(1) case. Interestingly, when the wrong d is imposed, F_0 tends to be mildly under-sized, and F_1 and F_1^j, $j \in \{a,b\}$, never reject their null hypotheses. Consequently, the empirical size remains under control when facing misspecification errors concerning the order of integration of the stochastic process. This characteristic leads the union statistics to retain a controlled empirical size regardless of d – F_U^j, $j \in \{a, b\}$, are either correctly sized ($c = 0$) or conservative ($c > 0$) statistics. Finally, it is worth noting that these features are obtained for both the OAAT and joint break dates estimation strategies.

Let us now focus on the performance of the HLT statistics. When the true d is assumed, the empirical size of S_1 approaches the 5% as T increases, although over-size distortions are observed in finite samples – this feature has to be considered when conducting the empirical power analysis. The empirical size of S_0 is close to the nominal one for $T = 100$, although S_0 becomes conservative for large T.[6] When the wrong d is assumed, the empirical size of S_d, $d \in \{0, 1\}$, tends to zero as T increases. The union statistic (U) inherits all these features: (i) it shows over-size distortions in finite samples when $y_t \sim I(1)$ – although the empirical size tends to the 5% as T gets large – and (ii) it becomes conservative as c and/or T increase when $y_t \sim I(0)$.

Table B.11 collects the results for the I(0) case. Now the empirical size of F_0 equals the nominal one regardless of T and ρ, whereas F_1 and F_1^j, $j \in \{a, b\}$, never reject their null hypotheses. This implies conservative F_U^j, $j \in \{a, b\}$, statistics. Note that these characteristics are observed irrespective of the break dates estimation strategy that is used. As above, the empirical size of S_0 is close to the nominal one for $T = 100$, but S_0 becomes conservative in large samples. Further, S_1 (almost) never rejects the null hypothesis, which implies a conservative U statistic.

Although it is difficult to establish a clear dominance of one set of statistics over the other, we can conclude that the BP sequential statistics present better overall performance in terms of empirical size. This statement is supported by the fact that the over-size distortions in finite samples featured by the HLT statistics are not present on the BP statistics for the I(1) case. In both cases, the corresponding union statistics become conservative under the NI(1) case, whereas for the I(0) case there is not a clear dominance

[6]Results available upon request show important size distortions for S_1 (0.40) and S_0 (0.26) when $T = 50$, which is not the case for F_1 (0.10), F_1^a (0.06), F_1^b (0.10) and F_0 (0.04) – between parenthesis, the empirical size under the respective null hypothesis.

of one approach over the other – the HLT union statistic only shows less under-size distortions than F_U^j, $j \in \{a, b\}$, in three out of fifteen situations that have been analyzed, whereas F_U^j, $j \in \{a, b\}$, are superior to the HLT union statistic in four out of fifteen situations; in the rest of situations the rejection rates are equivalent.

5.2.2 Empirical power

Table B.10 presents the empirical power for the one structural break case. When $y_t \sim I(1)$ and regardless of γ, the rejection rates of F_1, F_1^j, F_U^j, $j \in \{a, b\}$, S_1 and U tend to the nominal size as T increases. This is an expected result, since the fixed magnitude of the structural break becomes negligible in the limit. As for the statistics that assume a wrong d, the rejection rates of S_0 tend in the limit towards zero, whereas the ones for F_0 take values around the 5%, especially when the joint break dates estimation strategy is implemented. Consequently and in the worst scenario, the signal sent by S_0 might lead to reduce the empirical power of the union statistic, a feature that should not be expected for F_0.

Let us now analyze the performance of the statistics under the NI(1) scenario. In general, F_0 encompasses S_0, being the dominance more prominent when the joint break dates estimation strategy is applied. The empirical power of S_1 tends towards zero as T increases, whereas F_1 and F_1^j, $j \in \{a, b\}$, never reject their null hypotheses. The ability of S_1 to detect the presence of structural breaks in finite samples – although it should bear in mind the size distortions that have been found – leads U to outperform F_U^j, $j \in \{a, b\}$, when $c = 15$ and $T = 100$. The picture changes as c and γ increase, since the empirical power of F_U^j, $j \in \{a, b\}$, based on the joint break date estimation, is superior to the U one.

Table B.11 collects the empirical power for the one structural break case for the I(0) case. As predicted by the theory, F_1 and F_1^j, $j \in \{a, b\}$, never reject their null hypotheses regardless of γ and ρ. The empirical power of F_0 is non-negligible – with values that tend to one as γ and T increase – although it decreases as ρ gets large for given values of γ and T. The good performance of F_0 translates into the empirical power of F_U^j, $j \in \{a, b\}$, with values that approach to one as γ and T increase. The use of the joint break dates estimation strategy produces marginal improvements. As for the HLT statistics, the rejection rates of S_1 tend to zero as T increases, although this statistic shows some ability to detect the presence of structural breaks for small T. The empirical power of S_0 and U is non-negligible, with values that tend to one as γ and T increase, although it experiences a reduction as ρ increases for given values of γ and T. Finally, in general, we observe that F_U^j, $j \in \{a, b\}$, outperform U – the exception is found for $\gamma = 5$ with $\rho = 0.8$ and $T = 100$.

The results for $m = 2$ structural breaks are presented in Table B.15 and lead to similar conclusions. All in all, the simulation evidence indicates that the empirical power of F_U^j, $j \in \{a, b\}$, especially based on the joint break date estimation, is superior to the empirical power of the U statistic. Consequently, the robust BP-type sequential statistics that have been proposed in the paper can be used in empirical analyses to address the presence of multiple level shifts without prior knowledge about the order of integration of the time series.

5.2.3 Estimation of the number and position of the structural breaks

Table B.12 reports the frequency of the estimated number of structural breaks for the union statistics when $m = 1$ – results for the other statistics are available upon request. Let us first focus on the I(1) and NI(1) cases. As can be seen, over-estimation of the number of structural breaks is not an issue for any of the statistics, with values for the frequency of the estimated number of structural breaks that mimic the empirical power figures that have been discussed above. The U statistic outperforms F_U^j, $j \in \{a, b\}$, for small values of γ and finite T – mainly due to the over-size distortions that experience the former statistic – although the converse situation is found as γ, c and T increase. As expected, the frequency of the estimated number of structural breaks tends towards the 5% nominal significance level for large T. For the I(0) case, F_U^j, $j \in \{a, b\}$, outperform U in general – the exception is found for $T = 100$ and $\rho = 0.8$. The use of the joint break dates estimation strategy leads to over-estimate the number of structural breaks, so that the statistics that are based on the OAAT strategy would be the preferred ones from this point of view.

Figure B.1 collects the densities of the estimated break fraction when there is evidence of structural breaks for $m = 1$ and $\gamma = 5$. To ease comparison, we have excluded from the analysis the few occasions in which the procedures have detected more than one structural break. It would be possible to include those cases considering just one of the estimated break points – for instance, the first one for the OAAT estimation strategy and the earliest one for the joint estimation strategy – although this might introduce noise and difficult the interpretation of the results. For I(1) stochastic processes, the densities of the estimated break fraction associated to the three sequential testing procedures are centered around the true break fraction parameter. However, the estimation that is derived from the BP sequential statistic that is based on the joint estimation procedure outperforms the other ones – this is more evident as T increases. Similar conclusions are found for the NI(1) case, with the only exception when $T = 1000$ and $c = 30$, in which case the HLT-based estimate marginally outperforms the others. Finally, the simulation results for the I(0) case show that the three estimation procedures are nearly identical in terms of break date location – the densities are overlapped.

All in all, the BP sequential statistic that is proposed in this paper offers the possibility of conducting a robust analysis of the presence of structural breaks with desirable features. Simulation evidence indicates that the empirical size of the statistic is close to the nominal one, and the statistics have decent empirical power. Although the use of the joint estimation strategy to estimate the break dates might lead to over-estimate the number of structural breaks, the location of the structural breaks that is obtained is preferred to the OAAT-strategy-based statistics one.

6 Empirical illustration

The purchasing power parity hypothesis (PPP) has not ceased to arouse interest, both academically and from the point of view of policymakers.[7] For academics, PPP is a key assumption on which rely most theoretical open economy macro-models and a reference to assess the viability of currency unions and setting parities in monetary unions. For policymaking and policy design is crucial in measuring exchange rate misalignments and, furthermore, real exchange rate is a measure of competitiveness. The PPP hypothesis postulates that the exchange rate between two currencies should equate the two prices level if expressed in a common currency. If the nominal exchange rate is defined as the domestic price of a foreign currency, then the real exchange rate (RER) is the nominal exchange rate adjusted for national prices differences. Consequently, if PPP holds, the RER behavior should be constant over time and its movements represent deviations from the PPP.[8] The history of the empirical testing of the PPP has gone hand in hand with the spectacular development of econometric techniques and, in fact, the PPP has been a test bench for most econometric tools in the field of time series.

One of the most widespread ways in the literature to carry out the empirical validation of the PPP is to assess the order of integration of the real exchange rate. However, this approach soon faced the problem of the low power of unit root tests in finite samples, which was one of the causes of the loss of confidence in PPP during the eighties. The consideration of longer time periods might solve this drawback, although this increases the probability of the presence of structural breaks, a feature that is relevant for the implementation of popular unit root tests.[9] Therefore, the correct detection of structural

[7]This hypothesis has a long history in Economics since its early formulation in the sixteenth century in the school of Salamanca, later recovered by the English classical school in the nineteenth century and formally coined and developed throughout the twentieth century.

[8]Confidence in compliance of the PPP has undergone different stages over the years. In its early stages, the PPP faced difficulties for its correct measurement, while in the Bretton-Woods period RER was considered stable over long periods and real exchange rate constant. After the period of high volatility that followed the bankruptcy of the Bretton-Woods system, confidence in the PPP begins to break apart, although it was still accepted short-run variation in RER, but long-run stability. However, during eighties, there was strong evidence against PPP, and it will be in the nineties when long-run PPP revived. See Sarno and Taylor (2003) for a summary of the literature.

[9]Froot et al. (1995), Lothian and Taylor (1996, 2000) and Taylor (2002) are some examples of testing PPP with long historical databases.

changes in the trajectory of the real exchange rate becomes a key issue for the PPP analysis. Although there exists a flurry of methods for structural changes detection, the problem is tangled given that RER are very persistent series whose dynamics are very close to the unit root and that balance between the stationarity and the unit root region.[10] Therefore, the proposal that has been designed in this paper is of great relevance for PPP research. Following Bai and Perron (1998, 2003a), the empirical strategy first computes the $UDmax$ and $WDmax$ statistics. If these statistics indicate the presence of structural instabilities, the analysis proceeds to detect the number and location of structural changes by applying the sequential statistics. However, to compare the results of the BP and HLT sequential statistics, the second step will be carried out regardless of the results obtained in the first step. Finally, Appendix C summarizes the results of the unit root hypothesis testing and computes the degree of persistence of the different RER time series that have been used.

6.1 Databases

The analysis is carried out using two different databases. The first one is provided by Jordà, Schularick and Taylor (2018) – hereafter, JST database – and collects annual data from 1870 to 2020 for Austria, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom and United States of America – i.e., $N = 17$ countries.[11] The RER ($Q_{n,t}$) for country n at time t has been computed as $Q_{n,t} = P^*_{n,t}/(P_t E_{n,t})$, where $P^*_{n,t}$ are the foreign consumer price index (CPI), P_t the US CPI and $E_{n,t}$ the nominal exchange rate against the US dollar, $n = 1, \ldots, 16$. Taking logs we obtain the RER as $q_{n,t} = p^*_{n,t} - p_t - e_{n,t}$.

Figure C.1 shows the evolution of RER for all countries. Throughout the last century and a half, important changes in the international-monetary system have occurred. The first stage of the Gold standard era 1870-1914, characterized by stability and an increase in globalization; the international monetary system during the interwar period 1918-1939 distinguished by volatility and the failed attempt to recover the Gold standard; the

[10] This high persistence of the RER has been documented, for instance, by Rossi (2005).

[11] The original data end in 2016, but we have updated them until 2020 using the same sources.

Bretton-Woods system of fixed exchange rates (1944-1973) which led a long period of stability until the decade of the seventies; the return to the flotation in 1973 after the turbulences of the seventies and the end of the Bretton-Woods era; the emergence of local monetary integration zones among which the creation of the euro in 1999 stands out. All these events are potential sources of structural breaks.

The second database that is used is the Penn World Tables (PWT) extension by Feenstra, Inklaar and Timmer (2015). They construct PPP exchange rates with different prices with the purpose to convert gross domestic product (GDP) at national prices to a common currency (US dollars) making them comparable across countries. The database comprises a total of $N = 182$ countries and, in general, the time period goes from 1950 to 2019.[12] The revised PWT uses three definitions of PPP exchange rates based on different concepts of prices: consumption of households and government (RER-C), domestic absorption (real consumption plus investment, RER-A) and output-side GDP deflator (RER-O). Therefore, we define the RER as $q^l_{n,t} = p^{*l}_{n,t} - p^l_t$, where $p^{*l}_{n,t}$ is the CPI of country n at time t, at PPP level of US GDP, and p^l_t is the CPI of US, $n = 1, \ldots, N$. Superscript l, $l \in \{C, A, O\}$, refers to the different concepts that define the price series: Real consumption of households and government (C), Real domestic absorption (A) and output-side real GDP (O). Figure C.2 presents the evolution of these RER, which may also be affected by the events mentioned above.

6.2 Testing the presence of multiple level shifts

We begin by applying the $F_d(m|0)$, $d \in \{0, 1\}$, statistics described in Section 3.2.1 that test the null hypothesis of no structural break against the alternative that there are m structural breaks. Results available upon request indicate that the $UDmax_U$ and $WDmax_U$ statistics computed for the historical database only detect the presence of structural changes for Japan, regardless of the estimation break dates strategy.[13] The evidence for the PWT database depends on the definition of RER that is used. For

[12] Countries with less than 30 observations have been discarded, so that $N = 180$ (for the OAAT break dates estimation strategy) or $N = 157$ (for the joint break dates estimation strategy).

[13] The $UDmax_0$ also finds evidence of structural breaks in Belgium, the Netherlands and Switzerland at the 10% level of significance when the break dates are jointly estimated.

RER-C, the presence of structural breaks is found in 30 ($UDmax_U$) and 28 ($WDmax_U$) countries when the OAAT break dates estimation strategy is used, and in 13 ($UDmax_U$) and 8 ($WDmax_U$) countries when the joint break dates estimation strategy is applied. With RER-A the figures are 32 ($UDmax_U$) and 27 ($WDmax_U$), when the OAAT estimation procedure is implemented, and 19 ($UDmax_U$) and 13 ($WDmax_U$), for the joint break dates estimation procedure. Finally, for RER-O, we find evidence of structural breaks in 37 ($UDmax_U$) and 27 ($WDmax_U$) countries (OAAT), and in 25 ($UDmax_U$) and 15 ($WDmax_U$) countries (joint). Therefore, these results indicate scarce evidence of structural breaks affecting the RER time series that have been analyzed.

We have also computed the sequential statistics for all countries to compare the pictures that are obtained by the HLT and BP statistics. The first interesting feature concerns the difference in the number of breaks detected by both methods. While the HLT statistics find structural changes that are statistical significant at the 5% significance level in seven out of the sixteen analyzed countries,[14] the BP statistics only do so for Japan. Regarding the location of the structural breaks, all breaks detected by the HLT statistics are placed around the World War II, while the BP statistics only find a structural break for Japan in 1970 when the OAAT strategy is used, and two breaks (1945, 1971) when the joint break dates estimation method is applied. The striking difference in results might be due to the conservative feature that show the BP statistics, although a visual inspection of Figure C.1 might suggest that the structural changes detected by the HLT statistics might be due more to the presence of large outliers than to changes in the level of RER.

The analysis that is based on the PWT database produces similar conclusions. Due to the large number of countries, we present the results of the estimated break dates for RER-C in Figure C.3 for both the OAAT and joint break dates estimation strategies. Focusing on the results for the U and F_U^b statistics, we find 206 (U), 20 (F_U^b, OAAT) and 10 (F_U^b, joint) estimated break dates.[15] For the other RER definitions, the figures are

[14] To be specific, the countries and break dates are: Denmark (1945), Italy (1941), the Netherlands (1945), Norway (1945), Portugal (1919), Spain (1946) and UK (1945).

