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GRAU DE MATEMÀTIQUES

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The De Rham's Theorem

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Realitzat a: Departament de Matemàtiques i Informàtica
Barcelona, 13 de juny de 2023

Abstract

The main goal of this work is to prove the De Rham Theorem and highlight its meaning and relevance. It will provide the reader with the necessary concepts to prove the De Rham Theorem. To do that, this work presents a brief but comprehensive introduction to both homology and cohomology theory as well as to smooth manifolds.

2020 Mathematics Subject Classification. 14F40

Acknowledgements

First of all, I want to express my gratitude to my advisor, Dr. Javier José Gutiérrez for his guidance and suggestions. I am thankful to him for helping me structure this work, and writing it with the appropriate formality and meaningful layout. I would also like to thank my family and friends for their support during the endeavour of writing this work. With a special appreciation to all of those who read this work and noticed small errors and misprints. I also have to thank my professors in the courses "Topologia i geometria global de corbes i superficies" and "Geometria diferencial" since I gained knowledge to comprehend smooth manifolds and I used many of the notes I took during those classes to understand the basics of this work.

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Chapter 1

Introduction and motivation

Smooth manifolds are one of the mathematical structures most related to physics due to the intrinsic nature of space-time. In the many areas of study of smooth manifolds, this work revolves around a single theme, the De Rham Cohomolgy and one of its central results, the De Rham's theorem. Despite the many applications in electromagnetism and gauge theories in modern day particle physics, this work is purely mathematical and it will focus its scope in the mathematical tools to prove the Theorem. However in the last chapter, we will provide an intuitive notion of what the De Rham's theorem means, and we will also discuss its relevance in physics. First of all an introduction to homology and cohomology will be made to give us the homological tools for the theorem. Afterwards, an introduction to smooth manifolds will follow so we can combine differential forms and homological algebra. In the last chapter the work will present the De Rham Cohomology and a set of results needed to prove the theorem and the proof itself. The stimulus for the theorem will become more apparent but for now let us give the reader some motivation.

First of all, one defines a quocient group of closed forms modulus exact forms. This is the so called De Rham Cohomology. Studying these groups one finds a very remarkable result: De Rham cohomology groups are topological invariants (in fact they are homotopic invariants). The same De Rham groups are obtained for different differentiable structures, depending only on the topological manifold. This result leads to the following question: can one compute the De Rham cohomology of a smooth manifold in a purely topological way?. The De Rham theorem answers precisely that question. It establishes an isomorphism between the singular cohomology groups and the De Rham cohomology groups. In conclusion, this work aims to study a powerful theorem involving many branches of mathematics, relating topological and geometrical properties to algebraic structures. We will not go in detail in some aspects that could be interesting and we will skip proofs since they can easily be found in the bibliography and would require a whole book to be complete. All in all, it is a physics-inspired work but studied purely from the mathematical side, used to expand the knowledge of content not taught during the Mathematics degree.

Chapter 2

Introduction to homology and cohomology

Before getting in depth with cohomology of differential forms it is useful to introduce the basics of homology and cohomology.

2.1 Simplicial homology

Definition 1. Let $n \geq 0$ be a natural number. The (geometric) n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the convex hull of the standard basis vectors of $\mathbb{R}^{n+1}$ endowed with the subspace topology.

Let us denote these standard basis vectors by $e_0, \ldots, e_n$. Every point $v \in \Delta^n$ can uniquely be written as a convex linear combination of the e_i, i.e., there is a unique expression

$$v = \sum_{i=0}^{n} t_i e_i, \quad t_i \geq 0, \quad t_0 + \cdots + t_n = 1.$$

The coordinates t_i are the barycentric coordinates of the point v. Thus, to be completely specific, we have

$$\Delta^n = \left\{ (t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0,\, t_0 + \cdots + t_n = 1 \right\}.$$

Recall that a convex linear map is a map which sends convex linear combinations to convex linear combinations. It follows that a convex linear map

$$\alpha : \Delta^n \to \Delta^m$$

is uniquely determined by its values on $e_i \in \Delta^n$ for $i = 0, \ldots, n$.

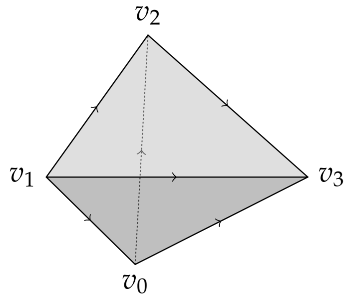

Figure 2.1: Representation of a 3-simplex Δ^3

Definition 3. Let X be a topological space.

(1) A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$.

(2) The singular n-chain group $C_n(X)$ is the free abelian group generated by the singular n-simplices in X. Its elements are called singular n-chains in X.

Let us recall the notion of a free abelian group generated by a set. As a motivation for the concept we include the following reminder.

Reminder 4. Let V be a finite-dimensional vector space with basis $b_1, \ldots, b_n \in V$ and let W be a further vector space over the same field. Then a linear map $f : V \to W$ is uniquely determined by the values $f(b_1), \ldots, f(b_n) \in W$.

Definition 5. Let S be a set. A free abelian group generated by S is a pair $(F(S), i_S)$ consisting of an abelian group $F(S)$ and a map of sets $i_S : S \to F(S)$ which satisfies the following universal property: Given a further pair $(A, j : S \to A)$ with A an abelian group and j a map of sets, then there is a unique group homomorphism $f : F(S) \to A$ such that $f \circ i_S = j$.

2.2 Singular homology

Recall that we have the i-th face map $d_i : \Delta^{n-1} \to \Delta^n$ for $0 \leq i \leq n$. Given a singular n-simplex $\sigma : \Delta^n \to X$ in a space X, we obtain a singular $(n-1)$-simplex $d_i(\sigma)$ in X by setting:

$d_i(\sigma) = \sigma \circ d_i : \Delta^{n-1} \xrightarrow{d_i} \Delta^n \xrightarrow{\sigma} X, 0 \leq i \leq n.$

By linear extension, this gives rise to a group homomorphism

$d_i : C_n(X) \to C_{n-1}(X), 0 \leq i \leq n,$

which will also be called the i-th face map. We proceed to give some essential definitions.

Definition 2.1. *Let X be a topological space.*

(1) The n-th singular boundary operator ∂ is given by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : C_n(X) \to C_{n-1}(X).$$

(2) The kernel $Z_n(X)$ of the boundary operator $\partial : C_n(X) \to C_{n-1}(X)$, i.e., the abelian group

$$Z_n(X) = ker(\partial : C_n(X) \to C_{n-1}(X)),$$

is the group of singular n-cycles in X. An element of $Z_n(X)$ is sometimes also referred to as a closed singular n-chain.

(3) The image $B_n(X)$ of the boundary operator $\partial : C_{n+1}(X) \to C_n(X)$, i.e., the abelian group

$$B_n(X) = im(\partial : C_{n+1}(X) \to C_n(X)),$$

is the group of singular n-boundaries in X.

These definitions state that a singular p-chain c is called a *cycle* if $\partial c = 0$, and a *boundary* if $c = \partial c'$ for some $(p+1)$-chain c'. We would like the reader to give special attention to this two objects. This is so later in the work the relationship between *cycles* and *boundaries* and what will be called *closed forms* and *exact forms* respectively, becomes apparent.
Since $B_n(X) \subset Z_n(X)$ we can define the nth homology group of X as the following quotient group

$$H_n(X) = Z_n(X)/B_n(X).$$

To every quocient group, there are equivalence classes. The equivalence classs in $H_p(X)$ of a singular p-cycle c is called the homology class of c and as usual is denoted $[c]$.

Definition 2.2. *A $chain\,complex$ is defined as a sequence of $\mathcal{A}$-modules and $\mathcal{A}$-linear maps, where $\mathcal{A}$ is a commutative ring,*

$$\cdots \to C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \to \cdots .$$

Definition 2.3. *Let M be a topological space, the sequence of abelian groups and homomorphisms*

$$\cdots \to C_{p+1}(M) \xrightarrow{\partial} C_p(M) \xrightarrow{\partial} C_{p-1}(M) \to \cdots .$$

is called the singular chain complex.

Proposition 2.4. *(Properties of Singular Homology Groups)*
(a) For any one-point space $\{q\}$, $H_0(\{q\})$ is the infinite cyclic group generated by the homology class of the unique singular 0-simplex mapping Δ_0 to q, and $H_p(\{q\}) = 0$ for all $p \neq 0$.

(b) Let $\{M_j\}$ be any collection of topological spaces, and let $M = \bigcup_j M_j$. The inclusion maps $i_j : M_j \to M$ induce an isomorphism $\bigoplus_j H_p(M_j) \xrightarrow{\sim} H_p(M)$.

(c) Homotopy equivalent spaces have isomorphic singular homology groups.

