Universitat de Barcelona
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GRAU DE MATEMÀTIQUES

Treball final de grau

Generalizing Pick's Theorem

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Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 13 de juny de 2023

Contents

Abstract

The main goal of this project is to analyze the function that counts the number of integer points inside the dilates of a lattice polytope. One of the first result in this area is Pick's Theorem [1] proven by the mathematician Georg Alexander Pick in 1899, which gives the area of a polygon by counting the points on its border and its interior. In the 1960s, the French mathematician Eugène Ehrhart further explored this field and proved that for lattice d-polytopes, these functions are polynomials of degree d [2]. For this reason, these polynomials are called Ehrhart polynomials. In the following years, with the work of Richard P. Stanley and Ian G. Macdonald, the fundations of this field, Ehrhart Theory, were stablished.

Our research into this topic started with [3], results from which are stated on Chapter 6.

In the first chapter, we present and prove Pick's Theorem. Moreover, we also show that no analogue to this formula that expresses the volume of a polyhedron as a function only of its numbers of interior and boundary points exists.

From the second to the fifth chapter, we will follow Ehrhart Theory as it was done in [4]. All of the proofs are inspired in the ones that appear in this book, with some changes to make them easier to understand and the exercises left to the reader solved. In particular, in the second chapter, we make the first definitions on which this field of mathematics revolves. In addition, we also enunciate Ehrhart's Theorem, the first grand theorem on the field. In the third chapter, we develop the necessary mathematical tools needed to prove the before mentioned theorem. Lastly, in chapters four and five, we analyze the properties of the coefficients of the Ehrhart polynomial without new tools and with the help of the Ehrhart-Macdonald reciprocity respectively.

In the sixth chapter, as we said before, we present the study done by Max Kölbl, explaining it and showing its results. Moreover, some proofs were added or reworked for their presentation in this work.

Finally, in the seventh chapter, we return to the title of this paper and prove an n-dimensional generalization for Pick's Theorem that we have arrived to ourselves.

2020 Mathematics Subject Classification: 12D10, 30C15, 52B20, 52C07

Chapter 1

Pick's Theorem

Let us define a simple polygon as a region of the plane homeomorphic to a disk, whose border is a union of finite segments.

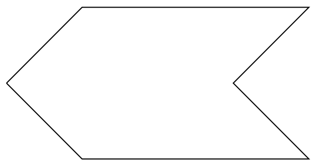

Figure 1.1: Simple polygon

With this definition, let us start with a classical result:

Theorem 1.1. (Pick's Theorem) *Given a simple polygon whose vertices have integer coordinates, let i be the number of integer points in thr interior to the polygon, and b the number of integer points on its boundary. Then the area of this polygon is equal to*

$$i + \frac{b}{2} - 1.$$

Proof. First, we will prove the result for rectangles with sides of length m and n parallel to the axes. It is easy to see that the number of interior points is $i = (m-1)(n-1)$, the number of boundary points $b = 2(m+n)$ and, the area $A = mn$. Then, we have that

$$i + \frac{b}{2} - 1 = mn - m - n + 1 + m + n - 1 = mn = A.$$

Thus, the Theorem holds.

Second, we will prove the result for right triangles with short sides of length m and n parallel to the axes. Suppose that there are d points on the hypotenuse, not counting the vertices. It is easy to see that we can construct a rectangle with two equal right triangles. Therefore, the number of interior points to the right triangle is $i = \frac{(m-1)(n-1)-d}{2}$, the number of boundary points $b = m + n + 1 + d$ and an area of $A = \frac{mn}{2}$. So,

$$i + \frac{b}{2} - 1 = \frac{mn - m - n + 1 - d}{2} + \frac{m + n + 1 + d}{2} - 1 = \frac{mn}{2} = A$$

and the claim is proven.

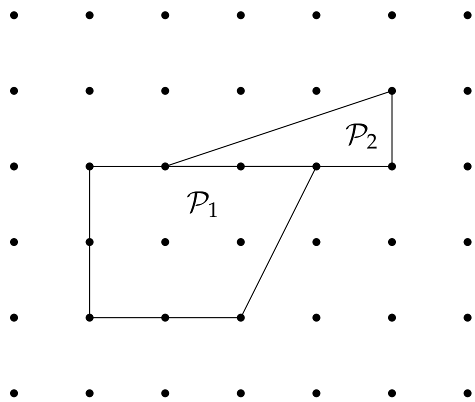

Figure 1.2: Two simple polygons whose union make another simple polygon

Next, we will prove that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two simple polygons following this rule with non-overlapping interiors and with a segment in common, then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ also follows it. Let l be number of boundary points in common. Therefore, $l \geq 2$ because they form an edge. Moreover, these points (except the vertices) will form part of the interior of $\mathcal{P}$. Let i, i_1, i_2 the number of interior points, b, b_1, b_2 the number of boundary points and A, A_1, A_2 the areas of $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Thus,

$$i = i_1 + i_2 + l - 2 \qquad b = b_1 + b_2 - 2l + 2$$

$$\begin{aligned} A = i + \frac{b}{2} - 1 &= i_1 + i_2 + l - 2 + \frac{b_1 + b_2}{2} - l + 1 - 1 = \\ &= i_1 + \frac{b_1}{2} - 1 + i_2 + \frac{b_2}{2} - 1 = A_1 + A_2. \end{aligned}$$

So, $\mathcal{P}$ also satisfies the statement.

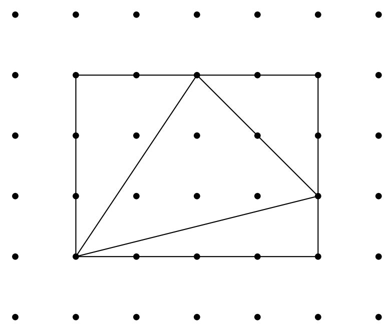

Figure 1.3: A general triangle and its bounding box

Now, let $\mathcal{P}$ be a general triangle and consider its bounding box, that is, the smallest rectangle with sides parallel to the axes that contains it. From this definition, we can see that all three of its vertices must be on the sides of the bounding box and, moreover, one of them must be on a corner. If that was not the case, there would be a side of the box not touching the triangle and so, we could find a smaller one. Therefore, the three sides of the triangle make, with the sides of the bounding box, at most three right triangles. Due to the fact that the right triangles and the rectangle of the bounding box all satisfy Pick's Theorem, the general triangle does too.

Finally, we will prove the result generally by using induction on the number of sides of the polygon. Suppose that Pick's Theorem applies to all simple polygons with integer vertices and less than k sides, and let $\mathcal{P}$ be a k sided polygon. If there is an interior diagonal, we can divide the polygon into two simple polygons with non-overlapping interiors with less than k sides each, and so, Pick's Theorem applies to them. Then due to the fact that the original polygon is the union of both of them, it also satisfies Pick's Theorem.

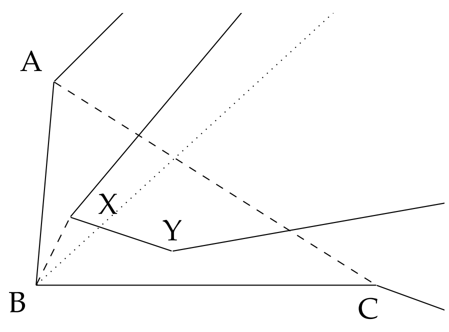

Figure 1.4: Proof of the existence of an interior diagonal

To prove that there is an interior diagonal, let A, B, C be three consecutive vertices such that they form an interior angle of less than $180°$. If AC is an interior diagonal we are done. If this is not the case, there is at least one vertex inside the triangle $\triangle ABC$ due to the fact that the polygon is simple. We take all these vertices and project them on the bisector of $\widehat{ABC}$. Then, the line from B to the vertex with the closest projection onto it is an interior diagonal (See Figure 1.4).

Thus, the induction hypothesis holds and the Theorem is proven. $\square$

Now, we will see that there is no analog to this formula that expresses the volume of a polyhedron as a function only of its numbers of interior and boundary points. For this reason, following [5], we make the following definition.

Definition 1.2. (Revee tetrahedra) *The Reeve tetrahedra is a family of polyhedra in three-dimensional space with vertices at* $(0,0,0), (1,0,0), (0,1,0)$ *and* $(1,1,h)$ *where h is a positive integer.*

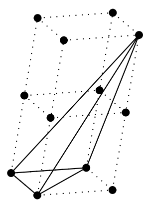

Figure 1.5: Reeve tetrahedron with $h = 2$

Proposition 1.3. Let $\mathcal{P}$ be a Revee tetrahedron. Then, we have that $\mathcal{P}$ has 0 interior points, 4 boundary points and its volume is $h/6$.

Proof. We can see that $\mathcal{P}$ is inside a 1×1 column of the grid, and so, it cannot have any interior points. Moreover, due to this fact, the only points that it can have are the 4 vertices on its boundary. Its volume comes from the formula

$$V = \frac{A_b h}{3} = \frac{h}{6}.$$

$\square$

Therefore, we can see that i and b don't depend on h, however, the volume does. So, the volume of a polyhedron cannot be a function of its numbers of interior and boundary points.

Chapter 2

Introduction to Ehrhart Polynomials

To find a result that encompasses Pick's Theorem, we have to look into something more general than Theorem 1.1. As we will see, this is highly complicated and, for this reason, we will restrict ourselves mainly to convex regions of $\mathbb{R}^n$.

First of all, we must define the generalization of a polygon. Informally, a polytope is a geometric object with flat sides, also called faces. The dimension of a polytope is the dimension of the affine space that it spans. An n-dimensional polytope is called n-polytope, where we have that 2-polytopes are polygons and 3-polytopes are polyhedra.

Definition 2.1. A convex polytope in $\mathbb{R}^d$ is the convex hull of finitely many points. In other words, for a finite set $\{\vec{v}_1, ..., \vec{v}_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is the smallest convex set containing them all. We have that

$$\mathcal{P} = \{\lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n \mid \lambda_k \geq 0 \; \forall \; k \in \{1, ..., n\}, \lambda_1 + \cdots + \lambda_n = 1\}.$$

The following notation will be used:

$$\mathcal{P} = \text{convex}(\{\vec{v}_1, ..., \vec{v}_n\}).$$

We will define the vertices of a convex polytope as the elements of the minimal subset V of $\{\vec{v}_1, ..., \vec{v}_n\}$ such that $\mathcal{P} = \text{convex}(V)$. A d-polytope with $d+1$ vertices is called a simplex.

Generally, Ehrhart Theory is defined over any lattice structure. However, in this text we will only work on the integer lattice $\mathbb{Z}^d$. The polytopes with vertices on a lattice are called lattice polytopes and, in particular, when we use the integer lattice they are also called integral polytopes.

Definition 2.2. Let $\mathcal{P}$ be a polytope and $t \in \mathbb{Z}^+$. We define the <u>t-dilate</u> of $\mathcal{P}$ as

$$t\mathcal{P} = \left\{ \vec{x} \;\middle|\; \frac{1}{t}\vec{x} \in \mathcal{P} \right\} = \{ t\vec{x} \mid \vec{x} \in \mathcal{P} \}$$

Definition 2.3. Let $\mathcal{P}$ be an integral polytope. We define the counting function for the points in the t-dilated polytope as

$$L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^d) = \# \left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right) \qquad \forall\, t \in \mathbb{Z}^+$$

The following is one of the main Theorem of this text and we will devote the next chapter to prove it.

Theorem 2.4. (Ehrhart's Theorem) *If $\mathcal{P}$ is an integral convex d-polytope, then $L_{\mathcal{P}}(t)$ is a polynomial of degree d in t. This polynomial is called the <u>Ehrhart polynomial</u> of $\mathcal{P}$.*

Chapter 3

Proving Ehrhart's Theorem

3.1 Triangulations

As we will see, most of the claims in Ehrhart Theory are easily shown on simplices. So, it would be a good idea if we could divide any convex polytope into simplices. For this matter, we make the following definitions:

Definition 3.1. A supporting hyperplane of a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is a hyperplane $H = \{\vec{x} \in \mathbb{R}^d \mid \vec{u} \cdot \vec{x} = w\}$ such that either $\vec{u} \cdot \vec{x} \geq w \ \forall \ \vec{x} \in \mathcal{P}$ or $\vec{u} \cdot \vec{x} \leq w \ \forall \ \vec{x} \in \mathcal{P}$, where $\vec{u} \in \mathbb{R}^d, w \in \mathbb{R}, \vec{u} \neq 0$ are constants of the hyperplane modulo multiplication by a nonzero scalar.

Definition 3.2. A face of a convex d-polytope is its intersection with one of its supporting hyperplanes. The $(d-1)$-dimensional faces are called facets.

These definitions give rise to an alternative definition of convex polytope.

Definition 3.3. A convex polytope in $\mathbb{R}^d$ is a compact set such that

$$\mathcal{P} = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \geq \vec{b}\}$$

where $\vec{b} \in \mathbb{R}^d$ and A is a matrix. Every component of the inequality corresponds to a single supporting hyperplane and so, it is not unique. Moreover, it can be proven that its interior is

$$\mathcal{P}^\circ = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} > \vec{b}\}.$$

In addition, the minimal A is when it is composed by the hyperplanes that define the facets of the polytope.

Definition 3.4. A triangulation of a d-polytope $\mathcal{P}$ is a finite collection T of d-simplices that has the following properties

- $\mathcal{P} = \bigcup\limits_{\Delta \in T} \Delta.$
- $\forall\ \Delta_1, \Delta_2 \in T, \Delta_1 \cap \Delta_2$ is a face common to both simplices.

Now, we will show that a triangulation is possible with the vertices of the polytope, and so, with integral simplices.

Theorem 3.5. (Existence of triangulation) *Every convex polytope can be triangulated so that every vertex of every simplex of the triangulation is a vertex of the polytope in question.*

The following proof is highly technical and it is only added for completion.

Proof. Without loss of generality, assume that $\mathcal{P}$ is full dimensional (it is not contained in a hyperplane) with vertices $\{\vec{v}_1, ..., \vec{v}_n\} \subset \mathbb{R}^d$. Let us randomly choose $h_1, ..., h_n \in \mathbb{R}$ and construct the polytope $\mathcal{Q} = \text{convex}(\{(\vec{v}_1, h_1), ..., (\vec{v}_n, h_n)\}) \subset \mathbb{R}^{d+1}$. We define a lower facet F of $\mathcal{Q}$ as a facet such that $(x_1, ..., x_{d+1} - \varepsilon) \notin \mathcal{Q}\ \forall\ \varepsilon > 0$, $(x_1, ..., x_{d+1}) \in F$.