[15] The results for the U and F_U^b (OAAT) statistics are based on a sample of 180 countries, whereas the F_U^b (joint) statistic is computed for 157 countries.

222/19/14 (RER-A) and 226/21/19 (RER-O). The percentages of cases in which the break dates that are estimated by the HLT and BP procedures coincide (totally or partially) are 97.23 (RER-C), 88.33 (RER-A) and 82.78 (RER-O) for the OAAT estimation strategy, and 94.91 (RER-C), 86.63 (RER-A) and 81.53 (RER-O) for the joint estimation strategy.

For RER-C and according to the HLT method, most of the break dates are located in the decade of the 80s followed by those of the 90s, first decade of the 21st century and, lastly, during the 70s. With the BP method, the scarce number of breaks are concentrated also on the decade of the 80s and in 1978, 1990, 1991 and 1996. Similar results are obtained when using the second definition of RER – the estimated breaks using HLT procedure resemble the ones obtained for RER-C; results for BP place the estimated breaks in the decades of 80s and 90s. Finally, in the case of RER-O the structural breaks are mainly found in the 80s with some breaks in the 60s and 90s by BP statistic, whereas the breaks detected by HLT have a similar location than the ones estimated with the previous price definitions. The highest number of breaks is mostly concentrated in underdeveloped countries.[16]

We have also studied the coincidences among the three definitions of RER. Let us first focus on HLT statistics. If we compare RER-C with RER-A based results, we can observe that the two price definitions coincide in detecting no structural breaks in 30% of cases, in 30% they coincide in the number and location of the breaks and in 36.11% they partially do so. Therefore, only in 3.99% of cases both definitions produce different results. Comparing RER-C and RER-O these figures are, respectively, 22.78%, 27.78% and 33.89% with a residual of 15.65%. Finally, comparing RER-A and RER-O provides the following percentages: 22.78%, 27.78% and 31.11%, with a residual of 18.33%. As for the BP statistics computed using the OAAT strategy, the results are 82.78%, 2.78% and 4.44% (RER-C vs. RER-A), 80.00%, 2.22% and 6.11% (RER-C vs. RER-O) and, finally, 79.44%, 4.44% and 2.22% (RER-A vs. RER-O) – the residuals are, respectively, 10,

[16]Countries that account for 4 or even 5 breaks are Azerbaijan, Bulgaria, Haiti, Lithuania, Latvia, Mongolia, New Zealand, Peru, Slovenia and Vietnam when using RER-C. For RER-A these countries are Grenada, Lithuania, Latvia, North Macedonia, Nepal, Peru, Slovenia, Tajikistan and Vietnam. Finally, for RER-O these countries are Azerbaijan, Croatia, Haiti, Israel, Latvia, North Macedonia, Peru, Philippines, Slovakia, Slovenia, British Virgin Islands and Vietnam.

11.67 and 14.9%. With the joint break dates estimation strategy we obtain the following figures: 87.9, 0.64 and 4.46% and a residual of 7% when comparing RER-C vs. RER-A, 84.08, 1.27 and 3.18% with a residual of 11.47% when comparing RER-C vs. RER-O and, finally, 82.17, 0.64 and 3.82% with a residual of 13.37% when comparing RER-A vs. RER-O. Therefore, the results with the three RER definitions are very similar, although RER-O is the one that yields the greatest differences.[17]

The empirical evidence that has been obtained so far allows us to extract some stylized facts. First, the most robust structural changes, detected in a large sample of countries and with both methodologies occur during the eighties, nineties and seventies, when the Bretton-Woods system goes bankrupt and the international monetary system enters mostly in a floating regime. Second, there is no significant effect of the inception of the euro on the behavior of the real exchange rate, since only Ireland presents a break in 2002 with the RER-C. Finally, the rest of the structural breaks that have been detected with the HLT method are found in underdeveloped countries, which suggests that they could be associated with specific crises instead of changes in the equilibrium of the RER.

Summing up, the HLT method tends to find many more breaks than the BP one. This might be due to the fact that the HLT method looks throughout the sample searching for local breaks inside a window, while the BP considers it globally.[18] The size distortions that have been found in the simulation exercise above suggests this explanation. This seems to indicate that the HLT method could interpret very local phenomena as true changes in the long-term equilibrium of the RER and, therefore, its results should be taken with caution. Otherwise, our confidence in the use of long historical databases to test the PPP would be adversely affected. Finally, based on the robust conclusion about the presence of multiple level shifts that has been obtained, the PPP hypothesis can be tested using the ADF unit root statistic with or without the inclusion of level shifts. This analysis is provided in Appendix C for completeness and shows that the PPP compliance is broad (around 81%) with the JST database, whereas evidence in favor of the PPP is

[17] The results obtained with the sequential procedure are coherent with the $F_d(m|0)$, $d \in \{0, 1\}$, ones.

[18] Another feature to consider concerns the amplitude of the window for the HLT statistics, which being smaller (0.1) than the one used for the BP statistics (0.15), generates more possibilities of structural breaks detection.

scarce (around 25%) for the PWT database.

7 Conclusions

The paper extends the methodology in Bai and Perron (1998) to the analysis of multiple level shifts affecting I(1) and NI(1) non-stationary stochastic processes. The proposal defines a unified framework where the same testing procedure can be used to detect the presence of multiple level shifts regardless of the order of integration of the time series. The paper derives the limiting distribution of the different statistics that have been proposed and provides the corresponding asymptotic critical values. A side contribution has involved the design of dynamic optimization algorithms that permit the feasible implementation of the statistics in empirical applications.

An extensive simulation experiment has been conducted to compare our proposal with other statistics available in the literature. We show that, in general, our approach offers better finite sample performance provided that the empirical size is controlled, and the empirical power is non-negligible. The use of the statistics is illustrated with the analysis of real exchange rates, considering the presence of multiple level shifts. The empirical application uses two different databases. The first one provides historical time series that allow us to cover a long time period, whereas the second one permits a worldwide perspective. Finally, we study the influence of the above analysis on the PPP test.

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Supplementary material
Appendix A. Mathematical appendix

A Mathematical appendix

A.1 Proof of Theorem 1

First, note that the SSR of the model under the null of no structural break is given by:

$$SSR\left(T\right)=\sum_{t=1}^{T}y_{t}-T\bar{y}^{2},$$

whereas the SSR under the alternative hypothesis of some fixed structural breaks is defined by:

$$SSR\left(T_{B,m}\right)=\sum_{t=1}^{T}y_{t}-\sum_{i=1}^{m+1}\left(T_{i}-T_{i-1}\right)\bar{y}_{i}^{2},$$

with the convention that $T_0 = 0$ and $T_{m+1} = T$. Provided that $\rho = 1 - c/T$, $0 \leq c < \infty$, in (2), we have that $T^{-1/2}y_t \Rightarrow \sigma_1 W_c\left(r\right)$, where $W_c\left(r\right) = \int_0^r e^{-(r-s)}dW\left(s\right)$ denotes a standard OU process and $W\left(s\right)$ a standard Brownian motion. Then, by the Functional Central Limit Theorem (FCLT), $\left(T_i/T - T_{i-1}/T\right)^{-1} T^{-3/2} \sum_{t=T_{i-1}+1}^{T_i} y_t \Rightarrow \sigma_1 \left(\lambda_i - \lambda_{i-1}\right)^{-1} \int_{\lambda_{i-1}}^{\lambda_i} W_c\left(s\right)ds$. Given that $\hat{\sigma}_1^2 \xrightarrow{p} \sigma_1^2$, we have that the $F_1\left(m|\,0\right)$ statistic is given in (6) converges towards:

$$\begin{aligned}
F_{1}\left(m|\,0\right) &= m^{-1}\hat{\sigma}_{1}^{-2}T^{-2}\max_{T_{B,m}\in T\Lambda(\epsilon)^{m}}\left[\sum_{i=1}^{m+1}\left(T_{i}-T_{i-1}\right)\bar{y}_{i}^{2}-T\bar{y}^{2}\right] \\
&\Rightarrow \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^{m}} m^{-1}\left[\sum_{i=1}^{m+1}\left(\lambda_{i}-\lambda_{i-1}\right)\left(\int_{\lambda_{i-1}}^{\lambda_{i}}W_{c}\left(s\right)ds\right)^{2}-\left(\int_{0}^{1}W_{c}\left(s\right)ds\right)^{2}\right] \\
&\equiv \sup_{\lambda_{B,m}\in\Lambda(\epsilon)^{m}} K_{c}\left(\lambda_{B,m}\right).
\end{aligned}$$

Finally, the derivation of the limit distribution for the double maximum statistics is straightforward and only requires taking the maximum of the (weighted) sequence of the limiting distribution of $F_1\left(m|\,0\right)$ for different values of $m \in \{1, 2, \ldots, m_{\max}\}$.

A.2 Proof of Theorem 2

Let us first focus on the limit of the expression:

$$A\left(T_{i-1}^0, T_i^0\right) = T^{-2}\hat{\sigma}_1^{-2}\left[\sum_{t=T_{i-1}^0+1}^{T_i^0}(y_t - \bar{y}_{(T_{i-1}^0, T_i^0)})^2 - \sum_{t=T_{i-1}^0+1}^{\tau}(y_t - \bar{y}_{(T_{i-1}^0, \tau)})^2 - \sum_{t=\tau+1}^{T_i^0}(y_t - \bar{y}_{(\tau, T_i^0)})^2\right]. \tag{A.1}$$

To simplify the notation, in what follows we use $\bar{y}_i = \bar{y}_{(T_{i-1}^0, T_i^0)}$, $\bar{y}_{i,1} = \bar{y}_{(T_{i-1}^0, \tau)}$ and $\bar{y}_{i,2} = \bar{y}_{(\tau, T_i^0)}$ to denote the mean using the whole segment, and the first and second subsegments defined by τ, respectively. The first element of $A\left(T_{i-1}^0, T_i^0\right)$ is defined by:

$$T^{-2}\hat{\sigma}_1^{-2}\sum_{t=T_{i-1}^0+1}^{T_i^0}(y_t - \bar{y}_i)^2 = T^{-2}\hat{\sigma}_1^{-2}\sum_{t=T_{i-1}^0+1}^{T_i^0}\left(y_t - \frac{1}{T_i^0 - T_{i-1}^0}\sum_{t=T_{i-1}^0+1}^{T_i^0} y_t\right)^2.$$

Provided that $\rho = 1 - c/T$, $0 \leq c < \infty$, in (2), by the FCLT we have that:

$$T^{-2}\hat{\sigma}_1^{-2}\sum_{t=T_{i-1}^0+1}^{T_i^0}(y_t - \bar{y}_i)^2 \Rightarrow \int_{\lambda_{i-1}^0}^{\lambda_i^0}\left(W_c(r) - \frac{1}{\lambda_i^0 - \lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_i^0} W_c(s)\,ds\right)^2 dr,$$

given that $\hat{\sigma}_1^2 \xrightarrow{p} \sigma_1^2$. The same applies to the other two elements of $A\left(T_{i-1}^0, T_i^0\right)$ so that we obtain:

$$\begin{aligned} A\left(T_{i-1}^0, T_i^0\right) \Rightarrow & \int_{\lambda_{i-1}^0}^{\lambda_i^0}\left(W_c(r) - \frac{1}{\lambda_i^0 - \lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_i^0} W_c(s)\,ds\right)^2 dr \\ & - \int_{\lambda_{i-1}^0}^{\lambda_\tau}\left(W_c(r) - \frac{1}{\lambda_\tau - \lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_\tau} W_c(s)\,ds\right)^2 dr \\ & - \int_{\lambda_\tau}^{\lambda_i^0}\left(W_c(r) - \frac{1}{\lambda_i^0 - \lambda_\tau}\int_{\lambda_\tau}^{\lambda_i^0} W_c(s)\,ds\right)^2 dr. \end{aligned}$$

Finally, by the FCLT:

$$
\begin{aligned}
F_1(m+1|m) \quad \Rightarrow \quad & \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)}\left[\int_{\lambda_{i-1}^0}^{\lambda_i^0}\left(W_c(r)-\frac{1}{\lambda_i^0-\lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_i^0}W_c(s)\,ds\right)^2 dr\right. \\
& -\int_{\lambda_{i-1}^0}^{\lambda_\tau}\left(W_c(r)-\frac{1}{\lambda_\tau-\lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_\tau}W_c(s)\,ds\right)^2 dr \\
& \left.-\int_{\lambda_\tau}^{\lambda_i^0}\left(W_c(r)-\frac{1}{\lambda_i^0-\lambda_\tau}\int_{\lambda_\tau}^{\lambda_i^0}W_c(s)\,ds\right)^2 dr\right]. \qquad \text{(A.2)}
\end{aligned}
$$

Note that this limiting distribution can be written as a function of independent functionals of demeaned OU processes. Let us denote the first term on the right-hand side of (A.2) as:

$$
A_0\left(\lambda_{i-1}^0,\lambda_i^0\right)=\int_{\lambda_{i-1}^0}^{\lambda_i^0}\left(W_c(r)-\frac{1}{\lambda_i^0-\lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_i^0}W_c(s)\,ds\right)^2 dr,
$$

which can also be expressed as:

$$
\begin{aligned}
A_0\left(\lambda_{i-1}^0,\lambda_i^0\right) &= \left(\lambda_i^0-\lambda_{i-1}^0\right)^2\int_0^1\left(W_c(a)-\int_0^1 W_c(b)\,db\right)^2 da \\
&= \left(\lambda_i^0-\lambda_{i-1}^0\right)^2\int_0^1 W_c^*(a)^2\,da,
\end{aligned}
$$

with $W_c^*(a)=W_c(a)-\int_0^1 W_c(b)\,db$, $a=\left(r-\lambda_{i-1}^0\right)/\left(\lambda_i^0-\lambda_{i-1}^0\right)$ and $b=\left(s-\lambda_{i-1}^0\right)/\left(\lambda_i^0-\lambda_{i-1}^0\right)$. The same applies to the second and third element on the right-hand side of (A.2), so that the limiting distribution can be alternatively expressed as:

$$
\begin{aligned}
F_1(m+1|m) \quad \Rightarrow \quad & \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)}\left(\lambda_i^0-\lambda_{i-1}^0\right)^2\left[\int_0^1 W_c^*(a)^2\,da\right. \qquad \text{(A.3)} \\
& \left.-\int_0^l W_{c,1}^*(a)^2\,da-\int_l^1 W_{c,2}^*(a)^2\,da\right] \\
\equiv \quad & \sup_{1\leq i\leq m+1}\sup_{\lambda_\tau\in\Lambda_i(\epsilon)} H_c\left(\lambda_{i-1}^0,\lambda_\tau,\lambda_i^0\right),
\end{aligned}
$$

with $l=\left(\lambda_\tau-\lambda_{i-1}^0\right)/\left(\lambda_i^0-\lambda_{i-1}^0\right)$, $W_{c,1}^*(a)=W_c(a)-\int_0^l W_c(b)\,db$ and $W_{c,2}^*(a)=W_c(a)-\int_l^1 W_c(b)\,db$.