Full proof of these properties can be found in chapter 16 of [2]

2.3 Cellular homology

Now, we discuss another kind of homology theory which, as we will see in the last theorem, will provide isomorphic groups to singular homology. This is very useful when dealing with explicit calculations.

First of all we need to define a new type of spaces, called CW-complexes. We will not go in depth, we are only going to present the key fundamentals of this complexes to understand cellular homology. We will follow the introduction in chapter IV of the Glen E Bredon book. [1].

A CW-complex is a space made up of "cells" attached in a nice way. The "C" in "CW" stands for "closure finite," and the "W" stands for "weak topology." It is possible to define these spaces intrinsically, but we prefer to do it by describing the process by which they are constructed. For the most part, we will be concerned only with "finite" complexes, meaning complexes having a finite number of cells, but we shall give the definition in general.

Let $K^(0)$ be a discrete set of points. These points are the 0-cells. If $K^{(n-1)}$ has been defined, let $\{f_{\partial\sigma}\}$ be a collection of maps $f^{n-1}_{\partial\sigma} : S^{n-1} \to K^{(n-1)}$, where σ ranges over some indexing set. Let Y be the disjoint union of copies D^n_σ of D^n, one for each σ, let B be the corresponding union of the boundaries S^{n-1}_σ of these disks, and put together the maps $f_{\partial\sigma}$ to produce a map $f : B \to K^{(n-1)}$. Then define

$$K^(n) = K^{(n-1)} \cup_f Y.$$

The map $f^{n-1}_{\partial\sigma}$ is called the "attaching map" for the cell σ. We refer to $K^{(n-1)} \cup_f Y$ as being obtained fron Y by attaching an n-cell.

If $K^(n)$ has been defined for all integers $n \geq 0$, let $K = \bigcup_n K^(n)$ with the "weak" topology that specifies that a set is open if and only if its intersection with each $K^(n)$ is open in $K^(n)$. (It follows that a set is closed if and only if its intersection with each $K^(n)$ is closed.)

For each σ, let $f_\sigma : D^n_\sigma \to K$ be the canonical map given by the attaching of the cell σ. This map is called the "characteristic map" of the cell σ. Let K_σ be the image of f^0_σ. This is a compact subset of K which will be called a "closed cell" (note, however, that this is generally not homeomorphic to D^n since there are identifications on the boundary). Denote by U_σ the image in K of the open disk $D^n_\sigma - S^{n-1}_\sigma$. This is homeomorphic to the interior of the standard n-disk (i.e., to $\mathbb{R}^n$). We shall refer to U_σ as an "open cell," but remember that this is usually not an open subset of K. It is open in $K^(n)$.

It is clear that the topology of each $K^(n)$, and hence of K itself, is characterized by the statement that a subset is open (closed) if and only if its inverse image under each f_σ is open (closed) if and only if its intersection with each K_σ is open (closed) in K^0_σ, where the topology of the latter is the topology of the quotient of D^n by the identifications made by $f_{\partial\sigma}$

We can as well define a *subcomplex*, which is simply a union of some of the closed cells which is itself a CW-complex with the same attaching maps.

Proposition 2.5. *(Properties of CW-complexes) If K is a CW-complex, then the following statements all hold:*

(1) If $A \subseteq K$ has no two points in the same open cell, then A is closed and discrete.

(2) If $C \subseteq K$ is compact, then C is contained in a finite union of open cells.

(3) Each cell of K is contained in a finite subcomplex of K.

For the computation of singular homology groups through cellular homology groups, we need to find an isomorphism between these two. This is expressed in the following theorem.

Theorem 2.6. *Let X be a CW complex. The abelian groups $C^\bullet_{cell}(X)$ can be turned into a chain complex, the homology of which is isomorphic to the singular homology $H_n(X)$ of X.*

We will not give proof of this theorem but the reader may find it in [6]

This theorem provides us with the possibility to compute the singular homology group of the n-sphere. We will do so by computing its cellular homology group.

The n-dimensional sphere $\mathbb{S}^n$ admits a CW structure with two cells: one 0-cell and one n-cell. The n-cell is attached by the constant mapping from S^{n-1} to the 0-cell.

Since the generators of the cellular chain groups $C_k(S^(_K n), S^n_{k-1})$ can be identified with the k-cells of $\mathbb{S}^n$, we have that $C_k(S^(_K n), S^n_{k-1}) = \mathbb{Z}$ for $k = 0, n$, and is otherwise trivial.

Hence, for $n > 1$, the resulting chain complex is:

$$\cdots \xrightarrow{\partial_{n+2}} 0 \xrightarrow{\partial_{n+1}} \mathbb{Z} \xrightarrow{\partial_n} 0 \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{Z} \to 0,$$

We can see that all the boundary maps are either to or from trivial groups, they must all be zero. This means that the cellular homology groups are equal to:

$$H_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n, \\ \{0\}, & \text{otherwise.} \end{cases}$$

2.4 Cohomology groups

In addition to the singular homology groups, for any topological space M and any abelian group G, one can define a closely related sequence of groups $H^p(M; G)$ called the singular cohomology groups with coefficients in G. The precise definition is unimportant for our purposes; we are only concerned with the special case $G = \mathbb{R}$, in which case it can be shown that $H^p(M; \mathbb{R})$ is a real vector space that is naturally isomorphic to the space $\mathrm{Hom}(H_p(M; \mathbb{R}), \mathbb{R})$. (For simplicity, let us take this as our definition of $H^p(M; \mathbb{R})$.) Any continuous map $F : M \to N$ induces a linear map $F^* : H^p(N; \mathbb{R}) \to H^p(M; \mathbb{R})$, defined by $F^*([\alpha]) = [F^*\alpha]$ for each $[\alpha] \in H^p(N; \mathbb{R})$ and each singular p-chain α in M. The functorial properties of F^* carry over to cohomology: $(G \circ F)^* = F^* \circ G^*$ and $(\mathrm{Id}_M)^* = \mathrm{Id}_{H^p(M;\mathbb{R})}$. It follows that p-th singular cohomology with coefficients in $\mathbb{R}$ defines a contravariant functor from the topological category to the category of real vector spaces and linear maps.

There is an important theorem of algebraic topology called the universal coefficient theorem, which shows how the singular cohomology groups with coefficients in an arbitrary group can be recovered from the singular homology groups. Thus, the cohomology groups besides their set having an algebra structure,they also organize the information in it in a different way that is more convenient for many purposes. In particular, the fact that the singular cohomology groups, like the De Rham cohomology groups, define contravariant functors makes it much easier to compare the two.

Proposition 2.7. *(Properties of Singular Cohomology).*

(a) For any one-point space $\{q\}$, $H^p(\{q\}; \mathbb{R})$ is trivial except when $p = 0$, in which case it is 1-dimensional.

(b) *If M_j is any collection of topological spaces and $M = \coprod_j M_j$, then the inclusion maps $i_j : M_j \to M$ induce an isomorphism from $H^p(M;\mathbb{R})$ to $\oplus_j H^p(M_j;\mathbb{R})$.*

(c) *Homotopy equivalent spaces have isomorphic singular cohomology groups.*

This properties can be deduced from the properties of Singular Homology in proposition 2.4.

The De Rham theorem establishes an isomorphism between this singular cohomology and the De Rham cohomology that we will define later in this work.

Chapter 3

Smooth manifolds and differential forms

Up to this point we have made an introduction to the homological side of the De Rham cohomology. In this chapter we will make a brief introduction to the differential geometry side of it. We need to discuss differential forms and from them, exact and closed forms in order to define de De Rham cohomology. This chapter requires a background in topology that the not familiarised reader can find in [5]

Definition 3.1. *Let M be a topological space. An atlas of class C^∞ and dimension m is a collection of pairs $\{(U_i, \varphi_i)\}_{i\in I}$, where U_i is an open set in M, and for all $i \in I$, φ_i is a homeomorphism from U_i to an open subset of $\mathbb{R}^m$. The following conditions are satisfied:*

i) $M = \bigcup_{i\in I} U_i$.

ii) For all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a C^∞ map between the two indicated open sets of $\mathbb{R}^m$.

Each pair (U_i, φ_i) is called a local chart of the atlas.

If we further require that $\varphi_j \circ \varphi_i^{-1}$ is a C^k differentiable map for all i, j, we will refer to it as a C^k atlas of dimension m. Similarly, if these functions are demanded to be analytic, we have an analytic atlas.

From now on, all considered atlases will be C^∞ atlases.

