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Renormalizability of Yang-Mills in $\mathbb{R}^4$

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Abstract

The present work is an attempt to introduce a novel axiomatic formulation of *Quantum Field Theory* proposed by Kevin Costello in [Cos11]. Far from being exhaustive, we aim to present the main results and constructions, followed by calculations in this formalism that match the physics literature. We try to give a pedagogical introduction, providing all the definitions that an undergraduate student would need to understand the key concepts. We start by defining the simplest kinds of quantum field theories and how to make sense of divergent quantities. As we move forward, we try to generalize these definitions to more and more general classes of theories. The endpoint of this work is to use Costello's machinery to define the Yang-Mills theory on $\mathbb{R}^4$ and prove that it is perturbatively renormalizable.

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Chapter 1

Introduction

Quantum Field Theory not only has been extremely successful as a framework to describe particle physics, but also had profound influence in the development of mathematics. However, there is still no consensus, at least in the mathematics community, as to what a quantum field theory actually is. Many attempts to axiomatize QFT arise from the Hamiltonian formulation of field theory, such as the Segal axioms [Seg99] or the Haag-Kastler axioms [Haa92]. In this work, we aim to present Kevin Costello's novel approach to formalizing perturbative quantum field theory, based on the Lagrangian formulation of field theory and the Wilsonian philosophy of *effective field theory*. The goal of this text is to prove in this framework that the Yang-Mills theory with semi-simple Lie algebra $\mathfrak{g}$ is perturbative renormalizable in $\mathbb{R}^4$.

In physics, a classical (field) theory with space of fields $\mathcal{E}$ is given by an action functional $S : \mathcal{E} \to \mathbb{R}$. All one wishes to know about that particular theory is encoded in the action S. For instance, the equations of motion are just a submanifold $EL(S) := \{\phi \in \mathcal{E} \big| dS(\phi) = 0\}$. Some requirements are imposed on the action, such as it has to be local, i.e. the integral of a Lagrangian (density). It is also imposed that is *invariant* under the action of some Lie group G, which is regarded as the symmetry group of the theory. A quantization S^q of a classical theory S^{cl} is just another theory involving a parameter $\hbar$, such that $\lim\limits_{\hbar \to 0} S^q = S^{cl}$. We will work in the *perturbative* regime, which means that we are working with quantizations which are infinitesimal deformations of the classical theory. This translates into treating $\hbar$ as a formal parameter. The action S^q will take values in $\mathbb{R}[[\hbar]]$, where $\mathbb{R}[[\hbar]]$ denotes the ring of formal power series in $\hbar$. It is defined as $I-$adic completion of the ring $\mathbb{R}[\hbar]$ with respect to the ideal $(\hbar)$, i.e. $\mathbb{R}[[\hbar]] = \lim\limits_{\leftarrow} \mathbb{R}[\hbar]/(\hbar)^n$.

The idea behind effective field theory is to think that we are limited by our detectors to measure phenomena that occur at energies below some energy Λ. All the phenomena that occur at energies equal or lower than Λ can be described by a scale Λ *effective action* $S^{eff}[\Lambda]$. If we want to restrict ourselves to even lower energies $\Lambda' < \Lambda$, we can infer a scale Λ' effective action from the knowledge of the scale Λ effective action. Given a process that occurs at energies lower than Λ' we should obtain the same predictions by doing calculations either with $S^{eff}[\Lambda]$ or with $S^{eff}[\Lambda']$. This will guide us to the renormalization group equation, which relates $S^{eff}[\Lambda]$ with $S^{eff}[\Lambda']$ for all $\Lambda' < \Lambda$. More specifically, observables can be thought of as the possible measurements one can make at a point x. In the perturbative regime, they are modelled as formal series on the fields and their derivatives at a point x. They form a rich structure known as a *factorization algebra*, which is the main topic of the sequels of this work [CG16] and [CG17].

The Feynman sum over-histories approach, tells us that given a quantum action S^q, the possible *states* our system can be in are superpositions of the fields weighted by $e^{iS^q/\hbar}$. Moreover, one can only measure expectation values of any observable, so $e^{iS/\hbar}$ may be thought of as some complex probability measure. The fundamental quantities one wants to compute are correlation functions of observables O_i, $\langle O_1 \ldots O_n \rangle = \int e^{iS/\hbar} O_1 \ldots O_n \mathcal{D}\phi$. We encounter the problem that there is no (non-zero) Lebesgue measure $\mathcal{D}\phi$ in an infinite-dimensional vector space. This is where the effective field philosophy comes into play. By restricting to phenomena below certain energies Λ, the integral above becomes finite-dimensional and we can therefore compute the expectation values as desired. Take for example the scalar field theory over a compact manifold M with space of fields $C^\infty(M)$ and action

$S(\phi) = -\frac{1}{2}\int \phi(D+m^2)\phi + I[\phi]$, where D denote the positive-definite Laplacian, $\phi \in C^\infty(M)$ and $I[\phi]$ is a function of the fields which is at least cubic in ϕ. Restricting ourselves to energies below Λ, the action functional describing our theory will be $S^{eff}[\Lambda](\phi)$, and the correlation functions will read $\langle O_1 \dots O_n\rangle = \int_{C^\infty(M)_{\leq\Lambda}} e^{iS^{eff}/\hbar} O_1 \dots O_n \mathcal{D}\phi$, where $C^\infty(M)_{\leq\Lambda}$ denotes the subspace of $C^\infty(M)$ of fields which are sums of eigenfunctions of the Laplacian whose eigenvalues are less than or equal to Λ. The requirement that we can describe a theory of energy $\Lambda' < \Lambda$ either with the $S^{eff}[\Lambda]$ or the $S^{eff}[\Lambda']$ effective action, translates into

$$\langle O_1 \dots O_n\rangle = \int_{C^\infty(M)_{\leq\Lambda}} e^{iS^{eff}[\Lambda]/\hbar} O_1 \dots O_n \mathcal{D}\phi = \int_{C^\infty(M)_{\leq\Lambda'}} e^{iS^{eff}[\Lambda']/\hbar} O_1 \dots O_n \mathcal{D}\phi,$$

which allows us to relate both effective actions through the renormalization group equation:

$$S^{eff}[\Lambda'](a) = \int_{\phi\in C^\infty(M)_{(\Lambda',\Lambda]}} e^{iS^{eff}[\Lambda](\phi+a)/\hbar}.$$

The infamous infinities in QFT arise when trying to describe physics at every scale, that is, trying to define the $S^{eff}[\infty]$ scale effective action. With this interpretation, infinities are actually to be expected, since there is no reason to believe we could describe particles with infinite energy.

We will also require that the effective actions $S^{eff}[\Lambda]$ become more and more local as $\Lambda \to \infty$, such that in the limit, interactions occur at points.

The definition of QFT given in this work tries to be as general as possible, since our only assumptions are:

1. The action principle: Physics at each scale is described by a Lagrangian, according to Feynman sum-over-histories approach
2. Locality

In Chapter 2 we will give a rigorous introduction to the ideas presented here, and we will prove that using this definition, there are as many QFT as there are Lagrangians. We will make this statement more precise. Throughout this project, we will work with a length scale cut-off based on the heat kernel, instead of the more intuitive energy picture given here. This is because locality is much simpler in the length scale picture, where the *length* L effective action $S^{eff}[L]$ describes all the physics that occur at scales greater or equal than L. We will also work in Euclidean (Riemannian) signature, instead of the more physical Lorentzian signature, for the sake of simplicity. This means that our integrals will be ordinary decaying exponentials instead of oscillatory complex exponentials $e^{iS/\hbar}$ by performing what is known as a *Wick rotation* $S \to -iS$. This gives quantum field theories an interpretation as statistical field theories, but we will not go into detail, see [Cos11, Chapter 1.2] for further discussion. One would need to analytically continue the results we will obtain to the Lorentzian signature.

In Chapter 3 we will see that for theories defined on $\mathbb{R}^n$ there is an action of $\mathbb{R}_{>0}$ on the space of theories, called the *local renormalization group flow*. This action allows us to talk about (perturbative) renormalizability, which is a very rich concept, leading to concepts such as universality, the β-function, asymptotic freedom etc. We don't have the time to introduce all of this, so for us, renormalizable theories will just be a natural way of picking a finite-dimensional subspace of "well-behaved" theories out of the infinite-dimensional space of possible theories.

In Chapter 4 we will see how to deal with theories that possess *gauge* symmetry. We will present the Batalin-Vilkovisky formalism to quantize gauge theories and sketch the principal results on this behalf. We will see that given a classical theory described by a classical action S^{cl} there may be obstructions to quantizing this theory, which will lie in certain cohomology groups. This will turn the problem of the existence of quantization of a given classical theory into an obstruction-deformation problem.

Finally, in Chapter 5, we will use all the machinery we have constructed to prove that Yang-Mills theory defined on $\mathbb{R}^4$ is perturbative renormalizable. We will sketch Costello's proof which relies only on the calculation of certain cohomology groups, without any Feynman graph manipulations. As Costello points out, the core of the proof relies on the fortuitous vanishing of $H^5(\mathfrak{su}(3))^{\mathrm{Out}(\mathfrak{su}(3))}$.

Chapter 2

Quantum Field Theory generalities

A Quantum Field Theory, as presented above consists of two elements: a free theory and an interaction, which deforms the free theory. In order to model a quantum field theory we will make extensive use of the following definitions:

Definition 2.0.1 (Fibre bundle)**.** Let $\pi : E \to M$ be a smooth map from a smooth manifold E to a smooth manifold M. We say that (E, π) is a fibre bundle with typical fibre F over M if there is a covering of M by open sets U_i and diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times F$, such that $\pi : \pi^{-1}(U_i) \to U_i$ is the composition of ϕ_i with the projection onto the first factor U_i in $U_i \times F$. The space E is called the total space and M is called the base.

Definition 2.0.2 (Vector bundle)**.** A fibre bundle $\pi : E \to M$ is a vector bundle if its typical fibre is a vector space F, and if the diffeomorphisms ϕ_i may be chosen in such a way that the diffeomorphisms $\phi_j \circ \phi_i^{-1} : \{x\} \times F \to \{x\} \times F$ are invertible linear maps of F for all $x \in U_i \cap U_j$.

A section s of a vector bundle E over M is a map $s : M \to E$, such that $\pi \circ s = \mathrm{id}_M$. The space of smooth sections is denoted $\Gamma(M, E)$.

Definition 2.0.3 (Metric)**.** A metric on a vector bundle $E \to M$ is a smooth family of positive definite inner products on the fibres of E. This means that for every $x \in M$, there is a positive definite inner product on the fibre E_x over x, $\langle -, - \rangle : E_x \times E_x \to \mathbb{R}$, such that for each smooth section $M \supset U \xrightarrow{s} E$, the map $x \mapsto \langle s(x), s(x) \rangle \in \mathbb{R}$ is a smooth function on U.

The free theory defines the particle(s) we are describing, together with its evolution in time when there is no interaction. We model it as a pair (B, S_{quad}), where B is the space of fields, which we regard as the space of sections of some vector bundle $B = \Gamma(M, E)$ over a manifold M, and S_{quad} is an action functional which is quadratic in the fields. The quantization of a free theory is a well-defined concept, and it can be made totally rigorous using the tools of functional analysis developed in the 20th century. In this theory, fields evolve under linear partial differential equations. As examples of free theories, consider

1. The scalar field theory on a Riemannian manifold (M, g), with space of fields $B = \Gamma(M, M \times \mathbb{R}) = C^\infty(M)$ and action $S_{quad} = -\int_M \frac{1}{2}\phi(D + m^2)\phi$, where $m > 0$ is the mass and D the positive-definite Laplacian for the metric g.

2. Chern-Simons theory over an oriented $3-$manifold M, with Lie algebra $\mathfrak{g}$, space of fields $\Omega^1(M) \otimes \mathfrak{g}$ and quadratic action $S_{quad} = \int_M \langle A, dA \rangle_{\mathfrak{g}}$, where $\langle -, - \rangle_{\mathfrak{g}}$ is some invariant pairing on $\mathfrak{g}$.

3. Yang-Mills theory over a $4-$manifold M, with Lie algebra $\mathfrak{g}$, space of fields $\Omega^1(M) \otimes \mathfrak{g}$ and action $S_{quad} = \int_M \langle dA, \star dA \rangle_{\mathfrak{g}}$, where $\langle -, - \rangle_{\mathfrak{g}}$ is some invariant pairing on $\mathfrak{g}$.

The propagator of a theory is the integral kernel for the operator of S_{quad}, so the kernel of $(D + m^2)^{-1}$ in the case of scalar field theory, and it allows us to describe completely the behaviour of the fields in the free theory. In section 2.3 we will see how it has the interpretation as the probability amplitude of a particle which travels from two spacetime points.

The interaction is what allows particles to affect each other (and even themself). With interactions, one can model scattering of particles, particle decays etc. It is where the complications of QFT arise, as it is not as well defined as the free theory. Since we will only work with interactions viewed as deformations of the free theory, we model interactions as formal series on the space of fields B, that is, interactions will be elements of the completed symmetric algebra over the dual space $\widehat{\mathrm{Sym}}^*(B^\vee) = \prod_{n\geq 0} \mathrm{Hom}(B^{\otimes n}, \mathbb{K})_{S_n}$. Elements of this algebra have as their Taylor components, continuous, S_n-invariant functionals $B^{\otimes n} \to \mathbb{K}$, where S_n denotes the symmetric group with n elements.

Feynman graphs appear as a combinatorial tool for evaluating integrals. In particular, they appear because in QFT, the Feynman sum-over-histories approach, tells us that in order to compute expectation values of an observable, one must use the measure given by the exponential of the action: $\langle \mathcal{O} \rangle \sim \int \exp(iS/\hbar)\mathcal{O}$. Computing this integral is equivalent, in finite dimensions, to summing over Feynman graphs. In infinite dimensions, we will see that the integral is ill-defined, but there is still an expansion into Feynman graphs, which will allow us to do computations.

2.1 Quantum Field Theory in finite dimensions

The famous problems that make it difficult to properly define QFTs arise only when dealing with spaces of fields of infinite dimension. Unluckily, every interesting field theory has an infinite-dimensional space of fields. Despite not being physically important, we will first define quantum field theories and all the objects we will need in finite dimensions, where everything is well defined. We will present the partition function, and how to calculate expectation values with combinatoric tricks and Feynman graphs expansions. We will then try to generalize this to the simplest quantum field theory: the scalar field theory on a compact manifold, where we will learn how to deal with divergent quantities and how to make sense of them. Finally, we will define QFT in the greatest generality we will need to properly define the Yang-Mills theory on $\mathbb{R}^4$.

The partition function is an essential quantity in QFT. The probabilistic nature of quantum mechanics introduces the need to compute expectation values of the quantities we want to know, as that is all we can expect to measure. All these expected values, computed through the Feynman sum-over histories approach, need to be normalized by the partition function $Z := \int \exp(iS/\hbar)$, for the answer to have a probabilistic interpretation. This means that for any observable, we will define its expected value as $\mathcal{O}$,

$$\langle \mathcal{O} \rangle := \frac{1}{Z} \int \exp(iS/\hbar)\, \mathcal{O}.$$

In finite dimensions one has the following data: A finite-dimensional vector space U over $\mathbb{R}$, a negative-definite quadratic form Φ on U and a function $I \in \widehat{\mathrm{Sym}}^*(U^\vee)$, such that the action is $S(\mathbf{x}) = \frac{1}{2}\Phi(\mathbf{x}, \mathbf{x}) + I(\mathbf{x})$, where $\mathbf{x} \in U$.

In this setting, the partition function reads:

$$\int_{\mathbf{x}\in U} \exp\left(\Phi(\mathbf{x}, \mathbf{x})/\hbar + I(\mathbf{x})/\hbar\right). \tag{2.1}$$

We will use the convention that the "measure" $\int_U \exp \Phi(\mathbf{x}, \mathbf{x})/\hbar$ is normalised to 1. If $U = \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\Phi(\mathbf{x}, \mathbf{x}) = -\frac{1}{2}\sum_{i,j} x^i A_{ij} x^j$, where A_{ij} is a positive definite symmetric matrix, then this means we are using the measure

$$\sqrt{\frac{\det A}{(2\pi)^n}} d^n\mathbf{x}.$$

One typical approach when calculating the partition function is to expand the exponential $\exp I(\mathbf{x})/\hbar$ as its power series so that it can be computed as an infinite sum

$$Z = \sum_{n=0}^{\infty} \int \exp\left(\frac{1}{2\hbar}\Phi(\mathbf{x}, \mathbf{x})\right) I^n(\mathbf{x})/\hbar^n.$$

Therefore, we will also be interested in computing integrals of the form,

$$\int_U \exp\left(-\frac{1}{2}\Phi(\mathbf{x},\mathbf{x})\right) F(\mathbf{x}),$$

where $F(\mathbf{x}) \in \widehat{\mathrm{Sym}}^*(U^\vee)$.

In the following section, we will see how to calculate these integrals using combinatorial methods through Feynman expansion.

2.1.1 Feynman Graphs

Feynman graphs are a nice graphical way of encoding integrals. They also have a natural interpretation as the trajectories taken by interacting particles. The idea is to define a graph with vertices, tails and edges, and attach fields to the tails, interactions to the vertices and propagators to the edges. Therefore, each Feynman graph will be encoding a way of contracting these three objects. The two main results of this section are that we can express the partition function as an infinite sum of Feynman graphs and that these Feynman graphs can be computed through combinatorial differentiation. Let's start by defining the type of graphs we will be using.

Definition 2.1.1 (Stable[1] graph)**.** A stable graph is a graph γ, possibly with external edges (or tails); and for each vertex v of γ and element $g(v) \in \mathbb{Z}_{\geq 0}$, called the genus of the vertex v; with the property that every vertex of genus 0 is at least trivalent, and every vertex of genus 1 is at least 1-valent.

If γ is a stable graph, the genus $g(\gamma)$ if γ is defined by:

$$g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g(v)$$

where $b_1(\gamma) = |E(\gamma)| - |V(\gamma)| + \#\text{connected components}$, is the first Betti number of γ.

It is determined by the following data:

i) A finite set $H(\gamma)$ of half-edges of γ

ii) A finite set $V(\gamma)$ of vertices of γ.

iii) An involution $\sigma : H(\gamma) \to H(\gamma)$. The set of fixed points of this involution is denoted $T(\gamma)$, and is called the set of tails of γ. The set of two elements orbits is denoted $E(\gamma)$, and is called the set of edges.

iv) A map $\pi : H(\gamma) \to V(\gamma)$, which sends a half-edge to the vertex to which is attached.

v) A map $g : V(\gamma) \to \mathbb{Z}_{\geq 0}$.

From this, we construct a topological space $|\gamma| := V(\gamma) \amalg \left(H(\gamma) \times [0, \frac{1}{2}]\right) / \sim$, where $\sim$ is the equivalence relation that identifies $(h, 0) \in H(\gamma) \times [0, \frac{1}{2}]$ with $\pi(h) \in V(\gamma)$, and $(h, \frac{1}{2})$ with $(\sigma(h), \frac{1}{2})$. We say γ is connected if $|\gamma|$ is.

We will be interested in automorphisms $g \in \mathrm{Aut}(\gamma)$ of stable graphs, which consist of a pair of maps $H(g) : H(\gamma) \to H(\gamma)$ and $V(g) : V(\gamma) \to V(\gamma)$, such that $H(g)$ commutes with σ and the following diagram

$$\begin{array}{ccc} H(\gamma) & \xrightarrow{H(g)} & H(\gamma) \\ \downarrow & & \downarrow \\ V(\gamma) & \xrightarrow{V(g)} & V(\gamma) \end{array}$$

commutes.

The involution allows us to distinguish between half-edges and tails, while the projection π tells us which half-edges (and tails) are attached to each vertex.

[1]The name stable comes from algebraic geometry, where this types of graphs are also considered.

Our objective is to expand the divergent integrals in terms of these graphs, so need to show how to attach to each graph operators and fields.

Let U be a finite-dimensional (super[2]) vector space, over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{O}(U)$ denote de completed symmetric algebra over the dual, that is $\mathscr{O}(U) := \widehat{\mathrm{Sym}}^* U^\vee$, and let $\mathscr{O}^+(U)[[\hbar]]$ denote those functionals which are at least cubic modulo $\hbar$. Each element $I \in \mathscr{O}^+(U)[[\hbar]]$ can be expressed as $I = \sum_{i,k} \hbar^i I_{i,k}$, where $I_{i,k}$ is homogeneous of degree k in the variable $u \in U$.

Each $I_{i,k}$ defines an S_k-invariant linear map, $D^k I_{i,k} \in (U^\vee)^{\otimes k}$:

$$\begin{aligned} D^k I_{i,k} &: U^{\otimes k} \to \mathbb{K} \\ u_1 \otimes \cdots \otimes u_k &\to \left(\frac{\partial}{\partial u_1} \cdots \frac{\partial}{\partial u_k} I_{i,k} \right)(0) \end{aligned}$$

Suppose we are given n fields $a_1, \ldots, a_n \in U$, a propagator $P \in \mathrm{Sym}^2 U$, an interaction $I \in \mathscr{O}^+(U)[[\hbar]]$, and an ordering of the set of tails $T(\gamma)$. Assigning to every stable graph with n tails, a propagator P on each internal edge and a_i on the i-th tail, gives an element of $U^{\otimes H(\gamma)}$ and putting $D^k I_{i,k}$ on each vertex of valency k and genus i gives an element of $\mathrm{Hom}(U^{\otimes H(\gamma)}, \mathbb{K})$. Define the weight $\omega_{\gamma,\phi}(P,I)(a_1, \ldots, a_n)$ as the contraction of these two elements.

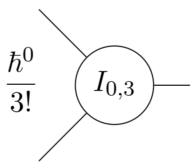
We also define $\omega_\gamma(P,I) \in \mathscr{O}(U)$ by $\omega_\gamma(P,I)(a) = \omega_{\gamma,\phi}(P,I)(a, \ldots, a)$, so that

$$\left(\frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_n} \omega_\gamma(P,I) \right)(0) = \sum_\phi \omega_{\gamma,\phi}(a_1, \ldots, a_n).$$

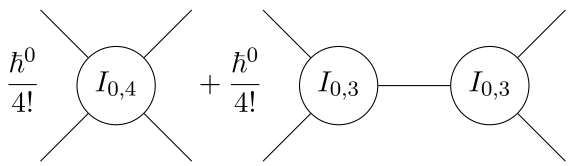
Where the sum runs over all possible orderings of the set of tails $T(\gamma)$. Arranging these functions into a formal power series we obtain

$$W(P,I) = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} \hbar^{g(\gamma)} \omega_\gamma(P,I) \in \mathscr{O}^+(U)[[\hbar]].$$

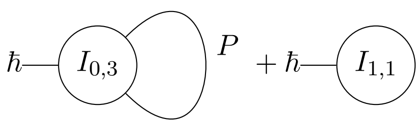
To illustrate, here are the first few terms that appear in the expansion:



Order $\hbar^0$ and valency 3.



Order $\hbar^0$ and valency 4.



Order $\hbar$ and valency 1.

We will see that this contraction on graphs reduces to just combinatoric differentiation. Given a propagator $P \in \mathrm{Sym}^2 U$, let us write it in a basis $P = \sum P' \otimes P''$. Define an order two differential operator $\partial_P : \mathscr{O}(U) \to \mathscr{O}(U)$ associated to P by $\partial_P = \frac{1}{2} \sum \frac{\partial}{\partial P'} \otimes \frac{\partial}{\partial P''}$. Then,

Lemma 2.1.1.

$$W(P,I)(a) = \hbar \log \{ \exp(\hbar \partial_P) \exp(I/\hbar)(a) \}$$

[2]As usual, the adjective super will refer to a $\mathbb{Z}_2-$graded space

Proof. First note that the identity holds for $P = 0$. The right-hand side is clearly equal to I. Now, the sum $W(0, I)$ will only have terms coming from graphs with no edges,

$$W(0,I) = \sum_{i,k} \frac{1}{|\mathrm{Aut}(v_i,k)|}\omega_{v_{i,k}}(0,I),$$

where $v_{i,k}$ denotes the graph with 1 vertex, genus i, valency k and no internal edges. The automorphism group of the graph $v_{i,k}$ is the symmetric group S_k which has $k!$ elements and the weight is simply $\omega_{v_{i,k}}(0,I) = k!I_{i,k}$. That is because

$$\begin{aligned} D^k\omega_{v_{i,k}}(a_1,\dots,a_k) = \left(\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\omega_{v_{i,k}}\right)(0) &= \sum_{\phi}\omega_{v_{i,k},\phi}(a_1,\dots,a_k) = k!\omega_{v_{i,k},\phi}(a_1,\dots,a_k) \\ &= k!D^kI_{i,k}(a_1,\dots,a_k). \end{aligned}$$

Thus, $W(0,I) = \sum_{i,k}\frac{\hbar^i}{k!}k!I_{i,k} = I$ by definition.