Now, we will prove that every lower facet of $\mathcal{Q}$ is a simplex. We choose any $d+1$ vertices of $\mathcal{P}$, without loss of generality, we assume that they are the first ones. These define the hyperplane with equation

$$\det \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \vec{v}_1 & \cdots & \vec{v}_{d+1} & \vec{x} \\ h_1 & \cdots & h_{d+1} & x_{d+1} \end{pmatrix} = 0.$$

For it to be a simplex, we need that, if we replace $(\vec{x}, x_{d+1})$ for any of the $(\vec{v}_j, h_j)$ $\forall\ j > d+1$, the equation isn't satisfied. This imposes a restriction on the value of h_j. Doing this for every set of $d+1$ vertices imposes a finite amount of restrictions but, due to the fact that $h_1, ..., h_n$ are selected randomly, this is possible.
Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ such that $\pi(x_1, ..., x_d, x_{d+1}) = (x_1, ..., x_d)$. We define

$$T = \{\pi(F) \mid F \text{ is a lower facet of } \mathcal{Q}\}.$$

Let us prove that this is a triangulation. By definition, we have that $\mathcal{P} \supseteq \bigcup_{\Delta \in T} \Delta$. To demonstrate the equality we will see that $\mathcal{P}^\circ \subseteq \bigcup_{\Delta \in T} \Delta$.

Let $\vec{x} \in \mathcal{P}^\circ$ and consider the line $L = \{\vec{x} + \lambda \vec{e}_{d+1} \mid \lambda \in \mathbb{R}\}$. Since $\vec{x} \in \mathcal{P}^\circ$, we have that $L \cap \mathcal{Q}^\circ \neq \varnothing$. Then $L \cap \mathcal{Q}$ is a segment with endpoints $(\vec{x}, y)$ and $(\vec{x}, z)$

where $y < z$. Since $(\vec{x}, y)$ is on the boundary of $\mathcal{Q}$, it is contained in some face of $\mathcal{Q}$. Thus, we can find a supporting hyperplane that defines F

$$H = \{\vec{v} \in \mathbb{R}^{d+1} \mid \vec{a} \cdot \vec{v} = b\}$$

such that

$$\mathcal{Q} \subseteq \{\vec{v} \in \mathbb{R}^{d+1} \mid \vec{a} \cdot \vec{v} \geq b\}.$$

We have to notice that $(\vec{x}, z)$ is not in H, otherwise, the whole segment would belong to the face, which is a contradiction with $L \cap \mathcal{Q}^\circ \neq \varnothing$. So $\vec{a} \cdot (\vec{x}, y) = b$ and $\vec{a} \cdot (\vec{x}, z) > b$, which gives that $a_{d+1}(z - y) > 0$ and so, $a_{d+1} > 0$. We will prove that this is a lower facet:

$$\forall\ \vec{x} \in F, \varepsilon > 0 \quad \vec{a} \cdot (\vec{x} - \varepsilon\, \vec{e}_{d+1}) = b - \varepsilon\, a_{d+1} < b.$$

Thus, it is proven that F is a lower facet with $\vec{x} \in \pi(F)$, and so, $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$. Moreover, due to the fact that $a_{d+1} \neq 0$, F is not aligned with the projection and, because it is a d-simplex, $\pi(F)$ is also a d-simplex. Therefore, T is a collection of d-simplices. In addition, we have that if F_1, F_2 are two lower facets, then $F_1 \cap F_2$ is a face common to both F_1 and F_2 and so, $\pi(F_1) \cap \pi(F_2)$ is a face common to both $\pi(F_1)$ and $\pi(F_2)$.

This concludes with the proof that T is a triangulation of $\mathcal{P}$. $\square$

3.2 Cones

Another helpful tool that we will use are cones. We will see later that a cone can encompass all of the dilates of a given polytope. For this reason, we start with the following definitions:

Definition 3.6. A pointed cone $\mathcal{K} \subseteq \mathbb{R}^d$ is a set of the form

$$\mathcal{K} = \{\vec{v} + \lambda_1 \vec{w}_1 + \cdots + \lambda_n \vec{w}_n \mid \lambda_1, ..., \lambda_n \geq 0\}$$

such that there exists a hyperplane H that satisfies that $\mathcal{K} \cap H = \{\vec{v}\}$. The point $\vec{v}$ is called the apex of $\mathcal{K}$ and the vectors $\vec{w}_1, ..., \vec{w}_n$ are its generators. The dimension of $\mathcal{K}$ is the dimension of the affine space spanned by it. If the dimension of $\mathcal{K}$ is d, we say that it is a d-cone. A d-cone is simplicial if it has exactly d linearly independent generators.

Definition 3.7. Let $\mathcal{P} \subset \mathbb{R}^d$ be a convex polytope with vertices $\vec{v}_1, ..., \vec{v}_n$. We define the cone over $\mathcal{P}$ as

$$\operatorname{cone}(\mathcal{P}) = \{\lambda_1 \vec{w}_1 + \cdots + \lambda_n \vec{w}_n \mid \lambda_1, ..., \lambda_n \geq 0\} \subset \mathbb{R}^{d+1}$$

where $\vec{w}_i = (\vec{v}_i, 1)\ \forall\ i \in \{1, ..., n\}$.

Remark 3.8. As we have already said, this structure is useful due to the fact that if we intersect the cone with the plane $x_{d+1} = t$, we get

$$\text{cone}(\mathcal{P}) \cap \{\vec{x} \in \mathbb{R}^{d+1} \mid x_{d+1} = t\} = \{(\vec{x}, t) \in \mathbb{R}^{d+1} \mid \vec{x} \in t\mathcal{P}\} = t\mathcal{P} \times \{t\},$$

that is, the t-dilated of $\mathcal{P}$ at height t.

Now, following the definitions in the previous section, we make the following:

Definition 3.9. A <u>supporting hyperplane</u> of a cone $\mathcal{K} \subset \mathbb{R}^d$ is a hyperplane $H = \{\vec{x} \in \mathbb{R}^d \mid \vec{u} \cdot \vec{x} = w\}$ such that either $\vec{u} \cdot \vec{x} \geq w \;\forall\, \vec{x} \in \mathcal{K}$ or $\vec{u} \cdot \vec{x} \leq w \;\forall\, \vec{x} \in \mathcal{K}$, where $\vec{u} \in \mathbb{R}^d, w \in \mathbb{R}, \vec{u} \neq 0$ are constants of the hyperplane modulo multiplication by a nonzero scalar.

Definition 3.10. A <u>face</u> of a d-cone is its intersection with one of its supporting hyperplanes. The $(d-1)$-dimensional faces are called <u>facets</u>.

As in the case of polytopes, it will be easier to work on simplicial cones. For this reason, we adapt the concept of triangulation to cones.

Definition 3.11. A <u>triangulation</u> of a d-cone $\mathcal{K}$ is a finite collection T of simplicial d-cones having the following properties

- $\mathcal{K} = \bigcup\limits_{\mathcal{S} \in T} \mathcal{S}$.
- $\forall\, \mathcal{S}_1, \mathcal{S}_2 \in T, \mathcal{S}_1 \cap \mathcal{S}_2$ is a face common to both cones.

Theorem 3.12. (Existence of triangulations for pointed cones) *Every pointed cone can be triangulated so that every generator of every simplicial cone of the triangulation is a generator of the cone in question.*

Proof. Let $\mathcal{K}$ be a d-cone. By definition, there exists a hyperplane H such that $\mathcal{K} \cup H = \{\vec{v}\}$, where $\vec{v}$ is the apex of $\mathcal{K}$. Let $\vec{w} \in \mathcal{K}^\circ$, then

$$\mathcal{P} = (\vec{w} - \vec{v} + H) \cap \mathcal{K}$$

is a $(d-1)$-dimensional convex polytope whose vertices are defined by the generators of $\mathcal{K}$. By Theorem 3.5, we can triangulate $\mathcal{P}$ using no new vertices. Then, each of the simplices Δ_i in this triangulation gives a simplicial cone

$$\mathcal{S}_i = \{\vec{v} + \lambda\vec{x} \mid \lambda \geq 0, \vec{x} \in \Delta_i\},$$

which, by construction, triangulate $\mathcal{K}$ with no new generator. $\square$

3.3 Integer-Point Transforms

To keep track of the points and simplifying the process of counting them, we can encode the integer points in a set using a sum of Laurent monomials.

Definition 3.13. Let $S \subseteq \mathbb{R}^d$ be a set. The integer-point transform of S is the following function

$$\sigma_S(\vec{z}) = \sum_{\vec{m} \in S \cap \mathbb{Z}^d} \vec{z}^{\vec{m}}$$

where $\vec{z}, \vec{m} \in \mathbb{R}^d$ and $\vec{z}^{\vec{m}} = z_1^{m_1} \cdots z_d^{m_d}$ is the element-wise exponentiation.

For simplicial cones, this function has the following form:

Theorem 3.14. *Suppose we have a simplicial d-cone*

$$\mathcal{K} = \{\lambda_1 \vec{w}_1 + \cdots + \lambda_d \vec{w}_d \mid \lambda_1, ..., \lambda_d \geq 0\}$$

with $\vec{w}_1, ..., \vec{w}_d \in \mathbb{Z}^d$. Then, for every $\vec{v} \in \mathbb{R}^d$,

$$\sigma_{\vec{v}+\mathcal{K}}(\vec{z}) = \frac{\sigma_{\vec{v}+\Pi}(\vec{z})}{(1 - \vec{z}^{\vec{w}_1}) \cdots (1 - \vec{z}^{\vec{w}_d})}$$

where Π is the half-open parallelepiped

$$\Pi = \{\lambda_1 \vec{w}_1 + \cdots + \lambda_d \vec{w}_d \mid 0 \leq \lambda_1, ..., \lambda_d < 1\}.$$

Proof. Let $\vec{m} \in (\vec{v} + \mathcal{K}) \cup \mathbb{Z}^d$. By definition, we can write

$$\vec{m} = \vec{v} + \lambda_1 \vec{w}_1 + \cdots + \lambda_d \vec{w}_d$$

for $\lambda_1, ..., \lambda_d \geq 0$. In particular, due to the fact that $\vec{w}_1, ..., \vec{w}_d$ is a basis of $\mathbb{R}^d$, this is done uniquely. Writing λ_i in term of its integer and fractional parts ($\lambda_i = \lfloor \lambda_i \rfloor + \{\lambda_i\}$), we have

$$\vec{m} = (\vec{v} + \{\lambda_1\}\vec{w}_1 + \cdots + \{\lambda_d\}\vec{w}_d) + \lfloor \lambda_1 \rfloor \vec{w}_1 + \cdots + \lfloor \lambda_d \rfloor \vec{w}_d$$

where $0 \leq \{\lambda_1\}, ..., \{\lambda_d\} < 1$, so

$$\vec{p} = \vec{v} + \{\lambda_1\}\vec{w}_1 + \cdots + \{\lambda_d\}\vec{w}_d \in \vec{v} + \Pi.$$

In particular, $\vec{p} \in \mathbb{Z}^d$ since $\vec{m}$ and $\lfloor \lambda_i \rfloor \vec{w}_i$ are all integer vectors. Thus, we can uniquely write

$$\vec{m} = \vec{p} + k_1 \vec{w}_1 + \cdots + k_d \vec{w}_d$$

where $\vec{p} \in \vec{v} + \Pi, k_1, ..., k_d \in \mathbb{Z}^+$, and so

$$\sigma_{\vec{v}+\mathcal{K}}(\vec{z}) = \sum_{\vec{m} \in (\vec{v}+\mathcal{K}) \cap \mathbb{Z}^d} \vec{z}^{\vec{m}} = \sum_{\vec{p} \in (\vec{v}+\Pi)} \vec{z}^{\vec{p}} \sum_{k_1 \geq 0} \vec{z}^{k_1 \vec{w}_1} \cdots \sum_{k_d \geq 0} \vec{z}^{k_d \vec{w}_d} =$$

$$= \sigma_{\vec{p} \in (\vec{v}+\Pi)}(\vec{z}) \frac{1}{1 - \vec{z}^{\vec{w}_1}} \cdots \frac{1}{1 - \vec{z}^{\vec{w}_d}} = \frac{\sigma_{\vec{v}+\Pi}(\vec{z})}{(1 - \vec{z}^{\vec{w}_1}) \cdots (1 - \vec{z}^{\vec{w}_d})}.$$

□

3.4 Proof of Ehrhart's Theorem

To finally prove Ehrhart's Theorem, we will need the following two lemmas:

Lemma 3.15. *Let f and g be functions such that*

$$\sum_{t\geq 0} f(t)z^t = \frac{g(z)}{(1-z)^{d+1}}.$$

Then, f is a polynomial of degree d, if and only if, g is a polynomial of degree at most d and $g(1) \neq 0$.

Proof. First, we will prove the $\Leftarrow$ implication. Using the Taylor expansion, we have that

$$\frac{1}{(1-z)^{d+1}} = \sum_{t\geq 0} \binom{d+t}{t} z^t = \sum_{t\geq 0} \binom{d+t}{d} z^t = \sum_{t\geq 0} \frac{(t+d)\cdots(t+1)}{d!} z^t$$

and $\forall\ a \geq 0$

$$\frac{z^a}{(1-z)^{d+1}} = \sum_{t\geq 0} \binom{d+t}{d} z^{t+a} = \sum_{t\geq 0} \binom{d-a+t}{d} z^t.$$

Assume now that $g(z) = \sum_{n=0}^{d} a_n z^n$. Then

$$\frac{g(z)}{(1-z)^{d+1}} = \sum_{t\geq 0} \left[\sum_{n=0}^{d} a_n \binom{d-n+t}{d} \right] z^t \Rightarrow f(t) = \sum_{n=0}^{d} a_n \binom{d-n+t}{d}$$

where we have that f is the sum of polynomials of degree d. If f has degree d, it must have non-zero leading coefficient. Thus, if we look only at the terms of degree d, we have

$$\sum_{n=0}^{d} a_n \frac{t^d}{d!} = \frac{t^d}{d!} \sum_{n=0}^{d} a_n = \frac{t^d}{d!} g(1) \neq 0 \Rightarrow g(1) \neq 0.$$

To prove the $\Rightarrow$ implication, we must solve the system of equations:

$$\begin{cases} a_0 \binom{d}{d} + a_1 \binom{d-1}{d} + \cdots + a_d \binom{0}{d} = a_0 = f(0) \\ a_0 \binom{d+1}{d} + a_1 \binom{d}{d} + \cdots + a_d \binom{1}{d} = (d+1)a_0 + a_1 = f(1) \\ \qquad\vdots \\ a_0 \binom{2d}{d} + \cdots + a_d \binom{d}{d} = f(d) \end{cases}$$

Which is a triangular system of linear equations with only one solution. The condition that $g(1) \neq 0$ is the same as in the previous implication. $\square$

Definition 3.16. The generating function of the Ehrhart polynomial is called the Ehrhart series. For an integral convex polytope $\mathcal{P}$ we have

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$$

Lemma 3.17. $\sigma_{\mathrm{cone}(\mathcal{P})}(1, ..., 1, z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \mathrm{Ehr}_{\mathcal{P}}(z)$.