Definition 3.2. *Let $\mathcal{A}$ be a differentiable atlas in a topological manifold M of dimension m. A maximal atlas is an atlas $\mathcal{A}$ such that there is no other atlas $\mathcal{B}$ satisfaying $\mathcal{A} \subset \mathcal{B}$ If $\mathcal{A}$ is maximal we say $\mathcal{A}$ is a smooth structure in M. We say that the pair (M,$\mathcal{A}$) is a smooth manifold of dimension m.*

Definition 3.3. *(Differentiable Map between Manifolds)Let $f : M \to N$ be a continuous map between differentiable manifolds M and N. We say that f is differentiable if for every chart (U, φ) of M and (V, ψ) of N, the map $\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$ is differentiable.*

Now let us define some concepts that will be needed for a proof in the Stokes Theorem. A very remarkable result regarding integration. A $domain\ of\ integration$ is a bounded subset of $\mathbb{R}^n$ whose boundary has n-dimensional measure zero.
Let M be a topological n-manifold with boundary. A *chart with corners* for M is a pair (U, ϕ), where U is an open subset of M and ϕ is a homeomorphism from U to a subset of $\bar{\mathbb{R}}^n_+$ conitaining part of its boundary. A *smooth structure with corners* on a topological manifold with boundary is a maximal collection of smoothly compatible charts with corners whose domains cover. We can now define a $smooth\ manifold\ with\ corners$ which is a topological manifold with boundary with a smooth structure.

Proposition 3.4. *Let N be the boundary of a a compact, oriented, smooth n-manifold with corners M.Let $E_1, \ldots, E_k$ be copact domains of integration in M; $D_1, \cdots, D_k$ are compact domains of integration in $\mathbb{R}^n$; and for $i = 1, \cdots, k$, $F_i \colon D_i \to M$ are smooth maps satisfying $i) F_i(D_i) = E_i and F_i|_{int D_i} is an orientation-preserving diffeomorphism from Int D_i onto Int E_i$ $ii) For each i \neq j, E_i and E_j intersect only on their boundaries.$*

Then for any n-form ω on M whose support is contained in $E_1 \cup \ldots \cup E_k$,

$$\int_M \omega = \sum_i \int_{D_i} F_i^* \omega \tag{3.1}$$

This is enough as far as integration is concerned.

We now define a special kind of function. Despite seeming a bit arbitrary at this point, it will be necessary in the proof of the De Rham Theorem.

Definition 3.5. *If M is a topological space, an exhaustion function for M is a continuous function $f : M \to \mathbb{R}$ with the property that the set $M_c = \{x \in M : f(x) \leq c\}$ is compact for each c.*

Proposition 3.6. *(Existence of Exhaustion functions) Every smooth manifold admits a smooth positive exhaustion function.*

Everything we need is this proposition. The proof requires some work with partitions of unity that extend more the content of this work and thus we will not go through it. The proof of this proposition can be found in chapter II of Lee's book [2].

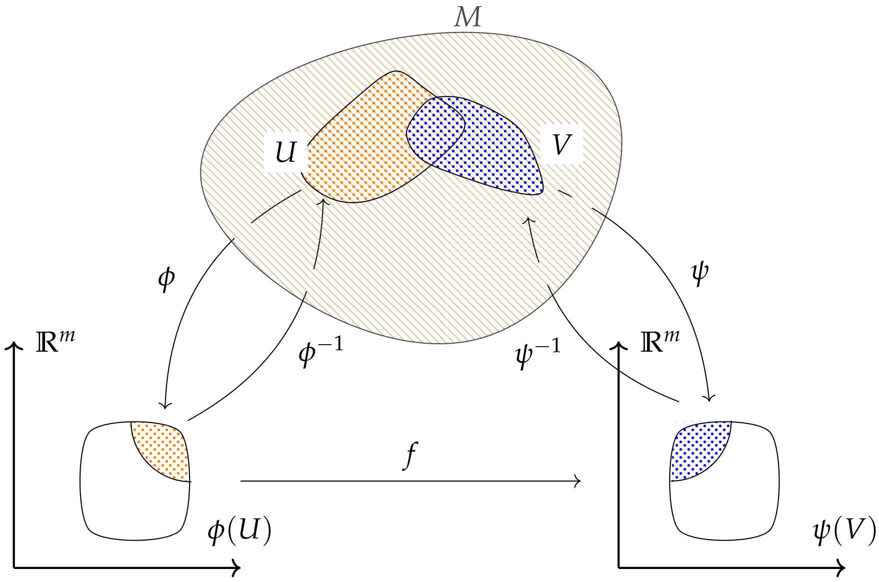

Figure 3.1: This figure represents a smooth function f:M→ M

3.1 Tangent space

In order to define differential forms we need to define the tangent space to a differential manifold.

Definition 3.7. *Let* M *be a smooth manifold, the set of smooth functions* $f\colon M \to \mathbb{R}$ *is denoted* $\mathcal{F}(M)$

Definition 3.8. *Let* M *be a differential manifold and* $p \in M$*. We will call derivation at* p*, any* $\mathbb{R}$*-lineal map* $\delta_p : \mathcal{F}(M) \to \mathbb{R}$ *such that for any* $f, g \in \mathcal{F}(M)$*,*
$\delta_p(f \cdot g) = \delta_p(f) \cdot g(p) + f(p) \cdot \delta_p(g)$.

The set of all the derivations at p *is an* $\mathbb{R}$*-vector space*

Definition 3.9. *This vector space is called the tangent space to* M *at* p *and it is denoted by* T_pM*. The elements of* T_pM*. are called tangent vectors to* M *at* p*. In local coordinates, the elements* $\left.\frac{\partial}{\partial x^i}\right|_p$ *form a basis of* T_pM

Given the concept of tangent space, we now can work our way to defining a smooth submanifold of a given manifold.

Definition 3.10. *Let* M *and* N *be smooth manifolds and let* $f\colon M \to N$ *be a differentiable map, we define the* $differential of f in p \in M$ *as the map*

$$d_pf\colon T_pM \to T_{f(p)}N$$

Definition 3.11. *Let M, N be smooth manifolds, $f\colon M \to N$ a smooth function. f is said to be an immersion if $d_p f$ is injective $\forall p \in M$.*

Definition 3.12. *An injective immersion $\phi\colon M \to N$ is said to be an embedding if ϕ is an homomorphism of M in its image $\phi(M)$ (with the subset topology from N).*

Proposition 3.13. *Let $f\colon M \to N$ be an embedding. $f(M)$ admits a differentiable structure such that $f\colon M \to f(M)$ is a diffeomorphism and the inclusion $i\colon f(M) \to N$ is an embedding.*

Proof. For each chart (U, ϕ) of M, where V is an open subset of N, we define $f(U)$ as the intersection of V and $f(M)$. We consider $(f(U), \phi \circ f^(-1))$ as a candidate chart for $f(M)$. To show that this gives a differentiable structure to $f(M)$, we need to verify that for any two charts (U_1, ϕ_1) and (U_2, ϕ_2) of M, the mapping:

$\phi_2 \circ f^{-1} \circ f \circ \phi_1^{-1}\Big|_{\phi_1(U_1 \cap U_2)}$

is differentiable, which is evident. This also implies that $f\colon M \to f(M)$ is a diffeomorphism.

To show that $i\colon f(M) \to N$ is an embedding, considering that the topology on $f(M)$ is induced by the subspace topology of N, we only need to verify that i is differentiable. Let $(f(U), \phi \circ f^{-1})$ be a chart of $f(M)$ and (W, δ) be a chart of N. Since $f(U) = V \cap f(M)$, we need to show that:

$\delta \circ f \circ \phi^{-1}\colon \phi(U \cap f^{-1}(W)) \to \delta(f(U) \cap W)$ is differentiable, which follows from the differentiability of f.

□

After this proposition it is possible to construct a submanifold of a given manifold.

Definition 3.14. *Let M and N be smooth manifolds, and let $f\colon M \to N$ be an embedding, we define a $smooth submanifold$ of M as the pair (M,f).*

Definition 3.15. *Let M be a differentiable manifold of dimension n. We define a vector field on M as the assignment of an element $X_p \in T_p M$ to each point $p \in M$. A vector field X is said to be differentiable if, for every $p \in M$ and any coordinate neighborhood $(U, (x_1, \ldots, x_n))$ around this point, the expression of X in this neighborhood is given by:*

$$X|_U = \sum_{i=1}^{n} \lambda_i(x_j) \frac{\partial}{\partial x_i}$$

where the functions λ_i are differentiable.

3.2 Tangent and cotangent bundles

3.3 Tensor fields

In order to define differential forms we first need to introduce the notion of a differentiable tensor field.

Definition 3.16. *Let E be a real vector space of finite dimension and E^* its dual vector space . A (k,l)-tensor is a multilinear map*

$$T : E \times ... \times E \times E^* \times ... \times E^* \to \mathbb{R} \tag{3.2}$$

We will denote the real vector space of (k,l) tensors as $Z_{(k,l)}(E)$

Now, as it is very common in mathematics we define a product between these mathematical objects.