We will prove that both sides satisfy the same differential equation. Given a parameter of square zero ε and $P' \in \mathrm{Sym}^2U$, it is clear that:

$$\exp\left(\hbar\partial_{P+\varepsilon P'}\right)\exp(I/\hbar) = (1+\varepsilon\hbar\partial_{P'})\exp(\hbar\partial_P)\exp(I/\hbar).$$

To verify that $\exp\left(W(P+\varepsilon P', I)/\hbar\right) = (1+\varepsilon\hbar\partial_{P'})\exp\left(W(P,I)/\hbar\right)$, we will consider, for $a_1,\dots,a_k \in U$:

$$\left(\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\exp(W(P,I)/\hbar)\right)(0) = \sum_{\gamma,\phi}\frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma,\phi)|}\omega_{\gamma,\phi}(a_1,\dots,a_k),$$

where the sum is over all possible disconnected stable graphs with an ordering $\phi : \{1,\dots,k\} \equiv T(\gamma)$ and the automorphism group preserves the ordering. When varying P to $P+\varepsilon P'$ we note that

$$\frac{d}{d\varepsilon}\left(\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\exp(W(P,I)/\hbar)\right)(0) = \sum_{\gamma,\phi}\frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma,e,\phi)|}\omega_{\gamma,e,\phi}(a_1,\dots,a_k),$$

where the sum is over possibly disconnected stable graphs with a distinguished edge $e \in E(\gamma)$. The weight is defined as usual, except for the distinguished edge, which is labelled by P' instead of P and the automorphism group now also preserves this distinguished edge.

Cutting this edge e in two, we obtain a stable graph with two more tails. If $P' = \sum u' \otimes u''$, where $u', u'' \in U$, the new tails are labelled by u' and u''. Thus,

$$\begin{aligned} \frac{d}{d\varepsilon}\left(\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\exp(W(P,I)/\hbar)\right)(0) &= \frac{1}{2}\sum_{\gamma,\phi}\frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma,\phi)|}\omega_{\gamma,\phi}(a_1,\dots,a_k) \\ &= \frac{1}{2}\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\frac{\partial}{\partial u'}\frac{\partial}{\partial u''}\exp\left(W(P,I)/\hbar\right)(0) \\ &= \frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_k}\partial_{P'}\exp\left(W(P,I)/\hbar\right)(0), \end{aligned}$$

where the sum is now over graphs with $k+2$ tails. □

The following lemma will prove our desired result of expressing our integrals as a sum of Feynman graphs. We will investigate only the case where $\mathbb{K} = \mathbb{R}$ for simplicity. Let U be a finite-dimensional vector space over $\mathbb{R}$ equipped with a negative defined, non-degenerate quadratic form, and let $P \in \mathrm{Sym}^2U$ be its inverse. Then,

Lemma 2.1.2.

$$W(P,I)(a) = \hbar\log\int_U \exp\left(\frac{1}{2\hbar}\Phi(\mathbf{x},\mathbf{x}) + \frac{1}{\hbar}I(\mathbf{x}+a)\right)$$

where the integral is thought of as an asymptotic expansion in $\hbar$ and the measure is normalized such that $\int_U \exp\left(\frac{\Phi(\mathbf{x},\mathbf{x})}{2\hbar}\right) = 1$.

Proof. First, we will define the Laplace transform of a function f on V as:

$$\mathcal{L}(f)(s) = \int_U dx \exp(s\mathbf{x}) f(\mathbf{x})$$

where $s \in U^\vee$. Note that the Laplace transform relates the bilinear form Φ with its dual P via:

$$\mathcal{L}(\exp\left(-\Phi(\mathbf{x},\mathbf{x})/2\hbar\right)) = \exp\left(\hbar P(s,s)/2\right)$$

This can be seen by completing the square. Next, note that, for f, F polynomial functions on U,

$$\mathcal{L}(fF) = \underline{f}(\mathcal{L}F)$$

where $\underline{f}$ denote the constant coefficient differential operator on $U^\vee$ associated to f. If f is expressed as $f = \sum_I c_I x^I$ where $I = (i_1, \ldots, i_n)$ is a multi-index, then $\underline{f}$ is defined as $\sum_I c_I \partial^I = c_{i_1,\ldots,i_n}\left(\frac{\partial}{\partial s_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial s_n}\right)^{i_n}$. This is a direct computation.

Finally, we will prove that $\int_U dx \mathcal{L}^{-1}(f) F = (\underline{f}F)(0)$.

$$\int_U dx \mathcal{L}^{-1}(f) F = \mathcal{L}\left[F\mathcal{L}^{-1}(f)\right](0) = \underline{F}\mathcal{L}\mathcal{L}^{-1}(f)(0) = \underline{F}f(0) = \underline{f}F(0)$$

Now

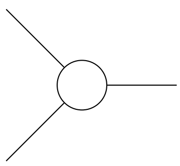
$$\begin{aligned}
\int_U \exp\left(\frac{1}{2\hbar}\Phi(x,x)\right)\exp\left(\frac{1}{\hbar}I(x+a)\right) &= \int_U \mathcal{L}^{-1}\mathcal{L}\left[\exp\left(\frac{1}{2\hbar}\Phi(x,x)\right)\right]\exp\left(\frac{1}{\hbar}I(x+a)\right) \\
&= \int_U \mathcal{L}^{-1}\left[\exp\left(\frac{1}{2\hbar}P(s,s)\right)\right]\exp\left(\frac{1}{\hbar}I(x+a)\right) \\
&= \underline{\exp\left(\frac{\hbar}{2}P(s,s)\right)}\exp\left(\frac{1}{\hbar}I(x)\right)(a) \\
&= \exp\left(\hbar\partial_P\right)\exp\left(I(x)/\hbar\right)(a)
\end{aligned}$$

□

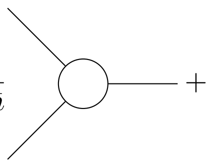
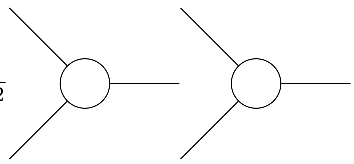
Example 1. . Let $U = \mathbb{R}$ and let $I(x) = \frac{x^3}{3!} \in \mathscr{O}(\mathbb{R})$. Our quadratic form will simply be $\Phi(x,x) = -x^2$, and the operator associated to the propagator will be $\partial_P = \frac{1}{2}\partial_x^2$. We will try to compute

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2\hbar}x^2\right)\exp\left(\frac{x^3}{3!\hbar}\right) = \exp\left(\hbar\partial_P\right)\exp\left(\frac{x^3}{3!\hbar}\right)\Big|_{x=0}$$

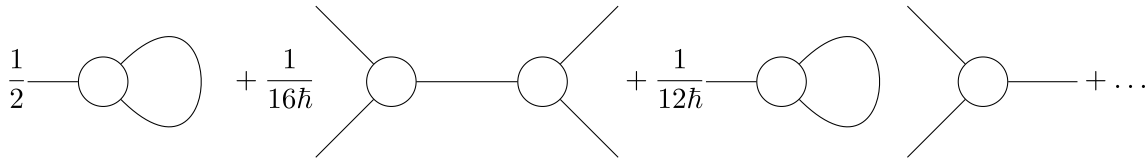
.

First, we associate a vertex with 3 tails for each x^3 

Therefore, $\exp(x^3/(3!\hbar))$ will be an infinite sum of the form:

$1 + \frac{1}{6\hbar}$  $+ \frac{1}{72\hbar^2}$  $+ \ldots$

The operator $\frac{1}{2}\hbar\partial_x^2$ acts by contracting a pair of tails. Applying it to the sum above we get:

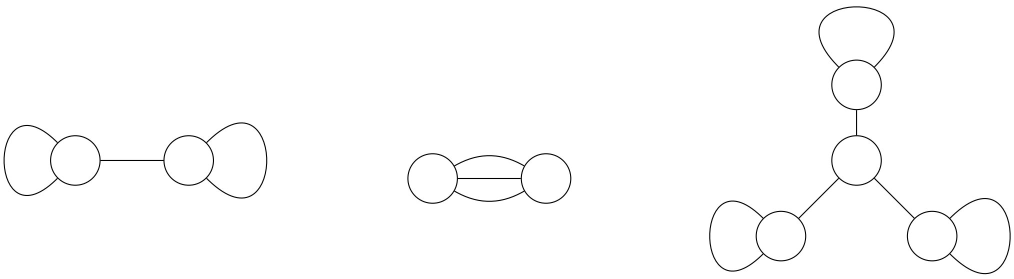


The next step is to act with $\frac{1}{4}\hbar^2\partial_x^4$, which contracts two pairs of tails, etc.

When computing $\exp(\hbar\partial_P)\exp(x^3/(3!\hbar))$, we get a factor of $\hbar$ for each pair of tails contracted, that is, for each edge, and a factor of $1/\hbar$ for each vertex. Furthermore, the factorial in the denominator of $\exp(\hbar\partial_P)$ will count the number of possible edges permutations and the factorial from $\exp(x^3/(3!\hbar))$ will count the number of tail permutation. When multiplying both we get the number of graph automorphisms. We see that we end up with

$$\exp(\hbar\partial_P)\exp(x^3/(3!\hbar)) = \sum_{\text{trivalent graphs } \gamma} \hbar^{-\chi(\gamma)} \frac{x^{\#\text{ tails}}}{|\mathrm{Aut}(\gamma)|}$$

where the sum is over all connected trivalent graphs and $\chi(\gamma)$ denotes the Euler characteristic. When evaluating at $x = 0$ we end up only with graphs with all its tails contracted. The asymptotic expansion of the integral $\int_{\mathbb{R}} \exp\left(-\frac{1}{2\hbar}x^2 + \frac{x^3}{6\hbar}\right)$ is, expanding the second exponential and integrating term by term, $\int_{\mathbb{R}} \exp\left(-\frac{1}{2\hbar}x^2 + \frac{x^3}{6\hbar}\right) \sim 1 + \frac{5}{24}\hbar + \frac{385}{1152}\hbar^2 + \ldots$. To reproduce the coefficient in $\hbar$ we would need to add all the contributions from all trivalent graphs with $\chi(\gamma) = -1$, that is, the graphs



And so on. We see that in finite dimensions, obtaining asymptotic expansions with Feynman graphs is not very useful.

2.2 Quantum Field Theory in infinite dimensions

When one tries to generalise this to the case where the space of fields is infinite-dimensional, like $C^\infty(M)$ for example, one encounters several barriers. The first one is that there is no (translational invariant) Lebesgue measure on any infinite-dimensional vector space, so we cannot define the integral 2.1 directly. Luckily, one can still define the expression $W(P, I)$, since one can still contract tensors if we are careful[3]

2.2.1 Nuclear spaces

In order to define properly theories in infinite dimensions, we need to introduce the type of spaces we will be using. We would like to have some notion of tensor product and to be able to define spaces of

[3]Singularities in Feynman graphs arise because the inverse of the quadratic forms we consider in infinite dimensions do not lie in the correct completed tensor product.

functions over these spaces $\widehat{\mathrm{Sym}}^*(U^\vee)$. This is the topic of [Cos11, Appendix 2], which we will briefly review. A nice complete review on this can be found in [Trè67].

It turns out that the spaces we will need are called *Nuclear* spaces. As Banach spaces, they are also a generalization of vector spaces, but in a completely different manner. In fact, an infinite-dimensional Banach space is never Nuclear.

We will start by recalling the definition of topological vector space.

Definition 2.2.1. A Topological Vector Space (or TVS) over $\mathbb{R}$ or $\mathbb{C}$ is a vector space equipped with a topology which makes scalar multiplication and addition continuous.

We will consider only Hausdorff topological vector spaces. A particularly important class of TVS, which contains both Banach and Nuclear spaces are *locally convex spaces.* Such a space has a basis for its topology given by convex sets. Finally, a Fréchet space is a complete, metrizable, locally convex space, that is, a locally convex space X whose topology can be induced by a metric, and which is complete with respect to any metric that defines its topology.

Subtleties in functional analysis often arise from the freedom to choose different topologies on the dual space. Let V be a TVS, we will denote $V^\vee$ the space of all continuous linear functionals equipped with the strong topology, also known as the topology of bounded convergence, which is the topology defined by the seminorms $|y|_B := \sup_{x\in B}|y(x)|$, for every bounded subset $B \subset V$ and $y \in V^\vee$. A basis of neighbourhoods of zero is then given by $\{y \in V^\vee | \, |y|_B < r\}$ for all bounded subsets B and $r > 0$. We will give the space $\mathrm{Hom}(E, F)$ of continuous linear maps from E to F the topology of bounded convergence, which given an open subset U of F and a bounded subset B of E, the basis of neighbourhood of zero is given by $\{f : E \to F | f(V) \subset U\}$. As long as E and F are locally convex Hausdorff spaces, so is $\mathrm{Hom}(E, F)$.

For tensor products, we will define the projective topology on $V \otimes_{alg} W$, where $\otimes_{alg}$ denotes the algebraic tensor product, as the finest, locally convex topology such that the canonical map

$$V \times W \to V \otimes_{alg} W$$

is continuous.

We will denote simply by $V \otimes W$ the completion of $V \otimes_{alg} W$with respect to the projective topology.

There is also a coarser topology on $V \otimes_{alg} W$, known as the injective topology. Denote the completion of $V \otimes_{alg} W$ with respect to the injective topology as $V \otimes_i W$. We will not recall the definition of injective topology as it is technical and not even Costello defines it. We will refer to [Trè67, Chapter 43] for details. A space V is said to be Nuclear if the canonical map $V \otimes W \to V \otimes_i W$ is an isomorphism for any locally convex Hausdorff space W. The importance of Nuclear spaces comes because most of the spaces we will consider are nuclear and they enjoy the properties we were looking for. Namely, examples of Nuclear spaces are:

i) The space of smooth functions on an open subset U in $\mathbb{R}^n$, $C^\infty(U)$, with the topology given by a set of neighbourhoods of 0 $\{f \in C^\infty(U) | \text{for any } K \subset U, \forall m \in \mathbb{N}, |f|_{m,K} < r\}$, as $r > 0$, where $|f|_{m,K} := \sup_{|p|\leq m}\left(\sup_{x\in K}\left|\left(\frac{\partial}{\partial x}\right)^p f\right|\right)$, $p = (p_1, \dots, p_n) \in \mathbb{N}^n$ and $|p| = p_1 + \cdots + p_n$.

ii) The space of functions with compact support on an open set or in a compact set, in $\mathbb{R}^n$ $C_c^\infty(U)$, with a similar topology as before.

iii) The space of distributions (the continuous dual space to compactly-supported smooth functions) on an open set U of $\mathbb{R}^n$, $\mathcal{D}(U)$, equipped with the strong topology.

iv) Distributions with compact support on a open set in $\mathbb{R}^n$, with the induced topology from *iii).*

v) Schwartz functions in $\mathbb{R}^n$ (we will define them in Chapter 3), $\mathscr{S}(\mathbb{R}^n)$, with the topology induced by the seminorms $|f|_{m,k} := \sup_{|p|\leq m}\left(\sup_{x\in\mathbb{R}^n}(1 + |x|)^k\left|\left(\frac{\partial}{\partial x}\right)^p f\right|\right)$.

vi) Schwartz distributions on $\mathbb{R}^n$, with the strong topology.

vii) Formal power series in n variables, with the topology induced by $|u|_m := \sup_{|p|\leq m} |u_p|$ for $m \in \mathbb{N}$, where $u = \sum\limits_{p=(p_1,\ldots,p_n)} u_p X^p$.

viii) Polynomials in n variables, with the topology of *vii)* but for $m = n$.

ix) Smooth sections of a vector bundle E over a compact manifold M, $\Gamma(M, E)$. The topology is defined by a basis of neighbourhoods of zero $\{f \in \Gamma(M, E) | \forall x \in U \subset M, f(x) \in V_x \subset E_x\}$, for all open subsets U of M and neighbourhoods of zero V_x in E_x, where E_x denotes the fibre over $x \in M$.

And they satisfy the following properties:

1. A closed linear subspace of a nuclear space is nuclear.
2. The quotient of a nuclear space by a closed linear subspace is nuclear.
3. A countable direct sum of nuclear spaces (equipped with the finest locally convex topology) is nuclear.
4. A direct product of nuclear spaces is nuclear.
5. The completed tensor product is nuclear.
6. If E is a nuclear Fréchet space, then $\mathscr{O}(E) = \prod_n \mathrm{Sym}^n E^\vee$ is nuclear.
7. Let E and F be nuclear Fréchet spaces, the following equalities hold:
   (a) $E \otimes F = \mathrm{Hom}(E^\vee, F)$
   (b) $E^\vee \otimes F = \mathrm{Hom}(E, F)$
   (c) $E^\vee \otimes F^\vee = (E \otimes F)^\vee$
   (d) $E^\vee \otimes F^\vee = \mathrm{Hom}(E, F^\vee)$

2.2.2 Feynman expansion

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, M a manifold, and E a (super) vector bundle on M over $\mathbb{K}$. Let $\mathcal{E} = \Gamma(M, E)$ denote the super nuclear Fréchet space of global sections of E. Let $\otimes$ be the completed projective tensor product, such that $\mathcal{E} \otimes \mathcal{E} = \Gamma(M^2, E \boxtimes E)$. Here $E \boxtimes E$ refers to the vector bundle $\mathrm{pr}_1^* E \otimes \mathrm{pr}_2^* E$ over $M \times M$, where pr_1 (pr_2) represents the projection of $M \times M$ onto the first (second) factor.

Let $\mathscr{O}(\mathcal{E}) = \widehat{\mathrm{Sym}}^*(\mathcal{E}^\vee)$ be the completed symmetric algebra over the (strong) dual space. Note that the structure of algebra is given by the direct product of distributions, which defines a map:

$$\mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{K}) \times \mathrm{Hom}(\mathcal{E}^{\otimes m}, \mathbb{K}) \longrightarrow \mathrm{Hom}(\mathcal{E}^{\otimes m+n}, \mathbb{K})$$

As before, let $\mathscr{O}^+(\mathcal{E})[[\hbar]] \subset \mathscr{O}(\mathcal{E})[[\hbar]]$ denote the subspace of functionals which are at least cubic modulo $\hbar$.

The construction is the same as before. The tensor products of interactions at vertices define an element of $\mathrm{Hom}(\mathcal{E}^{\otimes H(\gamma)}, \mathbb{K})$, while tensor products of propagators at edges define an element of $\mathcal{E}^{\otimes 2E(\gamma)}$. The weight $\omega_\gamma(P, I) \in \mathscr{O}(\mathcal{E})$ is defined as the contraction of these two elements, and arranging them in a formal power series we get:

$$W(P, I) = \sum_\gamma \frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma)|} \omega_\gamma(P, I) \in \mathscr{O}^+(\mathcal{E})[[\hbar]].$$

Again, define the differential operator $\partial_P : \mathscr{O}(\mathcal{E}) \longrightarrow \mathscr{O}(\mathcal{E})$, which acts on each direct factor by contraction with $P \in \mathrm{Sym}^2 \mathcal{E}$: $\mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{K}) \longrightarrow \mathrm{Hom}(\mathcal{E}^{\otimes n-2}, \mathbb{K})$.

As before, we have

Lemma 2.2.1.

$$W(P,I) = \hbar \log \{\exp(\hbar \partial_P) \exp(I/\hbar)\}$$

.

By analogy with the finite-dimensional case, one may attempt to define the formal identity:

$$W(P,I)(a) = \hbar \log \int_{\mathbf{x} \in \mathcal{E}} \exp\left(\frac{1}{2\hbar}\Phi(\mathbf{x},\mathbf{x}) + \frac{1}{\hbar} I(\mathbf{x}+a)\right)$$

The problem is that neither side of the equation is well-defined, since the propagator will not be in general smooth along the diagonal $M \times M$ and the integral is infinite-dimensional.

The key idea here is the effective theory philosophy explained in the Introduction. Infinities arise because we are trying to describe physics at infinite energy scales, so we will need to impose some cut-offs on the energy scales that we will like our theory to describe.

We use a regularization based on the heat kernel, which we will define shortly but first, we need some preliminary definitions.

Definition 2.2.2 ($s-$density bundle)**.** Given a Riemannian manifold (M,g), with atlas given by $\{V_\alpha, \phi_\alpha\}$, define the $s-$density bundle $|\Lambda|^s(M) \to M$ as the vector bundle such that for every $p \in M$, there is an open neighbourhood $U_\alpha \subset M$ of p such that there is a local trivialization map,

$$t_\alpha : |\Lambda|^s(M)\big|_{U_\alpha} \to \phi_\alpha(U_\alpha) \times \mathbb{R},$$

and whose transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^\times$ are given by:

$$t_{\alpha\beta} = |\det(d\phi_\alpha \circ d\phi_\beta^{-1})|^{-s}$$

When $s = 1$ we will denote $|\Lambda|(M)$ simply by $\mathrm{Dens}(M)$.

Definition 2.2.3 (Generalised Laplacian)**.** Let E ve a vector bundle over a Riemannian manifold (M,g). A generalized Laplacian on E is a second-order differential operator, H such that its principal symbol $\sigma(H)$ is $\sigma(H)(x,\xi) = |\xi|^2$. An equivalent way of saying this is that on any local coordinate system, with local coordinates $\mathbf{x} = (x_1, \ldots, x_n)$ and basis for the tangent bundle $\partial_1\big|_{\mathbf{x}}, \ldots, \partial_n\big|_{\mathbf{x}}$

$$H = -\sum_{i,j} g^{ij}(\mathbf{x}) \partial_i \partial_j + \text{first-order terms}$$

where $g^{ij}(\mathbf{x}) = (d\mathbf{x}^i, d\mathbf{x}^j)_x$ is the metric on the cotangent bundle T^*M.

Now, given a Riemannian manifold (M,g) and vector bundle E over M equipped with a generalised Laplacian H,

Definition 2.2.4 (Heat Kernel)**.** A heat kernel for a (generalised) Laplacian H is a continuous section $K_t(x,y)$ of the bundle $(E \otimes |\Lambda|^{1/2}) \boxtimes (E^* \otimes |\Lambda|^{1/2})$ over $\mathbb{R}_+ \times M \times M$, satisfying:

1. $K_t(x,y)$ is C^1 with respect to t.

2. $K_t(x,y)$ is C^2 with respect to x.

3. $K_t(x,y)$ satisfies the heat equation:

$$(\partial_t + H_x) K_t(x,y) = 0.$$

4. It satisfies the boundary condition at $t = 0$:

$$\lim_{t \to 0} K_t(x,y) = \delta(x,y).$$

Where $\delta(x.y)$ is the Dirac delta. More precisely, if s is a smooth section of $E \otimes |\Lambda|^{1/2}$ then,

$$\lim_{t \to 0} \int_{y \in M} K_t(x,y) s(y) = s(x), \quad \forall x \in M$$

where the limit is meant in the uniform norm, $||s||_0 = \sup_{x \in M} ||s(x)||$.

The motivation for using the heat kernel is threefold. First, it allows us to express the propagator as a one-dimensional integral

Proposition 2.2.1. *If the free term is* $H+m^2$*, the propagator,* $P(x,x') := (H+m^2)^{-1}$*, which satisfies* $(H+m^2)P(x,x') = \delta(x,x')$ *is expressed in terms of the heat kernel as*

$$P(x,x') = \int_0^\infty e^{-tm^2} K_t(x,x')dt$$

Proof.

$$\begin{aligned}\left(H+m^2\right)\int_0^\infty e^{-tm^2}K_t(x,x')dt &= m^2\int_0^\infty e^{-tm^2}K_t(x,x')dt - \int_0^\infty e^{-tm^2}\partial_t K_t(x,x')dt \\ &= m^2\int_0^\infty e^{-tm^2}K_t(x,x')dt - e^{-tm^2}K_t(x,x')\big|_0^\infty - m^2\int_0^\infty e^{-tm^2}K_t(x,x')dt \\ &= \lim_{t\to 0} K_t(x,x') \\ &= \delta(x,x')\end{aligned}$$

□

Therefore, we can impose cut-offs in a natural manner, by restricting the limits of integration by ε and L. Define the approximate or *regularized* propagator as $P(\varepsilon, L) := \int_\varepsilon^L K_t dt$, where ε will be referred to as the ultra-violet cut-off, and L will be referred to as the infrared cut-off. We note that K_t is a smooth function on $M \times M$ as long as $t > 0$. Letting $\varepsilon \to 0$ we will obtain the same singularities as before, and we will see in the next section see how to deal with them.

The second crucial aspect of the heat kernel is that it possesses an asymptotic expansion.

Proposition 2.2.2. *In* $\mathbb{R}^n$*, with the usual (negative-definite) Laplacian,* $D = -\sum_i \frac{\partial^2}{\partial x_i^2}$*, the heat kernel is given by:*

$$K_t(x,y) = (4\pi t)^{-n/2}\, e^{-||x-y||^2/4t}.$$

Proof. This is a direct computation. □

Proposition 2.2.3. *Let* (M,g) *be a Riemannian manifold, where* M *is a compact manifold. Let* E *be a vector bundle over* M *and* H *be a generalized Laplacian for* E*. The heat kernel,* $K_t(x,y)$ *has a small* t *asymptotic expansion:*

$$K_t(x,y) \simeq (4\pi t)^{-n/2}\, e^{-d(x,y)^2/4t} \sum_{i=0}^\infty t^i f_i(x,y).$$

Where $d(x,y)$ *denotes the geodesic distance between* x *and* y *and* $f_i(x,y)$ *are smooth sections of the bundle* $E \boxtimes E^*$*, which depend on the generalized Laplacian.*

The third one is that the heat kernel has an interpretation as the transition probability for a random path between two space-time points. Therefore, the propagator can be rewritten as

$$P(x,y) = \int_0^\infty e^{-tm^2} \int_{\substack{f:[0,t]\to M \\ f(0)=x, f(t)=y}} \exp\left(-\int_0^t ||df||^2\right)$$

This gives the interpretation that the propagator represents the probability that a particle starts at a point x in space-time and transitions to y through a random path.