Proof. This proof follows straightforward by evaluating the Integer-Point transform at $(1, ..., 1, z)$ given in Definition 3.13. □

Proof of Ehrhart's Theorem (Theorem 2.4). It will suffice to prove the theorem for simplices. This is due to the fact that any integral convex polytope can be triangulated in integral simplices (Theorem 3.5) without using any new vertices. Moreover, the intersection of these simplices are lower dimensional integral simplices too. Then, by Lemma 3.15, it suffices to prove that for an integral d-simplex

$$\mathrm{Ehr}_{\Delta}(z) = 1 + \sum_{t \geq 1} L_{\Delta}(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$$

for some polynomial g of degree at most d and $g(1) \neq 0$. In Lemma 3.17 we have seen that $\mathrm{Ehr}_{\Delta}(z) = \sigma_{\mathrm{cone}(\Delta)}(1, ..., 1, z)$. Thus, we will study the integer-point transform of $\mathrm{cone}(\Delta)$.

Let $\vec{v}_1, ..., \vec{v}_{d+1}$ be the vertices of Δ. Then, $\mathrm{cone}(\Delta) \subseteq \mathbb{R}^{d+1}$ is simplicial and has generators $\vec{w}_1 = (\vec{v}_1, 1), ..., \vec{w}_{d+1} = (\vec{v}_{d+1}, 1)$. By Theorem 3.14, we have that

$$\sigma_{\mathrm{cone}(\mathcal{P})}(1, ..., 1, z) = \frac{\sigma_{\Pi}(1, ..., 1, z)}{\prod_{n=1}^{d+1}(1-z)} = \frac{\sigma_{\Pi}(1, ..., 1, z)}{(1-z)^{d+1}}$$

where $\Pi = \{\lambda_1 \vec{w}_1 + \cdots + \lambda_{d+1} \vec{w}_{d+1} \mid 0 \leq \lambda_1, ..., \lambda_{d+1} < 1\}$. Due to the fact that Π is bounded, its integer-point transform is a Laurent polynomial, and because all of the x_{d+1} coordinates of the generators are 1, we have that $0 \leq \lambda_1 + \cdots + \lambda_{d+1} < d + 1$ and, because this sum must be an integer, its maximum is d. So, $\sigma_{\Pi}(1, ..., 1, z)$ is a polynomial of degree at most d in z. Moreover, $\sigma_{\Pi}(1, ..., 1, 1) = \#(\Pi \cup \mathbb{Z}^{d+1})$ and because the origin is always in Π, we have that $\sigma_{\Pi}(1, ..., 1, 1) \neq 0$. So, all the conditions for Lemma 3.15 and the Ehrhart polynomial is, indeed, a polynomial with degree d. □

Although we have only talked about convex polytopes as Ehrhart first did, we have that the finite union of polytopes, also called polytopal complexes or polytopes in general, also has a corresponding Ehrhart polynomial. In this way, other properties can also be generalized.

Chapter 4

Exploring the Ehrhart polynomial

4.1 Ehrhart h-polynomial

As we have seen, in some cases it is easier to extract the polynomial g from the Ehrhart series than to compute the Ehrhart polynomial. For this reason, it has a name of its own.

Definition 4.1. Let f be a polynomial of degree d. We define its $\underline{h\text{-polynomial}}$ as

$$h(z) = h_d z^d + \cdots + h_1 z + h_0 = (1-z)^{d+1} \sum_{t \geq 0} f(t) z^t$$

with $h(1) \neq 0$.

This is indeed a polynomial by Lemma 3.15.

Definition 4.2. Let $\mathcal{P}$ be an integral d-polytope. We define its $\underline{h\text{-polynomial}}$[1] as the h-polynomial of its Ehrhart polynomial

$$h_{\mathcal{P}}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^* = (1-z)^{d+1} \mathrm{Ehr}_{\mathcal{P}}(z)$$

From the work done in the previous chapter, we can deduce the following properties of the coefficients of both Ehrhart polynomial and its h-polynomial.

Proposition 4.3. Suppose Δ is an integral d-simplex with vertices $\vec{v}_1, ..., \vec{v}_{d+1}$ and let $\vec{w}_j = (\vec{v}_j, 1)\ \forall\, j \in \{1, ..., d+1\}$. Then, the k-th coefficient of the h-polynomial of Δ is equal to the number of integer points in

$$\{\lambda_1 \vec{w}_1 + \cdots + \lambda_d \vec{w}_d \ \mid\ 0 \leq \lambda_1, ..., \lambda_d < 1\}$$

with last coordinate equal to k.

[1] Also called h-vector or δ-vector/polynomial

Proof. By Theorem 3.14 and Lemma 3.17, we see that

$$h_\Delta(z) = (1-z)^{d+1}\sigma_{\text{cone}(\Delta)}(1,...,1,z) = \sigma_\Pi(1,...,1,z),$$

and from Definition 3.13 the coefficient of the term z^k is the number of integer points in Π with last coordinate equal to k. □

Proposition 4.4. Let $\mathcal{P}$ be an integral polytope with Ehrhart polynomial $L_\mathcal{P}(t) = c_d t^d + \cdots + c_1 t + c_0$ and h-polynomial $h_\mathcal{P}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^*$. Then $h_0^* = c_0$ and $h_1^* = \#(\mathcal{P}\cap\mathbb{Z}^m) - (d+1)c_0$.

Proof. The proof follows straightforward from the proof of Lemma 3.15, solving

$$\begin{cases} h_0^* = L_\mathcal{P}(0) = c_0 \\ (d+1)h_0^* + h_1^* = L_\mathcal{P}(1) = \#(\mathcal{P}\cap\mathbb{Z}^m) \end{cases}$$

□

We can also show that all the coefficients of the h-polynomial are non negative integers. However, the proof of this theorem is very technical and it is only added for completeness.

Theorem 4.5. (Stanley's nonnegativity theorem) *Suppose $\mathcal{P}$ is an integral convex d-polytope. Then, the coefficients of its h-polynomial are nonnegative integers.*

Proof. Triangulate cone$(\mathcal{P}) \subset \mathbb{R}^{d+1}$ into simplicial cones $\mathcal{K}_1,...,\mathcal{K}_m$. We will see that there exists a vector $\vec{v} \in \mathbb{R}^{d+1}$ such that

$$\text{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1} = (\vec{v} + \text{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1}$$

and no boundary of the cones $\vec{v} + \mathcal{K}_i$ contain any lattice point.

All these boundaries are collections of hyperplanes. Therefore, we will see which conditions are needed for a hyperplane to contain no integer points. Let $\vec{p} \in \mathbb{R}^{d+1}$ be the apex of the cone and $\vec{w}^1,..,\vec{w}^d \in \mathbb{Z}^{d+1}$ the generators that define it. Thus, we have that

$$H = \{\vec{p} + \lambda_1\vec{w}^1 + \cdots + \lambda_d\vec{w}^d \;\mid\; \lambda_1,...,\lambda_d \in \mathbb{R}\}$$

Let $\vec{z} \in H \cap \mathbb{Z}^{d+1}$, then

$$\begin{cases} z_1 = p_1 + \lambda_1\vec{w}_1^1 + \cdots + \lambda_d\vec{w}_1^d \\ \vdots \\ z_{d+1} = p_{d+1} + \lambda_1\vec{w}_{d+1}^1 + \cdots + \lambda_d\vec{w}_{d+1}^d \end{cases}$$

Since $\vec{w}^1, .., \vec{w}^d$ form a hyperplane in $\mathbb{R}^{d+1}$, they are linearly independent, and so, the first d equations give a unique result for $\lambda_1, ..., \lambda_d$ which is a rational number plus a fraction of $p_1, ..., p_d$ respectively. Then, if we pick $p_1, ..., p_d \in \mathbb{Q}$, we have that all $\lambda_1, ..., \lambda_d \in \mathcal{Q}$. Therefore, if p_{d+1} is irrational $\vec{z} \notin H$ because no $\vec{z} \in \mathbb{Z}^{d+1}$ can satisfy the last equation. For this reason, it is enough for all of the coordinates of $\vec{p}$ to be rational except one.

Now we will look to maintain the same integer points after the translation of the cone. Suppose the hyperplane of a facet is $H = \{\vec{x} \in \mathbb{R}^{d+1} \mid \vec{a} \cdot \vec{x} = b\}$ and that all of the cones are such that $\vec{a} \cdot \vec{x} \geq b \; \forall \; \vec{x} \in \text{cone}(\mathcal{P})$. Then, for the translation $\vec{v}$ we want

$$\forall \; \vec{x} \in \mathbb{Z}^{d+1} \mid \vec{a} \cdot \vec{x} \geq b \Rightarrow \vec{a} \cdot (\vec{v} + \vec{x}) > b \Rightarrow \vec{a} \cdot \vec{v} > 0$$

$$\forall \; \vec{x} \in \mathbb{Z}^{d+1} \mid \vec{a} \cdot \vec{x} < b \Rightarrow \vec{a} \cdot (\vec{v} + \vec{x}) < b \Rightarrow \vec{a} \cdot \vec{v} < b - \vec{a} \cdot \vec{x}$$

Due to the fact that the coordinate vectors $\vec{e}_1, ..., \vec{e}_{d+1}$ form a basis, one of these must extend $\vec{w}_1, ..., \vec{w}_d$ into a base of $\mathbb{R}^{d+1}$. Without loss of generality suppose that $\vec{e}_1$ does it. Let $\mu = \vec{e}_1 \cdot (\vec{w}_1 + \cdots + \vec{w}_d)$ and consider the parallelepiped Π generated by the vectors $\vec{w}_1, ..., \vec{w}_d, \mu\vec{e}_1$. These tile the space, with the lattice points included, due to the fact that all vectors are in $\mathbb{Z}^{d+1}$. Let two opposing corners of Π be elements of H, then, the nearest points to the hyperplane are in this parallelepiped or the adjacent ones. Between these, there is a finite number of points, and so, we can define the minimum of $b - \vec{a} \cdot \vec{x}$ which we will call $m_{\vec{a}}$. Then, we have that for this hyperplane $0 < \vec{a} \cdot \vec{v} < m_{\vec{a}}$ which is an open set. Because we have a finite amount of boundary hyperplanes, we have the intersection of a finite number of open sets, which is open. Additionally, consider M the minimum of all $m_{\vec{a}}$. Let $\vec{w} = \vec{w}_1 + \cdots + \vec{w}_n$ where $\vec{w}_1, ..., \vec{w}_n$ are all generators of $\text{cone}(\mathcal{P})$ and $\vec{p}$ its apex, then $\vec{a} \cdot \vec{w} > 0 \; \forall \; \vec{a}$. This is because if $\vec{w}_i$ is a generator of the hyperplane of $\vec{a}$, then $\vec{a} \cdot \vec{w}_i = 0$ and if it isn't, the point $\vec{p} + \vec{w}_i$ is in $\text{cone}(\mathcal{P})$ so $\vec{a} \cdot (\vec{p} + \vec{w}_i) > b$, and because $\vec{p}$ is on the boundary by definition $\vec{a} \cdot \vec{p} = b$, we have that $\vec{a} \cdot \vec{w}_i > 0$. Then the vector $\frac{M}{2|\vec{p}|}\vec{p}$ satisfies all inequalities, and so there is a solution. Moreover, because $\mathbb{Q}^d \times (\mathbb{R} \setminus \mathbb{Q})$ is dense in $\mathbb{R}^{d+1}$ and the solution space is open with at least one solution, there is a solution of this kind and so, there exists a vector $\vec{v} \in \mathbb{R}^{d+1}$ that satisfies all our conditions.

From this we have that every lattice point in $\vec{v} + \text{cone}(\mathcal{P})$ belongs to exactly one simplicial cone $\vec{v} + \mathcal{K}_j$

$$\text{cone}(\mathcal{P}) \cap \mathbb{Z}^{d+1} = (\vec{v} + \text{cone}(\mathcal{P})) \cap \mathbb{Z}^{d+1} = \bigcup_{j=1}^{m} ((\vec{v} + \mathcal{K}_j) \cap \mathbb{Z}^{d+1}),$$

which is a disjoint union. Thus, we have that

$$\sigma_{\text{cone}(\mathcal{P})}(z_1, ..., z_{d+1}) = \sum_{j=1}^{m} \sigma_{\vec{v}+\mathcal{K}_j}(z_1, ..., z_{d+1}).$$

From Proposition 4.3 we have that the coefficients of the h-polynomial of simplicial cones are nonnegative integers due to fact that they count points, and so, because the coefficients of the h-polynomial of $\mathcal{P}$ are sum of the nonnegative integers, they are too. □

This result, as Ehrhart's Theorem, can also be generalized to polytopes in general.

4.2 Ehrhart polynomial coefficients

Proposition 4.6. Suppose $\mathcal{P}$ is an integral convex d-polytope with h-polynomial $h_{\mathcal{P}}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^*$. Then $h_0^* = c_0 = 1$.

Proof. Following the proof of Stanley's nonnegativity theorem (Theorem 4.5) we have that the origin is in one and only one of the cones $\vec{v} + \mathcal{K}_j$ and so there is only one contribution to the constant term of the h-polynomial of $\mathcal{P}$. Moreover, the equality $h_0^* = c_0$ was proven in Proposition 4.4. □

This may seem redundant, due to the fact that the constant term of the Ehrhart series of a convex polytope was assumed to be 1. This term, however, corresponds to $L_{\mathcal{P}}(0) = c_0$ and so gives a geometric meaning to the assumption that $L_{\mathcal{P}}(0) = 1$.

Proposition 4.7. Suppose $\mathcal{P}$ is an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then, $d! c_k \in \mathbb{Z}$.

Proof. From Stanley's nonnegativity theorem (Theorem 4.5) we have that the coefficients of the h-polynomial of $\mathcal{P}$, $h_{\mathcal{P}}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^+$, are all integers. Now, using the proof of Lemma 3.15, we have that

$$L_{\mathcal{P}}(t) = h_d^* \binom{t}{d} + \cdots + h_1^* \binom{t+d-1}{d} + h_0^* \binom{t+d}{d}$$

where all of the binomial coefficients can be written as fractions with denominator $d!$ □

To continue our search for the values of the coefficients of the Ehrhart polynomial, we will need the following lemma.