Definition 3.17. *Let T and T' be a (k,l)-tensor and (k',l')-tensor respectively we can define the tensorial product $T \otimes T'$ as the (k+k',l+l')-tensor given by:*

$$(T \otimes T')((v, v')(w, w')) = T(v, w) \cdot T'(v', w') \tag{3.3}$$

Definition 3.18. *Let M be a smooth manifold, a (k,l) tensor field is a correspondence to each $p \in M$ an element $K_p \in T_{(k,l)}(T_pM)$*

$$p \in M \to \alpha_p \in Z_{(k,l)}(T_pM) \tag{3.4}$$

such that for every coordinate map U, in the local expression $q \in U$

Let U be an open set in M with coordinates $(x^i)_{i=1,...,n}$ using basis $\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}$ of T_pM and $\{d\partial x^j\}_{j=1,...,n}$ of T_p^*M. For every p$\in$ U,

$$K_p = \lambda^{j_1,...,j_l}_{i_1,...,i_k} dx^{i_1} \otimes ... \otimes ...dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes ...\frac{\partial}{\partial x^{j_l}} \tag{3.5}$$

A differentiable tensor field is a tensor field such that its restriction to any open set U, satisfies that $\lambda^{j_1,...,j_l}_{i_1,...,i_k}$ are differentiable functions in U.

3.4 Differential forms

Definition 3.19. *A differential k-form in M is an antisymmetric differential tensor field of type (k,0). Antisymmetric meaning that it changes sign whenever two of its arguments are interchanged.*

A (k,0)-tensor is said to be contravariant and a (0,k)-tensor is said to be covariant.

Definition 3.20. *The covariant k-tensor T on a finite-dimensional vector space V is said to be alternating if it satisfies*

$$T(X_1, ..., X_i, ..., X_j, ..., X_k) = -T(X_1, ..., X_j, ..., X_i, ..., X_k) \tag{3.6}$$

Definition 3.21. *Let $T^k(V)$ be the vector space of all contravariant k-tensor. Let V be a finite vector space, $\Lambda^k(V)$ the subspace of $T^k(V)$ consisting of alternating tensors, we define the alternating projector Alt:$T^k(V) \to \Lambda^k(V)$ as follows:*

$$(AltT)(X_1, ..., X_k) = \frac{1}{k!}\Sigma_\sigma(sgn\,\sigma)T(X_{\sigma(1)}, ..., X_{\sigma(k)}) \tag{3.7}$$

Where sgn is the sign of the permutation σ, satisfying $sgn\,\sigma = +1$ if the permutation is even, and $sgn\,\sigma = -1$ when the permutation is odd.

Definition 3.22. *If $\omega \in \Lambda^k(V)$and $\eta \in \Lambda^l(V)$, we deine the wedge product or exterior product of ω and η to be the alternating (k+l)-tensor*

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}Alt(\omega \otimes \eta) \tag{3.8}$$

Now we have all the formalism to define the exterior derivative. It is a generalisation of the differential of a function.

If $X_1, ..., X_n$ are vector fields on M and ω is a p-form then $\omega(X_1, \dots, X_p)$ is a smooth real valued function on M.

Definition 3.23. *The definition of the exterior derivative in coordinates is:*

$$d(\sum_J \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \tag{3.9}$$

3.5 Exact and closed forms

At this point we can make a distinction between what are called closed p-forms and exact p-forms.

Definition 3.24. *An exact p-form is a differential p-form α that is the exterior derivative of another differential form β. Using the notation introduced earlier, α is such that $\alpha = d\beta$*

Definition 3.25. *A closed p-form ω is a differential p-form that satisfies dω=0*

Proposition 3.26. *Every exact form is closed.*

Proof. We want to prove that that $dd = 0$. It is sufficient to check this on $W = f\, dx^1 \wedge \ldots \wedge dx^p$. We calculate

$$d\left(df \wedge dx^1 \wedge \ldots \wedge dx^p\right) = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^p.$$

If we rearrange the double sum so that it ranges over $j < i$, then we get two terms for each pair i, j. These terms are identical except for a $dx^j \wedge dx^i$ in one and a $dx^i \wedge dx^j$ in the other. Thus, it all cancels out.

□

This distinction is what will allow us to define the De Rham cohomology of M as a quocient group.

3.6 Smooth singular homology

In the first chapter we introduced homology and cohomology theories. Now that we have introduced smooth manifolds and differentiable forms we can combine singular homology with differentiable forms.

Definition 3.27. *If M is a smooth manifold, a smooth p-simplex in M is a smooth map $\sigma : \Delta_p \to M$. The subgroup of $C_p(M)$ generated by smooth p-simplices is denoted by $C_p^\infty(M)$ and called the smooth chain group in dimension p. Elements of this group are called smooth chains.*

Definition 3.28. *Let M be a smooth manifold. We define the pth smooth singular homology group of M as follows*

$$H_p^\infty = \frac{Ker[\partial : C_p^\infty(M) \to C_{p-1}^\infty(M)]}{Im[\partial : C_{p+1}^\infty(M) \to C_p^\infty(M)]} \tag{3.10}$$

Now that we have defined the p-th smooth singular homology group of a smooth manifold, we are interested in relating it to the singular homology group of M. Since $C_p^\infty(M) \subset C_p(M)$, there is a map on homology induced by inclusion too $i_* : H_p^\infty(M) \to H_p(M)$. This map gives us the relation between smooth singular homology and singular homology, it is expressed in the following theorem. Because our main goal is to prove the De Rham's theorem in this work, we will not write the prove to this theorem due to length contraints but it is of great use in showing that singular homology can be computed with smooth simplices.

Theorem 3.29. *(Smooth Singular vs. Singular Homology) For any smooth manifold M, the map $i_* : H_p^\infty(M) \to H_p(M)$ induced by inclusion is an isomorphism. The proof to this theorem will be skipped. Not because it is trivial but rather the opposite, it has many technical details. The reader can find the complete proof in Chapter 16 of [2].*

Chapter 4

The De Rham Cohomology

4.1 Definition and purpose

Now that we have introduced the closed and exact forms we can define the De Rham cohomology.

Definition 4.1. *Let M be a smooth Manifold, $\Omega^p(M)$ the vector space of all smooth p-forms on M and d the exterior derivative. The pth De Rham cohomology group of M is the real vector space*

$$H^p_{DR}(M) = \frac{ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{Im(d : \Omega^{p-1}(M) \to \Omega^p(M))}$$

$$H^p_{DR}(M) = \frac{closed\ p - forms}{exact\ p - forms} = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$$

We can now give an example that will help the reader understand the De Rham cohomology group, by computing it for spheres.

Theorem 4.2. *(Cohomology of Spheres). For $n \geq 1$, the De Rham cohomology groups of $\mathbb{S}^n$ are*

$$H^p_{dR}(\mathbb{S}^n) = \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{if } 0 < p < n. \end{cases}$$

Proof. For *p*=0, we can prove it for a more general case .

Proposition 4.3. *(Cohomology in Degree Zero). If M is a connected smooth manifold with or without boundary, then $H^0_{dR}(M)$ is equal to the space of constant functions and is therefore 1-dimensional.*

Proof. *Because there are no* 1*-forms,* $B^0(M) = 0$*. A closed* 0*-form is a smooth real-valued function* f *such that* $df = 0$*, and since* M *is connected, this is true if and only if* f *is constant. Therefore,* $H^0_{dR}(M) = Z^0(M) =$ *constants.*

For $p \geq 1$ we prove it by induction on n. For $n \geq 1$, note first that $\dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 1$. On the other hand, it can be proved that (see [2]) there is an injective linear map from $H^1_{\mathrm{dR}}(\mathbb{S}^n)$ into $\mathrm{Hom}(H_1(\mathbb{S}^1;\mathbb{R}),\mathbb{R})$, which is 1-dimensional. Thus, $H^1_{\mathrm{dR}}(S^1)$ has dimension exactly 1 and is spanned by the cohomology class of any orientation form.

Next, suppose $n \geq 2$ and assume by induction that the theorem is true for S^{n-1}. Because $\mathbb{S}^n$ is simply connected, $H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0$ due to the existence of of a linear map $\phi\colon H^1_{DR}(M) \to Hom(\pi_1(M,q),\mathbb{R}$ (see shapter 17 [2] for more details on this). For $p > 1$, we use the Mayer-Vietoris theorem as follows. Let N and S be the north and south poles in $\mathbb{S}^n$, respectively, and let $U = \mathbb{S}^n \times \{S\}$, $V = \mathbb{S}^n \times \{N\}$. By stereographic projection, both U and V are diffeomorphic to $\mathbb{R}^n$, and thus $U \cap V$ is diffeomorphic to $\mathbb{R}^n \times \{0\}$.