The existence and unicity of the heat kernel is a non-trivial matter. Luckily, in compact manifolds and in $\mathbb{R}^n$ a *unique* heat kernel exists [BGV92, Chapter 2]. For general non-compact manifolds, uniqueness is not guaranteed.

Example 2. Let's see for example the case of a free scalar field theory compact manifold with kinetic term $-\frac{1}{2}(D+m^2)$, where D denotes the non-negative Laplacian. In a compact manifold, there is a collection of eigenvalues, $m^2 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$, and an orthonormal basis $\{\phi_n\}_{n\geq 0}$ of $L^2(M)$, such that

$$D\phi_n = \lambda_n \phi_n.$$

The heat kernel[4] $K_t(x,y) \in C^\infty(M) \otimes C^\infty(M)$ then reads:

$$K_t(x,y) = e^{-tm^2} \sum_{n\geq 0} e^{-\lambda_n t} \phi_n(x) \otimes \phi_n(y).$$

Given $\varepsilon, L > 0$, the propagator reads:

$$P(\varepsilon, L) = \sum_{n\geq 0} \frac{e^{-\varepsilon(\lambda_n+m^2)} - e^{-L(\lambda_n+m^2)}}{\lambda_n + m^2} \phi_n(x) \otimes \phi_n(y)$$

We see that the *regularized* propagator damps down the high energy modes of the total propagator.

These cut-offs together with the regularized propagators will allow us to define, by analogy with the energy picture, "effective" interactions at each scale L which are related by the renormalization group equation.

Definition 2.2.5 (Renormalization group flow from length scale ε to L)**.** Is defined as the map:

$$\begin{aligned} \mathscr{O}^+(C^\infty(M))[[\hbar]] &\to \mathscr{O}^+(C^\infty(M))[[\hbar]] \\ I &\to W(P(\varepsilon, L), I) \end{aligned}$$

This definition has a nice physical interpretation. The scales ε and L are *length* scales and effective interactions at scale L encode all the physics that happens at scales smaller than L. Trying to describe interactions at all scales amounts to taking the limit $\varepsilon \to 0$ which recovers the singularities we avoided before.

Example 3. We will introduce the ϕ^3 scalar field theory as a toy model for illustrating all the definitions and machinery we have been building. The action for this theory is given by the usual free part $S_{quad}(\phi) = \int_M -\frac{1}{2}\phi(D+m^2)\phi$ and the interaction term is given by: $I(\phi) = \int_M \frac{1}{3!}\phi^3$

Our Feynman graph expansion will have vertices, labelled by the linear map:

$$\begin{aligned} C^\infty(M)^{\otimes 3} &\to \mathbb{R} \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \frac{\partial}{\partial\phi_1}\frac{\partial}{\partial\phi_2}\frac{\partial}{\partial\phi_3} I \\ &= \int_M \phi_1(x)\phi_2(x)\phi_3(x) \end{aligned}$$

Consider the following graphs:

.

The weight associated to the first graph will be

$$\omega_{\gamma_1}(P(\varepsilon, L), I)(a) = \int_\varepsilon^L \int_M a(x)^2 K_t(x,x) d\mathrm{Vol}_M dt,$$

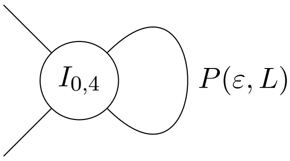
where $a \in C^\infty(M)$ and $d\mathrm{Vol}_M$ is the volume form associated to the metric on M.

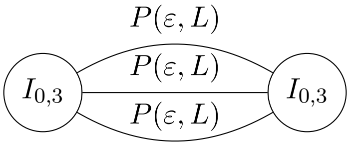
The heat kernel on the diagonal has the following form when t is small: $K_t(x,x) \sim t^{-n/2} +$ higher order terms, where $n = \dim M$. We see that the limit as $\varepsilon \to 0$ doesn't exist.

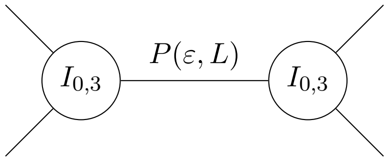
For γ_2, we have

$$\omega_{\gamma_2}(P(\varepsilon, L), I) = \int_{t_1,t_2,t_3 \in [\varepsilon, L]} \int_{x,y\in M} K_{t_1}(x,y) K_{t_2}(x,y) K_{t_3}(x,y) d\mathrm{Vol}_{M\times M} dt_1 dt_2 dt_3.$$

[4] We will omit the half-density notation for simplicity and to match notation from [Cos11]







γ_2 γ_3

γ_1

Again, using the small t asymptotic expansion, $k_t(x, y) \simeq t^{-n/2} e^{-d(x,y)^2/4t}$, at lowest order,

$$\int_{t_1,t_2,t_3 \in [\varepsilon, L]} \int_{x,y \in M} t_1^{-n/2} t_2^{-n/2} t_3^{-n/2} e^{-d(x,y)^2/t_1 - d(x,y)^2/t_2 - d(x,y)^2/t_3} d\mathrm{Vol}_{M \times M} dt_1 dt_2 dt_3,$$

which is again divergent for $\varepsilon \to 0$. For the third example, we have

$$\begin{aligned} \omega_{\gamma_3}(P(\varepsilon, L), I)(a) &= \int_{t=\varepsilon}^{L} \int_{x,y \in M} a(x)^2 K_t(x, y) a(y)^2 d\mathrm{Vol}_{M \times M} dt \\ &= \int_{x \in M} a(x)^2 \int_{y \in M} \int_{\varepsilon}^{L} e^{-t(D+m^2)} a(y)^2 d\mathrm{Vol}_{M \times M} dt \end{aligned}$$

Which admits a limit $\varepsilon \to 0$. This is in fact a general property of Feynman graphs without loops.

We are now prepared to give our first definition of QFT, which will be of a scalar field theory on a compact manifold. The core ideas are that we want our theory to be defined by a collection of interactions at each length scale L and that we can pass from one length scale to another by the renormalization group flow. Moreover, we want our interactions to become more and more local as $L \to 0$.

2.2.3 Scalar field theory on a compact manifold

Definition 2.2.6 (Perturbative scalar field theory for a compact manifold M)**.** A perturbative quantum field theory on a compact Riemannian manifold (M, g), with space of fields $C^\infty(M)$ and kinetic action $-\frac{1}{2}\int_M \phi(D + m^2)\phi$, is given by a set of effective interactions $I[L] \in \mathscr{O}^+(C^\infty(M))[[\hbar]]$ for all $L \in (0, \infty]$, such that:

1. The renormalization group equation,

$$I[L] = W(P(\varepsilon, L), I[\varepsilon])$$

is satisfied, for all $\varepsilon, L \in (0, \infty]$

2. For each i, k there is a small L asymptotic expansion

$$I_{i,k}[L] \simeq \sum_{r \in \mathbb{Z}_{\geq 0}} g_r(L) \Phi_r$$

where $g_r(L) \in C^\infty(0, \infty)_L$ and $\Phi_r \in \mathscr{O}_{loc}(C^\infty(M))$.

Let $\mathscr{T}^{(\infty)}(M)$ denote the set of scalar quantum field theories and let $\mathscr{T}^{(n)}(M)$ denote the set of theories defined modulo $\hbar^{n+1}$.

Here $\mathscr{O}_{loc}(C^\infty(M))$ is the subspace of $\mathscr{O}(C^\infty(M))$ of functionals that are the integral of some local Lagrangian. That is, if we decompose $I \in \mathscr{O}_{loc}(C^\infty(M))$ as $I = \sum_k I_k$, where each $I_k(a)$ is homogeneous of degree k in the variable $a \in C^\infty(M)$ then, each I_k must be the integral of some finite sum of differential operators: $I_k(a) = \sum_j \int_M D_{1,j}a \dots D_{k,j}a$, where $D_{i,j}$ are differential operators on M.

Without loss of generality, one can assume that the local functionals Φ_r are homogeneous of degree k in the field $a \in C^\infty(M)$. Then, the small L asymptotic expansion translates into the statement that there is a non-decreasing sequence $d_R \in \mathbb{Z}$, tending to infinity, such that for all $R \in \mathbb{Z}_{>0}$, and for all fields $a \in C^\infty(M)$,

$$\lim_{L \to 0} \frac{I_{i,k}[L](a) - \sum_{r=0}^{R} g_r(L)\Phi_r(a)}{L^{d_R}} = 0$$

in the TVS $\mathscr{O}_{loc}(C^\infty(M))$.

2.2.4 Removing singularities

One may ask why did we define a theory which involves singularities. The answer is that there is no canonical way of removing the singularities and therefore we regard their removal as an extra choice not part of the theory itself. All measurable quantities are *independent* of this choice, so it is merely an artefact to deal with infinite integrals.

The idea behind removing singularities is that given the space of functions $C^\infty((0,1)_\varepsilon)$, where the subscript refers to the name of the variable on which the functions take values, consider the subspace of functions which do have a limit when $\varepsilon \to 0$. We want to choose a complementary subspace of functions in $C^\infty((0,1)_\varepsilon)$, called the renormalization scheme **RS**, consisting of functions which don't have an $\varepsilon \to 0$ limit and consider them as our *purely singular* functions. We could then decompose the results of our calculations into the component which has a nice $\varepsilon \to 0$ limit and the purely singular component. Removing this purely singular component would yield a finite, measurable answer. With this idea we see that there is no canonical answer, each choice of complementary subspace will be equally valid.

Technical details show that the space of functions $C^\infty((0,1))$ is not suitable. Costello works instead in the linear subspace $\mathcal{P}((0,1)) \subset C^\infty((0,1))$ of periods.

The definition of period is rather technical, and we refer to [Cos11] Definition 2.9.1 for further details. Naively, a period is an integral of an algebraic differential form over an algebraic variety, and a period function $f \in \mathcal{P}((0,1))$ is just a smooth function $f \in C^\infty((0,1))$ whose values are periods. The restriction to the space of period functions is justified by Theorem 2.2.2, which shows that all Feynman weights have an asymptotic expansion in terms of periods.

Definition 2.2.7 (Renormalization Scheme)**.** Let $\mathcal{P}((0,1)_\varepsilon) \subset C^\infty((0,1)_\varepsilon)$ be the subalgebra of functions which are periods. A renormalization scheme is a complementary subspace $\mathcal{P}((0,1)_\varepsilon)_{<0}$, such that we have a direct sum decomposition: $\mathcal{P}((0,1)_\varepsilon) = \mathcal{P}((0,1)_\varepsilon)_{<0} \oplus \mathcal{P}((0,1)_\varepsilon)_{\geq 0}$, where $\mathcal{P}((0,1)_\varepsilon)_{\geq 0}$ denotes the subspace of periods which wave a $\varepsilon \to 0$ limit.

The need for periods and the ability to remove the singular part of functionals is encapsulated in the following theorem:

Theorem 2.2.2. *Let $I \in \mathscr{O}_{loc}(C^\infty(M))[[\hbar]]$ be a local functional and let γ be a connected stable graph.*

1. *There is a small ε asymptotic expansion*

$$\omega_\gamma(P(\varepsilon, L), I) \simeq \sum_{i=0}^{\infty} g_i(\varepsilon)\Psi_i$$

where $g_i \in \mathcal{P}((0,1)_\varepsilon)$ are periods, and $\Psi_i \in \mathscr{O}(C^\infty(M), C^\infty(0,\infty)_L)$[5]. Again, the small ε asymptotic expansions means that there is a non-decreasing sequence $d_R \in \mathbb{Z}$ indexed by $R \in \mathbb{Z}_{>0}$, such that $d_R \to \infty$ as $R \to \infty$, and such that for all R,

$$\lim_{\varepsilon \to 0} \frac{\omega_\gamma(P(\varepsilon, L), I) - \sum_{i=0}^{R} g_i(\varepsilon)\Psi_i}{\varepsilon^{d_R}} = 0$$

where the limit is taken in the TVS $\mathscr{O}(C^\infty(M), C^\infty(0,\infty)_L)$.

[5] As usual, a comma in this context means that the functions Ψ_i take values on $C^\infty(0,\infty)_L$

2. *The $g_i(\varepsilon)$ have a finite order pole at zero: for each i there is an integer k such that* $\lim_{\varepsilon\to 0} \varepsilon^k g_i(\varepsilon) = 0$

3. *Each Ψ_i appearing in the asymptotic expansion above has a small L asymptotic expansion of the form:*
$$\Psi_i \simeq \sum_{j=0}^{\infty} f_{i,j}(L)\Phi_{i,j}$$
where the $\Phi_{i,j}$ are local action functionals, i.e. elements of $\mathscr{O}_{loc}(C^\infty(M))$, and each function $f_{i,j}$ is a smooth function of L.

The proof of this theorem is the topic of the [Cos11, Appendix 1].

Definition 2.2.8 (Singular part)**.** Let $f \in \mathcal{P}((0,1))$, we define the singular part $\mathrm{Sing}(f)$ of f as the projection of f onto $\mathcal{P}((0,1))_{<0}$.

With these definitions, we will be able to extract the singular part of the Feynman graphs. Take $\omega_\gamma(P(\varepsilon, L), I)$. Theorem 2.2.2 showed that it has a small ε asymptotic expansion: $\omega_\gamma(P(\varepsilon,L),I) = \sum_{i=0}^{\infty} g_i(\varepsilon)\Phi_i$, where $g_i(\varepsilon) \in \mathcal{P}((0,1))$ and $\Phi_i \in \mathscr{O}(C^\infty(M), C^\infty(0,\infty)_L)$. It also showed that there exists $N \in \mathbb{Z}_{\geq 0}$ such that $g_n(\varepsilon)$ has a $\varepsilon \to 0$ limit when $n > N$. We therefore define the singular part of $\omega_\gamma((P(\varepsilon, L), I)$ as
$$\mathrm{Sing}_\varepsilon \omega_\gamma((P(\varepsilon,L),I) = \mathrm{Sing}_\varepsilon \Psi_N(\varepsilon) = \sum_{i=0}^{N} \left(\mathrm{Sing}_\varepsilon g_i(\varepsilon)\right)\Phi_i,$$
where $\Psi_N(\varepsilon) := \sum_{i=0}^{N} g_i(\varepsilon)\Phi_i$. Clearly, this definition is independent of N since increasing N would only add terms that have a $\varepsilon \to 0$ limit, so they wouldn't contribute.

The following theorem shows that the singular part has a nice expansion and the removal of the singular part makes the limit $\varepsilon \to 0$ exist. More explicitly

Lemma 2.2.3. *Let $I \in \mathscr{O}_{loc}(C^\infty(M))[[\hbar]]$ be a local functional, and let γ be a connected stable graph, then:*

1. $\mathrm{Sing}_\varepsilon \omega_\gamma(P(\varepsilon, L), I)$ *is a finite sum of the form*
$$\mathrm{Sing}_\varepsilon \omega_\gamma(P(\varepsilon,L),I) = \sum_i f_i(\varepsilon)\Phi_i$$
where $\Phi_i \in \mathscr{O}_{loc}(C^\infty(M), C^\infty(0,\infty)_L)$, and $f_i \in \mathcal{P}((0,1))_{<0}$ are purely singular periods.

2. *The limit*
$$\lim_{\varepsilon\to 0} \left(\omega_\gamma(P(\varepsilon,L),I) - \mathrm{Sing}_\varepsilon \omega_\gamma(P(\varepsilon,L),I)\right)$$
exists in the TVS $\mathscr{O}_{loc}(C^\infty(M), C^\infty(0,\infty)_L)$.

3. *Each Φ_i appearing in the finite sum above has a small L asymptotic expansion.*

To recap, we have shown that each term of our Feynman expansions will have an asymptotic expansion in terms of periods and that we have a well-defined notion of singular part which we can remove from the weights to obtain well-defined quantities which have $\varepsilon \to 0$ limits. This procedure involves calculating the singular part for each weight individually. Instead of this, we will show that there is a clever way of constructing *local counterterms* so that the removal of the singular part of the Feynman weights will be done by adding "corrections" to the interaction I.

Theorem 2.2.4 (Existence of local counter-terms)**.** *Given $I \in \mathscr{O}_{loc}(C^\infty(M))[[\hbar]]$, there exists a unique series of local counter-terms $I^{CT}_{i,k}(\varepsilon) \in \mathscr{O}_{loc}(C^\infty(M)) \otimes_{alg} \mathcal{P}((0,1))_{<0}$, for all $i > 0$, $k \geq 0$ such that $I^{CT}_{i,k}$ is homogeneous of degree k in the variable $a \in C^\infty(M)$ and, for all $L \in (0,\infty]$, the limit*
$$\lim_{\varepsilon\to 0} W\left(P(\varepsilon,L), I - \sum_{i,k} \hbar^i I^{CT}_{i,k}\right)$$
exists.

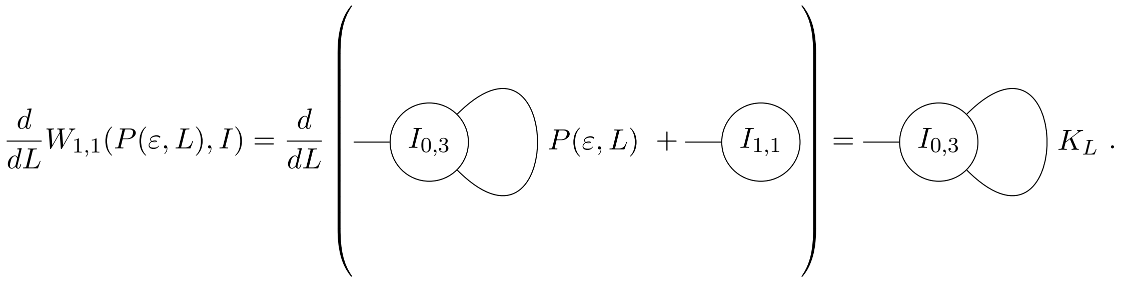
The first counter-term one needs to construct is from the graph with one with 1 loop and 1 external edge. Let us define $I_{1,1}^{CT} := \mathrm{Sing}_\varepsilon(W_{1,1}(P(\varepsilon,L),I)$. Let's check that $W_{1,1}(P(\varepsilon,L),I-\hbar I_{1,1}^{CT})$ has a nice limit and that $I_{1,1}^{CT}$ is local.

Clearly,

$$W_{1,1}(P(\varepsilon,L),I-\hbar I_{1,1}^{CT}) = W_{1,1}(P(\varepsilon,L),I) - I_{1,1}^{CT}$$

since for $\hbar I_{1,1}^{CT}$ only the graph with genus 0 and 1 tail will contribute. The $\varepsilon \to 0$ limit exists by construction.

To see that is local, we will first see that it is independent of L. By definition, $I_{1,1}^{CT} := \mathrm{Sing}_\varepsilon(W_{1,1}(P(\varepsilon,L),I)$, and $\frac{d}{dL}W_{1,1}$ is non-singular,



It follows that $\frac{d}{dL}I_{1,1}^{CT} = 0$. Finally, since $W_{1,1}$ has a small L asymptotic expansion by Theorem 2.2.2, we conclude that $I_{1,1}^{CT}$ is local.

We will construct the rest of the counter-terms by induction. Introduce a lexicographic order in $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$, such that: $(i,k)\prec(j,l)$ if $i<j$ or if $i=j$ and $k<l$.

Let us also introduce the notation: $W_{\prec(i,k)}(P,I) = \sum_{(j,l)\prec(i,k)} \hbar^j W_{j,l}(P,I)$. In terms of stable graphs:$W_{\prec(i,k)}(P,I) = \sum\limits_{\gamma\in\Gamma_{\prec(i,k)}} \frac{\hbar^{g(\gamma)}}{|\mathrm{Aut}(\gamma)|}\omega_\gamma(P,I)$ where $\Gamma_{\prec(i,k)}$ denote the set of stable graphs with genus smaller that i, or with genus equal to i and fewer that k external edges.

Assume we have constructed counter-terms for all $(j,l)\prec(i,k)$, and that:

i) For all L, the limit

$$\lim_{\varepsilon\to 0} W_{\prec(i,k)}\left(P(\varepsilon,L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon)\right)$$

exists.

ii) The counterterms $I_{(j,l)}^{CT}$ are local for $(j,l)\prec(i,k)$.

Defining the new counterterm as:

$$I_{i,k}^{CT}(\varepsilon,L) = \mathrm{Sing}_\varepsilon W_{i,k}\left(P(\varepsilon,L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I_{(j,l)}^{CT}(\varepsilon)\right) \tag{2.2}$$

It is immediate that the limit,

$$\lim_{\varepsilon\to 0} W_{\prec(i,k)}\left(P(\varepsilon,L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon) - \hbar^i I_{i,k}^{CT}(\varepsilon,L)\right)$$

exists, since

$$\lim_{\varepsilon\to 0} W_{\prec(i,k)}\left(P(\varepsilon,L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I_{j,l}^{CT}(\varepsilon)\right) - I_{i,k}^{CT}(\varepsilon,L).$$

We will again show that is local by first showing that it is independent of L.

Let $L' > L$, therefore:

$$\begin{aligned}
I^{CT}_{i,k}(\varepsilon, L') &= \mathrm{Sing}_\varepsilon W_{i,k}\left(P(\varepsilon, L'), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right) \\
&= \mathrm{Sing}_\varepsilon W_{i,k}\left(P(L, L'), W\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right)\right) \\
&= \mathrm{Sing}_\varepsilon W_{i,k}\left(P(L, L'), W_{\prec(i,k)}\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right)\right. \\
&\left. + \hbar^i W_{i,k}\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right)\right) \\
&= \mathrm{Sing}_\varepsilon W_{i,k}\left(P(L, L'), W_{\prec(i,k)}\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right)\right) \\
&+ \mathrm{Sing}_\varepsilon W_{i,k}\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right)
\end{aligned}$$

And since we know the first term is non-singular, the equation reduces to:

$$\begin{aligned}
I^{CT}_{i,k}(\varepsilon, L') &= \mathrm{Sing}_\varepsilon W_{i,k}\left(P(\varepsilon, L), I - \sum_{(j,l)\prec(i,k)} \hbar^j I^{CT}_{(j,l)}(\varepsilon)\right) \\
&= I^{CT}_{i,k}(\varepsilon, L).
\end{aligned}$$

Using the same argument as before we can show that it is local.

We are ready to state the main theorem of this chapter. It regards the questions, How many scalar field theories are there? Does every action functional describe a theory in the sense of Definition 2.2.12. How are quantizations at different orders in $\hbar$ related?

To answer these questions precisely, we will need the concept of principal bundle.

Definition 2.2.9 (Principal $G-$bundle)**.** A principal $G-$bundle , where G denotes any Lie group, is a fibre bundle $\pi : P \to X$ together with a smooth right action $P \times G \to P$ such that G preserve the fibres of P and acts freely and transitively on them in a way that for each $x \in X$ and $y \in P_x$, the map $G \to P_x$, sending g to gy is a diffeomorphism.

Theorem 2.2.5. *Let $\mathscr{T}^{(n)}$ denote the set of perturbative scalar field theories defined modulo $\hbar^{n+1}$. Then $\mathscr{T}^{(n+1)}$ is, in a canonical way, a principal bundle over $\mathscr{T}^{(n)}$ for the abelian group $\mathscr{O}_{loc}(C^\infty(M))$. Furthermore, $\mathscr{T}^{(0)}$ is canonically isomorphic to the space $\mathscr{O}^+_{loc}(C^\infty(M))$ of local action functionals which are at least cubic.*

There is a less natural way of stating this theorem, which is:

Theorem 2.2.6. *A choice of renormalization scheme leads to a section of each principal bundle $\mathscr{T}^{(n+1)} \longrightarrow \mathscr{T}^{(n)}$, and thus, an isomorphism between the space of theories defined modulo $\hbar^{n+1}$ and $\mathscr{O}^+_{loc}(C^\infty(M))[[\hbar]]/\hbar^{n+1}$, and an isomorphism between $\mathscr{T}^{(\infty)}$ and $\mathscr{O}^+_{loc}(C^\infty(M))[[\hbar]]$*

Even though the first theorem is more general and implies the second one, to prove them we will first prove the second one and infer the first one by analysing the dependence on the renormalization scheme.