Lemma 4.8. *Suppose $\mathcal{P}$ is a d-polytope. Then*

$$vol\ \mathcal{P} = \lim_{t\to\infty} \frac{\#(t\mathcal{P} \cap \mathbb{Z}^d)}{t^d}.$$

Proof. By definition, $\text{vol}\ \mathcal{P} = \int_{\mathcal{P}} d\vec{x}$, and by using the definition of the Riemann integral, we can think computing the volume by approximating it with d-dimensional boxes that get smaller. In this case, we will let the side of each of these boxes to be $1/t$ with center on the grid $(\frac{1}{t}\mathbb{Z})^d$. The value of each of this boxes would be one if its center is in $\mathcal{P}$ and zero otherwise. So

$$\text{vol}\ \mathcal{P} = \int_{\mathcal{P}} d\vec{x} = \lim_{t\to\infty} \frac{1}{t^d}\#\left(\mathcal{P} \cap \left(\frac{1}{t}\mathbb{Z}\right)^d\right) = \lim_{t\to\infty} \frac{\#(t\mathcal{P} \cap \mathbb{Z}^d)}{t^d}.$$

□

Corollary 4.9. Supose $\mathcal{P} \subset \mathbb{R}^d$ is an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then its volume vol $\mathcal{P}$ is equal to c_d.

Proof. Using Lemma 4.8 we have that

$$\text{vol}\ \mathcal{P} = \lim_{t\to\infty} \frac{c_d t^d + \cdots + c_1 t + c_0}{t^d} = c_d.$$

□

Corollary 4.10. Supose $\mathcal{P} \subset \mathbb{R}^d$ is an integral d-polytope with h-polynomial

$$h_{\mathcal{P}}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^*.$$

Then, $\text{vol}\ \mathcal{P} = \dfrac{h_d^* + \cdots + h_1^* + h_0^*}{d!}$.

Proof. From the proof of Theorem 3.15 and Lemma 4.8, we have that

$$\text{vol}\ \mathcal{P} = \lim_{t\to\infty} \frac{L_{\mathcal{P}}(t)}{t^d} = \lim_{t\to\infty} \frac{1}{t^d}\left[h_d^*\binom{t}{d} + \cdots + h_1^*\binom{t+d-1}{d} + h_0^*\binom{t+d}{d}\right] =$$

$$= \frac{1}{d!}\left[h_d^* + \cdots + h_1^* + h_0^*\right]$$

□

This result might look simple, but it is astonishing. From counting something discrete (the points inside a polytope) we can compute something continuous (its volume). Moreover, we could only count the points from the first $d+1$ dilates to find the volume of the polytope without need of integration. In addition, we have extended the notion of the Ehrhart polynomial, which first only had geometric meaning for positive integers, to include the number zero. In the next chapter, we will continue to extend its domain.

Corollary 4.11. Let $\mathcal{P}$ be an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then we have that

$$\operatorname{vol}\mathcal{P} \min_{0\le i\le d} \binom{t+d-i}{d}_k \le \frac{c_k}{d!} \le \binom{t+d}{d}_k \operatorname{vol}\mathcal{P} \qquad \forall\, k$$

where $\binom{t+d}{d}_k$ represents the coefficient of the k-th term of the binomial coefficient as a polynomial.

Proof. Let $h_d^* t^d + \cdots + h_1^* t + h_0^*$ be the h-polynomial of $\mathcal{P}$. Then, the lower bound is, using Corollary 4.10 and Stanley's nonnegative theorem (Theorem 4.5):

$$c_k = \sum_{i=0}^{d} h_i^* \binom{t+d-i}{d}_k \ge \sum_{i=0}^{d} h_i^* \min_i \binom{t+d-i}{d}_k = d! \operatorname{vol}\mathcal{P} \min_{0\le i\le d} \binom{t+d-i}{d}_k.$$

For the upper bound, on the other hand, we have to show that

$$\max_{0\le i\le d} \binom{t+d-i}{d}_k = \binom{t+d}{d}_k.$$

Knowing that we can write $\binom{t+d-i}{d} = (t+d-i)(t+d-i-1)\cdots(t-i+1)/d!$ we see that, in absolute value, we will get larger coefficients if all of them have the same sign and with larger absolute values the better. The factors we get are from the list $t+d, t+d-1, ..., t-d+1$, therefore, the greatest value will come from $(t+d)(t+d-1)\cdots(t+1)/d! = \binom{t+d}{d}$ and so

$$c_k = \sum_{i=0}^{d} h_i^* \binom{t+d-i}{d}_k \le \sum_{i=0}^{d} h_i^* \binom{t+d}{d}_k = \binom{t+d}{d}_k d! \operatorname{vol}\mathcal{P}.$$

□

These bounds are valid for all polytopes. However we get other bounds, which may be better, if we impose further restrictions in the above derivation, like convexity ($h_0^* = 1$).

Chapter 5

Ehrhart-Macdonald reciprocity

One of the most interesting characteristics of the Ehrhart polynomial is that, although we first defined a geometric meaning only for the positive integers, we will show that it has geometric meaning in all of the integers.

Theorem 5.1. (Ehrhart-Macdonald reciprocity) *Suppose $\mathcal{P}$ is an integral convex polytope. Then, the evaluation of the polynomial $L_{\mathcal{P}}$ at negative integers yields*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t).$$

The following results will prove the theorem.

Lemma 5.2. Let $S \subseteq \mathbb{R}^d$ be a set, and $-S = \{-\vec{x} \mid \vec{x} \in S\}$. Then

$$\sigma_{-S}(z_1, ..., z_d) = \sigma_S \left(\frac{1}{z_1}, ..., \frac{1}{z_d} \right)$$

Proof. By definition

$$\sigma_{-S}(z_1, ..., z_d) = \sum_{m \in -S \cap \mathbb{Z}^d} z_1^{m_1} \cdots z_d^{m_d} = \sum_{m \in S \cap \mathbb{Z}^d} z_1^{-m_1} \cdots z_d^{-m_d} =$$

$$= \sum_{m \in S \cap \mathbb{Z}^d} \left(\frac{1}{z_1} \right)^{m_1} \cdots \left(\frac{1}{z_d} \right)^{m_d} = \sigma_S \left(\frac{1}{z_1}, ..., \frac{1}{z_d} \right)$$

□

Proposition 5.3. Fix linearly independent vectors $\vec{w}_1, ..., \vec{w}_d \in \mathbb{Z}^d$ and let $\mathcal{K}$ be the simplicial cone generated by all of them. Then, if $\vec{v} \in \mathbb{R}^d$ is such that the boundary of the shifted cone $\vec{v} + \mathcal{K}$ contains no integer points,

$$\sigma_{\vec{v}+\mathcal{K}} \left(\frac{1}{z_1}, ..., \frac{1}{z_d} \right) = (-1)^d \sigma_{-\vec{v}+\mathcal{K}}(z_1, ..., z_d).$$

Proof. From Theorem 3.14 we have that

$$\sigma_{\vec{v}+\mathcal{K}}(\vec{z}) = \frac{\sigma_{\vec{v}+\Pi}(\vec{z})}{(1-\vec{z}^{\vec{w}_1})\cdots(1-\vec{z}^{\vec{w}_d})},$$

$$\sigma_{-\vec{v}+\mathcal{K}}(\vec{z}) = \frac{\sigma_{-\vec{v}+\Pi}(\vec{z})}{(1-\vec{z}^{\vec{w}_1})\cdots(1-\vec{z}^{\vec{w}_d})},$$

with

$$\Pi = \{\lambda_1\vec{w}_1 + \cdots + \lambda_d\vec{w}_d \ \mid \ 0 \leq \lambda_1, ..., \lambda_d < 1\}.$$

Due to the fact that the boundary contains no integer points, we can consider $\vec{v}+\Pi$ and $-\vec{v}+\Pi$ as open. Then, we have that

$$-(-\vec{v}+\Pi) + \vec{w}_1 + \cdots + \vec{w}_d =$$

$$= \{-(-\vec{v}+\lambda_1\vec{w}_1 + \cdots + \lambda_d\vec{w}_d) + \vec{w}_1 + \cdots + \vec{w}_d \ \mid \ 0 < \lambda_1, ..., \lambda_d < 1\} =$$

$$= \{\vec{v} + (1-\lambda_1)\vec{w}_1 + \cdots + (1-\lambda_d)\vec{w}_d \ \mid \ 0 < \lambda_1, ..., \lambda_d < 1\} =$$

$$= \{\vec{v} + \mu_1\vec{w}_1 + \cdots + \mu_d\vec{w}_d \ \mid \ 0 < \mu_1, ..., \mu_d < 1\} = \vec{v} + \Pi,$$

and so,

$$\sigma_{\vec{v}+\Pi}(\vec{z}) = \sigma_{-(-\vec{v}+\Pi)}(\vec{z})\vec{z}^{\vec{w}_1}\cdots\vec{z}^{\vec{w}_d} = \sigma_{-\vec{v}+\Pi}\left(\frac{1}{\vec{z}}\right)\vec{z}^{\vec{w}_1}\cdots\vec{z}^{\vec{w}_d}$$

where we have used Lemma 5.2 and $1/\vec{z} = (1/z_1, ..., 1/z_d)$. Then,

$$\sigma_{\vec{v}+\mathcal{K}}\left(\frac{1}{\vec{z}}\right) = \frac{\sigma_{\vec{v}+\Pi}(1/\vec{z})}{(1-\vec{z}^{-\vec{w}_1})\cdots(1-\vec{z}^{-\vec{w}_d})} = \frac{\sigma_{-\vec{v}+\Pi}(\vec{z})\vec{z}^{\vec{w}_1}\cdots\vec{z}^{\vec{w}_d}}{(1-\vec{z}^{-\vec{w}_1})\cdots(1-\vec{z}^{-\vec{w}_d})} =$$

$$= \frac{\sigma_{-\vec{v}+\Pi}(\vec{z})}{(\vec{z}^{\vec{w}_1}-1)\cdots(\vec{z}^{\vec{w}_d}-1)} = (-1)^d\frac{\sigma_{-\vec{v}+\Pi}(\vec{z})}{(1-\vec{z}^{\vec{w}_1})\cdots(1-\vec{z}^{\vec{w}_d})} = (-1)^d\sigma_{-\vec{v}+\mathcal{K}}(\vec{z})$$

□

Theorem 5.4. (Stanley reciprocity) *Suppose $\mathcal{K}$ is a d-cone with generators in $\mathbb{Z}^d$ and the origin as apex. Then*

$$\sigma_{\mathcal{K}}\left(\frac{1}{z_1}, ..., \frac{1}{z_d}\right) = (-1)^d\sigma_{\mathcal{K}^\circ}(z_1, ..., z_d)$$

Proof. We triangulate $\mathcal{K}$ into simplicial cones $\mathcal{K}_1, ..., \mathcal{K}_m$. Then, similarly to the proof of Stanley's nonnegativity theorem (Theorem 4.5), using a system of inequalities, we find that there exists a vector $\vec{v} \in \mathbb{R}^d$ such that

$$(\vec{v} + \mathcal{K}) \cap \mathbb{Z}^d = \mathcal{K}^\circ \cap \mathbb{Z}^d \tag{5.1}$$

and there are no lattice points on the boundary of any triangulation cone or the cones $-\vec{v} + \mathcal{K}_j$. Moreover, we will prove that

$$(-\vec{v} + \mathcal{K}) \cap \mathbb{Z}^d = \mathcal{K} \cap \mathbb{Z}^d \tag{5.2}$$

From (5.1) we have that

$$(\vec{v} + \mathcal{K}) \cap \mathbb{Z}^d = \{\vec{v} + \lambda_1\vec{w}_1 + \cdots + \lambda_n\vec{w}_n \in \mathbb{Z}^d \mid \lambda_1, ..., \lambda_n \geq 0\} =$$

$$= \{\lambda_1\vec{w}_1 + \cdots + \lambda_n\vec{w}_n \in \mathbb{Z}^d \mid \lambda_1, ..., \lambda_n > 0\} = \mathcal{K}^\circ \cap \mathbb{Z}^d.$$

Let $\vec{v} = \nu_1\vec{w}_1 + \cdots + \nu_n\vec{w}_m$. Then, from the previous equality, we have that $\forall\ \lambda_1, ..., \lambda_n > 0$ such that $\lambda_1\vec{w}_1 + \cdots + \lambda_n\vec{w}_n \in \mathbb{Z}^d, \lambda_i - \nu_i \geq 0\ \ \forall\ i \in \{1, ..., n\}$. This implies that $\mathcal{K} \cap \mathbb{Z}^d \subseteq (-\vec{v} + \mathcal{K}) \cap \mathbb{Z}^d$.

To prove the opposite inclusion, suppose that there exists a point $\vec{x} \in ((-\vec{v} + \mathcal{K}_k) \setminus \mathcal{K}_k) \cap \mathbb{Z}^d$. Without loss of generality, we assume that $\vec{w}_1, ..., \vec{w}_d \in \mathbb{Z}^d$ be the generators of $\mathcal{K}_k$. Then, we can uniquely write

$$\vec{x} = -\vec{v} + \lambda_1\vec{w}_1 + \cdots + \lambda_d\vec{w}_d = (\lambda_1 - \nu_1)\vec{w}_1 + \cdots + (\lambda_d - \nu_d)\vec{w}_d$$

with $\lambda_1, ..., \lambda_d \geq 0$. However, due to the fact that $\vec{x} \notin \mathcal{K}_k$, without loss of generality, we assume that $\lambda_d - \nu_d < 0$. Let $\vec{y} = \lceil\lambda_1\rceil\,\vec{w}_1 + \cdots + \lceil\lambda_{d-1}\rceil\,\vec{w}_{d-1} \in \overline{\mathcal{K}_k} \cap \mathbb{Z}^d$ and consider the point symmetric to $\vec{x}$ with respect to $\vec{y}$

$$\vec{z} = \vec{y} + (\vec{y} - \vec{x}) = \vec{v} + (2\lceil\lambda_1\rceil - \lambda_1)\vec{w}_1 + \cdots + (2\lceil\lambda_{d-1}\rceil - \lambda_{d-1})\vec{w}_{d-1} - \lambda_d\vec{w}_d =$$

$$= (2\lceil\lambda_1\rceil + \nu_1 - \lambda_1)\vec{w}_1 + \cdots + (2\lceil\lambda_{d-1}\rceil + \nu_{d-1} - \lambda_{d-1})\vec{w}_{d-1} + (\nu_d - \lambda_d)\vec{w}_d \in \mathbb{Z}^d$$

We can see that $\vec{z} \in (\mathcal{K}_k^\circ \setminus (\vec{v} + \mathcal{K}_k)) \cap \mathbb{Z}^d$ and, in particular, $\vec{z} \in (\mathcal{K}^\circ \setminus (\vec{v} + \mathcal{K})) \cap \mathbb{Z}^d$, which is a contradiction. For this reason, no such $\vec{x}$ exists and (5.2) holds.
By using Proposition 5.3 with (5.1) and (5.2), we have

$$\sigma_{\mathcal{K}}\left(\frac{1}{\vec{z}}\right) = \sigma_{-\vec{v}+\mathcal{K}}\left(\frac{1}{\vec{z}}\right) = \sum_{j=1}^{m} \sigma_{-\vec{v}+\mathcal{K}_j}\left(\frac{1}{\vec{z}}\right) =$$

$$= \sum_{j=1}^{m} (-1)^d \sigma_{\vec{v}+\mathcal{K}_j}(\vec{z}) = (-1)^d \sigma_{\vec{v}+\mathcal{K}}(\vec{z}) = (-1)^d \sigma_{\mathcal{K}^\circ}(\vec{z}).$$

□

Theorem 5.5. *Suppose that $\mathcal{P}$ is an integral convex polytope. Then, the evaluation of the rational function $Ehr_{\mathcal{P}}$ at $1/z$ yields*

$$Ehr_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^{\dim \mathcal{P}+1} Ehr_{\mathcal{P}^\circ}(z)$$

Proof. Suppose that $\mathcal{P}$ is a convex d-polytope. Then, by using Lemma 3.6 and Theorem 5.4, we have

$$\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \sigma_{\mathrm{cone}(\mathcal{P})}\left(\frac{1}{z}\right) = (-1)^{d+1}\sigma_{\mathrm{cone}(\mathcal{P})^\circ}(z) = (-1)^{d+1}\sigma_{\mathrm{cone}(\mathcal{P}^\circ)}(z) =$$

$$= (-1)^{d+1}\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$$

□

Before continuing with the next lemma we will make the following clarification. Although the following sums converge on different regions of the complex plane, in this lemma, we will consider the sums as their analytical continuation, in other words, as the rational functions they define.