It is possible to derive a simplified expression of the limiting distribution of the statistic if we note that (A.1) can be written as – we are in debt with one anonymous referee for pointing out this simplification:

$$\begin{aligned}
A\left(T_{i-1}^{0}, T_{i}^{0}\right) &= T^{-2}\hat{\sigma}_{1}^{-2}\left[(\bar{y}_{i,1}-\bar{y}_{i})\sum_{t=T_{i-1}^{0}+1}^{\tau}(y_{t}-\bar{y}_{i}-\bar{y}_{i,1})+(\bar{y}_{i,2}-\bar{y}_{i})\sum_{t=\tau+1}^{T_{i}^{0}}(y_{t}-\bar{y}_{i}-\bar{y}_{i,2})\right] \\
&= T^{-2}\hat{\sigma}_{1}^{-2}\left[(\bar{y}_{i,1}-\bar{y}_{i})^{2}(\tau-T_{i-1}^{0})+(\bar{y}_{i,2}-\bar{y}_{i})^{2}(T_{i}^{0}-\tau)\right].
\end{aligned}$$

Let $\Delta T_{i}^{0} = \left(T_{i}^{0}-T_{i-1}^{0}\right)$ and $\Delta\lambda_{i}^{0} = \left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)$, so that $\bar{y}_{i} = \frac{\tau-T_{i-1}^{0}}{\Delta T_{i}^{0}}\bar{y}_{i,1}+\frac{T_{i}^{0}-\tau}{\Delta T_{i}^{0}}\bar{y}_{i,2}$, $\bar{y}_{i,1}-\bar{y}_{i} = \frac{T_{i}^{0}-\tau}{\Delta T_{i}^{0}}(\bar{y}_{i,1}-\bar{y}_{i,2})$ and $\bar{y}_{i,2}-\bar{y}_{i} = -\frac{\tau-T_{i-1}^{0}}{\Delta T_{i}^{0}}(\bar{y}_{i,1}-\bar{y}_{i,2})$. Then,

$$\begin{aligned}
A\left(T_{i-1}^{0}, T_{i}^{0}\right) &= T^{-2}\hat{\sigma}_{1}^{-2}(\bar{y}_{i,1}-\bar{y}_{i,2})^{2}\left[\frac{(\tau-T_{i-1}^{0})(T_{i}^{0}-\tau)^{2}}{(\Delta T_{i}^{0})^{2}}+\frac{(T_{i}^{0}-\tau)(\tau-T_{i-1}^{0})^{2}}{(\Delta T_{i}^{0})^{2}}\right] \\
&= T^{-2}\hat{\sigma}_{1}^{-2}\frac{(\tau-T_{i-1}^{0})(T_{i}^{0}-\tau)}{\Delta T_{i}^{0}}(\bar{y}_{i,1}-\bar{y}_{i,2})^{2} \\
&= T^{-1}\hat{\sigma}_{1}^{-2}\frac{(\lambda_{\tau}-\lambda_{i-1}^{0})(\lambda_{i}^{0}-\lambda_{\tau})}{\Delta\lambda_{i}^{0}}(\bar{y}_{i,1}-\bar{y}_{i,2})^{2}. \qquad \text{(A.4)}
\end{aligned}$$

Since:

$$\begin{aligned}
T^{-1/2}(\bar{y}_{i,1}-\bar{y}_{i,2}) &\Rightarrow \sigma_{1}\left(\int_{\lambda_{i-1}^{0}}^{\lambda_{\tau}}W_{c}(s)\,ds-\int_{\lambda_{\tau}}^{\lambda_{i}^{0}}W_{c}(s)\,ds\right) \\
&= \sigma_{1}\left(\frac{\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)}{\left(\lambda_{\tau}-\lambda_{i-1}^{0}\right)}\int_{0}^{l}W_{c}(a)\,da-\frac{\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)}{\left(\lambda_{i}^{0}-\lambda_{\tau}\right)}\int_{l}^{1}W_{c}(a)\,da\right),
\end{aligned}$$

with $a=\left(s-\lambda_{i-1}^{0}\right)/\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)$ and $l=\left(\lambda_{\tau}-\lambda_{i-1}^{0}\right)/\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)$. Then,

$$\begin{aligned}
T^{-1/2}(\bar{y}_{i,1}-\bar{y}_{i,2}) &\Rightarrow \sigma_{1}\left(\frac{1}{l}\int_{0}^{l}W_{c}(a)\,da-\frac{1}{1-l}\int_{l}^{1}W_{c}(a)\,da\right) \\
&= \sigma_{1}\frac{\int_{0}^{l}W_{c}(a)\,da-l\int_{0}^{1}W_{c}(a)\,da}{l\,(1-l)},
\end{aligned}$$

so that,

$$
\begin{aligned}
A\left(T_{i-1}^{0},T_{i}^{0}\right) &\Rightarrow \frac{(\lambda_{\tau}-\lambda_{i-1}^{0})(\lambda_{i}^{0}-\lambda_{\tau})}{\Delta\lambda_{i}^{0}}\frac{\left(\int_{0}^{l}W_{c}\left(a\right)da-l\int_{0}^{1}W_{c}\left(a\right)da\right)^{2}}{l^{2}\left(1-l\right)^{2}} \\
&= \Delta\lambda_{i}^{0}l\left(1-l\right)\frac{\left(\int_{0}^{l}W_{c}\left(a\right)da-l\int_{0}^{1}W_{c}\left(a\right)da\right)^{2}}{l^{2}\left(1-l\right)^{2}} \\
&= \Delta\lambda_{i}^{0}\frac{\left(\int_{0}^{l}W_{c}\left(a\right)da-l\int_{0}^{1}W_{c}\left(a\right)da\right)^{2}}{l\left(1-l\right)}.
\end{aligned}
$$

Using these elements, it is straightforward to see that:

$$
F_{1}\left(m+1|m\right)\Rightarrow\sup_{1\leq i\leq m+1}\sup_{\lambda_{\tau}\in\Lambda_{i}(\epsilon)}\left[\Delta\lambda_{i}^{0}\frac{\left(\int_{0}^{l}W_{c}\left(a\right)da-l\int_{0}^{1}W_{c}\left(a\right)da\right)^{2}}{l\left(1-l\right)}\right]. \tag{A.5}
$$

As can be seen, the expressions (A.3) and (A.5) show that the limiting distribution of $F_{1}\left(m+1|m\right)$ depends on the length of the specific regime $\Delta\lambda_{i}^{0}$ into which the additional break date is searched.

A.3 Proof of Theorem 3

Using the elements in the previous proof, we can see that when $\rho=1-c/T$, $0\leq c<\infty$, in (2) the first element on the right-hand side of (16) converges towards:

$$
\begin{aligned}
&\left(T_{i}^{0}-T_{i-1}^{0}\right)^{-2}\hat{\sigma}_{1}^{-2}\sum_{t=T_{i-1}^{0}+1}^{T_{i}^{0}}\left(y_{t}-\bar{y}_{i}\right)^{2}=\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)^{-2}T^{-2}\hat{\sigma}_{1}^{-2}\sum_{t=T_{i-1}^{0}+1}^{T_{i}^{0}}\left(y_{t}-\bar{y}_{i}\right)^{2} \\
\Rightarrow\ &\left(\lambda_{i}^{0}-\lambda_{i-1}^{0}\right)^{-2}\int_{\lambda_{i-1}^{0}}^{\lambda_{i}^{0}}\left(W_{c}\left(r\right)-\frac{1}{\lambda_{i}^{0}-\lambda_{i-1}^{0}}\int_{\lambda_{i-1}^{0}}^{\lambda_{i}^{0}}W_{c}\left(s\right)ds\right)^{2}dr \\
\equiv\ &\int_{0}^{1}W_{c}^{*}\left(a\right)^{2}da,
\end{aligned}
$$

with $W_c^*(a)$ defined above. The same applies to the other elements in (16) so that:

$$\begin{aligned} F_1^a(m+1|m) &\Rightarrow \sup_{1\le i\le m+1}\sup_{\lambda_\tau\in\Lambda(\epsilon)}\left[\int_0^1 W_c^*(a)^2\,da\right. \\ &\quad \left. -\int_0^l W_c^*(a)^2\,da-\int_l^1 W_c^*(a)^2\,da\right] \\ &\equiv \sup_{1\le i\le m+1} J_c^a(x), \end{aligned} \tag{A.6}$$

As stated in Bai and Perron (1998), since over the different regimes the $WSSR$ are computed using non-overlapping observations, the weak limits in (A.6) are independent across regimes, which implies that the limit distribution of the $F_1^a(m+1|m)$ statistic can be computed as the maximum of $m+1$ independent random variables $J_c^a(x)$. Consequently,

$$P\left(F_1^a(m+1|m)\le x\right)=J_c^a(x)^{m+1},$$

and the critical values for the $F_1^a(m+1|m)$ statistic for different values of m and ϵ can be obtained from the distribution function $J_c^a(x)$.

Similar developments can be applied to obtain the limit distribution of the second modified statistic, so that:

$$F_1^b(m+1|m)\Rightarrow \sup_{1\le i\le m+1}\sup_{\lambda_\tau\in\Lambda(\epsilon)}\frac{\left(\int_0^l W_c(a)\,da-l\int_0^1 W_c(a)\,da\right)^2}{l(1-l)}\equiv \sup_{1\le i\le m+1} J_c^b(x).$$

A.4 Proof of Theorem 4

The key elements involved in the computation of the $F_0(m+1|m)$ are given by:

$$A\left(T_{i-1}^0,T_i^0\right)=\hat\sigma_0^{-2}\left[\sum_{t=T_{i-1}^0+1}^{T_i^0}(y_t-\bar y_i)^2-\sum_{t=T_{i-1}^0+1}^{\tau}(y_t-\bar y_{i,1})^2-\sum_{t=\tau+1}^{T_i^0}(y_t-\bar y_{i,2})^2\right], \tag{A.7}$$

an expression that is similar to (A.1) but without rescaling by T^{-2} and with $\hat\sigma_1^2$ replaced by $\hat\sigma_0^2$. Since $\rho=1-c/T$, $0\le c<\infty$, in (2), the rescaled elements inside the brackets in

(A.7) converge towards:

$$T^{-2}\sum_{t=T_{i-1}^0+1}^{T_i^0}(y_t-\bar{y}_i)^2 \Rightarrow \sigma_1^2\int_{\lambda_{i-1}^0}^{\lambda_i^0}\left(W_c(r)-\frac{1}{\lambda_i^0-\lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_i^0}W_c(s)\,ds\right)^2 dr$$
$$T^{-2}\sum_{t=T_{i-1}^0+1}^{\tau}(y_t-\bar{y}_{i,1})^2 \Rightarrow \sigma_1^2\int_{\lambda_{i-1}^0}^{\lambda_\tau}\left(W_c(r)-\frac{1}{\lambda_\tau-\lambda_{i-1}^0}\int_{\lambda_{i-1}^0}^{\lambda_\tau}W_c(s)\,ds\right)^2 dr$$
$$T^{-2}\sum_{t=\tau+1}^{T_i^0}(y_t-\bar{y}_{i,2})^2 \Rightarrow \sigma_1^2\int_{\lambda_\tau}^{\lambda_i^0}\left(W_c(r)-\frac{1}{\lambda_i^0-\lambda_\tau}\int_{\lambda_\tau}^{\lambda_i^0}W_c(s)\,ds\right)^2 dr.$$

Following the developments in Perron (1991), the (rescaled) non-parametric long-run variance estimator $\hat{\sigma}_0^2$ converges to:

$$(hT)^{-1}\,\hat{\sigma}_0^2 \Rightarrow \kappa\sigma_1^2\int_0^1 W_c^*(r,\lambda_{B,m}^0)^2 dr,$$

with the constant κ defined by $\kappa=\int_{-1}^1 K(s)\,ds$ where $K(j/h)$ is the kernel used in the computation of $\hat{\sigma}_0^2$,[19] and where $W_c^*(r)$ denotes the projection of $W_c(r)$ onto the space spanned by $\{1, 1(r>\lambda_1^0), 1(r>\lambda_2^0), \ldots, 1(r>\lambda_m^0)\}$. Andrews' (1991) AR(1)-based automatic bandwidth selection procedure that is applied is defined by:

$$\hat{h}_1^{\max}=1.1447\left(\frac{4\hat{a}^2T}{(1+\hat{a}^2)(1-\hat{a}^2)}\right)^{1/3},$$

where $\hat{a}$ is the OLS estimate of the autoregressive parameter of the AR(1) model that is estimated for the residuals from (24). Since $y_t\sim I(1)$, we have that $(1-\hat{a})=O_p(T^{-1})$ which implies that $\hat{h}_1^{\max}=O_p(T)$ and, hence,

$$T^{-2}\hat{\sigma}_0^2 \Rightarrow \kappa\sigma_1^2\int_0^1 W_c^*(r,\lambda_{B,m}^0)^2 dr.$$

[19] In this paper we use the quadratic spectral kernel, which implies $K(s)=\left(25/\left(12\pi^2s^2\right)\right)\left(\sin(6\pi s/5)/(6\pi s/5)-\cos(6\pi s/5)\right)$ and $\kappa=1.2931$.

Taking all these elements together, we have that (A.7) converges in the limit to:

$$\begin{aligned}
A\left(T_{i-1}^{0}, T_{i}^{0}\right) \;\Rightarrow\; & \left[\kappa \int_{0}^{1} W_{c}^{*}(r, \lambda_{B,m}^{0})^{2} dr\right]^{-1} \left[\int_{\lambda_{i-1}^{0}}^{\lambda_{i}^{0}} \left(W_{c}\left(r\right) - \frac{1}{\lambda_{i}^{0} - \lambda_{i-1}^{0}} \int_{\lambda_{i-1}^{0}}^{\lambda_{i}^{0}} W_{c}\left(s\right) ds\right)^{2} dr \right. \\
& - \int_{\lambda_{i-1}^{0}}^{\lambda_{\tau}} \left(W_{c}\left(r\right) - \frac{1}{\lambda_{\tau} - \lambda_{i-1}^{0}} \int_{\lambda_{i-1}^{0}}^{\lambda_{\tau}} W_{c}\left(s\right) ds\right)^{2} dr \\
& \left. - \int_{\lambda_{\tau}}^{\lambda_{i}^{0}} \left(W_{c}\left(r\right) - \frac{1}{\lambda_{i}^{0} - \lambda_{\tau}} \int_{\lambda_{\tau}}^{\lambda_{i}^{0}} W_{c}\left(s\right) ds\right)^{2} dr \right],
\end{aligned}$$

and, consequently,

$$F_{0}\left(m+1|m\right) \Rightarrow \sup_{1 \leq i \leq m+1} \sup_{\lambda_{\tau} \in \Lambda_{i}(\epsilon)} \left[\kappa \int_{0}^{1} W_{c}^{*}(r, \lambda_{B,m}^{0})^{2} dr\right]^{-1} H_{c}\left(\lambda_{i-1}^{0}, \lambda_{\tau}, \lambda_{i}^{0}\right).$$

Supplementary material
Appendix B. Tables of critical values and Monte Carlo simulation results

B Tables of critical values and Monte Carlo simulation results

B.1 Tables of critical values

Table B.1: Asymptotic critical values for the $F_d(m|0)$, $UDmax_d$ and $WDmax_d$ statistics, and the κ constant for the union statistics

$F_1(m|0)$ statistic

ϵ	$\xi \backslash m$	1	2	3	4	5	6	7	8	9	UDmax	WDmax
0.05	0.10	0.27	0.15	0.11	0.08	0.07	0.06	0.05	0.04	0.04	0.27	0.28
	0.05	0.37	0.21	0.14	0.11	0.09	0.07	0.06	0.06	0.05	0.37	0.38
	0.025	0.46	0.26	0.18	0.14	0.11	0.09	0.08	0.07	0.06	0.46	0.48
	0.01	0.60	0.34	0.23	0.18	0.14	0.12	0.10	0.09	0.08	0.60	0.61
0.15	0.10	0.27	0.15	0.11	0.08	0.06					0.27	0.28
	0.05	0.37	0.21	0.14	0.11	0.09					0.37	0.38
	0.025	0.46	0.26	0.18	0.14	0.11					0.46	0.48
	0.01	0.60	0.34	0.23	0.18	0.14					0.60	0.61
0.2	0.10	0.27	0.15	0.10							0.27	0.28
	0.05	0.37	0.21	0.14							0.37	0.38
	0.025	0.46	0.26	0.18							0.46	0.48
	0.01	0.60	0.34	0.23							0.60	0.61

$F_0(m|0)$ statistic

ϵ	$\xi \backslash m$	1	2	3	4	5	6	7	8	9	UDmax	WDmax
0.05	0.10	7.76	7.46	6.65	6.15	5.70	5.33	4.99	4.69	4.42	8.43	8.98
	0.05	9.31	8.39	7.36	6.75	6.23	5.80	5.41	5.08	4.77	9.76	10.63
	0.025	10.83	9.25	8.05	7.32	6.68	6.20	5.79	5.43	5.11	11.08	12.26
	0.01	12.80	10.38	8.94	8.01	7.32	6.77	6.28	5.87	5.51	12.94	14.28
0.15	0.10	6.86	5.95	4.96	4.15	3.21					7.24	8.08
	0.05	8.41	6.91	5.68	4.71	3.67					8.64	9.70
	0.025	9.96	7.79	6.34	5.23	4.07					10.09	11.33
	0.01	11.92	8.87	7.09	5.82	4.56					11.98	13.36
0.2	0.10	6.50	5.29	4.10							6.77	7.48
	0.05	8.03	6.25	4.78							8.20	9.09
	0.025	9.57	7.16	5.41							9.65	10.69
	0.01	11.55	8.24	6.21							11.60	12.72