Given a local action functional $I \in \mathscr{O}_{loc}(C^\infty(M))[[\hbar]]$ and a choice of renormalization scheme, we have shown how to construct local counter-terms, which allows us to define a scale L effective interaction:

$$I[L] := W^R(P(0, L), I) = \lim_{\varepsilon \to 0} W(P(\varepsilon, L), I - I^{CT}(\varepsilon))$$

The collection of these effective interactions satisfies all the axioms of Definition (2.2.6). Now we need to show that for each theory $\{I[L]\}$, there is a corresponding local action functional. We will prove it again, by induction. Let us assume that we have constructed local action functionals $I_{r,s}$ for $(r,s) \prec (i,k)$, such that

$$W^R_{a,b}\left(P(0,L), \sum_{(r,s)\prec(i,k)} \hbar^r I_{r,s}\right) = I_{a,b}[L]$$

for all $(a,b) \prec (i,k)$. The infinitesimal version of the renormalization group equation (see [Cos11] Fig.6) implies that

$$W^R_{\mathrm{i},k}\left(P(0,L), \sum_{(r,s)\prec(i,k)} \hbar^r I_{r,s}\right) - I_{i,k}[L]$$

is independent of L. The locality axiom for the theory $\{I[L]\}$ implies that this quantity is local. Thus, defining

$$I_{i,k} = I_{i,k}[L] - W^R_{\mathrm{i},k}\left(P(0,L), \sum_{(r,s)\prec(i,k)} \hbar^r I_{r,s}\right)$$

we arrive at

$$W^R_{a,b}\left(P(0,L), \sum_{(r,s)\preceq(i,k)} \hbar^r I_{r,s}\right) = I_{a,b}[L]$$

for all $(a.b) \preceq (i.k)$.

Thus far, we have shown that given a renormalization scheme, there is a bijection between $\mathscr{T}^{(\infty)}$ and functionals $I \in \mathscr{O}^+(C^\infty(M))[[\hbar]]$ and between $\mathscr{T}^{(n)}$ and functionals $I \in \mathscr{O}^+(C^\infty(M))[[\hbar]]/\hbar^{n+1}$. To show that $\mathscr{T}^{(n+1)} \to \mathscr{T}^{(n)}$ forms a principal bundle, note that the bijection constructed makes the map $\mathscr{T}^{(n+1)} \to \mathscr{T}^{(n)}$ surjective. Suppose that $\{I[L]\}$ and $\{J[L]\}$ are two theories defined modulo $\hbar^{n+2}$ and which agree modulo $\hbar^{n+1}$. Denote by $I_0[L] \in \mathscr{T}^{(0)}$ their classical theory. The tangent space of $\mathscr{T}^{(0)}$ at $I_0[L]$ consists of all infinitesimal deformations of the classical theory, which do not have to be at least cubic. More precisely, $T_{I_0[L]}\mathscr{T}^{(0)}$ is the set of $H[L] \in \mathscr{O}(C^\infty(M))$ such that $H[L]$ has a small L asymptotic expansion in terms of local action functionals and $I_0[L] + \delta H[L]$ satisfies the classical renormalization group equation modulo δ^2. We have a canonical isomorphisms of vector spaces $T_{I_0[L]} \cong \mathscr{O}_{loc}(C^\infty(M))$. Now note that

$$I_0[L] + \frac{1}{\hbar^{n+1}}\delta\left(I[L] - J[L]\right) \in \mathscr{O}(C^\infty(M))$$

satisfies the classical renormalization group equation modulo δ^2, and therefore, defines an element of $T_{I_0[L]} \cong \mathscr{O}_{loc}(C^\infty(M))$.

2.2.5 Vector-bundle valued Quantum Field Theories

We will see that the main theorems of the previous section, which showed that there is a bijection between local action functionals and theories hold for a more general class of theories. In this section, we will define theories whose fields are sections of some vector bundle over a manifold. Additionally, we will make them depend smoothly on an auxiliary supermanifold.

This additional data will be useful to include theories parametrised by $n-$forms over simplices, as we will need in Chapter 4. On a first read, this additional dependence can be omitted, as it doesn't give any insight on the problem of defining theories for vector bundles.

First, we need a preliminary definition:

Definition 2.2.10 (Sheaf)**.** A sheaf $\mathcal{F}$ over a topological space M is an assignment to each open set $U \subset M$ of a group $\mathcal{F}(U)$, known as the sections of $\mathcal{F}$ over U, which possesses the following two properties:

i) Given two such open sets U and V, with $U \subset V$ there exist restriction maps $r^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ which satisfy: $r^U_U = \mathrm{id}_U$ and if $U \subset V \subset W$, then $r^W_U = r^W_V \circ r^V_U$

ii) Let U be expressed as a union of open sets according to $U = \cup_i U_i$ then, given two sections $s_1, s_2 \in \mathcal{F}(U)$, $r^U_{U_i}(s_1) = r^U_{U_i}(s_2), \forall i$ implies $s_1 = s_2$. If $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j) \forall i, j$ then there is a unique $s \in \mathcal{F}(U)$ such that $r^U_{U_i} = s_i$.

Definition 2.2.11 (Nilpotent graded manifold)**.** A nilpotent graded manifold consists of:

i) A smooth manifold (possibly with) corners X, which means that X is locally modelled by $[0, \infty)^k \times \mathbb{R}^{n-k}$ instead of $\mathbb{R}^n$.

ii) A sheaf A of commutative super algebras over the sheaf of algebras C^∞_X,

such that they satisfy:

i) A is locally free of finite rank as a C^∞_X-module. In other words, A is the sheaf of sections of some super vector bundle on X.

ii) A is equipped with an ideal I such that $A/I = C^\infty_X$, and $I^k = 0$ for some $k > 0$. The ideal I, its powers I^l, and the quotient sheaves A/I^l, are all required to be locally free sheaves of C^∞_X-modules.

We will denote the algebra of C^∞ global sections of A, $\Gamma(X, A)$ by $\mathscr{A}$, and $\Gamma(X, I)$ by $\mathscr{I}$. Everything will depend on this auxiliary data $\mathscr{A}$.

Definition 2.2.12 (Free theory)**.** A free theory on a manifold M consists of:

i) A super vector bundle E (over the field $\mathbb{R}$ or $\mathbb{C}$) M, equipped with a direct sum decomposition $E = E_1 \oplus E_2$ into the spaces of propagating and non-propagating fields, respectively. Denote the space of smooth global sections of E or E_i by $\mathcal{E}$ and $\mathcal{E}_i$ respectively. Define $\mathcal{E}_1^! = \Gamma(M, E_1^\vee \otimes \mathrm{Dens}(M))$. Note that $\mathcal{E}_1^! \subset \mathcal{E}_1^\vee$.

ii) An even, $\mathscr{A}-$linear, order two differential operator

$$D_{\mathcal{E}_1} : \mathcal{E}_1 \otimes \mathscr{A} \to \mathcal{E}_1 \otimes \mathscr{A}$$

(where we are using the completed projective tensor product). $D_{\mathcal{E}_1}$ must be a generalized Laplacian which means that the symbol

$$\sigma(D_{\mathcal{E}_1}) \in \Gamma(T^*M, \mathrm{Hom}(E, E)) \otimes \mathscr{A}$$

must be the identity on E times a smooth family of Riemannian metrics

$$g \in C^\infty(T^*M) \otimes C^\infty(X).$$

Recall that $C^\infty(X) \subset \mathscr{A}$ is a subalgebra, as $\mathscr{A}$ is the global sections of a bundle of algebras on X.

iii) A differential operator

$$D' : \mathcal{E}_1^! \to \mathcal{E}_1,$$

which is symmetric. That is, equal to its formal adjoint:

$$D'^* : \mathcal{E}_1^! \to \mathcal{E}_1.$$

iv) $D' D^*_{\mathcal{E}_1} = D_{\mathcal{E}_1} D'$, where $D^*_{\mathcal{E}_1} : \mathcal{E}_1^! \to \mathcal{E}_1^!$ is the formal adjoint of $D_{\mathcal{E}_1}$.

We will usually abuse notation and denote the entirety of the data of a free theory on M as $\mathcal{E}$.

The use of graded vector bundles is essential to the Batalin-Vilkovisky formalism, and the introduction of the space of non-propagating fields $\mathcal{E}_2$ is introduced with an eye to applications on quantum gravity. It won't be used in the applications of this work.

A simple example is again the free scalar field theory. In this more general context, $\mathcal{E}_1 = C^\infty(M)$, $\mathcal{E}_2 = 0$. The operator $D_{\mathcal{E}_1}$ is the usual positive-definite Laplacian, whereas D' is the identity. The Riemannian volume element allows us to identify $\mathcal{E}^!$ with $\mathcal{E}$.

In order to introduce interactions, we need to make sense of the concept of effective interaction in this context. In the case that M is a compact manifold (or $\mathbb{R}^n$), there is a unique heat kernel $K_t \in \mathcal{E}_1^! \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathscr{A}$ for the operator $D_{\mathcal{E}_1}$. Composing it with D' gives an element

$$D' K_t \in \mathcal{E}_1 \otimes \mathcal{E}_1 \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathscr{A}$$

Which we see as an element of $\mathcal{E} \otimes \mathcal{E} \otimes C^\infty(\mathbb{R}_{>0}) \otimes \mathscr{A}$.

The propagator is defined as:

$$P(\varepsilon, L) = \int_\varepsilon^L D' K_t \in \mathcal{E} \otimes \mathcal{E} \otimes \mathscr{A}$$

We won't impose any positivity condition on $D_{\mathcal{E}_1}$ so K_∞ may not be defined. However, in most examples, it will be defined.

Define again the algebra of functionals on $\mathcal{E}$, with values in $\mathscr{A}$:

$$\mathscr{O}(\mathcal{E}, \mathscr{A}) = \prod_n \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathscr{A})_{S_n}$$

Local action functionals in this context will be defined as:

Definition 2.2.13 (Local action functional)**.** A functional $\Phi \in \mathscr{O}(\mathcal{E}, \mathscr{A})$ is said to be a local action functional if, when we expand Φ as a sum $\Phi = \sum \Phi_n$ of its homogeneous components, each $\Phi_n : \mathcal{E}^{\otimes n} \to \mathscr{A}$ can be written as a finite sum of the form:

$$\Phi_n(e_1, \dots, e_n) = \sum_j \int_M (D_{1,j} e_1) \dots (D_{n,j} e_n) d\mu$$

where $d\mu \in \mathrm{Dens}(M)$ and each $D_{i,j}$ is an $\mathscr{A}$-linear differential operator.

We want our interactions to be elements of $\mathscr{O}_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$, and if we want our interactions to have linear and quadratic terms, we will impose that they are accompanied by elements of our nilpotent ideal $\mathscr{I} \subset \mathscr{A}$.

We can now define the subset $\mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$ of local action functionals which are at least cubic modulo the ideal of $\mathscr{A}[[\hbar]]$ generated by $\hbar$ and $\mathscr{I}$.

The renormalization group operator:

$$W(P(\varepsilon, L), I) := \hbar \log\left(\exp(\hbar \partial_{P_{(\varepsilon,L)}}) \exp(I/\hbar)\right) : \mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A}) \longrightarrow \mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})$$

is well defined, but as we allowed for linear and quadratic terms in our interaction, we may now encounter univalent and bivalent genus 0 vertices. This will not result in an infinite sum as they are accompanied by the nilpotent ideal and so there will be only finitely many of such terms.

We are prepared to give the definition of a (interacting) theory:

Definition 2.2.14. Given a free theory $\mathcal{E}$, a (interacting) theory for this free theory is given by a collection of even elements

$$I[L] \in \mathscr{O}^+(\mathcal{E}, C^\infty(0, \infty)_L \otimes \mathscr{A})[[\hbar]]$$

such that:

1. They satisfy the renormalization group equation:

$$I[L'] = W(P(L, L'), I[L])$$

2. Each $I_{(i,k)}[L]$ has a small L asymptotic expansion:

$$I_{(i,k)}[L](e) \simeq \sum \Psi_r(e) f_r(L)$$

where $\Psi_r \in \mathscr{O}_{loc}(\mathcal{E})$ are local action functionals and $f_r(L) \in C^\infty(0,\infty)_L$.

Let $\mathscr{T}^{(\infty)}(\mathcal{E})$ denote the space of such theories, and $\mathscr{T}^{(n)}(\mathcal{E})$ denote the space of theories defined modulo $\hbar^{n+1}$, such that $\mathscr{T}^{(\infty)} = \varprojlim \mathscr{T}^{(n)}$

The main theorems in this context are:

Theorem 2.2.7. *The space $\mathscr{T}^{(n+1)}(\mathcal{E})$ has the structure of an $\mathscr{O}_{loc}(\mathcal{E}, \mathscr{A})$-principal bundle over $\mathscr{T}^{(n)}(\mathcal{E})$, in a canonical way. Further, $\mathscr{T}^{(0)}(\mathcal{E})$ is canonically isomorphic to the space $\mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})$ of $\mathscr{A}-$valued local action functionals on $\mathcal{E}$ which are at least cubic modulo the ideal $\mathscr{I} \subset \mathscr{A}$. The choice of renormalization scheme leads to a bijection between $\mathscr{T}^{(\infty)}(\mathcal{E})$ and the space $\mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$ of local action functionals with values in $\mathscr{A}$, which are at least cubic modulo the ideal in $\mathscr{A}[[\hbar]]$ generated by $\mathscr{I} \subset \mathscr{A}$ and $\hbar$.*

The proof is essentially the same as for the scalar field theory and won't be presented here. We refer to the reader to [Cos11, Chapter 2.13].

There is a further generalisation to theories defined on non-compact manifolds, where a unique heat kernel is not guaranteed to exist. It involves choosing a fake heat kernel. We won't go into detail since our goal is to define Yang-Mills theory in $\mathbb{R}^4$ and in $\mathbb{R}^n$ things are much simpler, as we will see.

Chapter 3

Theories on $\mathbb{R}^n$

Theories on $\mathbb{R}^n$ present extra difficulties with respect to the compact manifold case. Not only we have to deal with the previous $\varepsilon \to 0$ singularities, which we will call *ultraviolet* singularities, but also with *infrared* singularities, coming from the $L \to \infty$ limit. Luckily, theories on $\mathbb{R}^n$ are also better behaved than general theories on non-compact manifolds since we can use the ordinary heat kernel on $\mathbb{R}^n$ as long as we consider theories $I[L]$ with $L < \infty$.

These infrared divergences make the renormalization group operator only well defined for functionals I which are well-behaved, that is, the components $I_{i,k}$ are tempered distributions on $\mathbb{R}^{nk}$ of rapid decay away from the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$.

Again, every definition will depend on an auxiliary manifold X, a vector bundle A on X whose section we will denote $\mathscr{A}$, as in Definition 2.2.12.

The first definition we will need is that of Schwartz space and Schwartz function. These functions will be the ones whose derivatives are of rapid decay.

Definition 3.0.1 (Schwartz space)**.** The Schwartz space is the function space

$$\mathscr{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) | \forall \alpha, \beta \in \mathbb{N}^n, ||f||_{\alpha,\beta} < \infty\},$$

where $C^\infty(\mathbb{R}^n, \mathbb{C})$ denote the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and $||f||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta f)(x)|$, where we have used multi-index notation: $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $D^\beta := \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$

Denote by $\mathcal{D}(\mathbb{R}^n, \mathscr{A})$ the space of continuous linear maps $\mathscr{S}(\mathbb{R}^n) \to \mathscr{A}$, that is, the space of $\mathscr{A}-$valued tempered distributions on $\mathbb{R}^n$.

There is an $\mathscr{A}-$bilinear direct product map

$$\begin{aligned} \mathcal{D}(\mathbb{R}^n, \mathscr{A}) \boxtimes \mathcal{D}(\mathbb{R}^k, \mathscr{A}) &\longrightarrow \mathcal{D}(\mathbb{R}^{n+k}, \mathscr{A}) \\ (\Psi, \Phi) &\longmapsto \Psi \boxtimes \Phi. \end{aligned}$$

The direct product $\Psi \boxtimes \Phi$ is uniquely determined by the property that for all Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$, $g \in \mathscr{S}(\mathbb{R}^k)$ and $(\Psi \boxtimes \Phi)(f \boxtimes g) = \Psi(f)\Phi(g)$, where the product on the right is taken in the algebra $\mathscr{A}$ and $f \boxtimes g \in \mathscr{S}(\mathbb{R}^{n+k})$ is the usual exterior product of functions.

We will be interested in distributions of rapid decay on $\mathbb{R}^n$, more concretely we will impose that they decay as fast as $e^{-b||x||^2}$ for some $b > 0$. That will mean that they are continuous linear maps on spaces of functions whose growth is bounded by $e^{b||x||^2}$.

Definition 3.0.2. Let V, W be finite-dimensional vector spaces over $\mathbb{R}$. For all $a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{R}_{>0}$ and $l \in \mathbb{Z}_{\geq 0}$, define the norm $|| - ||_{a,b,l}$ on $\mathscr{S}(V \oplus W)$ by the formula

$$||f||_{a,b,l} = \sum_{|I| \leq l} \sup_{(v,w) \in V \oplus W} \left| (1 + ||v||^2)^a e^{-b||w||^2} \partial_I f \right|.$$

This formula may be extended to a map $C^\infty(V \oplus W) \to [0, \infty]$. Let $\mathscr{T}(V, W) \subset C^\infty(V \oplus W)$ be the subspace of those functions such that, for all a, b and l, $||f||_{a,b,l} < \infty$. Finally, give $\mathscr{T}(V, W)$ the topology induced by the seminorms $||f||_{a,b,l}$

We say that a continuous linear map $\Phi : \mathscr{S}(V \oplus W) \to \mathscr{A}$ is of rapid decay along W if it extends to a continuous linear map $\Phi : \mathscr{T}(V, W) \to \mathscr{A}$.

Let us also give $\mathscr{A}$ the topology induced by the seminorms $|| - ||_{K,D}$, where $K \subset X$ is a compact subset and $D : \mathscr{A} \to \mathscr{A}$ is a differential operator, given by taking the supremum over K of Da.

From these definitions, we obtain the following properties:

1. $\mathscr{S}(V \oplus W) \subset \mathscr{T}(V, W)$ is dense.

2. A continuous linear map $\Phi : \mathscr{S}(V \oplus W) \to \mathscr{A}$ extends to a continuous linear map $\mathscr{T}(V, W) \to \mathscr{A}$ if and only if, for all compact subsets $K \subset X$ and all differential operators $D : \mathscr{A} \to \mathscr{A}$, there exists some a, b, l and C such that:
$$||\Phi(f)||_{K,D} \leq C||f||_{a,b,l}$$

3. Let $\Phi : \mathscr{T}(V, W) \to \mathscr{A}$ and $\Psi : \mathscr{T}(V', W') \to \mathscr{A}$ be continuous linear maps. The direct product $\Psi \boxtimes \Phi : \mathscr{S}(V \oplus V' \oplus W \oplus W') \to \mathscr{A}$ extends to a continuous linear map $\mathscr{T}(V \oplus V', W \oplus W') \to \mathscr{A}$.

We can finally define our *good distributions*

Definition 3.0.3 (Good distribution)**.** A tempered distribution $\Phi : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{A}$ is good if it is translational invariant and, it satisfies the following property:

1. If we decompose $\mathbb{R}^{nk}$ as an orthogonal direct sum $\mathbb{R}^n \oplus \mathbb{R}^{n(k-1)}$, where $\mathbb{R}^n$ is the small diagonal, then Φ is of rapid decay along $\mathbb{R}^{n(k-1)}$. That is, Φ extends to a continuous linear map
$$\Phi : \mathscr{T}(\mathbb{R}^n, \mathbb{R}^{n(k-1)}) \to \mathscr{A}$$

Denote the space of good distributions as $\mathcal{D}_g(\mathbb{R}^{nk})$.

The importance of these definitions is that Feynman graphs still make sense when contracting good distributions and the result is also a good distribution.

Let γ be a connected graph. Let $H(\gamma), T(\gamma), V(\gamma)$ and $E(\gamma)$ denote the set of half-edges, tails vertices and internal edges, respectively.

For every $v \in V(\gamma)$, denote $H(v)$ the set of half-edges adjoining v and for every $e \in E(\gamma)$ denote $H(e)$ the pair of half-edges forming e. Suppose we have the following data:

1. For every $v \in V(\gamma)$, we have a good distribution
$$I_v \in \mathcal{D}_g(\mathbb{R}^{nH(v)}, \mathscr{A}).$$

2. For each edge $e \in E(\gamma)$, we have a function
$$P_e \in C^\infty(\mathbb{R}^{nH(e)}).$$
Let h_1, h_2 denote the two half edges of e, and let $x_{h_i} : \mathbb{R}^{nH(e)} \longrightarrow \mathbb{R}^n$ be the corresponding linear maps. Let us assume that P_e is invariant under the $\mathbb{R}^n$ action on $\mathbb{R}^{nH(e)}$; this amount to saying that P_e is independent of $x_{h_1} + x_{h_2}$. Let us further assume that for any multi-index I, there exists b such that $|\partial_I P_e| \leq e^{-b||x_{h_1} - x_{h_2}||^2}$.

Therefore, we can attempt to define the weights $\omega_\gamma(I_v, P_e)$ on $\mathbb{R}^{nT(\gamma)}$ by contracting the distributions I_v with the functions P_e. The following theorem states that this procedure works.

Theorem 3.0.1. *The distribution $\omega_\gamma(I_v, P_e)$ is well-defined, and it is a good distribution on $\mathbb{R}^{nT(\gamma)}$.*

We would like to give a definition of scalar field theory on $\mathbb{R}^n$ in the same lines as before and state the main theorem on the classification of theories in this context. We will define the space
$$\mathscr{O}(\mathscr{S}(\mathbb{R}^n)) = \prod_{k \geq 1} \mathcal{D}_g(\mathbb{R}^{nk})_{S_k}$$

as the space of formal power series on $\mathscr{S}(\mathbb{R}^n)$ whose Taylor components are good distributions. Note that it is not an algebra, and it doesn't have a constant term as we are considering only translational invariant theories.

A good distribution is *local* if it is supported on the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. Translational invariance implies that a local good distribution may be written as a finite sum:

$$f(x_1, \ldots, x_k) \mapsto \sum_I \int_{x \in \mathbb{R}^n} (\partial_I f)(x_1, \ldots x_k)$$

where $\partial_I : \mathscr{S}(\mathbb{R}^{nk}) \to \mathscr{S}(\mathbb{R}^{nk})$ are constant-coefficient differential operators corresponding to multi-indices $I \in (\mathbb{Z}_{\geq 0})^{nk}$.

Denote $\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))$ the subspace of functionals on $\mathscr{S}(\mathbb{R}^n)$ whose Taylor components are local elements of $\mathcal{D}_g(\mathbb{R}^n)$. Denote also by $\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \subset \mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$ the subspace of functionals which are at least cubic modulo $\hbar$.

We will again assume that our kinetic term is given by $-\frac{1}{2} \int \phi(D + m^2)\phi$, where D is the non-negative Laplacian and $m > 0$ is the mass.

The propagator of our theory will be [BGV92]

$$P(\varepsilon, L) = \int_\varepsilon^L e^{-tm^2} K_t dt = \int_\varepsilon^L t^{-n/2} e^{-tm^2} e^{-||x-y||^2/t} dt \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

The only differences between the definition of theory in $\mathbb{R}$‘n with respect to the compact manifold case are that we now require the effective interactions $I[L]$ to be good distributions on $\mathbb{R}^{nk}$ and that we can weaken the requirement that $I[L]$ has a small L asymptotic expansion. We only require that $I[L]$ tends to zero away from the diagonals in $\mathbb{R}^{nk}$, since with translational invariance this is enough to prove the bijection between theories and Lagrangians.

More specifically:

Definition 3.0.4. A scalar field theory on $\mathbb{R}^n$, with mass $m > 0$, is given by a collection of functionals $I[L] \in \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n), C^\infty(0, \infty)_L)[[\hbar]]$ where $L \in (0, \infty)$, such that:

1. The renormalization group equation

$$W(P(\varepsilon, L), I[\varepsilon]) = I[L]$$

holds.

2. The following locality axiom holds: Expanding $I[L]$ as $\sum \hbar^i I_{i,k}[L]$, where each $I_{i,k}[L]$ is a S_k invariant map. Regarding $I_{i,k}[L]$ as a distribution on $\mathbb{R}^{nk}$, let $C \subset \mathbb{R}^{nk}$ be a compact subset in the complement of the small diagonal. Then, for all functions $f \in \mathscr{S}(\mathbb{R}^{nk})$ with compact support on C,

$$\lim_{L \to 0} I_{i,k}[L](f) = 0.$$

Denote by $\mathscr{T}^{(\infty)}$ the set of theories and by $\mathscr{T}^{(n)}$ the set of theories defined modulo $\hbar^{n+1}$.

The main theorem in this context is as follows:

Theorem 3.0.2. *The space $\mathscr{T}^{(n+1)}$ is a principal bundle over $\mathscr{T}^{(n)}$ for the group $\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))$ in a canonical way, and $\mathscr{T}^{(0)}$ is canonically isomorphic to the subset of $\mathscr{O}_{loc}(\mathscr{S}(\mathbb{R}^n))$ of functionals which are at least cubic.*

The choice of renormalization scheme yields a section of each torsor $\mathscr{T}^{(n+1)} \longrightarrow \mathscr{T}^{(n)}$, and therefore a bijection between the set $\mathscr{T}^{(\infty)}$ of theories and the set $\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$ of $\hbar$-dependent translational invariant functionals on $\mathscr{S}(\mathbb{R}^n)$ which are at least cubic modulo $\hbar$.