Lemma 5.6. *Let $f(t)$ be a polynomial, and consider the rational functions arising from*

$$S^+(z) = \sum_{t\geq 0} f(t)z^t \qquad S^-(z) = \sum_{t<0} f(t)z^t,$$

then $S^+(z) + S^-(z) = 0$

Proof. Taking the negative powers of $z-1$ and calculating their Taylor expansion at $z=0$ and $z=+\infty$, we have

$$\frac{1}{(z-1)^n} = \frac{(-1)^n}{(1-z)^n} = (-1)^n \sum_{t\geq 0}\binom{n+t-1}{t} z^t = (-1)^n \sum_{t\geq 0}\binom{n+t-1}{n-1} z^t,$$

$$\frac{1}{(z-1)^n} = \frac{z^{-n}}{(1-z^{-1})^n} = z^{-n}\sum_{t\geq 0}\binom{n+t-1}{t} z^{-t} = \sum_{t\geq n}\binom{t-1}{n-1} z^{-t}$$

where

$$P_n(t) = \binom{n+t-1}{n-1} = \frac{(n+t-1)\cdots(t+1)}{(n-1)!},$$

$$Q_n(t) = \binom{t-1}{n-1} = \frac{(t-1)\cdots(t-n+1)}{(n-1)!},$$

$$Q_n(-t) = \frac{(-t-1)\cdots(-t-n+1)}{(n-1)!} = (-1)^{n-1}\frac{(t+1)\cdots(n-t-1)}{(n-1)!} = (-1)^{n-1}P_n(t).$$

So

$$\frac{1}{(z-1)^n} = (-1)^n \sum_{t\geq 0} P(t)z^t = \sum_{t\geq n} Q(t)z^{-t} = \sum_{t>0} Q(t)z^{-t} = (-1)^{n+1} \sum_{t<0} P(t)z^t.$$

It can be shown that $\{P_n\}_{0\leq n\leq d}$ is a basis of $\mathbb{R}$-vector space of the polynomials with degree at most d. Therefore, we have that $f(t) = \sum_{n=0}^{d} a_n P_n(t)$ and so

$$S^+(z) + S^-(z) = \sum_{t\geq 0} f(t)z^t + \sum_{t<0} f(t)z^t = \sum_{t\geq 0} f(t)z^t + \sum_{t>0} f(-t)z^{-t} =$$

$$= \sum_{t\geq 0}\sum_{n=0}^{d} a_n P_n(t)z^t + \sum_{t>0}\sum_{n=0}^{d} a_n P_n(-t)z^{-t} =$$

$$= \sum_{t\geq 0}\sum_{n=0}^{d} a_n P_n(t)z^t + (-1)^{n-1}\sum_{t>0}\sum_{n=0}^{d} a_n Q_n(-t)z^{-t} =$$

$$= \sum_{n=0}^{d} a_n \left(\sum_{t\geq 0} P_n(t)z^t + (-1)^{n-1}\sum_{t>0} Q_n(-t)z^{-t}\right) =$$

$$= \sum_{n=0}^{d} a_n \left(\frac{(-1)^n}{(z-1)^n} + \frac{(-1)^{n-1}}{(z-1)^n}\right) = 0$$

and we are done. □

With all these tools, we can now proceed to prove the Ehrhart-Macdonald reciprocity.

Proof of Ehrhart-Macdonald reciprocity (Theorem 5.1). Taking the Ehrhart series as a rational function and using Lemma 5.6 we have

$$\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \sum_{t\geq 0} L_{\mathcal{P}}(t)\left(\frac{1}{z}\right)^t = \sum_{t\leq 0} L_{\mathcal{P}}(-t)z^t = -\sum_{t\geq 1} L_{\mathcal{P}}(-t)z^t,$$

and using Theorem 5.5

$$\sum_{t\geq 1} L_{\mathcal{P}^\circ}(t)z^t = (-1)^{d+1}\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = (-1)^d \sum_{t\geq 1} L_{\mathcal{P}}(-t)z^t.$$

And the reciprocity follows from comparing the coefficients of the power series on the left and right hand-side of the equation. □

This result, can be extended to general polytopes too.

Lemma 5.7. Suppose that p is a degree d polynomial with h-polynomial $h_d z^d + \cdots + h_1 z + h_0$. Then, $h_d = h_{d-1} = \cdots = h_{k+1} = 0, h_k \neq 0$, if and only if, $p(-1) = p(-2) = \cdots = = p(-d+k) = 0, p(-d+k-1) \neq 0$

Proof. Suppose that $h_d = h_{d-1} = \cdots = h_{k+1} = 0, h_k \neq 0$. Then, the proof of Lemma 3.15 gives

$$p(t) = h_0 \binom{d+t}{d} + h_1 \binom{d+t-1}{d} + \cdots + h_k \binom{d+t-k}{d}$$

where we have that all binomial coefficients are zero for $t = -1, ..., -d+k$. Moreover, for $t = -d+k-1$ all of the binomial coefficients except the last one are zero, and since $h_k \neq 0$, it is not a root of p.
Conversely, suppose that $p(-1) = p(-2) = \cdots = p(-d+k) = 0, p(-d+k-1) \neq 0$, then

$$0 = p(-1) = h_0 \binom{d-1}{d} + h_1 \binom{d+t-1}{d} + \cdots + h_{d+1} \binom{0}{d} + h_d \binom{-1}{d} = h_d \binom{-1}{d}$$

and so $h_d = 0$. With this reasoning, we have iteratively that the root $-d+k$ gives that $h_{k+1} = 0$. To show that $h_k \neq 0$, we suppose that $h_k = 0$ following the same reasoning as in the first part, we would have that $p(-d+k-1) = 0$, which is a contradiction. $\square$

Theorem 5.8. *Suppose that $\mathcal{P}$ is an integral d-polytope with h-polynomial*

$$h_{\mathcal{P}}(z) = h_d^* z^d + \cdots + h_1^* z + h_0^*.$$

Then, $h_d^* = \cdots = h_{k+1}^* = 0, h_k^* \neq 0$, if and only if, $(d-k+1)\mathcal{P}$ is the smallest dilate of $\mathcal{P}$ that contains an interior lattice point.

Proof. Using Lemma 5.7 we have that h_k^* is the highest nonzero coefficient, if and only if, $L_{\mathcal{P}}(-1) = \cdots = L_{\mathcal{P}}(-d+k) = 0, L_{\mathcal{P}}(-d+k-1) \neq 0$, and using the Ehrhart-Macdonald reciprocity (Theorem 5.1) the claim follows. $\square$

Corollary 5.9. Let $\mathcal{P} \subset \mathbb{R}^m$ be an integral d-polytope with h-polynomial $h_{\mathcal{P}}(z) = h_d^* t^d + \cdots + h_1^* t + h_0^*$. Then $h_d^* = \#(\mathcal{P}^\circ \cap \mathbb{Z}^m)$.

Proof. From the Ehrhart-Macdonald reciprocity (Theorem 5.1) and the proof of Lemma 3.15 we have

$$\#(\mathcal{P}^\circ \cap \mathbb{Z}^m) = L_{\mathcal{P}^\circ}(1) =$$

$$= (-1)^d L_{\mathcal{P}}(-1) = (-1)^d \left[\binom{d-1}{d} h_0^* + \cdots + \binom{0}{d} h_1^* + \binom{-1}{d} h_d^* \right] = h_d^*$$

$\square$

5.1 Ehrhart Series of Reflexive Polytopes

There is a group of polytopes that have a very interesting property, their h-polynomial is palindromic. These are the reflexive polytopes.

Definition 5.10. A polytope $\mathcal{P}$ is reflexive if it is integral and it can be described as

$$\mathcal{P} = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq \vec{1}\}$$

where A is an integral matrix and $\vec{1}$ is a vector which has all components equal to 1.

Example 5.11. The d-cube $[-1,1]^d$ is a reflexive polytope. As we can see, it is equivalent to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Example 5.12. The standard reflexive simplex of dimension d, which is defined as

$$\Delta_{sr}^d = \text{convex}\left(\left\{e_1, ..., e_d, -\sum_{i=1}^{d} e_i\right\}\right)$$

is a reflexive polytope because it is equivalent to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -d & 1 & \cdots & 1 \\ 1 & -d & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Theorem 5.13. (Hibi's palindromic theorem) *Suppose $\mathcal{P}$ is an integral d-polytope that contains the origin in its interior. Then $\mathcal{P}$ is reflexive, if and only if, its h-vector is palindromic, that is, if $h_k^* = h_{d-k}^*$ $\forall\ 0 \leq k \leq d/2$.*

Proof. First, we will see that a $\mathcal{P}$ is reflexive if and only if

$$(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = t\mathcal{P} \cap \mathbb{Z}^d \qquad \forall\ t \in \mathbb{Z}_{\geq 0}.$$

Suppose that $\mathcal{P}$ is reflexive, then

$$t\mathcal{P} = \{t\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq \vec{1}\} = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq t\vec{1}\}$$

and

$$t\mathcal{P}^\circ = \{t\vec{x} \in \mathbb{R}^d \mid A\vec{x} < \vec{1}\} = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} < t\vec{1}\}.$$

So

$$(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = \{\vec{x} \in \mathbb{Z}^d \mid A\vec{x} < (t+1)\vec{1}\},$$
$$t\mathcal{P} \cap \mathbb{Z}^d = \{\vec{x} \in \mathbb{Z}^d \mid A\vec{x} \leq t\vec{1}\}.$$

Then, it is trivial that $t\mathcal{P} \cap \mathbb{Z}^d \subseteq (t+1)\mathcal{P}^\circ$. On the other hand, if $\vec{x} \in (t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d$, we have that $A\vec{x} < (t+1)\vec{1}$ but since A and $\vec{x}$ both have integer coefficients, the right-hand side must also have them, so $A\vec{x} \leq t\vec{1}$, and the equality holds.

Reciprocally, suppose that $(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = t\mathcal{P} \cap \mathbb{Z}^d$. Then if H is a facet of $\mathcal{P}$, there must be no lattice points between tH and $(t+1)H$. Indeed, assume that there was $t \in \mathbb{Z}$ such that there exists $\vec{x} \in \mathbb{Z}^d$ in between tH and $(t+1)H$. Let V be a vertex on H and $\vec{a}_1, ..., \vec{a}_{d-1}$ be a basis of generators of H with start at V and endpoints in vertices of $\mathcal{P}$ in H, and $\vec{a}_d$ the vector that goes from the origin to V. The first vectors form a parallelepiped that tile the hyperplane, and with $\vec{a}_d$, one that tiles all $\mathbb{R}^d$. Then, we can find an integer point congruent with $\vec{x}$ in the parallelepiped with origin in tA which has one face on tH and the opposite on $(t+1)H$. Moreover, due to the fact that $t+1 \geq 1$, this face is on a facet of $(t+1)\mathcal{P}$. Therefore, if we translate the parallelepiped once with $\vec{a}_d$, we have an integer point $\vec{p} \in \mathbb{Z}^d$ such that $\vec{p} \in (t+2)\mathcal{P}^\circ \cap \mathbb{Z}^d$ and $\vec{p} \notin (t+1)\mathcal{P} \cap \mathbb{Z}^d$, which is a contradiction.

Suppose $H = \{\vec{x} \in \mathbb{R}^d \mid a_1x_1 + \cdots + a_dx_d = b\}$ and let $\vec{v}^1, ..., \vec{v}^{d+1} \in \mathbb{Z}^d$ be points such that they uniquely define H. Due to the fact that the origin is in the interior of $\mathcal{P}$, we have $b \neq 0$. Suppose $b = 1$ and solve the system of linear equations given by $\vec{a} \cdot \vec{v}^i = b \; \forall \; 0 \leq i \leq d+1$. This system gives rational solutions for $a_1, ..., a_d$. Multiplying both sides of the equation by the least common divisor of their denominators, we have a solution such that $a_1, ..., a_d \in \mathbb{Z}^d, \gcd(a_1, ..., a_d) = 1$ and $b \in \mathbb{Z}^+$. Then we must have that

$$\{\vec{x} \in \mathbb{Z}^d \mid tb < a_1x_1 + \cdots + a_dx_d < (t+1)b \; \forall \; t \in \mathbb{Z}_{\geq 0}\} = \varnothing \tag{5.3}$$

Bézout's identity tells us that there exists $\vec{x} \in \mathbb{Z}^d$ such that $a_1x_1 + \cdots + a_dx_d = \gcd(a_1, ..., a_d) = 1$. Then if $b > 1$ we can find $z \in \mathbb{Z}$ such that $tb < z < (t+1)b$. This is a contradiction with (5.3) and, therefore, we have that $b = 1$. Arranging all of the equations of the facets in one matrix A, we have $A\vec{x} \le \vec{1}$ as we wanted to prove.