κ_ξ constant for the union statistics

$F_U(m|0)$ statistic

ϵ	$\xi \backslash m$	1	2	3	4	5	6	7	8	9	UDmax	WDmax
0.05	0.10	1.24	1.18	1.17	1.16	1.15	1.15	1.14	1.14	1.14	1.21	1.32
	0.05	1.19	1.15	1.14	1.13	1.13	1.12	1.12	1.12	1.12	1.18	1.22
	0.025	1.17	1.13	1.12	1.12	1.11	1.11	1.11	1.11	1.10	1.16	1.16
	0.01	1.14	1.11	1.11	1.10	1.10	1.09	1.09	1.09	1.09	1.13	1.12
0.15	0.10	1.25	1.20	1.19	1.19	1.19					1.23	1.30
	0.05	1.21	1.17	1.16	1.15	1.15					1.20	1.22
	0.025	1.18	1.14	1.13	1.13	1.14					1.17	1.17
	0.01	1.14	1.12	1.12	1.13	1.13					1.14	1.13
0.2	0.10	1.26	1.22	1.21							1.24	1.28
	0.05	1.21	1.18	1.17							1.20	1.22
	0.025	1.18	1.15	1.15							1.18	1.17
	0.01	1.15	1.13	1.13							1.15	1.13

Table B.2: Percentiles of the limiting distribution of the $F_1(m+1|m)$ test statistic under the null hypothesis of m structural breaks, with $\epsilon = 0.15$ trimming

λ_1	λ_2	90%	95%	97.5%	99%
0.0	0.0	0.271	0.367	0.463	0.595
0.2	0.0	0.174	0.233	0.299	0.380
0.3	0.0	0.133	0.179	0.228	0.295
0.4	0.0	0.101	0.133	0.170	0.220
0.5	0.0	0.091	0.117	0.144	0.178
0.6	0.0	0.101	0.134	0.167	0.219
0.7	0.0	0.132	0.178	0.227	0.294
0.8	0.0	0.172	0.232	0.297	0.383
0.2	0.6	0.056	0.073	0.090	0.113
0.2	0.7	0.067	0.091	0.115	0.148
0.2	0.8	0.098	0.132	0.167	0.213
0.3	0.7	0.041	0.057	0.072	0.094
0.3	0.8	0.068	0.091	0.115	0.148
0.4	0.8	0.057	0.073	0.091	0.113

Table B.3: Response surfaces to approximate asymptotic critical values for the $F_1(m+1|m)$ statistic

	Response surface for $\epsilon = 0.15$ trimming				Response surface for $\epsilon = 0.2$ trimming			
	$q_i(90)$	$q_i(95)$	$q_i(97.5)$	$q_i(99)$	$q_i(90)$	$q_i(95)$	$q_i(97.5)$	$q_i(99)$
Constant	0.19976	0.27097	0.33898	0.43429	0.23305	0.31849	0.39279	0.51608
$m+1$	0.069355	0.093165	0.12036	0.15654	0.035111	0.044852	0.065766	0.073822
λ_1^0	-1.0506	-1.4234	-1.7713	-2.2994	-0.76273	-1.0329	-1.327	-1.6473
λ_2^0	-0.91861	-1.2528	-1.5945	-2.0573	-1.3322	-1.7944	-2.2993	-2.8137
λ_3^0	-0.70643	-0.97381	-1.2634	-1.6337				
λ_4^0	-1.9091	-2.5557	-3.1133	-4.0907				
$(\lambda_2^0-\lambda_1^0)^2$	1.2826	1.7183	2.0755	2.7067	0.69502	0.94493	1.1982	1.4467
$(\lambda_3^0-\lambda_2^0)^2$	-0.77756	-1.0353	-1.2296	-1.6271				
$(\lambda_4^0-\lambda_3^0)^2$	1.0776	1.4462	1.7515	2.3082				
$(\lambda_2^0-\lambda_1^0)^3$	0.37166	0.49962	0.61979	0.79848	0.29151	0.37785	0.47938	0.57359
$(\lambda_3^0-\lambda_2^0)^3$	0.48756	0.65365	0.82408	1.0529				
$(\lambda_4^0-\lambda_3^0)^3$	0.5096	0.63972	0.77859	1.0099				
$(\lambda_2^0-\lambda_1^0)^4$	0.10593	0.17229	0.30655	0.37909	0.46269	0.60481	0.7926	1.0268
$(\lambda_3^0-\lambda_2^0)^4$	0.46473	0.61601	0.76995	0.99354				
$(\lambda_4^0-\lambda_3^0)^4$	0.47885	0.60557	0.74677	0.96125				
$\lambda_1^0\lambda_2^0$	4.9633	6.7998	8.4449	11.0321	6.8934	9.3944	11.9073	14.6075
$\lambda_1^0\lambda_3^0$	1.3883	1.8497	2.3827	3.118				
$\lambda_1^0\lambda_4^0$	-0.64643	-1.0735	-1.3225	-1.803				
$\lambda_2^0\lambda_3^0$	-0.7465	-0.85568	-0.84964	-1.2024				
$\lambda_2^0\lambda_4^0$	-1.0441	-0.87488	-1.1403	-0.92061				
$\lambda_3^0\lambda_4^0$	8.2439	10.7604	13.0505	16.8277				
$(\lambda_1^0\lambda_2^0)^2$	-7.9025	-11.3078	-14.2963	-19.0513	-29.2615	-40.0477	-50.4537	-62.1692
$(\lambda_1^0\lambda_3^0)^2$	-9.7196	-13.0662	-16.8865	-22.1729				
$(\lambda_1^0\lambda_4^0)^2$	3.9658	6.6857	8.081	11.0277				
$(\lambda_2^0\lambda_3^0)^2$	-1.0112	-1.9845	-2.7759	-3.4772				
$(\lambda_2^0\lambda_4^0)^2$	6.5205	6.1799	7.8481	7.212				
$(\lambda_3^0\lambda_4^0)^2$	-19.4391	-25.0053	-30.2275	-38.4969				
$(\lambda_1^0\lambda_2^0)^3$	13.6587	20.0593	25.6652	34.7477	74.9627	102.5975	129.2464	159.8464
$(\lambda_1^0\lambda_3^0)^3$	27.5466	37.3052	48.1805	63.7459				
$(\lambda_1^0\lambda_4^0)^3$	-10.5665	-18.0597	-21.4575	-29.3801				
$(\lambda_2^0\lambda_3^0)^3$	1.0398	2.6807	4.0275	5.0804				
$(\lambda_2^0\lambda_4^0)^3$	-15.9105	-15.9008	-19.9233	-19.1424				
$(\lambda_3^0\lambda_4^0)^3$	28.7593	36.978	44.7113	56.8095				
$(\lambda_1^0\lambda_2^0)^4$	-9.6853	-14.4132	-18.544	-25.3558	-69.0372	-94.4848	-119.1584	-147.5741
$(\lambda_1^0\lambda_3^0)^4$	-27.1268	-36.9519	-47.5982	-63.5589				
$(\lambda_1^0\lambda_4^0)^4$	10.4566	18.0688	21.1916	29.1702				
$(\lambda_2^0\lambda_3^0)^4$	-0.72268	-1.9049	-2.9781	-3.806				
$(\lambda_2^0\lambda_4^0)^4$	13.5808	14.0014	17.3964	17.1208				
$(\lambda_3^0\lambda_4^0)^4$	-16.0248	-20.6229	-24.9596	-31.6979				
R^2	0.98879	0.98839	0.98757	0.98687	0.98837	0.98805	0.98825	0.98692
$\bar{R}^2$	0.98733	0.98688	0.98596	0.98517	0.98308	0.98261	0.98292	0.98097

Table B.4: Asymptotic critical values for the $F_1^a(m+1|m)$ and $F_0(m+1|m)$ statistics, and the κ constant for the union statistic

Asymptotic critical values for the $F_1^a(m+1|m)$ statistic

ϵ	$\xi \backslash m$	0	1	2	3	4	5	6	7	8	9
0.05	0.10	0.33	0.44	0.50	0.55	0.58	0.62	0.64	0.66	0.68	0.70
	0.05	0.44	0.55	0.62	0.67	0.71	0.74	0.77	0.80	0.81	0.84
	0.025	0.56	0.68	0.74	0.80	0.84	0.86	0.89	0.91	0.93	0.95
	0.01	0.72	0.83	0.91	0.96	0.99	1.01	1.06	1.10	1.12	1.15
0.15	0.10	0.33	0.44	0.50	0.55	0.59	0.62	0.65	0.67	0.69	0.71
	0.05	0.44	0.55	0.62	0.68	0.72	0.75	0.77	0.79	0.81	0.83
	0.025	0.55	0.68	0.76	0.80	0.83	0.87	0.89	0.92	0.94	0.96
	0.01	0.73	0.85	0.94	0.99	1.03	1.06	1.09	1.11	1.12	1.14
0.2	0.10	0.33	0.44	0.50	0.54	0.58	0.61	0.63	0.66	0.69	0.70
	0.05	0.45	0.56	0.62	0.67	0.70	0.73	0.75	0.78	0.80	0.82
	0.025	0.57	0.68	0.74	0.79	0.82	0.86	0.88	0.90	0.93	0.95
	0.01	0.73	0.83	0.90	0.96	0.97	1.01	1.03	1.05	1.08	1.12

Asymptotic critical values for the $F_0(m+1|m)$ statistic

ϵ	$\xi \backslash m$	0	1	2	3	4	5	6	7	8	9
0.05	0.10	8.00	9.48	10.38	10.99	11.43	11.82	12.18	12.47	12.72	12.93
	0.05	9.47	10.93	11.75	12.43	12.98	13.35	13.66	13.98	14.24	14.53
	0.025	10.85	12.19	13.24	13.85	14.53	14.96	15.30	15.53	15.73	16.00
	0.01	12.73	14.23	15.27	15.93	16.42	16.75	17.11	17.46	17.78	18.10
0.15	0.10	7.03	8.53	9.36	9.99	10.46	10.86	11.15	11.48	11.71	11.93
	0.05	8.55	10.01	10.93	11.47	11.96	12.32	12.68	12.93	13.12	13.37
	0.025	9.90	11.32	12.19	12.85	13.36	13.75	14.25	14.57	14.76	15.09
	0.01	11.61	13.09	14.21	15.10	15.51	15.96	16.26	16.48	16.62	17.10
0.2	0.10	6.69	8.15	9.00	9.55	10.07	10.46	10.81	11.09	11.33	11.59
	0.05	8.20	9.61	10.54	11.09	11.62	12.01	12.33	12.61	12.80	13.01
	0.025	9.55	11.01	11.93	12.52	12.97	13.36	13.72	14.08	14.29	14.59
	0.01	11.22	12.77	13.73	14.61	15.22	15.59	15.88	16.20	16.39	16.60

$\kappa_\xi^{F(m+1|m)}$ constant for the union $F_U^a(m+1|m)$ statistic

ϵ	$\xi \backslash m$	0	1	2	3	4	5	6	7	8	9
0.05	0.10	1.23	1.18	1.17	1.16	1.16	1.15	1.15	1.15	1.15	1.15
	0.05	1.19	1.17	1.16	1.15	1.15	1.14	1.13	1.12	1.12	1.11
	0.025	1.17	1.15	1.14	1.12	1.11	1.10	1.11	1.11	1.11	1.12
	0.01	1.14	1.11	1.10	1.12	1.12	1.14	1.11	1.10	1.09	1.09
0.15	0.10	1.23	1.19	1.18	1.17	1.17	1.16	1.16	1.15	1.14	1.14
	0.05	1.19	1.17	1.15	1.14	1.13	1.13	1.13	1.14	1.14	1.13
	0.025	1.17	1.14	1.13	1.15	1.14	1.13	1.13	1.12	1.12	1.11
	0.01	1.14	1.16	1.14	1.11	1.11	1.11	1.11	1.11	1.11	1.12
0.2	0.10	1.27	1.21	1.19	1.18	1.18	1.16	1.16	1.15	1.14	1.14
	0.05	1.20	1.16	1.16	1.15	1.14	1.13	1.14	1.14	1.14	1.14
	0.025	1.16	1.14	1.13	1.14	1.14	1.13	1.13	1.12	1.12	1.12
	0.01	1.14	1.14	1.13	1.10	1.09	1.10	1.09	1.12	1.11	1.11

Table B.5: Asymptotic critical values for the $F_1^b(m+1|m)$ statistic and the κ constant for the union statistic

Asymptotic critical values for the $F_1^b(m+1\|m)$ statistic											
ϵ	$\xi\backslash m$	0	1	2	3	4	5	6	7	8	9
0.05	0.10	0.27	0.37	0.42	0.46	0.50	0.52	0.54	0.56	0.58	0.60
	0.05	0.37	0.47	0.53	0.56	0.60	0.63	0.66	0.68	0.69	0.71
	0.025	0.47	0.57	0.63	0.68	0.72	0.74	0.76	0.78	0.79	0.81
	0.01	0.61	0.71	0.77	0.81	0.85	0.87	0.90	0.92	0.95	0.98
0.15	0.10	0.28	0.37	0.43	0.47	0.50	0.53	0.55	0.57	0.59	0.60
	0.05	0.37	0.47	0.53	0.58	0.62	0.64	0.66	0.68	0.69	0.71
	0.025	0.47	0.58	0.64	0.69	0.71	0.74	0.76	0.78	0.80	0.82
	0.01	0.63	0.73	0.80	0.84	0.89	0.91	0.92	0.93	0.96	0.98
0.2	0.10	0.28	0.37	0.42	0.46	0.49	0.52	0.54	0.57	0.59	0.60
	0.05	0.39	0.47	0.53	0.57	0.60	0.63	0.65	0.67	0.69	0.70
	0.025	0.49	0.58	0.63	0.68	0.70	0.73	0.75	0.77	0.80	0.82
	0.01	0.63	0.71	0.77	0.82	0.84	0.86	0.88	0.91	0.94	0.96

$\kappa_\xi^{F(m+1\|m)}$ constant for the union $F_U^b(m+1\|m)$ statistic											
ϵ	$\xi\backslash m$	0	1	2	3	4	5	6	7	8	9
0.05	0.10	1.24	1.19	1.18	1.16	1.16	1.15	1.16	1.16	1.15	1.14
	0.05	1.19	1.16	1.15	1.15	1.15	1.14	1.13	1.13	1.12	1.12
	0.025	1.17	1.15	1.14	1.13	1.11	1.10	1.11	1.11	1.12	1.12
	0.01	1.14	1.11	1.11	1.11	1.11	1.11	1.12	1.11	1.10	1.09
0.15	0.10	1.24	1.20	1.18	1.17	1.17	1.16	1.16	1.15	1.14	1.14
	0.05	1.20	1.17	1.16	1.15	1.13	1.13	1.13	1.13	1.14	1.13
	0.025	1.17	1.14	1.13	1.15	1.15	1.13	1.13	1.12	1.12	1.11
	0.01	1.13	1.15	1.13	1.11	1.10	1.11	1.11	1.12	1.11	1.11
0.2	0.10	1.28	1.22	1.19	1.18	1.18	1.17	1.16	1.15	1.14	1.14
	0.05	1.20	1.17	1.16	1.15	1.14	1.13	1.14	1.13	1.14	1.14
	0.025	1.16	1.14	1.13	1.13	1.15	1.14	1.13	1.12	1.11	1.12
	0.01	1.14	1.15	1.12	1.11	1.10	1.10	1.10	1.12	1.12	1.12

B.2 Tables of Monte Carlo simulations

B.2.1 One structural break

Table B.6: Empirical size and power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ statistics, $j \in \{1, 0, U\}$. I(1) and NI(1) cases, $m = 1$. LRV estimated using the one-at-a-time break dates estimation strategy