Part of the Mayer-Vietoris sequence for $\{U, V\}$ reads

$$H^{p-1}_{\mathrm{dR}}(U) \to H^{p-1}_{\mathrm{dR}}(V) \to H^{p-1}_{\mathrm{dR}}(U \cap V) \to H^p_{\mathrm{dR}}(\mathbb{S}^n) \to H^p_{\mathrm{dR}}(U) \to H^p_{\mathrm{dR}}(V).$$

Because U and V are diffeomorphic to $\mathbb{R}^n$, the groups on both ends are trivial when $p > 1$, which implies that $H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{p-1}_{\mathrm{dR}}(U \cap V)$. Moreover, $U \cap V$ is diffeomorphic to $\mathbb{R}^n \times \{0\}$ and therefore homotopy equivalent to S^{n-1}. Thus, we conclude that $H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{p-1}_{\mathrm{dR}}(S^{n-1})$ for $p > 1$, and the desired result follows by induction. As in the $n = 1$ case, any smooth orientation form on $\mathbb{S}^n$ determines a nonzero cohomology class, which therefore spans $H^n_{\mathrm{dR}}(\mathbb{S}^n)$.

□

We now compare this computation with the singular cohomology of the sphere.

Consider the n-dimensional sphere $\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$.
Let $A = \{(x_0, \ldots, x_n) \in \mathbb{S}^n \mid x_0 > -\frac{1}{2}\}$ and $B = \{(x_0, \ldots, x_n) \in \mathbb{S}^n \mid x_0 < \frac{1}{2}\}$.
Of course, both A and B are open in $\mathbb{S}^n$, and their union is $\mathbb{S}^n$. Furthermore, it can be easily seen that their intersection can be contracted into the "big circle," i.e., $A \cap B$ has the homotopy type of S^{n-1}. Also, both A and B are contractible (they are homeomorphic to $\mathbb{R}^n$ via stereographic projection).

So, let's write part of a Mayer-Vietoris sequence (for the cohomology $H^m(X) = H^m(X, G)$, where G is a fixed Abelian group):

$$\cdots \to H^m(A) \oplus H^m(B) \to H^m(A \cap B) \to H^{m+1}(\mathbb{S}^n) \to H^{m+1}(A) \oplus H^{m+1}(B) \to \cdots$$

Since both A and B are contractible and $A \cap B$ is homotopic to S^{n-1}, we have the following short exact sequence:

$$0 \to H^m(S^{n-1}) \to H^{m+1}(\mathbb{S}^n) \to 0$$

This shows that $H^m(S^{n-1})$ is isomorphic to $H^m(\mathbb{S}^n)$ for every $n > 0$ and $m > 0$. Therefore, in order to calculate the cohomology groups of spheres, we only need to know the cohomology groups of S^1. And those can be also calculated if we once again apply the previous schema. Note that in the case of S^1, we have that $A \cap B$ has the homotopy type of a discrete space with two points. Therefore, all their cohomology groups are trivial, except for H^0 (which can be easily calculated to be equal to $H^0(*) \oplus H^0(*)$, where $*$ is a one-pointed space).

This schema can be used for other spaces like the torus (which can also be calculated using Kunneth's formula).

After defining this new cohomology we are now interested in proving that De Rham cohomology groups are topological invariants. This will be a consequence of a more general result, its homotopy invariance. The topological invariance is of great relevance because it is what motivates the isomorphism stated in the De Rham theorem. As we saw when we calculated the De Rham cohomology groups of spheres, one achieves the same result when computing the singular cohomology groups. This obviously is not a coincidence, this example illustrates the fact that there is some way of computing the De Rham cohomology with purely topological tools.
The topological invariance is what gives us hints of this possibility of computing the De Rham cohomolgy through purely topological methods. We are going to prove it in theorem 4.5, but first let us give the statement of a very known theorem.

Theorem 4.4. *(Whitney Approximation on Manifolds) Let M and N be smooth manifolds, and let $F\colon M \to N$ be a continuous map. Then F is homotopic to a smooth map $\tilde{F}\colon M \to N$. If F is smooth on a closed subset $A \subset M$, then the homotopy can be taken relative to A.*

This proof is not very technical but requires the use of embedded submanifolds, which we have not introduced in this work.

Theorem 4.5. *(Homotopy Invariance of De Rham Cohomology). If M and N are homotopy equivalent smooth manifolds with or without boundary, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ for each p. The isomorphisms are induced by any smooth homotopy equivalence $F\colon M \to N$.*

Proof. Suppose $F\colon M \to N$ is a homotopy equivalence, with homotopy inverse $G\colon N \to M$. By the Whitney approximation theorem there are smooth maps $\tilde{F}\colon M \to N$ homotopic to F and $\tilde{G}\colon N \to M$ homotopic to G.

Because homotopy is preserved by composition, it follows that $\tilde{F} \circ \tilde{G} \simeq F \circ G \simeq \mathrm{Id}_N$ and $\tilde{G} \circ \tilde{F} \simeq G \circ F \simeq \mathrm{Id}_M$; so $\tilde{F}$ and $\tilde{G}$ are homotopy inverses of each other.

It can be shown (see citar Lee) that if F and G are homotopic smooth maps, the induced cohomology maps $F^*, G^*\colon H^p_{DR}(N) \to H^p_{DR}(M)$ are equal. Since F and G are the inverse of each other, on cohomology,

$$(\tilde{F} \circ \tilde{G})^* = \tilde{G}^* \circ \tilde{F}^* = \mathrm{Id}^*_M = \mathrm{Id}_{H^p_{\mathrm{dR}}(M)}$$

where $(\tilde{F} \circ \tilde{G})^*$ denotes the induced map on cohomology. The same argument shows that $(\tilde{G} \circ \tilde{F})^*$ is also the identity, so $\tilde{F}^*\colon H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ is an isomorphism. □

Corollary 4.6. *(Topological Invariance of De Rham Cohomology). The De Rham cohomology groups are topological invariants: if M and N are homeomorphic smooth manifolds with or without boundary, then their De Rham cohomology groups are isomorphic.*

This is trivial because every homeomorphism is a homotopy equivalence.

4.2 Preparation for the theorem

In this section we present a series of previous results that are necessary to develop the proof of the De Rham Theorem.

For the following lemma we introduce the concept of an exact sequence. A sequence is said to be exact if for every map f_i in the sequence, $Ker(f_i) = Im(f_{i-1})$.

Lemma 4.7. *(The Five lemma): Consider the following commutative diagram of $\mathbb{R}$-modules and linear maps*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

If the horizontal rows are exact and $f_1, f_2, f_4 and f_5$ are isomorphisms, then f_3 is also an isomorphism.

Proof. This is quite an immediate proof. **(a)** Let $b_3 \in B_3$. Since f_4 is an isomorphism, it is surjective, there exists $a_4 \in A_4$ such that $f_4(a_4) = \beta_3(b_3)$. Because $\alpha_i\beta_i$ are exact, this implies that $\beta_4(\beta_3(b_3)) = 0 = f_4(\alpha_4(a_4)) = \alpha_5(\alpha_4(a_4))$. Therefore, $\alpha_4(a_4) = 0$ (since f_5 is injective).

Since $\alpha_4(a_4) = 0$, there exists $a_3 \in A_3$ such that $\alpha_3(a_3) = a_4$. Now, consider $b_3 - f_3(a_3)$. We have $\beta_3(b_3 - f_3(a_3)) = \beta_3(b_3) - \beta_3 f_3(a_3) = f_4(a_4) - f_4\alpha_3(a_3) = f_4(a_4) - f_4(a_4) = 0$. Thus, there exists $b_2 \in B_2$ such that $\beta_2(b_2) = b_3 - f_3(a_3)$.

Next, consider $a_2 \in A_2$ such that $f_2(a_2) = b_2$ (since f_2 is surjective). Now, let's compute $f_3(\alpha_2(a_2) + a_3)$:

$$f_3(\alpha_2(a_2)+a_3) = f_3\alpha_2(a_2)+f_3(a_3) = \beta_2 f_2(a_2)+f_3(a_3) = \beta_2(b_2)+f_3(a_3) = b_3 - f_3(a_3)+f_3(a_3) = b_3.$$

Hence, we have shown that for any $b_3 \in B_3$, there exist $b_2 \in B_2$ and $a_2 \in A_2$ such that $\beta_2(b_2) = b_3 - f_3(a_3)$ and $f_3(\alpha_2(a_2) + a_3) = b_3$.

(b) Now, let $a_3 \in A_3$ such that $f_3(a_3) = 0$. Since f_4 is injective, we have $f_4\alpha_3(a_3) = \beta_3 f_3(a_3) = 0$. Therefore, $\alpha_3(a_3) = 0$ (since f_4 is injective). This implies there exists $a_2 \in A_2$ such that $\alpha_2(a_2) = a_3$.