Moreover, Theorem 5.5 tells us that

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = (-1)^{d+1}\mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right) = \frac{h_0^* z^{d+1} + h_1^* z^d + \cdots + h_d^* z}{(1-z)^{d+1}}.$$

Using the previous result we have that $\mathcal{P}$ is reflexive if and only if

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = \sum_{t\ge 1} L_{\mathcal{P}^\circ}(t)z^t = \sum_{t\ge 1} L_{\mathcal{P}}(t-1)z^t = z\sum_{t\ge 0} L_{\mathcal{P}^\circ}(t)z^t = z\mathrm{Ehr}_{\mathcal{P}}(z) =$$

$$= \frac{h_d^* z^{d+1} + \cdots + h_1^* z^2 + h_0^* z}{(1-z)^{d+1}}$$

that is, if and only if, $h_k^* = h_{d-k}^*\ \forall\ 0 \le k \le d/2$. $\square$

Corollary 5.14. The roots of a reflexive d-polytope $\mathcal{P}$ are symmetrically distributed with respect to the so called critical line ($\mathrm{Re}(z) = -1/2$).

Proof. From the proof of Hibi's palindromic theorem (Theorem 5.13), we have that $(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d = t\mathcal{P}^\cap \mathbb{Z}^d$. By using Ehrhart-Macdonald reciprocity (Theorem 5.1), $L_{\mathcal{P}}(t) = L_{\mathcal{P}^\circ}(t+1) = (-1)^d L_{\mathcal{P}}(-t-1)$. Therefore, if t is a root of $L_{\mathcal{P}}$, we have that $-t-1$ is a root too. Thus, the claim is proven $\square$

Definition 5.15. Suppose $\mathcal{P}$ an integral d-polytope. We define its k-face polynomial as

$$F_{\mathcal{P}}^k(t) = c_{k,k}t^k + \cdots + c_{k,1}t + c_{k,0} = \sum_{\substack{\mathcal{F}\subseteq\mathcal{P}\\ \dim\mathcal{F}=k}} L_{\mathcal{F}}(t)$$

where $\mathcal{F}$ are the faces of $\mathcal{P}$.

Theorem 5.16. *Let $\mathcal{P}$ be an integral d-polytope. Then, we have the relation*

$$L_{\mathcal{P}}(t) = F_{\mathcal{P}}^d(t) = \sum_{j=0}^{d}(-1)^j F_{\mathcal{P}}^j(-t).$$

Proof. The first equality is trivial. We will prove the second equality by induction on the dimension of $\mathcal{P}$. If $d = 0$, we have that $L_{\mathcal{P}}$ is a constant equal to its number of points and so $L_{\mathcal{P}}(t) = L_{\mathcal{P}}(-t) = (-1)^0 F_{\mathcal{P}}^0(-t)$.
Suppose that the claim holds for $d-1$. Let $\vec{x} \in t\mathcal{P}$. If $\vec{x} \in t\mathcal{P}^\circ$ we are counting this

point in the term $L_{\mathcal{P}^\circ}(t) = (-1)^d L_{\mathcal{P}}(-t) = (-1)^d F^d_{\mathcal{P}}(-t)$. If not, then $\vec{x} \in \partial(t\mathcal{P}) = t\partial\mathcal{P}$ and so we are counting this point in the term $F^{d-1}_{\mathcal{P}}(t)$, but because it has dimension $d-1$, the claim holds. Moreover, due to the fact that $(t\mathcal{P}) \cap \partial(t\mathcal{P}) = \varnothing$,

$$L_{\mathcal{P}}(t) = (-1)^d F^d_{\mathcal{P}}(-t) + F^{d-1}_{\mathcal{P}}(t) = \sum_{j=0}^{d} (-1)^j F^j_{\mathcal{P}}(-t).$$

Then, the induction hypothesis holds and the result does too. $\square$

Theorem 5.17. *Suppose $\mathcal{P}$ an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then*

$$c_{d-1}t^{d-1} + c_{d-3}t^{d-3} + \cdots = \frac{1}{2}\sum_{j=0}^{d-1} (-1)^j F^j_{\mathcal{P}}(-t).$$

Proof. Using the Ehrhart-Macdonald reciprocity we have that

$$L_{\mathcal{P}^\circ}(t) = (-1)^d L_{\mathcal{P}}(-t) = (-1)^d F^d_{\mathcal{P}}(-t).$$

Then, from Theorem 5.16

$$L_{\mathcal{P}}(t) - L_{\mathcal{P}^\circ}(t) = \sum_{j=0}^{d-1} (-1)^j F^j_{\mathcal{P}}(-t). \tag{5.4}$$

On the other hand, by writing $L_{\mathcal{P}}, L_{\mathcal{P}^\circ}$ as polynomials using the Ehrhart-Macdonald reciprocity (Theorem 5.1)

$$L_{\mathcal{P}}(t) - L_{\mathcal{P}^\circ}(t) = 2c_{d-1}t^{d-1} + 2c_{d-3}t^{d-3} + \cdots. \tag{5.5}$$

The result follows from equating the right-hand side of (5.4) and (5.5). $\square$

Corollary 5.18. If k and d have different parities, then

$$c_k = \frac{1}{2}\sum_{j=0}^{d-1} (-1)^{j+k} c_{j,k}.$$

If k and d have the same parity, then

$$0 = \frac{1}{2}\sum_{j=0}^{d-1} (-1)^{j+k} c_{j,k}.$$

Proof. These equations follow from expanding the face polynomials and equating coefficients of equal degree in the previous theorem. $\square$

5.2 Relative volume and Euler characteristic

To find further relations between a polytope and the coefficients of its Ehrhart polynomial, we will use the following.

Lemma 5.19. *Suppose that $S \subset \mathbb{R}^d$ is an m-polytope. Then, its relative volume to the lattice is equal to*

$$vol\ S = \lim_{t \to \infty} \frac{\#(tS \cap \mathbb{Z}^d)}{t^m}.$$

Corollary 5.20. Suppose that $\mathcal{P} \subset \mathbb{R}^d$ an integral m-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_m t^m + \cdots + c_1 t + 1$. Then, its relative volume $\mathrm{vol}\,\mathcal{P}$ is equal to c_m.

Corollary 5.21. Suppose $\mathcal{P}$ is an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then

$$c_{d-1} = \frac{1}{2} \sum_{\substack{\mathcal{F} \subset \mathcal{P} \\ \dim \mathcal{F} = d-1}} \mathrm{vol}\,\mathcal{F}.$$

Proof. The proof follows easily from the definition of the k-face polynomials (Definition 5.15) and Theorem 5.17. □

Proposition 5.22. Suppose $\mathcal{P}$ is an integral d-polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = c_d t^d + \cdots + c_1 t + c_0$. Then, $c_{d-1} \geq \dfrac{d+1}{2(d-1)!}$.

Proof. We have that the minimum relative volume of an integral m-polytope is $1/m!$. Moreover, a d-polytope has at least $d+1$ facets. Then, its relative surface is at least $\dfrac{d+1}{(d-1)!}$ and the result follows. □

Definition 5.23. A simplicial complex $\mathcal{S}$ is a set of simplices that satisfy the following conditions:

- Every face of a simplex from $\mathcal{S}$ is also in $\mathcal{S}$.
- The non-empty intersection of any two simplices $\sigma_1, \sigma_2 \in \mathcal{S}$ is a face of both σ_1 and σ_2.

For a simplicial complex, its Euler characteristic is equal to the alternating sum

$$\chi = k_0 - k_1 + k_2 - k_3 + \cdots$$

where k_n denotes the number of n-simplices in the complex.

Example 5.24. A simplicial decomposition of the square $[0,1]^2$ is

$$\begin{aligned}\mathcal{S} = \{&\{(0,0)\}, \{(1,0)\}, \{(1,1)\}, \{(0,1)\},\\ &\{[0,1] \times \{0\}\}, \{[0,1] \times \{1\}\}, \{\{0\} \times [0,1]\},\\ &\{\{1\} \times [0,1] \times \{0\}\}, \{(t,t) \mid t \in [0,1]\},\\ &\{(x,y) \mid 0 \le x \le y \le 1\}, \{(x,y) \mid 0 \le y \le x \le 1\}\}.\end{aligned}$$

$$\chi(\mathcal{S}) = 4 - 5 + 2 = 1.$$

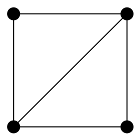

Figure 5.1: A simplicial complex of the square $[0,1]^2$ where the 4 vertices are 0-simplices, the edges and diagonal are 1-simplices and the top and bottom triangle are 2-simplices.

Corollary 5.25. Let $\mathcal{P}$ be an integral polytope. Then, the constant term of its Ehrhart polynomial is its Euler characteristic.

Proof. For integral convex polytopes, we have that the constant term is 1, which is its Euler characteristic. Let $\mathcal{P}$ be an integral polytope. If we consider its decomposition into a simplicial complex, we find that

$$F^j_{\mathcal{P}}(0) = \sum_{\substack{\Delta \in \mathcal{S} \\ \dim \Delta = j}} L_\Delta(0) = \sum_{\substack{\Delta \in \mathcal{S} \\ \dim \Delta = j}} 1 = k_j.$$

For this reason, by using Theorem 5.16, we have that

$$L_{\mathcal{P}}(0) = c_0 = \sum_{j=0}^{d} (-1)^j F^j_{\mathcal{P}}(0) = \sum_{j=0}^{d} (-1)^j k_j = \chi(\mathcal{P}).$$

□

Chapter 6

Ehrhart polynomials of reflexive polytopes

An important subclass of lattice polytopes are the reflexive polytopes, defined in the last chapter, which got attention after Batyrev noticed their connection to string theory in [6]. The Ehrhart polynomials of these polytopes have been shown to have the property that their roots are symmetrically distributed across the critical line:

$$CL = \left\{ z \in \mathbb{C} \ \mid \ \mathrm{Re}(z) = -\frac{1}{2} \right\}.$$

As in [3] we will restrict ourselves to those polytopes called <u>CL-polytopes</u> in [7].

Definition 6.1. An integral convex polytope is a CL-polytope if all of its roots lie on CL.

6.1 CL-polynomials

Definition 6.2. We denote by <u>CL-polynomials</u> as the class of polynomials of the form

$$f(z) = b(z)(z^2 + z + c_0) \cdots (z^2 + z + c_m) \in \mathbb{R}[z]$$

where $c_i \geq 1/4$ and, for a nonzero constant a, we have that $b(z) = a$ if the degree of f is even or $b(z) = a(2z + 1)$ otherwise.

Proposition 6.3. If f is a CL-polynomial, its h-polynomial has palindromic coefficients.

Proof. We can prove it inductively on the degree of f.

If f has degree zero, we have that its Ehrhart series is a scalar multiple of $1/(1-t)$. On the other hand, if it has degree one, we have that its Ehrhart polynomial is a scalar multiple of $(t+1)/(1-t)^2$. It is easy to see that both of them are palindromic.

By the inductive step, suppose that f is a polynomial of degree d with palindromic h-polynomial

$$\sum_{k=0}^{\infty} f(k)t^k = \frac{h^*(t)}{(1-t)^{d+1}}, \qquad h^*(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} h_i^* p_i^d(t).$$

Then,

$$\sum_{k=0}^{\infty}(k^2+k+c)f(k)t^k = t\left(t\left(\frac{h^*(t)}{(1-t)^{d+1}}\right)'\right)' + t\left(\frac{h^*(t)}{(1-t)^{d+1}}\right)' + c\frac{h^*(t)}{(1-t)^{d+1}} =$$

$$= t^2\left(\frac{h^*(t)}{(1-t)^{d+1}}\right)'' + 2t\left(\frac{h^*(t)}{(1-t)^{d+1}}\right)' + c\frac{h^*(t)}{(1-t)^{d+1}} =$$

$$= \sum_{i=0}^{d} h_i^*\left(t^2\left(\frac{t^i}{(1-t)^{d+1}}\right)'' + 2t\left(\frac{t^i}{(1-t)^{d+1}}\right)' + c\frac{t^i}{(1-t)^{d+1}}\right) =$$

$$= \sum_{i=0}^{d} h_i^*\left(\frac{(d-i+2)(d-i+1)t^{i+2} + 2i(d-i+2)t^{i+1} + i(i-1)t^i}{(1-t)^{d+3}} + \right.$$

$$\left. +2\frac{(d-i+1)t^{i+1} + it^i}{(1-t)^{d+2}} + c\frac{t^i}{(1-t)^{d+1}}\right) =$$

$$= \sum_{i=0}^{d} h_i^* \frac{(d^2-2di+i^2+d-i+c)t^{i+2} + 2(di-i^2+d-c+1)t^{i+1} + (i^2+i+c)t^i}{(1-t)^{d+3}} =$$

$$= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} h_i^* \frac{\alpha(t^{i+2}+t^{d-i}) + \beta(t^{i+1}+t^{d-i+1}) + \gamma(t^i+t^{d-i+2})}{(1-t)^{d+3}}$$

where we have used $\alpha = d^2-2di+i^2+d-i+c, \beta = 2(di-i^2+d-c+1)$, $\gamma = i^2+i+c$ as a shorthand. From this we can see that this polynomial is palindromic. Hence, the induction hypothesis holds. $\square$

Definition 6.4. Let

$$p_i^d(z) = d!\left(\binom{z+d-i}{d} + \binom{z+i}{d}\right) \qquad \forall\, 0 \le i < \frac{d}{2}$$

$$p_{\frac{d}{2}}^d(z) = d!\binom{z+\frac{d}{2}}{d}$$

Note that $p_0^d, p_1^d, ..., p_{\lfloor \frac{d}{2} \rfloor}^d$ define a basis for all polynomials whose h-polynomial is of degree d and palindromic.

Corollary 6.5. If f is a CL-polynomial of degree d

$$d!\sum_{t\ge 0} f(t)z^t = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} h_i p_i^d(z)$$

where h_i are the coefficients of the h-polynomial of f.