			$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
γ	c	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
0	0	100	.08	.08	.08	.08	.08	.05	.00	.00	.00	.00	.06	.06	.06	.06	.06	.08	.04	.02	.07	.08	.05
		300	.05	.06	.06	.06	.06	.03	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.02	.01	.04	.06	.04
		1000	.05	.05	.05	.05	.05	.02	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.02	.01	.04	.05	.03
	15	100	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.01	.00	.00	.00
		300	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.02	.01	.00	.00	.00
		1000	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00
	30	100	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.02	.01	.00	.00
		300	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.02	.01	.00	.00	.00
		1000	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00
1	0	100	.08	.08	.08	.08	.08	.05	.00	.00	.00	.00	.07	.06	.06	.06	.06	.08	.04	.07	.08	.02	.06
		300	.06	.06	.06	.06	.06	.04	.00	.00	.00	.00	.04	.04	.04	.05	.05	.06	.03	.04	.06	.01	.03
		1000	.05	.05	.05	.05	.05	.03	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.02	.04	.05	.01	.03
	15	100	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.01	.00	.01	.00
		300	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.01	.00	.01	.00
		1000	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00
	30	100	.00	.00	.00	.00	.00	.12	.02	.01	.00	.00	.04	.00	.00	.00	.00	.00	.10	.04	.00	.06	.00
		300	.00	.00	.00	.00	.00	.05	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.04	.01	.00	.02	.00
		1000	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.05	.02	.04	.05	.01	.03
5	0	100	.16	.16	.16	.16	.16	.07	.01	.00	.00	.00	.14	.13	.13	.13	.13	.16	.05	.14	.17	.03	.12
		300	.09	.09	.09	.09	.09	.04	.00	.00	.00	.00	.06	.06	.06	.06	.06	.09	.04	.06	.09	.01	.05
		1000	.07	.07	.06	.06	.06	.03	.00	.00	.00	.00	.04	.06	.05	.05	.05	.07	.02	.04	.06	.01	.04
	15	100	.00	.00	.00	.00	.00	.31	.01	.01	.00	.00	.13	.01	.00	.00	.00	.00	.28	.11	.00	.16	.00
		300	.00	.00	.00	.00	.00	.10	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.09	.02	.00	.04	.00
		1000	.00	.00	.00	.00	.00	.05	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.04	.01	.00	.01	.00
	30	100	.00	.00	.00	.00	.00	.85	.19	.04	.02	.02	.60	.06	.01	.00	.01	.00	.83	.57	.00	.67	.00
		300	.00	.00	.00	.00	.00	.54	.03	.00	.00	.00	.23	.01	.00	.00	.00	.00	.49	.20	.00	.30	.00
		1000	.00	.00	.00	.00	.00	.18	.01	.00	.00	.00	.05	.01	.00	.00	.00	.00	.16	.04	.00	.08	.00

Table B.7: Empirical size and power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ statistics, $j \in \{1, 0, U\}$. I(0) case, $m = 1$. LRV estimated using the one-at-a-time break dates estimation strategy

			$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
γ	ρ	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
0	0	100	.00	.00	.00	.00	.00	.04	.03	.02	.02	.01	.02	.01	.01	.01	.00	.00	.04	.02	.00	.03	.00
		300	.00	.00	.00	.00	.00	.05	.04	.04	.05	.04	.02	.02	.01	.01	.02	.00	.05	.02	.00	.04	.00
		1000	.00	.00	.00	.00	.00	.06	.06	.08	.07	.06	.02	.02	.03	.03	.02	.00	.06	.02	.00	.07	.00
	.5	100	.00	.00	.00	.00	.00	.04	.02	.01	.01	.01	.02	.00	.00	.00	.00	.00	.04	.01	.00	.03	.00
		300	.00	.00	.00	.00	.00	.05	.04	.03	.03	.02	.03	.01	.01	.01	.01	.00	.05	.03	.00	.04	.00
		1000	.00	.00	.00	.00	.00	.06	.05	.07	.06	.06	.03	.02	.03	.02	.02	.00	.06	.03	.00	.07	.00
	.8	100	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.00	.00	.01	.00
		300	.00	.00	.00	.00	.00	.04	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.01	.00	.02	.00
		1000	.00	.00	.00	.00	.00	.06	.04	.04	.03	.03	.02	.01	.00	.01	.00	.00	.05	.02	.00	.04	.00
1	0	100	.00	.00	.00	.00	.00	.99	.95	.88	.84	.84	.96	.86	.74	.64	.67	.00	.98	.96	.00	.96	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.5	100	.00	.00	.00	.00	.00	.43	.13	.06	.03	.04	.24	.04	.01	.01	.01	.00	.41	.22	.00	.28	.00
		300	.00	.00	.00	.00	.00	.95	.82	.68	.57	.61	.88	.63	.43	.30	.35	.00	.94	.86	.00	.91	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.8	100	.00	.00	.00	.00	.00	.05	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.05	.01	.00	.02	.00
		300	.00	.00	.00	.00	.00	.19	.04	.01	.01	.01	.08	.01	.00	.00	.00	.00	.17	.07	.00	.11	.00
		1000	.00	.00	.00	.00	.00	.73	.52	.36	.28	.28	.58	.30	.16	.11	.12	.00	.72	.56	.00	.63	.00
5	0	100	.00	.00	.00	.00	.00	1.0	.98	.95	.93	.94	1.0	.96	.93	.92	.92	.00	1.0	1.0	.00	1.0	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.5	100	.00	.00	.00	.00	.00	.99	.81	.46	.31	.36	.98	.55	.25	.15	.17	.00	.99	.97	.00	.98	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	.99	.99	1.0	1.0	.98	.98	.98	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.8	100	.00	.00	.00	.00	.00	.51	.04	.01	.01	.01	.23	.01	.00	.00	.00	.00	.46	.21	.00	.27	.00
		300	.00	.00	.00	.00	.00	1.0	.66	.23	.08	.12	.98	.31	.04	.01	.02	.00	1.0	.97	.00	.99	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00

Table B.8: Empirical size and power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ statistics, $j \in \{1, 0, U\}$. I(1) and NI(1) cases, $m = 1$. LRV estimated using the joint break dates estimation strategy

			$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
γ	c	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
0	0	100	.07	.07	.07	.07	.07	.09	.00	.00	.00	.00	.07	.05	.05	.05	.05	.07	.07	.04	.07	.07	.04
		300	.05	.06	.06	.06	.06	.05	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.04	.01	.04	.06	.03
		1000	.05	.05	.05	.05	.05	.04	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.04	.01	.04	.05	.03
	15	100	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.01	.01	.00	.00
		300	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.01	.00	.00	.00
		1000	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00
	30	100	.00	.00	.00	.00	.00	.04	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.02	.01	.00	.00
		300	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.03	.01	.00	.00	.00
		1000	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.02	.01	.00	.00	.00
1	0	100	.07	.08	.07	.07	.08	.08	.00	.00	.00	.00	.08	.06	.06	.06	.06	.07	.07	.07	.08	.04	.05
		300	.05	.06	.06	.06	.06	.05	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.04	.04	.06	.01	.03
		1000	.05	.05	.05	.05	.05	.04	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.04	.04	.05	.01	.03
	15	100	.00	.00	.00	.00	.00	.06	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.04	.01	.00	.02	.00
		300	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.01	.00	.01	.00
		1000	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00
	30	100	.00	.00	.00	.00	.00	.14	.03	.01	.00	.00	.06	.00	.00	.00	.00	.00	.13	.05	.00	.07	.00
		300	.00	.00	.00	.00	.00	.06	.01	.00	.00	.00	.02	.00	.00	.00	.00	.00	.05	.02	.00	.03	.00
		1000	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.00	.00	.00	.00	.05	.04	.04	.05	.01	.03
5	0	100	.16	.16	.16	.17	.17	.11	.01	.00	.00	.00	.14	.13	.13	.13	.13	.16	.09	.15	.17	.04	.12
		300	.09	.09	.09	.09	.09	.06	.00	.00	.00	.00	.07	.06	.06	.06	.06	.09	.06	.07	.09	.02	.05
		1000	.07	.07	.06	.06	.06	.05	.00	.00	.00	.00	.04	.06	.04	.05	.05	.07	.04	.04	.06	.01	.04
	15	100	.00	.00	.00	.00	.00	.41	.03	.01	.00	.00	.16	.01	.00	.00	.00	.00	.36	.14	.00	.20	.00
		300	.00	.00	.00	.00	.00	.15	.01	.00	.00	.00	.04	.00	.00	.00	.00	.00	.12	.03	.00	.06	.00
		1000	.00	.00	.00	.00	.00	.06	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.05	.01	.00	.01	.00
	30	100	.00	.00	.00	.00	.00	.90	.24	.07	.04	.04	.69	.09	.02	.01	.01	.00	.88	.67	.00	.76	.00
		300	.00	.00	.00	.00	.00	.60	.05	.01	.00	.00	.29	.01	.00	.00	.00	.00	.56	.26	.00	.36	.00
		1000	.00	.00	.00	.00	.00	.21	.01	.00	.00	.00	.06	.01	.00	.00	.00	.00	.18	.06	.00	.09	.00

Table B.9: Empirical size and power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ satistics, $j \in \{1, 0, U\}$. I(0) case, $m = 1$. LRV estimated using the joint break dates estimation strategy

			$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
γ	ρ	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
0	0	100	.00	.00	.00	.00	.00	.04	.03	.02	.02	.02	.02	.01	.01	.01	.01	.00	.04	.01	.00	.04	.00
		300	.00	.00	.00	.00	.00	.05	.04	.04	.04	.04	.02	.02	.01	.02	.02	.00	.05	.02	.00	.05	.00
		1000	.00	.00	.00	.00	.00	.06	.06	.08	.07	.06	.02	.02	.03	.03	.02	.00	.06	.02	.00	.07	.00
	.5	100	.00	.00	.00	.00	.00	.05	.02	.01	.01	.01	.02	.01	.00	.00	.00	.00	.04	.02	.00	.03	.00
		300	.00	.00	.00	.00	.00	.06	.04	.03	.03	.03	.03	.01	.01	.01	.01	.00	.05	.03	.00	.04	.00
		1000	.00	.00	.00	.00	.00	.06	.06	.07	.06	.06	.03	.02	.03	.02	.02	.00	.06	.03	.00	.07	.00
	.8	100	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.01	.00	.01	.00
		300	.00	.00	.00	.00	.00	.04	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.03	.01	.00	.02	.00
		1000	.00	.00	.00	.00	.00	.06	.04	.04	.03	.03	.02	.01	.00	.01	.00	.00	.05	.02	.00	.04	.00
1	0	100	.00	.00	.00	.00	.00	.99	.95	.89	.85	.86	.96	.88	.77	.67	.70	.00	.99	.96	.00	.97	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.5	100	.00	.00	.00	.00	.00	.46	.17	.08	.05	.06	.26	.07	.02	.01	.02	.00	.44	.24	.00	.31	.00
		300	.00	.00	.00	.00	.00	.95	.83	.70	.60	.63	.88	.65	.45	.33	.38	.00	.94	.87	.00	.91	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.8	100	.00	.00	.00	.00	.00	.07	.01	.00	.00	.00	.02	.00	.00	.00	.00	.00	.06	.02	.00	.03	.00
		300	.00	.00	.00	.00	.00	.20	.04	.01	.01	.01	.09	.01	.00	.00	.00	.00	.18	.08	.00	.12	.00
		1000	.00	.00	.00	.00	.00	.73	.52	.37	.29	.29	.59	.31	.17	.12	.13	.00	.72	.56	.00	.64	.00
5	0	100	.00	.00	.00	.00	.00	1.0	1.0	1.0	.99	.99	1.0	1.0	.99	.98	.98	.00	1.0	1.0	.00	1.0	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.5	100	.00	.00	.00	.00	.00	1.0	.86	.55	.37	.42	1.0	.63	.30	.20	.22	.00	1.0	.99	.00	1.0	.00
		300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
	.8	100	.00	.00	.00	.00	.00	.61	.07	.01	.01	.01	.29	.02	.00	.00	.00	.00	.57	.27	.00	.36	.00
		300	.00	.00	.00	.00	.00	1.0	.71	.28	.11	.15	.98	.35	.07	.02	.03	.00	1.0	.98	.00	.99	.00
		1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00

Table B.10: Empirical size and power of HLT and BP test statistics. I(1) and NI(1) cases, $m = 1$

						OAAT estimation						Joint estimation					
γ	c	T	S_1	S_0	U	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b
0	0	100	.14	.04	.15	.08	.06	.08	.05	.04	.07	.08	.06	.07	.08	.05	.07
		300	.08	.01	.09	.06	.04	.05	.03	.02	.04	.05	.04	.05	.04	.03	.04
		1000	.05	.00	.05	.05	.03	.05	.02	.02	.04	.05	.04	.05	.04	.02	.04
	15	100	.05	.06	.09	.00	.00	.00	.02	.01	.00	.00	.00	.00	.03	.01	.01
		300	.00	.01	.01	.00	.00	.00	.02	.00	.00	.00	.00	.00	.03	.01	.00
		1000	.00	.00	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.01	.00	.00
	30	100	.01	.05	.05	.00	.00	.00	.03	.01	.01	.00	.00	.00	.04	.01	.01
		300	.00	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.03	.00	.00
		1000	.00	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.02	.00	.00
1	0	100	.14	.04	.16	.08	.06	.06	.04	.04	.04	.07	.06	.06	.07	.06	.08
		300	.09	.01	.09	.06	.04	.04	.03	.02	.02	.06	.04	.04	.05	.03	.04
		1000	.05	.00	.05	.05	.04	.04	.02	.03	.03	.05	.04	.04	.04	.03	.04
	15	100	.05	.06	.09	.00	.00	.00	.03	.01	.01	.00	.00	.00	.05	.02	.01
		300	.00	.01	.01	.00	.00	.00	.02	.01	.01	.00	.00	.00	.02	.01	.01
		1000	.01	.00	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.01	.00	.00
	30	100	.02	.06	.07	.00	.00	.00	.11	.04	.04	.00	.00	.00	.14	.06	.05
		300	.00	.01	.01	.00	.00	.00	.04	.01	.01	.00	.00	.00	.05	.02	.02
		1000	.00	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.03	.00	.00
5	0	100	.53	.16	.54	.16	.13	.13	.06	.10	.10	.16	.13	.13	.10	.10	.15
		300	.15	.01	.15	.09	.06	.06	.04	.04	.04	.09	.06	.06	.06	.05	.07
		1000	.06	.01	.06	.07	.04	.04	.02	.03	.03	.07	.04	.04	.04	.03	.04
	15	100	.52	.34	.59	.00	.00	.00	.29	.13	.13	.00	.00	.00	.38	.16	.15
		300	.04	.04	.07	.00	.00	.00	.09	.03	.03	.00	.00	.00	.13	.04	.04
		1000	.01	.01	.02	.00	.00	.00	.04	.01	.01	.00	.00	.00	.06	.01	.01
	30	100	.56	.55	.67	.00	.00	.00	.83	.60	.60	.00	.00	.00	.89	.69	.68
		300	.03	.11	.11	.00	.00	.00	.51	.22	.22	.00	.00	.00	.57	.28	.28
		1000	.00	.02	.02	.00	.00	.00	.17	.05	.05	.00	.00	.00	.19	.06	.06

Table B.11: Empirical size and power of HLT and BP test statistics. I(0) case, $m = 1$