3.1 Vector bundle theories

There is again a generalization to vector bundle theories on $\mathbb{R}^n$. Everything will depend again on an auxiliary manifold X, equipped with a sheaf A of commutative graded algebras over the sheaf of algebras C^∞_X. The space of global sections $\Gamma(X, A)$ will be denoted $\mathscr{A}$. We have the following data:

i) A finite-dimensional super vector space E. Let

$$\mathcal{E} = E \otimes \mathscr{S}(\mathbb{R}^n).$$

Thus, $\mathcal{E}$ is the space of Schwartz sections of the trivial vector bundle $E \times \mathbb{R}^n$

ii) A degree zero symmetric element $K_t \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \otimes E \otimes E \otimes C^\infty((0,\infty)_t) \otimes \mathscr{A}$, playing the role of the heat kernel. We assume that in some basis e_i of E, K_t can be written as $K_t = \sum P_{i,j}(x-y, t^{-1/2}) e^{-||x-y||^2/t} e_i \otimes e_j$, where $P_{i,j} \in \mathscr{A}\left[x-y, t^{\pm 1/2}\right]$, are polynomials in the variables $x_k - y_k$ and $t^{\pm 1/2}$ with coefficients in $\mathscr{A}$.

The relevant spaces of functions will be

$$\mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A}) \subset \mathcal{D}(\mathbb{R}^{nk}, \mathscr{A})$$

as defined before, that is, the space of $\mathscr{A}-$valued translational invariant distributions of rapid decay away from the diagonal. Let

$$\mathscr{O}(\mathcal{E}, \mathscr{A}) = \prod_{k>0} \left(\mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A}) \otimes (E^\vee)^{\otimes k}\right)_{S_k}$$

be the space of $\mathscr{A}-$valued functionals on $\mathcal{E}$, whose Taylor components are given by local distributions. Since each component is required to be translational invariant, every element of $\mathscr{O}(\mathcal{E}, \mathscr{A})$ will be translational invariant.

Again, we will call a good distribution $\Phi \in \mathcal{D}_g(\mathbb{R}^{nk}, \mathscr{A})$ *local*, if it is supported on the small diagonal $\mathbb{R}^n \subset \mathbb{R}^{nk}$. As such, it may be written as a finite sum

$$f(x_1, \dots x_k) \mapsto \sum_I \int_{x \in \mathbb{R}^n} a_I (\partial_I f)(x_1, \dots x_k)$$

where $a_I \in \mathscr{A}$ and $I \in (\mathbb{Z}_{\geq 0})^{nk}$ are multi-indices.

We will denote by $\mathscr{O}^+(\mathcal{E}, \mathscr{A})[[\hbar]] \subset \mathscr{O}(\mathcal{E}, \mathscr{A})[[\hbar]]$ the set of functionals which are at least cubic modulo the ideal in $\mathscr{A}[[\hbar]]$ generated by $\mathscr{I} \subset \mathscr{A}$ and $\hbar$. $\mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$ is defined in the same way.

Define the renormalization group flow as:

$$\begin{aligned} \mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]] &\longrightarrow \mathscr{O}^+_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]] \\ I &\longmapsto W(P(\varepsilon, L), I) \end{aligned}$$

Which is well defined due to Theorem 3.0.1.

Given the data of a free theory, a vector bundle theory on $\mathbb{R}^n$ is defined as:

Definition 3.1.1. A family of theories on $\mathcal{E}$, over $\mathscr{A}$, is a collection

$$\{I[L] \in \mathscr{O}^+(\mathcal{E}, \mathscr{A})[[\hbar]]\}$$

of translational invariant effective interactions such that

1. Each $I[L]$ is of degree 0.

2. The renormalization group equation

$$I[L] = W(P(\varepsilon, L), I[\varepsilon])$$

is satisfied.

3. The following locality axiom holds: Expanding $I[L] = \sum \hbar^i I_{i,k}[L]$, where each $I_{i,k}[L]$ is a S_k-invariant map $\mathcal{E}^{\otimes k} \to \mathscr{A}$. We require that for all elements $f \in \mathcal{E}^{\otimes k}$ which are compactly supported away from the small diagonal in $\mathbb{R}^{nk}$, and for all $x \in X$:

$$\lim_{L \to 0} I_{i,k}[L](f)_x = 0.$$

This limit is taken in the finite-dimensional graded vector space A_x, where the subscript x denotes the restriction of an element of $\mathscr{A}$ to its value at the fibre A_x of A above $x \in X$.

The main theorem in this context is:

Theorem 3.1.1. *The space $\mathscr{T}^{(m+1)} \longrightarrow \mathscr{T}^{(m)}$ is a torsor for the space $\mathscr{O}^0_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$ of local action functionals of degree* 0 *on $\mathcal{E}$. Furthermore, $\mathscr{T}^{(0)}$ is canonically isomorphic to the space of degree 0 local action functionals on $\mathcal{E}$ which are at least cubic.*

If we choose a renormalization scheme, we find a section of each torsor $\mathscr{T}^{(m+1)} \longrightarrow \mathscr{T}^{(m)}$. Thus a renormalization scheme yields a bijection

$$\mathscr{T}^{(\infty)}(\mathcal{E}, \mathscr{A}) \cong \mathscr{O}^{+,0}_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$$

between the set of theories and the set of translational-invariant, local action functionals on $\mathcal{E}$ which are of degree 0, which are at least cubic modulo the ideal in $\mathscr{A}[[\hbar]]$ generated by $\mathscr{I} \subset \mathscr{A}$ and $\hbar$.

3.2 Renormalizability

The main theorems above show that there is an infinite-dimensional space of theories in $\mathbb{R}^n$. A fair assumption is that one wants a theory to be predictive, and that means that it can only have a finite amount of free parameters so that one can do a finite amount of experiments and be able to choose a particular theory from the space of all possible theories. As such, one aims to select from the infinite-dimensional moduli space of theories, a finite-dimensional subspace of *well-behaved* theories.

An old-fashioned definition of well-behaved theory is one that has only finitely many counterterms. We won't be using this definition as it relies on the renormalization scheme, which we regard as unnatural. We will define a theory $\{I[L]\}$ to be well-behaved or *renormalizable* if, roughly, it doesn't grow too fast as $L \to 0$, measured in units appropriate to the length scale L and the space of deformations which also satisfy this growth condition is finite-dimensional. This definition is possible in $\mathbb{R}^n$ as we have a natural action of $\mathbb{R}_{>0}$ on the space of theories.

Note that this definition may be regarded as a weaker, perturbative approximation of an ideal non-perturbative definition of renormalizability. The non-perturbative definition or *Kadanoff-Wilson renormalizability*, says that a theory is renormalizable if it converges to a fixed point under the *local renormalizatino group flow* $\mathcal{RG}_\ell$ as $\ell \to 0$ and the unstable manifold of this fixed point is finite-dimensional. Our first condition excludes theories that clearly don't converge to a fixed point but includes theories that don't converge to a fixed point in a more subtle way.

3.2.1 Scalar field theories

Firstly, let's introduce the local renormalization group flow on the space of scalar field theories on $\mathbb{R}^n$, which combines the renormalization group flow already introduced, a rescaling on $\mathbb{R}^n$ and a rescaling of the fields ϕ.

First, define the operation

$$R_\ell : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$$
$$R_\ell(\phi)(x) = \ell^{n/2-1}\phi(\ell x)$$

for $x \in \mathbb{R}^n$. The exponent $\ell^{n/2-1}$ may seem arbitrary. It is chosen for convenience so that the action functional of the massless free field is preserved under this change of coordinates, i,e $R_\ell \int \phi D\phi d\mathrm{Vol}_M = \int \phi D\phi d\mathrm{Vol}_M$. This action may be thought of as a change in the units of measurement.

Let's see how this rescaling acts on the propagator $P(\varepsilon, L) \in \mathrm{Sym}^2\mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned} R_\ell P(\varepsilon, L) &= \sum_i \int_\varepsilon^L t^{-n/2} e^{-\|x-y\|^2/4t} dt R_\ell\phi_i(x) \otimes R_\ell\phi_i(y) \\ &= \sum_i \ell^{n-2} \int_\varepsilon^L t^{-n/2} e^{-\ell^2\|x-y\|^2/4t} dt \phi_i(x) \otimes \phi_i(y) \\ &= \sum_i \int_{\ell^{-2}\varepsilon}^{\ell^{-2}L} u^{-n/2} e^{-\|x-y\|^2/4t} du \phi_i(x) \otimes \phi_i(y) \\ &= P(\ell^{-2}\varepsilon, \ell^{-2}L). \end{aligned}$$

For some eigenbasis of the Laplacian.

For functionals $I \in \mathscr{O}(\mathscr{S}(\mathbb{R}^n))$, define $R_\ell^*(I)(\phi) = I(R_{\ell^{-1}}\phi)$, such that $R_\ell^*(I)(\phi(x)) = I(\ell^{1-n/2}\phi(\ell^{-1}x))$. Note that the pairing between functionals and fields is invariant, i.e:

$$(R_\ell^* I)(R_\ell \phi) = I(\phi).$$

We are ready to define the local renormalization group flow. The idea will come again from the energy picture, where we want to measure the effective action as $\Lambda \to \infty$ measuring with units appropriate to the scale. That is if we measure $S^{eff}[1]$ in joules, we should measure $S^{eff}[10^3]$ in kilo-joules etc. When translating to the length scale picture, since in natural units energy is equal to length^{-2} we define the local renormalization group flow as,

Definition 3.2.1 (Local renormalization group flow)**.** Given a collection of effective interactions defining a theory $\{I[L]\}$, define the local renormalization group flow on the space of theories,

$$\mathcal{RG}_\ell : \mathscr{T}^{(\infty)} \to \mathscr{T}^{(\infty)},$$

by: $\mathcal{RG}_\ell(\{I[L]\}) = \{\mathcal{RG}_\ell(I[L])\}$, where $\mathcal{RG}_\ell(I[L]) = R_\ell^*(I[\ell^2 L])$.

The collection of effective interaction $\mathcal{RG}_\ell(I[L]) \in \mathscr{O}(\mathscr{S}(\mathbb{R}^n)), C^\infty((0,\infty)_\ell \otimes C^\infty((0,\infty)_L))$ defines a smooth family of theories parametrized by ℓ.

To check that $\mathcal{RG}_\ell(I[L])$ defines a theory, note that the locality axiom is immediate. For the renormalization group equation:

$$\begin{aligned} I[\ell^2 L] &= \hbar \log\{\exp\left(\hbar\partial_{P(\ell^2\varepsilon, \ell^2 L)}\right) \exp\left(I[\ell^2\varepsilon]/\hbar\right)\} \\ R_\ell^* I[\ell^2 L] &= R_\ell^* \hbar \log\{\exp\left(\hbar\partial_{P(\ell^2\varepsilon, \ell^2 L)}\right) \exp\left(I[\ell^2\varepsilon]/\hbar\right)\} \\ \mathcal{RG}_\ell(I[L]) &= R_\ell^* \hbar \log\{\exp\left(\hbar\partial_{P(\varepsilon, L)}\right) \exp\left(\mathcal{RG}_\ell(I[\varepsilon])/\hbar\right)\} \\ &= W\left(P(\varepsilon, L), \mathcal{RG}_\ell(I[\varepsilon])\right) \end{aligned}$$

Proposition 3.2.1. *Given a translational invariant theory on $\mathbb{R}^n$, $\{I[L]\}$, then $\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \otimes \mathbb{C}[\log\ell, \ell, \ell^{-1}]$.*

This allow us to classify each term in $\mathcal{RG}(I[L])$ as:

i) Relevant if it varies as $\ell^k \log^r \ell$ for some $k \geq 0$ and $r \in \mathbb{Z}_{\geq 0}$.

ii) Irrelevant if it varies as $\ell^k \log^r \ell$ for some $k < 0$ and $r \in \mathbb{Z}_{\geq 0}$.

iii) Marginal if it varies as $\log^r \ell$ for some $r \in \mathbb{Z}_{\geq 0}$.

Definition 3.2.2. A theory $\{I[L]\}$ is relevant if, for each L, $\mathcal{RG}_\ell(I[L])$ consists entirely of relevant terms; or in other words, if

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \otimes \mathbb{C}[\log\ell, \ell].$$

A theory $\{I[L]\}$ is marginal if, for each L, $\mathcal{RG}_\ell(I[L])$ consists entirely of marginal terms; or in other words, if

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \otimes \mathbb{C}[\log \ell].$$

Let $\mathscr{R}^{(n)} \subset \mathscr{T}^{(n)}$ $\left(\text{respectively } \mathscr{M}^{(n)} \subset \mathscr{T}^{(n)}\right)$ denote the subset of $\mathscr{T}^{(n)}$ of theories consisting of relevant (resp. marginal) theories defined modulo $\hbar^{n+1}$. Let $\mathscr{R}^{(\infty)} = \lim_{\leftarrow} \mathscr{R}^{(n)}$ and $\mathscr{M}^{(\infty)} = \lim_{\leftarrow} \mathscr{M}^{(n)}$ be the spaces of relevant and marginal theories defined to all orders in $\hbar$

We are ready to define a renormalizable theory.

Definition 3.2.3. A theory on $\mathbb{R}^n$ is renormalizable if it is relevant and, term by term in $\hbar$, has only finitely any relevant deformations. That is, a theory $\{I[L]\} \in \mathscr{T}^{(\infty)}$ is renormalizable if $\{I[L]\} \in \mathscr{R}^{(\infty)}$ and for all finite n, $T_{I[L]}\mathscr{R}^{(n)}$ is finite-dimensional. A theory is strictly renormalizable if it is renormalizable and it is marginal. A theory is strongly renormalizable if it is strictly renormalizable and all its relevant deformations are marginal, i.e, if $\{I[L]\} \in \mathscr{M}^{(\infty)}$ and $T_{I[L]}\mathscr{R}^{(\infty)} = T_{I[L]}\mathscr{M}^{(\infty)}$.

We know that a choice of renormalization scheme leads to a bijection between theories and local action functionals:

$$\mathscr{T}^{(\infty)} \cong \mathscr{O}^+_{loc}\left(\mathscr{S}(\mathbb{R}^n)\right)[[\hbar]].$$

Thus, the renormalization group flow translates into an $\mathbb{R}_{>0}$ action on the space of local action functionals.

For $I \in \mathscr{O}^+_{loc}\left(\mathscr{S}(\mathbb{R}^n)\right)[[\hbar]]$, denote by $\mathcal{RG}_\ell(I)$ the family of local action functionals arising from the action of the local renormalization group flow. It follows from the fact that $\mathcal{RG}_\ell(\{I[L]\})$ is a smooth family of theories and Theorem 3.0.2, that $\mathcal{RG}_\ell(I)$ is a smooth family of action functionals. That is

$$\mathcal{RG}_\ell(I) \in \mathscr{O}^+_{loc}\left(\mathscr{S}(\mathbb{R}^n), C^\infty(0,\infty)_\ell\right)[[\hbar]].$$

A functional is said to be of dimension k if $R^*_\ell I = \ell^k I$. As such, denote by $\mathscr{O}^+_{loc,k}\left(\mathscr{S}(\mathbb{R}^n)\right)$ the space of local action functionals of dimension k, and $\mathscr{O}^+_{loc,\geq 0}\left(\mathscr{S}(\mathbb{R}^n)\right)$ the subspace of functionals of non-negative dimension. The main theorem in this context is as follows:

Theorem 3.2.1. *The space $\mathscr{R}^{(m+1)}$ is, in a canonical way, a torsor over $\mathscr{R}^{(m)}$ for the abelian group $\mathscr{O}^+_{loc,\geq 0}\left(\mathscr{S}(\mathbb{R}^n)\right)$. Also, $\mathscr{R}^{(0)}$ is canonically isomorphic to the subspace of $\mathscr{O}^+_{loc,\geq 0}\left(\mathscr{S}(\mathbb{R}^n)\right)$ of functionals which are at least cubic.*

The space $\mathscr{M}^{(m+1)}$ is, in a canonical way, a torsor over $\mathscr{M}^{(m)}$ for the abelian group $\mathscr{O}^+_{loc,0}\left(\mathscr{S}(\mathbb{R}^n)\right)$. Also, $\mathscr{M}^{(0)}$ is canonically isomorphic to the subspace of $\mathscr{O}^+_{loc,0}\left(\mathscr{S}(\mathbb{R}^n)\right)$ of functionals which are at least cubic.

The choice of renormalization scheme leads to sections of the torsors $\mathscr{R}^{(m+1)} \longrightarrow \mathscr{R}^{(m)}$ and $\mathscr{M}^{(m+1)} \longrightarrow \mathscr{M}^{(m)}$ and thus, to bijections:

$$\mathscr{R}^{(\infty)} \cong \mathscr{O}^+_{loc,\geq 0}(\mathscr{S}(\mathbb{R}^n))[[\hbar]],$$

$$\mathscr{M}^{(\infty)} \cong \mathscr{O}^+_{loc,0}(\mathscr{S}(\mathbb{R}^n))[[\hbar]].$$

The following corollary is a known result in the physics literature.

Corollary 3.2.1.1. *Let us choose a renormalization scheme. Then, we find a bijection between (translational invariant) strictly renormalizable scalar field theories on $\mathbb{R}^n$ and Lagrangians of the form:*

i) The ϕ^3 theory on $\mathbb{R}^6$.

ii) The ϕ^4 theory on $\mathbb{R}^4$.

iii) The ϕ^6 theory on $\mathbb{R}^3$.

iv) The free field theory on $\mathbb{R}^n$ when $n = 5$ or $n > 6$.

Proof. This is an immediate result after noticing that, for $I = \frac{1}{k!}\int_{\mathbb{R}^n} \phi^k$,

$$R_\ell^*(I) = \ell^{n+k(1-\frac{1}{2}n)} I.$$

□

Let's study the action of the local renormalization group flow on the space of local action functionals.

Start by fixing a renormalization scheme, $\mathbf{RS}_0 := \mathcal{P}((0,1))_{<0}$, which we will use to identify $\mathscr{T}^{(\infty)}$ with $\mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$ and define

$$\mathbf{RS}_\ell = \{f \in \mathcal{P}((0,1)) | f(\ell^{-2}\varepsilon) \in \mathbf{RS}_0\}.$$

There is a change of renormalization scheme map,

$$\Phi_{\ell,0} = \Phi_{\mathbf{RS}_\ell \to \mathbf{RS}_0} : \mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]] \longrightarrow \mathscr{O}^+_{loc}(\mathscr{S}(\mathbb{R}^n))[[\hbar]]$$

By definition, a theory associated to the action functional I and renormalization scheme $\mathbf{RS}_\ell$ is equivalent to a theory associated to the action functional $\Phi_{\ell,0}$ and renormalization scheme $\mathbf{RS}_0$.

Lemma 3.2.2. *The local renormalization group flow $\mathcal{RG}_\ell$ is the composition $\mathcal{RG}_\ell = \Phi_{\ell,0} \circ R_\ell^*$.*

Proof. The theory associated to a local action functional I is defined by a collection of effective interactions:

$$W^R(P(0,L), I) = \lim_{\varepsilon \to 0} W(P(\varepsilon, L), I - I^{CT}(\varepsilon)).$$

By definition of the local renormalization group flow,

$$\begin{aligned} W^R(P(0,L), \mathcal{RG}_\ell(I)) &= R_\ell^* W^R(P(0,\ell^2 L), I) \\ &= \lim_{\varepsilon \to 0} R_\ell^* W(P(\varepsilon, \ell^2 L), I - I^{CT}(\varepsilon)) \\ &= \lim_{\varepsilon \to 0} W(R_\ell P(\varepsilon, \ell^2 L), R_\ell^* I - R_\ell^* I^{CT}(\varepsilon)) \\ &= \lim_{\varepsilon \to 0} R_\ell^* W(P(\ell^{-2}\varepsilon, L), R_\ell^* I - R_\ell^* I^{CT}(\ell^2\varepsilon)) \end{aligned}$$

Where in the last step we reparametrized the dummy variable ε. Since $R_\ell^* I^{CT}(\ell^2\varepsilon)$ is purely singular for the renormalization scheme $\mathbf{RS}_\ell$ and the above limit exists, it follows that $R_\ell^* I^{CT}(\ell^2\varepsilon)$ is the counterterm for $R_\ell^* I$ with this renormalization scheme.

We have that the collection of effective interactions:

$$\lim_{\varepsilon \to 0} R_\ell^* W(P(\ell^{-2}\varepsilon, L), R_\ell^* I - R_\ell^* I^{CT}(\ell^2\varepsilon)),$$

define a theory associated to $R_\ell^* I$ and renormalization scheme $\mathbf{RS}_\ell$. By definition of change of renormalization scheme map, this is the same as the theory associated to $\Phi_{\ell,0} R_\ell^* I$ for the renormalization scheme $\mathbf{RS}_0$. Thus,

$$W^R(P(0,L), \mathcal{RG}_\ell(I)) = W^R(P(0,L), \Phi_{\ell,0} R_\ell^* I).$$

□

3.2.2 Vector bundle theories

Let's see how all of this generalize to the more general context of vector bundle theories.

Let us assume that we have the data if a free theory from section 3.1. Also, assume that we have a direct sum decomposition of E into super vector spaces: $E = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} E_i$, where E_i is the space of elements of E of dimension i.

The direct sum decomposition induces an $\mathbb{R}_{>0}$ action on $\mathcal{E} = \mathscr{S}(\mathbb{R}^n) \otimes E$, by

$$R_\ell(f(x)e_i) = f(\ell x)\ell^i e_i$$

where $f(x) \in \mathscr{S}(\mathbb{R}^n)$ and $e_i \in E_i$.

We require that the propagator scales as $R_\ell(P(\varepsilon, L)) = P(\ell^{-2}\varepsilon, \ell^{-2}L)$.

As seen in Theorem 3.1.1, a choice of renormalization scheme leads to a bijection

$$\mathscr{T}^{(\infty)} \cong \mathscr{O}^{+,0}_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$$

between theories and local action functionals $I \in \mathscr{O}_{loc}(\mathcal{E}, \mathscr{A})[[\hbar]]$ which are at least cubic modulo $\hbar$. Everything depends on $\mathscr{A}$.

The local renormalization group flow acts on the space of theories, $\mathcal{RG}_\ell : \mathscr{T}^{(\infty)} \to \mathscr{T}^{(\infty)}$, and just as for scalar field theories,

Proposition 3.2.2. *For any theory $\{I[L]\} \in \mathscr{T}^{(\infty)}(\mathcal{E}, \mathscr{A})$,*

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \mathscr{A})[[\hbar]] \otimes \mathbb{C}[\ell, \ell^{-1}, \log \ell].$$

Which allows us to define the space of relevant $\mathscr{R}^{(\infty)}(\mathcal{E}, \mathscr{A})$ and marginal $\mathscr{M}^{(\infty)}(\mathcal{E}, \mathscr{A})$ theories as those theories such that:

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \mathscr{A})[[\hbar]] \otimes \mathbb{C}[\ell, \log \ell],$$
$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \mathscr{A})[[\hbar]] \otimes \mathbb{C}[\log \ell]$$

respectively. Define again the subspaces $\mathscr{O}_{loc,k}(\mathcal{E}, \mathscr{A})$ of local action functionals of dimension k, and $\mathscr{O}_{loc,\geq 0}(\mathcal{E}, \mathscr{A})$ of local action functionals of non-negative dimension.

Theorem 3.2.3. *The space $\mathscr{R}^{(n+1)}(\mathcal{E}, \mathscr{A})$ is a torsor over $\mathscr{R}^{(n)}(\mathcal{E}, \mathscr{A})$ for the abelian group $\mathscr{O}_{loc,\geq 0}(\mathcal{E}, \mathscr{A})$. The space $\mathscr{R}^{(0)}(\mathcal{E}, \mathscr{A})$ is canonically isomorphic to the subspace of $\mathscr{O}_{loc,\geq 0}(\mathcal{E}, \mathscr{A})$ of local action functionals which are at least cubic.*

The space $\mathscr{M}^{(n+1)}(\mathcal{E}, \mathscr{A})$ is a torsor over $\mathscr{M}^{(n)}(\mathcal{E}, \mathscr{A})$ for the abelian group $\mathscr{O}_{loc,0}(\mathcal{E}, \mathscr{A})$. The space $\mathscr{M}^{(0)}(\mathcal{E}, \mathscr{A})$ is canonically isomorphic to the subspace of $\mathscr{O}_{loc,0}(\mathcal{E}, \mathscr{A})$ of local action functionals which are at least cubic.