Lemma 6.6. *Let f be a CL-polynomial of degree d, then for every $z_0 \in CL$, $f(z_0) \in \mathbb{R} \cdot \sqrt{-1}^d$.*

Proof. Without loss of generality we will write $z_0 = -\frac{1}{2} + a_0\sqrt{-1}$ with $a_0 \ge 0$. Let $b_i^d(z)$ be the the expression $d!\binom{z+d-i}{d}$ viewed as a polynomial. Then, expressing everything in exponential form, we have

$$z + d - j = d - j - \frac{1}{2} + a_0\sqrt{-1} = r_j^d(z_0)\exp\left(\sqrt{-1}\left(\frac{\pi}{2} - \mu_j^d(z_0)\right)\right),$$

$$z + d - (2d - j - 1) = -d + j + \frac{1}{2} + a_0\sqrt{-1} = r_j^d(z_0)\exp\left(\sqrt{-1}\left(\frac{\pi}{2} + \mu_j^d(z_0)\right)\right),$$

$$b_i^d(z_0) = \prod_{j=0}^{d-1} r_{i+j}^d(z_0)\exp\left(\sqrt{-1}\left(\frac{d\pi}{2} - \sum_{j=0}^{d-1}\mu_{i+j}^d(z_0)\right)\right),$$

$$b_{d-i}^d(z_0) = \prod_{j=0}^{d-1} r_{d-i+j}^d(z_0)\exp\left(\sqrt{-1}\left(\frac{d\pi}{2} - \sum_{j=0}^{d-1}\mu_{d-i+j}^d(z_0)\right)\right) =$$

$$= \prod_{j=0}^{d-1} r_{2d-(i-j+d-1)-1}^d(z_0)\exp\left(\sqrt{-1}\left(\frac{d\pi}{2} - \sum_{j=0}^{d-1}\mu_{2d-(i-j+d-1)-1}^d(z_0)\right)\right) =$$

$$= \prod_{j=0}^{d-1} r_{i-j+d-1}^d(z_0)\exp\left(\sqrt{-1}\left(\frac{d\pi}{2} + \sum_{j=0}^{d-1}\mu_{i-j+d-1}^d(z_0)\right)\right) =$$

$$= \prod_{j=0}^{d-1} r_{i+j}^d(z_0) \exp\left(\sqrt{-1}\left(\frac{d\pi}{2} + \sum_{j=0}^{d-1} \mu_{i+j}^d(z_0)\right)\right).$$

Then, we can decompose $b_i^d(z_0) = x_i^d(z_0)y_i^d(z_0)$ and $b_{d-i}^d(z_0) = x_i^d(z_0)\overline{y_i^d(z_0)}$ with

$$x_i^d(z_0) = \prod_{j=0}^{d-1} r_{i+j}^d(z_0)\sqrt{-1}^d, \qquad y_i^d(z_0) = \exp\left(-\sum_{j=0}^{d-1} \mu_{i+j}^d(z_0)\sqrt{-1}\right),$$

and so $p_i^d(z_0) = x_i^d(z_0)(y_i^d(z_0) + \overline{y_i^d(z_0)}) \in \mathbb{R} \cdot \sqrt{-1}^d$, and because this is a basis of the palindromic polynomials, the claim holds for all CL-polynomials of degree d. ☐

Lemma 6.7. *Let d be a fixed parameter. If we regard $\mathbb{R} \cdot \sqrt{-1}^d$ as a totally ordered set with $a\sqrt{-1}^d \preceq b\sqrt{-1}^d$, if and only if, $a \leq b$.*
For every $p_i^d(z)$ there exists a unique positive real number a_i^d such that
$p_i^d(-\frac{1}{2} + a_i^d\sqrt{-1}) = 0$ and $p_i^d(-\frac{1}{2} + b\sqrt{-1}) \succ 0$, $\forall\ b > a_i^d$. Moreover, $i < k$ implies $a_i^d > a_k^d$.

Proof. First, we fix an integer $0 \leq i \leq \lfloor \frac{1}{2} \rfloor$, and consider $z = -\frac{1}{2} + a\sqrt{-1} \in CL$. From the previous proof, we have that $p_i^d(z) = x_i^d(z)(y_i^d(z) + \overline{y_i^d(z)})$. We can see that $x_i^d(z) \succ 0$.
Consider $0 \leq j < d$. Then, Re $(z + d - j) > 0$ and we have

$$\mu_j^d(z) = \frac{\pi}{2} - \arctan\left(\frac{a}{d - j - \frac{1}{2}}\right) \xrightarrow{a \to +\infty} 0.$$

On the other hand, if $j \geq d$. Then Re $(z + d - j) < 0$ and we have

$$\mu_j^d(z) = \frac{\pi}{2} + \arctan\left(\frac{a}{d - j - \frac{1}{2}}\right) \xrightarrow{a \to +\infty} 0,$$

and so we have that $y_i^d(z)$ goes to 1 as a goes to infinity. Then a_i^d is the number for which Re $(y_i^d(-\frac{1}{2} + a_i^d\sqrt{-1})) = 0$, Re $(y_i^d(-\frac{1}{2} + b\sqrt{-1})) > 0\ \forall\ b > a_i^d$.

Moreover, consider

$$\nu_i^d(z) = \mu_{d+i}^d(z) - \mu_i^d(z) = \arctan\left(\frac{a}{d - i - \frac{1}{2}}\right) - \arctan\left(\frac{a}{i - \frac{1}{2}}\right) \leq 0 \quad \forall\, 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$$

due to the fact that arc-tangent is an increasing function and that
$i - \frac{1}{2} \leq d - i - \frac{1}{2}$ with equality only at $i = \frac{d}{2}$. Then, if we have $0 \leq i < k \leq \left\lfloor \frac{d}{2} \right\rfloor$,

$$\arg(y_i^d(z)) = \arg(y_k^d(z)) + \sum_{n=i}^{k-1} \nu_n^d(z) < \arg(y_k^d(z)),$$

so if $z = -\frac{1}{2} + b\sqrt{-1}, b \geq a_i^d$ we have that

$$0 \leq \arg(y_i^d(z)) < \arg(y_k^d(z))$$

and so $a_i^d > a_k^d$. $\square$

6.2 CL-polytopes

Proposition 6.8. Suppose that $\mathcal{P}$ is an integral convex CL-polytope, then its Ehrhart polynomial is a CL-polynomial with an h-polynomial with nonnegative coefficients.

Proof. It follows trivially from the requirement on its roots and that the Ehrhart polynomial is real. The second part is due to Hibi's palindromic theorem. $\square$

Proposition 6.9. Let f be a CL-polynomial of degree d. Then for all of its roots $-\frac{1}{2} + \alpha\sqrt{-1}, \alpha \leq a_0^d$.

Proof. First, we notice that the coefficient of p_0^d cannot be zero. This is because all other p_i^d have roots at 0 and -1, meaning that without p_0, f wouldn't be a CL-polynomial. We will look at $f(z_0^d)$ where $z_0^d = -\frac{1}{2} + a_0^d\sqrt{-1}$. From Lemma 6.7 we have that $p_0^d(z_0^d) = 0$ and $p_i^d(z_0^d) > 0 \; \forall \; i > 0$. Moreover, if $z = -\frac{1}{2} + \alpha\sqrt{-1}, \alpha > a_0^d$, we have that $p_i^d(z_0^d) > 0 \; \forall \; i$. Then, z_0^d is the largest root that can be assumed. Moreover, it is assumed, if and only if, $f = p_0^d$. $\square$

However, we will show that no multiple of p_0^d is the Ehrhart polynomial of any convex d-polytope with $d > 1$.

Proposition 6.10. No multiple of p_0^d is the Ehrhart polynomial of any convex d-polytope with $d > 1$.

Proof. The constant term of $p_0^d = d!$, then if kp_0^d is the Ehrhart polynomial of a convex d-polytope, we have that $k = 1/d!$.

Expanding $p_0^d/d!$ up to its two greatest coefficients, we have that

$$\frac{p_0^d(z)}{d!} = \left(\binom{z+d}{d} + \binom{z}{d}\right) = \frac{2}{d!}z^d + z^{d-1}\sum_{i=-(d-1)}^{d}\frac{i}{d!} + \cdots =$$

$$= \frac{2}{d!}z^d + \frac{d}{d!}z^{d-1} + \cdots = \frac{2}{d!}z^d + \frac{1}{(d-1)!}z^{d-1} + \cdots$$

However, due to the fact that

$$c_{d-1} = \frac{1}{(d-1)!} < \frac{d+1}{2(d-1)!}$$

by Corollary 5.22, this is not the Ehrhart polynomial of any convex d-polytope.

□

Theorem 6.11. (Hibi's Lower Bound Theorem) *Let $\mathcal{P}$ be an integral d-polytope with h-polynomial $h(t) = \sum_{i=0}^{d} h_i^* t^i$ with $h_d^* \neq 0$. Then, the equality $h_1^* \leq h_i^*$ $\forall\, 1 \leq i < d$ holds.*

The proof can be found in [8].

Proposition 6.12. The h-polynomial of the standard reflexive simplex of dimension d is $h_{\Delta_{sr}^d}(t) = \sum_{i=0}^{d} t^i$.

Proof. By Proposition 4.3, the k-th coefficient of the h-polynomial of Δ_{sr}^d is equal to the number of integer points in

$$\begin{aligned}\Pi =\{&\lambda_1(1,0,...,0,1) + \lambda_2(0,1,...,0,1) + \cdots + \lambda_d(0,0,...,1,1) + \\ &+ \lambda_{d+1}(-1,-1,...,-1,1) \mid 0 \leq \lambda_1, ...\lambda_d < 1\}\end{aligned}$$

with last coordinate equal to k. We have that

$$\frac{k}{d+1}[(1,0,...,0,1) + (0,1,...,0,1) + \cdots + (-1,-1,...,-1,1)] = (0,0,...,0,k),$$

and so, there is at least one point with last coordinate k $\forall\, 0 \leq k \leq d$. Moreover, it is easy to see that $\Pi \cap (\mathbb{R}^d \times \{k\})$ is

$$\left[\operatorname{conv}\left(\left\{\sum_{j=1}^{k} \vec{w}_{i_j}\right\}_{1 \leq i_1 < ... < i_k \leq d+1}\right)\right]^{\circ}$$

where $\vec{w}_i = \vec{e}_i + \vec{e}_{d+1}\ \forall\, 1 \leq i \leq d$ and $\vec{w}_{d+1} = (-1, ..., -1, 1)$ and $\vec{e}_i$ are the standard basis vectors. Then, all of the components first d components of these points are $-1, 0, 1$ and so $(0, ..., 0, k)$ is the only point that can be in $\Pi \cap (\mathbb{R}^d \times \{k\})$. Therefore,

$$h_{\Delta^d_{sr}}(t) = \sum_{i=0}^{d} t^i.$$

$\square$

Theorem 6.13. *Let $a_{sr} \in \mathbb{R}_{\geq 0}$ denote the number such that $-\frac{1}{2} + a^d_{sr}\sqrt{-1}$ is the extremal root of the Ehrhart polynomial of Δ^d_{sr} in the upper half plane. Then every CL-polytope of dimension $d \leq 9$ whose extremal root of the Ehrhart polynomial in the upper half complex plane is $-\frac{1}{2} + \beta\sqrt{-1}$ satisfies $\beta \leq a^d_{sr}$.*

Proof. In the case $d \leq 5$ it can be showed computationally using Mathematica that $a^d_1 < a^d_{sr} < a^d_0$

$$p_0\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) \prec 0, \qquad p_i\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) \succ 0 \quad \forall\, 1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor,$$

$$\left(p_0 + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} p_i\right)\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) = 0.$$

Let us consider another CL-polytope of dimension d with Ehrhart polynomial $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} h^*_i p_i$. Because $h^*_0 = 1$ always and the h-vector is palindromic, we have that $h^*_d = 1$ and Theorem 6.11 holds. Moreover, since $h^*_d = 1$ it has an interior point and it must have at least $d+1$ vertices so $h^*_i \geq 1\ \forall\, 1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$. Let $a \geq a^d_{sr}$, then

$$\left|p_0\left(-\frac{1}{2} + a\sqrt{-1}\right)\right| \leq \left|\left(\sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} p_i\right)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right| \leq \left|\left(\sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} h^*_i p_i\right)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right|$$

where the equality holds if, and only if, $a = a^d_{sr}$ and $h^*_i = 1\ \forall\, i$.

When $6 \leq d \leq 9$ it can be showed computationally that $a^d_2 < a^d_{sr} < a^d_1 < a^d_0$, which implies that.

$$p_0\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) \prec 0, \qquad p_1\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) \prec 0,$$

$$p_i\left(-\frac{1}{2} + a^d_{sr}\sqrt{-1}\right) \succ 0 \quad \forall\, 2 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor,$$

$$\left(p_0 + p_1 + \sum_{i=2}^{\left\lfloor \frac{d}{2} \right\rfloor} p_i\right)\left(-\frac{1}{2} + a_{sr}^d\sqrt{-1}\right) = 0.$$

Assume that $h_1^* = k \geq 1$ so $h_i^* \geq k \; \forall \; 2 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$. Let $a \geq a_{sr}^d$, then

$$\left|(p_0 + kp_1)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right| \leq \left|(kp_0 + kp_1)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right| \leq$$

$$\leq \left|\left(\sum_{i=2}^{\left\lfloor \frac{d}{2} \right\rfloor} kp_i\right)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right| \leq \left|\left(\sum_{i=2}^{\left\lfloor \frac{d}{2} \right\rfloor} h_i^* p_i\right)\left(-\frac{1}{2} + a\sqrt{-1}\right)\right|$$

where the equality holds if, and only if, $a = a_{sr}^d$ and $h_i^* = 1 \; \forall \; i$. □

Chapter 7

Generalizing Pick's Theorem

Knowing the values we get from the first, second and last coefficient, we can think that all the coefficients of the Ehrhart polynomial are linear functions of $V_d, ..., V_0, \chi$ where V_k are the relative volumes of the faces of dimension k.

Remark 7.1. Let $\mathcal{P}$ be an integral d-polytope. Then, not all coefficients of the Ehrhart polynomial of $\mathcal{P}$ are linear functions of $V_d, ..., V_0, \chi(\mathcal{P})$.

Proof. Let $\mathcal{P}$ be a Reeve tetrahedron with height h. Then, we have

$$L_{\mathcal{P}}(t) = \frac{h}{6}t^3 + t^2 + \alpha t + 1,$$

$$-L_{\mathcal{P}}(-1) = 0 = \frac{h}{6} - 1 + \alpha t - 1 \Rightarrow \alpha = 2 - \frac{h}{6}.$$

Let us suppose that $\alpha = aV + bA + cL + dv + e\chi(\mathcal{P})$ where v is the number of vertices. Then

$$2 - \frac{h}{6} = \alpha = a\frac{h}{6} + 2b + 6c + 4d + e \Rightarrow a = -1.$$

Let $\mathcal{P}$ be a cube of side l. Then, we have that

$$L_{\mathcal{P}}(t) = (lt+1)^3 = l^3t^3 + 3l^2t^2 + 3lt + 1,$$

$$3l = \alpha = -l^3 + 6l^2b + 12lc + 8d + e.$$

Which is unsolvable with a, b, c, d, e constants. $\square$

First, we will try to recover Pick's Theorem.