						OAAT estimation						Joint estimation					
γ	ρ	T	S_1	S_0	U	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b
0	0	100	.00	.05	.05	.00	.00	.00	.04	.02	.02	.00	.00	.00	.04	.02	.02
		300	.00	.03	.02	.00	.00	.00	.05	.02	.02	.00	.00	.00	.05	.02	.02
		1000	.00	.03	.03	.00	.00	.00	.06	.02	.02	.00	.00	.00	.06	.02	.02
	.5	100	.01	.05	.05	.00	.00	.00	.04	.02	.02	.00	.00	.00	.04	.02	.02
		300	.00	.01	.01	.00	.00	.00	.05	.03	.03	.00	.00	.00	.05	.03	.03
		1000	.00	.02	.02	.00	.00	.00	.06	.03	.03	.00	.00	.00	.06	.03	.03
	.8	100	.04	.05	.07	.00	.00	.00	.03	.01	.01	.00	.00	.00	.04	.01	.01
		300	.00	.01	.01	.00	.00	.00	.03	.01	.01	.00	.00	.00	.04	.01	.01
		1000	.00	.01	.01	.00	.00	.00	.05	.02	.02	.00	.00	.00	.05	.02	.02
1	0	100	.00	.10	.10	.00	.00	.00	.99	.96	.96	.00	.00	.00	.99	.96	.96
		300	.00	.24	.24	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.00	.95	.94	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.5	100	.01	.06	.06	.00	.00	.00	.42	.23	.23	.00	.00	.00	.45	.25	.25
		300	.00	.02	.02	.00	.00	.00	.94	.87	.87	.00	.00	.00	.95	.88	.88
		1000	.00	.18	.18	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.8	100	.04	.06	.07	.00	.00	.00	.05	.01	.01	.00	.00	.00	.07	.02	.02
		300	.00	.01	.01	.00	.00	.00	.18	.08	.08	.00	.00	.00	.19	.09	.09
		1000	.00	.01	.01	.00	.00	.00	.72	.57	.57	.00	.00	.00	.72	.58	.58
5	0	100	.70	.97	.97	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		300	.26	.97	.97	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.00	.97	.97	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.5	100	.63	.85	.87	.00	.00	.00	.99	.98	.98	.00	.00	.00	1.0	1.0	1.0
		300	.11	.99	.99	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.00	1.0	1.0	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.8	100	.53	.41	.61	.00	.00	.00	.48	.23	.23	.00	.00	.00	.58	.29	.29
		300	.01	.42	.42	.00	.00	.00	1.0	.98	.98	.00	.00	.00	1.0	.98	.98
		1000	.00	.94	.93	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0

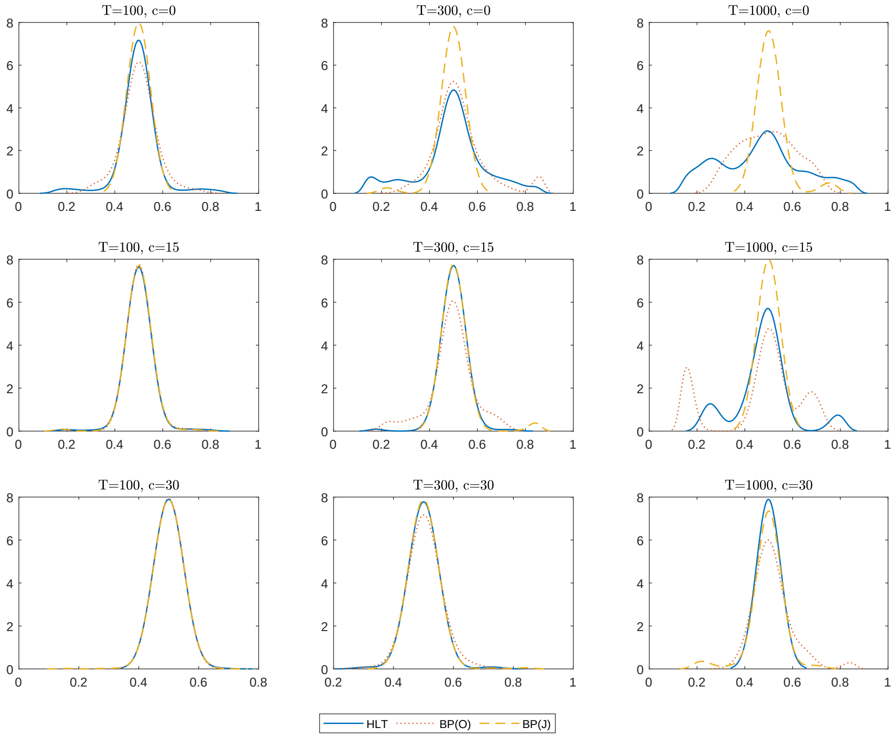
Table B.12: Frequency of estimated number of structural breaks. Union statistics, $m = 1$

I(1) and NI(1) cases

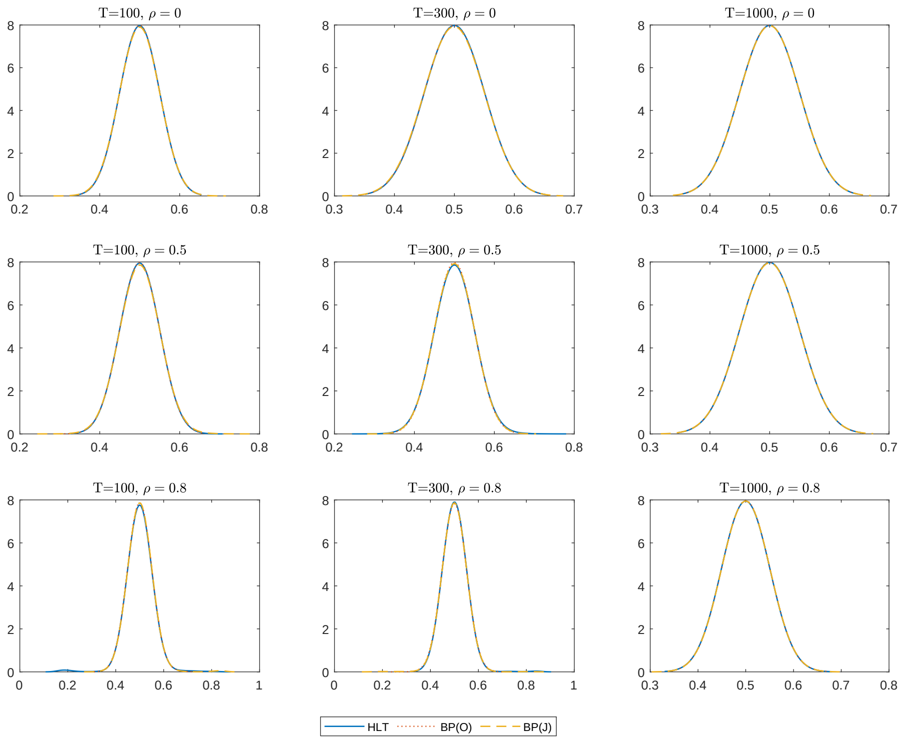
			OAAT estimation									Joint estimation					
			U			F_U^a			F_U^b			F_U^a			F_U^b		
γ	c	$T\backslash m$	1	2	>2	1	2	>2	1	2	>2	1	2	>2	1	2	>2
1	0	100	.14	.02	.00	.04	.00	.00	.07	.00	.00	.05	.01	.00	.07	.00	.00
		300	.09	.00	.00	.02	.00	.00	.04	.00	.00	.02	.00	.00	.04	.00	.00
		1000	.05	.00	.00	.02	.00	.00	.04	.00	.00	.02	.00	.00	.04	.00	.00
	15	100	.07	.02	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00
		300	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00
		1000	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
	30	100	.06	.01	.00	.04	.00	.00	.04	.00	.00	.04	.01	.00	.04	.01	.00
		300	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00
		1000	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	0	100	.48	.06	.00	.09	.01	.00	.14	.01	.00	.09	.01	.00	.14	.01	.00
		300	.14	.01	.00	.04	.00	.00	.06	.00	.00	.04	.00	.00	.07	.00	.00
		1000	.06	.00	.00	.03	.00	.00	.04	.00	.00	.02	.00	.00	.04	.00	.00
	15	100	.54	.03	.00	.12	.00	.00	.12	.00	.00	.15	.01	.00	.14	.01	.00
		300	.07	.00	.00	.03	.00	.00	.03	.00	.00	.04	.00	.00	.04	.00	.00
		1000	.02	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00
	30	100	.63	.03	.00	.59	.01	.00	.59	.01	.00	.67	.02	.00	.66	.02	.00
		300	.11	.00	.00	.22	.00	.00	.21	.00	.00	.27	.00	.00	.27	.00	.00
		1000	.02	.00	.00	.05	.00	.00	.05	.00	.00	.06	.00	.00	.06	.00	.00

I(0) case

			One-at-a-time estimation									Joint estimation					
			U			F_U^a			F_U^b			F_U^a			F_U^b		
γ	ρ	$T\backslash m$	1	2	>2	1	2	>2	1	2	>2	1	2	>2	1	2	>2
1	0	100	.09	.01	.00	.95	.00	.00	.95	.00	.00	.53	.27	.16	.53	.27	.16
		300	.24	.00	.00	.99	.01	.00	.99	.01	.00	.27	.32	.41	.27	.32	.41
		1000	.93	.02	.00	.99	.01	.00	.99	.01	.00	.15	.26	.59	.15	.26	.59
	.5	100	.05	.01	.00	.23	.00	.00	.23	.00	.00	.19	.05	.01	.19	.05	.01
		300	.02	.00	.00	.87	.01	.00	.86	.01	.00	.55	.22	.10	.56	.23	.10
		1000	.18	.00	.00	.99	.01	.00	.99	.01	.00	.30	.33	.38	.30	.33	.37
	.8	100	.06	.01	.00	.01	.00	.00	.01	.00	.00	.01	.01	.00	.01	.01	.00
		300	.01	.00	.00	.08	.00	.00	.08	.00	.00	.07	.02	.01	.07	.02	.01
		1000	.01	.00	.00	.57	.00	.00	.56	.00	.00	.43	.12	.04	.42	.12	.03
5	0	100	.93	.04	.00	.99	.01	.00	.99	.01	.00	.84	.10	.06	.84	.10	.06
		300	.95	.01	.00	.99	.01	.00	.99	.01	.00	.73	.13	.14	.73	.13	.14
		1000	.95	.02	.00	.99	.01	.00	.99	.01	.00	.60	.17	.23	.60	.17	.23
	.5	100	.83	.03	.00	.97	.01	.00	.97	.01	.00	.95	.03	.01	.95	.03	.01
		300	.99	.00	.00	.99	.01	.00	.99	.01	.00	.92	.05	.03	.92	.05	.03
		1000	.99	.01	.00	.99	.01	.00	.99	.01	.00	.83	.09	.08	.83	.09	.08
	.8	100	.57	.03	.00	.23	.00	.00	.22	.00	.00	.28	.01	.00	.27	.01	.00
		300	.42	.00	.00	.97	.00	.00	.97	.00	.00	.95	.03	.01	.95	.03	.01
		1000	.93	.01	.00	.98	.01	.00	.98	.01	.00	.92	.05	.03	.92	.05	.03



(a) The I(1) and NI(1) cases



(b) The I(0) case

Figure B.1: Densities of the estimated break fraction, $m = 1$, $\gamma = 5$

B.2.2 Two structural breaks

This section provides the empirical power analysis with two structural breaks using the one-at-a-time (OAAT) and joint break dates estimation strategies.

Table B.13: Empirical power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ statistics, $j \in \{1, 0, U\}$. I(1) and NI(1) cases, $m = 2$

				$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
Method	γ	c	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
OAAT	1	0	100	.08	.08	.08	.08	.08	.04	.01	.00	.00	.00	.07	.06	.06	.06	.06	.08	.04	.07	.08	.02	.05
			300	.06	.06	.06	.06	.06	.03	.00	.00	.00	.00	.04	.04	.05	.05	.05	.06	.02	.04	.06	.01	.03
			1000	.05	.05	.05	.05	.05	.03	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.02	.04	.05	.01	.03
		15	100	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.04	.01	.00	.02	.00
			300	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.01	.00	.01	.00
			1000	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.01	.00	.00	.00	.00
		30	100	.00	.00	.00	.00	.00	.15	.02	.01	.00	.01	.05	.00	.00	.00	.00	.00	.14	.05	.00	.07	.00
			300	.00	.00	.00	.00	.00	.07	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.05	.02	.00	.03	.00
			1000	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.00	.00	.00	.00	.00	.05	.02	.04	.05	.01	.03
	5	0	100	.17	.20	.20	.20	.19	.05	.00	.00	.00	.00	.14	.16	.16	.16	.16	.17	.05	.13	.19	.02	.14
			300	.08	.09	.09	.09	.09	.03	.00	.00	.00	.00	.05	.07	.07	.07	.07	.08	.02	.05	.09	.00	.06
			1000	.07	.07	.07	.07	.07	.03	.00	.00	.00	.00	.04	.07	.04	.05	.05	.07	.02	.04	.07	.01	.04
		15	100	.00	.00	.00	.00	.00	.14	.03	.01	.00	.00	.04	.01	.00	.00	.00	.00	.12	.04	.00	.06	.00
			300	.00	.00	.00	.00	.00	.10	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.08	.02	.00	.02	.00
			1000	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.01	.00	.00	.00	.00	.00	.04	.01	.00	.02	.00
		30	100	.00	.00	.00	.00	.00	.46	.24	.06	.03	.03	.17	.09	.02	.01	.01	.00	.41	.16	.00	.22	.00
			300	.00	.00	.00	.00	.00	.30	.05	.00	.00	.00	.07	.00	.00	.00	.00	.00	.26	.07	.00	.11	.00
			1000	.00	.00	.00	.00	.00	.18	.01	.00	.00	.00	.04	.00	.00	.00	.00	.00	.15	.04	.00	.06	.00
Joint	1	0	100	.08	.08	.08	.08	.08	.09	.00	.00	.00	.00	.07	.06	.06	.05	.06	.08	.07	.07	.08	.03	.05
			300	.06	.06	.06	.06	.06	.06	.00	.00	.00	.00	.04	.04	.04	.04	.05	.06	.04	.04	.06	.01	.03
			1000	.05	.05	.05	.05	.05	.05	.00	.00	.00	.00	.04	.04	.04	.04	.04	.05	.04	.04	.05	.01	.03
		15	100	.00	.00	.00	.00	.00	.07	.00	.00	.00	.00	.02	.00	.00	.00	.00	.08	.07	.07	.08	.03	.05
			300	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.01	.00	.00	.00	.00	.06	.04	.04	.06	.01	.03
			1000	.00	.00	.00	.00	.00	.02	.00	.00	.00	.00	.00	.00	.00	.00	.00	.05	.04	.04	.05	.01	.03
		30	100	.00	.00	.00	.00	.00	.18	.02	.01	.01	.01	.07	.01	.00	.00	.00	.08	.07	.07	.08	.03	.05
			300	.00	.00	.00	.00	.00	.07	.00	.00	.00	.00	.02	.00	.00	.00	.00	.06	.04	.04	.06	.01	.03
			1000	.00	.00	.00	.00	.00	.04	.00	.00	.00	.00	.00	.00	.00	.00	.00	.05	.04	.04	.05	.01	.03
	5	0	100	.16	.18	.18	.18	.18	.12	.01	.00	.00	.00	.14	.14	.14	.15	.14	.16	.09	.13	.18	.05	.12
			300	.08	.09	.09	.09	.09	.08	.00	.00	.00	.00	.06	.06	.06	.07	.06	.08	.07	.06	.09	.02	.05
			1000	.07	.06	.06	.06	.06	.06	.00	.00	.00	.00	.04	.06	.04	.04	.05	.07	.05	.04	.07	.01	.04
		15	100	.00	.00	.00	.00	.00	.21	.06	.01	.01	.01	.07	.02	.00	.00	.00	.16	.09	.13	.18	.05	.12
			300	.00	.00	.00	.00	.00	.13	.00	.00	.00	.00	.03	.00	.00	.00	.00	.08	.07	.06	.09	.02	.05
			1000	.00	.00	.00	.00	.00	.07	.00	.00	.00	.00	.02	.00	.00	.00	.00	.07	.05	.04	.07	.01	.04
		30	100	.00	.00	.00	.00	.00	.56	.31	.09	.05	.06	.23	.12	.03	.01	.02	.16	.09	.13	.18	.05	.12
			300	.00	.00	.00	.00	.00	.35	.07	.01	.00	.00	.11	.01	.00	.00	.00	.08	.07	.06	.09	.02	.05
			1000	.00	.00	.00	.00	.00	.21	.01	.00	.00	.00	.06	.01	.00	.00	.00	.07	.05	.04	.07	.01	.04

Table B.14: Empirical power for the $F_j(m|0)$, $UDmax_j$ and $WDmax_j$ statistics, $j \in \{1, 0, U\}$. I(0) case, $m = 2$