Chapter 4

Gauge theories

In general, a theory that possesses some kind of symmetry means that there is an action of a Lie group G on the space of fields such that the action is $G-$invariant. The elements of $EL(S)$ are regarded as physically equivalent if they are related by an element of the group. A typical example is a relativistic point particle and the group G is the Poincaré group. The action of the Poincaré group is viewed as a change in inertial frame of reference, so if two solutions are related by a Poincaré transformation it means that they are describing the same physical problem. Gauge symmetry on the contrary is a *local* symmetry. A theory is said to be gauge invariant if there is a space-time-dependent transformation of the fields which leaves the action invariant. The canonical example is classical electromagnetism, where Maxwell equations (or the Maxwell action) expressed in terms of the electric $\phi(\mathbf{x}, t)$ and magnetic potential $\mathbf{A}(\mathbf{x}, t)$ are invariant under the *gauge* transformations $\phi \mapsto \phi + \frac{\partial}{\partial t} f(\mathbf{x}, t)$ and $\mathbf{A} \mapsto \mathbf{A} + \mathbf{grad}\ g(\mathbf{x}, t)$.

A gauge theory is mathematically modelled as a principal $G-$bundle, $P \to M$ where M is regarded as spacetime and G is the structure group. A gauge field is a connection on this principal bundle and the gauge group is the group of diffeomorphisms of the principal $G-$bundle P.

It is convenient to recall the difference between gauge theories and field theories equipped with some symmetry group. A gauge group is not a group of symmetries of the theory. The theory does not make any sense before taking the quotient by the gauge group. More precisely, if the space of gauge fields is B and the structure group is $\mathcal{G}$, the "physical" space of field configurations is not B but $B/\mathcal{G}$. In this sense, the presence of a gauge symmetry may be regarded as a redundancy in our description of the fields.

In order to quantize a gauge theory, one uses the Batalin-Vilkovisky formalism, which is regarded as the most general way to quantize gauge theories. Classically, this amounts to introducing extra fields: ghosts, anti-ghosts and anti-fields, and writing an extended classical action on the extended space of fields, which encodes: the original action, the Lie bracket on the space of infinitesimal gauge symmetries and the action of this algebra on the space of fields. In homological algebra, this space describes the *derived* moduli space of solutions to the Euler-Lagrange equations of the theory.

One wishes to quantize this theory by imposing that the action satisfies the *quantum master equation.* Here one encounters the same divergences as in the non-gauge case. The Wilsonian philosophy is needed to circumvent this problem.

When dealing with graded algebras with differentials on it, we will demand that this differential respects the grading of the algebra, more precisely we define

Definition 4.0.1 (Differential graded algebra)**.** A differential graded algebra (dga) is a graded algebra

$$\mathscr{A} = \bigoplus_{k \geq 0} \mathscr{A}_k$$

with differential $d : \mathscr{A} \to \mathscr{A}$ of degree $+1$, such that

1. $\mathscr{A}$ is graded commutative, i.e. $x \cdot y = (-1)^{kl} y \cdot x$ for $x \in \mathscr{A}^k, y \in \mathscr{A}^l$.

2. d is a derivation, i,e $d(x \cdot y) = dx \cdot y + (-1)^k x \cdot dy$, for $x \in \mathscr{A}^k$.

3. $d^2 = 0$.

We also define a differential graded (dg) manifold as a smooth (graded) manifold whose algebra of functions is a dga.

4.1 The Batalin-Vilkovisky formalism

We will start by doing a crash course on the BV formalism in the finite-dimensional case.

Given a finite-dimensional classical gauge theory, consisting of a vector space of fields V, treated as a formal manifold near $0 \in V$, a Lie algebra $\mathfrak{g}$ acting on V, which integrates to an action of the formal Lie group G associated to $\mathfrak{g}$, and an action functional $f \in \mathscr{O}(V)$, which is $G-$invariant and having a critical point in $0 \in V$.

We would like to make sense of integrals of the form:

$$\int_{V/G} \exp\left(f/\hbar\right)$$

over the quotient space V/G.

Our first problem is that the quotient V/G is not a manifold in general, so we cannot make sense of this integral perturbatively through a stationary phase expansion. The idea is to interpret this quotient in a homological fashion, by considering the *derived* quotient instead of the naive quotient. The *derived* invariants for the action of $\mathfrak{g}$ on the algebra $\mathscr{O}(V)$ is the Chevalley-Eilenberg complex:

$$C^*(\mathfrak{g}, \mathscr{O}(V)),$$

whose elements are alternating $\mathbb{R}-$multilinear functions $\mathrm{Hom}_{\mathbb{R}}(\bigwedge^* \mathfrak{g}, \mathscr{O}(V))$, with differential being defined as the dual of the Lie bracket (extended by the graded Leibniz rule).

We can now define the derived quotient of V by G as the object whose algebra of functions is $C^*(\mathfrak{g}, \mathscr{O}(V))$. That is $\mathfrak{g}[1] \oplus V$, where $[1]$ refers to a degree shift by 1, so $\mathfrak{g}$ is in degree -1. This is because, under the natural isomorphism $\mathrm{Sym}^* V[1] \cong \bigwedge V$, it is clear that $\widehat{\mathrm{Sym}}^*(\mathfrak{g}[1] \oplus V)^\vee = \bigwedge^* \mathfrak{g}^* \otimes \mathscr{O}(V)$.

Therefore, the derived quotient is a dg manifold, where the (Chevalley-Eilenberg) differential can be regarded as a degree 1 vector field X on $\mathfrak{g}[1] \oplus V$, which is of square 0.

The first step, known as the BRST construction is to try and replace our original integral by

$$\int_{\mathfrak{g}[1]\oplus V} \exp\left(f/\hbar\right).$$

We can attempt to make sense of this integral perturbatively, since we are integrating over a formal dg manifold. However, f is highly degenerate on $\mathfrak{g}[1] \oplus V$ since f is independent of $\mathfrak{g}[1]$ and is constant along $G-$orbits on V, and we can only perform the stationary phase expansion around non-degenerate critical points.

This is the problem that the BV formalism was meant to solve, and it does so in a beautiful way. Let $E := T^*[-1](\mathfrak{g}[1] \oplus V)$ denote the shifted cotangent bundle of $\mathfrak{g}[1] \oplus V$, so that

$$E = \underbrace{\mathfrak{g}[1]}_{\text{ghosts}} \oplus \underbrace{V}_{\text{fields}} \oplus \underbrace{V^\vee[-1]}_{\text{anti-fields}} \oplus \underbrace{\mathfrak{g}^\vee[-2]}_{\text{anti-ghosts}},$$

where we have named each summand according to the usual physics notation. The cotangent bundle has a natural symplectic structure so we will need some concepts from symplectic geometry which we will briefly review.

Definition 4.1.1 (Symplectic manifold)**.** A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed non-degenerate two form.

Definition 4.1.2 (Hamiltonian vector field)**.** Given a symplectic manifold (M, ω), the non-degeneracy of the symplectic form ω gives a canonical, fibre-wise linear isomorphism between the tangent and cotangent bundle. Therefore, for each smooth function $H : M \to \mathbb{R}$ there is a unique vector field X_H called the *Hamiltonian vector field of* H satisfying $\omega(X_H, Y) = dH(Y)$, for all vector fields Y.

Definition 4.1.3 (Poisson bracket)**.** A Poisson bracket on a symplectic manifold (M, ω) is a bilinear operation

$$\{-,-\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

defined by $\{f, g\} = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields of f and g respectively.

Definition 4.1.4 (Lagrangian subspace)**.** Let (M, ω) be a symplectic manifold. We say $L \subset M$ is a Lagrangian submanifold if $\omega \mid_L = 0$ and $\dim L = \frac{1}{2} \dim M$.

The function $f \in \mathfrak{g}[1] \oplus V$ pulls back naturally to a function on E via the projection $\pi : E \to \mathfrak{g}[1] \oplus V$ and by abuse of notation we will still call $\pi^* f$ by f. The vector field X on $\mathfrak{g}[1] \oplus V$ also induces a vector field on E and by inclusion , which we will also continue to call X. Since $[X, X] = 0$ on $\mathfrak{g}[1] \oplus V$, as it comes from a differential, the same holds on E, and since f is preserved by X in $\mathfrak{g}[1] \oplus V$, the same is true on E.

E is a dg manifold and since it is a (shifted) cotangent bundle, it comes equipped with a (odd) symplectic form of degree -1. Since X preserves the symplectic form, we know from symplectic geometry that there exists –at least locally– a unique function h_X on E whose Hamiltonian vector field is X, and which vanishes at the origin. That is, $\omega(X, \cdot) = dh_X(\cdot)$, which means that h_X is of degree 0.

Since every symplectic manifold is a Poisson manifold, the degree -1 symplectic form induces a degree 1 Poisson bracket, which translates the statements $[X, X] = 0$, $Xf = 0$ and (trivially) $[f, f] = 0$ into $\{h_X, h_X\} = 0$, $\{h_X, f\} = 0$ and $\{f, f\} = 0$ respectively. These identities tells us that the function $S_{BV} := f + h_X$, known as the BV action, satisfy the *BV classical master equation*:

$$\{f + h_X, f + h_X\} = 0.$$

As usual, the actions we will consider split into kinetic and interacting terms as $S_{BV}(e) = \frac{1}{2}\langle e, Qe\rangle + I_{BV}(e)$, where $Q : E \to E$, is a degree 1 map, skew-self adjoint for the degree -1 pairing $\langle -, -\rangle$, and I_{BV} a function which is at least cubic.

With this splitting, the classical master equation implies that $Q^2 = 0$ and

$$QI_{BV} + \frac{1}{2}\{I_{BV}, I_{BV}\} = 0$$

The BV formalism tells us to replace the original integral by:

$$\int_{e \in L} \exp\left(S_{BV}(e)/\hbar\right) \tag{4.1}$$

where $L \subset E$ is a small generic Lagrangian perturbation of the zero section $\mathfrak{g}[1] \oplus V \subset E$.

If the complex (E, Q) has 0 cohomology, that is: $H^*(E, Q) = 0$, the pairing $\langle e, Qe\rangle$ will be non-degenerate on a generic Lagrangian L, which will allow us to perform the above integral perturbatively around $0 \in L$. This condition may be relaxed as we will see. We would want this integral to not depend on the Lagrangian submanifold we choose, that is, to be invariant under deformations of the Lagrangian submanifold. This motivates the definition of the *quantum master equation* (QME):

$$QI + \frac{1}{2}\{I, I\} + \hbar \Delta I = 0,$$

where Δ is the linear order two differential operator corresponding to $\omega^\vee$. That is, since ω provides an isomorphism $E[-1] \cong E^\vee$, ω will be an element of $\bigwedge^2 E^\vee \cong \left(\mathrm{Sym}^2 E\right)[-2]$, meaning that $\omega^\vee \in \mathrm{Sym}^2 E$.

$$\Delta : \mathscr{O}(E) \to \mathscr{O}(E)$$

will be the order two differential operator that on $\mathrm{Sym}^2 E^\vee$ is given by contraction with $\omega^\vee$.

The following lemma tells us that the integral (4.1) is unchanged under perturbations of L if the action S_{BV} satisfies the QME.

Lemma 4.1.1. *Let $L \subset E$ be a Lagrangian on which the pairing $\langle e, Qe\rangle$ is non-degenerate. (Such a Lagrangian exists if and only if $H^*(E,Q) = 0$). Suppose that $I \in \mathscr{O}(E)[[\hbar]]$ is an $\hbar-$dependent function on E satisfying the QME. Then, for $a \in E$, the integral:*

$$\int_{e\in L} \exp\left(\frac{1}{2\hbar}\langle e, Qe\rangle + \frac{1}{\hbar} I(e+a)\right)$$

is unchanged under deformations of L.

Following the BV construction, I_{BV} automatically satisfies the classical master equation $QI_{BV} + \frac{1}{2}\{I_{BV}, I_{BV}\} = 0$ and therefore if $\Delta I_{BV} = 0$ we can quantize the gauge theory. If not, we would try to "correct" I by $I + \sum_i \hbar^i I_i$ such that it satisfies the QME order by order in $\hbar$.

The operator Δ can be thought of as a divergence associated to a translational invariant measure on E. Let $I \in \mathscr{O}(E)$ and let X_I be the associated vector field defined by $X_I f = \{I, f\}$. Δ satisfies: $\mathcal{L}_{X_I}\mu = (\Delta I)\,\mu$, where $\mathcal{L}_{X_I}$ denotes the Lie derivative. Therefore, ΔI is the infinitesimal change in volume corresponding to X_I. In this picture, the equation $\Delta I = 0$ says that X_I is measure-preserving. The case where $I \in \mathscr{O}(E)[[\hbar]]$ is a little more involved and doesn't have such a direct interpretation.

We now proceed to see how can we relax the condition of $H^*(E,Q) = 0$ to perform the integral. Let $L \subset E$ be a Lagrangian subspace and let $Q : L \to \mathrm{Im}\, Q$ be an isomorphism. Let also $\mathrm{Ann}(L) = \{e \in E | \langle e, \ell\rangle = 0, \forall \ell \in L\}$ be the set of vectors which pair to zero with any element of L. Thus, we can identify $H^*(E,Q) = \mathrm{Ann}(L) \cap \ker Q$.

This allows for a direct sum decomposition: $E = L \oplus H^*(E,Q) \oplus \mathrm{Im}\, Q$, where $H^*(E,Q)$ inherits a degree -1 symplectic pairing, and thus a BV operator $\Delta_{H^*(E,Q)}$ acting on functions on $H^*(E,Q)$.

In this setting we have the following generalization of the BV formulation:

Lemma 4.1.2. *Let $I \in \mathscr{O}(E)[[\hbar]]$ be an $\hbar-$dependent function on E satisfying the QME. The function on $H^*(E,Q)$ defined by:*

$$a \mapsto \hbar \log \int_{e\in L} \exp\left(\frac{1}{2\hbar}\langle e, Qe\rangle + \frac{1}{\hbar} I(e+a)\right) \tag{4.2}$$

satisfy the QME. Furthermore, small perturbations of the isotropic subspace L change this solution of the QME on $H^*(E,Q)$ to a homotopic solution of the QME.

Note that his integral is the renormalization group flow. Let's be precise about what we mean by homotopy between solutions of the QME.

Definition 4.1.5 (Homotopy of solutions of QME)**.** Given two solutions of the QME f_0 and f_1 on $H^*(E,Q)$ we say that they are homotopic if there exists an element $F \in \mathscr{O}(H^*(E,Q)) \otimes \Omega^*([0,1])[[\hbar]]$ which satisfies the QME: $d_{dR}F + \frac{1}{2}\{F,F\} + \hbar\Delta_{H^*(E,Q)}F = 0$ and restricts to f_0 and f_1 when valuated at 0 and 1. d_{dR} denotes the de Rham differential on the commutative differential algebra $\Omega^*([0,1])$.

If we explicitly write $F \in \mathscr{O}(H^*(E,Q)) \otimes \Omega^*([0,1])[[\hbar]]$ as $F(t, dt) = A(t) + dtB(t)$, the QME imposed on F is

$$\frac{1}{2}\{A(t), A(t)\} + \hbar\Delta_{H^*(E,Q)}A(t) = 0$$

$$\frac{d}{dt}A(t) + \{A(t), B(t)\} + \hbar\Delta_{H^*(E,Q)}B(t) = 0.$$

4.2 Free BV theory on a compact manifold

When trying to generalize these definitions to infinite dimensions things go through little change, at least classically, i.e. before introducing the quantum master equation. We refer to [Cos11, Chapter 5.3] for further details.

The first example of a quantum BV theory will be in a compact manifold. As always, the free theory will be easy to define and will be well-behaved. Issues will arise when introducing interactions.

Definition 4.2.1 (Free BV theory)**.** A free theory on a compact manifold M consists of the following data.

i) A $\mathbb{Z}-$graded vector bundle E on M, over $\mathbb{R}$ or $\mathbb{C}$, whose space of sections is $\mathcal{E} = \Gamma(M, E)$.

ii) E is equipped with anti-symmetric map of vector bundles on M, of cohomological degree -1,

$$\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M).$$

Such that it is non-degenerate on each fibre. This pairing induces, by integration, a pairing in the space of global sections:

$$\langle -, - \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathbb{C}$$
$$\langle e_1, e_2 \rangle = \int_M \langle e_1, e_2 \rangle_{loc}.$$

iii) A differential operator $Q : \mathcal{E} \to \mathcal{E}$ of cohomological degree one, which squares to zero and is skew self-adjoint for the pairing, i.e: $\langle e_1, Qe_2 \rangle = -\langle Qe_1, e_2 \rangle$. Furthermore, $(\mathcal{E}, Q)$ must be an elliptic complex, which means that the complex of vector bundles $(\pi^* E, \sigma(Q))$ on $T^*M \backslash M$ is exact.

Definition 4.2.2 (Gauge fixing operator on free BV theory)**.** A gauge fixing operator on a free BV theory $\mathcal{E}$ is given by an operator $Q^{GF} : \mathcal{E} \to \mathcal{E}$, such that:

i) Q^{GF} is of cohomological degree -1, of square zero and self-adjoint for the pairing $\langle -, - \rangle$.

ii) The commutator $D = [Q, Q^{GF}]$ is required to be a generalized Laplacian.

We will now see how a free BV theory together with a choice of gauge fixing operator gives a free theory in the sense of Definition 1.2.10.

Let $\mathcal{E}^!$ be the space of sections of $E^\vee \otimes \mathrm{Dens}(M)$ on M. The pairing on E gives rise to an isomorphism: $\mathcal{E}^! \to \mathcal{E}$. Composing it with $Q^{GF} : \mathcal{E} \to \mathcal{E}$ define an operator $D' : \mathcal{E}^! \to \mathcal{E}$. This operator, together with $D : \mathcal{E} \to \mathcal{E}$, define a free theory in the sense of Definition 2.2.12.

To define the heat kernel we will need a convolution operator: $\star : \mathcal{E} \otimes \mathcal{E} \to \mathrm{End}(\mathcal{E})$, such that, for any $K \in \mathcal{E} \otimes \mathcal{E}$, and $e \in E$, $K \star e := (-1)^{|K|}(1 \otimes \langle -, - \rangle)(K \otimes e)$, where $\langle -, - \rangle$ is the pairing on $\mathcal{E}$. The convolution operator leaves invariant the first factor of K and contracts the second one with $e \in \mathcal{E}$. The reason of the sign choice is that $(QK) \star e = [Q, K \star]e$. (Here Q is referring to $1 \otimes Q + Q \otimes 1$).

There is a heat kernel $K_\ell \in \mathcal{E} \otimes \mathcal{E}$ for the operator $e^{-\ell D}$ characterized by

$$K_\ell \star e = e^{-\ell D} e.$$

The propagator of our free theory will be:

$$P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_\ell d\ell.$$

It is immediate to see that, for any $e \in \mathcal{E}$, $P(\varepsilon, L) \star e = Q^{GF} \int_\varepsilon^L e^{-\ell D} e d\ell$.

We see that in infinite dimensions we require an extra object, the gauge fixing operator, to define a free theory. It allows us to define the propagator of our theory, and all the subspaces over which we will integrate are of the form Im Q_{GF}. This coincides with the notion in Physics of gauge fixing, where an equation is imposed over the gauge fields to remove the redundancy in the description of the fields.

4.3 Interacting (pre-) theories on a compact manifold

We will now introduce interaction with the same philosophy of effective field theory. The notion of theory will be reserved for those pre-theories which satisfy the QME.

Definition 4.3.1 (Pre-theory on compact manifold)**.** A pre-theory is a collection of effective interactions $\{I[L]\}$ satisfying the renormalization group equation and the locality axiom as defined before. More precisely:

i) Each $I[L] \in \mathscr{O}^{+,0}(\mathcal{E})[[\hbar]]$ is of degree 0 and at least cubic modulo $\hbar$.

ii) The renormalization group equation

$$I[L] = W(P(\varepsilon, L), I[\varepsilon])$$

is satisfied.

iii) Each $I_{i,k}[L]$ has a small L asymptotic expansion in terms of local action functionals.

Denote $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E})$ the set of pre-theories, and $\widetilde{\mathscr{T}}^{(n)}(\mathcal{E})$ the set of theories defined modulo $\hbar^{n+1}$

Unfortunately, the QME in infinite dimensions is ill-defined. If one tries to define, naively a BV Laplacian on the space of functionals on $\mathcal{E}$, by letting $\Delta_\ell = -\partial_{K_\ell} : \mathscr{O}(\mathcal{E}) \to \mathscr{O}(\mathcal{E})$ be the order two differential operator associated to the heat kernel, one finds that the BV Laplacian would be $\Delta_0 I = -\lim_{\ell\to 0} \partial_{K_\ell} I$, which has the same singularities as the one-loop Feynman diagrams.

We will circumvent this problem as always, with the philosophy that the fundamental objects are the effective interactions. That is, we want to impose a QME on each scale L.

This motivates the following definition:

Definition 4.3.2 (Scale L quantum master equation)**.** Let $I \in \mathscr{O}^{+}(\mathcal{E})[[\hbar]]$. We say it satisfy the scale L quantum master equation if

$$QI + \frac{1}{2}\{I, I\}_L + \hbar\Delta_L I = 0,$$

where the scale L BV bracket $\{-,-\}_L : \mathscr{O}(\mathcal{E}) \otimes \mathscr{O}(\mathcal{E}) \to \mathscr{O}(\mathcal{E})$ is defined as: $\{I, J\}_L = \Delta_L(IJ) - \Delta_L(I)J - (-1)^{|I|} I\Delta_L(J)$, and $\Delta_L := -\partial_{K_L} : \mathscr{O}(\mathcal{E}) \to \mathscr{O}(\mathcal{E})$.

In practice, it is almost always impossible to check directly given a collection of $I[L]$ if they satisfy the QME. In the next section, we will describe how to use obstruction/deformation methods to deal with this problem.

The compatibility condition that makes it possible to combine the BV formalism with the effective field theory philosophy is guaranteed by the following lemma:

Lemma 4.3.1. *A functional $I[\varepsilon] \in \mathscr{O}^{+}(\mathcal{E})[[\hbar]]$ satisfies the scale ε QME if and only if $W(P(\varepsilon, L), I[\varepsilon])$ satisfies the scale L QME.*

That is, the renormalization group flow takes solutions of the scale ε QME to solutions of the scale L QME. The proof may be found in [Cos11], Lemma 9.2.2.

Definition 4.3.3 (Theory in the BV formalism)**.** A pre-theory $\{I[L]\}$ such that each $I[L]$ satisfies the scale L QME is called a theory. The set of theories will be denoted $\mathscr{T}^{(\infty)}$ and the set of theories defined modulo $\hbar^{n+1}$ will be denoted $\mathscr{T}^{(n)}$.

4.4 BV Theories as simplicial sets

We have seen that in order to define a theory we need a gauge fixing condition which defines the isotropic space over which we integrate. The change of this gauge fixing condition changes the solution of the QME into a homotopic one. We want to keep track of exactly how our notion of theory depends on this gauge fixing condition. To do this we will make use of a useful construction, the simplicial set, which is an abstract generalization of the simplicial complex.

Definition 4.4.1 (Simplicial set)**.** A simplicial set consists of a sequence of sets $X_0, X_1, \ldots$ and, for each $n \geq 0$, functions $d_i : X_n \longrightarrow X_{n-1}$ and $s_i : X_n \longrightarrow X_{n+1}$ for each i with $0 \geq i \geq n$ called face maps and degeneracy maps respectively, such that:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j \\
d_i s_j &= s_{j-1} d_i && \text{if } i < j \\
d_j s_j &= d_{j+1} s_j = \mathrm{id} && \\
d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j
\end{aligned}$$

The idea will be to define theories parametrised by the $n-$simplex, so that theories will be attached to points, homotopies between theories will be attached to edges, homotopies between homotopies will be attached to faces etc. In this way, we will keep track of the particular dependence on the gauge fixing condition. Citing Costello and Gwilliams in [CG17]: "We forewarn the reader that our definitions and constructions involve a heavy use of functional analysis and (perhaps more surprisingly) simplicial sets, which is our preferred way of describing a space of field theories. Making a quantum field theory typically requires many choices, and as mathematicians, we wish to pin down precisely how the quantum field theory depends on these choices. The machinery we use gives us very precise statements, but statements that can be forbidding at first sight."

In Definition 4.1.5, we defined the notion of homotopy between solutions of the QME. For any I satisfying the QME, the formula

$$I_H(a) = \hbar \log \int_{e \in L} \exp\left(\frac{1}{2\hbar}\langle e, Qe\rangle + \frac{1}{\hbar} I(e+a)\right) \tag{4.3}$$

gives us a solution of the QME on $H^*(\mathcal{E}, Q)$. If we vary this isotropic space L, the solution of the QME on $H^*(\mathcal{E}, Q)$ changes by a homotopy. Conversely, if we vary I by a homotopy, the corresponding solution I_H on $H^*(E, Q)$ varies by a homotopy.

This can be phrased as the set of gauge fixing conditions has an enrichment to a simplicial set, where the $n-$simplices are just smooth families of isotropic subspaces $L \subset E$ parametrized by the $n-$simplex, satisfying the equation 4.3.

Denote this simplicial set by $\mathscr{GF}(E, Q)$. The set of solutions of the QME also has an enrichment to a simplicial set, whose $1-$simplices are homotopies, $2-$simplicies are homotopies between homotopies etc. Denote this set by $\mathscr{QME}(E, Q)$.