Theorem 7.2. (Generalized Pick's Theorem) *Let $\mathcal{P}$ be an integral polygon, A its area, i the number of interior points and b the number of boundary points. Then*

$$A = i + \frac{b}{2} - \chi(\mathcal{P}).$$

Proof. From all that we know of Ehrhart polynomials, we have that

$$L_{\mathcal{P}}(t) = At^2 + \alpha t + \chi(\mathcal{P}).$$

In particular

$$L_{\mathcal{P}}(-1) = i = A - \alpha + \chi(\mathcal{P}) \Rightarrow \alpha = A + \chi(\mathcal{P}) - i,$$

$$L_{\mathcal{P}}(1) = i + b = A + \alpha + \chi(\mathcal{P}) = 2A + 2\chi(\mathcal{P}) - i,$$

and so we have the result

$$A = i + \frac{b}{2} - \chi(\mathcal{P}).$$

□

Observe that for the case of a simple polygon, we have that $\chi(\mathcal{P}) = 1$ and so we recover Pick's Theorem.

However, if we want to generalize it to higher dimensions we need more information due to the fact that a polynomial of degree d needs $d + 1$ points to be fully defined. Moreover, this information must come from higher dilates of the polytope.

Theorem 7.3. (3D Pick's Theorem) *Let $\mathcal{P}$ be an integral polyhedra, V its volume, i the number of interior points, b the number of boundary points and I the number of interior points of its 2-dilate. Then*

$$V = \frac{I + b - 2i - 3\chi(\mathcal{P})}{6}.$$

Proof. As in the previous proof we have

$$L_{\mathcal{P}}(t) = Vt^3 + \beta t^2 + \alpha t + \chi(\mathcal{P}).$$

Then, in particular we have

$$-L_{\mathcal{P}}(-1) = i = V - \beta + \alpha - \chi(\mathcal{P}) \Rightarrow \beta = V + \alpha - \chi(\mathcal{P}) - i,$$

$$L_{\mathcal{P}}(1) = i + b = V + \beta + \alpha + \chi(\mathcal{P}) = 2V + 2\alpha - i \Rightarrow \alpha = i - V + \frac{b}{2},$$

$$\beta = V + \alpha - \chi(\mathcal{P}) - i = \frac{b}{2} - \chi(\mathcal{P}).$$

Furthermore,

$$-L_{\mathcal{P}}(-2) = I = 8V - 4\beta + 2\alpha - \chi(\mathcal{P}) = 6V + 2i - b + 3\chi(\mathcal{P}).$$

And so

$$V = \frac{I + b - 2i - 3\chi(\mathcal{P})}{6}$$

with

$$\alpha = i - V + \frac{b}{2} = \frac{8i + 2b - I + 3\chi(\mathcal{P})}{6}.$$

□

Theorem 7.4. (4D Pick's Theorem) *Let $\mathcal{P}$ be an integral 4-polytope, V its volume, i the number of interior points, b the number of boundary points, and I and B the number of interior and boundary points of its 2-dilate, respectively. Then*

$$H = \frac{2I + B - 8i - 4b + 6\chi(\mathcal{P})}{24}$$

where H represents its 4-volume.

Proof. As in the previous proof we have

$$L_{\mathcal{P}}(t) = Ht^4 + \gamma t^3 + \beta t^2 + \alpha t + \chi(\mathcal{P}).$$

Then, in particular we have

$$L_{\mathcal{P}}(-1) = i = H - \gamma + \beta - \alpha + \chi(\mathcal{P}) \Rightarrow \gamma = H - i + \beta - \alpha + \chi(\mathcal{P}),$$
$$L_{\mathcal{P}}(1) = i + b = H + \gamma + \beta + \alpha + \chi(\mathcal{P}) = 2H - i + 2\beta + 2\chi(\mathcal{P}),$$
$$\beta = i - H - \chi(\mathcal{P}) + \frac{b}{2} \Rightarrow \gamma = -\alpha + \frac{b}{2}.$$

Furthermore,

$$L_{\mathcal{P}}(-2) = I = 16H - 8\gamma + 4\beta - 2\alpha + \chi(\mathcal{P}) = 12H - 2b + 4i + 6\alpha - 3\chi(\mathcal{P}),$$
$$\alpha = \frac{I + 2b - 4i - 12H + 3\chi(\mathcal{P})}{6} \Rightarrow \gamma = \frac{12H - I + 4i + b - 3\chi(\mathcal{P})}{6},$$
$$L_{\mathcal{P}}(2) = I + B = 16H + 8\gamma + 4\beta + 2\alpha + \chi(\mathcal{P}) = 24H - I + 8i + 4b - 6\chi(\mathcal{P}).$$

And so

$$H = \frac{2I + B - 8i - 4b + 6\chi(\mathcal{P})}{24}$$

with

$$\alpha = \frac{I + 2b - 4i - 12H + 3\chi(\mathcal{P})}{6} = \frac{8b - B}{12},$$
$$\beta = i - H - \chi(\mathcal{P}) + \frac{b}{2} = \frac{32i + 16b - 2I - B - 30\chi(\mathcal{P})}{24},$$
$$\gamma = -\alpha + \frac{b}{2} = \frac{B - 2b}{12}.$$

□

After having done this two examples, we will generalize this formulas up to arbitrary dimension. With the knowledge we have today about Ehrhart polynomials, we have that the minimum number of dilates to count is

$$d+1 \leq 2n+1 \Rightarrow n \geq \frac{d}{2} \Rightarrow n = \left\lceil \frac{d}{2} \right\rceil.$$

This is due to the fact that a polynomial of degree d has $d+1$ coefficients and, apart from the Euler characteristic, from each dilate we get two variables, the amount of points in the interior and on the boundary.

Theorem 7.5. **(n-dimensional Pick's theorem)** *Let $\mathcal{P}$ be an integral d-polytope, V_d its d-volume, and i_k and b_k the number of interior points and boundary points of its k-dilate, respectively. Then, if $d = 2n$ is even, we have*

$$V_{2n} = \frac{1}{(2n)!}\left[\sum_{j=1}^{n} 2(-1)^{n-j}\binom{2n}{n-j} i_j + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n}{n+j} b_j + (-1)^n \binom{2n}{n}\chi\right],$$

and if $d = 2n+1$ is odd,

$$\begin{aligned} V_{2n+1} =& \frac{1}{(2n+1))!}\left[i_{n+1} + \sum_{j=1}^{n}(-1)^{n-j-1}\left[\binom{2n+1}{n-j+1} - \binom{2n+1}{n-j}\right] i_j + \right. \\ &\left. + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n+1}{n-j} b_j + (-1)^{n+1}\binom{2n+1}{n+1}\chi\right]. \end{aligned}$$

To prove this theorem we will first need to calculate the determinant of the following matrix.

Lemma 7.6. *Let M be the square matrix*

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Then its determinant is

$$\det M = \Pi_{0 \leq i < j \leq n}(x_j - x_i).$$

This matrix is called a Vandermonde matrix.

Proof. We will prove it using induction on the size of the matrix.

For $n = 0$, we have that $\det M = 1 = \prod_{0 \le i < j \le n}(x_j - x_i)$.

By the inductive step, suppose that the formula holds for $n-1$. Then, subtracting to each column the previous one multiplied by x_0

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & \cdots & x_1^{n-1}(x_1 - x_0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_0 & x_n(x_n - x_0) & \cdots & x_n^{n-1}(x_n - x_0) \end{vmatrix} =$$

$$= \begin{vmatrix} x_1 - x_0 & x_1(x_1 - x_0) & \cdots & x_1^{n-1}(x_1 - x_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_0 & x_n(x_n - x_0) & \cdots & x_n^{n-1}(x_n - x_0) \end{vmatrix} =$$

$$= \prod_{1 \le j \le n} (x_n - x_0) \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le j \le n} (x_j - x_0) \prod_{1 \le i < j \le n} (x_j - x_i) =$$

$$= \prod_{0 \le i < j \le n} (x_j - x_i)$$

And so, the inductive hypothesis holds. □

Proof of the n-dimensional Pick's theorem (Theorem 7.5). Let us first write down the system of equations to find the coefficients of the Ehrhart polynomial.

$$\begin{cases} L_{\mathcal{P}}\left(\left\lfloor -\frac{d}{2} \right\rfloor\right) = \left\lfloor -\frac{d}{2} \right\rfloor^d c_d + \cdots + c_0 = (-1)^d i_{-\left\lfloor -\frac{d}{2} \right\rfloor} \\ \vdots \\ L_{\mathcal{P}}(-1) = (-1)^d c_d + \cdots + c_0 = (-1)^d i_1 \\ L_{\mathcal{P}}(0) = c_0 = \chi \\ L_{\mathcal{P}}(1) = c_d + \cdots + c_0 = i_1 + b_1 \\ \vdots \\ L_{\mathcal{P}}\left(\left\lfloor \frac{d}{2} \right\rfloor\right) = \left\lfloor \frac{d}{2} \right\rfloor^d c_d + \cdots + c_0 = i_{\left\lfloor \frac{d}{2} \right\rfloor} + b_{\left\lfloor \frac{d}{2} \right\rfloor} \end{cases}$$

For simplicity, let us define

$$W_d(j) = \begin{cases} (-1)^d i_{-j} & \text{if } j < 0 \\ \chi & \text{if } j = 0 \\ i_j + b_j & \text{if } j > 0 \end{cases}$$

Knowing that the term of degree d corresponds to the d-volume V_d, we can solve it using Cramer's rule and the previous lemma:

$$V_d = \frac{\begin{vmatrix} W_d(-\left\lfloor -\frac{d}{2}\right\rfloor) & \left\lfloor -\frac{d}{2}\right\rfloor^{d-1} & \left\lfloor -\frac{d}{2}\right\rfloor^{d-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ W_d(-1) & (-1)^{d-1} & (-1)^{d-2} & \cdots & 1 \\ W_d(0) & 0 & 0 & \cdots & 1 \\ W_d(1) & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_d(\left\lfloor \frac{d}{2}\right\rfloor) & \left\lfloor \frac{d}{2}\right\rfloor^{d-1} & \left\lfloor \frac{d}{2}\right\rfloor^{d-2} & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \left\lfloor -\frac{d}{2}\right\rfloor^{d} & \left\lfloor -\frac{d}{2}\right\rfloor^{d-1} & \left\lfloor -\frac{d}{2}\right\rfloor^{d-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ (-1)^{d} & (-1)^{d-1} & (-1)^{d-2} & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left\lfloor \frac{d}{2}\right\rfloor^{d} & \left\lfloor \frac{d}{2}\right\rfloor^{d-1} & \left\lfloor \frac{d}{2}\right\rfloor^{d-2} & \cdots & 1 \end{vmatrix}} =$$

$$= \frac{\sum_{j=0}^{d+1} (-1)^{d(d-1)/2+j} W_d\left(\left\lfloor -\frac{d}{2}\right\rfloor + j\right) \prod_{\substack{\left\lfloor -\frac{d}{2}\right\rfloor \leq i<k\leq \left\lfloor \frac{d}{2}\right\rfloor \\ i,k\neq \left\lfloor -\frac{d}{2}\right\rfloor + j}} (k-i)}{(-1)^{(d+1)d/2} \prod_{\left\lfloor -\frac{d}{2}\right\rfloor \leq i<j\leq \left\lfloor \frac{d}{2}\right\rfloor} (j-i)} =$$

$$= \frac{\prod_{n=1}^{d} n! \sum_{j=0}^{d+1} (-1)^{d+j} W_d\left(\left\lfloor -\frac{d}{2}\right\rfloor + j\right) / \left[j!\left(\left\lfloor \frac{d}{2}\right\rfloor - \left\lfloor -\frac{d}{2}\right\rfloor - j\right)!\right]}{\prod_{i=1}^{d} i!} =$$

$$= \frac{\prod_{n=1}^{d} n! \sum_{j=0}^{d+1} (-1)^{d+j} W_d\left(\left\lfloor -\frac{d}{2}\right\rfloor + j\right) / [j!(d-j)!]}{\prod_{i=1}^{d} i!} =$$

$$= \frac{1}{d!} \sum_{j=0}^{d+1} (-1)^{d+j} \binom{d}{j} W_d\left(\left\lfloor -\frac{d}{2}\right\rfloor + j\right)$$

We have

$$
\begin{aligned}
V_d =& \frac{1}{d!}\left[\sum_{j=1}^{-\left\lfloor -\frac{d}{2}\right\rfloor}(-1)^{\left\lfloor -\frac{d}{2}\right\rfloor - j}\binom{d}{-\left\lfloor -\frac{d}{2}\right\rfloor - j} i_j + \right.\\
&\left. + \sum_{j=1}^{\left\lfloor \frac{d}{2}\right\rfloor}(-1)^{d+\left\lfloor -\frac{d}{2}\right\rfloor + j}\binom{d}{-\left\lfloor -\frac{d}{2}\right\rfloor + j}(i_j + b_j) + (-1)^{d+\left\lfloor -\frac{d}{2}\right\rfloor}\binom{d}{-\left\lfloor -\frac{d}{2}\right\rfloor}\chi\right].
\end{aligned}
$$

So, if $d = 2n$ is even,

$$
\begin{aligned}
V_{2n} =& \frac{1}{(2n)!}\left[\sum_{j=1}^{n}(-1)^{n-j}\binom{2n}{n-j} i_j + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n}{n+j}(i_j + b_j) + (-1)^n\binom{2n}{n}\chi\right] =\\
=& \frac{1}{(2n)!}\left[\sum_{j=1}^{n}2(-1)^{n-j}\binom{2n}{n-j} i_j + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n}{n+j} b_j + (-1)^n\binom{2n}{n}\chi\right],
\end{aligned}
$$

and if $d = 2n+1$ is odd,

$$
\begin{aligned}
V_{2n+1} =& \frac{1}{(2n+1))!}\left[\sum_{j=1}^{n+1}(-1)^{n-j-1}\binom{2n+1}{n-j+1} i_j + \right.\\
&\left. + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n+1}{n+j+1}(i_j + b_j) + (-1)^n\binom{2n+1}{n+1}\chi\right] =\\
=& \frac{1}{(2n+1))!}\left[i_{n+1} + \sum_{j=1}^{n}(-1)^{n-j-1}\left[\binom{2n+1}{n-j+1} - \binom{2n+1}{n-j}\right] i_j + \right.\\
&\left. + \sum_{j=1}^{n}(-1)^{n+j}\binom{2n+1}{n-j} b_j + (-1)^{n+1}\binom{2n+1}{n+1}\chi\right].
\end{aligned}
$$

□

This result is of theoretical importance due to the fact that, with what is known today of the coefficients of the Ehrhart polynomial, we can solve the volume in the minimum theoretical number of dilates. However, computationally, it would be easy to select as dilate a highly composite number so that with only one loop, one can calculate the maximum number of dilates possible, its divisors.

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