				$F_1(m\|0)$					$F_0(m\|0)$					$F_U(m\|0)$					$UDmax_j$			$WDmax_j$		
Method	γ	c	$T\backslash m$	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	0	U	1	0	U
OAAT	1	0	100	.00	.00	.00	.00	.00	1.0	.99	.98	.97	.96	.99	.96	.93	.90	.90	.00	1.0	.98	.00	.99	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.5	100	.00	.00	.00	.00	.00	.50	.18	.09	.05	.06	.26	.07	.03	.02	.02	.00	.46	.24	.00	.32	.00
			300	.00	.00	.00	.00	.00	.98	.94	.89	.84	.82	.93	.85	.71	.59	.60	.00	.98	.93	.00	.95	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.8	100	.00	.00	.00	.00	.00	.06	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.05	.02	.00	.03	.00
			300	.00	.00	.00	.00	.00	.26	.05	.01	.01	.01	.11	.01	.00	.00	.00	.00	.23	.09	.00	.15	.00
			1000	.00	.00	.00	.00	.00	.86	.70	.61	.52	.51	.72	.46	.35	.27	.27	.00	.85	.71	.00	.77	.00
	5	0	100	.00	.00	.00	.00	.00	.99	.99	.98	.96	.96	.98	.98	.95	.94	.94	.00	.99	.98	.00	.98	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.5	100	.00	.00	.00	.00	.00	.95	.88	.55	.38	.42	.75	.64	.30	.18	.20	.00	.93	.72	.00	.83	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	.99	.99	1.0	1.0	.98	.97	.98	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.8	100	.00	.00	.00	.00	.00	.21	.06	.01	.01	.01	.06	.02	.00	.00	.00	.00	.18	.06	.00	.09	.00
			300	.00	.00	.00	.00	.00	.93	.78	.32	.16	.19	.63	.42	.08	.02	.03	.00	.91	.58	.00	.73	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
Joint	1	0	100	.00	.00	.00	.00	.00	1.0	.99	.98	.98	.97	.99	.97	.94	.92	.91	.00	1.0	.99	.00	.99	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.5	100	.00	.00	.00	.00	.00	.53	.21	.12	.07	.08	.29	.09	.04	.02	.03	.00	.50	.26	.00	.35	.00
			300	.00	.00	.00	.00	.00	.98	.95	.89	.84	.83	.94	.85	.73	.62	.62	.00	.98	.93	.00	.95	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.8	100	.00	.00	.00	.00	.00	.09	.01	.00	.00	.00	.03	.00	.00	.00	.00	.00	.08	.03	.00	.04	.00
			300	.00	.00	.00	.00	.00	.27	.06	.01	.01	.01	.13	.01	.00	.00	.00	.00	.25	.11	.00	.16	.00
			1000	.00	.00	.00	.00	.00	.86	.70	.62	.53	.52	.73	.48	.36	.28	.27	.00	.85	.71	.00	.77	.00
	5	0	100	.00	.00	.00	.00	.00	1.0	1.0	.99	.97	.97	1.0	.99	.95	.92	.93	.00	1.0	1.0	.00	1.0	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.5	100	.00	.00	.00	.00	.00	.95	.88	.58	.42	.47	.79	.68	.33	.22	.24	.00	.94	.75	.00	.84	.00
			300	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.96	.97	.00	1.0	1.0	.00	1.0	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00
		.8	100	.00	.00	.00	.00	.00	.31	.10	.03	.02	.02	.10	.03	.01	.00	.01	.00	.25	.09	.00	.12	.00
			300	.00	.00	.00	.00	.00	.93	.78	.33	.16	.20	.64	.43	.09	.03	.04	.00	.91	.60	.00	.73	.00
			1000	.00	.00	.00	.00	.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	.00

Table B.15: Empirical power of HLT and BP test statistics, $m = 2$

I(1) and NI(1) cases

						One-at-a-time estimation						Joint estimation					
γ	c	T	S_1	S_0	U	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b
1	0	100	.14	.04	.16	.08	.06	.06	.04	.05	.07	.08	.05	.05	.08	.06	.07
		300	.08	.01	.09	.06	.04	.04	.03	.02	.04	.06	.04	.04	.04	.03	.04
		1000	.05	.00	.06	.05	.04	.04	.02	.02	.04	.05	.04	.04	.04	.03	.04
	15	100	.05	.06	.09	.00	.00	.00	.04	.01	.01	.00	.00	.00	.06	.02	.02
		300	.00	.01	.01	.00	.00	.00	.02	.01	.01	.00	.00	.00	.03	.00	.00
		1000	.00	.00	.01	.00	.00	.00	.01	.00	.00	.00	.00	.00	.02	.00	.00
	30	100	.02	.06	.06	.00	.00	.00	.14	.05	.05	.00	.00	.00	.17	.07	.06
		300	.00	.01	.00	.00	.00	.00	.06	.02	.02	.00	.00	.00	.07	.02	.02
		1000	.00	.00	.00	.00	.00	.00	.03	.00	.00	.00	.00	.00	.04	.00	.00
5	0	100	.70	.16	.71	.17	.12	.12	.05	.09	.13	.16	.11	.11	.10	.10	.13
		300	.20	.02	.21	.08	.06	.06	.03	.03	.06	.08	.05	.05	.07	.04	.06
		1000	.07	.00	.07	.07	.04	.04	.02	.03	.04	.07	.04	.04	.05	.03	.04
	15	100	.71	.40	.74	.00	.00	.00	.12	.04	.04	.00	.00	.00	.19	.07	.07
		300	.06	.05	.10	.00	.00	.00	.08	.02	.02	.00	.00	.00	.12	.03	.03
		1000	.01	.00	.01	.00	.00	.00	.04	.01	.01	.00	.00	.00	.06	.02	.01
	30	100	.75	.68	.82	.00	.00	.00	.43	.17	.17	.00	.00	.00	.53	.22	.21
		300	.03	.15	.16	.00	.00	.00	.27	.07	.07	.00	.00	.00	.32	.11	.10
		1000	.00	.01	.01	.00	.00	.00	.17	.04	.04	.00	.00	.00	.19	.05	.05

I(0) case

						One-at-a-time estimation						Joint estimation					
γ	ρ	T	S_1	S_0	U	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b	F_1	F_1^a	F_1^b	F_0	F_U^a	F_U^b
1	0	100	.00	.13	.12	.00	.00	.00	1.0	.98	.98	.00	.00	.00	1.0	.99	.99
		300	.00	.40	.40	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.00	.99	.99	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.5	100	.01	.07	.07	.00	.00	.00	.48	.26	.26	.00	.00	.00	.51	.28	.28
		300	.00	.05	.05	.00	.00	.00	.98	.93	.93	.00	.00	.00	.98	.94	.94
		1000	.00	.28	.27	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.8	100	.04	.06	.08	.00	.00	.00	.06	.03	.03	.00	.00	.00	.08	.03	.03
		300	.00	.01	.01	.00	.00	.00	.24	.11	.11	.00	.00	.00	.25	.13	.13
		1000	.00	.01	.01	.00	.00	.00	.85	.72	.72	.00	.00	.00	.85	.73	.73
5	0	100	.86	.99	.99	.00	.00	.00	.99	.98	.98	.00	.00	.00	1.0	1.0	1.0
		300	.46	.98	.98	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.04	.95	.95	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.5	100	.81	.94	.95	.00	.00	.00	.94	.75	.75	.00	.00	.00	.95	.78	.78
		300	.21	1.0	1.0	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
		1000	.00	1.0	1.0	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0
	.8	100	.72	.49	.77	.00	.00	.00	.20	.06	.06	.00	.00	.00	.27	.10	.10
		300	.02	.61	.61	.00	.00	.00	.92	.62	.62	.00	.00	.00	.92	.63	.63
		1000	.00	.99	.99	.00	.00	.00	1.0	1.0	1.0	.00	.00	.00	1.0	1.0	1.0

Table B.16: Frequency of estimated number of structural breaks. Union statistics, $m = 2$

I(1) and NI(1) cases

			One-at-a-time estimation									Joint estimation					
			U			F_U^a			F_U^b			F_U^a			F_U^b		
γ	c	$T\backslash m$	1	2	>2	1	2	>2	1	2	>2	1	2	>2	1	2	>2
1	0	100	.14	.01	.01	.05	.00	.00	.07	.00	.00	.05	.00	.00	.07	.00	.00
		300	.09	.00	.00	.02	.00	.00	.04	.00	.00	.02	.00	.00	.04	.00	.00
		1000	.05	.00	.00	.02	.00	.00	.04	.00	.00	.03	.00	.00	.04	.00	.00
	15	100	.06	.02	.01	.01	.00	.00	.01	.00	.00	.02	.00	.00	.02	.00	.00
		300	.01	.00	.00	.01	.00	.00	.01	.00	.00	.00	.00	.00	.00	.00	.00
		1000	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
	30	100	.05	.01	.01	.05	.00	.00	.05	.00	.00	.06	.01	.00	.06	.01	.00
		300	.00	.00	.00	.02	.00	.00	.02	.00	.00	.02	.00	.00	.02	.00	.00
		1000	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	0	100	.45	.23	.01	.07	.02	.00	.13	.00	.00	.07	.03	.00	.11	.03	.00
		300	.20	.01	.00	.03	.00	.00	.06	.00	.00	.03	.01	.00	.05	.01	.00
		1000	.07	.00	.00	.02	.00	.00	.04	.00	.00	.02	.00	.01	.03	.01	.00
	15	100	.42	.30	.01	.03	.01	.00	.03	.01	.00	.03	.04	.00	.02	.04	.00
		300	.09	.01	.00	.01	.00	.00	.01	.00	.00	.03	.01	.00	.03	.01	.00
		1000	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00	.01	.00	.00
	30	100	.38	.41	.01	.01	.16	.00	.01	.15	.00	.01	.20	.01	.01	.19	.01
		300	.14	.02	.00	.06	.02	.00	.06	.02	.00	.06	.05	.00	.06	.04	.00
		1000	.01	.00	.00	.04	.00	.00	.04	.00	.00	.04	.01	.00	.04	.01	.00

I(0) case

			One-at-a-time estimation									Joint estimation					
			U			F_U^a			F_U^b			F_U^a			F_U^b		
γ	ρ	$T\backslash m$	1	2	>2	1	2	>2	1	2	>2	1	2	>2	1	2	>2
1	0	100	.11	.01	.00	.58	.40	.00	.58	.40	.00	.34	.31	.34	.34	.31	.33
		300	.35	.04	.00	.00	.98	.02	.00	.98	.01	.00	.16	.84	.00	.16	.84
		1000	.12	.87	.00	.00	1.0	.00	.00	1.0	.00	.00	.14	.86	.00	.14	.86
	.5	100	.05	.01	.00	.25	.01	.00	.24	.01	.00	.23	.04	.02	.22	.04	.02
		300	.05	.00	.00	.72	.21	.00	.72	.21	.00	.49	.26	.19	.49	.26	.19
		1000	.26	.01	.00	.01	.98	.02	.01	.98	.02	.00	.22	.78	.00	.22	.78
	.8	100	.06	.01	.00	.03	.00	.00	.03	.00	.00	.02	.01	.00	.02	.01	.00
		300	.01	.00	.00	.11	.00	.00	.10	.00	.00	.09	.02	.01	.09	.03	.01
		1000	.01	.00	.00	.69	.03	.00	.69	.03	.00	.54	.12	.07	.54	.12	.06
5	0	100	.01	.93	.00	.00	.98	.00	.00	.98	.00	.01	.78	.21	.01	.78	.21
		300	.02	.95	.00	.00	1.0	.00	.00	1.0	.00	.00	.67	.33	.00	.67	.33
		1000	.01	.94	.00	.00	1.0	.00	.00	1.0	.00	.00	.53	.47	.00	.53	.47
	.5	100	.19	.72	.00	.00	.74	.01	.00	.73	.01	.00	.73	.05	.00	.72	.05
		300	.01	.98	.00	.00	1.0	.00	.00	1.0	.00	.00	.88	.12	.00	.88	.12
		1000	.00	.99	.00	.00	1.0	.00	.00	1.0	.00	.00	.78	.22	.00	.78	.22
	.8	100	.42	.32	.01	.03	.04	.00	.03	.04	.00	.02	.07	.00	.02	.07	.00
		300	.40	.21	.00	.06	.56	.00	.06	.55	.00	.04	.57	.02	.04	.57	.02
		1000	.12	.87	.00	.00	1.0	.00	.00	1.0	.00	.00	.88	.12	.00	.88	.12

Supplementary material
Appendix C. Empirical illustration: Figures, tables and purchasing power parity hypothesis testing

C Empirical illustration

This section provides the pictures of the time series that are analized and the figures (hit maps) that summarize the estimated break dates for the PWT database. We include the discussion about testing the unit root hypothesis on RER, which is usually implemented in international economics as a way to test the (quasi) PPP hypothesis. The results also include the estimation of half-life of shocks.