The integral (4.2) defines a map of simplicial sets:

$$\begin{aligned}
&\mathscr{GF}(E, Q) \times \mathscr{QME}(E, Q) \longrightarrow \mathscr{QME}(H, 0) \\
&\qquad (Q^{GF}, I) \mapsto I_H = \hbar \log \int_{e \in \mathrm{Im}\ Q^{GF}} \exp\left(\frac{1}{2\hbar}\langle e, Qe\rangle + \frac{1}{\hbar} I(e+a)\right)
\end{aligned}$$

In finite dimensions, two theories are equivalent if they are homotopic or related by non-linear change of coordinates. More precisely,

Lemma 4.4.1. *Given a finite-dimensional graded vector space V equipped with a symplectic pairing of degree -1 $\langle -, - \rangle$ and a differential Q preserving the pairing. Then, two solutions $I_0, I_1 \in \mathscr{O}(V)$ of the QME are homotopic if and only if there is a symplectic diffeomorphism $\phi : V \to V$, in the connected component of the identity, such that*

$$e^{\langle v, Qv\rangle + I_1/\hbar} dV = \phi^* \left(e^{\langle v, Qv\rangle + I_0/\hbar} dV\right)$$

Where $v \in V$ and dV is the Lebesgue measure associated to V.

In infinite dimensions, only the homotopy picture holds, since it is not possible to take into account the change in the non-existent Lebesgue measure, so we will use it to talk about equivalent theories. We will see that the dependence of the notion of theory on the choice of gauge fixing condition is only up to homotopy so a theory will be independent of the choice of gauge fixing condition when the space of gauge fixing condition is contractible.

Fix a Free BV theory $(\mathcal{E}, Q, \langle -, - \rangle)$ on M.

Definition 4.4.2 (Family of gauge fixing conditions)**.** A family of gauge fixing conditions for $\mathcal{E}$, over $\Omega^*(\Delta^n)$, is an $\Omega^*(\Delta^n)$ linear differential operator

$$Q^{GF} : \mathcal{E} \otimes \Omega^*(\Delta^n) \longrightarrow \mathcal{E} \otimes \Omega^*(\Delta^n)$$

of cohomological degree -1, with the following properties:

i) Q^{GF} is self-adjoint for the $\Omega^*(\Delta^n)$ linear pairing $\mathcal{E} \otimes \mathcal{E} \otimes \Omega^*(\Delta^n) \to \Omega^*(\Delta^n)$.

ii) $\left(Q^{GF}\right)^2 = 0$.

iii) The operator $D = \left[Q + d_{dR}, Q^{GF}\right]$ is a generalized Laplacian. This means that the symbol of D is a smooth family of Riemannian metrics on the bundle T^*M, parametrized by Δ^n.

Fixing a family of gauge fixing condition over $\Omega^*(\Delta^n)$ allows us to define families of pre-theories over $\Omega^*(\Delta^n)$. Such a family is given by a collection of effective interactions $\{I[L]\}$, where $I[L] \in \mathscr{O}^+(\mathcal{E}, \Omega^*(\Delta^n))[[\hbar]]$ is of cohomological degree 0 accounting for both the cohomological grading of $\mathcal{E}$ and that of $\Omega^*(\Delta^n)$. Again it is imposed that $\{I[L]\}$ must satisfy the renormalization group equation: $I[L] = W(P(\varepsilon, L), I[\varepsilon])$, as well as the locality axiom.

Given any face or degeneracy maps $\Delta^m \to \Delta^n$, one can pull back a family of pre-theories or gauge fixing conditions, making them simplicial sets. The simplicial set of pre-theories $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q)$ has as $n-$simplices families of gauge fixing conditions over $\Omega^*(\Delta^n)$, together with a family of pre-theories for this family of gauge fixing conditions.

There is a natural map of simplicial sets: $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q) \to \mathscr{GF}(\mathcal{E}, Q)$, which assigns to each pre-theory, the family of gauge fixing conditions it has assigned. Once a renormalization scheme is chosen, we find an isomorphism of simplicial sets: $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q) \cong \mathscr{GF}(\mathcal{E}, Q) \times \mathscr{O}^{+,0}_{loc}(\mathcal{E})[[\hbar]]$

In this context, the QME is defined as:

Definition 4.4.3 (QME for interactions parametrized by $\Omega^*(\Delta^n)$)**.** A functional $I \in \mathscr{O}(\mathcal{E}, \Omega^*(\Delta^n))[[\hbar]]$ satisfies the scale L QME if

$$(Q + d_{dR} + \hbar\Delta_L)e^{I/\hbar} = 0,$$

where d_{dR} is the de Rham differential on $\Omega^*(\Delta^n)$, and Δ_L is the operator contraction with the heat kernel $K_L \in \mathcal{E} \otimes \mathcal{E} \otimes \Omega^*(\Delta^n)$. Everything is linear over $\Omega^*(\Delta^n)$.

As before, the renormalization group equation turns solutions of the QME into solutions of the QME. A theory is a pre-theory where each $I[L]$ satisfies the scale L QME. The simplicial set of theories will be denoted $\mathscr{T}^{(\infty)}(\mathcal{E}, Q)$. A homotopy between two theories is a $1-$simplex of $\mathscr{T}^{(\infty)}(\mathcal{E}, Q)$ connecting the two $0-$simplex. This coincides with our notion that two theories are equivalent if they are related by a homotopy, which we regard as a change of coordinates. The set of theories $\mathscr{T}^{(\infty)}(\mathcal{E}, Q)$ is a fibration over $\mathscr{GF}(\mathcal{E}, Q)$. A fibration is a generalization of the concept of fibre bundle. The precise definition is as follows.

Definition 4.4.4 (Fibration)**.** A fibration between two topological vector spaces E and B is a continuous map $p : E \to B$, such that the homotopy lifting property is satisfied for all spaces X. That is, for every homotopy $h : X \times [0,1] \to B$ and for every lift $\tilde{h}_0 : X \to E$ lifting $h\big|_{X\times 0} = h_0$, there exists a homotopy $\tilde{h} : X \times [0,1] \to E$ lifting h with $\tilde{h}_0 = \tilde{h}_{X\times 0}$. That is to say that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "h_0"] \arrow[d, hook, "i"] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

The fibres over each connected component of a fibration are homotopy equivalent and in most examples, the set of gauge fixing conditions is contractible, making the notion of theory independent of the choice of point in $\mathscr{GF}(\mathcal{E}, Q)$.

Remark. These definitions motivate the need for including the auxiliary nilpotent manifold in the previous sections. Being explicit, our manifold with corners X is Δ^n, the sheaf of commutative super algebras is $A = \bigwedge^* T^*\Delta^n$, which is the sheaf of sections of the super vector bundle $T[1]\Delta^n$, and whose sheaf of sections $\mathscr{A} = \Gamma(X, A) = \Omega^*(\Delta^n)$. Therefore, theories as simplicial sets are included in the previous definitions and we can use the results obtained before.

4.5 Obstruction theory

The simplicial set of pre-theories defined modulo $\hbar^{n+1}$, $\widetilde{\mathscr{T}}^{(n+1)}(\mathcal{E}, Q)$ forms a $\mathscr{O}^0_{loc}(\mathcal{E}, \Omega^*(\Delta^n))-$principal bundle over $\widetilde{\mathscr{T}}^{(n)}(\mathcal{E}, Q)$. If one tries to obtain a similar result for theories, one finds that there is a cohomological obstruction to extending a point of $\mathscr{T}^{(n)}(\mathcal{E}, Q)$ to a point in $\mathscr{T}^{(n+1)}(\mathcal{E}, Q)$.

More concretely, let us fix a free BV theory $(\mathcal{E}, Q, \langle -, - \rangle)$ on a compact manifold M and let $\{I[L]\} \in \mathscr{T}^{(n)}(\mathcal{E}, Q)[k]$ denote a $k-$simplex in the space of theories defined modulo $\hbar^{n+1}$. Let us lift, arbitrarily $\{I[L]\}$ to an element of $\{\widetilde{I}[L]\} \in \widetilde{\mathscr{T}}^{(n+1)}(\mathcal{E}, Q)[k]$. Define the scale L obstruction by

$$O_{n+1}[L] = \frac{1}{\hbar^{n+1}} \left(Q\widetilde{I}[L] + \frac{1}{2}\{\widetilde{I}[L], \widetilde{I}[L]\}_L + \hbar\Delta_L \widetilde{I}[L] \right).$$

This expression is independent of $\hbar$, since $\widetilde{I}[L]$ satisfies the QME modulo $\hbar^{n+1}$. The following lemma and corollary tell us that $O_{n+1}[L]$ is an obstruction to lifting a theory defined modulo $\hbar$ to a theory defined modulo $\hbar^{n+1}$ and that the possible lifts are those elements that kill the obstruction.

Lemma 4.5.1. *Let ε be a parameter of square zero and cohomological degree -1. Let $I_0[L]$ be $I[L]$ modulo $\hbar$. Then, $I_0[L] + \varepsilon O_{n+1}[L]$ satisfies both the scale L classical master equation and the classical renormalization group equation. Thus, it defines a classical theory in the BV formalism. The set of lifts of $\{I[L]\}$ to a $k-$simplex of $\mathscr{T}^{(n+1)}(\mathcal{E}, Q)$ is the set of degree 0 elements $J[L] \in \mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^n)$ such that $I_0[L] + \delta J[L]$ satisfies the classical renormalization group equation and locality axiom modulo δ^2, and such that $QJ[L] + \{I_0[L], J[L]\} = O_{n+1}[L]$.*

Corollary 4.5.1.1. *Let $\{I[L]\} \in \mathscr{T}^{(n)}(\mathcal{E}, Q)[k]$. Then, there is an obstruction $O_{n+1} \in \mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^k))$ which is a closed, degree 1 element, where the cochain complex has the differential $Q + \{I_0, -\}$. The set of lifts of $\{I[L]\}$ to a $k-$simplex of $\mathscr{T}^{(n+1)}(\mathcal{E}, Q)$ is the set of degree 0 elements $J \in \mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^k))$ making O_{n+1} exact.*

The obstruction defines a map $\mathscr{T}^{(n)}(\mathcal{E}, Q) \to \mathscr{O}_{loc}(\mathcal{E}, Q)[1]$, where $\mathscr{O}_{loc}(\mathcal{E}, Q)[1]$ denotes the simplicial set whose elements are closed, degree 1 elements of $\mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^k))$.

Theorem 4.5.2. *There is a homotopy-fibre diagram of simplicial sets:*

$$\begin{tikzcd} \mathscr{T}^{(n+1)}(\mathcal{E}, Q) \arrow[r] \arrow[d] & \mathscr{T}^{(n)}(\mathcal{E}, Q) \arrow[d, "O_{n+1}"] \\ 0 \arrow[r] & \mathscr{O}_{loc}(\mathcal{E}, Q)[1] \end{tikzcd}$$

Which says that a point of $\mathscr{T}^{(n+1)}(\mathcal{E}, Q)$ is the same as a point of $\mathscr{T}^{(n)}(\mathcal{E}, Q)$ together with a homotopy between the obstruction and zero. More precisely, the homotopy fibre is a way of assigning a *fibration* to any topological map. Given the map $O_{n+1} : \mathscr{T}^{(n)}(\mathcal{E}, Q) \to \mathscr{O}_{loc}(\mathcal{E}, Q)[1]$, the diagram above tells us that $\mathscr{T}^{(n+1)}(\mathcal{E}, Q) = \mathrm{Hofibre}(O_{n+1}) = \{(T, \gamma) | T \in \mathscr{T}^{(n)}(\mathcal{E}, Q), \gamma : [0, 1] \to \mathscr{O}_{loc}(\mathcal{E}, Q)[1], \gamma(0) = O_{n+1}(T), \gamma(1) = 0\}$.

Example 4. Let's introduce Chern-Simons theory on an oriented $3-$manifold M with a compact Lie group G. Let us denote $\mathfrak{g}$ its Lie algebra and fix an invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. Note that this is not the same as the Lie algebra in section 4.1, i.e. the Lie algebra that acts on the space of fields. The Chern-Simons field is a $\mathfrak{g}$-valued 1 form. Assuming we are perturbing around the trivial flat connection on the trivial $\mathfrak{g}-$bundle on M we can identify the space of fields with $\Omega^1(M) \otimes \mathfrak{g}$. The gauge group is, as usual, $\mathscr{G} = \mathrm{Maps}(M, G)$ of smooth maps from M to G. The Lie algebra of infinitesimal gauge symmetries will be $\Omega^0(M) \otimes \mathfrak{g}$ which is what we had been calling $\mathfrak{g}$ in section 4.1. The action of the Lie algebra on the space of fields is the usual affine-linear action:

$$A \mapsto [X, A] + dX$$

where $A \in \Omega^1(M) \otimes \mathfrak{g}$ and $X \in \Omega^0(M) \otimes \mathfrak{g}$. Denote by $\langle -, - \rangle$ the pairing on $\Omega^*(M) \otimes \mathfrak{g}$ defined by

$$\langle \omega_1 \otimes E_1, \omega_2 \otimes E_2 \rangle = \int_M \omega_1 \wedge \omega_2 \langle E_1, E_2 \rangle_g$$

The Chern-Simons action reads:

$$S_{CS}(A) = \frac{1}{2} \int_M \langle A, dA \rangle + \frac{1}{6} \langle A, [A, A] \rangle.$$

In the BV formalism, the extended space of fields is:

$$\begin{array}{ll} \Omega^0(M) \otimes \mathfrak{g} & \text{ghosts, degree } -1 \\ \Omega^1(M) \otimes \mathfrak{g} & \text{fields, degree } 0 \\ \Omega^2(M) \otimes \mathfrak{g} & \text{anti-fields, degree } 1 \\ \Omega^3(M) \otimes \mathfrak{g} & \text{anti-ghosts, degree } 2 \end{array}$$

That is, the extended space of fields may be identified with $\mathcal{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$.

The Batalin-Vilkovisky action $S = S_{CS} + S_{gauge}$ reads simply:

$$S(e) = \frac{1}{2} \langle e, de \rangle + \frac{1}{6} \langle e, [e, e] \rangle.$$

A gauge fixing condition for the Chern-Simons theory is d^*, which means that we integrate over the isotropic subspace $\mathrm{Im}\ d^* \subset \mathcal{E}$. The choice of gauge fixing condition is equivalent to the choice of a metric on M, since the operator d^* is uniquely characterized by a metric. Therefore the space of gauge fixing conditions is $\mathrm{Met}(M)$, which is contractible so Chern-Simons theory is independent of the choice of gauge fixing condition.

We now present BV theories on $\mathbb{R}^n$, whose main difference with respect to compact manifolds is that we can talk about renormalizability. This requires including an extra grading by dimension.

4.6 Batalin-Vilkovisky theories on $\mathbb{R}^n$

Definition 4.6.1 (BV free theory on $\mathbb{R}^n$)**.** A free theory on $\mathbb{R}^n$ in the BV formalism consists of the following data:

i) A bi-graded vector space E. The first grading is called the cohomological grading, and the second is the dimension grading. We will think of E as a trivial vector bundle on $\mathbb{R}^n$, and let $\mathcal{E} = E \otimes \mathscr{S}(\mathbb{R}^n)$ be the space of Schwarz sections. The grading by dimension on $\mathcal{E}$ induces an $\mathbb{R}_{>0}$ action on $\mathcal{E} = E \otimes \mathscr{S}(\mathbb{R}^n)$, by $R_\ell(e \otimes f(x)) = \ell^i e \otimes f(\ell x)$, where $f(x) \in \mathscr{S}(\mathbb{R}^n)$ and e is of dimension i.

ii) E is equipped with a non-degenerate degree -1 anti-symmetric pairing $\langle -, - \rangle : E \otimes E \longrightarrow \det(\mathbb{R}^n)$ which respects dimension, so that E_i pair with E_{n-i}. The pairing on E induces an integrating pairing

$$\begin{aligned} \langle -, - \rangle &: \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathbb{C} \\ \langle f_1 e_1, f_2 e_2 \rangle &= \int_{R^n} f_1 f_2 \langle e_1, e_2 \rangle \end{aligned}$$

where $e_i \in E$ and $f_i \in \mathscr{S}(\mathbb{R}^n)$. This pairing is of dimension 0.

iii) A differential operator $Q : \mathcal{E} \longrightarrow \mathcal{E}$ which is translation invariant, of cohomological degree one, preservers dimension, is of square zero and is skew self-adjoint for the pairing.

Definition 4.6.2. Let $(\mathcal{E}, Q)$ be a free BV theory. A family of gauge fixing operators on $\mathcal{E}$, parametrized by $\Omega^*(\Delta^m)$, is an $\Omega^*(\Delta^m)$ linear differential operator

$$Q^{GF} : \mathcal{E} \otimes \Omega^*(\Delta^m) \longrightarrow \mathcal{E} \otimes \Omega^*(\Delta^m)$$

such that:

i) Q^{GF} is of cohomological degree -1, translational invariant, of square zero, and self-adjoint for the pairing $\langle -, - \rangle$.

ii) Q^{GF} is of dimension -2.

iii) The commutator $D = [Q + d_{dR}, Q^{GF}]$ is a sum of two terms $D = D' + D''$, where D' is the tensor product of the Laplacian on $\mathbb{R}^n$ with the identity on E and D'' is a nilpotent operator commuting with D'.

Again, let

$$\mathscr{O}(\mathcal{E}, \Omega^*(\Delta^m)) = \Pi_{k>0} \operatorname{Hom}(\mathcal{D}_g(\mathbb{R}^{nk}, \Omega^*(\Delta^m)) \otimes E^k)_{S_k},$$

where $\mathcal{D}_g(\mathbb{R}^{nk}, \Omega^*(\Delta^m))$ refers to the space of $\Omega^*(\Delta^m)-$valued distributions on $\mathbb{R}^{nk}$, which are invariant under the action of $\mathbb{R}^n$ by translation, and of rapid decay away from the diagonal.

There is a heat kernel $K_\ell \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \otimes E \otimes E \otimes \Omega^*(\Delta^m)$, for the operator $\exp(-\ell D)$. If we pick a basis e_i for E, the heat kernel is: $K_\ell = \sum \Phi_{i,j}(x, y, \ell) e_i \otimes e_j$ where, for all basis elements e_k and functions $g(x) \in \mathscr{S}(\mathbb{R}^n)$,

$$(-1)^{|e_k|} \sum_{i,j} \int_{y \in \mathbb{R}^n} \Phi_{i,j}(x, y, \ell) g(y) e_i \langle e_j, e_k \rangle = \exp(-\ell D)\, g(x) e_k.$$

Where the functions $\Phi_{i,j}(x, y, \ell)$ are of the form $\Psi(\ell^{1/2}, \ell^{-1/2}, x - y) e^{-||x-y||^2/\ell}$, and Ψ are of the form $\Psi \in \Omega^*(\Delta^m)[\ell^{1/2}, \ell^{-1/2}, x - y]$.

The propagator is then given by: $P(\varepsilon, L) = \int_\varepsilon^L (Q^{GF} \otimes 1) K_\ell d\ell$.

Definition 4.6.3. As on a compact manifold, a pre-theory is a collection of effective interactions $\{I[L]\}$ satisfying the renormalization group flow and the locality axiom as defined before. Define the simplicial set $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q)$ of pre-theories, where an m-simplex consists of:

i) An m-simplex Q^{GF} of the simplicial set of gauge operators.

ii) A collection $I[L] \in \mathscr{O}^{+,0}(\mathcal{E}, \Omega^*(\Delta^m))[[\hbar]]$ of effective interactions, which are of cohomological degree 0 and at least cubic modulo $\hbar$.

And such that:

i) The renormalization group equation $I[L] = W(P(\varepsilon, L), I[\varepsilon])$ is satisfied.

ii) The locality axiom holds. If we consider $I_{i,k}[L]$ as an element $I_{i,k}[L] \in \mathcal{D}_g(\mathbb{R}^{nk}, \Omega^*(\Delta^m)) \otimes (\mathcal{E}^\vee)^{\otimes k}$, then, if $e \in \mathscr{S}(R^{nk}) \otimes E^{\otimes k}$ has compact support away from the small diagonal, $I_{i,k}[L](e) \to 0$ as $L \to 0$.

Since we are working over $\mathbb{R}^n$ there is a natural action of $\mathbb{R}_{>0}$ on families of theories over the $n-$simplex, thus on the simplicial set $\widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q)$.

If $\{I[L]\} \in \widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q)[m]$ is an $m-$simplex in the space of pre-theories, then

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \Omega^*(\Delta^m))[[\hbar]] \otimes \mathbb{C}[\ell, \ell^{-1}, \log \ell].$$

We will let $\widetilde{\mathscr{R}}^{(\infty)}(\mathcal{E}, Q)$ and $\widetilde{\mathscr{M}}^{(\infty)}(\mathcal{E}, Q)$ denote the sub simplicial sets of relevant and marginal pre-theories respectively. As before, a $m-$simplex will be relevant if

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \Omega^*(\Delta^m))[[\hbar]] \otimes \mathbb{C}[\ell, \log \ell],$$

and marginal if

$$\mathcal{RG}_\ell(I[L]) \in \mathscr{O}^+(\mathcal{E}, \Omega^*(\Delta^m))[[\hbar]] \otimes \mathbb{C}[\log \ell].$$

Again, one can define the QME the same way as in the compact case, allowing us to define the set of theories $\mathscr{T}^{(\infty)}(\mathcal{E}, Q) \subset \widetilde{\mathscr{T}}^{(\infty)}(\mathcal{E}, Q)$ as the sub simplicial set of pre-theories where a $m-$simplex is a $m-$simplex in $\widetilde{\mathscr{R}}^{(\infty)}(\mathcal{E}, Q)$, described by a collection of effective interactions $\{I[L]\} \in \mathscr{O}^+(\mathcal{E}, \Omega^*(\Delta^m))[[\hbar]]$, such that each $I[L]$ satisfies the scale L QME. We will define the sets $\mathscr{R}^{(\infty)}(\mathcal{E}, Q)$ and $\mathscr{M}^{(\infty)}(\mathcal{E}, Q)$ of relevant and marginal theories as:

$$\mathscr{R}^{(\infty)}(\mathcal{E}, Q) = \widetilde{\mathscr{R}}^{(\infty)}(\mathcal{E}, Q) \cap \mathscr{T}^{(\infty)}(\mathcal{E}, Q)$$

$$\mathscr{M}^{(\infty)}(\mathcal{E}, Q) = \widetilde{\mathscr{M}}^{(\infty)}(\mathcal{E}, Q) \cap \mathscr{T}^{(\infty)}(\mathcal{E}, Q)$$

Again, given an $m-$simplex $\{I[L]\}$ in the space $\mathscr{T}^{(n)}(\mathcal{E}, Q)$, there is an obstruction $O_{n+1}(\{I[L]\}) \in \mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^m))$ to lifting $\{I[L]\}$ to a $m-$simplex of $\mathscr{T}^{(n+1)}(\mathcal{E}, Q)$. This obstruction is closed, $QO_{n+1} + d_{dR}O_{n+1} + \{I_0, O_{n+1}\} = 0$, and of cohomological degree 1.

Recall that the obstruction $O_{n+1}(\{I[L]\})$ depends on a lift of $\{I[L]\}$ to a pre-theory defined modulo $\hbar^{n+2}$, whereas its cohomology is independent of this lift.

It can be shown that if $\{I[L]\}$ is a relevant theory defined modulo $\hbar^{n+1}$, and we lift it to a relevant pre-theory defined modulo $\hbar^{n+2}$, the obstruction $O_{n+1}(\{I[L]\})$ associated to this lift is of dimension ≥ 0.

For the marginal case, a similar result holds, where the obstruction is of dimension zero.

From these results, it follows that a lift of an element $\{I[L]\} \in \mathscr{R}^{(n)}(\mathcal{E}, Q)$ to an element of $\mathscr{R}^{(n+1)}(\mathcal{E}, Q)$ is equivalent as giving a local action functional $J \in \mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^m))$ of cohomological degree 0 and dimension ≥ 0, such that it kills the obstruction $O_{n+1}(\{I[L]\})$:

$$(Q + d_{dR})\, J + \{I_0, J\} = O_{n+1}.$$

We get a similar result for marginal theories.

Finally, let us denote by $\mathscr{O}_{loc}(\mathcal{E}, Q)$ the simplicial abelian group whose $m-$simplices are closed, degree 0 elements of $\mathscr{O}_{loc}(\mathcal{E}, \Omega^*(\Delta^m))$. Similarly, denote $\mathscr{O}^{\geq 0}_{loc}(\mathcal{E}, Q)$ and $\mathscr{O}^0_{loc}(\mathcal{E}, Q)$ the simplicial abelian groups whose $m-$simplices are closed degree 0 elements of $\mathscr{O}^{\geq 0}_{loc}(\mathcal{E}, \Omega^*(\Delta^m))$ and $\mathscr{O}^0_{loc}(\mathcal{E}, \Omega^*(\Delta^m))$ respectively.