Next, consider $b_2 \in B_2$ such that $\beta_2(b_2) = f_2(a_2)$ (by surjectivity of f_2). Then there exists $a_1 \in A_1$ such that $f_1(a_1) = b_1$ (by surjectivity of f_1). Now, observe:

$$\begin{aligned}
\beta_1(b_1) &= \beta_1 f_1(a_1) \\
&= f_2\alpha_1(a_1) \\
&= f_2(a_2).
\end{aligned}$$

Since $\beta_2(b_2) = f_2(a_2)$, we have $\beta_2(b_2) = \beta_1(b_1)$. By the injectivity of β_1, we can conclude that $b_2 = b_1$. Hence, we have shown that for any $a_3 \in A_3$ with $f_3(a_3) = 0$, there exists $b_2 \in B_2$ such that $\beta_2(b_2) = f_2(a_2)$.

Therefore, we have proved both directions, and the map f_3 is both surjective and injective.

$\square$

Let us now state a general theorem that can be used to compute the De Rham cohomology groups of many manifolds, by expressing them as unions of open submanifolds with simpler cohomology.

Theorem 4.8. *(Mayer-Vietoris). Let M be a smooth manifold with or without boundary, and let U and V be open subsets of M whose union is M. We denote the inclusion of submanifolds by $i\colon U \cap V \to U$, $j\colon U \cap V \to V$, $k\colon U \to M$ and $l\colon V \to M$. For each p, there is a linear map $\delta : H^p_{dR}(U \cap V) \to H^{p+1}_{dR}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta} H^p_{PR}(M) \xrightarrow{k^* \oplus l^*} H^p_{DR}(U) \oplus H^p_{DR}(V) \xrightarrow{i^* - j^*} H^p_{DR}(U \cap V) \xrightarrow{\delta} H^{p+1}_{DR}(M) \xrightarrow{k^* \oplus l^*} \cdots$$

Where the "" symbol indicates the pullback maps induced in differential forms. For instance $i^*\colon \Omega^p(U) \to \Omega^p(U \cap V)$.*

The proof of this theorem is not straight-forward, we are only going to give the statement of a necessary lemma for the proof. A short exact sequence of complexes consists of three complexes A^*, B^*, C^*, together with cochain maps:

$$0 \to A \xrightarrow{F} B \xrightarrow{G} C \to 0,$$

such that each sequence

$$0 \to A^p \xrightarrow{F} B^p \xrightarrow{G} C^p \to 0$$

is exact.

Lemma 4.9. *(The ZigZag Lemma). Given a short exact sequence of complexes as above, for each p there is a linear map*

$$\delta\colon H^p(C^*) \to H^(p+1)(A^*),$$

called the connectinghomomorphism, such that the following sequence is exact:

$$\ldots \xrightarrow{\delta} H^p(A^*) \xrightarrow{F^*} H^p(B^*) \xrightarrow{G^*} H^p(C^*) \xrightarrow{\delta} H^{p+1}(A^*) \xrightarrow{F^*} \ldots$$

The proof of this lemma as well as the whole proof of the Mayer-Vietoris theorem can be foun in Chapter 15 of [2]

Theorem 4.10. *(Stokes's Theorem for Chains) If c is a smooth p-chain in a smooth manifold M, and ω is a smooth $(p-1)$-form on M, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$

Proof. It suffices to prove the theorem when c is just a smooth simplex σ. Since p is a manifold with corners, Stokes's theorem says that

$$\int_\sigma d\omega = \int_{\Delta_p} \sigma^* d\omega = \int_{\Delta_p} d\sigma^* \omega =$$

The maps $F_{i,p}\colon i = 0, 1, \ldots, p$ are parametrizations of the boundary faces of σ satisfying the conditions of proposition 3.4, except possibly that they might not be orientation-preserving. To check the orientations, note that $F_{i,p}$ is the restriction to $\Delta_p \cup \partial \mathbb{H}^p$ of the affine diffeomorphism sending the simplex $[e_0, \ldots, e_p]$ to $[e_0, \ldots, e_i, \ldots, e_p, \sigma_i]$. This is easily seen to be orientation-preserving if and only if $[\sigma_0, \ldots, \sigma_{b_i}, \ldots, \sigma_p, \sigma_i]$ is an even permutation of $[e_0, \ldots, e_p]$, which is the case if and only if $p - i$ is even. Since the standard coordinates on $\partial \mathcal{H}^p$ are positively oriented if and only if p is even, then $F_{i,p}$ is orientation-preserving for $\partial \mathcal{H}^p$ if and only if i is even. Thus, by Proposition 3.4,

$$\int_{\partial \Delta_p} \sigma^* \omega = \sum_{i=0}^{p} (-1)^i \int_{\sigma \circ F_{i,p}}^{*} \omega.$$

By definition of the singular boundary operator, this is equal to $\int_{\partial \mathcal{H}^{p-1}} d\omega$. ∎

Allow us to make a brief incision in this theorem and its relevance. The Stokes theorem provides a powerful tool for relating the boundary of a chain to the integral of its derivative over its interior. If ω is exact, then the integral of ω over any compact submanifold without boundary is zero. This has a very well-known relative in physics. Consider a physical system described by a conservative force, such as gravity or an electrostatic field. In these cases, the force can be derived from a scalar potential function, and the corresponding vector field can be represented by a differential 1-form ω which is exact, i.e., $\omega = d\phi$ for some scalar function ϕ. This

We now define a homomorphism using this theorem. This is going to be the linear map that will in fact, be an isomorphism between $H^p_{dR}(M)$ and $H^p(M; \mathbb{R})$. Let us define a natural linear map $\mathcal{J} : H^p_{dR}(M) \to H^p(M; \mathbb{R})$, called the De Rham homomorphism, as follows. For any $[\omega] \in H^p_{dR}(M)$ and $[c] \in H^p(M)$, we define

$$\mathcal{J}\,[\omega]\,[[c]] = \int_c \omega,$$

where c is any smooth p-cycle representing the homology class $[c]$. This is well defined because if c and c_0 are smooth cycles representing the same homology class, then by the isomorphism between smooth singular and singular homology stated in theorem 3.29 guarantees that $c - c_0 = \partial b$ for some smooth $(p+1)$-chain b, which implies

$$\int_c \omega - \int_{c_0} \omega = \int_{\partial b} \omega = \int_b d\omega$$

Which is zero because $d\omega$=0 since $d\omega$ represents a cohomology class. while if $\omega = d\alpha$ is exact, then

$$\int_c \omega = \int_c d\alpha = \int_{\partial c} \alpha$$

Now, this integral is zero because $\partial c = 0$, since c represents a homology class. Clearly, $\mathcal{J}\,[\omega]\,[[c+c_0]] = \mathcal{J}\,[\omega]\,[[c]] + \mathcal{J}\,[\omega]\,[[c_0]]$, and the resulting homomorphism $\mathcal{J}\,[\omega]$ depends linearly on ω. Thus, $\mathcal{J}\,[\omega]$ is a well-defined element of $\mathrm{Hom}(H^p(M), \mathbb{R}) \cong H^p_{dR}(M)$.

Lemma 4.11. *(Naturality of the De Rham Homomorphism). For a smooth manifold M and nonnegative integer p, let $\mathcal{J} : H^p_{dR}(M) \to H^p(M;\mathbb{R})$ denote the De Rham homomorphism.*

(a) If $F : M \to N$ is a smooth map, then the following diagram commutes:

$$\begin{array}{ccc} H^p_{dR}(N) & \xrightarrow{F^*} & H^p_{dR}(M) \\ \big\downarrow \mathcal{J} & & \big\downarrow \mathcal{J} \\ H^p(N;\mathbb{R}) & \xrightarrow{F^*} & H^p(M;\mathbb{R}) \end{array}$$

(b) If M is a smooth manifold and U, V are open subsets of M whose union is M, then the following diagram commutes:

$$\begin{array}{ccc} H^{p-1}_{dR}(U \cap V) & \xrightarrow{\delta} & H^p_{dR}(M) \\ \big\downarrow \mathcal{J} & & \big\downarrow \mathcal{J} \\ H^{p-1}(U \cap V;\mathbb{R}) & \xrightarrow{\partial^*} & H^p(M;\mathbb{R}) \end{array}$$

Proof. Directly from the definitions, if σ is a smooth p-simplex in M and ω is a smooth p-form on N, we have

$$\int_\sigma F^*\omega = \int_{\Delta_p} \sigma^* F^* \omega = \int_{\Delta_p} (F \circ \sigma)^* \omega = \int_{F \circ \sigma} \omega.$$

This implies $\mathcal{J}(F^*[\omega])[\sigma] = \mathcal{J}[\omega][F \circ \sigma] = \mathcal{J}[\omega](F_*[\sigma]) = F^*(\mathcal{J}[\omega])[\sigma]$. Therefore, the diagram in part (a) commutes.