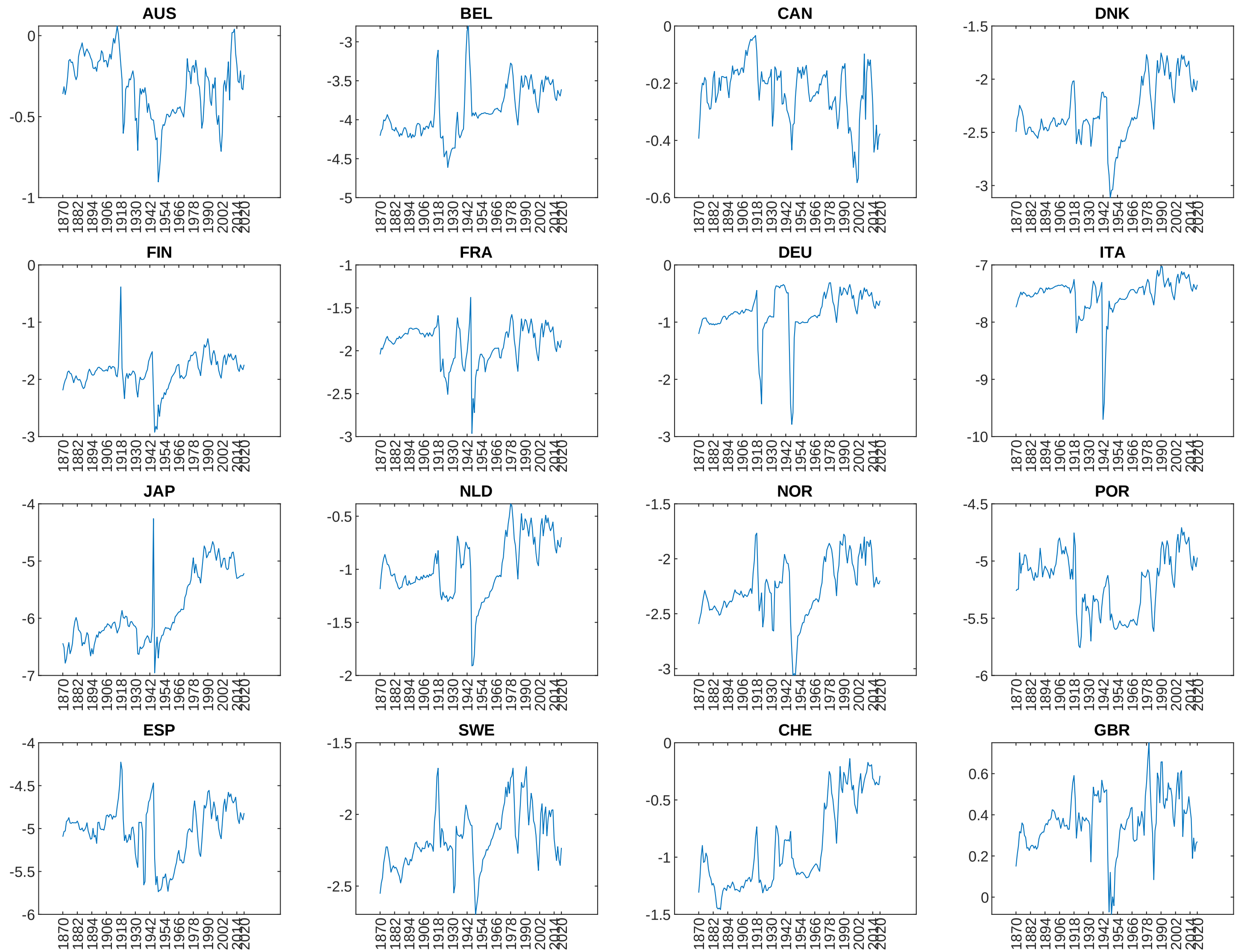


Figure C.1: Real exchange rates, JST database

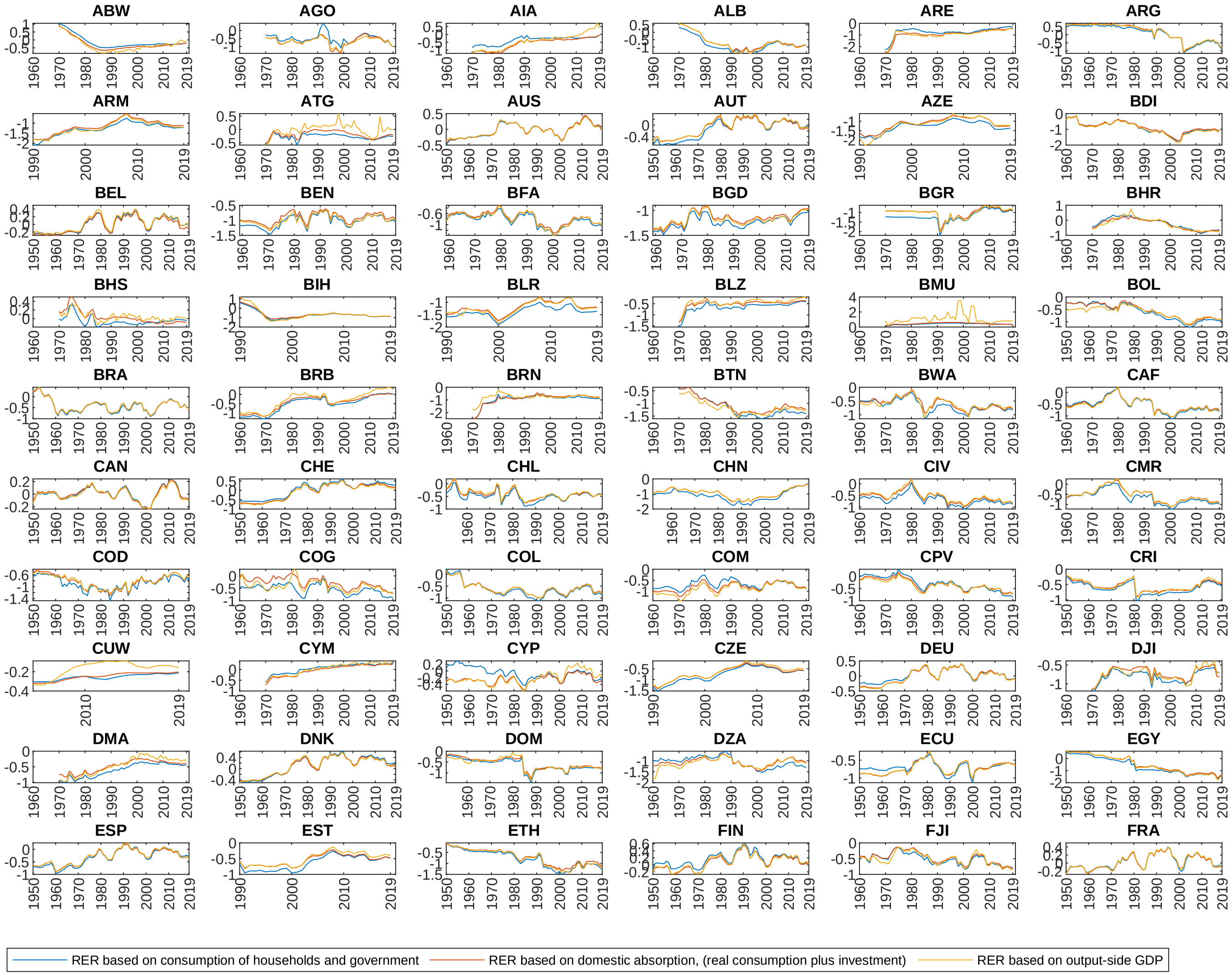
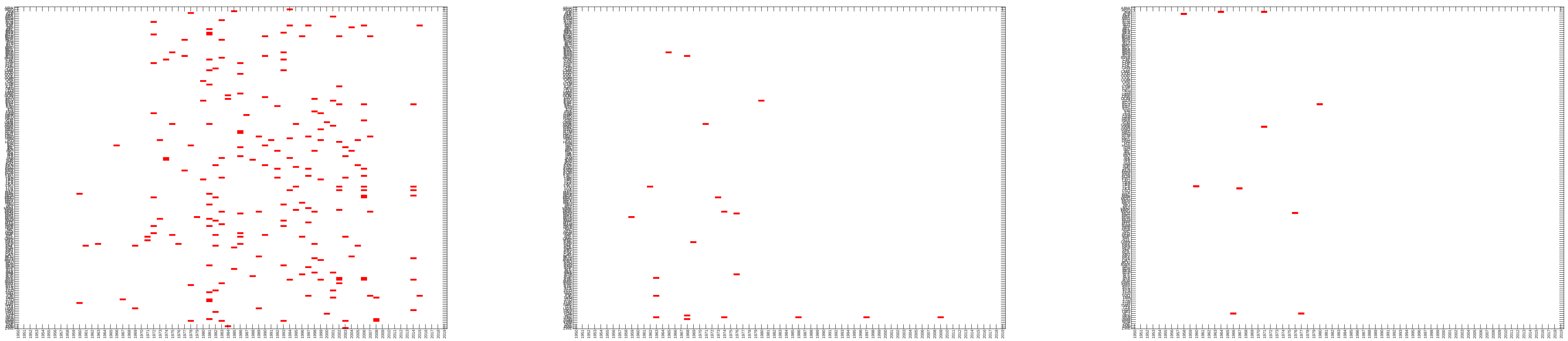


Figure C.2: Real exchange rates, PWT database

Figure C.2: Real exchange rates, PWT database

Figure C.2: Real exchange rates, PWT database



(a) HLT

(b) BP (one-at-a-time)

(c) BP (joint)

Figure C.3: Dates of breaks, RER based on consumption of households and government

C.1 Testing the PPP hypothesis

This part of the empirical application goes beyond the illustration of the statistical procedures that have been proposed in the paper, although we consider that it might be of interest to test the PPP hypothesis for the databases that have been collected. To do so, we have computed the ADF test statistic, which null hypothesis is the existence of a unit root against the alternative hypothesis of mean-reverting RER. The structural breaks robust analysis described in this paper, performed using either the HLT or BP approaches, determines the specification of the type of ADF regression equation that needs to be estimated. When no structural break is detected, the standard ADF statistic is computed. When evidence of structural breaks is found, the ADF regression equation is modified to include the detected structural breaks, as suggested by Perron (1990,1989) – note that we use the additive outlier specification:

$$\begin{aligned} q_{n,t} &= \mu_n + \sum_{i=1}^{m_n} \gamma_{n,i} DU_{n,i,t} + u_{n,t} && \text{(C.8)} \\ \Delta u_{n,t} &= \sum_{i=1}^{m_n} \sum_{j=0}^{k_n} \theta_{n,i,j} D\left(T_{n,i}\right)_{t-j} + \alpha_n u_{n,t-1} + \sum_{j=1}^{k_n} \delta_{n,j} \Delta u_{n,t-j} + \varepsilon_{n,t}, && \text{(C.9)} \end{aligned}$$

where $DU_{n,i,t} = 1$ for $t > T_{n,i}$, 0 otherwise, and $D\left(T_{n,i}\right)_t = 1$ for $t = T_{n,i} + 1$, 0 otherwise, $n = 1, \ldots, N$. In those cases for which no structural breaks have been detected $\gamma_{n,i} = \theta_{n,i,j} = 0 \ \forall i, j$ in (C.8) and (C.9). Critical values are computed by simulation, taking into account the specific sample size, the vector of m_i structural breaks and the number of lags of the parametric correction (k_n) in the ADF regression equation is selected using the BIC with a maximum of $k_{\max} = \left\lfloor 4(T/100)^{1/4} \right\rfloor$ lags.

As is customary in the literature, it is also of interest to measure the shock persistence of RER. Usually, persistence is measured by computing impulse-response functions (IRFs), half lives (HLs) and cumulative impulse-response functions (CIR). The IRF measures the effect of a shock of size one at time t on h future values of the variable of interest. Following Andrews and Chen (1994), IRFs functions can be calculated from the infinite-order moving average representation of an autoregressive process of order p_n for

$q_{n,t}$ from $q_{n,t} = (1-\gamma_{n,1}L-\cdots-\gamma_{n,p_n}L^{p_n})^{-1}\varepsilon_{n,t} = \sum_{h=0}^{\infty} c_{n,h}\varepsilon_{n,t-h}$, so that $IRF_n(h) = c_{n,h}$ for $h = 0, 1, ...$, where L is the lag operator, $n = 1, \ldots, N$. From this expression, it is straightforward to define other popular measures of shock persistence such as the half life, defined as the number of periods that it takes until half the effect of a shock dissipates, and the cumulated impulse response $CIR_n = \sum_{h=0}^{\infty} IRF_n(h)$, which measures the total cumulative effect of a shock over time. This is a scalar measure of persistence that summarizes the information contained in the IRF.

C.1.1 Simulation experiment

As mentioned above, this final part of the empirical application is beyond the statistical procedures that have been designed in the paper, and its validity might depend on the ability of the different statistics to detect the presence of structural breaks to obtain meaningful conclusions about the PPP hypothesis compliance. In order to address this issue, we have conducted a small scale simulation experiment to assess the performance of this analysis. The DGP is given in (26), with $\rho \in \{0.84, 1\}$, $\gamma \in \{0, 1, 5\}$, $\lambda_{B,1}^{0} = 0.5$, $T = 150$ and $m_{\max} = 5$ – the value of $\rho = 0.84$ corresponds to the mean of the estimated parameters in the empirical results for the historical and PWT databases. The rest of the specification of the simulation experiment is defined in Section 5. The experiment allows us to investigate different issues. First, we analyze the results of ignoring the presence of structural breaks in the computation of the ADF unit root test statistic. In case that the null hypothesis of unit root is rejected – i.e., evidence of PPP is found – we proceed to compute the HL of a shock following the procedure described above. Second, we conduct the robust structural break analysis that has been described in this paper using the U, F_U^b (OAAT) and F_U^b (joint) statistics.[20] Depending on the outcome of these statistics, the standard ADF (without structural breaks) or Perron's (1990) ADF (with multiple level shifts) unit root test statistics are computed. If the null hypothesis of unit root is rejected, the HL of shocks is obtained as described above.

Results in Table C.1 show that when the time series are I(1), the rejection rates of the

[20]The results obtained with the $F_d(m|0)$ and the $UDmax_U$ statistics do not substantially modify the conclusions.

ADF statistic are close to the nominal size of 5% when the structural break is ignored. This result is in accordance with the theory, since fixed break magnitudes have negligible effects in the limit. This is also found for the BP-based ADF statistic, whereas for the HLT-based one we observe a mild over-size distortion – the empirical size is 0.069 for HLT-based ADF, 0.061 for the BP (OAAT) based ADF and 0.065 for the BP (joint) based ADF statistics. Therefore, we can conclude that prior information about the presence of structural breaks does not affect the finite sample performance of the ADF, especially if the BP statistics are used. As for the HL estimates that are obtained when the unit root is rejected, the model that ignores the structural breaks always produces the largest estimates, followed by the BP-based ones and, finally, the HLT-based ones – note that the true HL is infinite. Knowledge about the presence of structural breaks is relevant when the time series are I(0), since unaccounted structural breaks decrease the empirical power of the ADF statistic. As can be seen, the HLT and BP based ADF unit root test statistics show similar performance for $\gamma = 1$, although the HLT and BP (joint) based ADF unit statistics outperform the BP (OAAT) based ADF ones when the break magnitude increases to $\gamma = 5$ – in this case the larger rejection rates shown by the HLT-based ADF statistics might be due to the over-size distortions discussed above. Finally, the HL estimates that are computed without considering the possibility of structural breaks tend to over-estimate the true HL as the magnitude of the structural break increases, whereas the ones that are based on the robust structural break analysis show mild under-estimation biases regardless of the break magnitude.

In all, we can conclude that the prior information that is obtained from the robust structural breaks analysis can be helpful to test the PPP hypothesis and measure shock persistence.

C.1.2 Empirical results

Let us first focus on the countries from the historical time series provided by Jordà, Schularick and Taylor (2018). The PPP holds in 13 out of 16 countries with HLT structural-breaks-based-results, and in 14 (OAAT) or 13 (joint) out of 16 countries with the BP structural-breaks-based ones – the degree of coincidence between both methods is 81% (OAAT) and 88% (joint). The empirical evidence in favor of the PPP is much weaker with the PWT database, although there are also no large differences between the use of one or another method of structural breaks detection. With RER-C we only do not reject PPP compliance in 43 (U) and 35 (F_U^b, OAAT) out of 180 countries, and in 33 (F_U^b, joint) out of 157 countries, with coincidence ratios of 77% (OAAT) and 73% (joint). With RER-A the figures are 49 (U), 31 (F_U^b, OAAT) and 33 (F_U^b, joint), with coincidence ratios of 78% (OAAT) and 79% (joint). Finally, with RER-O the figures are 48 (U), 38 (F_U^b, OAAT) and 36 (F_U^b, joint), with coincidence ratios of 79% (OAAT) and 72% (joint).

In spite of the great differences found in the number and position of the structural breaks with both methods, this fact does not seem to have a great influence on the PPP compliance. The greater number of structural breaks found with the HLT method might be explained by the size distortions in finite samples detected in the simulations above. But although this implies introducing a greater number of parameters in the ADF specification, this does not seem to have reduced the power of the test statistic. In summary, while with historical data, PPP compliance is broad (around 81%), in the case of PWT database it is much smaller (around 25%). It should be noted that although the PWT database theoretically runs from 1950 to the present, for many countries the available information defines shorter samples, starting in 1970 or even later in some cases. The smaller sample size reduces the possibilities of finding support for PPP due to the low power of the unit root test tests in finite samples and, also, because the highly persistent RER needs time to return to its level of equilibrium. In fact, since Rogoff (1996) called attention to the so-called "Rogoff puzzle" a broad body of literature has focused on studying the persistence of RER with more interest than the PPP compliance itself. According to these studies, half lives of deviations from parity usually fall in the range of 3 to 5 years.

Results available upon request show that for the historical database, the average HL with HLT-based-results is 4.24, 4.37 with BP (OAAT) and 4.12 with BP (joint). For the PWT database these values are 2.81/3.77/3.76, 2.55/3.84/3.81 and 2.16/2.73/2.75 for

RER-C, RER-A and RER-O, respectively. Therefore, we find very similar values with the different databases and methods, which offer shocks persistence estimates that lay within the range of values of the so-called "Rogoff puzzle".

Table C.1: Results of ADF experiment

		I(1)					I(0)				
				Structural breaks					Structural breaks		
	γ	True	No breaks	HLT	BP(O)	BP(J)	True	No breaks	HLT	BP(O)	BP(J)
PPP	0		0.060	0.069	0.061	0.065		0.960	0.959	0.960	0.960
	1		0.059	0.072	0.060	0.063		0.923	0.920	0.925	0.924
	5		0.047	0.090	0.053	0.059		0.098	0.372	0.266	0.330
HL	0	∞	7.040	6.659	6.969	6.715	3.728	3.591	3.572	3.584	3.585
	1	∞	6.997	6.376	6.918	6.715	3.395	3.767	3.726	3.736	3.727
	5	∞	6.804	5.317	6.351	5.592	3.377	5.446	3.376	3.104	2.909

Notes: The columns labelled as BP(O) and BP(J) denote the results that are based on the BP statistics computed using the one-at-a-time and the joint break dates estimation strategies, respectively