The obstructions are maps of simplicial sets:

$$\begin{aligned} O_{n+1} &: \mathscr{T}^{(n)}(\mathcal{E}, Q) \longrightarrow \mathscr{O}_{loc}(\mathcal{E}, Q)[1] \\ O_{n+1} &: \mathscr{R}^{(n)}(\mathcal{E}, Q) \longrightarrow \mathscr{O}^{\geq 0}_{loc}(\mathcal{E}, Q)[1] \\ O_{n+1} &: \mathscr{M}^{(n)}(\mathcal{E}, Q) \longrightarrow \mathscr{O}^{0}_{loc}(\mathcal{E}, Q)[1] \end{aligned}$$

And just as for the compact case,

Theorem 4.6.1. *There are homotopy Cartesian diagrams:*

$$\begin{tikzcd} \mathscr{T}^{(n+1)}(\mathcal{E}, Q) \arrow[r] \arrow[d] & \mathscr{T}^{(n)}(\mathcal{E}, Q) \arrow[d, "O_{n+1}"] \\ 0 \arrow[r] & \mathscr{O}_{loc}(\mathcal{E}, Q)[1] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{R}^{(n+1)}(\mathcal{E}, Q) \arrow[r] \arrow[d] & \mathscr{R}^{(n)}(\mathcal{E}, Q) \arrow[d, "O_{n+1}"] \\ 0 \arrow[r] & \mathscr{O}^{\geq 0}_{loc}(\mathcal{E}, Q)[1] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{M}^{(n+1)}(\mathcal{E}, Q) \arrow[r] \arrow[d] & \mathscr{M}^{(n)}(\mathcal{E}, Q) \arrow[d, "O_{n+1}"] \\ 0 \arrow[r] & \mathscr{O}^{0}_{loc}(\mathcal{E}, Q)[1] \end{tikzcd}$$

Chapter 5

Yang-Mills in $\mathbb{R}^4$

Yang-Mills theory has been of great relevance in physics as well in mathematics. From the physics point of view, it was the first gauge theory discovered and lots of phenomena can be described by a Yang-Mills theory with different structure groups. For instance, 3 of the 4 known fundamental interactions can be accurately described as Yang-Mills theories, namely, electromagnetism can be described with a $U(1)$ Yang-Mills theory, Quantum Chromodynamics by a $SU(3)$ Yang-Mills and electroweak interaction can be modelled with $U(1) \times SU(2)$. In mathematics, Yang-Mills led to a series of developments in areas such as algebraic geometry, topology, calculus of variations etc. The work of Atiyah, Bott, Donaldson and others led to some extremely important results, such as the index theorem or the exotic differentiable structures on $\mathbb{R}^4$. It also summed great relevance to the more general study of mathematical gauge theory, where the main topic of study is the moduli space of connections on a principal $G-$bundle modulo gauge equivalence. It is one of the canonical examples where the interplay between mathematics and physics led to great developments in both of these fields. The usual formulation of Yang-Mills is over an oriented Riemannian 4-manifold (M, g), with semi-simple Lie algebra $\mathfrak{g}$ and fixed invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, . Recall that a semi-simple Lie algebra is the sum of simple Lie algebras, i.e. non-abelian Lie algebras without any non-zero proper ideal. The gauge group is $\mathcal{G} = \mathrm{Maps}(M, G)$, where G is the compact Lie group associated to $\mathfrak{g}$, and the Lie algebra of infinitesimal gauge transformations is $\Omega^0(M) \otimes \mathfrak{g}$, acting on the fields by

$$A \mapsto [A, X] + dX$$

where $X \in \Omega^0(M) \otimes \mathfrak{g}$ and $A \in \Omega^1(M) \otimes \mathfrak{g}$. The pairing on $\Omega^*(M) \otimes \mathfrak{g}$ is given by

$$\langle \omega_1 \otimes E_1, \omega_2 \otimes E_2 \rangle = \int_M \omega_1 \wedge \omega_2 \langle E_1, E_2 \rangle_{\mathfrak{g}}$$

The Yang-Mills action is just $S_{YM}(A) = \langle \star F(A), F(A) \rangle$, where $\star$ denotes the Hodge stars operator $\star : \Omega^2(M) \to \Omega^2(M)$ and $F(A) = dA + A \wedge A$ is the curvature ($\mathfrak{g}-$valued) two-form. We will call this formulation the second-order formulation of Yang-Mills. In this formulation, there is no clear gauge fixing condition that allows us to use the results previously presented. Instead, we will use a different, but equivalent formulation of Yang-Mills.

5.1 Yang-Mills in the BV formalism

The drawback of the second-order formulation is that the quadratic part is second-order, and there is no clear gauge fixing operator which makes the operator D defined in Definition 2.2.12 a generalized Laplacian. Therefore, by performing a change of variables, Costello shows in Chapter 6.3 that one can make an equivalent formulation of Yang-Mills with the quadratic part of the action being first-order. This is the formulation of Yang-Mills we will use to prove renormalizability.

Let $\mathfrak{g}$ be a Lie algebra equipped with an invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ and let $\Omega^i(\mathbb{R}^n)$ denote the Schwartz $i-$forms on $\mathbb{R}^n$. The first-order formulation has two fields: a connection $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$, which under the action of the infinitesimal Lie algebra of gauge symmetries $\Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$ transforms as

$A \mapsto [X, A] + dX$, where $X \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$, and a self-dual two form $B \in \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$, which transforms as $B \mapsto [X, B]$.

The action reads: $S(A, B) = \langle F(A), B\rangle + c\langle B, B\rangle$, where $F(A)$ is the curvature and $\langle -, -\rangle$ is the inner product on $\Omega^*(\mathbb{R}^4) \otimes \mathfrak{g}$ defined above.

We want to integrate over the quotient space $\Omega^1(\mathbb{R}^4) \otimes \mathfrak{g} \oplus \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$ over $\mathscr{G}$, where $\mathscr{G} = \mathrm{Maps}(\mathbb{R}^4, G)$ is the gauge group.

Applying the BV formalism described before we arrive at the following extended space of fields $\mathcal{E}$:

$$
\begin{array}{rr}
\Omega^0(\mathbb{R}^4) \otimes \mathfrak{g} & \text{ghosts, degree } -1 \\
\Omega^1(\mathbb{R}^4) \otimes \mathfrak{g} \oplus \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g} & \text{fields, degree } 0 \\
\Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g} \oplus \Omega^3(\mathbb{R}^4) \otimes \mathfrak{g} & \text{anti-fields, degree } 1 \\
\Omega^4(\mathbb{R}^4) \otimes \mathfrak{g} & \text{anti-ghosts, degree } 2
\end{array}
$$

There is a natural odd symplectic structure, where ghosts pair with anti-ghosts and fields with anti-fields.

Let us denote by $X \in \Omega^0(\mathbb{R}^4) \otimes \mathfrak{g}$ a ghost variable, $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$ and $B \in \Omega^2_+(\mathbb{R}^4) \otimes \mathfrak{g}$ the field variables, $A^\vee \in \Omega^3(\mathbb{R}^4) \otimes \mathfrak{g}$ and $B^\vee \in \Omega^2(\mathbb{R}^4) \otimes \mathfrak{g}$ the anti-field variables, and $X^\vee \in \Omega^4(\mathbb{R}^4) \otimes \mathfrak{g}$ the anti-ghosts.

The extended action reads:

$$
S_{FO}(c) = \frac{1}{2}\langle [X, X], X^\vee\rangle + \langle [X, B], B^\vee\rangle + \langle dX, A^\vee\rangle + \langle [X, A], A^\vee\rangle + \langle F(A), B^\vee\rangle + c\langle B, B\rangle. \tag{5.1}
$$

Where the first term encodes the Lie bracket on the Lie algebra of infinitesimal symmetries, the next three encode the action of this Lie algebra on the space of fields, and the last two are the original first-order Yang-Mills action. The auxiliary dga manifold used in previous definitions comes into play when working with Yang-Mills theory, where it is convenient to describe the BV space of fields with the auxiliary dga:

$$
\begin{tikzcd}
\mathscr{Y}^0 & \mathscr{Y}^1 & \mathscr{Y}^2 & \mathscr{Y}^3 \\
\Omega^0(\mathbb{R}^4) \arrow[r, "d"] & \Omega^1(\mathbb{R}^4) \arrow[r, "d_+"] & \Omega^2_+(\mathbb{R}^4) & \\
& \oplus & \oplus & \\
& \Omega^2_+(\mathbb{R}^4) \arrow[r, "d"] \arrow[uur, "2c\,\mathrm{Id}"] & \Omega^3(\mathbb{R}^4) \arrow[r, "d"] & \Omega^4(\mathbb{R}^4)
\end{tikzcd}
$$

where $d_+ : \Omega^1(\mathbb{R}^4) \longrightarrow \Omega^2_+(\mathbb{R}^4)$ denotes the de Rham differential composed with the projection onto the self-dual subspace of $\Omega^2(\mathbb{R}^4)$.

$\mathscr{Y} = \mathscr{Y}_0 \oplus \mathscr{Y}_1 \oplus \mathscr{Y}_2 \oplus \mathscr{Y}_3$ is a dga algebra, which has a trace $\mathrm{Tr} : \mathscr{Y} \to \mathbb{R}$ of degree -3 defined by

$$
\mathrm{Tr}(a) = \int_{\mathbb{R}^4} a
$$

if $a \in \mathscr{Y}^3 = \Omega^4(\mathbb{R}^4)$ and 0 otherwise. Identifying $\mathscr{Y} \otimes \mathfrak{g}[1]$ with the BV space of fields, we note that the trace map gives $\mathscr{Y} \otimes \mathfrak{g}[1]$ an odd symplectic pairing, given by

$$
\langle a \otimes E, a' \otimes E'\rangle = \mathrm{Tr}(aa')\langle E, E'\rangle_{\mathfrak{g}}.
$$

Thus, we are in the setting of the Batalin-Vilkovisky formalism. Applying the procedure presented before, we arrive at the following action for the first-order formulation of the Yang-Mills theory.

Lemma 5.1.1. *The Batalin-Vilkovisky action for the first-order Yang-Mills theory is a Chern-Simons-type action*

$$S_{FO}(a\otimes E)=\frac{1}{2}\langle a\otimes E,Qa\otimes E\rangle+\frac{1}{6}\langle a\otimes E,[a\otimes E,a\otimes E]\rangle$$

Proof. This is a direct computation. In local coordinates $(x_1,\dots,x_n)$, given a basis T_a for the Lie algebra $\mathfrak{g}$ and using Einstein summation convention, we find that $X=c^aT_a$, $A=A^aT_a$, $B=B^aT_a$, $A^\vee=A^{\vee a}T_a$, $B^\vee=B^{\vee a}T_a$ and $X^\vee=X^{\vee a}T_a$, where

$$A^a=A^a_\mu dx^\mu \qquad B^a=\frac{1}{2}B^a_{\mu\nu}dx^\mu\wedge dx^\nu \qquad B^{\vee a}=\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}B^{a\mu\nu}dx^\lambda\wedge dx^\sigma$$
$$A^{\vee a}=\frac{1}{6}\epsilon_{\mu\nu\lambda\sigma}A^{a\mu}dx^\nu\wedge dx^\lambda\wedge dx^\sigma \qquad X^{\vee a}=\frac{1}{24}\epsilon_{\mu\nu\lambda\sigma}dx^\mu\wedge dx^\nu\wedge dx^\lambda\wedge dx^\sigma$$

For the quadratic part,

$$\frac{1}{2}\langle e,de\rangle=\frac{1}{2}\langle X,dA^\vee\rangle+\frac{1}{2}\langle A,dB^\vee\rangle+\frac{1}{2}\langle B,dA\rangle+\frac{1}{2}\langle B,2cB\rangle$$

plus permutations. Firstly

$$\langle X,dA^\vee\rangle=\int X^adA^{\vee b}\langle T_a,T_b\rangle_{\mathfrak{g}}=-\int dA^{\vee b}X^a\langle T_a,T_b\rangle_{\mathfrak{g}}=\int d\left(A^{\vee b}X^a\right)\langle T_a,T_b\rangle_{\mathfrak{g}}+\int A^{\vee b}dX^a\langle T_a,T_b\rangle_{\mathfrak{g}},$$

where we are having into account both the cohomological degree and the form degree. Therefore $\langle X,dA^\vee\rangle=\langle A^\vee,X\rangle$ up to total derivatives. The same argument holds for $\langle dA,B\rangle$.

For $\langle B^\vee,dA\rangle=\int B^{\vee a}dA^b\langle T_a,T_b\rangle_{\mathfrak{g}}=-\int dA^bB^{\vee a}\langle T_a,T_b\rangle_{\mathfrak{g}}=\int d\left(A^bB^{\vee a}\right)\langle T_a,T_b\rangle_{\mathfrak{g}}-\int A^bdB^{\vee a}\langle T_a,T_b\rangle_{\mathfrak{g}}$. We see that $\langle B^\vee,dA\rangle=-\langle A,dB^\vee\rangle$ up to total derivatives.

The quadratic part is therefore, $\langle dX,A^\vee\rangle+\langle d_+A,B\rangle+c\langle B.B\rangle$.

For the cubic term, we have

$$\langle X^\vee,[X,X]\rangle+\langle B,[A,A]\rangle+\langle B^\vee,[A,A]\rangle+\langle X,[A,A^\vee]\rangle+\langle X,[B,B]\rangle+\langle X,[B,B^\vee]\rangle+\langle X,[B^\vee,B^\vee]\rangle$$

plus permutations.

Note that

$$\langle X,[B,B]\rangle=\int X^aB^bB^c\langle T_a,[T_b,T_c]\rangle_{\mathfrak{g}}=\int X^aB^cB^b\langle T_a,[T_c,T_b]\rangle_{\mathfrak{g}}=-\int X^aB^bB^c\langle T_a,[T_b,T_c]\rangle_{\mathfrak{g}}=0.$$

Also, the other permutations $\langle B,[X,B]\rangle=-\langle B,[B,X]\rangle$, therefore the contribution of this term and its permutations is zero.

A similar argument shows that the contributions from $\langle X,[B^\vee,B^\vee]\rangle$ and $\langle B^\vee,[A,A]\rangle$ is zero.

Now for terms such as $\langle X^\vee,[X,X]\rangle$, they all three give the same contribution

$$\int X^{\vee a}X^bX^c\langle T_a,[T_b,T_c]\rangle_{\mathfrak{g}}=-\int X^bX^{\vee a}X^c\langle T_a,[T_b,T_c]\rangle_{\mathfrak{g}}=\int X^bX^{\vee a}X^c\langle T_b,[T_a,T_c]\rangle_{\mathfrak{g}}=\langle X,[X^\vee,X]\rangle.$$

A similar argument shows that $\langle X,[A^\vee,A]\rangle$, $\langle B,[A,A]\rangle$ and $\langle X,[B^\vee,B]\rangle$ have the same value for all its permutations. Dividing by 6 and counting all the permutations, we see that the interaction part is: $\frac{1}{2}\langle X^\vee,[X,X]\rangle+\langle A^\vee,[X,A]\rangle+\langle B^\vee,[A,A]\rangle+\langle B^\vee,[X,B]\rangle$. □

The gauge fixing for our first-order Yang-Mills theory $Q^{GF}:\mathscr{Y}\to\mathscr{Y}$ will be defined by the following diagram:

$$\begin{array}{ccccc}
\Omega^0(\mathbb{R}^4) & \xleftarrow[d^*]{} & \Omega^1(\mathbb{R}^4) & \xleftarrow[2d^*]{} & \Omega^2_+(\mathbb{R}^4) & & \\
 & & \oplus & & \oplus & & \\
 & & \Omega^2_+(\mathbb{R}^4) & \xleftarrow[2d^*_+]{} & \Omega^3(\mathbb{R}^4) & \xleftarrow[d^*]{} & \Omega^4(\mathbb{R}^4)
\end{array}$$

The Laplacian-type operator will be $D = [Q, Q^{GF}]$, which is clearly of order two. Note that D can be decomposed as $D = D' + 4cD''$, where both D' and D'' are independent of the coupling constant. D' is the Laplacian in the space of forms $dd^* + d^*d$, while D'' is given by the following diagram:

$$\begin{array}{cccc} \Omega^0(\mathbb{R}^4) & \Omega^1(\mathbb{R}^4) & \Omega^2_+(\mathbb{R}^4) & \\ & \uparrow{\scriptstyle d^*} & \uparrow{\scriptstyle d^*_+} & \\ & \Omega^2_+(\mathbb{R}^4) & \Omega^3(\mathbb{R}^4) & \Omega^4(\mathbb{R}^4) \end{array}$$

They satisfy

$$[D', D''] = 0 \quad \text{and} \quad (D'')^2 = 0$$

Both of these properties assure that it satisfies the technical conditions from Definition 4.2.1, and therefore we know that there is a propagator given by the integral of a heat kernel, satisfying various properties.

5.2 Renormalizability

In this section, we will sketch Costello's proof of the following theorem

Theorem 5.2.1. *Pure Yang-Mills theory on $\mathbb{R}^4$ or on any compact four manifold with flat metric, and coefficients in any semi-simple Lie algebra $\mathfrak{g}$ is perturbatively renormalizable.*

First, split the classical action S_{FO} in quadratic and interacting terms:

$$S_{FO}(a \otimes E) = \langle a \otimes E, Qa \otimes E \rangle + I^{(0)}(a \otimes E),$$

The interaction term $I^{(0)} \in \mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])$ is cubic, of cohomological degree zero, and satisfies the classical master equation: $QI^{(0)} + \frac{1}{2}\{I^{(0)}, I^{(0)}\} = 0$.

At the classical level, Yang-Mills is conformally invariant, which means that $I^{(0)} \in \mathscr{M}^{(0)}$. Our goal is to classify all lifts of $I^{(0)}$ to elements of $\mathscr{R}^{(\infty)}$.

Define the complex

$$\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}, Q + \{I^{(0)}, -\}\right)$$

of translational invariant local action functionals, with differential $Q + \{I^{(0)}, -\}$. Let

$$H^{i,j}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right)$$

denote the cohomology of this complex, in cohomological degree i and scaling dimension j.

Suppose we have already lifted our theory to an element $I^{(n)} \in \mathscr{R}^{(n)}$ of relevant theories defined modulo $\hbar^{n+1}$. In Chapter 4.6, we saw that the obstruction to lifting $I^{(n)}$ to $\mathscr{R}^{(n+1)}$ is an element of

$$H^{1,\geq 0}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right).$$

If the obstruction vanishes, the moduli space of lifts up to equivalence is a quotient of $H^{0,\geq 0}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right)$ by some action of the space $H^{-1,\geq 0}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right)$, which is thought of as the space of symmetries. If the latter also vanishes the space of lifts up to equivalence will be parametrized by $H^{0,\geq 0}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right)$.

Therefore, renormalizability of Yang-Mills in $\mathbb{R}^4$ reduces to the study of the cohomology groups $H^{i,j}\left(\mathscr{O}_{loc}(\mathscr{Y} \otimes \mathfrak{g}[1])^{\mathbb{R}^4}\right)$ for $i = -1, 0, 1$ and $j \geq 0$. In fact, as the classical theory is also $SO(4)-$invariant, we will restrict ourselves to study quantizations which are also $SO(4)-$invariant.

Theorem 5.2.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. For any non-zero value of the coupling constant c, there are natural isomorphisms*

$$H^{i,0}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)\cong\begin{cases}0 & \text{if } i<0\\ H^0\left(\mathfrak{g},\mathrm{Sym}^2\mathfrak{g}\right) & \text{if } i=0\\ H^5(\mathfrak{g}) & \text{if } i=1\end{cases}$$

Furthermore,

$$H^{i,j}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)\cong 0 \text{ if } j>0 \text{ and } i\leq 2$$

We see that the cohomology groups $H^{0,j}\left(\mathscr{O}_{loc}(\mathscr{Y}\otimes\mathfrak{g}[1])^{\mathbb{R}^4\ltimes SO(4)}\right)$ vanish for $j>0$, which implies that any relevant lift is equivalent to a marginal lift. The potential obstructions to constructing a marginal lift lie in the cohomology group $H^5(\mathfrak{g})$. Unluckily, this group is non-zero if the semi-simple Lie algebra contains a factor of $\mathfrak{su}(n)$ for $n\geq 3$.

The key idea is to note that the classical Yang-Mills action is invariant under another symmetry, so we will restrict ourselves to quantizations which also have this symmetry.

Let $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$ be the decomposition of $\mathfrak{g}$ into simple factors, and let $H\subset\mathrm{Out}\mathfrak{g}$ be the subset of outer automorphisms, i.e. automorphisms that don't come from conjugation, which respect the decomposition. $I^{(0)}$ is $H-$invariant and if a quantization $I^{(n)}\in\mathscr{M}^{(\infty)}$ is $H-$invariant then, the obstruction $O_{n+1}(I^{(n)})$ will also be $H-$invariant. We have the following result.

Lemma 5.2.3. *For any semi-simple Lie algebra $\mathfrak{g}$,*

$$H^5(\mathfrak{g})^H=0.$$

Thus, if $I^{(n)}$ is $H-$invariant, the obstruction $O_{n+1}(I^{(n)})$ vanishes.

With this, it is concluded that renormalizable quantizations of pure Yang-Mills theory on $\mathbb{R}^4$ exist at any level since the obstructions vanish at every level. The set of renormalizable quantizations of pure Yang-Mills is characterized by the following corollary.

Corollary 5.2.3.1. *Let $\mathscr{M}^{(\infty)}_{YM}$ and $\mathscr{R}^{(\infty)}_{YM}$ denote the sub-simplicial set of marginal (respectively, relevant) theories which coincide at the classical level with Yang-Mills theory. The inclusion*

$$\mathscr{M}^{(\infty)}_{YM}\hookrightarrow\mathscr{R}^{(\infty)}_{YM}$$

is an isomorphism on π_0 where π_0 denotes connected components. In particular, there is a (non-canonical) bijection

$$\pi_0\left(\mathscr{M}^{(\infty)}_{YM}\right)\cong H^0\left(\mathfrak{g},\mathrm{Sym}^2\mathfrak{g}\right)\otimes\hbar\mathbb{R}[[\hbar]].$$

Thus, the set of renormalizable quantizations of pure Yang-Mills is the set of deformations of the chosen pairing on $\mathfrak{g}$ to a symmetric invariant pairing

$$\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}[[\hbar]]$$

which, modulo $\hbar$, is the original pairing.

Furthermore, Costello continues and proves that these quantizations are *universal*, that is, every quantization of Yang-Mills is equivalent, in the low-energy limit. We see that the proof of the Renormalizability of Yang-Mills didn't rely on any Feynman graph manipulation as is usual. The key result is the fortuitous vanishing of the cohomology group $H^5(\mathfrak{su}(3))^{\mathrm{Out}(\mathfrak{su}(3))}$. As Costello points out, a more direct or intuitive proof is desirable.

Conclusion

In this work we have introduced a novel formulation of quantum field theory, presenting the main results and reproducing some well-known results in the physics literature. Far from being far-fetched, this formalism has proven to be successful for obtaining new mathematical structures [CLL15][GGW16],[LL16], and also for obtaining physics results from the mathematical point of view [EWY17] for example.

It also led to the construction of the notion of factorization algebra, a very rich concept for which two volumes have been dedicated[CG16],[CG17].

It remains to translate rigorously all these results to Lorentzian signature. In these lines, we remark that the results, as usual in physics, are obtained as infinitesimal deformations of the classical theory. A complete, rigorous formulation of non-perturbative QFT is desirable.

Bibliography

[Cos11] Kevin Costello, *Renormalization and Effective Field Theory*, volume 170 of of Mathematical Surveys and Monographs. AMS, 2011

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quauntum Field Theory*, volume I. Number 31 in New Mathematical Monographs. Cambridge University Press, 2016.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quauntum Field Theory*, volume II.

[EWY17] Chris Elliott, Brian Williams, Philsang Yoo, *Asymptotic Freedom in the BV Formalism*, arXiv:1702.05973.

[Pas17] James Pascaleff, *Perturbative Quantization and Master Equation.* Notes from reading group “Higher Genus B-Model and Quantization of BCOV Theory". Institute for Advanced Study, 2017.

[Fol08] Gerald B. Folland. *Quantum Field Theory. A tourist guide for mathematicians.*, volume 149 of Mathematical Surveys and Monographs. AMS, 2008.

[BGV92] Nicole Berline, Ezra Getzler, Michèle Vergne. *Heat Kernels and Dirac Operators.* Principles of mathematical sciences 298, Springer Verlag, 1992.

[Fre91] Daniel S. Freed, *Five Lectures on Supersymmetry.* American Mathematical Society, 1991.

[BM94] John C. Baez, Javier P. Muniain.*Gauge Fields, Knots and Gravity.* World Scientific Publishing Company, 1994.

[Trè67] François Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

[Seg99] Graeme Segal, *Notes on quantum field theory.*

[Haa92] Rudolph Haag, *Local quantum physics.* Springer, 1992.

[CLL15] Kwokwai Chan, Naichung Conan Leung, and Qin Li. *A mathematical foundation of Rozansky-Witten theory.* arXiv preprint arXiv:1502.03510, 2015.

[GGW16] Vassily Gorbounov, Owen Gwilliam, and Brian Williams. *Chiral differential operators via Batalin- Vilkovisky quantization.* arXiv preprint arXiv:1610.09657, 2016.

[LL16] Qin Li and Si Li. *On the B-twisted topological sigma model and Calabi-Yau geometry.* J. Differential Geom., 102(3):409–484, 2016.