Now we prove part (b). Asking for the commutativity of this diagram is equivalent to asking for the following equation:

$$\mathcal{J}(\delta[\omega])[e] = (\partial^* \mathcal{J}[\omega])[e] = \mathcal{J}[\omega](\partial_*[e]) \tag{4.1}$$

for any $[\omega] \in H^{p-1}_{DR}(U \cap V)$ and any $[e] \in H_p(M)$.

Using the identification of $H^p_{DR}(M)$ with $\mathrm{Hom}(H_p(M;\mathbb{R}), \mathbb{R})$, we can rewrite this as

$$I[\iota(\omega)] = \iota_*[I(\omega)].$$

Let σ be a p-simplex representing $\iota(\omega)$ and let c be a $(p-1)$-chain representing @$(\iota(\omega))$. Then we have

$$I[\iota(\omega)](\sigma) = \int_\sigma \iota(\omega) = \int_\sigma \omega$$

and

$$\iota_*[I(\omega)](c) = \int_c I(\omega) = \int_c \omega.$$

Therefore, we have $\int_\sigma \omega = \int_c \omega$, which proves the commutativity of the diagram. $\square$

4.3 The De Rham Theorem

Let us begin this section with the statement in the theorem. We subsequently give some definitions for the prove and prove necessary lemmas.

Theorem 4.12. *(De Rham's Theorem) Let M be an arbitrary smooth manifold. The Homomorphism $\mathcal{J} : H^p_{DR}(M) \to H^p(M; R)$ is an isomorphism.*

A smooth manifold is a De Rham manifold if $\mathcal{J} : H^p_{DR}(M) \to H^p(M; R)$ is an isomorphism for each integer p$\geq$0.

If M is an arbitrary smooth manifold, an open cover $\{U_i\}$ of M is called a De Rham cover if each open set U_i is a De Rham manifold, and every finite intersection $U_{i_1} \cap ... \cap U_{i_k}$ is De Rham. If such De Rham cover is a basis for the topology of M, it is called a *De Rham basis* for M.

We now prove the Poincaré Lemma, which is a particular case of the De Rham Theorem. Later, we will use this lemma to prove the end result.

Lemma 4.13. *(Poincaré Lemma). The De Rham Theorem is true for any convex subset U of* $\mathbb{R}^{n+1}$.

Proof. We can assume that U contains the origin. We must show:
i)that any closed p-form ω, $p \geq 1$, on U is exact
ii)that any smooth function f on U with $df = 0$ is constant.

The reason this is sufficient to prove the lemma is because the De Rham map takes a constant function with value r to the constant 0-cocycle taking value r on each 0-simplex.

Since df=0, every $\frac{\partial \omega_I}{\partial x^i}$ is 0 and thus f is is constant in every chart. Now, because in the Lemma U is convex, U is also connected and the locally constant f is constant in U. This proves ii).

To prove i), we take U $\subset \mathbb{R}^{n+1}$ with coordinates $x_0, ..., x_n$. For p ≥ 0, we define

$$\phi : \Omega^{p+1} \to \Omega^p \tag{4.2}$$

as follows. If $\omega = f(x_0, ..., x_n)dx_{j_0} \wedge ... \wedge dx_{j_p}$ then

$$\phi(\omega) = (\int_0^1 t^p f(tx)dt)\eta, \tag{4.3}$$

where

$$\eta = \sum_{i=0}^{p} (-1)^i x_{j_i} dx_{j_0} \wedge ... \wedge d\hat{x}_{j_i} \wedge ... \wedge dx_{j_p} \tag{4.4}$$

Then, using D_k to denote the partial derivative with respect to the kth variable,

$$d\phi(\omega) = \sum_{k=0}^{n} (\int_0^1 t^{p+1} D_k f(tx)dt)dx_k \wedge \eta + (\int_0^1 t^p f(tx)dt)d\eta = S + T \tag{4.5}$$

where S is the term that contains the sum and T the rest. Also

$$d\omega = \sum_{k=0}^{n} D_k f(x)dx_k \wedge dx_{j_0} \wedge ... \wedge dx_{j_p}, \tag{4.6}$$

so that

$$\begin{aligned}
\phi(d\omega) = \sum_{k=0}^{n} (\int_0^1 t^{p+1} D_k f(tx)dt)(x_k dx_{j_0} \wedge x_k dx_{j_0} - dx_k \wedge \eta) \\
= \sum_{k=0}^{n} x_k (\int_0^1 t^{p+1} D_k f(tx)dt)dx_{j_0} \wedge dx_{j_p} - S \\
= (\int_0^1 t^{p+1} \frac{d}{dt} f(tx)dt)dx_{j_0} \wedge dx_{j_p} - S \\
= \{t^{p+1} f(tx)]_0^1 - (p+1) \int_0^1 t^p f(tx)dt\} dx_{j_0} \wedge dx_{j_p} - S \\
= \omega - T - S
\end{aligned}$$

since d$\eta = (p+1)dx_{j_0} \wedge ... \wedge dx_{j_p}$. Then $d\phi(\omega) + \phi(d\omega) for \omega \in \Omega^p(\mathbb{R}^{n+1}), p \geq$ 1 This means that if ω is a closed p-form, which means that $d\omega$=0, then $\omega = d(\phi(\omega))$, which means that ω is an exact form on U. We have thus proved the lemma. □

After proving the Poincaré Lemma we now have that every convex open subset of $\mathbb{R}$ is De Rham.

Lemma 4.14. *If M has a finite De Rham cover, then M is De Rham.*

Proof. Let$\{U_i\}_{i=1,...,k}$ be a cover of M such that each U_i is De Rham. We will prove the lemma by induction on k. Suppose first that M has a De Rham cover consisting of two sets U,V. We have (explained before i should, form Mayer-Vitoris) the following commutative diagram

$$\begin{tikzcd}
(U) \oplus H^{p-1}_{DR}(V) \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & H^{p-1}_{DR}(U \cap V) \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & H^p_{DR}(M) \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & {} \\
H^{p-1}(U;\mathbb{R}) \oplus H^{p-1}(V;\mathbb{R}) \arrow[r, "\beta_1"] & H^{p-1}(U \cap V;\mathbb{R}) \arrow[r, "\beta_2"] & H^p(M;\mathbb{R}) \arrow[r, "\beta_3"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\alpha_3"] & H^p_{DR}(U) \oplus H^p_{DR}(V) \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & H^p_{DR}(U \cap V) \arrow[d, "f_5"] \\
{} \arrow[r, "\beta_3"] & H^{p-1}(U;\mathbb{R}) \oplus H^{p-1}(V;\mathbb{R}) \arrow[r, "\beta_4"] & H^p(U \cap V;\mathbb{R})
\end{tikzcd}$$

Since U and V and the intersection are De Rham, $f_1, f_2, f_4 and f_5$ are isomorphisms, it follows from the Five Lemma that f_3 is also an isomorphism. This proves that M is De Rham.

We now make the induction hypothesis that M has a De Rham cover with k ≥ 2 sets and suppose $U_1, ..., U_{k+1}$ is a De Rham cover of M. By putting U=$U_1 \cup ... \cup U_k$ and V=U_{k+1} the hypothesis implies that U, V and $U \cap V$ are De Rham. Therefore $U \cup V$ is also De Rham by the argument above.

□

Lemma 4.15. *If $\{M_j\}$ is any countable collection of De Rham manifolds, then their disjoint union is De Rham.*

Proof. Let $\{M_j\}$ be a countable collection of manifolds where $\psi\colon H^p_{DR}(M_j) \to H^p(M_j)$ is an isomorphism for each j. Let $M = \bigcup_j M_j$ be the disjoint union of these manifolds. Denote the inclusion maps by $i_j : M_j \to M$. Then the map $i = (i_1, i_2, \ldots)$ induces isomorphisms between $\oplus_j H^p_{DR}(M_j)$ and $H^p_{DR}(M)$ as well as $\oplus_j H^p(M_j; \mathbb{R})$ and $H^p(M; \mathbb{R})$. For each p, $\oplus_j(\psi\colon (M_j) \to M)$ is an isomorphism between the direct product of the De Rham and singular cohomology groups. Therefore, by the naturality of the De Rham homorphism stated in Lemma 4.11, $\psi\colon H^p_{DR}(M) \to H^p(M)$ must also be an isomorphism.

□

The next step in this proof is to show that if M has a De Rham basis, then M is De Rham.

Suppose U_j is a De Rham basis for M. Let $f : M \to \mathbb{R}$ be an exhaustion function. We can guarantee its existence using Proposition 3.6. For each integer m, define subsets A_m and A'_m of M by

$$A_m = \{q \in M \mid m < f(q) \leq m + 1\},$$

$$A'_m = \{q \in M \mid m - \frac{1}{2} < f(q) < m + \frac{3}{2}\}.$$

For each point $q \in A_m$, there is a basis open subset containing q and contained in A'_m. The collection of all such basis sets is an open cover of A_m. Since f is an exhaustion function, A_m is compact, and therefore it is covered by finitely many of these basis sets. Let B_m be the union of this finite collection of sets. This is a finite De Rham cover of B_m, so because of Lemma 4.15, B_m is De Rham.

Observe that $B_m \subset A'_m$, so B_m can have nonempty intersection with $B_{\tilde{m}}$ only when $\tilde{m} = m - 1$, m, or $m + 1$. Therefore, if we define

$$U = \bigcup_{m \text{ odd}} B_m, \quad V = \bigcup_{m \text{ even}} B_m,$$

then U and V are disjoint unions of De Rham manifolds, and so they are both De Rham by Step 1. Finally, $U \cap V$ is De Rham because it is the disjoint union of the sets $B_m \cap B_{m+1}$ for $m \in \mathbb{Z}$, each of which has a finite De Rham cover consisting of sets of the form $U_i \cap U_k$, where U_i and V_j are basis sets used to define B_m and B_{m+1}, respectively. Thus, $M = U \cup V$ is De Rham by Lemma 4.15.

If U is an open subset of $\mathbb{R}^n$ for some integer n, then U has a basis consisting of Euclidean balls. Because each ball is convex, it is De Rham, and because any finite intersection of balls is convex, finite intersections are also De Rham. This means that U has a De Rham basis and consequently U is De Rham.

Any smooth manifold has a basis given by smooth charts, each of these is diffeomorphic to an open subset of $\mathbb{R}^n$ as are their finite intersection. Since we proved that any subset of $\mathbb{R}^n$ is De Rham and if M has a De Rham basis M is De Rham, it follows that is De Rham. We have then proved the theorem.

Chapter 5

Conclusions

We are going to use this section to provide the reader with an intuitive meaning of the De Rham theorem. Since we started this work, with homological algebra, we began with abstract concepts so it is only natural that the theorem seems somehow too far from intuition. The expert reader may skip this section since one can find it overly simplistic, however the goal here is not to be very formal but to help visualize this whole work.

De Rham cohomology is a mathematical tool that allows us to study the topology of a smooth manifold using differential forms. It provides a way to measure the "holes" or "twists" in a manifold that cannot be detected by considering only its homology.

The main idea behind De Rham cohomology is to associate cohomology classes to closed differential forms on a manifold. These cohomology classes capture the information about the topology of the manifold in a way that is independent of the specific choice of coordinates.

The cohomology groups in De Rham cohomology, denoted as $H^p(M)$, classify the closed forms up to exact forms. Each cohomology group $H^p(M)$ represents a different "degree" of non-exactness or non-triviality. For example, $H^0(M)$ counts the number of globally constant functions on the manifold, while $H^1(M)$ counts the number of non-trivial closed 1-forms that are not exact.

The dimension of each cohomology group $H^p(M)$ corresponds to the number of independent non-trivial closed p-forms on the manifold. If the dimension is zero, it means that all closed p-forms on the manifold are exact, indicating a lack of "holes" or "twists" in that degree. Let us take a more detailed look at this fact.

Homology groups can be used as a tool to count how many holes there are in a manifold. Cohomology in general does not have such a direct correspondence to the "counting of holes". Nevertheless, for simple enough manifolds (that is for

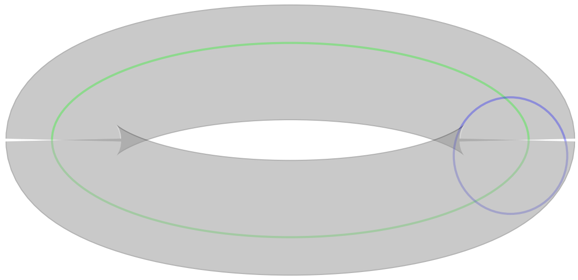

Figure 5.1: Visual representation of a 3D torus

imstance submanifolds of $\mathbb{R}^n$), cohomology gives the same intuition as homology. Taking this into account let us illustrate a fairly easy example.
We can give an intuitive meaning to the cohomology of this torus. In very informal terms, the p-cohomology group counts, through its generators how many independent p-spheres cannot be retracted to a single point.

The green line in figure 5 cannot be contracted to a single point so we have what is called a 1-dimensional hole. The purple line cannot be contracted to a single point nor can it be transformed to the green loop. Thus the first cohomology group will have two generators.

Let us now consider a n-sphere, since we computed their cohomology groups it is interesting to check how our intuition matches the results. For n=0, the sphere is a point and so its only non trivial cohomology group is $H^0(\mathbb{S}^0)$. For n=1, we can not contract any two points to a single point because they are not connected, so $H^0(\mathbb{S}^1)$ is non trivial, furthermore in a loop, we cannot contract a loop to a single point so $H^1(\mathbb{S}^1)$ is non trivial. In a 2-sphere, again since we cannot contract any two different given points, $H^0(\mathbb{S}^2)$. However, in a 2-sphere we can obviously contract any loop to a point (see figure 5), then it follows $H^1(\mathbb{S}^2)$=0. Now again, in a 2-sphere we cannot contract 2-spheres so $H^2(\mathbb{S}^2)$ is non trivial.

In the same logic it is quite clear that in any n-sphere we cannot contract any two points nor an n-sphere, however we can contract any n-1 sphere. Even more, since all possible two-points or n-sphere are non contractible in $\mathbb{S}^2$, there are infinite many generators so this groups are simply $\mathbb{R}$. Writing these results we obtain

$$H^p_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{if } 0 < p < n. \end{cases}$$

As we proved in Theorem 4.2.

Let us now give an intuition to the closed and exact forms. Recall that in

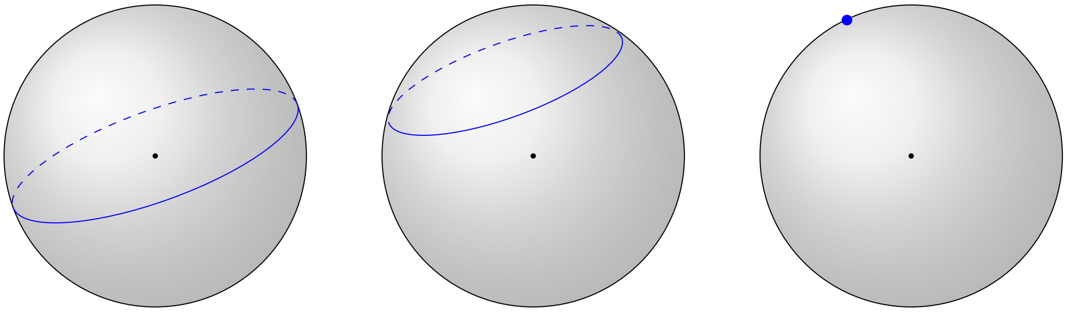

Figure 5.2: Visual representation of contracting a 1-dimensional loop to a single point in a 3D-sphere

chapter 2, in section 2.2 we mentioned a relationship between a cycle and a closed form and between a boundary and an exact form. Intuitively this translates to a closed form being a hole with no boundary and an exact form being a hole is a form that is the differential of another form.
In physics, what one can measure are forces and their associated fields. It is thus very common to calculate the potential given a field. Consider for instance Maxwell's equations in the vacuum in compact notation

$$dF = 0$$

This expression of Maxwell's equations states, simply put, that F must be a closed form. Furthermore, as we discussed this is equivalent to demanding that F is a cycle. Seeking a potential, is basically finding a form A such that $F = dA$. Obviously this is implying that F is an exact form.Which again, as we discussed, is equivalent to demanding that F is a boundary.

If there are no p-dimensional holes in the manifold, its pth De Rham cohomology group is trivial, so this must mean that every exact form is closed.
In physical terms, if there are no holes in the space-time manifold, every field F derived from a potential must satisfy $dF = 0$. One could think of charges as holes in this space-time and so in the absence of these charges, the Faraday tensor satisfies $dF = 0$. This is a huge simplification, but it provides some sense of understanding and above all we hope it gives the reader the sense of relevance of the De Rham cohomology. The magnificent beauty of the De Rham's Theorem lies in the encoding of the information of a given smooth manifold. Independently of the choice of charts, one can contain the relationship between exact and closed forms (very relevant in physics) in a purely algebraic object, the singular cohomology group which is isomorphic to the De Rham cohomology group.
Applications of the theorem could be a whole thesis on its own, in spite of this,

we aimed to give a brief insight. Hopefully the reader is struck with the same feeling of amusement as I did while studying this topic and realizing its potential significance in physics.

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